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A topological approach to modal logics

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1 Introduction

Studies of interpretations using topological spaces in modal logics have been started long time before. Tarski interpreted modal operator $\Box$ by interior operator of a topological space, and show that it gives a very natural semantics in S4. Recently, several studies related to them have appeared in the field under the names, *modal logics of space* and *spatial reasoning*. Among others, it is shown that *Cantor space* and *Euclid space* is complete in some modal logics. These studies make clear whether an well-known topological space is complete in some modal logic, or not.

In ordinary life we consider that two or more things with very near certain concept can make us think these things might be equivalent, parhaps might be equal. Thus we need to formalise the concept of "closeness".

Any topological space has concept of neighbourhood. So it will be interesting to see how the concept of neighbourhood in topological sense is related to some logical properties.

In this study, we will discuss two topological semantics for modal logics. The first is called *topological model* which interprets modal operator $\Box$ by interior operator, and the second is called *subset model* two modal operators using points and open subsets.

2 Topological models

For modal logic S4, topological models is used for the semantics. A topological model consists of a triple $\langle X, O, \nu \rangle$ with a set X, a system of open sets O(on X), and a valuation ν. In a topological model the modal operator $\Box$ is interpreted as interior operator.

A topo-bisimulation $\rightleftharpoons$ is a relation between two models $\langle X, O, \nu \rangle$ and $\langle X', O', \nu' \rangle$. A topo-bisimulation is defined as follows. If $x \rightleftharpoons x'$

(i) $x \in \nu(P) \Leftrightarrow x' \in \nu'(P)$,

(ii) if $x \in u \in O$, then there is $u' \in O'$ such that $x' \in u'$ and for an arbitrary $y' \in u'$ and there exists $y \in u$ such that $y \rightleftharpoons y'$,

(iii) if $x' \in u' \in O'$, then there is $u \in O$ such that $x \in u$ and for an arbitrary $y \in u$ and there exists $y' \in u'$ such that $y \rightleftharpoons y'$.

The fact which S4 is complete for Cantor space and Real line is proved by Aiello[1] by using topo-bisimulaiton. We note that the completeness of S4 with respect to the class of all topological spaces with metric and separable dense-in-itself in topological sense was proved by McKinsey and Tarski by using algebraic proof.

3 Subset space logic and completeness

Subset space logic is described by using bi-modal language with modal operators K and $\Box$, which is finitely axiomatizable. The semantics uses a model $\langle X, O, \nu \rangle$ with a set X, the class O of subsets of X, and valuation ν, it defined as follows.

Let $M = \langle X, O, \nu \rangle$ be a subset model. if $x \in u \in O$,

$x, u \models P \Leftrightarrow x \in \nu(P)$ for all $P \in Prop$
$x, u \models \varphi \wedge \chi \Leftrightarrow x, u \models \varphi$ and $x, u \models \chi$
$x, u \models \neg\varphi \Leftrightarrow x, u \not\models \varphi$
$x, u \models \Box\varphi \Leftrightarrow$ for all $v \in O$, if $v \subseteq u$ then $x, v \models \varphi$
$x, u \models K\varphi \Leftrightarrow$ for all $y \in u,\ y, u \models \varphi$

Which implies that they behave essentially in the sameway as topological models.

An interpretation of $x, u \models \Box\varphi$ says that whatever it changes a view at the field u containing x, the sentence φ is always true at a point x.

An interpretation of $x, u \models K\varphi$ says that the sentence φ is always true at any point of u.

It is shown that, subset space logic is complete with respect to this semantics, but the *Truth lemma* which is used this proof has a bit special.

4 Subset space logic and its decidability

The subset space logic is complete with respect to the subset modess, but we cannot show the finite model property with respect to them [2]. So to get the finite modes property we need a class of models larger than the class of all subset models. Cross axiom models are introduced for this purpose, which is larger than the class of all subset models, defined as bellow.

$\langle J, \stackrel{L}{\rightarrow}, \stackrel{\Diamond}{\rightarrow} \rangle$ is a cross axiom model if J is a set, $\stackrel{L}{\rightarrow}$ is a non-empty equivalence relation on J and $\stackrel{\Diamond}{\rightarrow}$ is a non-empty quasi-order relation on J and they satisfies following conditions:

For all $i, j, k \in J$,

$$\text{if } i \stackrel{\Diamond}{\rightarrow} j \stackrel{L}{\rightarrow} k, \text{ then there is } l \in J \text{ such that } i \stackrel{L}{\rightarrow} l \stackrel{\Diamond}{\rightarrow} k$$

and ν is a valuation with the following conditions:

For arbitrary $i, j \in J$ and all $P \in Prop$,

$$\text{if } i \stackrel{\Diamond}{\rightarrow} j, \text{ then } i \in \nu(P) \Leftrightarrow j \in \nu(P).$$

Then, we can show the finite model property of subset space logic with respect to the cross axiom models. To show this we apply filtration to the canonical model which becomes a cross axiom model. Therefore subset space logic is shown to the decidable.

5 Further considerations

We try to make clear a connection with between topo-bisimulation and the system of opens of given two topological models.

Let $g : O \to O'$ be surjective. For an arbitrary $y \in X$, we define that

$A := \{P \in Prop \mid y \in \nu(P)\}$,
$D := \{v \in O \mid y \in v\}$

and

$$E(y) := \bigcap_{P \in A} \nu'(P) \cap \bigcap_{P \notin A} \nu'(P)^c \cap \bigcap_{v \in D} g(v) \cap \bigcap_{v \notin D} g(v)^c$$

Similarly, for an arbitrary $y' \in X'$, we define that

$A' := \{P \in Prop \mid y' \in \nu'(P)\}$,
$D' := \{v \in O \mid y' \in g(v)\}$

$$E'(y') := \bigcap_{P \in A'} \nu(P) \cap \bigcap_{P \notin A'} \nu(P)^c \cap \bigcap_{v \in D'} v \cap \bigcap_{v \notin D'} v^c.$$

We define $x \rightleftharpoons x'$, as the relation between X and X', by the condition that

for all $P \in Prop \;\; x \in \nu(P) \Leftrightarrow x' \in \nu'(P)$, and
for an arbitrary $v \in O \;\; x \in v \Leftrightarrow x' \in g(v)$.

Then, we can prove the next proposition.

Let $g : O \to O'$ be a surjective map. If both $E(y)$ and $E'(y')$ are nonempty for all $y \in X$ and all $y' \in X'$, then $\rightleftharpoons$ becomes a total topo-bisimulation.

This makes clear that the concept of bi-simulation is determined by points of intersections among open sets and images of valuation.

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