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Investigations into Intuitionistic and Other Negations

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Chapter 1

Introduction

1.1 Background of the thesis

Around a century ago, formal logic was undergoing its formative years. Many important logical systems have their origins in the first three decades of the century, including many-valued logic, modal logic and intuitionistic logic. There were many competing views of what the correct logic should be. Enquiries concerning the interrelationship of such logics also grew along with these developments.

Two of the well-known examples of such enquiries would be the so-called *negative translations* [73, 52, 48, 76] of classical logic into intuitionistic logic, as well as the translation of intuitionistic logic into the modal logic S4 [51, 81]. It was perhaps not coincidental that both investigations concerned intuitionistic logic; the system has since been known for its close relationship with many logics, the whole framework of *substructural logics* [37, 46, 107] being one prime example. Intuitionistic logic is therefore established to be one of the reference points when considering various logical systems.

When we turn our attention to intuitionistic logic itself, we are soon made aware of the essential rôle negation plays in its characterisation. One may, for example, understand the difference between classical and intuitionistic logic as that of whether it allows a *sui generis* status to a double negation of a proposition. In classical logic, a double negation of a proposition is equivalent to the proposition itself, whereas in intuitionistic logic, a double negation of a proposition is in general strictly weaker than the proposition. This may be contrasted with a triple negation, which is equivalent to a single negation even in intuitionistic logic, hence no further complications arise. This difference between classical and intuitionistic negations is essentially capitalised in the above-mentioned negative translations, where atomic formulae are replaced with their double negations.

Negation therefore becomes a pivotal point in the analysis of intuitionistic logic. And here the subject can be tackled from many different angles. One can investigate the properties of negation comparatively, with references to other logics such as classical logic. It is also possible to take a more revisionary approach; one may for example look at intuitionistic-type logics with stronger or weaker negations. Or one can introduce a new negation to intuitionistic logic that is of fundamentally different character. There have been countless contributions from these perspectives in the investigation of intuitionistic logic, but there still remain many aspects which are left unexplored. A fuller understanding of intuitionistic negation and its variants has the potential to lead to fruitful applications in mathematics, philosophy, linguistics and computer science.

The present thesis therefore aims to fill some of these unexplored aspects of intuitionistic and related logics, with a particular emphasis on negation. Although each topic we shall treat in this thesis is mostly independent of others, the topics do share a similar motif. It may be explained as the focus on the rôle of logical principles (or axioms) in logical systems. Such principles often have decisive importance in judging whether a logic is acceptable from a philosophical point of view. Although this thesis will not make extensive discussions on philosophical matters, our investigation should be of non-negligible value in this regard as well.

1.2 Structure of the thesis

We have the following structure for the present thesis.

In Chapter 2, I shall introduce the proof theory (Hilbert system, sequent calculus) and semantics (Kripke semantics, Beth semantics). We shall also discuss the notion of negation in intuitionistic logic, and describe negative translations.

In Chapter 3, we investigate the relationship between classical and intuitionistic logic. Our main concern is the problem posed by Ishihara [68], namely what class of atomic instances of the *law of excluded middle* is sufficient to conserve classical theorems in intuitionistic logic. Such a class helps us to better understand certain aspects of the relationship between classical and intuitionistic negation than is provided by negative translations. In particular, in some cases we can identify classical propositions which are also intuitionistically derivable, by inspecting their shapes. One solution to the above problem was offered by Ishii [69]. This solution was obtained through Glivenko's theorem, and is in general not comparable to the class obtained by Ishihara [69]. In the first part of the chapter, we shall note that the class given by the solution has a room for improvement, by considering weaker logical principles than the law of excluded middle. Such a direction in addition will allow us to extend the range of analysis to logics weaker than intuitionistic logics. This is particularly important, as such logics are often contended to be the alternative constructive logics to intuitionistic logic. This chapter also discusses expansion of this result to predicate logic (with additional non-constructive axioms), as well as to minimal logic by considering an additional class of atomic instances of double negation of *ex falso quodlibet*. Overall, the results of this chapter shall significantly broaden our understanding of the relationship between non-constructive principles and constructive logics. The contents of this chapter are based on the author's work [88].

In Chapter 4, we shall discuss the framework of *subminimal* negation [24, 25] by Almudena Colacito, Dick de Jongh and Ana Lucia Vargas, which gives a general method to obtain logics with weaker negation than minimal logic. A motivation in considering such negations can be argued as the fact that they are generally paraconsistent, in the sense that not all, and desirably not any counterintuitive, propositions follow from a contradiction. Thus subminimal negation is purported to be a framework that makes constructivity and paraconsistency compatible. Subminimal negation will be utilised to analyse the logic **SUBMIN** [126, 127] of Dimiter Vakarelov. The logic is identical to intuitionistic logic, except for the negation. The sense of the negation may be interpreted as that one cannot assert a negation unless he has in advance another negation at hand. In other words, a negation can be asserted only relatively to other negations. Such a negation can be helpful if one is sceptical of the demonstrability of the truth of a negation except via another negation. In the first section of this chapter, we shall formulate a system $\mathbf{An^{-}PC}$ corresponding to **SUBMIN** in the framework of subminimal negation. Then the Kripke semantics of the two approaches are compared and translations between them are defined. As a result of these enquiries, we shall acquire a clearer view of the interrelationship between the two frameworks. This is followed by the formulation of a sequent calculus corresponding to $\mathbf{An^{-}PC}$; it will be shown that the calculus enjoys the admissibility of the Cut rule, and consequently it satisfies properties like decidability and Craig Interpolation Property. This enquiry complements the knowledge about the proof-theoretic properties of Vakarelov's logic, which was not very-well understood. Finally, we introduce a new countable class of logics with subminimal negation, which shall be of interest to the study of the structure of logics with subminimal negation. The contents of this chapter are based on the author's publication [91].

In Chapter 5, we shall study negations which support inferences underivable with the intuitionistic negation. The benefit of such negations is that they provide more flexibility to intuitionistic systems. We first look at empirical negation, which is defined by the negation at the root of a kripke model. Informally, the root is understood to represent the present moment. Empirical negation is then to be viewed as the negation in non-mathematical contexts, in contrast with intuitionistic negation, which is more suited to mathematical contexts. Empirical negation defines a logic called $\mathbf{IPC}^{\sim}$ [29, 30], with respect to which we shall observe that when a different relational semantics of Evert Willem Beth [6] is considered, the resulting logic changes to another system $\mathbf{TCC}_{\omega}$ by Andrei Borisovich Gordienko [54]. This shall point to a the significance of Beth semantics as an alternative semantics to Kripke semantics. We shall then look at the proof-theory of $\mathbf{IPC}^{\sim}$ and $\mathbf{TCC}_{\omega}$. There exists a problem in comparing the pair in that these systems are axiomatised with different rules of inference. To improve this situation, we shall give uniform axiomatisations for the two logics as well as the logic **daC** of Graham Priest [100] and $\mathbf{CC}_{\omega}$ of Richard Sylvan [116]. This also allows to extract frame conditions for these axioms with the Kripke semantics of $\mathbf{CC}_{\omega}$ as the basis. The chapter then concludes with the formulation of labelled sequent calculi [84], extensions of the standard sequent calculi, of the above systems. We shall then prove the admissibility of cut for the calculi, and as a consequence show that the calculi correspond to the axiomatic systems. The contents of this chapter are based on the author's publications [89, 90].

Chapter 6 discusses what happens when empirical negation is seen from the opposite viewpoint. We shall introduce an operator that signifies the notion of actuality in intuitionistic setting. Such an operator is useful when one wishes to assign a privileged status to a world in a Kripke model. Actuality has in the past been considered in the classical setting [26, 28, 55]. In addition, a semantical idea for intuitionistic actuality is sketched in [66], but a proof system was lacking. The first objective of this chapter is to obtain a proof system that becomes sound and complete with the semantics. We shall introduce the kripke semantics described in [66], and then introduce an axiomatic system which we shall call $\mathbf{IPC}^{@}$. We shall show the strong completeness of the logic with the Kripke semantics. This is followed by comparisons with various logical systems, including the system of empirical negation, classical actuality [26], projection operator in fuzzy logic [2], global intuitionistic logic [118]. The contents of this chapter are based on a joint work of the author with Hitoshi Omori [92].

The thesis will be concluded by offering some remarks about the possible future directions and prospectives.

Chapter 2

Intuitionistic logic and the role of negation

2.1 Overview of intuitionistic logic

The Mathematical movement of intuitionism was initiated by the Dutch mathematician Luitzen Egbertus Jan Brouwer (1881-1966)[1]. The central tenet of intuitionism states that mathematics is concerned with mental construction [121, p.4]. In particular mathematical objects are not to be understood as some mind-independent entities; this is for instance illustrated by the following passage of Brouwer's high disciple Arend Heyting (1898-1980):

> The idea of an existence of mathematical entities outside our minds must not enter into the proofs (...). Maybe they would better do avoid completely the words "to exist"; if they continue, nevertheless, to use them, these words would have no other meaning for them than "to be constructed by reason". [61]

Another important viewpoint of intuitionism is that logic is dependent of mathematics, rather than the other way around [16, pp.72–75]. Logic is therefore not to be presupposed before mathematics. Brouwer asserts:

> [T]he function of the logical principles is not to guide arguments concerning experience subtended by mathematical systems, but to describe regularities which are subsequently observed in the language of the arguments. [14]

Consequently we cannot use logic blindly as rules for mathematical activity; we at least have to be aware that such rules turn out to be inadequate upon closer inspection. This point in particular lead Brouwer to arrive at the conclusion that *the law of excluded middle* (LEM), which states that every proposition is either true or false, has to be rejected in mathematics when infinity is involved, on the ground that unsolvable mathematical problem may exist in such cases [14].

LEM is a logical principle that has been long-accepted, and is valid in the standard formalisation of classical logic [96]. This meant that in order to formalise the intuitionistic mathematics one would need another type of formal logic. Brouwer

[1] For a detailed accounts of Intuitionism and Brouwer, c.f. [67, 130, 131].

himself, however, did not proceed to an explicit formalisation of such a system, probably stemming from his above-mentioned view regarding the relationship between logic and mathematics, in addition to his scepticism towards language and formalisation [15, 18]. Therefore the task fell to the hands of other people to formulate a logical system that reflects the insights offered by Brouwer. This has been carried out [129] by Alexsander Nikolaevich Kolmogorov (1903-1987)[73], Ivan Efremovich Orlov (1886-1936?) [97], Valery Ivanovich Glivenko (1897-1940) [50] and A. Heyting [60] albeit with slight differences in the treatment of negation, a point which we shall discuss in more detail later. The details of the resulting *intuitionistic logic*, based essentially on Heyting's formalisation, are explained in the next section.

Concurrently to the formalisation of intuitionistic logic, an informal interpretation for its logical connectives has been explored [72, 61], eventually crystallising into what is now known as the *Brouwer-Heyting-Kolmogorov interpretation* [121], or BHK interpretation for short. In BHK interpretation, the meaning of each proposition is explicated by its proof condition, so that

- A proof of $A \wedge B$ is a pair of a proof of A and a proof of B.
- A proof of $A \vee B$ is either a proof of A or a proof of B.
- A proof of $A \rightarrow B$ is a construction which transforms a proof of A into a proof of B.
- Nothing is regarded as as a proof of $\bot$.

BHK interpretation is not to be taken as the formal semantics for intuitionistic logic; however this interpretation is made precise later by the *realizability interpretation* of Stephen Cole Kleene (1909-1994) [71] and Nels David Nelson (1918-2003) [85]. we shall see in a later section some other semantics for intuitionistic logic.

2.2 Proof theory of intuitionistic logic

We shall now give a formal account for intuitionistic logic, first proof theoretically (i.e. in terms of syntactic derivation of formulae by rules) and then semantically (i.e. in terms of validity of formulae based on assignments of truth). In order to perform this, we need to specify the *language* to be used in the formalisation of logical propositions (*formulae*) as the initial step.

- $p, q, r \ldots$: countable supply of propositional variables.
- $\bot$: a constant.
- $A, B, C, \ldots$: metavariables for formulae.
- $\Gamma, \Delta, \ldots$: metavatiables for sets of formulae.

We shall occasionally call a propositional variable as an *atomic formula*. A formula that is either atomic or $\bot$ will be called a *prime formula*. A non-atomic formula will be called a *compound formula*. There are some possible choices in how to construct a compound formula; in this thesis, we shall mainly use the following two propositional languages $\mathcal{L}_\bot$ and $\mathcal{L}_\neg$, respectively given by:

$$A ::= p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \bot,$$
$$A ::= p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \neg A.$$

In practice, we shall drop parentheses whenever it aids the readability. In $\mathcal{L}_\bot$, we shall use $\neg A$ as an abbreviation for $A \to \bot$. In addition, in both $\mathcal{L}_\bot$ and $\mathcal{L}_\neg$ we shall use the abbreviation $A \leftrightarrow B$ for $(A \to B) \wedge (B \to A)$.

2.2.1 Hilbert-style presentation of intuitionistic logic

There are several alternatives in the types of proof systems for intuitionistic logic, each with distinct advantage. We shall first look at the so-called *Hilbert* system for intuitionistic logic, named after David Hilbert (1862-1943) (though he is not the sole originator of this approach [120, p.57]), which consists of many *axiom (schemata)* and one rule of *Modus Ponens*. We shall call the system **IPC**.

Definition 2.2.1. The below gives the axiomatisation of **IPC**.

- $A \to (B \to A)$
- $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
- $A \to (B \to (A \wedge B))$
- $(A_1 \wedge A_2) \to A_i$
- $A_i \to (A_1 \vee A_2)$
- $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
- $\bot \to A$
- $\dfrac{A \qquad A \to B}{B}$ (MP)

where $i \in \{1, 2\}$. We say a formula A is *provable* (or *derivable*) from a set of *assumptions* Γ (denoted $\Gamma \vdash A$), if there is a finite $\Gamma_0 \subseteq \Gamma$ and a finite list $A_1, \ldots, A_n$ of formulae such that each A_i is either:

- an axiom;
- a member of Γ_0; or
- obtained from A_j and A_k for $j, k < i$ by (MP).

and $A_n \equiv A$. When $\Gamma = \emptyset$, we write $\vdash A$.

When we need to be explicit, we use the notation $\vdash_i$ as well. On the other hand, whenever the context makes it clear, the derivability for other systems will be denoted by $\vdash$ as well.

Occasionally, we shall look at the predicate extension **IQC** of **IPC**. For this purpose we introduce some preliminary notions. First, to the language of **IPC** we add quantifiers $\forall$,$\exists$ and

- countable supply of *variables* $x_1, x_2, \ldots$.
- countable supply of n-ary *functions* $f_1^n, f_2^n, \ldots$ for each n.
- countable supply of n-ary *relations* $R_1^n, R_2^n, \ldots$ for each n.

We shall call an 0-ary function as a *constant* and denote by meta-variables $c_1, c_2 \ldots$. Propositional variables will be identified with 0-ary relations.

A *term*, denoted by $t_1, t_2, \ldots$, is defined by the next clauses.

1. A variable is a term.
2. If $t_1, \ldots, t_n$ are terms, then $f^n(t_1, \ldots, t_n)$ is a term.

Then an *atomic formula* has the form $R^n(t_1, \ldots, t_n)$.
We next define the notion of the *free occurrence* of a variable x in a formula A, by the following clauses.

1. If A is atomic, then any occurrence of x in A is free.
2. If $A \equiv \bot$, then x does not occur free in A.
3. if $A \equiv B \circ C$ where $\circ \in \{\wedge, \vee, \rightarrow\}$, then x occurs free in A if it does so in B or C.
4. if $A \equiv QyB$ where $Q \in \{\forall, \exists\}$, then x occurs free in A if it does so in B and $x \not\equiv y$.

We shall then denote the result of substituting all free occurrences of x in A by t as $A[x/t]$.

Definition 2.2.2. The axiomatisation of **IQC** is defined by adding the next axioms and a rule to that of **IPC**.

- $\forall x(B \rightarrow A) \rightarrow (B \rightarrow \forall y A[x/y])$ where x does not occur free in B, and either y does not occur free in A or $y \equiv x$.
- $\forall x A \rightarrow A[x/t]$
- $A[x/t] \rightarrow \exists x A$
- $\forall x(A \rightarrow B) \rightarrow (\exists y A[x/y] \rightarrow B)$ where x does not occur free in B, and either y does not occur free in A or $y \equiv x$.
- $\dfrac{A}{\forall x A}$ (Gen) where x does not occur free in the assumption.

The notion of deduction in **IQC** also follows that of **IPC**, except that we allow formulae obtained by (Gen) in the sequence along with (MP).

When required, we shall use the notation $\vdash_{qi}$ for the derivability.

Hilbert systems for classical logic is obtained from **IPC** or **IQC** by adding $A \vee \neg A$ as an axiom. We shall henceforth call them **CPC** and **CQC**. We shall use $\vdash_c$ and $\vdash_{qc}$ for the explicit notations for the systems.

The following property holds in **IPC** (**IQC**) and **CPC** (**CQC**).

Theorem 2.2.1 (Deduction Theorem)**.** $\Gamma, A \vdash B$ if and only if $\Gamma \vdash A \rightarrow B$.

Proof. By induction on the depth of deduction. □

2.2.2 Sequent calculus

Another type of proof system is the sequent calculus introduced by Gerhard Gentzen (1909-1945) [47]. In this type of system, the basic unit in an inference is not a formula but a sequent, which is a pair of a finite multiset and a singleton of formulae, expressed as $\Gamma \Rightarrow C$. (note here and afterwards, Γ, Δ and denote multisets when talking about sequent calculi.) Γ and C are to be called the *antecedent* and *succedent* of the sequent, respectively.

Here we shall introduce the sequent calculus **G3i** for intuitionistic logic [120, p.77]. **G3i** is in the style of so-called **G3**-systems, whose characteristic is the lack of *structural rules* which introduce or eliminate 'excessive' formulae. This feature allows the analysis of deductions in sequent calculus much easier.

Definition 2.2.3 (**G3i**)**.**

$$\Gamma, p \Rightarrow p \ (Ax) \qquad\qquad \bot, \Gamma \Rightarrow C \ (\mathrm{L}\bot)$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ (\mathrm{L}\wedge) \qquad\qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\mathrm{R}\wedge)$$

$$\frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ (\mathrm{L}\vee) \qquad\qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_i \vee A_2}\ (\mathrm{R}\vee)$$

$$\frac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C}\ (\mathrm{L}\to) \qquad\qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\mathrm{R}\to)$$

where $i \in \{1, 2\}$.

A proof/derivation/deduction in **G3i** is a tree whose leaves are instances of (Ax) and (L$\bot$), and each node is a sequent obtained by the previous nodes by applying one of the nodes. We shall denote the derivability in **G3i** by $\vdash \Gamma \Rightarrow C$. When necessary, we shall also denote it $\vdash_{g3i}$, in order to make explicit the system we are working with. A *principal* formula in the conclusion of a rule is a formula not in Γ, C; formulae in $\Gamma \cup \{C\}$ are in turn called the *contexts*.

In using the sequent calculus, it is practical to use the next structural rules.

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ (\mathrm{LW}) \qquad\qquad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ (\mathrm{LC})$$

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}\ (\mathrm{Cut})$$

Although these rules are not *derivable* in the sense that one can obtain the conclusions from the corresponding premises within the same derivation, they are still *admissible*, meaning that the derivability of the premises imply the derivability of the conclusion. (essentially this amounts to the constructibility of a separate derivation which proves the conclusion). We refer to [120, Chapter 3,4] for the details. The admissibility of (Cut) in particular has many useful consequences; for example, the following *subformula property* holds in **G3i**.

Theorem 2.2.2 (subformula property)**.** If $\vdash \Gamma \Rightarrow A$, then all the formulae occurring in the derivation occurs in $\Gamma \cup \{A\}$ as a subformula.

Proof. Cf. [120, Proposition 4.2.1]. □

A consequence of subformula property is the decidability of **G3i**: given a sequent $\Gamma \Rightarrow A$, one can decide whether **G3i** derives the sequent.

Corollary 2.2.1 (decidability of **G3i**)**. G3i** is decidable.

Proof. Cf. [120, Theorem 4.2.6]. □

Finally we observe that the derivability in **IPC** and **G3i** coincide, as desired. Hence the two calculi represent the same logic.

Proposition 2.2.1. $\Gamma \vdash A$ in **IPC** iff $\vdash \Gamma \Rightarrow A$ in **G3i**.

Proof. By induction on the derivation of A. For the left-to-right direction, we practically need to appeal to the admissibility of (Cut). For details, cf. [120]. □

2.3 Semantic of intuitionistic logic

Next we shall look at the semantics of intuitionistic logic, which is the other side of the coin in understanding a logic. The archetypical example of a semantics is truth-table for classical logic, according to which each propositional variable is assigned a *truth-value* of either T (true) or F (false). Then the truth-value of compound formulae is determined inductively. If for any assignment of truth-values a formula turns out to be true, then the formula is said to be valid. (Cf. for instance [79, Chapter 2.3] for details.)

For intuitionistic logic, we would wish to capture it similarly by validity stemming from the assignment of truth-values to each formula. This approach however is not directly transferable; as the famous result of Kurt Gödel shows, intuitionistic logic cannot be characterised by many-valued truth-tables [53]. Indeed, such a requirement necessitates the so-called Gödel-Dummett axiom $(A \to B) \vee (B \to A)$ to be provable, Cf. [38, 136].

A modification of truth-table approach is to consider multiple truth assignment at the same time, with each assignment related by some ordering. This type of semantics is called *relational semantics*. Here we shall look at two relational semantics, *Kripke semantics* [74] and *Beth semantics* [6].

2.3.1 Kripke semantics

Let $(W, \leq)$ be an inhabited pre-ordered set, i.e. a reflexive and transitive set with $\exists w(w \in W)$. We may require $\leq$ to be anti-symmetric, in other words a partial ordering, but the difference is inessential. We shall call each element of W as a *world* or a *state*. $(W, \leq)$ gives us an (intuitionistic) *Kripke frame* or simply a *frame*. Frames allow us to construct a (Kripke) *model* $\mathcal{M} = (\mathcal{F}, \mathcal{V})$, where $\mathcal{F}$ is a frame and $\mathcal{V}$ is an assignment which assigns a subset $\mathcal{V}(p) \subseteq W$ for each propositional variable. This assignment has a restriction that

$$w \in \mathcal{V}(p) \text{ and } w \leq w' \text{ implies } w' \in \mathcal{V}(p).$$

That is to say, $\mathcal{V}(p)$ is *upward closed* or *monotone*. $\mathcal{V}$ then uniquely determines the *forcing* of a formula A at a world w, denoted by $w \Vdash A$, with the following clauses.

$$\begin{aligned} w \Vdash p &\Leftrightarrow w \in \mathcal{V}(p). \\ w \Vdash A \wedge B &\Leftrightarrow w \Vdash A \text{ and } w \Vdash B. \\ w \Vdash A \vee B &\Leftrightarrow w \Vdash A \text{ or } w \Vdash B. \\ w \Vdash A \to B &\Leftrightarrow \forall w' \geq w (w' \Vdash A \text{ implies } w' \Vdash B). \\ w \Vdash \bot &\Leftrightarrow \text{ never.} \end{aligned}$$

In certain cases, we may denote the model explicitly, and write $\mathcal{M}, w \Vdash A$ or $(\mathcal{F}, \mathcal{V}), w \Vdash A$.

If for a model $\mathcal{M}$, A is forced at all worlds in W, then we write $\mathcal{M} \vDash A$ and say A is *valid* in $\mathcal{M}$. If $\mathcal{M} \vDash B$ holding for $B \in \Gamma$ implies $\mathcal{M} \vDash A$ then we write $\Gamma \vDash A$, and say A is a *logical consequence* of Γ. If Γ is empty, then we simply write $\vDash A$ and say A is *valid*. We will often restrict $\mathcal{M}$ relative to a fixed frame $\mathcal{F}$; in such a case $\mathcal{F} \vDash A$ will mean A is valid in all $\mathcal{M}$ whose frame is $\mathcal{F}$.

Proposition 2.3.1. If $w \Vdash A$ and $w' \geq w$ then $w' \Vdash A$.

Proof. By induction on the complexity of A. □

It turns out that if a formula is provable (under assumption) in **IPC** then it is valid, and *vice versa*: we call these relations *soundness* and *(strong) completeness*. Since soundness is usually easier to prove than completeness, the word "completeness" is often abused to signify that both relations hold.

Theorem 2.3.1 (completeness of **IPC** with respect to Kripke semantics.)**.** $\Gamma \vdash A$ in **IPC** if and only if $\Gamma \vDash A$.

Proof. Cf. [121, section 2.6]. □

2.3.2 Beth semantics

Beth semantics differs from kripke semantics chiefly in the extra conditions the frame and the model satisfy, and the forcing of disjunction. Informally, the difference may be understood as temporal, of whether one can stay in a world indefinitely (Kripke) or not (Beth). We shall discuss on thi point in Chapter 5. More formally, one can see Beth semantics as a generalisation of Kripke semantics, as the embedding we shall use in the following completeness proof should clarify.

Before introducing Beth semantics, we need some preliminary definition. We shall employ the following notations (taken from [121, Chapter 4.1] with slight alternations) for sequences and related notions.

- $\alpha, \beta, \ldots$: infinite sequences of the form $\langle b_1, b_2, \ldots \rangle$ of natural numbers.
- $\langle \rangle$: the empty sequence.
- $b, b', \ldots$: finite sequences of the form $\langle b_1, \ldots, b_n \rangle$ of natural numbers.
- $b * b'$: b concatenated with b'.
- $lh(b)$: the length of b.

- $b \preceq b'$: $b * b'' = b'$ for some b''.
- $b \prec b'$: $b \preceq b'$ and $b \neq b'$.
- $\bar{\alpha}n$: α's initial segment up to the nth element.
- $\alpha \in b$: b is α's initial segment.

We define a *tree* to be a set T of finite sequences of natural number such that $\langle\rangle \in T$ and $b \in T \wedge b' \prec b \rightarrow b' \in T$. We call each finite sequence in T a *node* and $\langle\rangle$ the *root*. A *successor* of a node b is a node of the form $b * \langle x \rangle$. By *leaves* of T, we mean the nodes of T which do not have a successor, i.e. nodes b such that $\neg\exists x(b * \langle x \rangle) \in T$. A *spread* then is a tree whose nodes always have a successor, i.e. $\forall b \in T \exists x(b * \langle x \rangle \in T)$.

A clarification: whilst $\langle b, b, \ldots \rangle$ denotes an *infinite* sequence consisting just of bs, $\langle b, \ldots, b \rangle$ denotes a *finite* sequence consisting just of bs.

Definition 2.3.1 (Beth model)**.** A *Beth frame* is a pair $\mathcal{F} = (W, \preceq)$ that defines a spread. Then A *Beth model* $\mathcal{M}$ is a pair $(\mathcal{F}, \mathcal{V})$, where $\mathcal{V}$ is an assignment of propositional variables to the nodes such that:

$$b \in \mathcal{V}(p) \Leftrightarrow \forall \alpha \in b \exists m(\bar{\alpha}m \in \mathcal{V}(p)). \text{ [covering]}$$

(The left-to-right direction is trivial, and it is straightforward to see that a covering assignment is monotone.)

The forcing relation $\Vdash A$ for a Beth model is defined by the following clauses.

$$\begin{array}{lcl} b \Vdash p & \Longleftrightarrow & b \in \mathcal{V}(p). \\ b \Vdash A \wedge B & \Longleftrightarrow & b \Vdash A \text{ and } b \Vdash B. \\ b \Vdash A \vee B & \Longleftrightarrow & \forall \alpha \in b \exists n(\bar{\alpha}n \Vdash A \text{ or } \bar{\alpha}n \Vdash B). \\ b \Vdash A \rightarrow B & \Longleftrightarrow & \text{for all } b' \succeq b, \text{ if } b' \Vdash A \text{ then } b' \Vdash B. \\ b \Vdash \bot & \Leftrightarrow & \text{never.} \end{array}$$

Proposition 2.3.2.
(i) $b \Vdash A$ if and only if $\forall \alpha \in b \exists n(\bar{\alpha}n \Vdash A)$. (covering property)
(ii) $b' \succeq b$ and $b \Vdash A$ implies $b' \Vdash A$. (monotonicity)

Proof. Cf. [122, Lemma 13.1.2]. □

We first look at how to embed Kripke models into Beth models, in accordance with the method outlined in [122].[2]

Given a Kripke model $\mathcal{M}_K = (W_K, \leq, \mathcal{V}_K)$, we construct a corresponding Beth model $\mathcal{M}_B = (W_B, \preceq, \mathcal{V}_B)$ with the following stipulation.

- W_B is the set of finite nondecreasing sequences in $(W_K, \leq)$ of length ≥ 0.
- $\preceq$ is defined accordingly.
- Define an auxiliary valuation $\bar{\mathcal{V}}_B(p)$ s.t. $\langle w_0, \ldots w_n \rangle \in \bar{\mathcal{V}}_B(p)$ if and only if $w_n \in \mathcal{V}_K(p)$.

[2] Also cf. [119] for some corrections.

- Then $\mathcal{V}_B(p) = \bar{\mathcal{V}}_B(p) \cup \{\langle\rangle\}$ if $\mathcal{V}_K(p) = W_K$; otherwise $\mathcal{V}_B(p) = \bar{\mathcal{V}}_B(p)$.

Lemma 2.3.1 (embeddability of Kripke models to Beth model)**.**
(i) $\mathcal{M}_B$ is indeed a Beth model.
(ii) $\mathcal{M}_K \vDash A$ if and only if $\mathcal{M}_B \vDash A$.

Proof. In the following, we shall occasionally write $\langle b_0, \ldots, b_{-1}\rangle$ to mean $\langle\rangle$. (This is purely a conventional notation to simplify the exposition, and should not be confused with the notation in the definition of $\bar{\mathcal{V}}_B(p)$, in which n cannot be -1.)

(i) We need to show that the assignment is covering. Suppose $\langle b_0, \ldots, b_n\rangle \in \mathcal{V}_B(p)$. If $n = -1$, then $\langle\rangle \in \mathcal{V}_B(p)$. So by definition of $\mathcal{V}_B$, $w \in \mathcal{V}_K(p)$ for all $w \in W_K$. Hence for each $\alpha = \langle w, \ldots\rangle \in \langle\rangle$, $\langle w\rangle \in \mathcal{V}_B(p)$; so $\exists m(\bar{\alpha}m \in \mathcal{V}_B(p))$. If $n > -1$, then $\langle b_0, \ldots, b_n\rangle \in \mathcal{V}_B(p)$ immediately implies $\forall\alpha \in \langle b_0, \ldots, b_n\rangle\exists m(\bar{\alpha}m \in \mathcal{V}_B(p))$.

Conversely, suppose $\forall\alpha \in \langle b_0, \ldots, b_n\rangle\exists m(\bar{\alpha}m \in \mathcal{V}_B(p))$. If $n = -1$, then for any $w \in W_K$, $\langle w, w, \ldots\rangle \in \langle\rangle$. By our supposition, either $\langle\rangle \in \mathcal{V}_B(p)$ or $\langle w, w, \ldots, w\rangle \in \mathcal{V}_B(p)$. In both cases, $w \in \mathcal{V}_K(p)$. Hence $W_K = \mathcal{V}_K(p)$. Thus $\langle\rangle \in \mathcal{V}_B(p)$, as required. If $n > -1$, then $\langle b_0, \ldots, b_n, b_n, \ldots\rangle \in \langle b_0, \ldots, b_n\rangle$. So either $\langle\rangle \in \mathcal{V}_B(p)$, $\langle b_0, \ldots, b_i\rangle \in \mathcal{V}_B(p)$ for $i < n$, or $\langle b_0, \ldots, b_n, b_n, \ldots, b_n\rangle \in \mathcal{V}_B(p)$. In the first case, $b_n \in \mathcal{V}_K(p)$. In the second case, $b_i \in \mathcal{V}_K(p)$, so by the monotonicity of $\mathcal{V}_K$, $b_n \in \mathcal{V}_K$. In the last case, $b_n \in \mathcal{V}_K(p)$. So in any case, $\langle b_0, \ldots, b_n\rangle \in \mathcal{V}_B(p)$.

(ii) It suffices to show:

1. $\langle\rangle \Vdash A$ if and only if $\mathcal{M}_K \vDash A$.

2. $\langle b_0, \ldots, b_n\rangle \Vdash A$ if and only if $b_n \Vdash A$. (where $n > -1$)

We prove these by simultaneous induction on the complexity of A.

If $A \equiv p$, then 1. and 2. follow by definition. If $A \equiv \bot$, then A is never forced in either of the models, so the statement holds vacuously.

If $A \equiv A_1 \wedge A_2$, then for 1. $\langle\rangle \Vdash A_1 \wedge A_2$ if and only if $\langle\rangle \Vdash A_1$ and $\langle\rangle \Vdash A_2$ By I.H. this is equivalent to $\mathcal{M}_K \vDash A_1$ and $\mathcal{M}_K \vDash A_2$, which in turn is equivalent to $\mathcal{M}_K \vDash A_1 \wedge A_2$. For 2., $\langle b_0, \ldots, b_n\rangle \Vdash A_1 \wedge A_2$ if and only if $\langle b_0, \ldots, b_n\rangle \Vdash A_1$ and $\langle b_0, \ldots, b_n\rangle \Vdash A_2$. By I.H. this is equivalent to $b_n \Vdash A_1$ and $b_n \Vdash A_2$, which in turn is equivalent to $b_n \Vdash A_1 \wedge A_2$.

If $A \equiv A_1 \vee A_2$, then for 1., $\langle\rangle \Vdash A_1 \vee A_2$ if and only if $\forall\alpha \in \langle\rangle\exists m(\bar{\alpha}m \Vdash A_1$ or $\bar{\alpha}m \Vdash A_2)$. For each $w \in W_K$, $\langle w, w, \ldots\rangle \in \langle\rangle$, so either $\langle\rangle \Vdash A_1$, $\langle\rangle \Vdash A_2$, $\langle w, \ldots, w\rangle \Vdash A_1$ or $\langle w, \ldots, w\rangle \Vdash A_2$. If one of the former two cases holds, then by I.H. $\mathcal{M}_K \vDash A_i$, for one of $i \in \{1, 2\}$; so $w \Vdash A_1 \vee A_2$. If one of the latter two cases hold, then by I.H. $w \Vdash A_i$ for one of $i \in \{1, 2\}$; so $w \Vdash A_1 \vee A_2$. Hence we conclude $w \Vdash A_1 \vee A_2$ for all $w \in W_K$, i.e. $\mathcal{M}_K \vDash A_1 \vee A_2$. For the converse direction, assume $\mathcal{M}_K \vDash A_1 \vee A_2$ and let $\alpha = \langle w, \ldots\rangle \in \langle\rangle$. Then since $w \Vdash A_1$ or $w \Vdash A_2$, $\langle w\rangle \Vdash A_1$ or $\langle w\rangle \Vdash A_2$ by I.H.. Thus $\forall\alpha \in \langle\rangle\exists m(\bar{\alpha}m \Vdash A_1$ or $\bar{\alpha}m \Vdash A_2)$. Hence $\langle\rangle \Vdash A_1 \vee A_2$.

For 2. If $\langle b_0, \ldots, b_n\rangle \Vdash A_1 \vee A_2$, then for all $\alpha \in \langle b_0, \ldots, b_n\rangle$ there exists m s.t. $\bar{\alpha}m \Vdash A_1$ or $\bar{\alpha}m \Vdash A_2$. As $\langle b_0, \ldots, b_n, b_n, \ldots\rangle \in \langle b_0, \ldots, b_n\rangle$, we have, for $i \in \{1, 2\}$, either $\langle\rangle \Vdash A_i$, $\langle b_0, \ldots, b_l\rangle \Vdash A_i$ for $l \leq n$, or $\langle b_0, \ldots, b_n, b_n, \ldots, b_n\rangle \Vdash A_i$. In each case $b_n \Vdash A_i$ by I.H.; so $b_n \Vdash A_1 \vee A_2$. Conversely, if $b_n \Vdash A_1 \vee A_2$, then $b_n \Vdash A_1$ or $b_n \Vdash A_2$. So by I.H. $\langle b_0, \ldots, b_n \rangle \Vdash A_1$ or $\langle b_0, \ldots, b_n \rangle \Vdash A_2$. Hence immediately $\forall \alpha \in \langle b_0, \ldots, b_n \rangle \exists m (\bar{\alpha} m \Vdash A_1$ or $\bar{\alpha} m \Vdash A_2)$, i.e. $\langle b_0, \ldots, b_n \rangle \Vdash A_1 \vee A_2$.

If $A \equiv A_1 \rightarrow A_2$, then for 1., suppose $\langle\rangle \Vdash A_1 \rightarrow A_2$. Let $w \in W_K$ and $w' \geq w$. If $w' \Vdash A_1$, then $\langle w' \rangle \Vdash A_1$ by I.H.. So $\langle w' \rangle \Vdash A_2$ and thus $w' \Vdash A_2$. Consequently $w \Vdash A_1 \rightarrow A_2$ and so $\mathcal{M}_K \models A_1 \rightarrow A_2$. Conversely, suppose $\mathcal{M}_K \models A_1 \rightarrow A_2$. Let $\langle b_0, \ldots, b_n \rangle \Vdash A_1$. If $n = -1$, then by I.H. $\mathcal{M}_K \models A_1$, so $\mathcal{M}_K \models A_2$. Hence $\langle b_0, \ldots, b_n \rangle \Vdash A_2$ again by I.H.. If $n > -1$, then $b_n \Vdash A_1$, so $b_n \Vdash A_2$. Hence $\langle b_0, \ldots, b_n \rangle \Vdash A_2$. Thus $\langle\rangle \Vdash A_1 \rightarrow A_2$.

For 2., suppose $\langle b_0, \ldots, b_n \rangle \Vdash A_1 \rightarrow A_2$ and let $b_{n'} \geq b_n$. If $b_{n'} \Vdash A_1$, then by I.H. $\langle b_0, \ldots, b_n, b_{n'} \rangle \Vdash A_1$; so $\langle b_0, \ldots, b_n, b_{n'} \rangle \Vdash A_2$. Thus $b_{n'} \Vdash A_2$. Hence $b_n \Vdash A_1 \rightarrow A_2$. Conversely, suppose $b_n \Vdash A_1 \rightarrow A_2$. Assume $\langle b_0, \ldots, b_n, \ldots, b_{n'} \rangle \Vdash A_1$. Then $b_n \leq b_{n'}$ and $b_{n'} \Vdash A_1$. So $b_{n'} \Vdash A_2$. Thus $\langle b_0, \ldots, b_n, \ldots, b_{n'} \rangle \Vdash A_2$. Therefore $\langle b_0, \ldots, b_n \rangle \Vdash A_1 \rightarrow A_2$.

□

This lemma immediately leads the weak completeness (i.e. completeness in cases of no assumptions) of **IPC** with respect to **IPC**.

Theorem 2.3.2 (weak completeness of **IPC** with Beth semantics.)**.** $\vdash A$ in **IPC** if and only if $\models A$ in Beth model.

Proof. It is routine to check that the soundness holds. Then the weak completeness readily follows from the previous lemma and the Kripke completeness of **IPC**. □

2.4 Negation in intuitionistic logic

As have been mentioned, the rejection of the law of excluded middle plays a pivotal role in the characterisation of intuitionistic logic. Hence, one may view that the main difference between classical and intuitionistic logic consists in the difference in negation. [3] From this perspective it is of little surprise that the early controversies surrounding intuitionistic logic — as the correct formalisation of Brouwer's ideas — stem from the disagreements in its treatment of negation. Indeed, such disputes often initiated the formal, less opinionated investigation today of negation in intuitionistic contexts. Here we shall introduce a few important variants of intuitionistic logic with different negation.

One objection to intuitionistic negation is that the notion of negation is at tension with the notion of construction, as proposed by George François Cornelis Griss (1898-1953) in e.g. [56]. [4] The central idea of the critique is the following [58, 59]: if the concept of a mathematical object is given by its construction, then it is unclear what concepts those mathematical objects that do not have a construction possess. In particular, when it comes to negation, to assert $\neg P(a)$ for a mathematical object a and a predicate P, in the orthodox view one has to be able to construct (as supposition) a that satisfies P, and then derive a contradiction. But if the above critique is correct, then it is not apparent what this supposition amounts

[3] However, this is not the whole story, as even for the negation-less fragment, classical logic is distinguished from intuitionistic logic by e.g. Pierce's formula $((A \rightarrow B) \rightarrow A) \rightarrow A$.

[4] Cf. also [42, 57] for related discussions.

to. For instance, in order to demonstrate the proposition *a square circle does not exist*, one ought to first give a (mental) construction of a square circle; is this really possible? On this ground Griss espoused the rejection of negation altogether from mathematics, giving the alternative *negationless intuitionistic mathematics*. The logic for this mathematics has been studied by a number of people in later years: cf. [49, 137, 128, 86, 87, 80].

Another type of objection to intuitionistic logic concerns more specifically the validity of the formula called *ex falso qoudlibet* (EFQ): $\bot \to A$. The rejection of EFQ is motivated by dubious inferences it allows, like $\neg A \to (A \to B)$ [134]. The elimination of EFQ from the list of axioms in **IPC** leads to the system christened *minimal logic* (**MPC**) by Ingebrigt Johansson (1904-1987) [70], which was partially anticipated by the formalisation of intuitionism by Kolmogorov [73]. In consequence of the rejection of EFQ, the absurdity symbol $\bot$, having no axiom related to it, behaves like a propositional variable. Yet the strength of **MPC** is not too decreased by such a change. Indeed, the famous Gödel-Gentzen translation $()^g$ [48, 52] and again the pre-dating Kolmogorov translation $()^k$ [73], each given by the following clauses, give faithful embeddings of classical logic into minimal logic.

Definition 2.4.1 (Gödel-Gentzen and Kolmogorov translation)**.**

$$
\begin{array}{rlrl}
p^g & \equiv \neg\neg p & p^k & \equiv \neg\neg p \\
\bot^g & \equiv \bot & \bot^k & \equiv \bot \\
(A \wedge B)^g & \equiv A^g \wedge B^g & (A \wedge B)^k & \equiv \neg\neg(A^k \wedge B^k) \\
(A \vee B)^g & \equiv \neg(\neg A^g \wedge \neg B^g) & (A \vee B)^k & \equiv \neg\neg(A^k \vee B^k) \\
(A \to B)^g & \equiv A^g \to B^g & (A \to B)^k & \equiv \neg\neg(A^k \to B^k)
\end{array}
$$

Let us write Γ^g for $\{A^g : A \in \Gamma\}$: similarly for Γ^k. **MPC** (**MQC**) is defined from **IPC** (**IQC**)by eliminating the axiom $\bot \to A$. We will use the notations $\vdash_m$ and $\vdash_{qm}$ when we try to be explicit. Then the translations achieve the following equivalences.

Proposition 2.4.1. $\Gamma \vdash A$ in **CPC** iff $\Gamma^g \vdash A^g$ in **MPC** iff $\Gamma^k \vdash A^k$ in **MPC**.

Proof. Cf. [121, Theorem 2.3.5, Proposition 2.3.8] □

The class of logics between classical and intuitionistic logic, called *intermediate* or *superintuitionistic* logics[5] are also investigated, and these logics have much to do with negations as well. An intermediate logic is defined by additional axioms to **IPC**, which usually corresponds semantically to a class of Kripke frames which satisfy certain conditions. For instance, the axiom $\neg A \vee \neg\neg A$ (WLEM) corresponds to Kripke frames that satisfy the property $\forall w, x, y(x \geq w \wedge y \geq w \to \exists z(z \geq x \wedge z \geq y))$. For more information on intermediate logics, cf. for example [63, 22].

Similarly to intermediate logics, one may also consider the logics between minimal and intuitionistic logic. Such logics have been studied by Krister Segerberg. [109] (also Cf. [94]). What is of particular interest for us is a logic named **JP'** (called *Gliveko's logic* in [94]), which is defined by adding $\neg\neg(\bot \to A)$ to **MPC**. An

[5]A slight difference between the two is that superintuitionistic logic include **CPC**; this framework is introduced by Toshio Umezawa [123, 124]

important characteristic of this logic is that it is the smallest extension of **MPC**, in which Glivenko's theorem — $\Gamma \vdash \neg\neg A$ in the system if $\Gamma \vdash A$ in **CPC** — holds. This fact will be relevant in the investigation of Chapter 3.

Chapter 3

Conservation of classical propositions and non-constructive principles

Ishihara [68] proposed the following problem for propositional logic:

> *If* $\Gamma \vdash A$ *classically, then what class* V *of propositional variables are sufficient to conclude* $\mathcal{E}_V, \Gamma \vdash A$ *intuitionistically, where* $\mathcal{E}_V = \{p \vee \neg p : p \in V\}$*?*

One answer to the problem, offered in the same paper, is to take

$$V = (\mathcal{V}^-(\Gamma) \cup \mathcal{V}^+(A)) \cap (\mathcal{V}^+_{ns}(\Gamma) \cap \mathcal{V}^-(A)).$$

Here $\mathcal{V}^+, \mathcal{V}^-$ and $\mathcal{V}^+_{ns}$ designate the sets of positive/negative/non-strictly positive propositional variables in formulae. This result is shown via induction on the depth of deduction in classical cut-free sequent calculus.

Another answer to the problem was given by Ishii [69]. In this case, the class is any $V \in V^*(A)$, where $V^*(A)$ is defined inductively, by:

$$\begin{aligned}
V^*(p) &= \{\{p\}\}, \\
V^*(\bot) &= \{\emptyset\}, \\
V^*(A \wedge B) &= \{V_1 \cup V_2 : V_1 \in V^*(A), V_2 \in V^*(B)\}, \\
V^*(A \vee B) &= \{V_1 \cup \mathsf{V}(B) : V_1 \in V^*(A)\} \cup \{\mathsf{V}(A) \cup V_2 : V_2 \in V^*(B)\}, \\
V^*(A \to B) &= V^*(B).
\end{aligned}$$

(where $\mathsf{V}(A)$ denotes the set of all propositional variables in A.)

In this section, we shall concentrate on refining this latter class by Ishii. The refinement proceeds in two directions. One is to consider not the instances of LEM, but of weaker *weak excluded middle* (WLEM) $\neg\neg A \vee \neg A$ and *double negation elimination* (DNE) $\neg\neg A \to A$. This allows us to avoid invoking the full LEM in some cases where it is not required. For the second direction, we observe the alternation to WLEM and DNE also allows to extend Ishii-style result to a weaker system, namely to Glivenko's logic, which as we recall is defined by weakening the intuitionistic axiom of *ex falso quodlibet* (EFQ) $\bot \to A$ into its double negation $\neg\neg(\bot \to A)$. We will call the axiom *Avoidability of Q* (AVQ) for a reason we shall explain. We shall also extend our analysis to minimal logic, by additionally considering the class of atomic AVQ/EFQ.

Although Ishihara and Ishii treated the problem for propositional logic, we can generalise it into predicate logic, by asking the class of sufficient atomic formulae rather than propositional variables. Our main concern remains propositional, but for the sake of a wider scope, we shall treat predicate case at the same time. This however requires the additional axioms of

- *double negation shift* (DNS) $\forall x \neg\neg A(x) \to \neg\neg\forall x A(x)$
- *constant domain* (CD) $\forall x(A(x) \vee C) \to (\forall x A(x) \vee C)$ [x not free in C]

We note that DNS is the axiom required for Glivenko's theorem for predicate logic [121, p.106], and CD is provable once we add the missing connective of co-implication to intuitionistic logic [104]. If we look at the propositional fragments, then our proofs do apply to intuitionistic/Glivenko's/minimal logic, without any extra axioms. We shall use $\vdash_{qi^+}, \vdash_{qg^+}$ and $\vdash_{qm^+}$ for the derivability in the predicate logics with DNS and CD, and $\vdash_{qi}, \vdash_{qg}$ and $\vdash_{qm}$ for the logics without DNS and CD, as we introduced in the previous chapter. When we talk about propositional logic, we shall write $\vdash_i, \vdash_g$ and $\vdash_m$. Recall also that classical derivability is denoted by $\vdash_{qc}$ and $\vdash_c$.

3.1 Preliminaries

Here we shall define some basic classes of formulae for later use. For our purpose, in this paper we shall only concern *literal* subformulae [120, pp.4-5]; so the subformulae of $\forall x A/\exists x A$ are those of A, and not those of all substitution instances of A.

We define the sets $\mathcal{V}^+(A)/\mathcal{V}^-(A)$ of atomic formulae occurring *positively/negatively* in A by the following clauses.

$$
\begin{array}{rlrl}
\mathcal{V}^+(P) &= \{P\} & \mathcal{V}^-(P) &= \emptyset \\
\mathcal{V}^+(\bot) &= \emptyset & \mathcal{V}^-(\bot) &= \emptyset \\
\mathcal{V}^+(A \wedge B) &= \mathcal{V}^+(A) \cup \mathcal{V}^+(B) & \mathcal{V}^-(A \wedge B) &= \mathcal{V}^-(A) \cup \mathcal{V}^-(B) \\
\mathcal{V}^+(A \vee B) &= \mathcal{V}^+(A) \cup \mathcal{V}^+(B) & \mathcal{V}^-(A \vee B) &= \mathcal{V}^-(A) \cup \mathcal{V}^-(B) \\
\mathcal{V}^+(A \to B) &= \mathcal{V}^-(A) \cup \mathcal{V}^+(B) & \mathcal{V}^-(A \to B) &= \mathcal{V}^+(A) \cup \mathcal{V}^-(B) \\
\mathcal{V}^+(\forall x A) &= \mathcal{V}^+(A) & \mathcal{V}^-(\forall x A) &= \mathcal{V}^-(A) \\
\mathcal{V}^+(\exists x A) &= \mathcal{V}^+(A) & \mathcal{V}^-(\exists x A) &= \mathcal{V}^-(A)
\end{array}
$$

We shall also write $\mathsf{V}(A)$ to denote the set of atomic formulae that occur in A. Further, we define the sets $\mathcal{V}^{s+}(A)/\mathcal{V}^{s+}_{nd}(A)$ of propositional variables occurring *strictly positively/non-deterministic strictly positively* in A by the following clauses.

$$
\begin{array}{rlrl}
\mathcal{V}^{s+}(P) &= \{P\} & \mathcal{V}^{s+}_{nd}(P) &= \{P\} \\
\mathcal{V}^{s+}(\bot) &= \emptyset & \mathcal{V}^{s+}_{nd}(\bot) &= \emptyset \\
\mathcal{V}^{s+}(A \wedge B) &= \mathcal{V}^{s+}(A) \cup \mathcal{V}^{s+}(B) & \mathcal{V}^{s+}_{nd}(A \wedge B) &= \mathcal{V}^{s+}_{nd}(A) \cup \mathcal{V}^{s+}_{nd}(B) \\
\mathcal{V}^{s+}(A \vee B) &= \mathcal{V}^{s+}(A) \cup \mathcal{V}^{s+}(B) & \mathcal{V}^{s+}_{nd}(A \vee B) &= \mathcal{V}^{s+}_{nd}(A) \text{ or } \mathcal{V}^{s+}_{nd}(B) \\
\mathcal{V}^{s+}(A \to B) &= \mathcal{V}^{s+}(B) & \mathcal{V}^{s+}_{nd}(A \to B) &= \mathcal{V}^{s+}_{nd}(B) \\
\mathcal{V}^{s+}(\forall x A) &= \mathcal{V}^{s+}(A) & \mathcal{V}^{s+}_{nd}(\forall x A) &= \mathcal{V}^{s+}_{nd}(A) \\
\mathcal{V}^{s+}(\exists x A) &= \mathcal{V}^{s+}(A) & \mathcal{V}^{s+}_{nd}(\exists x A) &= \mathcal{V}^{s+}_{nd}(A)
\end{array}
$$

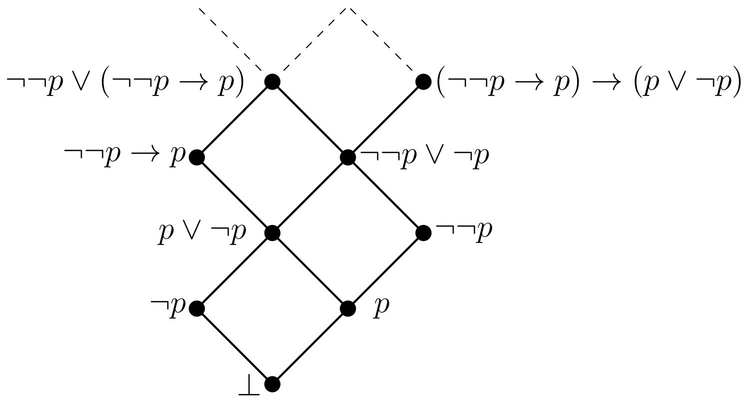


Figure 3.1: Rieger-Nishimura lattice

In the clause for disjunction in $\mathcal{V}_{nd}^{s+}$, the choice between $\mathcal{V}_{nd}^{s+}(A)$ and $\mathcal{V}_{nd}^{s+}(B)$ can be arbitrarily made, and we may assume there is a stipulation beforehand regarding each of the choices. If we replace the clause for atomic formulae $P, Q, R \ldots$ by a clause for propositional variables $p, q, r \ldots$, and drop the clauses for quantifiers, then we get the clauses for propositional language. The same remark applies to classes that appear later.

3.2 Ishii's class and Glivenko's logic

The argument of Ishii runs as follows: If $\Gamma \vdash_c A$, then by Glivenko's theorem $\Gamma \vdash_i \neg\neg A$; and the class V suffices for $\mathcal{E}_V \vdash_i \neg\neg A \to A$. One thing to note in the latter step is that Ishii's V collects every propositional variable occurring strictly positively in a formula, in order to derive DNE for the variables, i.e. to show $p \vee \neg p \vdash_i \neg\neg p \to p$. However for this purpose we can just assume $\neg\neg p \to p$ itself, which is weaker. (As an axiom schema, on the other hand, $\neg\neg A \to A$ and $A \vee \neg A$ are equivalent.) This hints that some instances of LEM in Ishii's class can be replaced with weaker axioms.

If we recall Rieger-Nishimura lattice [108, 93], the lattice of one-propositional variable formulae in intuitionistic propositional logic, we can see that $p \vee \neg p$ is the meet of $\neg\neg p \vee \neg p$ and $\neg\neg p \to p$. Hence it would seem reasonable to modify Ishihara's problem to ask what classes of atomic WLEM and DNE are sufficient for the preservation of classical theorems.

We shall give an answer to this modified problem, and we shall look at the preservation for Glivenko's logic (with DNS+CD for predicate case). To recall, it is defined by adding AVQ: $\neg\neg(\bot \to A)$ to minimal logic. It is the smallest extension of minimal logic for which Glivenko's theorem holds. It is easy to check that the addition of DNS still extends the theorem to the predicate logic. Semantically, it is characterised by Kripke frames (for minimal logic) in which one can always find a path which *avoids* entering the set Q of worlds in which $\bot$ is forced. That is to say, frames s.t. $\forall x \forall y \geq x(y \notin Q \Rightarrow \exists z \geq y \forall t \geq z(t \notin Q))$. For more information on Glivenko's logic, cf. [109, 138, 33, 94].

Let $\mathcal{W}_A$ to be the universal closure of instances of WLEM for atomic formulae occurring in A. That is, $\mathcal{W}_A = \{\forall\vec{x}(\neg\neg P \vee \neg P) : P \in \mathsf{V}(A)\}$. (if we are to concentrate on propositional logic, just take atomic instances of WLEM.)

Lemma 3.2.1. $\mathcal{W}_A \vdash_{qg^+} \neg\neg A \vee \neg A$.

Proof. We prove by induction on the complexity of A. For the atomic case where $A \equiv P$, we have $\forall\vec{x}(\neg\neg P \vee \neg P) \vdash_{qg^+} \neg\neg P \vee \neg P$. For $A \equiv \bot$, we have $\vdash_{qg^+} \neg\neg\bot \vee \neg\bot$.

When $A \equiv B \wedge C$, by I.H.

$$\mathcal{W}_B \vdash_{qg^+} \neg\neg B \vee \neg B \text{ and } \mathcal{W}_C \vdash_{qg^+} \neg\neg C \vee \neg C.$$

Now $\mathcal{W}_A = \mathcal{W}_B \cup \mathcal{W}_C$ and

$$\vdash_{qg^+} \neg B \vee \neg C \to \neg(B \wedge C) \text{ and } \vdash_{qg^+} \neg\neg B \wedge \neg\neg C \to \neg\neg(B \wedge C).$$

So $\mathcal{W}_A \vdash_{qg^+} \neg\neg(B \wedge C) \vee \neg(B \wedge C)$.

When $A \equiv B \vee C$, the argument is similar to the last case, but we appeal to

$$\vdash_{qg^+} \neg B \wedge \neg C \to \neg(B \vee C) \text{ and } \vdash_{qg^+} \neg\neg B \vee \neg\neg C \to \neg\neg(B \vee C)$$

instead. Then we conclude $\mathcal{W}_A \vdash_{qg^+} \neg\neg(B \vee C) \vee \neg(B \vee C)$.

When $A \equiv B \to C$, the argument is again analogous, and we appeal to

$$\vdash_{qg^+} \neg\neg B \wedge \neg C \to \neg(B \to C) \text{ and } \vdash_{qg^+} (\neg B \vee \neg\neg C) \to \neg\neg(B \to C).$$

For the latter, we recall $\vdash_{qm^+} \neg\neg(\bot \to C) \to (\neg B \to \neg\neg(B \to C))$ and so $\vdash_{qg^+} \neg B \to \neg\neg(B \to C)$. Thus $\mathcal{W}_A \vdash_{qg^+} \neg\neg(B \to C) \vee \neg(B \to C)$.

When $A \equiv \forall x B$, by I.H. $\mathcal{W}_B \vdash_{qg^+} \neg\neg B \vee \neg B$ and $\mathcal{W}_A = \mathcal{W}_B$. Thus $\mathcal{W}_A \vdash_{qg^+} \neg\neg B \vee \exists x\neg B$ and so $\mathcal{W}_A \vdash_{qg^+} \forall x(\neg\neg B \vee \exists x\neg B)$ because we took $\mathcal{W}_A$ to be a set of universal closure. Hence by CD, $\mathcal{W}_A \vdash_{qg^+} \forall x\neg\neg B \vee \exists x\neg B$ and by DNS, $\mathcal{W}_A \vdash_{qg^+} \neg\neg\forall x B \vee \exists x\neg B$. Therefore $\mathcal{W}_A \vdash_{qg^+} \neg\neg\forall x B \vee \neg\forall x B$.

When $A \equiv \exists x B$, we have the same I.H. and again $\mathcal{W}_A = \mathcal{W}_B$. Then $\mathcal{W}_A \vdash_{qg^+} \forall x(\exists x\neg\neg B \vee \neg B)$ and by CD, $\mathcal{W}_A \vdash_{qg^+} \exists x\neg\neg B \vee \forall x\neg B$. Since $\vdash_{qm^+} \exists x\neg\neg B \to \neg\neg\exists x B$, we conclude $\mathcal{W}_A \vdash_{qg^+} \neg\neg\exists x B \vee \neg\exists x B$. □

One may observe that the argument above does not work for LEM, because then in the case for implication we would need $\neg B \to (B \to C)$, which is not available in Glivenko's logic. So there is another benefit of considering classes of WLEM (and DNE) instead of LEM. As a further note, it will turn out to be important that this lemma, along with Glivenko's theorem, are the only instances where we require non-minimal axiom; the rest of this section can be argued in minimal logic (+DNS+CD), once we assume Lemma 3.2.1.

Now we are ready to recreate Ishii's result with finer classes in terms of WLEM and DNE. Given $\Gamma \vdash_{qc} A$, then by Glivenko's theorem $\Gamma \vdash_{qg^+} \neg\neg A$. Ishii's method would then show DNE for A with classes of WLEM and DNE as additional assumptions. We shall use an alternative method[1] to better visualise what is going on. We will push the double negation in front of A inside, until it reaches in front of atomic formulae occurring strictly positively in A, with the aid of the above lemma. Then the instances of DNE for strictly positive atomic formulae will let us regain the original A.

[1] We thank Hajime Ishihara for the suggestion.

Lemma 3.2.2.
(i) $\vdash_{qm^+} (\neg\neg A \vee \neg A) \to (\neg\neg(A \vee B) \to (\neg\neg A \vee \neg\neg B))$.
(ii) $\vdash_{qm^+} \forall x(\neg\neg A \vee \neg A) \to (\neg\neg\exists x A \to \exists x \neg\neg A)$.

Proof. (i) follows from $\vdash_{qm^+} \neg\neg A \to \neg\neg A$ and $\vdash_{qm^+} \neg A \to (\neg B \to \neg(A \vee B))$. For (ii), we first have $\vdash_{qm^+} \forall x(\neg\neg A \vee \neg A) \to (\exists x \neg\neg A \vee \forall x \neg A)$ using CD; then from both disjuncts one can deduce $\neg\neg\exists x A \to \exists x \neg\neg A$. In particular, using $\vdash_{qm^+} \forall x \neg A \to \neg\exists x A$, one can deduce $\forall x \neg A \to (\neg\neg\exists x A \to \neg\neg A)$, because in minimal logic any negation can be derived from a contradiction. Thence $\vdash_{qm^+} \forall x \neg A \to (\neg\neg\exists x A \to \exists x \neg\neg A)$. □

We define a new class of atomic WLEM from $\mathcal{W}_A$.

Definition 3.2.1. We define $\widetilde{\mathcal{W}}_A$ inductively.

$$\begin{aligned}
\widetilde{\mathcal{W}}_P &= \widetilde{\mathcal{W}}_\perp = \emptyset \\
\widetilde{\mathcal{W}}_{A\wedge B} &= \widetilde{\mathcal{W}}_A \cup \widetilde{\mathcal{W}}_B \\
\widetilde{\mathcal{W}}_{A\vee B} &= \widetilde{\mathcal{W}}_A \cup \mathcal{W}_B \text{ or } \mathcal{W}_A \cup \widetilde{\mathcal{W}}_B \\
\widetilde{\mathcal{W}}_{A\to B} &= \widetilde{\mathcal{W}}_B \\
\widetilde{\mathcal{W}}_{\forall x A} &= \widetilde{\mathcal{W}}_A \\
\widetilde{\mathcal{W}}_{\exists x A} &= \mathcal{W}_A
\end{aligned}$$

In the clause for disjunction, again the choice is arbitrary and can be assumed to be pre-determined. Also note that $\mathcal{W}_A \supseteq \widetilde{\mathcal{W}}_A$, because the former takes all atomic formulae of A.

We now make a modified use of [121, Definition 3.24, Lemma 3.25].

Definition 3.2.2 (multiple formula contexts)**.** Let $*_1, *_2, \ldots$ be a countable set of symbols. The class $\mathcal{F}$ of *multiple formula contexts* is defined inductively as follows. (where $F, F' \in \mathcal{F}$ and A a formula.)
(i) $*_n, \perp, \forall x F, \exists x F, A \to F \in \mathcal{F}$.
(ii) Assume no $*_n$ occurs in both F and F'. Then $F \wedge F', F \vee F' \in \mathcal{F}$.

We shall write $F[*_1, \ldots, *_n]$ to denote the occurrences of $*_1, \ldots, *_n$ in $F \in \mathcal{F}$. By convention we include in the notation when no $*_n$ occurs, in which case we write $F[*_1, \ldots, *_{-1}]$. The result of substituting $*_1, \ldots, *_n$ with $A_1, \ldots, A_n$ will be denoted $F[A_1, \ldots, A_n]$. It is straightforward to see that each $*_i$ points a position in F at which an atomic formula would occur strictly positively; i.e. $P_1, \ldots, P_n \in \mathcal{V}^{s+}(F[P_1, \ldots, P_n])$. Further, each formula A can be written as $F[P_1, \ldots, P_n]$, where $\{P_1, \ldots, P_n\} = \mathcal{V}^{s+}(A)$. In below, we shall abbreviate $F[(\neg\neg)P_1, \ldots, (\neg\neg)P_n]$ occasionally as $F[(\neg\neg)P_{1,\ldots,n}]$.

Proposition 3.2.1. Let $F[*_1, \ldots, *_n] \in \mathcal{F}$. Then

$$\widetilde{\mathcal{W}}_{F[P_1,\ldots,P_n]} \vdash_{qg^+} \neg\neg F[P_1, \ldots, P_n] \to F[\neg\neg P_1, \ldots, \neg\neg P_n].$$

Proof. We prove by induction on the construction of F.

- When $F \equiv *_n$, $F[P_n] \equiv P_n$, and $\vdash_{qg^+} \neg\neg P_n \to \neg\neg P_n$.

- When $F \equiv \bot$, then $\vdash_{qg^+} \neg\neg\bot \to \bot$.
- When $F \equiv \forall x F'$, then by I.H.,

$$\widetilde{\mathcal{W}}_{F'[P_{1,\ldots,n}]} \vdash_{qg^+} \neg\neg F'[P_{1,\ldots,n}] \to F'[\neg\neg P_{1,\ldots,n}].$$

Also by definition, we have $\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} = \widetilde{\mathcal{W}}_{F'[P_{1,\ldots,n}]}$. Thus

$$\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} \vdash_{qg^+} \forall x \neg\neg F'[P_{1,\ldots,n}] \to \forall x F'[\neg\neg P_{1,\ldots,n}].$$

As $\vdash_{qm^+} (\neg\neg\forall x C \to \neg\exists x \neg C) \wedge (\neg\exists x \neg C \to \forall x \neg\neg C)$, we conclude

$$\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} \vdash_{qg^+} \neg\neg\forall x F'[P_{1,\ldots,n}] \to \forall x F'[\neg\neg P_{1,\ldots,n}].$$

- When $F \equiv \exists x F'$, then $\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} = \mathcal{W}_{F'[P_{1,\ldots,n}]} \supseteq \widetilde{\mathcal{W}}_{F'[P_{1,\ldots,n}]}$. So we can apply the I.H.

$$\widetilde{\mathcal{W}}_{F'[P_{1,\ldots,n}]} \vdash_{qg^+} \neg\neg F'[P_{1,\ldots,n}] \to F'[\neg\neg P_{1,\ldots,n}].$$

This enables us to assert

$$\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} \vdash_{qg^+} \exists x \neg\neg F'[P_{1,\ldots,n}] \to \exists x F'[\neg\neg P_{1,\ldots,n}]. \tag{3.1}$$

In addition, by Lemma 3.2.1 we have $\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} \vdash_{qg^+} \neg\neg F'[P_{1,\ldots,n}] \vee \neg F'[P_{1,\ldots,n}]$; thus $\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} \vdash_{qg^+} \forall x(\neg\neg F'[P_{1,\ldots,n}] \vee \neg F'[P_{1,\ldots,n}])$. So by Lemma 3.2.2 (ii),

$$\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} \vdash_{qg^+} \neg\neg\exists x F'[P_{1,\ldots,n}] \to \exists x \neg\neg F'[P_{1,\ldots,n}].$$

Combine this with (3.1) to conclude

$$\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} \vdash_{qg^+} \neg\neg\exists x F'[P_{1,\ldots,n}] \to \exists x F'[\neg\neg P_{1,\ldots,n}].$$

- When $F \equiv A \to F'$, by I.H.

$$\widetilde{\mathcal{W}}_{F'[P_{1,\ldots,n}]} \vdash_{qg^+} \neg\neg F'[P_{1,\ldots,n}] \to F'[\neg\neg P_{1,\ldots,n}].$$

Now $\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} = \widetilde{\mathcal{W}}_{F'[P_{1,\ldots,n}]}$ and also $\vdash_{qm^+} \neg\neg(C \to D) \to (C \to \neg\neg D)$; hence we conclude

$$\widetilde{\mathcal{W}}_{F[P_{1,\ldots,n}]} \vdash_{qg^+} \neg\neg(A \to F'[P_{1,\ldots,n}]) \to (A \to F'[\neg\neg P_{1,\ldots,n}]).$$

When $F \equiv F_1 \wedge F_2$, we have $F[P_{1,\ldots,m}, P_{m+1,\ldots,n}] \equiv F_1[P_{1,\ldots,m}] \wedge F_2[P_{m+1,\ldots,n}]$. Also by I.H. it follows that

$$\widetilde{\mathcal{W}}_{F_1}[P_{1,\ldots,m}] \vdash_{qg^+} \neg\neg F_1[P_{1,\ldots,m}] \to F_1[\neg\neg P_{1,\ldots,m}]$$

and

$$\widetilde{\mathcal{W}}_{F_2}[P_{m+1,\ldots,n}] \vdash_{qg^+} \neg\neg F_2[P_{m+1,\ldots,n}] \to F_2[\neg\neg P_{m+1,\ldots,n}].$$

Thus

$$\widetilde{\mathcal{W}}_{F} \vdash_{qg^+} \neg\neg(F_1 \wedge F_2)[P_1, \ldots, P_n]) \to F_1 \wedge F_2[\neg\neg P_{1,\ldots,n}].$$

- When $F \equiv F_1 \vee F_2$ with $F[P_{1,\dots,m}, P_{m+1,\dots,n}] \equiv F_1[P_{1,\dots,m}] \vee F_2[P_{m+1,\dots,n}]$, and $\widetilde{\mathcal{W}}_{F[P_{1,\dots,n}]}$ is either $\widetilde{\mathcal{W}}_{F_1[P_{1,\dots,m}]} \cup \mathcal{W}_{F_2[P_{m+1,\dots,n}]}$ or $\mathcal{W}_{F_1[P_{1,\dots,m}]} \cup \widetilde{\mathcal{W}}_{F_2[P_{m+1,\dots,n}]}$. Without loss of generality, assume the former. Then by Lemma 3.2.1

$$\widetilde{\mathcal{W}}_{F[P_{1,\dots,n}]} \vdash_{qg^+} \neg\neg F_2[P_{m+1,\dots,n}] \vee \neg F_2[P_{m+1,\dots,n}].$$

  Hence by lemma 3.2.2 (i),

$$\widetilde{\mathcal{W}}_{F[P_{1,\dots,n}]} \vdash_{qg^+} \neg\neg(F_1 \vee F_2)[P_{1,\dots,n}] \to \neg\neg F_1[P_{1,\dots,m}] \vee \neg\neg F_2[P_{m+1,\dots,n}].$$

  Moreover, $\widetilde{\mathcal{W}}_{F[P_{1,\dots,n}]} \supseteq \widetilde{\mathcal{W}}_{F_1[P_{1,\dots,m}]} \cup \widetilde{\mathcal{W}}_{F_2}[P_{m+1,\dots,n}]$ and so we can apply the same I.H. as in the case for conjunction. Thus we conclude

$$\widetilde{\mathcal{W}}_{F[P_{1,\dots,n}]} \vdash_{qg^+} \neg\neg(F_1 \vee F_2)[P_{1,\dots,n}] \to (F_1 \vee F_2)[\neg\neg P_{1,\dots,n}].$$

□

Next, let $\mathcal{D}_A := \{\forall x(\neg\neg P \to P) : P \in \mathcal{V}^{s+}(A)\}$. Then we have the following.

Proposition 3.2.2. Let $F[*_1, \dots, *_n] \in \mathcal{F}$. Then

$$\mathcal{D}_{F[P_1,\dots,P_n]} \vdash_{qg^+} F[\neg\neg P_1, \dots, \neg\neg P_n] \to F[P_1, \dots, P_n].$$

Proof. We argue by induction on the construction of F.

- When $F \equiv *_n$, we have $\forall x(\neg\neg P \to P) \vdash_{qg^+} \neg\neg P \to P$.
- When $F \equiv \bot$, then $\vdash_{qg^+} \neg\neg\bot \to \bot$.
- When $F \equiv \forall x F'$, then by I.H.

$$\mathcal{D}_{F'[P_{1,\dots,n}]} \vdash_{qg^+} F'[\neg\neg P_{1,\dots,n}] \to F'[P_{1,\dots,n}].$$

  Thus

$$\mathcal{D}_{F[P_{1,\dots,n}]} \vdash_{qg^+} \forall x F'[\neg\neg P_{1,\dots,n}] \to \forall x F'[P_{1,\dots,n}].$$

- When $F \equiv \exists x F'$, similar to the previous case.
- When $F \equiv A \to F'$, then $\mathcal{D}_{F[P_{1,\dots,n}]} = \mathcal{D}_{F'[P_{1,\dots,n}]}$. Also by I.H.

$$\mathcal{D}_{F'[P_{1,\dots,n}]} \vdash_{qg^+} F'[\neg\neg P_{1,\dots,n}] \to F'[P_{1,\dots,n}].$$

  Thus

$$\mathcal{D}_{F[P_{1,\dots,n}]} \vdash_{qg^+} (A \to F')[\neg\neg P_{1,\dots,n}] \to (A \to F')[P_{1,\dots,n}].$$

- When $F \equiv F_1 \wedge F_2$, with $F[P_{1,\dots,m}, P_{m+1,\dots,n}] \equiv F_1[P_{1,\dots,m}] \wedge F_2[P_{m+1,\dots,n}]$, by I.H. we have

$$\mathcal{D}_{F_1[P_{1,\dots,m}]} \vdash_{qg^+} F_1[\neg\neg P_{1,\dots,m}] \to F'[P_{1,\dots,m}]$$

  and

$$\mathcal{D}_{F_2[P_{m+1,\dots,n}]} \vdash_{qg^+} F_2[\neg\neg P_{m+1,\dots,n}] \to F_2[P_{m+1,\dots,n}].$$

  Hence

$$\mathcal{D}_{F[P_{1,\dots,n}]} \vdash_{qg^+} (F_1 \wedge F_2)[\neg\neg P_{1,\dots,n}] \to (F_1 \wedge F_2)[P_{m+1,\dots,n}].$$

- When $F \equiv F_1 \vee F_2$, similar to the previous case.

□

Therefore, from the preceding two propositions we conclude:

Corollary 3.2.1. For $F[*_1, \dots, *_n] \in \mathcal{F}$,

$$\widetilde{\mathcal{W}}_{F[P_1,\dots,P_n]}, \mathcal{D}_{F[P_1,\dots,P_n]} \vdash_{qg^+} \neg\neg F[P_1, \dots, P_n] \to F[P_1, \dots, P_n].$$

Thus the desired result follows.

Theorem 3.2.1. If $\Gamma \vdash_{qc} A$, then $\widetilde{\mathcal{W}}_A, \mathcal{D}_A, \Gamma \vdash_{qg^+} A$.

To reiterate the comparison with Ishii's result, our class always takes the same occurrence of atomic formulae as his class, but we assume only WLEM or DNE of the formulae, unlike his class, which contains instances of LEM. In addition, our result reaches to weaker logic than Ishii's.

As we noted, the definition of $\widetilde{\mathcal{W}}$ contains non-determinism in the clause for $\vee$. In the clause, one has to take all the atomic WLEM for one of the disjuncts. One way to choose the disjunct is to pick one which is negated. Then we can potentially avoid assuming any instances of LEM for the preservation. For instance, for the next classical theorem

$$\vdash_c \neg\neg(p \vee q) \to (\neg\neg p \vee q),$$

we can take $\widetilde{\mathcal{W}}_{\neg\neg(p\vee q)\to(\neg\neg p\vee q)} = \{\neg\neg p \vee \neg p\}$ and $\mathcal{D}_{\neg\neg(p\vee q)\to(\neg\neg p\vee q)} = \{\neg\neg q \to q\}$, s.t.

$$\neg\neg p \vee \neg p, \neg\neg q \to q \vdash_g \neg\neg(p \vee q) \to (\neg\neg p \vee q).$$

With the same choice of disjuncts, Ishii's class gives $\{p \vee \neg p, q \vee \neg q\}$, which is stronger than ours as a set of assumptions. With a different choice, our method and his method give equivalent classes $\{\neg\neg q \vee \neg q, \neg\neg q \to q\}$ and $\{q \vee \neg q\}$, respectively. Hence his best choice is also available to us, and our method moreover provides another class incomparable to it. In addition, our class has the merit of using weaker logical principles. So in some cases our class is a strict improvement over his, in addition to the fact it is a generalisation to Glivenko's logic.

3.3 Extension to minimal logic

As we remarked after Lemma 3.2.1, in the last section we relied on the axiom AVQ in two places. Firstly it was required for Glivenko's theorem for Glivenko's logic; secondly it was required in the case for implication in the lemma.

Here it might be hoped that by assuming atomic instances of AVQ, we can overcome the reliance. For this purpose, we shall first look at the usage of AVQ in the 3.2.1. Let $\mathcal{Q}_A$ be the class of the universal closure of all atomic formulae in A, i.e. $\mathcal{Q}_A = \{\forall\vec{x}\neg\neg(\bot \to P) : P \in \mathsf{V}(A)\}$. With this we shall modify Lemma 3.2.1.

Lemma 3.3.1.
(i) $\mathcal{Q}_A \vdash_{qm^+} \neg\neg(\bot \to A)$.
(ii) $\mathcal{Q}_A, \mathcal{W}_A \vdash_{qm^+} \neg\neg A \vee \neg A$.

Proof. For (i), define $\mathcal{Q}'_A = \{\forall\vec{x}\neg\neg(\bot \to P) : P \in \mathcal{V}^{s+}_{nd}(A)\}$. We shall show a stronger result that $\mathcal{Q}'_A \vdash_{qm^+} \neg\neg(\bot \to A)$. We argue by induction on the complexity of A. It is helpful to recall $\vdash_{qm^+} (C \to D) \to (\neg\neg C \to \neg\neg D)$.

- It is immediate that $\mathcal{Q}'_P \vdash_{qm^+} \neg\neg(\bot \to P)$ and $\vdash_{qm^+} \neg\neg(\bot \to \bot)$.
- When $A \equiv B \wedge C$, then by I.H., $\mathcal{Q}'_B \vdash_{qm^+} \neg\neg(\bot \to B)$ and $\mathcal{Q}'_C \vdash_{qm^+} \neg\neg(\bot \to C)$. Thus $\mathcal{Q}'_A \vdash_{qm^+} \neg\neg(\bot \to B \wedge C)$.
- When $A \equiv B \vee C$, w.l.o.g. assume $\mathcal{V}^{s+}_{nd}(B \vee C) = \mathcal{V}^{s+}_{nd}(B)$. Then by I.H., $\mathcal{Q}'_B \vdash_{qm^+} \neg\neg(\bot \to B)$ and so $\mathcal{Q}'_A \vdash_{qm^+} \neg\neg(\bot \to B \vee C)$.
- When $A \equiv B \to C$, then by I.H., $\mathcal{Q}'_C \vdash_{qm^+} \neg\neg(\bot \to C)$. So $\mathcal{Q}'_A \vdash_{qm^+} \neg\neg(\bot \to (B \to C))$.
- When $A \equiv \forall x B$, then by I.H. $\mathcal{Q}'_B \vdash_{qm^+} \neg\neg(\bot \to B)$. Thus $\mathcal{Q}'_A \vdash_{qm^+} \forall x\neg\neg(\bot \to B)$, and by DNS, $\mathcal{Q}'_A \vdash_{qm^+} \neg\neg\forall x(\bot \to B)$. Therefore $\mathcal{Q}'_A \vdash_{qm^+} \neg\neg(\bot \to \forall x B)$.
- When $A \equiv \exists x B$, then by I.H. $\mathcal{Q}'_B \vdash_{qm^+} \neg\neg(\bot \to B)$. So $\mathcal{Q}'_A \vdash_{qm^+} \neg\neg(\bot \to \exists x B)$.

For (ii), the proof is almost identical to Lemma 3.2.1. We only need to be careful that $\mathcal{Q}_A$ is large enough to apply I.H.. Here we look at the case for implication. When $A \equiv B \to C$, $\mathcal{Q}_A = \mathcal{Q}_B \cup \mathcal{Q}_C$ and $\mathcal{W}_A = \mathcal{W}_B \cup \mathcal{W}_C$. So we can apply I.H. that $\mathcal{Q}_B, \mathcal{W}_B \vdash_{qm^+} \neg\neg B \vee \neg B$ and $\mathcal{Q}_C, \mathcal{W}_C \vdash_{qm^+} \neg\neg C \vee \neg C$. In addition, by (i) $\mathcal{Q}_C \vdash_{qm^+} \neg\neg(\bot \to C)$ and so $\mathcal{Q}_C \vdash_{qm^+} \neg B \to \neg\neg(B \to C)$. Then we argue as in Lemma 3.2.1. □

We may note $\mathcal{Q}_A$ and $\mathcal{W}_A$ collect instances from the same set of atomic formulae; so we can merge the two classes into one class $\mathcal{R}_A := \{\forall\vec{x}(\neg\neg(\bot \leftrightarrow P) \vee \neg\neg P) : P \in \mathsf{V}(A)\}$, since $\vdash_{qm^+} ((\neg\neg P \vee \neg P) \wedge \neg\neg(\bot \to P)) \leftrightarrow (\neg\neg(\bot \leftrightarrow P) \vee \neg\neg P)$. We then redefine $\widetilde{\mathcal{W}}_A$ into $\widetilde{\mathcal{R}}_A$.

Definition 3.3.1. We define $\widetilde{\mathcal{R}}_A$ inductively.

$$\begin{aligned}
\widetilde{\mathcal{R}}_P &= \widetilde{\mathcal{R}}_\bot = \emptyset \\
\widetilde{\mathcal{R}}_{A\wedge B} &= \widetilde{\mathcal{R}}_A \cup \widetilde{\mathcal{R}}_B \\
\widetilde{\mathcal{R}}_{A\vee B} &= \widetilde{\mathcal{R}}_A \cup \mathcal{R}_B \text{ or } \mathcal{R}_A \cup \widetilde{\mathcal{R}}_B \\
\widetilde{\mathcal{R}}_{A\to B} &= \widetilde{\mathcal{R}}_B \\
\widetilde{\mathcal{R}}_{\forall x A} &= \widetilde{\mathcal{R}}_A \\
\widetilde{\mathcal{R}}_{\exists x A} &= \mathcal{R}_A
\end{aligned}$$

Then we can proceed exactly as in the last section. The only difference is that we now assume a bigger class $\widetilde{\mathcal{R}}_A$ instead of $\widetilde{\mathcal{Q}}_A$ in order to use instances of WLEM. Therefore:

Proposition 3.3.1. If $\Gamma \vdash_{qm^+} \neg\neg A$, then $\widetilde{\mathcal{R}}_A, \mathcal{D}_A, \Gamma \vdash_{qm^+} A$.

Let us next turn our attention to Glivenko's theorem. We define another class of atomic instances of AVQ, which we shall call $\widetilde{\mathcal{Q}}$.

Definition 3.3.2. We define $\widetilde{\mathcal{Q}}_A$ inductively by the following clauses.

$$\begin{aligned}
\widetilde{\mathcal{Q}}_P = \widetilde{\mathcal{Q}}_\bot &= \emptyset \\
\widetilde{\mathcal{Q}}_{A\wedge B} &= \widetilde{\mathcal{Q}}_A \cup \widetilde{\mathcal{Q}}_B \\
\widetilde{\mathcal{Q}}_{A\vee B} &= \widetilde{\mathcal{Q}}_A \cup \widetilde{\mathcal{Q}}_B \\
\widetilde{\mathcal{Q}}_{A\to B} &= \widetilde{\mathcal{Q}}_A \cup \mathcal{Q}_B \\
\widetilde{\mathcal{Q}}_{\forall xA} &= \widetilde{\mathcal{Q}}_A \\
\widetilde{\mathcal{Q}}_{\exists xA} &= \widetilde{\mathcal{Q}}_A
\end{aligned}$$

Note here $\mathcal{Q}_A \supseteq \widetilde{\mathcal{Q}}_A$ for any A. We shall write $\widetilde{\mathcal{Q}}_\Gamma$ to mean $\bigcup_{A\in\Gamma} \widetilde{\mathcal{Q}}_A$.

We shall modify a couple of classes of formulae in [121, Definition 3.14], which originally comes from [78]. First, recall *Gödel-Gentzen translation* of classical logic into minimal logic. A useful theorem related to the translation is the following.

Theorem 3.3.1. $\vdash_{qm} \neg\neg A^g \leftrightarrow A^g$.

Proof. Cf. for instance [121, Lemma 1.3.3]. □

What we would like to do is to convert Γ^g and A^g back into Γ and A with the aid of extra assumptions of AVQ. Towards this goal we introduce the following classes.

Definition 3.3.3 (Q-spreading, Q-isolating)**.** Given a formula A, we say it is *Q-spreading* if $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} A \to A^g$, and *Q-isolating* if $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} A^g \to \neg\neg A$.

Then we obtain the following result.

Proposition 3.3.2. Let A be a formula. Then A is both Q-spreading and Q-isolating.

Proof. We argue by induction on the complexity of A.

- When $A \equiv P$, then we have $\vdash_{qm^+} P \to \neg\neg P$ and $\vdash_{qm^+} \neg\neg P \to \neg\neg P$.
- When $A \equiv \bot$, then we have $\vdash_{qm^+} \bot \to \bot$ and $\vdash_{qm^+} \bot \to \neg\neg\bot$.
- When $A \equiv B \wedge C$, then by I.H., $\widetilde{\mathcal{Q}}_B \vdash_{qm^+} B \to B^g$ and $\widetilde{\mathcal{Q}}_C \vdash_{qm^+} C \to C^g$. Also $\widetilde{\mathcal{Q}}_A = \widetilde{\mathcal{Q}}_B \cup \widetilde{\mathcal{Q}}_C$ and $A^g \equiv B^g \wedge C^g$. Thus $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} A \to A^g$, so A is Q-spreading. Similarly, by I.H. $\widetilde{\mathcal{Q}}_B \vdash_{qm^+} B^g \to \neg\neg B$ and $\widetilde{\mathcal{Q}}_C \vdash_{qm^+} C^g \to \neg\neg C$; thus $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} A^g \to \neg\neg A$ and so A is Q-isolating.
- When $A \equiv B \vee C$, then by I.H., $\widetilde{\mathcal{Q}}_B \vdash_{qm^+} B \to B^g$ and $\widetilde{\mathcal{Q}}_C \vdash_{qm^+} C \to C^g$. Also $\widetilde{\mathcal{Q}}_A = \widetilde{\mathcal{Q}}_B \cup \widetilde{\mathcal{Q}}_C$ and $A^g \equiv \neg(\neg B^g \wedge \neg C^g)$. Thus $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} (B \vee C) \to \neg(\neg B^g \wedge \neg C^g)$. Similarly, by I.H. $\widetilde{\mathcal{Q}}_B \vdash_{qm^+} B^g \to \neg\neg B$ and $\widetilde{\mathcal{Q}}_C \vdash_{qm^+} C^g \to \neg\neg C$; hence $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \neg B \wedge \neg C \to \neg B^g \wedge \neg C^g$ and consequently $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \neg(\neg B^g \wedge \neg C^g) \to \neg\neg(B \vee C)$.

- When $A \equiv B \to C$, By I.H. $\widetilde{\mathcal{Q}}_B \vdash_{qm^+} B^g \to \neg\neg B$ and $\widetilde{\mathcal{Q}}_C \vdash_{qm^+} C \to C^g$. we have $\widetilde{\mathcal{Q}}_A = \widetilde{\mathcal{Q}}_B \cup \mathcal{Q}_C$. Since $\mathcal{Q}_C \supseteq \widetilde{\mathcal{Q}}_C$, we can apply the I.H. Now $(B \to C)^g \equiv B^g \to C^g$ and $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} (B \to C) \to (B^g \to \neg\neg C^g)$. Thus by Theorem 3.3.1, $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} (B \to C) \to (B^g \to C^g)$. To see A is Q-isolating, by I.H. $\widetilde{\mathcal{Q}}_B \vdash_{qm^+} B \to B^g$ and $\widetilde{\mathcal{Q}}_C \vdash_{qm^+} C^g \to \neg\neg C$. So
$$\widetilde{\mathcal{Q}}_A \vdash_{qm^+} (B^g \to C^g) \to (\neg\neg B \to \neg\neg C). \tag{3.2}$$
Next note by Lemma 3.3.1 (i), $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \neg\neg(\bot \to C)$ and so $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \neg B \to \neg\neg(B \to C)$; hence
$$\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \neg(B \to C) \to \neg\neg B. \tag{3.3}$$
Combining (3.2) and (3.3), we obtain
$$\widetilde{\mathcal{Q}}_A \vdash_{qm^+} (B^g \to C^g) \to (\neg(B \to C) \to \neg\neg C).$$
Then note $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \neg(B \to C) \to \neg C$ to conclude $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} (B^g \to C^g) \to \neg\neg(B \to C)$.

- When $A \equiv \forall x B$, then by I.H. $\widetilde{\mathcal{Q}}_B \vdash_{qm^+} B \to B^g$. Since $\widetilde{\mathcal{Q}}_A = \widetilde{\mathcal{Q}}_B$ and $(\forall x B)^g = \forall x B^g$, we have $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \forall x B \to \forall x B^g$. To see A is Q-isolating, by I.H. $\widetilde{\mathcal{Q}}_B \vdash_{qm^+} B^g \to \neg\neg B$. Thus $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \forall x B^g \to \forall x \neg\neg B$ and by DNS, $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \forall x B^g \to \neg\neg\forall x B$.

- When $A \equiv \exists x B$, then by I.H. $\widetilde{\mathcal{Q}}_B \vdash_{qm^+} B \to B^g$. Since $\widetilde{\mathcal{Q}}_A = \widetilde{\mathcal{Q}}_B$ and $(\exists x B)^g = \neg\forall x \neg B^g$, we have $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \exists x B \to (\exists x B)^g$. To see A is Q-isolating, $\widetilde{\mathcal{Q}}_B \vdash_{qm^+} B^g \to \neg\neg B$. Thus $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \exists x B^g \to \exists x \neg\neg B$, which implies $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \exists x B^g \to \neg\neg\exists x B$ and consequently $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \neg\neg\exists x B^g \to \neg\neg\exists x B$. Therefore $\widetilde{\mathcal{Q}}_A \vdash_{qm^+} \neg\forall x \neg B^g \to \neg\neg\exists x B$.

□

Combining Proposition 3.3.1 and Proposition 3.3.2, we finally obtain an answer to Ishihara's problem for minimal logic (+DNS+CD).

Theorem 3.3.2. If $\Gamma \vdash_{qc} A$, then $\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}}, \widetilde{\mathcal{R}}_A, \mathcal{D}_A, \Gamma \vdash_{qm^+} A$.

Proof. If $\Gamma \vdash_{qc} A$, then $\Gamma^g \vdash_{qm^+} A^g$. So there is some finite $\Delta \subseteq \Gamma$ such that $\Delta^g \vdash_{qm^+} A^g$. Then by Proposition 3.3.2, $\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}}$ assures $D \to D^g$ for all $D \in \Delta$ and $A^g \to \neg\neg A$; thus $\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}}, \Delta \vdash_{qm^+} \neg\neg A$. Hence by Proposition 3.3.1, $\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}}, \widetilde{\mathcal{R}}_A, \mathcal{D}_A, \Gamma \vdash_{qm^+} A$. □

Let us write $\mathsf{V}(\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}})$, $\mathsf{V}(\widetilde{\mathcal{R}}_A)$ and $\mathsf{V}(\mathcal{D}_A)$ for atomic formulae occurring in the sets. Then we further obtains the next result.

Corollary 3.3.1. If $\Gamma \vdash_{qc} A$ and $\mathsf{V}(\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}}) \subseteq \mathsf{V}(\widetilde{\mathcal{R}}_A) \cup \mathsf{V}(\mathcal{D}_A)$, then $\widetilde{\mathcal{R}}_A, \mathcal{D}_A, \Gamma \vdash_{qm^+} A$.

Proof. Trivially, if an element of $\mathsf{V}(\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}})$ is in $\mathsf{V}(\widetilde{\mathcal{R}}_A)$ then the instance of AVQ is already assumed by the latter collection. In addition, because $\vdash_{qm^+} (\neg\neg P \to P) \to (\bot \to P)$, *a fortiori* atomic instances of DNE subsume the instances of AVQ. So we can eliminate $\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}}$ from the assumption. □

The derivability of EFQ from DNE might make one wonder of the possibility that perhaps $\mathcal{V}^{s+}(A)$ always gives sufficient instances of EFQ (via instances of DNE) for the preservation to minimal logic. In other words, can we drop the condition $\mathsf{V}(\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}}) \subseteq \mathsf{V}(\widetilde{\mathcal{R}}_A) \cup \mathsf{V}(\mathcal{D}_A)$ from the above corollary? The answer is in the negative, because for $\vdash_{qc} \neg\neg(\bot \to P)$, $\widetilde{\mathcal{R}}_A = \mathcal{D}_A = \emptyset$ and $\nvdash_{m^+} \neg\neg(\bot \to P)$.

In the above, we proceeded with two steps: of obtaining $\neg\neg A$ from A^g and of obtaining A from $\neg\neg A$. We can make them into a single step by considering a *wiping* class ($\vdash_{qm^+} A^g \to A$) instead of an isolating class ($\vdash_{qm^+} A^g \to \neg\neg A$). Then it turns out we would need to assume not a class of AVQ but classes of DNE and WLEM for the analogue of Proposition 3.3.2 to work. These classes, let us call them $\mathcal{D}'_A$ and $\widetilde{\mathcal{W}}'_A$, turn out to be larger than $\mathcal{D}_A$ and $\widetilde{\mathcal{W}}_A$, respectively. To be precise, $\mathcal{D}'_A := \{\forall x(\neg\neg P \to P) : P \in \mathsf{V}(A)\}$ (which, as commented above implies contains $\widetilde{\mathcal{Q}}_A$) and $\widetilde{\mathcal{W}}'_A$ is obtained by changing the clause for implication to $\widetilde{\mathcal{W}}'_{A\to B} = \widetilde{\mathcal{W}}'_A \cup \widetilde{\mathcal{W}}'_B$. We need bigger classes, essentially because in order to apply I.H. we have to make sure the assumptions in the I.H. need to be included in the current assumption. Hence our two-step approach in terms of an isolating class gives a better outcome than one-step approach using a wiping class.

For propositional logic, there is an alternative method to obtain a class of AVQ different from $\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}}$. The argument appeals to derivations in **G3i**. We note that in **G3i**, the C in the rule (L$\bot$) can be assumed (and we shall assume in this chapter) to be either a propositional variable or $\bot$, since $\bot, \Gamma \Rightarrow C$ for general C can be derived cut-free with the instances. If we restrict it to $\bot$ alone, then we get the calculus for minimal logic **G3m**. We shall denote the derivability in the calculi and $\vdash_{g3m}$, respectively. By the equivalence with the Hilbert-type system, $\Gamma \vdash_m A$ if and only if $\vdash_{g3m} \Gamma \Rightarrow A$.

Now we shall recall the subformula property [120, Proposition 4.2.1] for **G3i**: instead of the full statement, we extract the part we will rely on.

Proposition 3.3.3 (subformula property)**.** If a sequent $\Gamma \Rightarrow p$ occurs in a derivation in **G3i** of $\Gamma' \Rightarrow C$, then $p \in \mathcal{V}^-(A)$ for some $A \in \Gamma'$, or $p \in \mathcal{V}^+(C)$.

This means if $\vdash_{g3i} \Gamma \Rightarrow C$, then all the propositional variables in the application of (L$\bot$) occur either negatively in Γ or positively in C. Hence the propositional instances $\bot \to p$ of EFQ for $p \in \mathcal{V}^-(\Gamma) \cup \mathcal{V}^+(C)$ suffice as assumption set $\mathcal{A}_{\Gamma\cup\{C\}}$ to derive $\vdash_{g3m} \mathcal{A}_{\Gamma\cup\{C\}}, \Gamma \Rightarrow C$.

For our purpose, $\mathcal{A}$ can be weakened to a set of instances of AVQ; that is, given $\Gamma \vdash_i A$, we define $\mathcal{B}_{\Gamma\cup\{A\}} := \{\neg\neg(\bot \to p) : p \in \mathcal{V}^-(\Gamma) \cup \mathcal{V}^+(A)\}$. Then:

Theorem 3.3.3. if $\Gamma \vdash_c A$, then $\mathcal{B}_{\Gamma\cup\{A\}}, \widetilde{\mathcal{R}}_A, \mathcal{D}_A, \Gamma \vdash_m A$.

Proof. If $\Gamma \vdash_c A$, then by Glivenko's theorem $\Gamma \vdash_i \neg\neg A$. So by the above observation, $\mathcal{A}_{\Gamma\cup\{\neg\neg A\}}, \Gamma \vdash_m \neg\neg A$. Thus for some finite $\mathcal{A}' \subseteq \mathcal{A}_{\Gamma\cup\{\neg\neg A\}}$, $\mathcal{A}', \Gamma \vdash_m \neg\neg A$. Consequently, for $\neg\neg\mathcal{A}' := \{\neg\neg(\bot \to p) : p \in \mathcal{A}'\}$ we obtain $\neg\neg\mathcal{A}', \Gamma \vdash_m \neg\neg A$, by contraposing finite many times.[2] Therefore $\mathcal{B}_{\Gamma\cup\{A\}}, \Gamma \vdash_m \neg\neg A$ (note $\mathcal{V}^+(\neg\neg C) = \mathcal{V}^+(C)$ for any C). Then we argue as in Theorem 3.3.2 to reach the conclusion. □

For the predicate case, we have additional axioms, and so the same strategy via cut-free calculus does not apply. To compare our two methods for propositional logic, for $A \equiv (\bot \to p) \vee \neg\neg q$ we see $\widetilde{\mathcal{Q}}_A = \{p\}$ but $\mathcal{B}_A = \{p, q\}$; on the other hand, for $A \equiv \bot \to (q \to p)$ we have $\widetilde{\mathcal{Q}}_A = \{p, q\}$ but $\mathcal{B}_A = \{p\}$. Hence it is not the case that one of $\widetilde{\mathcal{Q}}_A$ and $\mathcal{B}_A$ is always smaller than the other.

3.4 Discussion

The present enquiry extended the results of Ishii (i) to Glivenko's logic by considering the classes of WLEM and DNE instead of LEM (ii) to minimal logic by assuming additional classes of AVQ, and (iii) to predicate logic enhanced with intermediate axioms DNS and CD.

An apparent future direction would be to contrast the present study with the methodology of Ishihara [68]. Although his method does not have a direct connection to Glivenko's theorem, and so Glivenko's logic may not play as vital a role, it is still of interest whether the class of assumptions can be weakened with those of WLEM and DNE. It is also desirable to obtain the conservativity result for minimal logic using AVQ. However, Ishihara's method in its original form appears to make crucial use of LEM and EFQ, hence some new techniques would be required. It also has to be carefully investigated how strong the predicate fragment has to be in order to extend Ishihara's result to predicate logic.

From a different perspective, the enquiries so far all used proof-theoretical methodology. It would be interesting to consider whether a semantical viewpoint can offer another route to obtain an improvement to the known results.

[2] This argument is suggested by Hajime Ishihara, and the idea is related to the one in [39].

Chapter 4

Relationship among logics with weak negation

4.1 Minimal logic and subminimal negation

One topic we mentioned in the Chapter 2 is how important a role negation plays in the characterisation of intuitionistic logic. We also made reference to some alternative formalisations of the ideas of intuitionism. One of them was Johansson's minimal logic, which rejected the *ex falso quodlibet* principle of the intuitionistic logic.

We shall look into two generalisations of minimal logic in the direction of yet weaker negation. One of them is the logic introduced by Dimiter Vakarelov [126, 127]. Vakarelov's system, which we shall call by the name $\mathbf{An^{-}PC}$, is characterised by a negation which is "relative", in the sense that no theorem of the form $\neg A$ is derivable in the system.

The other approach is the class of logics endowed with what is called *submnimal negation*. This family of logics was introduced by Almudena Colacito, Dick de Jongh and Ana Lucia Vargas [25, 24]. The only axiom about negation that is assumed in all of the logics is:

$$(A \leftrightarrow B) \rightarrow (\neg A \leftrightarrow \neg B).$$

We next move on to the main objective of this chapter: investigation of $\mathbf{An^{-}PC}$ by applying the results and techniques of subminimal negation. We shall begin with capturing $\mathbf{An^{-}PC}$ in terms of the semantics of subminimal negation, thereby investigate its semantic properties as well as clarify its relationship with the logics with subminimal negation. Aided by this insight we shall then move on to formulate the sequent calculus for $\mathbf{An^{-}PC}$, prove cut-elimination and some of its consequences. Finally, as a separate topic, we shall formulate a new countable class of logics with subminimal negation.

4.2 Vakarelovs logic and subminimal logic: semantics

In this section, we shall introduce two types of Kripke semantics for $\mathbf{An^{-}PC}$. We shall then establish the translations between the semantics. From here we use the language $\mathcal{L}_{\neg,\top}$, which is obtained from $\mathcal{L}_{\neg}$ with an additional constant $\top$. Intuitively,

$\top$ stands for a tautology that always holds.

Vakarelov [126, 127] formulates an extension of intuitionistic Kripke semantics for his system, which we shall call (F, G)-*semantics*. Here we slightly modify it for $\mathbf{An^{-}PC}$. It uses two upward closed sets of worlds F and G. Intuitively, they each stands for the set of worlds in which all/some negations are forced. Then the forcing of negation is modified from that of intuitionistic logic to reflect this aspect: we require a negation to be forced in a world only if it is in G. Hence we have the following definition.

Definition 4.2.1. An (F, G)-*frame* $\mathcal{F}$ for $\mathbf{An^{-}PC}$ is a quadruple $(W, \leq, F, G)$ where $(W, \leq)$ is an inhabited pre-ordered set, and F,G are upward closed subsets of W such that $F \subseteq G$. An (F, G)-*model* $\mathcal{M}$ for $\mathbf{An^{-}PC}$ is a pair $(\mathcal{F}, \mathcal{V})$ where $\mathcal{F}$ is an (F, G)-frame and $\mathcal{V}$ is a mapping assigning an upward closed set of worlds $\mathcal{V}(p)$ to each propositional variable p. The valuation $(\Vdash_{(F,G)})$ of formulae in a world is inductively defined as follows.

$$
\begin{array}{lll}
w \Vdash_{(F,G)} \top. & & \\
w \Vdash_{(F,G)} p & \Leftrightarrow & w \in \mathcal{V}(p). \\
w \Vdash_{(F,G)} A \wedge B & \Leftrightarrow & w \Vdash_{(F,G)} A \text{ and } w \Vdash_{(F,G)} B. \\
w \Vdash_{(F,G)} A \vee B & \Leftrightarrow & w \Vdash_{(F,G)} A \text{ or } w \Vdash_{(F,G)} B. \\
w \Vdash_{(F,G)} A \to B & \Leftrightarrow & \text{for all } w' \geq w[w' \Vdash_{(F,G)} A \text{ implies } w' \Vdash_{(F,G)} B]. \\
w \Vdash_{(F,G)} \neg A & \Leftrightarrow & \text{for all } w' \geq w[w' \Vdash_{(F,G)} A \text{ implies } w' \in F] \\
 & & \text{and } w \in G.
\end{array}
$$

The Kripke semantics for subminimal negation in comparison allows a more general framework. The main idea is to refine the notion of negation to its essential characteristic. That restricts us to regard negation as just an operator such that the (upward closed) set of worlds forcing a formula determines the set of worlds forcing the negation of formulae. It is also required that this is consistent with the upward closure of forcing, which necessitates the imposition of a restriction of *locality* on such a determination.

Definition 4.2.2. An N-*frame* is a triple $(W, \leq, N)$, where $(W, \leq)$ is a pre-ordered set and $N : \mathcal{U}(W) \to \mathcal{U}(W)$ is a mapping from the set $\mathcal{U}(W)$ of upward closed subsets of W to itself. N must satisfy the condition of *locality*:

$$\forall U \in \mathcal{U}(W) \forall w \in W (w \in N(U) \Leftrightarrow w \in N(U \cap \mathcal{R}(w))),$$

where $\mathcal{R}(w) := \{v : v \geq w\}$. When it comes to the valuation $(\Vdash_N)$, only the valuation for negation is different, which is given by

$$w \Vdash_N \neg A \Leftrightarrow w \in N(\mathcal{V}(A)),$$

for $\mathcal{V}(A) := \{w : w \Vdash_N A\}$.

$\mathbf{NPC}$ is sound and complete with respect to N-semantics [25]. We now consider the relationship between the two semantics over $\mathbf{An^{-}PC}$. We will make use of a fact from [24, Lemma 4.3.2] that $\mathcal{R}(w) \cap N(U) = \mathcal{R}(w) \cap N(U \cap \mathcal{R}(w))$ for any U and w.

As we shall look into more closely in the next section, the following formula plays a characteristic role for $\mathbf{An^-PC}$

$$[\mathrm{An}^-]\ (A \to \neg A) \to (\neg B \to \neg A).$$

We wish to capture semantically the validity of this formula. Hence we introduce the next class of frames.

Definition 4.2.3. We define an *NAn^--frame* as an N-frame with the property

$$\forall U, V \in \mathcal{U}(W)[U \subseteq N(U) \Rightarrow N(V) \subseteq N(U)].$$

As the name suggests, the condition of $N\mathrm{An}^-$-frame directly corresponds to the validity of An^-.

Lemma 4.2.1. Let $\mathcal{F}$ be an N-frame. The next conditions are equivalent.

(i) $\mathcal{F} \vDash_N (A \to \neg A) \to (\neg B \to \neg A)$ for all A and B.

(ii) $\mathcal{F}$ is an $N\mathrm{An}^-$-frame.

Proof. To see (i) implying (ii), we shall prove by contraposition. Assume $\mathcal{F}$ to be a non-$N\mathrm{An}^-$-frame. That is to say, $\exists U, V \in \mathcal{U}(W)[U \subseteq N(U) \wedge \neg(N(V) \subseteq N(U))]$. Then let $\mathcal{V}$ be such that $\mathcal{V}(p) = U$ and $\mathcal{V}(q) = V$. By assumption we can find $w \in N(V)$ such that $w \notin N(U)$. Now for all $w' \geq w$, if $(\mathcal{F}, \mathcal{V}), w' \Vdash_N p$ then $w' \in U \subseteq N(U)$. Thus $(\mathcal{F}, \mathcal{V}), w' \Vdash_N \neg p$ and so $(\mathcal{F}, \mathcal{V}), w \Vdash_N p \to \neg p$. But $(\mathcal{F}, \mathcal{V}), w \nVdash_N \neg q \to \neg p$ by our choice of w. Hence $(\mathcal{F}, \mathcal{V}), w \nVdash_N (p \to \neg p) \to (\neg q \to \neg p)$. Therefore $\mathcal{F} \nvDash_N (p \to \neg p) \to (\neg q \to \neg p)$.

To see (ii) implying (i), let $\mathcal{V}$ and w be arbitrary. Assume for $v \geq u \geq w$, $(\mathcal{F}, \mathcal{V}), u \Vdash_N A \to \neg A$ and $(\mathcal{F}, \mathcal{V}), v \Vdash_N \neg B$. Then $\mathcal{V}(A) \cap \mathcal{R}(u) \subseteq N(\mathcal{V}(A)) \cap \mathcal{R}(u)$ by the former assumption. Now $N(\mathcal{V}(A)) \cap \mathcal{R}(u) = N(\mathcal{V}(A) \cap \mathcal{R}(u)) \cap \mathcal{R}(u) \subseteq N(\mathcal{V}(A) \cap \mathcal{R}(u))$. Thus $\mathcal{V}(A) \cap \mathcal{R}(u) \subseteq N(\mathcal{V}(A) \cap \mathcal{R}(u))$, so we infer from the condition of $N\mathrm{An}^-$-frame that $N(\mathcal{V}(B)) \subseteq N(\mathcal{V}(A) \cap \mathcal{R}(u))$. Then by the latter assumption $v \in N(\mathcal{V}(A) \cap \mathcal{R}(u))$, which by locality implies $v \in N(\mathcal{V}(A) \cap \mathcal{R}(u) \cap \mathcal{R}(v))$. But $\mathcal{V}(A) \cap \mathcal{R}(u) \cap \mathcal{R}(v) = \mathcal{V}(A) \cap \mathcal{R}(v)$. So by locality again, $v \in N(\mathcal{V}(A))$. Thus $(\mathcal{F}, \mathcal{V}), v \Vdash_N \neg A$. So $(\mathcal{F}, \mathcal{V}), w \Vdash_N (A \to \neg A) \to (\neg B \to \neg A)$; thus $\mathcal{F} \vDash_N \mathrm{An}^-$. $\square$

From now on we shall denote the valuation/validity with respect to the class of $N\mathrm{An}^-$-frames by $\Vdash_{NAn^-}$ and $\vDash_{NAn^-}$.

Given an $N\mathrm{An}^-$-frame, we define an (F, G)-frame in the following way.

Theorem 4.2.1. Let $\mathcal{F} = (W, \leq, N)$ be an $N\mathrm{An}^-$-frame, and $\mathcal{V}$ be a valuation. Take

$$F := \bigcap_{U \in \mathcal{U}(W)} N(U) \text{ and } G = \bigcup_{U \in \mathcal{U}(W)} N(U).$$

Then $\mathcal{M} := (W, \leq, F, G, \mathcal{V})$ defines an (F,G)-model for $\mathbf{An^-PC}$ such that

$$(\mathcal{F}, \mathcal{V}), w \Vdash_{NAn^-} A \Leftrightarrow \mathcal{M}, w \Vdash_{(F,G)} A.$$

Proof. Throughout the proof, $\mathcal{V}(A)$ denotes $\{w : w \Vdash_{NAn^-} A\}$. First note that $F \subseteq G$, and so F and G are well-defined. We use induction on the complexity of A. It is sufficient to consider the case for negation, i.e. to show

$$(\mathcal{F}, \mathcal{V}), w \Vdash_{NAn^-} \neg A \Leftrightarrow \forall w' \geq w[\mathcal{M}, w' \Vdash_{(F,G)} A \Rightarrow w' \in F] \text{ and } w \in G.$$

For the left-to-right direction, if $(\mathcal{F}, \mathcal{V}), w \Vdash_{NAn^-} \neg A$, then $w \in G$. Further, if $\mathcal{M}, w' \Vdash_{(F,G)} A$ for $w' \geq w$, then by I.H. $(\mathcal{F}, \mathcal{V}), w' \Vdash_{NAn^-} A$, so $(\mathcal{F}, \mathcal{V}), w' \Vdash_{NAn^-} A \wedge \neg A$. We must show $w' \in N(V)$ for each $V \in \mathcal{U}(W)$. Let q be a variable not occurring in A, and take an arbitrary $V \in \mathcal{U}(W)$. Then let $\mathcal{V}' := \mathcal{V}$ except that $\mathcal{V}'(q) = V$. Then we can show by induction that $(\mathcal{F}, \mathcal{V}), u \Vdash_{NAn^-} B \Leftrightarrow (\mathcal{F}, \mathcal{V}'), u \Vdash_{NAn^-} B$ for arbitrary u and B not containing q. Hence $(\mathcal{F}, \mathcal{V}'), w' \Vdash_{NAn^-} A \wedge \neg A$. Also, by Lemma 4.2.1, $(\mathcal{F}, \mathcal{V}'), w' \Vdash_{NAn^-} (A \wedge \neg A) \to \neg q$. So $(\mathcal{F}, \mathcal{V}'), w' \Vdash_{NAn^-} \neg q$. Hence $w' \in N(\mathcal{V}'(q)) = N(V)$. As V is arbitrary, this procedure can be done for any U in $\mathcal{U}(W)$, i.e. we can always pick a fresh variable r not occurring in A and a valuation V'' with $V'' := V$ except $V''(r) = U$ such that $w' \in N(U)$. Therefore $w' \in \bigcap_{U \in \mathcal{U}(W)} N(U) = F$.

For the right-to-left direction, by assumption and I.H., $\mathcal{R}(w) \cap \mathcal{V}(A) \subseteq F$. Also by definition, $F \subseteq N(\mathcal{V}(A))$. So,

$$\mathcal{R}(w) \cap \mathcal{V}(A) \subseteq \mathcal{R}(w) \cap N(\mathcal{V}(A)) = \mathcal{R}(w) \cap N(\mathcal{R}(w) \cap \mathcal{V}(A)).$$

Hence $\mathcal{R}(w) \cap \mathcal{V}(A) \subseteq N(\mathcal{R}(w) \cap \mathcal{V}(A))$. Thus by the frame property of $\mathcal{F}$, $N(V) \subseteq N(\mathcal{R}(w) \cap \mathcal{V}(A))$ for any V. So $G \subseteq N(\mathcal{R}(w) \cap \mathcal{V}(A))$. As $w \in G$, $w \in N(\mathcal{R}(w) \cap \mathcal{V}(A))$. So $w \in N(\mathcal{V}(A))$ by locality. Hence $(\mathcal{F}, \mathcal{V}), w \Vdash_{NAn^-} \neg A$. □

We have a similar theorem addressing the opposite direction, starting from an (F, G)-model to define an $N\mathrm{An}^-$-models such that they satisfy the same formulae.

Theorem 4.2.2. Let $\mathcal{F} = (W, \leq, F, G)$ be an (F, G)-frame for $\mathbf{An^-PC}$ and $\mathcal{V}$ be a valuation. Define $N : \mathcal{U}(W) \to \mathcal{U}(W)$ by

$$w \in N(U) \Leftrightarrow (\mathcal{R}(w) \cap U \subseteq F) \wedge w \in G.$$

Then $\mathcal{M} := (W, \leq, N, \mathcal{V})$ defines an $N\mathrm{An}^-$-model such that

$$(\mathcal{F}, \mathcal{V}), w \Vdash_{(F,G)} A \Leftrightarrow \mathcal{M}, w \Vdash_{NAn^-} A.$$

Proof. First we check that N is well-defined and an $N\mathrm{An}^-$-frame. We show

(i) $w \in N(U)$ and $w' \geq w \Rightarrow w' \in N(U)$;

(ii) $w \in N(U) \Leftrightarrow w \in N(U \cap \mathcal{R}(w))$;

(iii) $U \subseteq N(U) \Rightarrow N(V) \subseteq N(U)$.

For (i), if $w \in N(U)$, then by definition $\mathcal{R}(w) \cap U \subseteq F$ and $w \in G$. So $\mathcal{R}(w') \cap U \subseteq F$ and $w' \in G$ for any $w' \geq w$. Hence $w' \in N(U)$. For (ii), by definition, $w \in N(U)$ if and only if $(\mathcal{R}(w) \cap U \subseteq F) \wedge w \in G$; but this latter condition is equivalent to $(\mathcal{R}(w) \cap \mathcal{R}(w) \cap U \subseteq F) \wedge w \in G$. This is equivalent to $w \in N(\mathcal{R}(w) \cap U)$ again by definition. For (iii), assume $U \subseteq N(U)$ and let $w \in N(V)$. Then $(\mathcal{R}(w) \cap V \subseteq F) \wedge w \in G$. Now if $w' \in \mathcal{R}(w) \cap U$, then $w' \in N(U)$ by assumption. So $\mathcal{R}(w') \cap U \subseteq F$.

Hence $w' \in F$. Thus $(\mathcal{R}(w) \cap U \subseteq F) \wedge w \in G$. That is, $w \in N(U)$.

The equivalence of valuation is shown by induction on the complexity of A. For negation,

$$(\mathcal{F}, \mathcal{V}), w \Vdash_{(F,G)} \neg A \text{ if and only if } \forall w' \geq w[(\mathcal{F}, \mathcal{V}), w' \Vdash_{(F,G)} A \Rightarrow w' \in F] \wedge w \in G.$$

By I.H., this is equivalent to $\forall w' \geq w[\mathcal{M}, w' \Vdash_{NAn^-} A \Rightarrow w' \in F] \wedge w \in G$, and this in turn is equivalent to $(\mathcal{R}(w) \cap \mathcal{V}(A) \subseteq F) \wedge w \in G$ and hence to $w \in N(\mathcal{V}(A))$, that is $\mathcal{M}, w \Vdash_{NAn^-} \neg A$. □

Let us name the mapping from an $N\mathrm{An}^-$-frame to an (F, G)-frame defined in Theorem 4.2.1 as Φ, and the mapping from an (F, G)-frame to an $N\mathrm{An}^-$-frame defined in Theorem 4.2.2 as Ψ.

Theorem 4.2.3. The mappings Φ and Ψ are inverse mappings; i.e.

(i) $\Psi \circ \Phi(\mathcal{F}) = \mathcal{F}$ for $N\mathrm{An}^-$-frame $\mathcal{F}$.

(ii) $\Phi \circ \Psi(\mathcal{F}) = \mathcal{F}$ for (F, G)-frame $\mathcal{F}$.

Proof. For (i), given $\mathcal{F} = (W, \leq, N)$, we have to show

$$w \in N(U) \Leftrightarrow [\mathcal{R}(w) \cap U \subseteq \bigcap_{V \in \mathcal{U}(W)} N(V)] \wedge w \in \bigcup_{V \in \mathcal{U}(W)} N(V).$$

For the left-to-right direction, We argue by contraposition. Assume

$$\neg[\mathcal{R}(w) \cap U \subseteq \bigcap_{V \in \mathcal{U}(W)} N(V)] \vee w \notin \bigcup_{V \in \mathcal{U}(W)} N(V).$$

If $w \notin \bigcup_{V \in \mathcal{U}(W)} N(V)$, then immediately $w \notin N(U)$. If $\neg[\mathcal{R}(w) \cap U \subseteq \bigcap_{V \in \mathcal{U}(W)} N(V)]$, take $u \in \mathcal{R}(w) \cap U$ such that $u \notin \bigcap_{V \in \mathcal{U}(W)} N(V)$. Now if $w \in N(U)$, then take $\mathcal{V}$ such that $\mathcal{V}(p) = U$ and $\mathcal{V}(q) = V$ for certain V. Then $w \in N(U)$ implies $(\mathcal{F}, \mathcal{V}), w \Vdash_{NAn^-} \neg p$. Thus $(\mathcal{F}, \mathcal{V}), u \Vdash_{NAn^-} p \wedge \neg p$. As $(\mathcal{F}, \mathcal{V}), u \Vdash_{NAn^-} (p \wedge \neg p) \to \neg q$, $(\mathcal{F}, \mathcal{V}), u \Vdash_{NAn^-} \neg q$. So $u \in N(V)$. As V is arbitrary, $u \in \bigcap_{V \in \mathcal{U}(W)} N(V)$, a contradiction. For the right-to-left direction, if $[\mathcal{R}(w) \cap U \subseteq \bigcap_{V \in \mathcal{U}(W)} N(V)] \wedge w \in \bigcup_{V \in \mathcal{U}(W)} N(V)$ then $\mathcal{R}(w) \cap U \subseteq \bigcap_{V \in \mathcal{U}(W)} N(V) \subseteq N(U)$. So

$$\mathcal{R}(w) \cap U \subseteq \mathcal{R}(w) \cap N(U) = \mathcal{R}(w) \cap N(U \cap \mathcal{R}(w)) \subseteq N(U \cap \mathcal{R}(w)).$$

By the frame property, $N(V) \subseteq N(U \cap \mathcal{R}(w))$ for any V. So $\bigcup_{V \in \mathcal{U}(W)} N(V) \subseteq N(U \cap \mathcal{R}(w))$. Hence $w \in N(U)$ by locality.

For (ii), given $\mathcal{F} = (W, \leq, F, G)$, define N as in Theorem 4.2.2. We have to show $\bigcap_{U \in \mathcal{U}(W)} N(U) = F$ and $\bigcup_{U \in \mathcal{U}(W)} N(U) = G$. To see the former, for the left-to-right direction, if $w \in \bigcap_{U \in \mathcal{U}(W)} N(U)$ then in particular $w \in N(\mathcal{R}(w))$. By definition, this means $(\mathcal{R}(w) \cap \mathcal{R}(w) \subseteq F) \wedge w \in G$. Hence $w \in F$. For the right-to-left direction, if $w \in F$ then $(\mathcal{R}(w) \cap U \subseteq F) \wedge w \in G$ for all U. So $w \in \bigcap_{U \in \mathcal{U}(W)} N(U)$ by definition of N. Next, to see the latter, for the left-to-right direction, if $w \in \bigcup_{U \in \mathcal{U}(W)} N(U)$ then $w \in N(V)$ for some V. Hence $w \in G$. For the right-to-left direction, if $w \in G$ then $(\mathcal{R}(w) \cap F \subseteq F) \wedge w \in G$. Hence $w \in N(F)$, so $w \in \bigcup_{U \in \mathcal{U}(W)} N(U)$. □

Corollary 4.2.1.

(i) If CL_A is the class of $N\text{An}^-$-frames validating a formula A, then $\{\Phi(\mathcal{F}) : \mathcal{F} \in CL_A\}$ is the class of (F, G)-frames validating A.

(ii) If CL_A is the class of (F, G)-frames validating a formula A, then $\{\Psi(\mathcal{F}) : \mathcal{F} \in CL_A\}$ is the class of $N\text{An}^-$-frames validating A.

Proof. For (i), by Theorem 4.2.1, If $\mathcal{F} \vDash_{NAn^-} A$ then $\Phi(\mathcal{F}) \vDash_{(F,G)} A$. Conversely, if $\mathcal{G} \vDash_{(F,G)} A$, then $\Psi(\mathcal{G}) \vDash_{NAn^-} A$, so $\Psi(\mathcal{G}) \in CL_A$. Therefore by the last theorem, $\mathcal{G} = \Phi \circ \Psi(\mathcal{G}) \in \{\Phi(\mathcal{F}) : \mathcal{F} \in CL_A\}$. (ii) is analogous. □

4.3 Vakarelovs logic and subminimal logic: proof theory

Now we turn our attention to the proof theory of **An$^-$PC**. First we introduce Hilbert systems of minimal logic in $\mathcal{L}_{\neg,\top}$ and the eponymous system of **An$^-$PC**, which we shall confirm as the counterpart of the $\bot$-less fragment of Vakarelov's **SUBMIN**. We shall then show the soundness and completeness with the (F, G)-semantics. This is followed by the formulation of a sequent calculus **GAn$^-$**, and show cut-elimination and its consequences.

So let us begin with minimal logic. An important characteristic of **MPC** in $\mathcal{L}_\bot$ was that there is no axiom specific to $\bot$. In $\mathcal{L}_{\neg,\top}$, however, this turns out to be insufficient; we need an axiom expressing the property of negation in the system. We shall denote the system by **MPC$_\neg$**. It in addition contains $\top$; this addition is inessential but for the sake of comparison with the original system of Vakarelov.

Definition 4.3.1 (**MPC$_\neg$**)**.** We axiomatise **MPC$_\neg$** in by the axioms of **MPC** and

- $A \to \top$
- $(A \to B) \wedge (A \to \neg B) \to \neg A$

Then **An$^-$PC** is obtained by weakening the latter axiom of negation.

Definition 4.3.2. (**An$^-$PC**) We define a Hilbert system of **An$^-$PC** by replacing $(A \to B) \wedge (A \to \neg B) \to \neg A$ in **MPC$_\neg$** with the next axioms.

$$\text{N} : (A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$$
$$\text{An}^- : (A \to \neg A) \to (\neg B \to \neg A)$$

We shall use $\vdash_{\mathbf{An}^-}$ for the derivability in **An$^-$PC**. It is easy to check the deduction theorem holds for **An$^-$PC**.

The removal of An$^-$ defines the logic **NPC** taken to be basic in [10, 24, 25] (therein called N). The deducibility in **NPC** will be denoted by $\vdash_{\mathbf{N}}$. If we add An: $(A \to \neg A) \to \neg A$ to **NPC** (note An is a strengthening of An$^-$), we obtain back **MPC$_\neg$** [10, 24].[1] Therefore An$^-$ be understood as a restriction on An.

Let us now see as an example that $\vdash_{\mathbf{An}^-} (A \to B) \to (\neg B \to \neg A)$. First we observe that $\vdash_{\mathbf{An}^-} [(A \to B) \wedge A] \to (A \leftrightarrow B)$; so by N, $\vdash_{\mathbf{An}^-} [(A \to B) \wedge A] \to (\neg B \to \neg A)$. Hence $\vdash_{\mathbf{An}^-} [(A \to B) \wedge \neg B] \to (A \to \neg A)$; by An$^-$ we obtain

[1] A weaker axiom $(A \wedge \neg A) \to (B \to \neg B)$ in place of N suffices for this purpose. That this axiom is derivable in **NPC** shows the limitation of paraconsistency in subminimal logics.

$\vdash_{\mathbf{An}^-} [(A \to B) \wedge \neg B] \to (\neg B \to \neg A)$. Therefore $\vdash_{\mathbf{An}^-} (A \to B) \to (\neg B \to \neg A)$.

We can similarly check that $(A \wedge \neg A) \to \neg B$ and $\neg A \leftrightarrow \neg\neg\neg A$ are derivable in $\mathbf{An^-PC}$. We shall refer to $(A \to B) \to (\neg B \to \neg A)$ and $(A \wedge \neg A) \to \neg B$ as Co and NeF. The addition of Co (in this case N is redundant) or NeF to **NPC** each defines logics **CoPC** and **NeFPC** (CoPC and NeF in [10, 24, 25]).

We now check that $\mathbf{An^-PC}$ is equivalent to the $\bot$-less fragment of **SUBMIN**, which is defined by $\mathbf{CoPC}+\neg A \to \neg\neg\top$. We shall call the fragment as $\mathbf{SUBMIN}^-$.

Proposition 4.3.1. $\mathbf{An^-PC}$ is equal to $\mathbf{SUBMIN}^-$.

Proof. We have to show (i) $\mathbf{An^-PC} \vdash$ Co and $\mathbf{An^-PC} \vdash \neg A \to \neg\neg\top$; as well as (ii) $\mathbf{SUBMIN}^- \vdash$ N and $\mathbf{SUBMIN}^- \vdash \mathrm{An}^-$.

For (i), the former is already discussed. For the latter, $\vdash_{\mathbf{An}^-} \neg A \to (\neg\top \to \top)$ and so $\vdash_{\mathbf{An}^-} \neg A \to (\neg\top \to \neg\neg\top)$ from Co. Then by An^-, $\vdash_{\mathbf{An}^-} \neg A \to (\neg A \to \neg\neg\top)$ and so $\vdash_{\mathbf{An}^-} \neg A \to \neg\neg\top$.

For (ii), the former follows from the presence of Co. For the latter, we first note that NeF is derivable from Co. Since $\mathbf{SUBMIN}^- \vdash (A \to \neg A) \to (A \to (A \wedge \neg A))$, we have $\mathbf{SUBMIN}^- \vdash (A \to \neg A) \to (A \to \neg B)$ by NeF. Hence $\mathbf{SUBMIN}^- \vdash (A \to \neg A) \to (\neg\neg B \to \neg A)$ by another application of Co. [2] In particular, $\mathbf{SUBMIN}^- \vdash (A \to \neg A) \to (\neg\neg\top \to \neg A)$. Finally use $\neg B \to \neg\neg\top$ to replace $\neg\neg\top$ with $\neg B$. □

Now we check the soundness and completeness of $\mathbf{An^-PC}$ with respect to (F, G)-semantics.

Theorem 4.3.1 (Soundness of $\mathbf{An^-PC}$)**.** If $\Gamma \vdash_{\mathbf{An}^-} A$, then $\Gamma \vDash_{(F,G)} A$.

Proof. We argue by induction on the depth of deduction. Here we look at the cases for the negative axioms. Let $(\mathcal{F}, \mathcal{V})$ and $w \in W$ be arbitrary.

$(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$

Suppose $(\mathcal{F}, \mathcal{V}), u \Vdash_{(F,G)} A \leftrightarrow B$ and $(\mathcal{F}, \mathcal{V}), v \Vdash_{(F,G)} \neg A$ for $v \geq u \geq w$. We want to show $(\mathcal{F}, \mathcal{V}), v \Vdash_{(F,G)} \neg B$. The latter supposition implies $v \in G$ from Definition 4.2.1. Also, if $(\mathcal{F}, \mathcal{V}), s \Vdash_{(F,G)} B$ for $s \geq v$, we have $(\mathcal{F}, \mathcal{V}), s \Vdash_{(F,G)} A$ from the former supposition. So $s \in F$. Therefore $(\mathcal{F}, \mathcal{V}), v \Vdash_{(F,G)} \neg B$, as desired. So $(\mathcal{F}, \mathcal{V}), u \Vdash_{(F,G)} \neg A \to \neg B$. Similarly, $(\mathcal{F}, \mathcal{V}), u \Vdash_{(F,G)} \neg B \to \neg A$. So $(\mathcal{F}, \mathcal{V}), w \Vdash_{(F,G)} (A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$. Since $(\mathcal{F}, \mathcal{V})$ and w are arbitrary, $\vDash_{(F,G)} (A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$.

$(A \to \neg A) \to (\neg B \to \neg A)$

Suppose $(\mathcal{F}, \mathcal{V}), u \Vdash_{(F,G)} A \to \neg A$ and $(\mathcal{F}, \mathcal{V}), v \Vdash_{(F,G)} \neg B$ for $v \geq u \geq w$. We want to show $(\mathcal{F}, \mathcal{V}), v \Vdash_{(F,G)} \neg A$. If $(\mathcal{F}, \mathcal{V}), s \Vdash_{(F,G)} A$ for $s \geq v$, then $(\mathcal{F}, \mathcal{V}), s \Vdash_{(F,G)} \neg A$. Thus by the semantic definition, $(\mathcal{F}, \mathcal{V}), t \Vdash_{(F,G)} A$ implies $t \in F$ for $t \geq s$. In particular $s \in F$, and since $\neg B$ holds in v, $v \in G$ follows. So $(\mathcal{F}, \mathcal{V}), v \Vdash_{(F,G)} \neg A$ as desired. Thus $(\mathcal{F}, \mathcal{V}), u \Vdash_{(F,G)} \neg B \to \neg A$ and so $(\mathcal{F}, \mathcal{V}), w \Vdash_{(F,G)} (A \to \neg A) \to (\neg B \to \neg A)$. Since $(\mathcal{F}, \mathcal{V})$ and w are arbitrary, $\vDash_{(F,G)} (A \to \neg A) \to (\neg B \to \neg A)$. □

[2] Analogously, **CoPC** derives this weaker version of An^-. Since **CoPC** is strictly contained in $\mathbf{MPC}_\neg$ [25], this implies $\mathbf{CoPC} \nvdash A \to \neg\neg A$, as otherwise $\mathbf{CoPC} \vdash$ An.

Next we shall look at the completeness of $\mathbf{An^-PC}$ with respect to (F,G)-semantics. In what follows, we call a set of formulae Δ *saturated*, if (a) $\Delta \vdash_{\mathbf{An}^-} A \Rightarrow A \in \Delta$ and (b) $\Delta \vdash_{\mathbf{An}^-} A \vee B \Rightarrow \Delta \vdash_{\mathbf{An}^-} A$ or $\Delta \vdash_{\mathbf{An}^-} B$. A saturated set is also sometimes called a *theory with disjunction property*.

Theorem 4.3.2 (Completeness of $\mathbf{An^-PC}$)**.** If $\Gamma \models_{(F,G)} A$, then $\Gamma \vdash_{\mathbf{An}^-} A$.

Proof. The argument is analogous to the one for intuitionistic logic; cf. for instance [121]. Given $\Gamma \nvdash_{\mathbf{An}^-} A$, we construct a saturated $\Gamma_0 \supseteq \Gamma$ s.t. $\Gamma_0 \nvdash_{\mathbf{An}^-} A$. Then the canonical model $\mathcal{M} = (W, \leq, F, G, \mathcal{V})$ with respect to Γ_0 is defined standardly. For F and G we take $F := \{\Delta : \neg B \in \Delta$ for all $B\}$ and $G := \{\Delta : \neg B \in \Delta$ for some $B\}$. Note that F, G are upward closed and $F \subseteq G$. Now it is sufficient to show $B \in \Delta \Leftrightarrow \mathcal{M}, \Delta \Vdash_{(F,G)} B$ for any $\Delta \in W$. We argue this by induction on the complexity of B. We consider the case when B has the form $\neg C$: other cases are standard.

For the left-to-right direction, assume $\neg C \in \Delta$. Then $\Delta \in G$. Further if for $\Delta' \geq \Delta$ we have $\mathcal{M}, \Delta' \Vdash_{(F,G)} C$, then by I.H. $C \in \Delta'$. As Δ' is saturated, $C \wedge \neg C \in \Delta'$, and so $\neg D \in \Delta'$ for any D, because $\vdash_{\mathbf{An}^-} (p \wedge \neg p) \to \neg q$. Hence $\Delta' \in F$. Consequently $\mathcal{M}, \Delta \Vdash_{(F,G)} \neg C$.

For the right-to-left direction, assume $\mathcal{M}, \Delta \Vdash_{(F,G)} \neg C$. Then $\Delta \in G$, and for all $\Delta' \geq \Delta$, if $\mathcal{M}, \Delta' \Vdash_{(F,G)} C$ then $\Delta' \in F$. Now suppose $\Delta \nvdash_{\mathbf{An}^-} C \to \neg C$. Then $\Delta \cup \{C\} \nvdash_{\mathbf{An}^-} \neg C$. Then there is a saturated $\Delta_0 \supseteq \Delta \cup \{C\}$ such that $\Delta_0 \nvdash_{\mathbf{An}^-} \neg C$. Thus $\neg C \notin \Delta_0$. But by I.H., $C \in \Delta_0$ implies $\mathcal{M}, \Delta_0 \Vdash_{(F,G)} C$. Thus $\Delta_0 \in F$ and so $\neg C \in \Delta_0$, a contradiction. Therefore $\Delta \vdash_{\mathbf{An}^-} C \to \neg C$, and so $\Delta \vdash_{\mathbf{An}^-} \neg D \to \neg C$ for any D by An$^-$. As $\Delta \in G$, there is E s.t. $\neg E \in \Delta$. Hence $\Delta \vdash_{\mathbf{An}^-} \neg C$, so $\neg C \in \Delta$. □

4.4 Sequent calculus

We shall consider the following system $\mathbf{GAn}^-$ of sequent calculus. This system is based on ones given in [10, 24] for subminimal logics, and is in the style of the **G3**-systems of Troelstra and Schwichtenberg [120].

Our goal in this section is to show the cut-elimination theorem, which enables in-detail analysis of proofs, and thus to obtain important theorems such as decidability and Craig's interpolation theorem.

The formulation of $\mathbf{GAn}^-$ is obtained from that of **G3i** by replacing (L⊥) with a couple of axioms for negation, and adding an axiom for ⊤.

Definition 4.4.1 ($\mathbf{GAn}^-$)**.** We define $\mathbf{GAn}^-$ by (Ax), (R∘), (L∘) for $\circ \in \{\wedge, \vee, \to\}$ and the next rules.

$$\Gamma \Rightarrow \top \ (\mathrm{R}\top)$$

$$\frac{\Gamma, \neg A, A \Rightarrow B \qquad \Gamma, \neg A, B \Rightarrow A}{\Gamma, \neg A \Rightarrow \neg B}\ (\mathrm{N}) \quad \frac{\Gamma, \neg B, A \Rightarrow \neg A}{\Gamma, \neg B \Rightarrow \neg A}\ (\mathrm{An}^-)$$

We denote the addition of (Cut) to $\mathbf{GAn}^-$ as $\mathbf{GAn}^-$+Cut. We shall call the formulae in position of A in the Cut rule as the *cutformulae*. If we eliminate the rule An$^-$, we obtain the system **G3n** of [24], which is shown to be equivalent to **NPC** (also cf. [10]). *Principal* formulae are defined as in **G3i**. Note however that in (N) and (An^-), both $\neg A$ and $\neg B$ in the conclusion are principal.

We shall first see that there is a correspondence between $\mathbf{GAn}^-$ and $\mathbf{An}^-\mathbf{PC}$.

Lemma 4.4.1. $\vdash_{\mathbf{GAn}^-} \Gamma, A \Rightarrow A$ for all A.

Proof. Given the statement for **G3i**, it suffices to check for the cases where $A \equiv \top$ and $A \equiv \neg B$. For $A \equiv \top$, we immediately see that $\Gamma, \top \Rightarrow \top$ is an instance of ($\mathrm{R}\top$). For the case $A \equiv \neg B$, as for **G3n**, (N) guarantees the statement. That is:

$$\dfrac{\Gamma, \neg B, B \Rightarrow B \qquad \Gamma, \neg B, B \Rightarrow B}{\Gamma, \neg B \Rightarrow \neg B}\,(\mathrm{N})$$

□

Proposition 4.4.1. $\Gamma \vdash_{\mathbf{An}^-} A$ if and only if $\vdash_{\mathbf{GAn}^- + \mathrm{Cut}} \Gamma \Rightarrow A$.

Proof. It suffices to consider the case for An^-. For the left-to-right direction, we proceed by induction on the depth of deduction in $\mathbf{An}^-\mathbf{PC}$. An^- is derivable in $\mathbf{GAn}^- + \mathrm{Cut}$ with the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg B, A, A \to \neg A \Rightarrow A \qquad \dfrac{\neg B, A, \neg A, A \Rightarrow A \qquad \neg B, A, \neg A, A \Rightarrow A}{\neg B, A, \neg A \Rightarrow \neg A}\,(\mathrm{N})}{\neg B, A, A \to \neg A \Rightarrow \neg A}\,(\mathrm{L}\!\to)}{\neg B, A \to \neg A \Rightarrow \neg A}\,(\mathrm{An}^-)}{A \to \neg A \Rightarrow \neg B \to \neg A}\,(\mathrm{R}\!\to)}{\Rightarrow (A \to \neg A) \to (\neg B \to \neg A)}\,(\mathrm{R}\!\to)$$

For the right-to-left direction, we proceed by induction on the depth of deduction of $\mathbf{GAn}^- + \mathrm{Cut}$. In the case for ($\mathrm{An}^-$), suppose the deduction end with an instance $\dfrac{\Gamma, \neg B, A \Rightarrow \neg A}{\Gamma, \neg B \Rightarrow \neg A}$ of (An^-). By I.H. we have $\Gamma, \neg B, A \vdash_{\mathbf{An}^-} \neg A$. Use deduction theorem and instances of (MP) to conclude $\Gamma, \neg B \vdash_{\mathbf{An}^-} \neg A$. □

In [127] it is shown using completeness that no formula of the form $\neg A$ is provable in **SUBMIN**. Here we give a different proof for $\mathbf{An}^-\mathbf{PC}$ by syntactic means, via a condition formulae in a provable sequent observes with respect to two classes of formulae F^+ and F^-. This separates $\mathbf{An}^-\mathbf{PC}$ from $\mathbf{MPC}_\neg$, which proves $\neg\neg(p \to p)$. In general, we shall write $\mathbf{XPC} \subseteq \mathbf{YPC}$ if $\Gamma \vdash_{\mathbf{X}} A$ implies $\Gamma \vdash_{\mathbf{Y}} A$ for all Γ, A. So the above is expressed as $\mathbf{MPC}_\neg \supsetneq \mathbf{An}^-\mathbf{PC}$. A syntactic argument may be preferred over a semantic argument, from a constructive point of view, because we only need to consider finite objects of proofs.

Definition 4.4.2. We define the following classes of formulae.
$F^+ ::= p \mid \top \mid P_1 \wedge P_2 \mid P \vee A \mid A \vee P \mid A \to P \mid N \to A$
$F^- ::= \neg A \mid N \wedge A \mid A \wedge N \mid N_1 \vee N_2 \mid P \to N$
(where $P \in F^+, N \in F^-, A \in F^+ \cup F^-$)

Note that all formulas belong to one and only one of the classes.

Proposition 4.4.2.

(i) If $\vdash_{\mathbf{GAn}^- + \mathrm{Cut}} \Gamma \Rightarrow A$ and $A \in F^-$, then Γ contains a formula in F^-.

(ii) $\nvdash_{\mathbf{GAn}^- + \mathrm{Cut}} \Rightarrow \neg A$ for any A;.

Proof. For (i), we proceed by induction on the depth of deduction. We consider the case for (L→); others cases are simpler. Suppose the deduction ends with an instance

$$\frac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C} \text{(L}\to\text{)}$$

If $C \in F^-$, check if Γ has a formula in F^-. If it does, then the statement follows. If not, then by I.H., $B \in F^-$. Also we see $A \in F^+$, as otherwise $A \to B \in F^+$, contradicting I.H.. So $A \to B \in F^-$. (ii) is then an immediate consequence. □

Another advantage of this syntactic approach is that the above condition allows one to pick out a concrete formula in the class F^- directly from a proof. In any case, it is an interesting question to ask whether this approach of studying provable sequents using classes of formulae can be generalised for further applications.

The system **G3copc** corresponding to **CoPC** is obtained from **G3n** by replacing N with the rule

$$\cdot \ \frac{\Gamma, \neg B, A \Rightarrow B}{\Gamma, \neg B \Rightarrow \neg A} \text{(Co)}$$

This **G3copc** is proved in [10, 24] to be decidable, and using this we can separate **An⁻PC** from **CoPC** syntactically, which we prefer by the same reason as above.

Proposition 4.4.3. $\mathbf{An^-PC} \supsetneq \mathbf{CoPC}$.

Proof. Recall that in the proof of Proposition 4.3.1, we have already seen that $\mathbf{An^-PC}$ contains **CoPC**.
We show that The instance $(p \to \neg p) \to (\neg q \to \neg p)$ of An^- is not a theorem of **CoPC**. We search the possible proofs of $\Rightarrow (p \to \neg p) \to (\neg q \to \neg p)$ in **G3copc**. The last rule has to be R→, so the premise must be $p \to \neg p \Rightarrow \neg q \to \neg p$. Then the rule above it must be (L→) or (R→). For (L→) a non-theorem $p \to \neg p \Rightarrow p$ is a premise. So the rule above must be (R→) with the premise $p \to \neg p, \neg q \Rightarrow \neg p$. Then the rule above must be (L→) or (Co). If (L→) one of the premises must be a non-theorem $p \to \neg p, \neg q \Rightarrow p$, so (Co) must be applied with the premise $p \to \neg p, \neg q, p \Rightarrow q$. Then (L→) is the only possible rule above; but then a non-theorem $\neg q, \neg p, p \Rightarrow q$ is a premise. This exhausts the possible proofs. □

Next we prove the cut-elimination theorem for $\mathbf{GAn^-}$, similarly to the systems for other subminimal negation in [10, 24]. We write $\vdash_k \Gamma \Rightarrow A$ when there is a derivation of the sequent with depth less than or equal to k. We say a rule is *depth-preserving admissible (dp-admissible)* if the derivability of the premises within a certain depth implies that of the conclusion within the same depth. If the condition on depth is lifted, we will simply say a rule is *admissible*. We see examples of a dp-admissible rules in the following.

Lemma 4.4.2. (LW) and (LC) are dp-admissible in $\mathbf{GAn^-}$.

Proof. For (LW), we show by induction on $\vdash_k$. If $k = 0$, then the sequent is obtained by an instance of an axiom, and so is the result of Weakening. If $k = n + 1$, then we have to consider which rule is applied in the last step. If it is an instance of (An^-), for instance, then the last sequent is of the form $\vdash_{k+1} \Gamma, \neg B \Rightarrow \neg A$. Hence $\vdash_k \Gamma, \neg B, A \Rightarrow \neg A$; so by I.H., $\vdash_k C, \Gamma, \neg B, A \Rightarrow \neg A$. Thus $\vdash_{k+1} C, \Gamma, \neg B \Rightarrow \neg A$.

The cases for other rules are similar.

For (LC), we show by induction on $\vdash_k$ with subinduction on the complexity of the contraction formula. Here we shall look at the case where the rule applied is (An$^-$), and the contraction formula is principal.

$$\dfrac{\vdash_k \Gamma, \neg B, \neg B, A \Rightarrow \neg A}{\vdash_{k+1} \Gamma, \neg B, \neg B \Rightarrow \neg A}\,(\mathrm{An}^-)$$

Then by I.H. $\vdash_k \Gamma, \neg B, A \Rightarrow \neg A$. So the following derivation is possible.

$$\dfrac{\vdash_k \Gamma, \neg B, A \Rightarrow \neg A}{\vdash_{k+1} \Gamma, \neg B \Rightarrow \neg A}\,(\mathrm{An}^-)$$

For some other cases, we need to appeal to the so-called *inversion lemma* [120, Proposition 3.5.4], but those cases are identical to the ones for intuitionistic logic. Cf. [24, Theorem 5.2.2], [120, Proposition 3.5.5.]. □

Now we are ready to prove the cut-elimination theorem for $\mathbf{GAn}^-$; namely, if a sequent is provable in $\mathbf{GAn}^- + \mathrm{Cut}$, then so is it in the system without cut, i.e. $\mathbf{GAn}^-$. This is shown by observing that cut is an admissible rule in $\mathbf{GAn}^-$.

Theorem 4.4.1. If $\vdash_{\mathbf{GAn}^- + \mathrm{Cut}} \Gamma \Rightarrow A$ then $\vdash_{\mathbf{GAn}^-} \Gamma \Rightarrow A$

Proof. Our proof is by induction on the the complexity of cutformula, with subinduction on the *level* (sum of the depths of the premises) of the deduction. By the theorem for **NPC** [10, Theorem 4.1] [24, Theorem 6.1.1] it suffices to treat cases involving (An$^-$).

Case 1: Cutformula is not principal on the right.

The case depends on whether an application of (An$^-$) appears on the left or the right premise. If it appears on the left, then it further depends on whether the right premise is an axiom or a rule. For the former, if the right premise is an instance of (Ax), then the conclusion is also an instance of (Ax):

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdash_{k-1} \Gamma, \neg B, A \Rightarrow \neg A\end{matrix}}{\vdash_k \Gamma, \neg B \Rightarrow \neg A}\,(\mathrm{An}^-) \qquad \vdash_0 \neg A, \Gamma', p \Rightarrow p}{\vdash \Gamma, \Gamma', \neg B, p \Rightarrow p}\,(\mathrm{Cut})$$

Similarly when we have (R⊤) instead of (Ax). For the latter, we have

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdash_{k-1} \Gamma, \neg B, A \Rightarrow \neg A\end{matrix}}{\vdash_k \Gamma, \neg B \Rightarrow \neg A}\,(\mathrm{An}^-) \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ \vdash_{k'-1} \neg A, \Gamma'' \Rightarrow C'\end{matrix}}{\vdash_{k'} \neg A, \Gamma' \Rightarrow C}\,(\mathrm{Rule})}{\vdash \Gamma, \Gamma', \neg B \Rightarrow C}\,(\mathrm{Cut})$$

if the right premise is a 1-premise rule. We can elevate the cut and reduce the level of the cut to be applied. By I.H. the particular cut is known to be admissible, so we obtain the following derivation.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdash_{k-1} \Gamma, \neg B, A \Rightarrow \neg A\end{matrix}}{\vdash_k \Gamma, \neg B \Rightarrow \neg A}\,(\mathrm{An}^-) \qquad \begin{matrix}\mathcal{D}_2\\ \vdash_{k'-1} \neg A, \Gamma'' \Rightarrow C'\end{matrix}}{\vdash \Gamma, \Gamma'' \neg B \Rightarrow C'}\,(\mathrm{I.H.})}{\vdash \Gamma, \Gamma', \neg B \Rightarrow C}\,(\mathrm{Rule})$$

The case for a 2-premise rule is analogous. If (An^-) appears on the right, then the cutformula must reside in the context on the right premise.

$$\dfrac{\overset{\mathcal{D}_1}{\vdash_k \Gamma \Rightarrow C} \qquad \dfrac{\overset{\mathcal{D}_2}{\vdash_{k'-1} \Gamma', C, \neg B, A \Rightarrow \neg A}}{\vdash_{k'} \Gamma', C, \neg B \Rightarrow \neg A}\,(\mathrm{An}^-)}{\vdash \Gamma, \Gamma', \neg B \Rightarrow \neg A}\,(\mathrm{Cut})$$

Then we can elevate the cut so that I.H. becomes applicable.

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdash_k \Gamma \Rightarrow C} \qquad \overset{\mathcal{D}_2}{\vdash_{k'-1} \Gamma', C, \neg B, A \Rightarrow \neg A}}{\vdash \Gamma, \Gamma', \neg B, A \Rightarrow \neg A}\,(\mathrm{I.H.})}{\vdash \Gamma, \Gamma', \neg B \Rightarrow \neg A}\,(\mathrm{An}^-)$$

Case 2: Cutformula is not principal on the left.

The left premise cannot be an application of (An^-); so the right premise must be. The case depends on whether the cutformula is principal on the right. If not, the last case covers this case. If it is, the left premise must be an instance of a left rule. For a 1-premise rule, we have the next deduction.

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdash_{k-1} \Gamma'' \Rightarrow \neg B}}{\vdash_k \Gamma' \Rightarrow \neg B}\,(\mathrm{Rule}) \qquad \dfrac{\overset{\mathcal{D}_2}{\vdash_{k'-1} \Gamma, \neg B, A \Rightarrow \neg A}}{\vdash_{k'} \Gamma, \neg B \Rightarrow \neg A}\,(\mathrm{An}^-)}{\vdash \Gamma, \Gamma' \Rightarrow \neg A}\,(\mathrm{Cut})$$

This can be turned into the following deduction.

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdash_{k-1} \Gamma'' \Rightarrow \neg B} \qquad \dfrac{\overset{\mathcal{D}_2}{\vdash_{k'-1} \Gamma, \neg B, A \Rightarrow \neg A}}{\vdash_{k'} \Gamma, \neg B \Rightarrow \neg A}\,(\mathrm{An}^-)}{\vdash \Gamma, \Gamma'' \Rightarrow \neg A}\,(\mathrm{I.H.})}{\vdash \Gamma, \Gamma' \Rightarrow \neg A}\,(\mathrm{Rule})$$

Case 3: Cutformula is principal on both premises.

Each premise must be an application of (N) or (An^-). So we have one of:

- *Subcase 3.1: (An^-) in both premises.*

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdash_{k-1} \Gamma, \neg C, B \Rightarrow \neg B}}{\vdash_k \Gamma, \neg C \Rightarrow \neg B}\,(\mathrm{An}^-) \qquad \dfrac{\overset{\mathcal{D}_2}{\vdash_{k'-1} \Gamma', \neg B, A \Rightarrow \neg A}}{\vdash_{k'} \Gamma', \neg B \Rightarrow \neg A}\,(\mathrm{An}^-)}{\vdash \Gamma, \Gamma', \neg C \Rightarrow \neg A}\,(\mathrm{Cut})$$

Then we can elevate the cut in the following manner.

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdash_{k-1} \Gamma, \neg C, B \Rightarrow \neg B}}{\vdash_k \Gamma, \neg C \Rightarrow \neg B}\,(\mathrm{An}^-) \qquad \overset{\mathcal{D}_2}{\vdash_{k'-1} \Gamma', \neg B, A \Rightarrow \neg A}}{\vdash \Gamma, \Gamma', \neg C, A \Rightarrow \neg A}\,(\mathrm{I.H.})}{\vdash \Gamma, \Gamma', \neg C \Rightarrow \neg A}\,(\mathrm{An}^-)$$

- *Subcase 3.2: (An^-) on the left and (N) on the right.*

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdash_{k-1} \Gamma, \neg B, A \Rightarrow \neg A}}{\vdash_k \Gamma, \neg B \Rightarrow \neg A}\,(\mathrm{An}^-) \qquad \dfrac{\overset{\mathcal{D}_2}{\vdash_{k'} \Gamma', \neg A, A \Rightarrow C} \qquad \overset{\mathcal{D}_3}{\vdash_{k''} \Gamma', \neg A, C \Rightarrow A}}{\vdash_{max(k',k'')+1} \Gamma', \neg A \Rightarrow \neg C}\,(\mathrm{N})}{\vdash \Gamma, \Gamma', \neg B \Rightarrow \neg C}\,(\mathrm{Cut})$$

Then we perform two cuts that are admissible because of reduced level.

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdash_{k-1} \Gamma, \neg B, A \Rightarrow \neg A}}{\vdash_k \Gamma, \neg B \Rightarrow \neg A}\,(\text{An}^-) \qquad \overset{\mathcal{D}_3}{\vdash_{k''} \Gamma', \neg A, C \Rightarrow A}}{\vdash \Gamma, \Gamma', \neg B, C \Rightarrow A \cdots \text{(a)}}\,(\text{I.H.})$$

$$\dfrac{\overset{\mathcal{D}_1}{\vdash_{k-1} \Gamma, \neg B, A \Rightarrow \neg A} \qquad \dfrac{\overset{\mathcal{D}_2}{\vdash_{k'} \Gamma', \neg A, A \Rightarrow C} \qquad \overset{\mathcal{D}_3}{\vdash_{k''} \Gamma', \neg A, C \Rightarrow A}}{\vdash_{max(k',k'')+1} \Gamma', \neg A \Rightarrow \neg C}\,(\text{N})}{\vdash \Gamma, \Gamma', \neg B, A \Rightarrow \neg C \cdots \text{(b)}}\,(\text{I.H.})$$

Next we combine the two with a cut of reduced *complexity*.

$$\dfrac{\overset{\text{(a)}}{\vdash \Gamma, \Gamma', \neg B, C \Rightarrow A} \qquad \overset{\text{(b)}}{\vdash \Gamma, \Gamma', \neg B, A \Rightarrow \neg C}}{\vdash \Gamma, \Gamma, \Gamma', \Gamma', \neg B, \neg B, C \Rightarrow \neg C \cdots \text{(c)}}\,(\text{I.H.})$$

Now we utilise the admissibility of Contraction.

$$\dfrac{\dfrac{\overset{\text{(c)}}{\vdash \Gamma, \Gamma, \Gamma', \Gamma', \neg B, \neg B, C \Rightarrow \neg C}}{\vdash \Gamma, \Gamma', \neg B, C \Rightarrow \neg C}\,(\text{LC})}{\vdash \Gamma, \Gamma', \neg B \Rightarrow \neg C}\,(\text{An}^-)$$

- *Subcase 3.3: N on the left and* An^- *on the right.*

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdash_k \Gamma, \neg A, A \Rightarrow B} \qquad \overset{\mathcal{D}_2}{\vdash_{k'} \Gamma, \neg A, B \Rightarrow A}}{\vdash_{max(k,k')+1} \Gamma, \neg A \Rightarrow \neg B}\,(\text{N}) \qquad \dfrac{\overset{\mathcal{D}_3}{\vdash_{k''-1} \Gamma', \neg B, C \Rightarrow \neg C}}{\vdash_{k''} \Gamma', \neg B \Rightarrow \neg C}\,(\text{An}^-)}{\vdash \Gamma, \Gamma', \neg A \Rightarrow \neg C}\,(\text{Cut})$$

The following derivation reduces the level of (Cut).

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdash_k \Gamma, \neg A, A \Rightarrow B} \qquad \overset{\mathcal{D}_2}{\vdash_{k'} \Gamma, \neg A, B \Rightarrow A}}{\vdash_{max(k,k')+1} \Gamma, \neg A \Rightarrow \neg B}\,(\text{N}) \qquad \overset{\mathcal{D}_3}{\vdash_{k''-1} \Gamma', \neg B, C \Rightarrow \neg C}}{\vdash \Gamma, \Gamma', \neg A, C \Rightarrow \neg C}\,(\text{I.H.})}{\vdash \Gamma, \Gamma', \neg A \Rightarrow \neg C}\,(\text{An}^-)$$

□

As cut is absent in ($\mathbf{GAn}^-$), the formulae occurring in a derivation of $\Gamma \Rightarrow A$ are limited to those that are subformulae of a formula in $\Gamma \cup \{A\}$ (*Subformula property*). This allows the decidability of $\mathbf{An^-PC}$ like other subminimal logics [24].

Another consequence is the disjunction property: *If* Γ *does not contain a disjunction, then* $\vdash_{\mathbf{GAn}^-} \Gamma \Rightarrow A \vee B$ *implies* $\vdash_{\mathbf{GAn}^-} \Gamma \Rightarrow A$ *or* $\vdash_{\mathbf{GAn}^-} \Gamma \Rightarrow B$. The proof is as in [24], and An^- does not affect the argument.

We next consider a further consequence called Craig's interpolation theorem. Recall the sets of propositional variables in Γ and A are denoted as $\mathsf{V}(\Gamma)$ and $\mathsf{V}(A)$, respectively.

Definition 4.4.3. Given a sequent $\Gamma_1, \Gamma_2 \Rightarrow A$, we associate a *split sequent* $\Gamma_1; \Gamma_2 \Rightarrow A$. Then an *interpolant* of $\Gamma_1; \Gamma_2 \Rightarrow A$ is a formula I s.t. $\vdash_{\mathbf{GAn}^-} \Gamma_1 \Rightarrow I$ and $\vdash_{\mathbf{GAn}^-} \Gamma_2, I \Rightarrow A$, with $\mathsf{V}(I) \subseteq \mathsf{V}(\Gamma_1) \cap \mathsf{V}(\Gamma_2 \cup \{A\})$.

Theorem 4.4.2 (Craig's Interpolation)**.** If $\vdash_{\mathbf{GAn}^-} \Gamma \Rightarrow B$, then each split sequent $\Gamma_1; \Gamma_2 \Rightarrow B$ of the sequent has an interpolant.

Proof. We argue it by induction on the depth of deduction. All cases except for An^- are treated in [10, 24]. Now if the deduction ends with an instance of

$$\frac{\Gamma, \neg B, A \Rightarrow \neg A}{\Gamma, \neg B \Rightarrow \neg A}\,(\mathrm{An}^-)$$

then there are two ways of splitting the endsequent.

1. The endsequent is split as $\Gamma_1, \neg B; \Gamma_2 \Rightarrow \neg A$. We have to find I such that

$$\vdash_{\mathbf{GAn}^-} \Gamma_1, \neg B \Rightarrow I,\ \vdash_{\mathbf{GAn}^-} \Gamma_2, I \Rightarrow \neg A$$

for $\mathsf{V}(I) \subseteq \mathsf{V}(\Gamma_1 \cup \{\neg B\}) \cap \mathsf{V}(\Gamma_2 \cup \{\neg A\})$. By I.H. an I' satisfies

$$\vdash_{\mathbf{GAn}^-} \Gamma_1, \neg B \Rightarrow I' \text{ and } \vdash_{\mathbf{GAn}^-} \Gamma_2, A, I' \Rightarrow \neg A.$$

We define $I := \neg\neg I' \wedge I'$. Then,

$$\begin{aligned} \mathsf{V}(\neg\neg I' \wedge I') &\subseteq \mathsf{V}(\Gamma_1 \cup \{\neg B\}) \cap \mathsf{V}(\Gamma_2 \cup \{A, \neg A\}) \text{ [by I.H.]} \\ &= \mathsf{V}(\Gamma_1 \cup \{\neg B\}) \cap \mathsf{V}(\Gamma_2 \cup \{\neg A\}) \end{aligned}$$

as required. Now we note that $\vdash_{\mathbf{GAn}^-} \neg A \Rightarrow \neg\neg\neg A$ and also the rule $\dfrac{\Gamma, \neg B, A \Rightarrow B}{\Gamma, \neg B \Rightarrow \neg A}\,(\mathrm{Co})$ is admissible. Then we can show $\vdash_{\mathbf{GAn}^-} \Gamma_1, \neg B \Rightarrow \neg\neg I' \wedge I'$ and $\vdash_{\mathbf{GAn}^-} \Gamma_2, \neg\neg I' \wedge I' \Rightarrow \neg A$ by the following deductions.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\neg B \Rightarrow \neg\neg\neg B} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}'}{\Gamma_1, \neg B \Rightarrow I'}}{\Gamma_1, \neg I', \neg B \Rightarrow I'}\,(\mathrm{LW})}{\Gamma_1, \neg I' \Rightarrow \neg\neg B}\,(\mathrm{Co})}{\Gamma_1, \neg\neg\neg B, \neg I' \Rightarrow \neg\neg B}\,(\mathrm{LW})}{\Gamma_1, \neg\neg\neg B \Rightarrow \neg\neg I'}\,(\mathrm{Co})}{\Gamma_1, \neg B \Rightarrow \neg\neg I'}\,(\mathrm{Cut}) \qquad \dfrac{\mathcal{D}'}{\Gamma_1, \neg B \Rightarrow I'}}{\Gamma_1, \neg B \Rightarrow \neg\neg I' \wedge I'}\,(\mathrm{R}\wedge)$$

Note that the appeal to the admissibility of cut in the proof. The other deduction is much simpler.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma_2, A, I' \Rightarrow \neg A}}{\Gamma_2, \neg\neg I', A, I' \Rightarrow \neg A}\,(\mathrm{LW})}{\Gamma_2, \neg\neg I', I' \Rightarrow \neg A}\,(\mathrm{An}^-)}{\Gamma_2, \neg\neg I' \wedge I' \Rightarrow \neg A}\,(\mathrm{L}\wedge)$$

2. The endsequent is split as $\Gamma_1; \neg B, \Gamma_2 \Rightarrow \neg A$. We have to find I such that

$$\vdash_{\mathbf{GAn}^-} \Gamma_1 \Rightarrow I,\ \vdash_{\mathbf{GAn}^-} \Gamma_2, \neg B, I \Rightarrow \neg A$$

for $\mathsf{V}(I) \subseteq \mathsf{V}(\Gamma_1) \cap \mathsf{V}(\Gamma_2 \cup \{\neg B, \neg A\})$. By I.H. an I' satisfies

$$\vdash_{\mathbf{GAn}^-} \Gamma_1 \Rightarrow I' \text{ and } \vdash_{\mathbf{GAn}^-} \Gamma_2, \neg B, A, I' \Rightarrow \neg A.$$

Define $I := I'$. Then,

$$\begin{aligned}\mathsf{V}(I') &\subseteq \mathsf{V}(\Gamma_1) \cap \mathsf{V}(\Gamma_2 \cup \{\neg B, A, \neg A\}) \text{ [by I.H.]}\\ &= \mathsf{V}(\Gamma_1) \cap \mathsf{V}(\Gamma_2 \cup \{\neg B, \neg A\})\end{aligned}$$

as required. It remains to show $\vdash_{\mathbf{GAn^-}} \Gamma_1 \Rightarrow I'$ and $\vdash_{\mathbf{GAn^-}} \Gamma_2, \neg B, I' \Rightarrow \neg A$. But they immediately follow from I.H..

□

4.5 Countable classes of logics with subminimal negation

It is known that there are uncountably many subsystems of minimal logic with subminimal negation [8]. In this section, we shall give more insights into the structure of the class of such logics, by describing an additional family of countably many logics. This class will be obtained by considering weaker variants of the axiom An.In this sense it is connected to the study of $\mathbf{An^-PC}$ as studies of weaker versions of An. As observed previously, the presence of An has a huge influence on weak negations, and this type of study contributes to the understanding of this mechanism.

The countable family will be investigated using N-frames. For $i \geq 0$, we define the axiom An_i by $(\neg^i A \to \neg^{i+1} A) \to \neg^{i+1} A$, where

$$\begin{aligned}\neg^0 A &:\equiv A\\ \neg^{i+1} A &:\equiv \neg(\neg^i A).\end{aligned}$$

(Note $\mathrm{An}_0 \equiv \mathrm{An}$.) Then for $i \geq 0$, $\mathbf{An_iPC}$ is defined by $\mathbf{NPC} + \mathrm{An}_i$. We shall use $\vdash_{\mathbf{An_i}}$ for the derivability in $\mathbf{An_iPC}$.

Proposition 4.5.1. Let $\mathcal{F}$ be an N-frame. The next conditions are equivalent.

(i) $\mathcal{F} \vDash_N (\neg^i A \to \neg^{i+1} A) \to \neg^{i+1} A$ for all A

(ii) $\forall U \in \mathcal{U}(W) \forall w \in W [\mathcal{R}(w) \cap N^i(U) \subseteq N^{i+1}(U) \Rightarrow \mathcal{R}(w) \subseteq N^{i+1}(U)]$

Proof. We first see (i) implies (ii). Let $U \in \mathcal{U}(W)$ and $w \in W$. Assume

$$\mathcal{R}(w) \cap N^i(U) \subseteq N^{i+1}(U).$$

Let $\mathcal{V}(p) = U$. Now for $w' \geq w$, if $(\mathcal{F}, \mathcal{V}), w' \Vdash_N \neg^i p$, then $w' \in N^i(\mathcal{V}(p))$, and consequently $w' \in \mathcal{R}(w) \cap N^i(U) \subseteq N^{i+1}(U)$. So

$$(\mathcal{F}, \mathcal{V}), w' \Vdash_N \neg^i p \Rightarrow (\mathcal{F}, \mathcal{V}), w' \Vdash_N \neg^{i+1} p.$$

Thus $(\mathcal{F}, \mathcal{V}), w \Vdash_N \neg^i p \to \neg^{i+1} p$, and so $(\mathcal{F}, \mathcal{V}), w \Vdash_N \neg^{i+1} p$ by (i). Hence $\mathcal{R}(w) \subseteq N^{i+1}(U)$.

Next we see (ii) implies (i). Let $\mathcal{V}$ be a valuation and $w \in W$. Suppose

$$(\mathcal{F}, \mathcal{V}), w' \Vdash_N \neg^i A \to \neg^{i+1} A \text{ for } w' \geq w.$$

Then for $w'' \geq w'$, $(\mathcal{F}, \mathcal{V}), w'' \Vdash_N \neg^i A$ implies $(\mathcal{F}, \mathcal{V}), w'' \Vdash_N \neg^{i+1} A$. That is,

$$\mathcal{R}(w') \cap N^i(\mathcal{V}(A)) \subseteq N^{i+1}(\mathcal{V}(A)).$$

Thus by (ii), $\mathcal{R}(w') \subseteq N^{i+1}(\mathcal{V}(A))$ and so $(\mathcal{F}, \mathcal{V}), w' \Vdash_N \neg^{i+1} p$. Hence

$$(\mathcal{F}, \mathcal{V}), w \Vdash_N (\neg^i A \to \neg^{i+1} A) \to \neg^{i+1} A;$$

so $\mathcal{F} \vDash_N (\neg^i A \to \neg^{i+1} A) \to \neg^{i+1} A$. □

Definition 4.5.1. We define an $\mathbf{An_i}$-*frame* as an N-frame satisfying:

$$[\mathbf{P_{An_i}}]\ \forall U \in \mathcal{U}(W) \forall w \in W[\mathcal{R}(w) \cap N^i(U) \subseteq N^{i+1}(U) \Rightarrow \mathcal{R}(w) \subseteq N^{i+1}(U)].$$

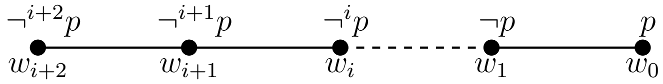
We shall use symbols $\Vdash_{\mathbf{An_i}}$ for valuation at a world and $\vDash_{\mathbf{An_i}}$ for validity, with respect to these frames.

By Proposition 4.5.1, it follows that if $\Gamma \vdash_{\mathbf{An_i}} A$ then $\Gamma \vDash_{\mathbf{An_i}} A$. We want to see that each $\mathbf{An_iPC}$ is distinct, and for this we first need the following lemma.

Lemma 4.5.1. There is an $\mathbf{An_{i+1}}$-frame refuting An_i: i.e. $\nvDash_{\mathbf{An_{i+1}}} \mathrm{An}_i$.

Proof. Let $\mathcal{F} := (W, \leq, N) = (\{w_0, \ldots, w_{i+2}\}, \{(w_j, w_k) : j \geq k\}, N)$,

$$\text{where: } N(U) = \begin{cases} \mathcal{R}(w_{i+2}) & \text{if } U = \emptyset \text{ or } U = \mathcal{R}(w_{i+2}), \\ \mathcal{R}(w_{j+1}) & \text{if } U = \mathcal{R}(w_j) \text{ for } 0 \leq j \leq i+1. \end{cases}$$



Note that N is a mapping between upward closed sets, as required. Now let $\mathcal{V}$ be such that $\mathcal{V}(p) = \{w_0\}$. We first check $\mathcal{F}$ is an $\mathbf{An_{i+1}}$-frame. For $\mathcal{F}$ to be an N-frame, it has to satisfy locality. As stated in [25], in a linear frame this condition becomes (for $U \neq \emptyset$):

$$\forall w, v \in W[w \in N(\mathcal{R}(v)) \Leftrightarrow w \in N(\mathcal{R}(v) \cap \mathcal{R}(w))]$$

If $w \leq v$, or for $U = \emptyset$, no condition is imposed. If $w > v$, this becomes

$$w \in N(\mathcal{R}(v)) \Leftrightarrow w \in N(\mathcal{R}(w)) \Leftrightarrow w \in \{u | u \in N(\mathcal{R}(u))\}$$

In our frame, for each $w \in W$, $w \in N(\mathcal{R}(w))$. So $W = \{u | u \in N(\mathcal{R}(u))\}$. Hence $\mathcal{F}$ must satisfy $w \in N(\mathcal{R}(v))$ for all w, v such that $w > v$. This is satisfied, as $w > v$ implies $w \in \mathcal{R}(v)$, and $\mathcal{R}(v) \subseteq N(\mathcal{R}(v))$. $\mathcal{F}$ must also satisfy

$$\mathbf{P_{An_{i+1}}} : \forall U \forall w[\mathcal{R}(w) \cap N^{i+1}(U) \subseteq N^{i+2}(U) \Rightarrow \mathcal{R}(w) \subseteq N^{i+2}(U)].$$

But as $N^{i+2}(U) = \mathcal{R}(w_{i+2})$ for all U, $\mathbf{P_{An_{i+1}}}$ always holds.

We argue $(\mathcal{F}, \mathcal{V})$ falsifies An_i. We have $\mathcal{V}(\neg^i p) = N^i(\mathcal{V}(p)) = \mathcal{R}(w_i)$ and $\mathcal{V}(\neg^{i+1} p) = \mathcal{R}(w_{i+1})$. So

$$(\mathcal{F}, \mathcal{V}), w_{i+2} \Vdash_{\mathbf{An_{i+1}}} \neg^i p \to \neg^{i+1} p$$

but

$$(\mathcal{F}, \mathcal{V}), w_{i+2} \nVdash_{\mathbf{An_{i+1}}} \neg^{i+1} p.$$

Hence $(\mathcal{F}, \mathcal{V}), w_{i+2} \nVdash_{\mathbf{An_{i+1}}} (\neg^i p \to \neg^{i+1} p) \to \neg^{i+1} p$. □

Theorem 4.5.1. $\mathbf{MPC_\neg} \equiv \mathbf{An_0PC} \supsetneq \mathbf{An_1PC} \supsetneq \mathbf{An_2PC} \ldots \supset \mathbf{NPC}$.

Proof. It is immediate to see $\mathbf{An_iPC} \supseteq \mathbf{An_{i+1}PC}$. Then by Lemma 4.5.1 $\nvDash_{\mathbf{An_{i+1}}} \mathrm{An}_i$ and so $\nvdash_{\mathbf{An_{i+1}}} \mathrm{An}_i$ by soundness. Thus $\mathbf{An_iPC} \nsubseteq \mathbf{An_{i+1}PC}$. □

4.6 Discussion

We have observed that $\mathbf{An^{-}PC}$ is a logic with subminimal negation which satisfies many of the standard properties enjoyed by the systems investigated in [24, 25]. $\mathbf{An^{-}PC}$ strictly contains these systems, and the addition of a single proposition of the form $\neg A$ reduces it to minimal logic. Thus $\mathbf{An^{-}PC}$ appears to be quite strong among the logics with subminimal negation. To extend the result of this section, therefore, it would be interesting whether there exists a stronger subsystem of minimal logic than $\mathbf{An^{-}PC}$. In particular, it should have a significant consequence to study whether there is a maximal such subsystem.

As for the infinite class of logics with subminimal negation we treated at the end, in the proof we used linear models, which may be seen as many-valued truth tables. Therefore it would be intriguing to investigate further what sort of negations are defined in this setting. Furthermore, the completeness proofs for $\mathbf{An_iPC}$ are lacking for $i \neq 0$, which is another gap that needs to be filled.

Chapter 5

Analyses of modal, empirical and co-negations

5.1 Introduction

The philosophy of Intuitionism has long acknowledged that there is more to negation than the customary, *reduction to absurdity*. Brouwer [17] has already introduced the notion of *apartness* as a positive version of inequality, such that from two apart objects (e.g. points, sequences) one can learn not only they are unequal, but also how much or where they are different. (Cf. [132, pp.319-320]). He also introduced the notion of *weak counterexample*, in which a statement is reduced to a constructively unacceptable principle, to conclude we cannot expect to prove the statement [121].

Another type of negation was discussed in the dialogue of Heyting [59, pp.17-19]. In it *mathematical negation* characterised by reduction to absurdity is distinguished from *factual negation*, which concerns the present state of our knowledge. In the dialogue it is emphasised that only the former type of negation has a part in mathematics, on the ground that the latter does not have the form of a mathematical assertion, i.e. assertion of a mental construction. Nevertheless it remains the case that factual negation has a place in his theoretical framework.

One formalisation of logic with this "negation at the present stage of knowledge" was given by M. De [29] and axiomatised by De and H. Omori [30], under the name of *empirical negation*. The central idea of their logic $\mathbf{IPC}^{\sim}$ is semantic: the Kripke semantics of $\mathbf{IPC}^{\sim}$ is taken to be rooted, with the root being understood as representing the present moment. Then the empirical negation $\sim A$ is defined to be forced at a world, if A is not forced at the root.

Yet another type of negation in the intuitionistic framework is *co-negation* introduced by C. Rauszer [102, 103]. Seen from Kripke semantics, a co-negation $\sim A$ is forced at a world, if there is a preceding world in which A is not forced. This is dual to the forcing of intuitionistic negation $\neg A$, which requires A not being forced at all succeeding nodes. Co-negation was originally defined in terms of co-implication, but the co-negative fragment was extracted by G. Priest [100], to define a logic named **daC**.

In both empirical and co- negation, the semantic formulation arguably gives a more fundamental motivation than the syntactic formulation. In particular, in case of empirical negation, it is of essential importance that a Kripke frame can be understood as giving the progression of growth of knowledge. It may be noted, however, that Kripke semantics is not the only semantics to give this kind of picture. *Beth semantics* is another semantics whose frames represent the growth of knowledge. It then appears a natural question to ask, whether the same forcing condition of empirical/co- negation gives rise to the same logic. That is to say, whether $\mathbf{IPC}^{\sim}$ and **daC** will be sound and complete with respect to Beth semantics. Indeed, for co-implication, a similar question was asked by Restall [106]. There it was found out that one needs to alter the forcing condition to get a complete semantics.

In this chapter, we shall observe that another logic called $\mathbf{TCC}_\omega$, introduced by A.B. Gordienko [54], becomes sound and complete with Beth models with the forcing conditions of empirical and co- negation (which turn out to coincide). This is of significant interest for those who advocate empirical or co- negation from a semantic motivation, as it will provide a choice in the logic to which they should adhere.

This is followed by another observation about the axiomatisation of $\mathbf{IPC}^{\sim}$ and **daC**, which employ the disjunctive syllogism rule (RP). In contrast, the axiomatisation of of $\mathbf{TCC}_\omega$ and a related system $\mathbf{CC}_\omega$ of R. Sylvan [116], which is a subsystem of the other three, use the contraposition rule (RC). We shall observe that this difference in rules can be eliminated, by replacing (RP) with (RC) and an additional axiom. This will give a completeness proof of **daC** with respect to the semantics of $\mathbf{CC}_\omega$, and thus the semantics of Došen [34]. It will also provide a more unified viewpoint of the logics related to $\mathbf{CC}_\omega$ as defined by extra axioms with no change in rules.

5.2 Empirical negation in Kripke Semantics

In this chapter, we shall consider the following propositional language $\mathcal{L}^{\sim}$:

$$A ::= p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid {\sim}A,$$

where $\sim$ is used to emphasize the contra-intuitionistic character of the negation.

Definition 5.2.1 (Kripke model for $\mathbf{IPC}^{\sim}$)**.** A *Kripke Frame* $\mathcal{F}_{\mathcal{K}}^{\sim}$ for $\mathbf{IPC}^{\sim}$ is an inhabited pre-ordered set $(W, \leq)$ with a *root* $r \in W$ such that $r \leq w$ for all $w \in W$. A *Kripke model* $\mathcal{M}_{\mathcal{K}}^{\sim}$ for $\mathbf{IPC}^{\sim}$ is a pair $(\mathcal{F}_{\mathcal{K}}^{\sim}, \mathcal{V})$, where $\mathcal{V}$ is a mapping that assigns a set of worlds $\mathcal{V}(p) \subseteq W$ to each propositional variable p. We assume $\mathcal{V}$ to be *monotone*, viz. $w \in \mathcal{V}(p)$ and $w' \geq w$ implies $w' \in \mathcal{V}(p)$. To denote a model, we shall use both $\mathcal{M}_{\mathcal{K}}^{\sim}$ and $(\mathcal{F}_{\mathcal{K}}^{\sim}, \mathcal{V})$ interchangeably. Similar remarks apply to different notions of model in the later sections.

Given $\mathcal{M}_{\mathcal{K}}^{\sim}$, the *forcing* (or *valuation*) of a formula in a world, denoted $w \Vdash_{\mathcal{K}} A$, is inductively defined as follows.

$$\begin{aligned}
w \Vdash_{\mathcal{K}} p &\iff w \in \mathcal{V}(p).\\
w \Vdash_{\mathcal{K}} A \wedge B &\iff w \Vdash_{\mathcal{K}} A \text{ and } w \Vdash_{\mathcal{K}} B.\\
w \Vdash_{\mathcal{K}} A \vee B &\iff w \Vdash_{\mathcal{K}} A \text{ or } w \Vdash_{\mathcal{K}} B.\\
w \Vdash_{\mathcal{K}} A \to B &\iff \text{ for all } w' \geq w, \text{ if } w' \Vdash_{\mathcal{K}} A \text{ then } w' \Vdash_{\mathcal{K}} B.\\
w \Vdash_{\mathcal{K}} {\sim}A &\iff r \nVdash_{\mathcal{K}} A.
\end{aligned}$$

If $\mathcal{M}_{\mathcal{K}}^{\sim}, w \Vdash_{\mathcal{K}} A$ for all $w \in W$ (or equivalently, $\mathcal{M}_{\mathcal{K}}^{\sim}, r \Vdash_{\mathcal{K}} A$), we write $\mathcal{M}_{\mathcal{K}}^{\sim} \vDash_{\mathcal{K}} A$ and say *A is valid in $\mathcal{M}_{\mathcal{K}}^{\sim}$*. For a set of formulae Γ, if $\mathcal{M}_{\mathcal{K}}^{\sim} \vDash_{\mathcal{K}} C$ for all $C \in \Gamma$ implies

$\mathcal{M}^{\sim}_{\mathcal{K}} \vDash_{\mathcal{K}} A$, then we write $\Gamma \vDash_{\mathcal{K}} A$ and say A *is a logical consequence of* Γ. If Γ is empty, we simply write $\vDash_{\mathcal{K}} A$ and say A *is valid* (in **IPC$^{\sim}$**).

A Hilbert-style proof system for **IPC$^{\sim}$** is established in [30], which we identify here with the logic itself for convenience, and denote it simply as **IPC$^{\sim}$**. We shall apply the same convention to other logics in later sections.

Definition 5.2.2 (**IPC$^{\sim}$**)**.**
The logic **IPC$^{\sim}$** is defined by adding the next axiom and rule to **IPC**.

- $A \lor {\sim}A$
- ${\sim}A \to ({\sim}{\sim}A \to B)$
- $\dfrac{A \lor B}{{\sim}A \to B}$ (RP)

we shall denote by the derivability in **IPC$^{\sim}$** by $\Gamma \vdash_{\sim} A$. Then it has been shown by De and Omori that **IPC$^{\sim}$** is sound and complete with the Kripke semantics.

Theorem 5.2.1 (Kripke completeness of **IPC$^{\sim}$**)**.** $\Gamma \vdash_{\sim} A \iff \Gamma \vDash_{\mathcal{K}} A$.

Proof. Cf. [30]. □

5.3 Empirical Negation in Beth Semantics

5.3.1 Beth Semantics and De's logic

Let us turn our attention to Beth models in this section. Our formalisation will be based on that of [119, 122]. If we apply to the forcing of $\sim$ the same criterion as to the Kripke semantics above, then we obtain the following semantics.

Definition 5.3.1 (Beth model)**.** A *Beth frame* $\mathcal{F}_{\mathcal{B}}$ is a pair $(W, \preceq)$ that defines a spread. Then A *Beth model* $\mathcal{M}_{\mathcal{B}}$ is a pair $(\mathcal{F}_{\mathcal{B}}, \mathcal{V})$, where $\mathcal{V}$ is a covering assignment of propositional variables.

The forcing relation $\Vdash_{\mathcal{B}} A$ for a Beth model is defined by the following clauses.

$$
\begin{aligned}
b \Vdash_{\mathcal{B}} p &\iff b \in \mathcal{V}(p).\\
b \Vdash_{\mathcal{B}} A \land B &\iff b \Vdash_{\mathcal{B}} A \text{ and } b \Vdash_{\mathcal{B}} B.\\
b \Vdash_{\mathcal{B}} A \lor B &\iff \forall\alpha \in b \exists n(\bar{\alpha}n \Vdash_{\mathcal{B}} A \text{ or } \bar{\alpha}n \Vdash_{\mathcal{B}} B).\\
b \Vdash_{\mathcal{B}} A \to B &\iff \text{ for all } b' \succeq b, \text{ if } b' \Vdash_{\mathcal{B}} A \text{ then } b' \Vdash_{\mathcal{B}} B.\\
b \Vdash_{\mathcal{B}} {\sim}A &\iff \langle\rangle \nVdash_{\mathcal{B}} A.
\end{aligned}
$$

Proposition 5.3.1.
(i) $b \Vdash_{\mathcal{B}} A$ if and only if $\forall\alpha \in b\exists n(\bar{\alpha}n \Vdash_{\mathcal{B}} A)$. (covering property)
(ii) $b' \succeq b$ and $b \Vdash_{\mathcal{B}} A$ implies $b' \Vdash_{\mathcal{B}} A$. (monotonicity)

Proof. We prove (i) by induction on the complexity of formulae. If $b \Vdash_{\mathcal{B}} A$, then trivially $\forall\alpha \in b\exists n(\bar{\alpha}n \Vdash_{\mathcal{B}} A)$. For the converse direction, we show by induction on the complexity of A. Because (i) holds in Beth models for intuitionistic logic, it suffices to check the case where $A \equiv {\sim}B$. If $\forall\alpha \in b\exists n(\bar{\alpha}n \Vdash_{\mathcal{B}} {\sim}B)$, then by definition $\forall\alpha \in b\exists n(\langle\rangle \nVdash_{\mathcal{B}} B)$; i.e. $\langle\rangle \nVdash_{\mathcal{B}} B$. Thus by definition again, $b \Vdash_{\mathcal{B}} {\sim}B$.

(ii) is an immediate consequence of (i). □

How does this semantics relate to $\mathbf{IPC}^{\sim}$? In considering this question, we first look at how to embed Kripke models into Beth models, in accordance with the method outlined in [122].

Given a Kripke model $\mathcal{M}^{\sim}_{\mathcal{K}} = (W_K, \leq, \mathcal{V}_K)$ for $\mathbf{IPC}^{\sim}$, we construct a corresponding Beth model $\mathcal{M}_{\mathcal{B}} = (W_B, \preceq, \mathcal{V}_B)$ with the following stipulations.

- W_B is the set of finite nondecreasing sequences of worlds (i.e. each w in a sequence is followed by w' s.t. $w \leq w'$) from the root r in $(W_K, \leq)$ with length > 0.
- $\preceq$ is defined accordingly.
- $\langle w_0, \ldots, w_n\rangle \in \mathcal{V}_B(p)$ if and only if $w_n \in \mathcal{V}_K(p)$.

The resulting W_B is a spread, because the reflexivity of $\leq$ assures that $\langle w_0, \ldots, w_n\rangle \in W_B$ implies $\langle w_0, \ldots, w_n, w_n\rangle \in W_B$. Note that w_0 is always the root r in $\mathcal{M}^{\sim}_{\mathcal{K}}$, and $\langle w_0\rangle$ is the root of $\mathcal{M}_{\mathcal{B}}$. The latter slightly differs from our definition of Beth model: we can fit the model to the definition if we reinterpret the sequences as mere labels for the tree, and the actual tree is constructed in such a way that $\langle w_0\rangle$ is the label for the node $\langle\rangle$, $\langle w_0, w_1, \ldots, w_n\rangle$ is the label for the node $\langle w_1, \ldots, w_n\rangle$. We can also adopt a different embedding, which we shall see later.

For any Kripke model, because we can concatenate the same element indefinitely many times, we can also consider infinite nondecreasing sequences of worlds. This fact will be used in the next lemma.

Lemma 5.3.1 (embeddability of Kripke models for $\mathbf{IPC}^{\sim}$)**.**
(i) $\mathcal{M}_{\mathcal{B}}$ is indeed a Beth model.
(ii) $\mathcal{M}^{\sim}_{\mathcal{K}} \vDash_{\mathcal{K}} A$ if and only $\mathcal{M}_{\mathcal{B}} \vDash_{\mathcal{B}} A$.

Proof. For (i), we need to check that $\mathcal{V}_B$ is a covering assignment. If

$$\forall \alpha \in \langle w_0, \ldots, w_n\rangle \exists m(\bar{\alpha} m \in \mathcal{V}_B(p)),$$

then in particular, $\alpha_0 := \langle w_0, \ldots, w_n\rangle * \langle w_n, w_n, \ldots\rangle \in \langle w_0, \ldots, w_n\rangle$. So there is an m such that $\bar{\alpha}_0 m \in \mathcal{V}_B(p)$. If $m \leq n+1 = lh(\langle w_0, \ldots, w_n\rangle)$, then by the monotonicity of $\mathcal{V}_B$ (which follows from that of $\mathcal{V}_K$, and the fact that $\mathcal{V}_B$ only looks at the last element of a sequence) we have $\langle w_0, \ldots, w_n\rangle \in \mathcal{V}_B(p)$. Otherwise, by definition of $\mathcal{V}_B$, $w_n \in \mathcal{V}_K(p)$; hence $\langle w_0, \ldots, w_n\rangle \in \mathcal{V}_B(p)$.

For (ii), it suffices to show $w_n \Vdash_{\mathcal{K}} A \Leftrightarrow \langle w_0, \ldots, w_n\rangle \Vdash_{\mathcal{B}} A$. We prove this by induction on the complexity of formulae. Given the result for intuitionistic logic, we only need to check for $A \equiv {\sim}B$. In this case,

$$w_n \Vdash_{\mathcal{K}} {\sim}B \Leftrightarrow w_0 \nVdash_{\mathcal{K}} B \Leftrightarrow \langle w_0\rangle \nVdash_{\mathcal{B}} B \Leftrightarrow \langle w_0, \ldots, w_n\rangle \Vdash_{\mathcal{B}} {\sim}B.$$

□

Let Q be the class of Beth models obtained by the above embedding. We shall denote Beth validity with respect to Q as $\vDash_{\mathsf{Q}}$.

Theorem 5.3.1 (Beth completeness of $\mathbf{IPC}^{\sim}$ with respect to Q)**.** $\Gamma \vdash_{\sim} A$ if and only if $\Gamma \vDash_{\mathsf{Q}} A$.

Proof. Because of Theorem 5.2.1, $\Gamma \vdash_{\sim} A$ if and only if $\Gamma \vDash_{\mathcal{K}} A$. Also by the preceding lemma, $\Gamma \vDash_{\mathcal{K}} A$ if and only if $\Gamma \vDash_{\mathsf{Q}} A$. □

5.3.2 Beth semantics and Gordienko's logic

The above theorem shows that $\mathbf{IPC}^{\sim}$ is sound and complete with respect to a certain class of Beth models. The question remains, however, of whether it is sound and complete with respect to all Beth models. A problem lies in the soundness direction, of the validity of (RP). In a Beth model, it is possible that a disjunction is forced at a world whilst neither of the disjuncts is.

This is contrastable with an admissible [30] rule of $\mathbf{IPC}^{\sim}$:

$$. \frac{A \to B}{\sim B \to \sim A} \text{(RC)}$$

Given any Beth model and assuming $A \to B$ is valid, if $\sim B$ is forced at a node $b' \succeq b$ given an arbitrary b, then $\langle\rangle$ does not force B, so $\langle\rangle$ cannot force A either; thus we can conclude b' forces $\sim A$ and so b forces $\sim B \to \sim A$, i.e. $\sim B \to \sim A$ is valid.

This admissibility of (RC) in Beth models motivates us to consider a variant of $\mathbf{IPC}^{\sim}$ in which (RP) is replaced with (RC). As already mentioned in [30], such a logic is known under the name $\mathbf{TCC}_{\omega}$, formulated by Gordienko in [54].

Definition 5.3.2 ($\mathbf{TCC}_{\omega}$)**.** The logic $\mathbf{TCC}_{\omega}$ is defined by adding the next axiom and rule to **IPC**.

- $A \vee \sim A$
- $\sim A \to (\sim\sim A \to B)$
- $\frac{A \to B}{\sim B \to \sim A}$ (RC)

We shall denote the provability in $\mathbf{TCC}_{\omega}$ by $\vdash_t$. We shall prove the soundness and completeness of $\mathbf{TCC}_{\omega}$ with respect to all Beth models. Again we want to embed Kripke models into Beth models; but as we see below, the Kripke models for $\mathbf{TCC}_{\omega}$ are not necessarily rooted. So we shall embed models in a slightly different way.

Definition 5.3.3 (Kripke model for $\mathbf{TCC}_{\omega}$)**.** A *Kripke Frame* $\mathcal{F}^t_{\mathcal{K}} = (W, \leq)$ for $\mathbf{TCC}_{\omega}$ is a non-empty pre-ordered set. A *Kripke model* $\mathcal{M}^t_{\mathcal{K}}$ for $\mathbf{TCC}_{\omega}$ is a pair $(\mathcal{F}^t_{\mathcal{K}}, \mathcal{V})$, where $\mathcal{V}$ is a monotone mapping that assigns a set of worlds $\mathcal{V}(p) \subseteq W$ for each propositional variable p.

Given $\mathcal{M}^t_{\mathcal{K}}$, The *forcing* of a formula in a world, denoted $w \Vdash_{\mathcal{K}t} A$, is inductively defined as follows.

$$\begin{aligned}
w \Vdash_{\mathcal{K}t} p &\iff w \in \mathcal{V}(p).\\
w \Vdash_{\mathcal{K}t} A \wedge B &\iff w \Vdash_{\mathcal{K}t} A \text{ and } w \Vdash_{\mathcal{K}t} B.\\
w \Vdash_{\mathcal{K}t} A \vee B &\iff w \Vdash_{\mathcal{K}t} A \text{ or } w \Vdash_{\mathcal{K}t} B.\\
w \Vdash_{\mathcal{K}t} A \to B &\iff \text{ for all } w' \geq w, \text{ if } w' \Vdash_{\mathcal{K}t} A \text{ then } w' \Vdash_{\mathcal{K}t} B.\\
w \Vdash_{\mathcal{K}t} \sim A &\iff w' \nVdash_{\mathcal{K}t} A \text{ for some } w'.
\end{aligned}$$

Theorem 5.3.2 (Kripke completeness for $\mathbf{TCC}_{\omega}$)**.** $\vdash_t A$ if and only if $\vDash_{\mathcal{K}t} A$.

Proof. Cf. [54] ∎

Given a Kripke model $\mathcal{M}^t_{\mathcal{K}} = (W_K, \leq, \mathcal{V}_K)$ for $\mathbf{TCC}_\omega$, we construct a corresponding Beth model $\mathcal{M}_{\mathcal{B}} = (W_B, \preceq, \mathcal{V}_B)$ in the same way we did in Lemma 2.3.1. This shall again enable the next embedding.

Lemma 5.3.2 (embeddability of Kripke models for $\mathbf{TCC}_\omega$)**.** $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} A$ if and only $\mathcal{M}_{\mathcal{B}} \vDash_{\mathcal{B}} A$.

Proof. By Lemma 2.3.1, it suffices to consider the case for $\sim A$:

1. $\langle\rangle \Vdash_{\mathcal{B}} \sim A$ if and only if $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} \sim A$.

2. $\langle b_0, \ldots, b_n\rangle \Vdash_{\mathcal{B}} \sim A$ if and only if $b_n \Vdash_{\mathcal{K}t} \sim A$. (where $n > -1$)

If $A \equiv \sim A_1$, then for 1., suppose $\langle\rangle \Vdash_{\mathcal{B}} \sim A_1$. Then $\langle\rangle \nVdash_{\mathcal{B}} A_1$. So $\mathcal{M}^t_{\mathcal{K}} \nvDash_{\mathcal{K}t} A_1$ by I.H.. Hence $w \nVdash_{\mathcal{K}t} A_1$ for some $w \in W_K$. Thus $u \Vdash_{\mathcal{K}t} \sim A$ for all $u \in W_K$. Thus $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} \sim A$. Conversely, suppose $\mathcal{M}^t_{\mathcal{K}} \vDash_{\mathcal{K}t} \sim A$. Take $w \in W_K$. Then $w \Vdash_{\mathcal{K}t} \sim A$, so $u \nVdash_{\mathcal{K}t} A$ for some $u \in W_K$. Hence $\mathcal{M}^t_{\mathcal{K}} \nvDash_{\mathcal{K}t} A$, so $\langle\rangle \nVdash_{\mathcal{B}} A$ by I.H.. Therefore $\langle\rangle \Vdash_{\mathcal{B}} \sim A$.

For 2., suppose $\langle b_0, \ldots, b_n\rangle \Vdash_{\mathcal{B}} \sim A$. Then $\langle\rangle \nVdash_{\mathcal{B}} A$. So $\mathcal{M}^t_{\mathcal{K}} \nvDash_{\mathcal{K}t} A$. Hence for some $w \in W_K$, $w \nVdash_{\mathcal{K}t} A$. Therefore $b_n \Vdash_{\mathcal{K}t} \sim A$. Conversely, if $b_n \Vdash_{\mathcal{K}t} \sim A$, then $w \nVdash_{\mathcal{K}t} A$ for some $w \in W_K$. By I.H. $\langle w\rangle \nVdash_{\mathcal{B}} A$. Thus $\langle\rangle \nVdash_{\mathcal{B}} A$. Therefore $\langle b_0, \ldots, b_n\rangle \Vdash_{\mathcal{B}} \sim A$. □

Theorem 5.3.3 (soundness and weak completeness of $\mathbf{TCC}_\omega$ with Beth semantics)**.** $\vdash_t A$ if and only if $\vDash_{\mathcal{B}} A$.

Proof. We first show the soundness by induction on the depth of deductions. We check $A \vee \sim A$, $\sim A \to (\sim\sim A \to B)$ and (RC). Let $\mathcal{M}_{\mathcal{B}} = (W_B, \preceq, \mathcal{V}_B)$ be a Beth model. By monotonicity, it suffices to check the root.

For $A \vee \sim A$, either

$$\langle\rangle \Vdash_{\mathcal{B}} A \text{ or } \langle\rangle \nVdash_{\mathcal{B}} A.$$

If the latter, $\langle\rangle \Vdash_{\mathcal{B}} \sim A$. So in either case, $\langle\rangle \Vdash_{\mathcal{B}} A \vee \sim A$.

For $\sim A \to (\sim\sim A \to B)$, if

$$b \Vdash_{\mathcal{B}} \sim A \text{ for } b \succeq \langle\rangle,$$

then if $b' \Vdash_{\mathcal{B}} \sim\sim A$ for $b' \succeq b$, we have

$$\langle\rangle \nVdash_{\mathcal{B}} \sim A \text{ and } \langle\rangle \nVdash_{\mathcal{B}} A.$$

But the former implies $\langle\rangle \Vdash_{\mathcal{B}} A$, a contradiction. Therefore $b' \Vdash_{\mathcal{B}} B$; so

$$\langle\rangle \Vdash_{\mathcal{B}} \sim A \to (\sim\sim A \to B).$$

For (RC), by I.H., $\vDash_{\mathcal{B}} A \to B$ and in particular, $\mathcal{M}_{\mathcal{B}} \vDash_{\mathcal{B}} A \to B$. If for $b \succeq \langle\rangle$ we have $b \Vdash_{\mathcal{B}} \sim B$, then $\langle\rangle \nVdash_{\mathcal{B}} B$. Now if $\langle\rangle \Vdash_{\mathcal{B}} A$, then as $\langle\rangle \Vdash_{\mathcal{B}} A \to B$, $\langle\rangle \Vdash_{\mathcal{B}} B$, a contradiction. Thus $\langle\rangle \nVdash_{\mathcal{B}} A$; hence $b \Vdash_{\mathcal{B}} \sim A$. So

$$\langle\rangle \Vdash_{\mathcal{B}} \sim B \to \sim A.$$

The completeness follows from the previous lemma and the Kripke completeness of $\mathbf{TCC}_\omega$ [54, Theorem 4.5]. □

5.3.3 Classical logic and Gordienko's logic

The fact that Kripke and Beth semantics differ on the forcing of disjunction is well-reflected in the following translation of **CPC** into $\mathbf{TCC}_\omega$.

Definition 5.3.4 ($()^t$)**.** We inductively define $()^t$ to be a mapping between formulae in $\mathcal{L}$.

$$\begin{aligned} p^t &\equiv p \\ (A \wedge B)^t &\equiv A^t \wedge B^t. \\ (A \vee B)^t &\equiv {\sim}{\sim}A^t \vee {\sim}{\sim}B^t. \\ (A \to B)^t &\equiv {\sim}{\sim}A^t \to {\sim}{\sim}B^t. \\ ({\sim}A)^t &\equiv {\sim}A^t. \end{aligned}$$

Beth-semantically speaking, $()^t$ restricts our attention to the root world, when it comes to disjunction and implication. This is related to the connection between empirical negation (of $\mathbf{IPC}^{\sim}$) and classical negation, as observed in [29] and [30]. A new point for $\mathbf{TCC}_\omega$ is that the restriction applies not only to implication but also to disjunction. This corresponds to the fact that in Beth semantics, both disjunction and implication look at other worlds, whereas in Kripke semantics, only the latter does so.

In the following, we make a heavy use of easily checkable equivalences in Beth semantics.

- $b \Vdash_{\mathcal{B}} {\sim}{\sim}A \iff \langle\rangle \Vdash_{\mathcal{B}} A$.
- $b \Vdash_{\mathcal{B}} {\sim}{\sim}A \vee {\sim}{\sim}B \iff \langle\rangle \Vdash_{\mathcal{B}} A$ or $\langle\rangle \Vdash_{\mathcal{B}} B$.
- $b \Vdash_{\mathcal{B}} {\sim}{\sim}A \to {\sim}{\sim}B \iff \langle\rangle \Vdash_{\mathcal{B}} A$ implies $\langle\rangle \Vdash_{\mathcal{B}} B$.

Let us use the notation $\Gamma^t := \{B^t : B \in \Gamma\}$. We shall henceforth abbreviate ${\sim}{\sim}A$ as ${\approx}A$. Metalinguistic 'implies' ($\Rightarrow$) should not be confused with $\to$ in the proof below.

Proposition 5.3.2 (faithful embedding of **CPC** into $\mathbf{TCC}_\omega$)**.** $\Gamma \vdash_c A$ if and only if $\Gamma^t \vdash_t A^t$.

Proof. The left-to-right direction is shown by induction on the depth of deductions. If A is an assumption, then correspondingly $A^t \in \Gamma^t$.

If A is an axiom, we exemplify by the case for the axiom

$$(A \to C) \to ((B \to C) \to (A \vee B \to C)).$$

$((A \to C) \to ((B \to C) \to (A \vee B \to C)))^t$ is

$${\approx}({\approx}A^t \to {\approx}C^t) \to {\approx}({\approx}({\approx}B^t \to {\approx}C^t) \to {\approx}({\approx}({\approx}A^t \vee {\approx}B^t) \to {\approx}C^t)).$$

Using Beth completeness, it is sufficient to show,

$$b \Vdash_{\mathcal{B}} {\approx}({\approx}A^t \to {\approx}C^t) \to {\approx}({\approx}({\approx}B^t \to {\approx}C^t) \to {\approx}({\approx}({\approx}A^t \vee {\approx}B^t \to {\approx}C^t)))$$

holds for any b in an arbitrary Beth model. This is equivalent to

$$\begin{array}{c}\langle\rangle \Vdash_{\mathcal{B}} \approx A^t \to \approx C^t \\ \text{implies } \langle\rangle \Vdash_{\mathcal{B}} \approx(\approx B^t \to \approx C^t) \to \approx(\approx(\approx A^t \vee \approx B^t) \to \approx C^t)\end{array}$$

by one of the above equivalences; this is further equivalent to

$$\begin{array}{c}\langle\rangle \Vdash_{\mathcal{B}} A^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t \\ \text{implies } (\langle\rangle \Vdash_{\mathcal{B}} \approx B^t \to \approx C^t) \Rightarrow (\langle\rangle \Vdash_{\mathcal{B}} \approx(\approx(\approx A^t \vee \approx B^t) \to \approx C^t))\end{array}$$

and to

$$\begin{array}{c}\langle\rangle \Vdash_{\mathcal{B}} A^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t \\ \text{implies } (\langle\rangle \Vdash_{\mathcal{B}} B^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t) \Rightarrow (\langle\rangle \Vdash_{\mathcal{B}} \approx A^t \vee \approx B^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t)\end{array}$$

and to

$$\begin{array}{c}(\langle\rangle \Vdash_{\mathcal{B}} A^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t) \text{ and } (\langle\rangle \Vdash_{\mathcal{B}} B^t \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t) \\ \text{implies } (((\langle\rangle \Vdash_{\mathcal{B}} A^t \text{ or } \langle\rangle \Vdash_{\mathcal{B}} B^t) \Rightarrow \langle\rangle \Vdash_{\mathcal{B}} C^t))\end{array}$$

and this holds. Here, if it were the case that $(A \vee B)^t \equiv (A^t \vee B^t)$, then we would get $\langle\rangle \Vdash_{\mathcal{B}} A^t \vee B^t$ instead of $\langle\rangle \Vdash_{\mathcal{B}} \approx A^t \vee \approx B^t$, and the formula fails to hold.

If the deduction ends with an application of

$$, \frac{B \qquad B \to A}{A} \text{ (MP)}$$

then by I.H., $\Gamma^t \vdash_t B^t$ and $\Gamma^t \vdash_t {\sim\sim} B^t \to {\sim\sim} A^t$. In [30, Lemma 2.8] the rule

$$\frac{A}{{\sim\sim} A} \text{ (RD)}$$

is shown to be derivable from (RC) in $\mathbf{IPC}^{\sim}$. The proof appeals to (RP) only non-essentially (it is used to derive ${\sim\sim} A \to A$, which is obtainable from $A \vee {\sim} A$ and ${\sim} A \to ({\sim\sim} A \to B)$ alone), and so (RD) is also derivable in $\mathbf{TCC}_\omega$. Thus we obtain $\Gamma^t \vdash_t {\sim\sim} B^t$. So by (MP), $\Gamma^t \vdash_t {\sim\sim} A^t$; hence $\Gamma^t \vdash_t A^t$ by double negation elimination.

The right-to-left direction follows from the easily noticeable equivalence that $\vdash_c A \leftrightarrow A^t$. □

Before moving on, we shall mention that there exists another reading of the negation in the Beth semantics for $\mathbf{TCC}_\omega$. Because the models are rooted, for any b,

$$\exists b' \leq b (b' \nVdash A) \Leftrightarrow \langle\rangle \nVdash A.$$

From this viewpoint the negation of $\mathbf{TCC}_\omega$ can be understood as co-negation as well. For Kripke semantics, the logic of co-negation is the logic **daC** of Priest [100]. A Hilbert-style axiomatisation of **daC** was first formulated by Castiglioni et al. [21]. This axiomatisation is obtained from that of $\mathbf{IPC}^{\sim}$ by removing the axiom ${\sim} A \to ({\sim\sim} A \to B)$. If we further replace (RP) with (RC), and add an axiom ${\sim\sim} A \to A$ (a theorem of **daC**), we obtain the logic $\mathbf{CC}_\omega$ of Sylvan [116]. Note $\mathbf{CC}_\omega$ can be strengthened to $\mathbf{TCC}_\omega$ by adding ${\sim} A \to ({\sim\sim} A \to B)$ as an axiom and dropping ${\sim\sim} A \to A$, which becomes redundant.

5.4 Eliminating (RP)

The last section made clear that the negations of $\mathbf{IPC}^{\sim}$ and $\mathbf{TCC}_\omega$ are characterised by the same valuation, but with respect to different semantics: Kripke and Beth. We may understand them as representing different types of experience, and thus different empirical negations. We can make an analogous remark for co-negation. This case is perhaps more interesting, for $\mathbf{TCC}_\omega$ and **daC** are not comparable [99]. In any case, these curious effects of "same forcing-condition in two similar semantics" encourage a further analysis.

Proof-theoretically, however, there is an obstacle in comparing the logics, in that $\mathbf{TCC}_\omega$ and $\mathbf{CC}_\omega$ employ the rule (RC), whereas **daC** and $\mathbf{IPC}^{\sim}$ employ the stronger (RP).

We would like, therefore, to have a new axiomatisation of $\mathbf{IPC}^{\sim}$ and **daC** with (RC), rather than (RP). We can expect such conversion would allow us to analyse and understand the logics from a more unified perspective.

We shall start such an attempt with $\mathbf{IPC}^{\sim}$, using a provable formula of $\mathbf{IPC}^{\sim}$, $({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B)$ [30, Proposition 2.14].

Proposition 5.4.1. The addition of $({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B)$ to $\mathbf{TCC}_\omega$ derives (RP).

Proof. In $\mathbf{TCC}_\omega$, assuming $(A \vee B)$ we can derive ${\sim}{\sim}(A \vee B)$ by (RD). So we have

$${\sim}B \to ({\sim}A \to {\sim}{\sim}(A \vee B)).$$

Also we infer from ${\sim}B \to ({\sim}A \to ({\sim}A \wedge {\sim}B))$ and $({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B)$ that

$${\sim}B \to ({\sim}A \to {\sim}(A \vee B)).$$

Thus

$${\sim}B \to ({\sim}A \to ({\sim}(A \vee B) \wedge {\sim}{\sim}(A \vee B))).$$

Next note ${\sim}(A \vee B) \to ({\sim}{\sim}(A \vee B) \to B)$ is an instance if one of the axioms. Combine the two and we obtain

$${\sim}B \to ({\sim}A \to B).$$

Then as $B \to ({\sim}A \to B)$ follows from intuitionistic logic, and $B \vee {\sim}B$ is an axiom, we conclude ${\sim}A \to B$. □

Hence we have obtained an alternative axiomatisation of $\mathbf{IPC}^{\sim}$ with (RC).

It is stated in [30] that $\mathbf{TCC}_\omega$ is a strict subsystem of $\mathbf{IPC}^{\sim}$, but no specific example is shown. As a side remark, we can use $({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B)$ to observe the following.

Proposition 5.4.2. $({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B)$ is underivable in $\mathbf{TCC}_\omega$.

Proof. We prove it via Beth completeness. Let $\mathcal{F}_\mathcal{B} = (W, \preceq)$ be the set of finite binary sequences ordered by the initial segment relation. Let $\mathcal{M}_\mathcal{B} = (\mathcal{F}_\mathcal{B}, \mathcal{V})$ be a model such that

$$b \in \mathcal{V}(p) \Leftrightarrow \langle 0 \rangle \preceq b \text{ and } b \in \mathcal{V}(q) \Leftrightarrow \langle 1 \rangle \preceq b.$$

Then it is straightforward to see that this assignment is covering: e.g. if $\forall\alpha \in b\exists m(\bar{\alpha}m \Vdash_{\mathcal{B}} p)$, then clearly $\langle 0\rangle \preceq b$. Now

$$\mathcal{M}_{\mathcal{B}}, \langle\rangle \nVdash_{\mathcal{B}} p \text{ and } \mathcal{M}_{\mathcal{B}}, \langle\rangle \nVdash_{\mathcal{B}} q,$$

so $\mathcal{M}_{\mathcal{B}}, \langle\rangle \Vdash_{\mathcal{B}} {\sim}p \wedge {\sim}q$; but since

$$\forall\alpha \in \langle\rangle(\bar{\alpha}1 \Vdash_{\mathcal{B}} p \text{ or } \bar{\alpha}1 \Vdash_{\mathcal{B}} q),$$

we have $\mathcal{M}_{\mathcal{B}}, \langle\rangle \Vdash_{\mathcal{B}} p\vee q$, i.e. $\mathcal{M}_{\mathcal{B}}, \langle\rangle \nVdash_{\mathcal{B}} {\sim}(p\vee q)$. Therefore $\mathcal{M}_{\mathcal{B}}, \langle\rangle \nVdash_{\mathcal{B}} ({\sim}p\wedge{\sim}q) \rightarrow {\sim}(p \vee q)$. □

Corollary 5.4.1 (failure of soundness for $\mathbf{IPC}^{\sim}$ with all Beth models)**.**
$\vdash_{\sim} A \not\Rightarrow \vDash_{\mathcal{B}} A$.

Proof. Otherwise $\vdash_{\sim} A \Rightarrow \vDash_{\mathcal{B}} A \Leftrightarrow \vdash_t A$, which is absurd. □

Ferguson [40, Theorem 2.3] gives the frame property of $({\sim}A \wedge {\sim}B) \rightarrow {\sim}(A \vee B)$ with respect to **daC**. We just mention a quite similar observation can be made for the Kripke models for $\mathbf{CC}_\omega$.

Definition 5.4.1 (Semantics of $\mathbf{CC}_\omega$)**.** A *Kripke frame* $\mathcal{F}^c_{\mathcal{K}}$ for $\mathbf{CC}_\omega$ is a triple $(W, \leq, S)$, where $S \subseteq W \times W$ is a reflexive and symmetric (accessibility) relation such that $u \leq v$ and uSw implies vSw, i.e. S is upward closed. A *Kripke model* $\mathcal{M}^c_{\mathcal{K}}$ for $\mathbf{CC}_\omega$ is defined as usual, except for the forcing condition ($\Vdash_{\mathcal{K}cc}$) of negation, which is

$$w \Vdash_{\mathcal{K}cc} {\sim}A \iff w' \nVdash_{\mathcal{K}cc} A \text{ for some } w' \text{ such that } wSw'.$$

Note if $S = W \times W$, then a $\mathbf{CC}_\omega$-frame (model) is a $\mathbf{TCC}_\omega$-frame (model) [54]. Indeed, what is shown in [54] is that $\mathbf{TCC}_\omega$ is sound and complete with the class of $\mathbf{CC}_\omega$-frames where S is transitive, and in particular the frames with $S = W \times W$ is sufficient for this. We shall occasionally denote uSv also by $vS^{-1}u$. As S is symmetric in $\mathbf{CC}_\omega$, this distinction is not quite necessary. This however clarifies appeals to symmetry in proofs, which becomes significant in a broader context.

Proposition 5.4.3. Let $\mathcal{F}^c_{\mathcal{K}}$ be a $\mathbf{CC}_\omega$-frame. Then the following conditions are equivalent:

(i) $\mathcal{F}^c_{\mathcal{K}} \vDash_{\mathcal{K}cc} ({\sim}A \wedge {\sim}B) \rightarrow {\sim}(A \vee B)$ for all A, B.

(ii) $\mathcal{F}^c_{\mathcal{K}}$ satisfies $\forall u, v, w(uSv$ and uSw implies $\exists xS^{-1}u(v \geq x$ and $w \geq x)$.

Proof. We shall first see (i) implies (ii). Suppose uSv and uSw. Let

$$\mathcal{V}(p) = \{x : v \ngeq x\} \text{ and } \mathcal{V}(q) = \{x : w \ngeq x\}.$$

Now if $w \in \mathcal{V}(p)$ and $x' \geq x$, then $v \geq x'$ implies $v \geq x$, a contradiction. So $v \ngeq x'$, and thus $x' \in \mathcal{V}(p)$. Hence $\mathcal{V}(p)$ is upward closed. Similarly $\mathcal{V}(q)$ is upward closed. Now since $v \geq v$ and $w \geq w$,

$$v \nVdash_{\mathcal{K}t} p \text{ and } w \nVdash_{\mathcal{K}t} q.$$

So $u \Vdash_{\mathcal{K}cc} {\sim}p \wedge {\sim}q$. Hence by assumption $u \Vdash_{\mathcal{K}cc} {\sim}(p \vee q)$. So there is an $xS^{-1}u$ such that

$$x \nVdash_{\mathcal{K}cc} p \text{ (i.e. } v \geq x) \text{ and } x \nVdash_{\mathcal{K}cc} q \text{ (i.e. } w \geq x),$$

as we desired.

Next we shall see (ii) implies (i). Assume $\mathcal{F}^c_{\mathcal{K}}$ satisfies (ii) and $\mathcal{V}$, u_0 be arbitrary. If for $u \geq u_0$

$$(\mathcal{F}^t_{\mathcal{K}}, \mathcal{V}), u \Vdash_{\mathcal{K}cc} {\sim}A \wedge {\sim}B,$$

then there are $vS^{-1}u$ and $wS^{-1}u$ such that

$$v \nVdash_{\mathcal{K}cc} A \text{ and } w \nVdash_{\mathcal{K}cc} B.$$

By (ii), there is $xS^{-1}u$ such that $v \geq x$ and $w \geq x$. Now $x \nVdash_{\mathcal{K}cc} A \vee B$. Hence $u \Vdash_{\mathcal{K}cc} {\sim}(A \vee B)$. So

$$(\mathcal{F}^t_{\mathcal{K}}, \mathcal{V}), u_0 \Vdash_{\mathcal{K}cc} ({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B).$$

Since w and $\mathcal{V}$ are arbitrary, $\mathcal{F}^c_{\mathcal{K}} \vDash_{\mathcal{K}cc} ({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B)$. □

Given a Kripke frame for **IPC**$^{\sim}$, we can regard it as a frame of **TCC**$_\omega$ with $S = W \times W$; i.e. there is an embedding. Then it is immediately seen that such a frame satisfies the above condition, because it is rooted. This means the class of Kripke frames for **TCC**$_\omega$ satisfying the above condition is complete with respect to **IPC**$^{\sim}$, for if a formula is validated by each such frame, then it must be validated by each frame of **IPC**$^{\sim}$.

Next we consider **daC**. The formula ${\sim}A \wedge {\sim}B \to {\sim}(A \vee B)$ used for **IPC**$^{\sim}$ cannot be used for **daC**, because it is not a theorem of **daC** [98, Table 3]. We instead have to look at another formula ${\sim}({\sim}(A \vee B) \vee A) \to B$.

Proposition 5.4.4. **CC**$_\omega$ $+ {\sim}({\sim}(A \vee B) \vee A) \to B =$ **daC**.

Proof. It has been observed in [98, Theorem 3.13] that ${\sim}({\sim}(A \vee B) \vee A) \to B$ is a theorem of **daC**. So we only have to check (RP) is admissible in **CC**$_\omega$ $+ {\sim}({\sim}(A \vee B) \vee A) \to B$. We first note

$$\frac{A}{{\sim}A \to B}$$

is derivable in **CC**$_\omega$ by the same argument as in [99, Theorem 4.3]. Assuming $A \vee B$ is derivable, from this we see ${\sim}(A \vee B) \to A$ is derivable. By intuitionistic logic, we can infer $({\sim}(A \vee B) \vee A) \to A$, and then by (RC), ${\sim}A \to {\sim}({\sim}(A \vee B) \vee A)$. On the other hand, ${\sim}({\sim}(A \vee B) \vee A) \to B$ is the added axiom. Thus we conclude ${\sim}A \to B$. □

${\sim}({\sim}(A \vee B) \vee A) \to B$ is used in [98, theorem 3.13] to establish that **daC** strictly contains another logic **daC'**, axiomatised by replacing (RP) with a weaker rule

$$. \frac{A \vee {\sim}B}{{\sim} A \to {\sim} B} \text{ (wRP)}$$

We shall note (wRP) in **daC**$'$ is similarly reducible to an axiom ${\sim}({\sim}(A \vee {\sim}B) \vee A) \to {\sim}B$.

Proposition 5.4.5. **CC**$_\omega$ $+ {\sim}({\sim}(A \vee {\sim}B) \vee A) \to {\sim}B =$ **daC**$'$

Proof. It has been observed in [99, Lemma 3.2] that $\sim(\sim(A \vee \sim B) \vee A) \to \sim B$ is a theorem of **daC′**. So we only have to check (wRP) is admissible in $\mathbf{CC}_\omega + \sim(\sim(A \vee \sim B) \vee A) \to \sim B$. This is proved as in the previous proposition, except that we infer $\sim A \to \sim(\sim(A \vee \sim B) \vee A)$ and $\sim(\sim(A \vee \sim B) \vee A) \to \sim B$ to conclude $\sim A \to \sim B$. □

Next, we turn our attention to the semantic side. Our goal will be to establish a connection between the Kripke semantics of $\mathbf{CC}_\omega$ and **daC**. For this we shall first consider the frame condition for $\sim(\sim(A \vee B) \vee A) \to B$.

Proposition 5.4.6. Let $\mathcal{F}^c_\mathcal{K}$ be a $\mathbf{CC}_\omega$-frame. Then the following conditions are equivalent:

(i) $\mathcal{F}^c_\mathcal{K} \models_{\mathcal{K}cc} \sim(\sim(A \vee B) \vee A) \to B$ for all A, B.

(ii) $\mathcal{F}^c_\mathcal{K}$ satisfies $\forall u, v(uSv \to \exists wS^{-1}v(w \leq u \text{ and } w \leq v))$.

Proof. We shall first see (i) implies (ii). We shall show the contrapositive. So suppose for some u and v, uSv holds but $\neg\exists wS^{-1}v(w \leq u \text{ and } w \leq v)$. Choose $\mathcal{V}$ s.t.

$$\mathcal{V}(p) = \{w : w \not\leq v\} \text{ and } \mathcal{V}(q) = \{w : w \not\leq u\}.$$

It is straightforward to see $\mathcal{V}(p)$ and $\mathcal{V}(q)$ are upward closed. Now since $\forall wS^{-1}v(w \not\leq u \text{ or } w \not\leq v)$, we have $\forall wS^{-1}v(w \Vdash_{\mathcal{K}cc} p \text{ or } w \Vdash_{\mathcal{K}cc} q)$. So $v \nVdash_{\mathcal{K}cc} \sim(p \vee q)$. In addition, $v \leq v$ means $v \nVdash_{\mathcal{K}cc} p$. Thus

$$u \Vdash_{\mathcal{K}cc} \sim(\sim(p \vee q) \vee p).$$

On the other hand, $u \leq u$ implies $u \nVdash_{\mathcal{K}cc} q$. Thus $u \nVdash_{\mathcal{K}cc} \sim(\sim(p \vee q) \vee p) \to q$. Therefore $\mathcal{F}^c_\mathcal{K} \not\models_{\mathcal{K}cc} \sim(\sim(p \vee q) \vee p) \to q$.

Next we shall see (ii) implies (i). Assume

$$\forall u, v(uSv \to \exists wS^{-1}v(w \leq u \text{ and } w \leq v)).$$

Let $\mathcal{V}$ and u be arbitrary, and for $v \geq u$, suppose $(\mathcal{F}^c_\mathcal{K}, \mathcal{V}), v \Vdash_{\mathcal{K}cc} \sim(\sim(A \vee B) \vee A)$. Then for some $wS^{-1}v$, $w \nVdash_{\mathcal{K}cc} \sim(A \vee B) \vee A$. Thus

$$w \nVdash_{\mathcal{K}cc} A \text{ and } \forall xS^{-1}w(x \Vdash_{\mathcal{K}cc} A \vee B).$$

Now by assumption, from vSw we infer $\exists yS^{-1}w(y \leq v \text{ and } y \leq w)$. From our observation above, we know $y \Vdash_{\mathcal{K}cc} A \vee B$. If $y \Vdash_{\mathcal{K}cc} A$, then $y \leq w$ implies $w \Vdash_{\mathcal{K}cc} A$, a contradiction. So $y \Vdash_{\mathcal{K}cc} B$, which with $y \leq v$ implies $v \Vdash_{\mathcal{K}cc} B$. Thus

$$(\mathcal{F}^c_\mathcal{K}, \mathcal{V}), u \Vdash_{\mathcal{K}cc} \sim(\sim(A \vee B) \vee A) \to B.$$

Since $\mathcal{V}$ and u are arbitrary, $\mathcal{F}^c_\mathcal{K} \models_{\mathcal{K}cc} \sim(\sim(A \vee B) \vee A) \to B$. □

Note that in the proof no appeal is made to neither the reflexivity nor symmetry of S. Thus we see the correspondence holds for a weaker setting of one of Došen's systems in [34, p.81-83] (under what he calls *condensed* frames). It has the same forcing condition, but the accessibility relation there is not assumed to be reflexive nor symmetric.

With the frame condition at hand, we can now translate back and forth the frames of $\mathbf{CC}_\omega$ and **daC**.

Definition 5.4.2 (semantics of **daC**)**.** A *Kripke frame* $\mathcal{F}^d_{\mathcal{K}}$ for **daC** is a pair $(W, \leq)$, and a *Kripke model* $\mathcal{M}^d_{\mathcal{K}}$ for **daC** is defined as usual, except for the forcing condition ($\Vdash_{\mathcal{K}cc}$) of negation, which is

$$\mathcal{M}^d_{\mathcal{K}}, w \Vdash_{\mathcal{K}d} {\sim}A \iff \mathcal{M}^d_{\mathcal{K}}, w' \nVdash_{\mathcal{K}d} A \text{ for some } w' \leq w.$$

Proposition 5.4.7.

(i) Let $\mathcal{F}^c_{\mathcal{K}} = (W, \leq, S)$ be a frame of $\mathbf{CC}_\omega$ satisfying

$$\forall u, v(uSv \to \exists wS^{-1}v(w \leq u \text{ and } w \leq v)).$$

Define $\Phi(\mathcal{F}^c_{\mathcal{K}}) = (W, \leq)$. Then for any $\mathcal{V}$ and w,

$$(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}cc} A \Leftrightarrow (\Phi(\mathcal{F}^c_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}d} A.$$

(ii) Let $\mathcal{F}^d_{\mathcal{K}}$ be a frame of **daC**. Define

$$S = \{(u, v) : \exists w(w \leq u \text{ and } w \leq v))\}.$$

and $\Psi(\mathcal{F}^d_{\mathcal{K}}) = (W, \leq, S)$. Then for any $\mathcal{V}$ and w,

$$(\mathcal{F}^d_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}d} A \Leftrightarrow (\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}cc} A.$$

(iii) $\Psi = \Phi^{-1}$ for the above Φ and Ψ.

Note the S defined in (ii) is well-defined: it is easy to check it is reflexive, symmetric and satisfies $\forall u, v(uSv \to \exists wS^{-1}v(w \leq u \text{ and } w \leq v))$.

Proof. In (i) and (ii), we only have to consider the case for negation.

For (i), if $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}cc} {\sim}A$, then for some $w'S^{-1}w$, $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w' \nVdash_{\mathcal{K}cc} A$. By the frame condition, there is $xS^{-1}w$ such that $x \leq w$ and $x \leq w'$. Because of the latter, $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), x \nVdash_{\mathcal{K}cc} A$. By I.H., $(\Phi(\mathcal{F}^c_{\mathcal{K}}), \mathcal{V}), x \nVdash_{\mathcal{K}d} A$. Since $x \leq w$, $(\Phi(\mathcal{F}^c_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}d} {\sim}A$. For the converse direction, if $(\Phi(\mathcal{F}^c_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}d} {\sim}A$ then for some $w' \leq w$, $(\Phi(\mathcal{F}^c_{\mathcal{K}}), \mathcal{V}), w' \nVdash_{\mathcal{K}d} A$. By I.H., $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w' \nVdash_{\mathcal{K}cc} A$. Here, since $w'Sw'$ by reflexivity and $w' \leq w$, we have $w'Sw$, so by symmetry wSw'. Thus $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}cc} {\sim}A$.

For (ii), if $(\mathcal{F}^d_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}d} {\sim}A$, then for some $w' \leq w$, $(\mathcal{F}^d_{\mathcal{K}}, \mathcal{V}), w' \nVdash_{\mathcal{K}d} A$. By I.H., $(\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), w' \nVdash_{\mathcal{K}cc} A$. Now as $w' \leq w$ and $w'Sw'$, wSw'. So $(\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}cc} {\sim}A$. For the converse direction, if $(\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), w \Vdash_{\mathcal{K}cc} {\sim}A$, then for some $w'S^{-1}w$, $(\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), w' \nVdash_{\mathcal{K}cc} A$. Thus there is an x such that $x \leq w$ and $x \leq w'$. We have $(\Psi(\mathcal{F}^d_{\mathcal{K}}), \mathcal{V}), x \nVdash_{\mathcal{K}cc} A$ by the latter. By I.H., $(\mathcal{F}^d_{\mathcal{K}}, \mathcal{V}), x \nVdash_{\mathcal{K}d} A$. Therefore $(\mathcal{F}^d_{\mathcal{K}}, \mathcal{V}), w \Vdash_{\mathcal{K}d} {\sim}A$.

For (iii), it is immediate to see that $\Phi(\Psi(\mathcal{F}^d_{\mathcal{K}})) = \mathcal{F}^d_{\mathcal{K}}$, as the mappings do not alter $(W, \leq)$. As for $\Psi(\Phi(\mathcal{F}^c_{\mathcal{K}})) = \mathcal{F}^c_{\mathcal{K}}$, we need to check the original S in $\mathcal{F}^c_{\mathcal{K}}$ and the defined S' in $\Psi(\Phi(\mathcal{F}^c_{\mathcal{K}}))$. It is easy from the frame condition that $S \subseteq S'$. Further, if $\exists x(x \leq w \text{ and } x \leq w')$, then xSw' by reflexivity, symmetry and upward closure of S. Thus again by upward closure of S, wSw'; so $S \supseteq S'$. □

This allows us to conclude the following completeness of **daC** with respect to the frames of $\mathbf{CC}_\omega$: let us denote the derivability in **daC** by $\vdash_d$.

Corollary 5.4.2. $\vdash_d A$ if and only if $\mathcal{F}^c_{\mathcal{K}} \models_{\mathcal{K}cc} A$ for all $\mathcal{F}^c_{\mathcal{K}}$ satisfying

$$\forall u, v(uSv \to \exists w S^{-1} v(w \leq u \text{ and } w \leq v)).$$

Proof. The last proposition established a bijection of frames agreeing in forcing. Thus the statement follows from the completeness of **daC** with respect to its models [100]. □

We now look at the frame condition for $\sim(\sim(A \vee \sim B) \vee A) \to \sim B$.

Proposition 5.4.8. Let $\mathcal{F}$ be a $\mathbf{CC}_\omega$-frame. Then the following conditions are equivalent.

(i) $\mathcal{F} \models_{\mathcal{K}cc} \sim(\sim(A \vee \sim B) \vee A) \to \sim B$ for all A, B.

(ii) $\mathcal{F}$ satisfies $\forall u, v(uSv \to \exists w S^{-1} v(w \leq v \text{ and } \forall x(wSx \to uSx)))$.

Proof. We shall first see (i) implies (ii). We show this by contraposition. Assume uSv but

$$\neg \exists w S^{-1} v(w \leq v \text{ and } \forall x(wSx \to uSx)).$$

Choose $\mathcal{V}$ such that

$$\mathcal{V}(p) = \{w : w \nleq v\} \text{ and } \mathcal{V}(q) = \{w : uSw\}.$$

Again the former set is upward closed, and the latter set is upward closed because of symmetry and upward closure of S. Now since

$$\forall w S^{-1} v(w \nleq v \text{ or } \neg \forall x(wSx \to uSx)),$$

if the former disjunct holds then $w \in \mathcal{V}(p)$. And if the latter disjunct holds, then $\exists x(wSx \text{ and } \neg uSx)$. So if $x \Vdash_{\mathcal{K}cc} q$, then uSx, a contradiction. Thus $x \nVdash_{\mathcal{K}cc} q$ and consequently, $w \Vdash_{\mathcal{K}cc} \sim q$. Thus $\forall w S^{-1} v(w \Vdash_{\mathcal{K}cc} p \text{ or } w \Vdash_{\mathcal{K}cc} \sim q)$. Also if $v \Vdash_{\mathcal{K}cc} p$, then $v \nleq v$, a contradiction. So $v \nVdash_{\mathcal{K}cc} p$; hence $u \Vdash_{\mathcal{K}cc} \sim(\sim(p \vee \sim q) \vee p)$. But if $u \Vdash_{\mathcal{K}cc} \sim q$, then $\exists x S^{-1} u(x \nVdash_{\mathcal{K}cc} q)$. So $\neg uSx$, a contradiction. Hence $u \nVdash_{\mathcal{K}cc} \sim q$. Thus

$$u \nVdash_{\mathcal{K}cc} \sim(\sim(p \vee \sim q) \vee p) \to \sim q.$$

Therefore $\nvDash_{\mathcal{K}cc} \sim(\sim(p \vee \sim q) \vee p) \to \sim q$.

To see (ii) implies (i), let $v \geq u$ for arbitrary and assume

$$v \Vdash_{\mathcal{K}cc} \sim(\sim(A \vee \sim B) \vee A).$$

We want to show $v \Vdash_{\mathcal{K}cc} \sim B$. By definition, $\exists w S^{-1} v(w \nVdash_{\mathcal{K}cc} \sim(A \vee \sim B) \vee A)$. So

$$\forall x S^{-1} w(x \Vdash_{\mathcal{K}cc} A \vee \sim B) \ (*)$$

and $w \nVdash_{\mathcal{K}cc} A$. By the frame condition, there is $xS^{-1}w$ such that $x \leq w$ and $\forall y(xSy \to vSy)$. From (*) we infer $x \Vdash_{\mathcal{K}cc} A$ or $x \Vdash_{\mathcal{K}cc} \sim B$. If the former, then $w \Vdash_{\mathcal{K}cc} A$, a contradiction. So $x \Vdash_{\mathcal{K}cc} \sim B$. But then for some $yS^{-1}x$, $y \nVdash_{\mathcal{K}cc} B$. Thus vSy by the frame condition. So $v \Vdash_{\mathcal{K}cc} \sim B$. Hence

$$u \Vdash_{\mathcal{K}cc} \sim(\sim(A \vee \sim B) \vee A) \to \sim B.$$

Since u is arbitrary, $\models_{\mathcal{K}cc} \sim(\sim(A \vee \sim B) \vee A) \to \sim B$. □

Note that contrary to the last case, in this proof we appealed to the symmetry of S in $\mathbf{CC}_\omega$.

5.5 Labelled sequent calculus

In this section, we define a labelled sequent calculus for some of the logics we have treated ($\mathbf{CC}_\omega, \mathbf{TCC}_\omega, \mathbf{daC}, \mathbf{IPC}^{\sim}$), with the aid of the insights obtained in the last section regarding their relationship. We shall show the admissibility of cut and the correspondence with the Hilbert-style system.

A *labelled formula* is an expression of the form $x : A$, where A is a formula and x is a *label*. We shall use $x, y, z \ldots$ for labels. We shall additionally consider *relational atoms*, which either have the form xSy, or $x \leq y$. An *item* is either a labelled formula or a relational atom. We denote items by $\alpha, \beta, \ldots$. A *sequent* has the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of items.

We shall consider the following calculus $\mathbf{G3cc}_\omega$.

Definition 5.5.1 ($\mathbf{G3cc}_\omega$)**.**

$$x \leq y, x : p, \Gamma \Rightarrow \Delta, y : p \ (\text{Ax1}) \qquad x \leq y, \Gamma \Rightarrow \Delta, x \leq y \ (\text{Ax2})$$

$$xSy, \Gamma \Rightarrow \Delta, xSy \ (\text{Ax3})$$

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta} (\text{L}\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B} (\text{R}\wedge)$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta} (\text{L}\vee) \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B} (\text{R}\vee)$$

$$\frac{x \leq y, x : A \to B, \Gamma \Rightarrow \Delta, y : A \qquad x \leq y, x : A \to B, y : B, \Gamma \Rightarrow \Delta}{x \leq y, x : A \to B, \Gamma \Rightarrow \Delta} (\text{L}\to)$$

$$\frac{x \leq y^*, y^* : A, \Gamma \Rightarrow \Delta, y^* : B}{\Gamma \Rightarrow \Delta, x : A \to B} (\text{R}\to)$$

$$\frac{xSy^*, \Gamma \Rightarrow \Delta, y^* : A}{x : {\sim}A, \Gamma \Rightarrow \Delta} (\text{L}{\sim}) \qquad \frac{xSy, y : A, \Gamma \Rightarrow \Delta, x : {\sim}A}{xSy, \Gamma \Rightarrow \Delta, x : {\sim}A} (\text{R}{\sim})$$

$$\frac{x \leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} (\text{Ref}) \qquad \frac{x \leq z, x \leq y, y \leq z, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta} (\text{Trans})$$

$$\frac{xSx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} (\text{Ref}_S) \qquad \frac{xSy, ySx, \Gamma \Rightarrow \Delta}{xSy, \Gamma \Rightarrow \Delta} (\text{Sym}_S)$$

$$\frac{x \leq y, xSz, ySz, \Gamma \Rightarrow \Delta}{x \leq y, xSz, \Gamma \Rightarrow \Delta} (\text{Up})$$

A proof (derivation/deduction) of a sequent $\Gamma \Rightarrow \Delta$ in $\mathbf{G3cc}_\omega$ (to be denoted $\vdash_{Gcc} \Gamma \Rightarrow \Delta$) is a tree whose root is the sequent, whose nodes are applications of rules, and whose leaves are axioms (0-premise rules).

In the rules, variables indicated by * are *eigenvariables*, meaning that they cannot occur in the conclusion of the rules. Γ, Δ are called *contexts*, and non-context items in the conclusion are called *principal*. The calculus is in a large part an amalgamation of the labelled calculus for modal logic [82] and intuitionistic logic [83]. It has the rules (Ref_S) and (Sym_S) corresponding to the reflexivity and symmetry in $\mathbf{CC}_\omega$. Additionally, it has the rule (Up) corresponding to the condition for upward closure in $\mathbf{CC}_\omega$.[1]

We shall also consider the following additional rules, corresponding to the additional frame conditions for **daC**, $\mathbf{TCC}_\omega$ and $\mathbf{IPC}^\sim$, to $\mathbf{G3cc}_\omega$.

Definition 5.5.2.

$$\frac{xSy, ySz^*, z^* \leq x, z^* \leq y, \Gamma \Rightarrow \Delta}{xSy, \Gamma \Rightarrow \Delta}\ (\mathrm{Pr}) \qquad \frac{xSz, xSy, ySz, \Gamma \Rightarrow \Delta}{xSy, ySz, \Gamma \Rightarrow \Delta}\ (\mathrm{Trans}_S)$$

$$\frac{xSy, xSz, t^* \leq y, t^* \leq z, xSt^*, \Gamma \Rightarrow \Delta}{xSy, xSz, \Gamma \Rightarrow \Delta}\ (\mathrm{De})$$

Where, as before, labels indicted with * are eigenvariables. The intention is that the addition of (Pr) should correspond to **daC**, (Trans_S) to $\mathbf{TCC}_\omega$, and (Trans_S) and (De) to $\mathbf{IPC}^\sim$. We shall denote the addition of some of these rules to $\mathbf{G3cc}_\omega$ by $\mathbf{G3cc}_\omega^+$, and the deducibility is denoted by $\vdash_{Gcc+}$.

We shall later observe how the sequent calculi correspond to the Hilbert-style systems. We now proceed with checking some standard properties of the calculi.

Proposition 5.5.1. $\vdash_{Gcc} x \leq y, x : A, \Gamma \Rightarrow \Delta, y : A$

Proof. By [84, Lemma 12.25], we only have to consider the case for $\sim$. When $A \equiv {\sim}B$,

$$\cfrac{\cfrac{\cfrac{\cfrac{z \leq z, z : B, xSz, x \leq y, ySz, \Gamma \Rightarrow \Delta, y : {\sim}B, z : B}{z : B, xSz, x \leq y, ySz, \Gamma \Rightarrow \Delta, y : {\sim}B, z : B}\ (\mathrm{Ref})}{xSz, x \leq y, ySz, \Gamma \Rightarrow \Delta, y : {\sim}B, z : B}\ (\mathrm{R}{\sim})}{xSz, x \leq y, \Gamma \Rightarrow \Delta, y : {\sim}B, z : B}\ (\mathrm{Up})}{x \leq y, x : {\sim}B, \Gamma \Rightarrow \Delta, y : {\sim}B}\ (\mathrm{L}{\sim})$$

where the first line is obtained from the inductive hypothesis. □

Definition 5.5.3 (substitution of labels)**.** We define the substitution of a label by another label $x[z/w]$, substitution for an item $\alpha[z/w]$ and for a multiset $\Gamma[z/w]$ by the following clauses. ($\circ \in \{\leq, S\}$)

$$\begin{aligned}
&x[z/w] \equiv w \text{ if } x \equiv z.\\
&x[z/w] \equiv x \text{ if } x \not\equiv z.\\
&\alpha[z/w] \equiv x[z/w] \circ y[z/w] \text{ if } \alpha \equiv x \circ y.\\
&\alpha[z/w] \equiv x[z/w] : A \text{ if } \alpha \equiv x : A.\\
&\Gamma[z/w] \equiv \{\alpha[z/w] : \alpha \in \Gamma\}
\end{aligned}$$

We shall denote instances of substitution by (Sub). In addition, we shall write $\mathbf{G3cc}_\omega^+ \vdash_n \Gamma \Rightarrow \Delta$ if the sequent has a derivation whose depth is less than n. We say a rule is *depth-preserving admissible* (*dp-admissible*) if and only if: if there are derivations of the premises of the rule each with the depth less than n, then there exists a derivation of the conclusion with the depth less than n. If the depth is not preserved, we just say the rule is *admissible*. We shall indicate an application of an admissible rule by a dashed line.

[1]Note that in the calculus for $\mathbf{IPC}^\sim$, we do not need a syntactic notion corresponding to the root world. This is because we are using here a larger (not necessarily rooted) class of Kripke frames than the class of rooted frames, for which $\mathbf{IPC}^\sim$ is still sound and complete.

Proposition 5.5.2 (dp-admissibility of substitution)**.**
The rule $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma[z/w] \Rightarrow \Delta[z/w]}$ (Sub) is dp-admissible in $\mathbf{G3cc}^{+}_{\omega}$.

Proof. We argue by induction on the depth of deduction. The intuitionistic rules are already treated in [84, Lemma 12.26]. For others, the case for (Ax2) is immediate, since the result of the substitution is also an instance of (Ax2). The case for (L$\sim$) and (R$\sim$) are similar to those of R$\Box$ and L$\Box$ in modal calculi, respectively; cf. [84, Lemma 11.4]. The other rules are instances of either *the scheme for mathematical rules* or *the geometric rule scheme* [84, pp.98, 134], so can be dealt with by the methodology of [84, Lemma 11.4]. □

We shall now move on to consider structural rules.

Definition 5.5.4 (structural rules)**.**

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\text{(LW)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\text{(RW)}$$

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\text{(LC)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha}\text{(RC)}$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \qquad \alpha, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\text{(Cut)}$$

Our goal is to prove that (Cut) is admissible. For this purpose we check that the rules of Weakening (LW,RW) and Contraction (LC,RC) are dp-admissible. We start with Weakening.

Proposition 5.5.3 (dp-admissibility of Weakening)**.**
(LW) and (RW) are dp-admissible in $\mathbf{G3cc}^{+}_{\omega}$.

Proof. The proof is by induction on the depth of deduction. In cases of applications of (Ax1)-(Ax3), the result of Weakening is again an instance of the axiom. For other rules, we apply the inductive hypothesis to the premises of the rule, and thereafter apply the rule to obtain the desired sequent; however for rules involving eigenvariables, we first need to apply dp-admissible substitution (Proposition 5.5.2) to substitute the eigenvariable with a fresh variable, so as to avoid the clash of variables. Then we apply the above procedure. □

For Contraction, we first need to demonstrate that the rules of $\mathbf{G3cc}^{+}_{\omega}$ are *dp-invertible*: that is, given a derivation of the conclusion of a rule, we can find a depth-preserving derivation of the premises.

Lemma 5.5.1. The rules of $\mathbf{G3cc}^{+}_{\omega}$ are dp-invertible.

Proof. We argue by induction on the depth of deduction. For the intuitionistic rules, we refer to [84, Theorem 12.28]. For the rules (R$\sim$), (Up), (Ref$_S$), (Sym$_S$), (Trans$_S$), (Pr) and (De), we can invert the sequent by dp-admissible weakening.

The case for (L$\sim$) is quite similar to that of R$\Box$ for modal logic [84, Lemma 11.7]. If $\vdash_0 x : \sim A, \Gamma \Rightarrow \Delta$, then the derivation is an instance of (Ax1), (Ax2) or (Ax3). In each case, $xSy, \Gamma \Rightarrow \Delta, y : A$ is also an instance of the same axiom. If $\vdash_{n+1} x : \sim A, \Gamma \Rightarrow \Delta$, then if it is obtained by (L$\sim$) with $x : \sim A$ principal, i.e. it is of the form

$$\frac{\vdash_n xSz, \Gamma \Rightarrow \Delta, z : A}{\vdash_{n+1} x : \sim A, \Gamma \Rightarrow \Delta} \text{(L}\sim\text{)}$$

where z does not occur in the conclusion; then by dp-admissible substitution, $\vdash_n xSy, \Gamma \Rightarrow \Delta, y : A$ (where y is a fresh variable).

We exemplify with (R$\to$) the case where $x : \sim A$ is obtained by a rule with eigenvalue condition in which it is not principal

$$\frac{\vdash_n z \leq t, t : C, x : \sim A, \Gamma' \Rightarrow \Delta', t : D}{\vdash_{n+1} x : \sim A, \Gamma' \Rightarrow \Delta', z : C \to D} \text{(R}\to\text{)}$$

then by dp-substitution, $\vdash_n z \leq t', t' : C, x : \sim A, \Gamma' \Rightarrow \Delta', t' : D$, where $t' \not\equiv y$. (Note that y is fixed beforehand.) By I.H., $\vdash_n z \leq t', t' : C, xSy, \Gamma' \Rightarrow \Delta', y : A, t' : D$. So by (R$\to$), $\vdash_{n+1} xSy, \Gamma' \Rightarrow \Delta', y : A, z : C \to D$.

If it is obtained by a rule without eigenvariable condition, then apply I.H. to the premise and apply the same rule. □

Proposition 5.5.4 (dp-admissibility of Contraction)**.**
(LC) and (RC) are dp-admissible in $\mathbf{G3cc}^+_\omega$.

Proof. We argue by simultaneous induction ((LC),(RC)) on the depth of the deduction. General outline is as in [84, Theorem 12.28]. As an example, suppose $\vdash_{n+1} x : \sim A, x : \sim A, \Gamma \Rightarrow \Delta$ and the last step is an instance of (L$\sim$) with $x : \sim A$ principal.

$$\frac{\vdash_n xSy, x : \sim A, \Gamma \Rightarrow \Delta, y : A}{\vdash_{n+1} x : \sim A, x : \sim A, \Gamma \Rightarrow \Delta} \text{(L}\sim\text{)}$$

Then by dp-admissible invertibility of (L$\sim$),

$$\vdash_n xSy, xSy, \Gamma \Rightarrow \Delta, y : A, y : A$$

So by I.H.

$$\vdash_n xSy, \Gamma \Rightarrow \Delta, y : A$$

Thus by (L$\sim$)

$$\vdash_{n+1} x : \sim A, \Gamma \Rightarrow \Delta$$

□

We are now ready to prove the admissibility of (Cut). We shall call the item to be eliminated in (Cut) the *cut-item*.

Theorem 5.5.1 (admissibility of Cut)**.** (Cut) is admissible in $\mathbf{G3cc}^+_\omega$.

Proof. We argue by induction on the complexity of cut-items, with a subinduction on the *level* (the sum of the depths of the deductions of the premises) of (Cut). Again the outline is the same as that of the intuitionistic case [84, Theorem 12.30]. In particular, rules that are mathematical or geometric are treated similarly to those of intermediate axioms.

Here we shall consider the case where the cut-item is principal in both of the premises, and has the form $x : \sim A$. We have

$$\frac{\dfrac{\vdash_{m-1} xSy, y : A, \Gamma \Rightarrow \Delta, x : \sim A}{\vdash_m xSy, \Gamma \Rightarrow \Delta, x : \sim A} \text{(R}\sim\text{)} \quad \dfrac{\vdash_{n-1} xSz, \Gamma' \Rightarrow \Delta', z : A}{\vdash_n x : \sim A, \Gamma' \Rightarrow \Delta'} \text{(L}\sim\text{)}}{\vdash xSy, \Gamma\Gamma' \Rightarrow \Delta\Delta'} \text{(Cut)}$$

Then

$$\dfrac{\vdash_{n-1} xSz, \Gamma' \Rightarrow \Delta', z : A}{\vdash_{n-1} xSy, \Gamma' \Rightarrow \Delta', y : A}\ \text{(Sub)}$$

(note that z is an eigenvariable, so cannot occur in Γ', Δ'). Moreover, by I.H. the following cut of a lower level ($m + n - 1 < m + n$) is admissible:

$$\dfrac{\vdash_{m-1} xSy, y : A, \Gamma \Rightarrow \Delta, x : {\sim}A \qquad \vdash_{n} x : {\sim}A, \Gamma' \Rightarrow \Delta'}{\vdash xSy, y : A, \Gamma\Gamma' \Rightarrow \Delta\Delta'}\ \text{(I.H.)}$$

From these, and a cut of lower complexity (admissible by I.H.), we obtain

$$\dfrac{\dfrac{\vdash_{n-1} xSy, \Gamma' \Rightarrow \Delta', y : A \qquad \vdash xSy, y : A, \Gamma\Gamma' \Rightarrow \Delta\Delta'}{\vdash xSy, xSy, \Gamma\Gamma'\Gamma' \Rightarrow \Delta\Delta'\Delta'}\ \text{(I.H.)}}{\vdash xSy, \Gamma\Gamma' \Rightarrow \Delta\Delta'}\ \text{(LC,RC)}$$

□

Next we observe that $\mathbf{G3cc}_\omega$ and calculi in $\mathbf{G3cc}^+_\omega$ indeed correspond to $\mathbf{CC}_\omega$, **daC**, $\mathbf{TCC}_\omega$ and $\mathbf{IPC}^{\sim}$.

Proposition 5.5.5.

(i) $\vdash_{cc} A$ implies $\vdash_{Gcc} \Rightarrow x : A$.

(ii) If we add (Pr)/(trans$_S$) to the calculus, then the axioms of $\mathbf{daC}/\mathbf{TCC}_\omega$ become derivable. If we add (trans$_S$) and (De), the axioms of $\mathbf{IPC}^{\sim}$ become derivable.

Proof.
(i) For $\mathbf{CC}_\omega$, the positive axioms can be shown to be derivable as in the intuitionistic case. We need to check $A \vee {\sim}A$, ${\sim}{\sim}A \to A$ and (RC).

<u>$A \vee {\sim}A$</u>

$$\dfrac{\dfrac{\dfrac{\dfrac{x \leq x, xSx, x : A \Rightarrow x : A, x : {\sim}A}{xSx, x : A \Rightarrow x : A, x : {\sim}A}\ \text{(Ref)}}{xSx \Rightarrow x : A, x : {\sim}A}\ \text{(R}{\sim}\text{)}}{\Rightarrow x : A, x : {\sim}A}\ \text{(Ref}_S\text{)}}{\Rightarrow x : A \vee {\sim}A}\ \text{(R}\vee\text{)}$$

<u>${\sim}{\sim}A \to A$</u>

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y \leq y, x \leq y, ySz, zSy, y : A \Rightarrow z : {\sim}A, y : A}{x \leq y, ySz, zSy, y : A \Rightarrow z : {\sim}A, y : A}\ \text{(Ref)}}{x \leq y, ySz, zSy \Rightarrow z : {\sim}A, y : A}\ \text{(R}{\sim}\text{)}}{x \leq y, ySz \Rightarrow z : {\sim}A, y : A}\ \text{(Sym}_S\text{)}}{x \leq y, y : {\sim}{\sim}A \Rightarrow y : A}\ \text{(L}{\sim}\text{)}}{\Rightarrow x : {\sim}{\sim}A \to A}\ \text{(R}{\to}\text{)}$$

<u>(RC)</u>
We first observe that $x : A \to B, x : A \Rightarrow x : B$ is derivable:

$$\dfrac{\dfrac{x \leq x, x : A \to B, x : A \Rightarrow x : A \qquad x \leq x, x : A \to B, x : B \Rightarrow x : B}{x \leq x, x : A \to B, x : A \Rightarrow x : B}\ \text{(L}{\to}\text{)}}{x : A \to B, x : A \Rightarrow x : B}\ \text{(Ref)}$$

Then

$$
\begin{array}{cl}
\Rightarrow x : A \to B \qquad x : A \to B, x : A \Rightarrow x : B & \\
\hdashline
x : A \Rightarrow x : B & (\text{Cut}) \\
\hdashline
y \leq z, zSx, x : A \Rightarrow x : B, z : {\sim}A & (\text{LW,RW}) \\
\hline
y \leq z, zSx \Rightarrow x : B, z : {\sim}A & (\text{R}{\sim}) \\
\hline
y \leq z, z : {\sim}B \Rightarrow z : {\sim}A & (\text{L}{\sim}) \\
\hline
\Rightarrow y : {\sim}B \to {\sim}A & (\text{R}{\to}) \\
\hdashline
\Rightarrow x : {\sim}B \to {\sim}A & (\text{Sub})
\end{array}
$$

(ii) We need to check each of the additional axioms are derivable in the corresponding calculi.

daC

First we apply (L$\vee$) to

$$x \leq y, ySz, zSt, t \leq y, t \leq z, t : A \Rightarrow z : {\sim}(A \vee B), z : A, y : B;$$
$$x \leq y, ySz, zSt, t \leq y, t \leq z, t : B \Rightarrow z : {\sim}(A \vee B), z : A, y : B$$

to obtain

$$x \leq y, ySz, zSt, t \leq y, t \leq z, t : A \vee B \Rightarrow z : {\sim}(A \vee B), z : A, y : B$$

Then,

$$
\begin{array}{cl}
x \leq y, ySz, zSt, t \leq y, t \leq z, t : A \vee B \Rightarrow z : {\sim}(A \vee B), z : A, y : B & \\
\hline
x \leq y, ySz, zSt, t \leq y, t \leq z \Rightarrow z : {\sim}(A \vee B), z : A, y : B & (\text{R}{\sim}) \\
\hline
x \leq y, ySz \Rightarrow z : {\sim}(A \vee B), z : A, y : B & (\text{Pr}) \\
\hline
x \leq y, ySz \Rightarrow z : {\sim}(A \vee B) \vee A, y : B & (\text{R}\vee) \\
\hline
x \leq y, y : {\sim}({\sim}(A \vee B) \vee A) \Rightarrow y : B & (\text{L}{\sim}) \\
\hline
\Rightarrow x : {\sim}({\sim}(A \vee B) \vee {\sim}A) \to B & (\text{R}{\to})
\end{array}
$$

TCC$_\omega$

$$
\begin{array}{cl}
u \leq u, y \leq z, ySu, zSv, zSu, vSz, vSu, u : A \Rightarrow u : A, v : {\sim}A & \\
\hline
y \leq z, ySu, zSv, zSu, vSz, vSu, u : A \Rightarrow u : A, v : {\sim}A & (\text{Ref}) \\
\hline
y \leq z, ySu, zSv, zSu, vSz, vSu \Rightarrow u : A, v : {\sim}A & (\text{R}{\sim}) \\
\hline
y \leq z, ySu, zSv, zSu, vSz \Rightarrow u : A, v : {\sim}A & (\text{Trans}_S) \\
\hline
y \leq z, ySu, zSv, zSu \Rightarrow u : A, v : {\sim}A & (\text{Sym}_S) \\
\hline
y \leq z, ySu, zSv \Rightarrow u : A, v : {\sim}A & (\text{Up}) \\
\hline
y \leq z, ySu, z : {\sim}{\sim}A \Rightarrow u : A & (\text{L}{\sim}) \\
\hline
y \leq z, y : {\sim}A, z : {\sim}{\sim}A \Rightarrow & (\text{L}{\sim}) \\
\hdashline
x \leq y, y \leq z, y : {\sim}A, z : {\sim}{\sim}A \Rightarrow z : B & (\text{LW,RW}) \\
\hline
x \leq y, y : {\sim}A \Rightarrow y : {\sim}{\sim}A \to B & (\text{R}{\to}) \\
\hline
x : {\sim}A \to ({\sim}{\sim}A \to B) & (\text{R}{\to})
\end{array}
$$

IPC$^{\sim}$

First we apply (L$\vee$) to

$$ySz, ySt, w : A, w \leq z, w \leq t, ySw \Rightarrow z : A, t : B, y : {\sim}(A \vee B);$$
$$ySz, ySt, w : B, w \leq z, w \leq t, ySw \Rightarrow z : A, t : B, y : {\sim}(A \vee B)$$

to obtain

$$ySz, ySt, w : A \vee B, w \leq z, w \leq t, ySw \Rightarrow z : A, t : B, y : {\sim}(A \vee B)$$

Then

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{ySz, ySt, w : A \vee B, w \leq z, w \leq t, uSw \Rightarrow z : A, t : B, y : {\sim}(A \vee B)}{ySz, ySt, w \leq z, w \leq t, ySw \Rightarrow z : A, t : B, y : {\sim}(A \vee B)}\ (\mathrm{R}{\sim})}{ySz, ySt \Rightarrow z : A, t : B, y : {\sim}(A \vee B)}\ (\mathrm{De})}{ySz, y : {\sim}B \Rightarrow z : A, y : {\sim}(A \vee B)}\ (\mathrm{L}{\sim})}{y : {\sim}A, y : {\sim}B \Rightarrow y : {\sim}(A \vee B)}\ (\mathrm{L}{\sim})}{y : {\sim}A \wedge {\sim}B \Rightarrow y : {\sim}(A \vee B)}\ (\mathrm{L}\wedge)}{x \leq y, y : {\sim}A \wedge {\sim}B \Rightarrow y : {\sim}(A \vee B)}\ (\mathrm{LW})}{\Rightarrow x : ({\sim}A \wedge {\sim}B) \to {\sim}(A \vee B)}\ (\mathrm{R}{\to})$$

□

For the converse direction, we argue via completeness of the Hilbert-style systems with respect to Kripke semantics. For this purpose the notion of valuation has to be modified in the style of [84, Definition 11.25-26], to accommodate labelled formulae and relational atoms.

Definition 5.5.5 (modified valuation for $\mathbf{CC}_\omega$-model)**.** Let $\mathcal{F}^c_\mathcal{K}$ be a $\mathbf{CC}_\omega$-frame. A *modified valuation* $\mathcal{V}_m$ is a pair $(\mathcal{V}, l)$, where $\mathcal{V}$ is the ordinary valuation for $\mathcal{F}^c_\mathcal{K}$ introduced before, and l maps each label x into to a world $l(x)$of $\mathcal{F}^c_\mathcal{K}$.

We say an item α is valid in a *modified model* $\mathcal{M}^{mc}_\mathcal{K} = (\mathcal{F}^c_\mathcal{K}, \mathcal{V}_m)$ (denoted $\mathcal{M}^{mc}_\mathcal{K} \Vdash^m_{\mathcal{K}cc} \alpha$), when

- $l(x) \leq l(y)$ (or $l(x)Sl(y)$) in $\mathcal{F}^c_\mathcal{K}$, if $\alpha \equiv x \leq y$ (or xSy).
- $(\mathcal{F}^c_\mathcal{K}, \mathcal{V}), l(x) \Vdash_{\mathcal{K}cc} A$, if $\alpha \equiv x : A$.

We say a sequent $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}^{mc}_\mathcal{K}$ (denoted $\mathcal{M}^{mc}_\mathcal{K} \vDash^m_{\mathcal{K}cc} \Gamma \Rightarrow \Delta$), if $\mathcal{M}^{mc}_\mathcal{K} \Vdash^m_{\mathcal{K}cc} \alpha$ for all $\alpha \in \Gamma$ implies $\mathcal{M}^{mc}_\mathcal{K} \Vdash^m_{\mathcal{K}cc} \beta$ for some $\beta \in \Delta$. If $\mathcal{M}^{mc}_\mathcal{K}$ is arbitrary, we say $\Gamma \Rightarrow \Delta$ is valid and write $\vDash^m_{\mathcal{K}cc} \Gamma \Rightarrow \Delta$.

Note that $\vDash^m_{\mathcal{K}cc} \Rightarrow x : A$ is valid if and only if $\vDash_{\mathcal{K}cc} A$. Similar statements hold when we restrict the class of frames. We now wish to demonstrate the following.

Proposition 5.5.6.

(i) $\vdash_{Gcc} \Gamma \Rightarrow \Delta$ implies $\vDash^m_{\mathcal{K}cc} \Gamma \Rightarrow \Delta$.

(ii) (Pr)/(Trans$_S$)/(De) become sound if we restrict consideration to the corresponding classes of frames .

Proof.
(i) We argue by induction on the depth of deductions. The cases for the intuitionistic rules follow straightforwardly from the definition of intuitionistic Kripke models. We shall look at the cases for (L∼), (R∼). The other cases are straightforward. In each case, we consider an arbitrary modified model $\mathcal{M}^{mc}_\mathcal{K} = (\mathcal{F}^c_\mathcal{K}, \mathcal{V}_m)$ with $\mathcal{V}_m = (\mathcal{V}, l)$.

(L$\sim$) Suppose $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^m_{\mathcal{K}cc} \alpha$ for all $\alpha \in \{x : \sim A\} \cup \Gamma$. Then $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), l(x) \Vdash_{\mathcal{K}cc} \sim A$. So there is w such that $l(x)Sw$ and $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), w \nVdash_{\mathcal{K}cc} A$. Take $\mathcal{V}'_m = (\mathcal{V}, l')$ where $l' = l$ except $l'(y) = w$. Note, since y does not occur in Γ and Δ, l and l' evaluate them in the same way. Thus $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}'_m) \Vdash^m_{\mathcal{K}cc} \alpha$ for all $\alpha \in \{xSy\} \cup \Gamma$. So by I.H., $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}'_m) \Vdash^m_{\mathcal{K}cc} \beta$ for some $\beta \in \{y : A\} \cup \Delta$. If it validates $y : A$, however, then $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}), l'(y) \Vdash_{\mathcal{K}cc} A$, a contradiction. Therefore $(\mathcal{F}^c_{\mathcal{K}}, \mathcal{V}_m) \Vdash^m_{\mathcal{K}cc} \beta$ for some $\beta \in \Delta$. Since $\mathcal{M}^{mc}_{\mathcal{K}}$ is arbitrary, $\vDash^m_{\mathcal{K}cc} x : \sim A, \Gamma \Rightarrow \Delta$.

(R$\sim$) Suppose $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^m_{\mathcal{K}cc} \alpha$ for all $\alpha \in \{xSy\} \cup \Gamma$. If $l(y) \nVdash_{\mathcal{K}cc} A$, then $l(x) \Vdash_{\mathcal{K}cc} \sim A$. Otherwise, $l(y) \Vdash_{\mathcal{K}cc} A$, so by I.H., $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^m_{\mathcal{K}cc} \alpha$ for all $\alpha \in \{xSy, y : A\} \cup \Gamma$. So in either case (the latter with I.H.), $\mathcal{M}^{mc}_{\mathcal{K}} \Vdash^m_{\mathcal{K}cc} \beta$ for some $\beta \in \Delta \cup \{x : \sim A\}$.

(ii) The case for (Trans$_S$) is straightforward and (Pr), (De) are similar to the case for (L$\sim$); one needs to appeal to the frame condition to pick out a world satisfying the desired order relation; then define a new modified valuation which is identical to the original except it assigns the world to the eigenvariable; then the rest follows as in the case for (L$\sim$). □

This allows us to conclude the other direction.

Corollary 5.5.1.

(i) $\vdash_{Gcc} \Rightarrow x : A$ implies $\vdash_{cc} A$.

(ii) $\vdash_{Gcc+}$ is sound with respect to the corresponding logics (**daC**,$\mathbf{TCC}_\omega$,$\mathbf{IPC}^\sim$).

Proof.
(i) If $\vdash_{Gcc} \Rightarrow x : A$, then by the previous proposition, $\vDash^m_{\mathcal{K}cc} \Rightarrow x : A$. Then as we remarked before, $\vDash_{\mathcal{K}cc} A$. Thus by the completeness of $\mathbf{CC}_\omega$, $\vdash_{cc} A$.
(ii) Similar. □

5.6 Discussion

We have seen that $\mathbf{TCC}_\omega$ can be regarded as the logic of empirical and co-negation for Beth semantics, which differs from $\mathbf{IPC}^\sim$ and **daC** for Kripke semantics. According to the interpretation in [122, p.679], the difference between Kripke and Beth semantics is the treatment of time. A node in Kripke models signifies a state of information, whereas in Beth models it signifies a moment in time. So for instance, to decide the forcing of a disjunction in a Kripke model, one can stay in a world as much as one likes, until one learns which of the disjuncts is true. In comparison, in Beth models this waiting time is expressed by posterior nodes, so we need to refer to those other worlds to decide the forcing of the disjunction in the original world. The two kinds of empirical and co-negation can be interpreted similarly.

The question remains, however, which empirical (or co-) negation one actually means in an assertion of negation. For example, if one says "There is no proof of P=NP", does it mean there is no proof at the present state of information, or there is no proof at the present moment?

Changing perspective, from an intuitionistic viewpoint there is a certain advantage in considering Beth semantics. There is a relatively simple proof of intuitionistic completeness (proving completeness with only intuitionistically accepted principles) for intuitionistic logic [43, 122]. The intuitionistic completeness proof for Kripke semantics [135] gives a more refined result, but is comparatively more involved. A possible future direction is to show the intuitionistic completeness for $\mathbf{TCC}_\omega$. An obstacle would be the treatment of excluded middle, but classical logic also has an intuitionistic completeness proof [75], so possibly this may be overcome. An intuitionistic completeness would be desirable if one is a full-fledged intuitionist, especially when the logic is motivated from the semantics, rather than from the syntax.

Chapter 6

Actuality in intuitionistic logic

6.1 Introduction

In the last chapter, we looked at a heterodox negation named empirical negation, obtained through rooted Kripke models. Now, a simple calculation therein reveals that double empirical negation of A is forced at a point iff A is forced at the base point. In other words, double empirical negation can be seen as an actuality operator explored by John N. Crossley, Lloyd Humberstone, Martin Davies and more. This then gives rise to a natural question of exploring an expansion of intuitionistic logic enriched by actuality operator. The aim of this chapter is twofold, and the first aim is to address this question. Although the notion of actuality has been discussed in classical settings (see our brief overview below), few attempts are known to discuss the notion of actuality based on intuitionistic logic.[1] The only exception known to us is the system suggested by Humberstone in [66, pp.75–76], whose semantics is based on the same idea as above. The proof theory for the system was not spelled out in [66], and our enquiry will serve to fill the gap. It is also of significant interest how we can incorporate the notion along the philosophical foundation of Dummett-Tennant-De.

The second aim is to draw some connections to closely related systems. This enables us to uncover links with other logical concepts, such as empirical negation and globality. For this purpose, we shall adopt a language that includes absurdity and therefore negation. Nonetheless we shall also observe how the notion of actuality is independent of that of negation, which is an advantage over an approach that defines actuality in terms of empirical negation. Before moving further, let us briefly review some of the developments in the literature related to our aim.

Actuality The notion of actuality has been studied in modal logic for a long time, and various conceptualisations have been introduced. Even at an early period, Crossley, Humberstone and Davies [26, 28] already introduced two different actuality operators, A and $\mathscr{F}$ (read *fixedly*). Each model $\mathcal{M}$ has a distinguished world w^*, and $\mathsf{A}\varphi$ is true at w iff φ is true at w^*. On the other hand, $\mathscr{F}\varphi$ is true at w iff for every model $\mathcal{M}'$, φ is true at $\mathcal{M}'$'s distinguished world w'. These two operators represent different intuitions about whether 'the actual world' is necessarily so or not.

[1] Note that there is a recent work on the notion of actuality based on relevant logics by Shawn Standefer in [114].

Another example for flexible actuality is that of Dominic Gregory [55], whose semantics includes a mapping @, which maps a world w to *its* actual world $@(w)$ in the same model, with a couple of conditions on @. This in particular allows there being more than one actual worlds in a model.[2]

Baaz' LGP and Titatni's GI Recall that Gödel-Dummett logic, introduced in [38] by Dummett, is an extension of intuitionistic logic with the linearity axiom:

$$(A \to B) \vee (B \to A). \tag{Lin}$$

Semantically, this logic is characterised by linear Kripke frames, which enables us to see it as a fuzzy logic in intuitionistic setting.

Then, in [2], Matthias Baaz expanded Gödel-Dummett logic by an additional operator, $\triangle$, which he called a *projection* modality, also later known as Baaz' Delta. The resulting logic is named **LGP**. Semantically, a formula of the form $\triangle A$ attains either the value 1 or 0, and it attains the value 1 iff A has the value 1.[3] In other words, $\triangle A$ is true iff A is valid in the model. Baaz in the same paper also mentions an operator equivalent to empirical negation in the setting of Gödel-Dummett logic (cf. [2, p.33]).

A logic closely related to **LGP** of Baaz is Satoko Titani's *global intuitionistic logic* **GI**, introduced in [118]. This logic, formulated as a sequent calculus, is defined by adding to intuitionistic logic an operator $\square$ of *globalization*. From a semantic perspective, in terms of algebraic semantics, $\square$ has the same interpretation as $\triangle$. There is also a fuzzy extension of **GI** called *fuzzy intuitionistic logic with globalization* **GIF** proposed by Gaisi Takeuti and Satoko Titani in [117], whose propositional fragment is equivalent to **LGP** (cf. [23, Remark 3]).

Note here that global intuitionistic logic can be regarded as an instance of intuitionistic modal logics which are equipped with at least two accessibility relations, intuitionistic $\leq$ and modal R. This is studied since 1948 by Frederic B. Fitch in [41], followed by Arthur N. Prior's [101] and R. A. Bull's papers [19, 20], and later major developments include [9, 11, 36, 95, 110, 111, 113, 125]. Some close connections of global intuitionistic logic to intuitionistic modal logics are studied by Hiroshi Aoyama in [1].

Based on these, this paper is structured as follows. We first introduce intuitionistic logic with actuality operator, called $\mathbf{IPC}^{@}$, both in terms of semantics and proof system, in §6.2. Then, in §6.3, we establish the soundness and strong completeness of $\mathbf{IPC}^{@}$. This is followed by a comparison of $\mathbf{IPC}^{@}$ with related systems in §6.4 and §6.5. More specifically, $\mathbf{IPC}^{@}$ is compared with intuitionistic logic with empirical negation as well as logic of actuality of Crossley and Humberstone in §6.4. We then turn to compare $\mathbf{IPC}^{@}$ with **LGP** of Baaz and **GI** of Titani in §6.5. The paper concludes with a brief summary of our main results and some directions for future research in §6.6.

[2]For more discussions on actuality, see, for instance, [44, 65, 115].

[3]This condition is closely related to the framework of *simple monadic Heyting algebra* which is explored in detail in [7] by Guram Bezhanishvili.

6.2 Semantics and Proof system

In this section, we first introduce the semantics for intuitionistic actuality using a rooted Kripke frame. This is then followed by an axiomatic system, whose completeness with respect to the semantics will be shown in the next section.

We shall use the following language $\mathcal{L}_\bot^@$ in describing our logic:

$$A ::= p \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid \bot \mid @A$$

Then we consider the following Kripke semantics.

Definition 6.2.1. A model $\mathcal{M}^@$ for the language $\mathcal{L}_\bot^@$ is defined as for a model of $\mathbf{IPC}^{\sim}$; so we shall consider frames with the root r. The forcing ($\Vdash_a A$) of formulae of the form $@A$ is given by the next clause.

$$\mathcal{M}^@, w \Vdash_a @A \iff \mathcal{M}^@, r \Vdash_a A.$$

Semantic consequence is now defined in terms of truth preservation at r: $\Gamma \vDash_a A$ iff for all models $\mathcal{M}^@$, $\mathcal{M}^@, r \Vdash_a A$ if $\mathcal{M}^@, r \Vdash_a B$ for all $B \in \Gamma$.

We claim that the next Hilbert system corresponds to the above semantics.

Definition 6.2.2. The system $\mathbf{IPC}^@$ is defined with the axiom schemata of **IPC** and the following axiom schemata and a rule of inference.

$$@(A \rightarrow B) \rightarrow (@A \rightarrow @B) \tag{K}$$

$$@A \rightarrow A \tag{T}$$

$$@A \rightarrow @@A \tag{4}$$

$$@A \vee (@A \rightarrow B) \tag{aLEM}$$

$$@(A \vee B) \rightarrow (@A \vee @B) \tag{aDIS}$$

$$\frac{A}{@A} \tag{RN}$$

We shall use $\vdash_a$ to denote the derivability in $\mathbf{IPC}^@$.

Because of the rule (RN), $\mathbf{IPC}^@$ does not enjoy the deduction theorem in the usual form. Yet the theorem turns out to be available in a different form, similarly to some modal logics.

Proposition 6.2.1. If $\Gamma, A \vdash_a B$ then $\Gamma \vdash_a @A \rightarrow B$.

Proof. We argue by the induction on the depth n of deduction.

If $n = 0$, then there are a few possibilities.

- If B an axioms or $B \in \Gamma$, then we have $\Gamma \vdash_a B$ and so $\Gamma \vdash_a @A \rightarrow B$.
- If $B = A$, then $@A \rightarrow A$ is an instance of the axiom (T); hence $\Gamma \vdash_a @A \rightarrow B$.

If $n = k + 1$, we have to consider deductions ending with one of the rules.

- For (MP), the premises have the form $\Gamma, A \vdash_a C$ and $\Gamma, A \vdash_a C \rightarrow B$. Thus, by I.H. we have $\Gamma \vdash_a @A \rightarrow C$ and $\Gamma \vdash_a @A \rightarrow (C \rightarrow B)$. Therefore it follows that $\Gamma \vdash_a @A \rightarrow B$.

- For (RN), the premise has the form $\Gamma, A \vdash_a C$ and $B \equiv @C$. Then by I.H. we have $\Gamma \vdash_a @A \rightarrow C$. By (K) and (RN), we have $\Gamma \vdash_a @@A \rightarrow @C$. Then an application of (4) gives us $\Gamma \vdash_a @A \rightarrow @C$, i.e. $\Gamma \vdash_a @A \rightarrow B$.

□

Next we look at the opposite direction.

Proposition 6.2.2. If $\Gamma \vdash_a @A \rightarrow B$ then $\Gamma, A \vdash_a B$.

Proof. If $\Gamma \vdash_a @A \rightarrow B$, then we use the fact that $\Gamma, A \vdash_a @A$ by (RN). Then $\Gamma, A \vdash_a B$ by (MP). □

Therefore we conclude:

Theorem 6.2.1. $\Gamma, A \vdash_a B$ iff $\Gamma \vdash_a @A \rightarrow B$.

Let us mention a corollary of the deduction theorem which shall prove vital for the completeness theorem.

Corollary 6.2.1. If $A \vdash_a C$ and $B \vdash_a C$, then $A \vee B \vdash_a C$.

Proof. If $A \vdash_a C$ and $B \vdash_a C$, then by deduction theorem $\vdash_a @A \rightarrow C$ and $\vdash_a @B \rightarrow C$. Thus $\vdash_a (@A \vee @B) \rightarrow C$; now use (aDIS) to deduce $\vdash_a @(A \vee B) \rightarrow C$. By deduction theorem again, we conclude $A \vee B \vdash_a C$. □

6.3 Soundness and completeness

We now turn to prove the soundness and the strong completeness. The proofs are in large part analogous to those of [30, 31] which build on [105]. First we look at the soundness.

Theorem 6.3.1. If $\Gamma \vdash_a A$ then $\Gamma \vDash_a A$.

Proof. By induction on the depth of the deduction. □

Next, we shall show the completeness. In below we introduce some concepts used in the argument for completeness.

(i) $\Sigma \vdash_\Pi A$ iff $\Sigma \cup \Pi \vdash_a A$.
(ii) Σ is a Π-*theory* iff:

(a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$.

(b) if $\vdash_\Pi A \rightarrow B$ then (if $A \in \Sigma$ then $B \in \Sigma$).

(iii) Σ is *prime* iff (if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$).
(iv) $\Sigma \vdash_\Pi \Delta$ iff for some $D_1, \ldots, D_n \in \Delta$, $\Sigma \vdash_\Pi D_1 \vee \ldots \vee D_n$.
(v) $\vdash_\Pi \Sigma \rightarrow \Delta$ iff for some $C_1, \ldots, C_n \in \Sigma$ and $D_1, \ldots, D_m \in \Delta$:

$$\vdash_\Pi C_1 \wedge \cdots \wedge C_n \rightarrow D_1 \vee \cdots \vee D_m.$$

(vi) Σ is Π-*deductively closed* iff (if $\Sigma \vdash_\Pi A$ then $A \in \Sigma$).
(vii) $\langle \Sigma, \Delta \rangle$ is a Π-*partition* iff:

(a) $\Sigma \cup \Delta = \mathsf{Form}$

(b) $\nvdash_\Pi \Sigma \to \Delta$

(viii) Σ is *non-trivial* iff $A \notin \Sigma$ for some formula A.

Lemma 6.3.1. If Γ is a non-empty Π-theory, then $\Pi \subseteq \Gamma$.

Proof. Take $A \in \Pi$. Then, we have $\Pi \vdash_a A$. Now since Γ is non-empty, we may pick a formula $C \in \Gamma$. With respect to this C, it holds that $\Pi \vdash_a C \to A$, in other words $\vdash_\Pi C \to A$. Now using the assumptions $C \in \Gamma$ and that Γ is Π-theory, we conclude that $A \in \Gamma$. □

We now introduce a number of lemmas concerning extensions of sets with various properties. For the proofs, cf. [30, §2] which are based on [105].

Lemma 6.3.2. If $\langle \Sigma, \Delta \rangle$ is a Π-partition then Σ is a prime Π-theory.

Lemma 6.3.3. If $\nvdash_\Pi \Sigma \to \Delta$ then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a Π-partition.

Corollary 6.3.1. Let Σ be a non-empty Π-theory, Δ be closed under disjunction, and $\Sigma \cap \Delta = \emptyset$. Then there is $\Sigma' \supseteq \Sigma$ such that $\Sigma' \cap \Delta = \emptyset$ and Σ' is a prime Π-theory.

Lemma 6.3.4. If $\Sigma \nvdash_a \Delta$ then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a partition, and Σ' is deductively closed.

We shall mention that the proof of this lemma relies on Corollary 6.2.1, and consequently on (aDIS). Hence the same argument cannot be directly imitated by a logic lacking this axiom, such as **GIPC** in §6.5.

Corollary 6.3.2. If $\Sigma \nvdash_a A$ then there are $\Pi \supseteq \Sigma$ such that $A \notin \Pi$, Π is a prime Π-theory and is Π-deductively closed.

Lemma 6.3.5. If Δ is a Π-theory and $A \to B \notin \Delta$, then there is a prime Π-theory $\Gamma \supseteq \Delta$, such that $A \in \Gamma$ and $B \notin \Gamma$.

Proof. Let $\Sigma = \{C : A \to C \in \Delta\}$. We check that Σ is a Π-theory. First, if $C_1, C_2 \in \Sigma$ then $A \to C_1, A \to C_2 \in \Delta$. Since $\vdash_a (A \to C_1 \wedge A \to C_2) \to (A \to (C_1 \wedge C_2))$ and Δ a Π-theory, we have $A \to (C_1 \wedge C_2) \in \Delta$. Thus $C_1 \wedge C_2 \in \Sigma$. Now suppose that $\vdash_\Pi C \to D$ and $C \in \Sigma$. Then $\vdash_\Pi (A \to C) \to (A \to D)$ and $A \to C \in \Delta$; so $A \to D \in \Delta$ and hence $D \in \Sigma$.

Clearly, $\Sigma \supseteq \Delta$. Moreover, it is straightforward to check that $A \in \Sigma$ and $B \vee \cdots \vee B \notin \Sigma$. Hence let Δ' be the closure of $\{B\}$ under disjunction. Then $\Sigma \cap \Delta' = \emptyset$, and we can apply Corollary 6.3.1 to conclude the desired result. □

Note that, since Σ is non-trivial, the obtained Γ is non-trivial as well.

We are now ready to prove the completeness.

Theorem 6.3.2. If $\Gamma \vDash_a A$ then $\Gamma \vdash_a A$.

Proof. We prove the contrapositive. Suppose that $\Gamma \nvdash_a A$. Then, by Corollary 6.3.2, there is a $\Pi \supseteq \Gamma$ such that Π is a prime Π-theory, Π-deductively closed and $A \notin \Pi$. Define the countermodel $\mathcal{M}^@ = (X, \leq, \mathcal{V})$ with Π as the root, where $X = \{\Delta : \Delta \text{ is a non-trivial prime } \Pi\text{-theory}\}$, $\Delta \leq \Sigma$ iff $\Delta \subseteq \Sigma$ and $\mathcal{V}$ is defined thus. For every state Σ and propositional parameter p:

$$\mathcal{M}^{@}, \Sigma \Vdash_a p \text{ iff } p \in \Sigma$$

We show by induction on B that $\mathcal{M}^{@}, \Sigma \Vdash_a B$ iff $B \in \Sigma$. We concentrate on the cases where B has the form $@C$ and $C \to D$.

When $B \equiv @C$, if $\mathcal{M}^{@}, \Sigma \Vdash_a @C$ then by definition $\mathcal{M}^{@}, \Pi \Vdash_a C$. By IH this is equivalent to $C \in \Pi$. Then $C \in \Sigma$ as $\Pi \subseteq \Sigma$ and also $\vdash_\Pi @C$ by (RN); hence $\vdash_\Pi C \to @C$. Now as Σ is a Π-theory, $C \in \Sigma$ implies $@C \in \Sigma$. For the other direction, it suffices to show $@C \in \Sigma$ implies $C \in \Pi$. First note $@C \vee @C \to D \in \Pi$ for all D because Π is Π-deductively closed. Then as Π is a prime theory, for each D either $@C \in \Pi$ or $@C \to D \in \Pi$. That is, either $@C \in \Pi$ or for all D, $@C \to D \in \Pi$. But if the latter, because Σ is a Π-theory, that $\Pi \subseteq \Sigma$ and $\vdash (@C \wedge (@C \to D)) \to D$ imply $D \in \Sigma$ for all D. This contradicts the non-triviality of Σ, so it must be that $@C \in \Pi$. But then $C \in \Pi$ by (T) and Π being a Π-theory.

When $B \equiv C \to D$, by IH $\mathcal{M}^{@}, \Sigma \Vdash_a C \to D$ iff for all Δ s.t. $\Sigma \subseteq \Delta$, if $C \in \Delta$ then $D \in \Delta$. Hence it suffices to show that this latter condition is equivalent to $C \to D \in \Sigma$. For the forward direction, we argue by contraposition; so assume $C \to D \notin \Sigma$. Then by Lemma 6.3.5 we can find find a non-trivial prime Π-theory Σ' such that $C \in \Sigma'$ but $D \notin \Sigma'$. For the backward direction, assume $C \to D \in \Sigma$ and $C \in \Delta$ for any Δ s.t. $\Sigma \subseteq \Delta$. Then $C \to D \in \Delta$ as well, and so $D \in \Delta$ since Δ is a Π-theory.

It now suffices to observe that $B \in \Pi$ for all $B \in \Gamma$ and $A \notin \Pi$, which in view of the above means $\Gamma \nVdash_a A$. This completes the proof. □

6.4 Comparison (I)

In this section, we give some comparisons of **IPC**$^{@}$ with **IPC**$^{\sim}$, as given in [29, 30], and **S5A** of Crossley and Humberstone, as given in [26].

6.4.1 Empirical negation and actuality

We start with recalling from the previous chapter that the semantics for **IPC**$^{\sim}$ is almost identical to that of **IPC**$^{@}$, except for the valuation of formulae of the form $\sim A$, which is given by:

$$w \Vdash_{\mathcal{K}} \sim A \text{ iff } r \nVdash_{\mathcal{K}} A.$$

remark 6.4.1. Note that Kosta Došen, in papers [35, 33, 34], considered negative modalities in models with two relations between worlds, like the models for intuitionistic modal logics, and one of them has the following condition:

$$w \Vdash \sim A \text{ iff for some } w' \in W, wRw' \text{ and } w' \nVdash A.$$

Although the modal relation R is absorbed by the intuitionistic relation $\leq$, empirical negation can be seen as having this type of valuation. Interestingly, Došen considered this sort of absorption is a necessary condition for a negative modality to be deemed a 'negation' (cf. [34, p.85]). For a recent discussion on negation understood as negative modality, see [4, 5, 32]. See also [62] for an up-to-date survey on negation, as well as negative modalities, in general.

remark 6.4.2. There are two more things to note with this valuation. First, intuitionistic $\bot$ and consequently the intuitionistic negation $\neg$ is definable in $\mathbf{IPC}^{\sim}$ by setting $\bot := \sim(A \to A)$. Second, since $w \Vdash_{\mathcal{K}} \sim\sim A$ iff $r \Vdash_{\mathcal{K}} A$, we see @ is also definable in $\mathbf{IPC}^{\sim}$ by $@A := \sim\sim A$.

A natural question then would be whether we can go the opposite direction, namely, is $\sim$ definable in $\mathbf{IPC}^{@}$? It turns out that this also holds. Since we have $\bot$ in $\mathcal{L}^{@}_{\bot}$, we readily see: $w \Vdash_a \neg @A$ iff $r \nVdash_a A$. The situation changes once we drop $\bot$ from the language. Let $\mathbf{IPC}^{@+}$ be defined in the language $\mathcal{L}^{@}$ that excludes $\bot$, with the corresponding omission of the axiom $\bot \to A$. The completeness for $\mathbf{IPC}^{@+}$ with respect to Kripke models with the base state is readily obtainable by an analogous means to that of $\mathbf{IPC}^{@}$.

Proposition 6.4.1. $\sim$ is not definable in $\mathbf{IPC}^{@+}$.

Proof. If $\sim$ is definable in $\mathbf{IPC}^{@+}$, then as we have seen $\bot$ is also definable as $\sim(A \to A)$. Let F be such a formula. Now choose a model such that $w \Vdash p$ for all p and $w \in W$. Then by induction on formula we can establish $w \Vdash A$ for all A and $w \in W$. So in particular, $w \Vdash F$ for all $w \in W$, a contradiction. □

Therefore $\mathbf{IPC}^{@+}$ may be seen as an intuitionistic system with actuality operator that is independent of negation. This system consequently has an advantage over $\mathbf{IPC}^{@}$ and $\mathbf{IPC}^{\sim}$ when a non-standard notion of negation is espoused. Moreover it offers a suitable starting point for combining intuitionism in empirical discourse and the school of intuitionism which eschews negation altogether, as a result of scepticism towards unrealised concepts (cf. [58]).

6.4.2 Classical actuality and constructive actuality

We now turn to compare $\mathbf{IPC}^{@}$ to **S5A** of Crossley and Humberstone. To this end, we first review the basics of **S5A**, with a slightly difference in the notation to replace A, for actuality, by @. Then the system is described by the language $\mathcal{L}^{@}_{m}$:

$$A ::= p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \bot \mid @A \mid \Box A$$

Definition 6.4.1 (Crossley & Humberstone)**.** An **S5A**-model for the language $\mathcal{L}^{@}_{m}$ is a triple $(W, r, \mathcal{V})$, where W is a non-empty set (of states); $r \in W$ (the base state); and $\mathcal{V}$ an assignment.

- $w \Vdash p$ iff $w \in \mathcal{V}(p)$;
- $w \nVdash \bot$;
- $w \Vdash \Box A$ iff for all $w' \in W$, $w' \Vdash A$;
- $w \Vdash @A$ iff $r \Vdash A$;
- $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$;
- $w \Vdash A \vee B$ iff $w \Vdash A$ or $w \Vdash B$;
- $w \Vdash A \to B$ iff $w \nVdash A$ or $w \Vdash B$.

Then, **S5A**-validity is defined in terms of truth at all $w \in W$: $\vDash_{\mathbf{S5A}} A$ iff for all **S5A**-models, $w \Vdash A$ for all $w \in W$.

Definition 6.4.2 (Crossley and Humberstone)**.** The axiomatic proof system for **S5A** consists of the following axioms in addition to any axiomatisation of **S5**:

$$@(@A \to A) \quad \text{(A1)} \qquad @A \leftrightarrow \neg @\neg A \quad \text{(A3)} \qquad @A \to \Box @A \quad \text{(A5)}$$
$$@(A \to B) \to (@A \to @B) \quad \text{(A2)} \qquad \Box A \to @A \quad \text{(A4)}$$

We refer to the derivability in **S5A** as $\vdash_{\mathbf{S5A}}$.

Based on these, Crossley and Humberstone established the following result.

Theorem 6.4.1 (Crossley and Humberstone)**.** $\vDash_{\mathbf{S5A}} A$ iff $\vdash_{\mathbf{S5A}} A$.

The above axiomatisation seen in view of $\mathbf{IPC}^@$ is problematic since the right-to-left direction of (A3) is *not* valid/derivable. However, a slightly different axiomatisation will allow us to compare **S5A** and $\mathbf{IPC}^@$ more easily.

Proposition 6.4.2. Let $\vdash_{\mathbf{S5A'}}$ be the derivability in a system obtained from the axiomatic proof system for **S5A** by replacing (A3) by the following two axioms:

$$@A \to \neg @\neg A \quad \text{(A3.1)} \qquad @(A \vee B) \to (@A \vee @B) \quad \text{(A3.2)}$$

Then, $\vdash_{\mathbf{S5A'}} A$ iff $\vdash_{\mathbf{S5A}} A$.

Proof. For the left-to-right direction, it suffices to check that (A3.2) is derivable in **S5A**. In view of (A3), (A3.2) is derivable iff $\vdash_{\mathbf{S5A}} (@\neg A \wedge @\neg B) \to @(\neg A \wedge \neg B)$. But this is obvious since @ is an extension of **K**-modality.

For the other way around, it suffices to prove $\vdash_{\mathbf{S5A'}} @A \vee @\neg A$. Since we have classical tautologies, we have $\vdash_{\mathbf{S5A'}} A \vee \neg A$, and by the rule of necessitation, we have $\vdash_{\mathbf{S5A'}} \Box(A \vee \neg A)$. This implies $\vdash_{\mathbf{S5A'}} @(A \vee \neg A)$ in view of (A4), and finally we make use of (A3.2) to obtain the desired result. □

remark 6.4.3. Note first that even though we do not have the necessity operator in $\mathbf{IPC}^@$, the actuality operator also enjoys the following condition:

$$w \Vdash_a @A \text{ iff } w \Vdash_a A \text{ for all } w \in W$$

This is because the base point is the root. Thus, if we regard □ as @ in the above axiomatisation of **S5A**, then we can see that all the axiom schemata and rules of inference related to □ and @ in **S5A** are derivable in $\mathbf{IPC}^@$.

Therefore, there is a sense in which $\mathbf{IPC}^@$ is a generalisation of **S5A**. But there is also a sense in which this generalisation is not simple. More specifically, we obtain the following result.

Proposition 6.4.3. $\mathbf{IPC}^@$ plus Peirce's law collapses into **Triv** based on **CPC**.

Proof. In view of (T), it suffices to prove $A \to @A$ in the extension. Note first that $A \vee (A \to B)$ is still derivable from an instance of Peirce's law, namely $(((A \vee (A \to B)) \to A) \to (A \vee (A \to B))) \to (A \vee (A \to B))$. Then as before we obtain $@A \vee @(A \to B)$, which entails $(@A \to @B) \to @(A \to B)$. Take $B \equiv @A$ and we then have $(@A \to @@A) \to @(A \to @A)$. By (4) and (T), we obtain $A \to @A$. □

remark 6.4.4. The above proof does not rely on the existence of $\bot$ in the language, and thus also applies to $\mathbf{IPC}^{@+}$.

6.5 Comparison (II)

In this section, we offer further comparisons of $\mathbf{IPC}^@$ with **LGP** of Baaz, as given in [2], and **GIPC** of Titani, as given in [118].

6.5.1 Baaz delta and actuality

As we mentioned in the introduction, Baaz' logic **LGP** is Gödel-Dummet logic equipped with a projection modality $\triangle$. Let us first look at the precise formulation in [2]. (For the sake of simplicity, we shall hereafter use $\mathcal{L}^@_\bot$ to describe the system, so @ will be used instead of $\triangle$.)

Definition 6.5.1 (Baaz)**.** Let $V \subseteq [0,1]$ be a set of *truth values* containing 0 and 1. A *valuation* $\mathfrak{V}$ based on V assigns a truth value in V to each propositional variable. $\mathfrak{V}$ is extended to all propositions by the clauses:

- $\mathfrak{V}(\bot) = 0$
- $\mathfrak{V}(A \wedge B) = min(\mathfrak{V}(A), \mathfrak{V}(B))$
- $\mathfrak{V}(A \vee B) = max(\mathfrak{V}(A), \mathfrak{V}(B))$
- $\mathfrak{V}(A \to B) = \begin{cases} \mathfrak{V}(B) & \text{if } \mathfrak{V}(A) > \mathfrak{V}(B) \\ 1 & \text{if } \mathfrak{V}(A) \leq \mathfrak{V}(B) \end{cases}$
- $\mathfrak{V}(@A) = \begin{cases} 1 & \text{if } \mathfrak{V}(A) = 1 \\ 0 & \text{if } \mathfrak{V}(A) \neq 1 \end{cases}$

Then $\mathbf{GP}(V) := \{A : \mathfrak{V}(A) = 1 \text{ for every } \mathfrak{V} \text{ based on } V\}$.

Definition 6.5.2. **LGP** is axiomatized by adding the following axiom to $\mathbf{IPC}^@$.

$$(A \to B) \vee (B \to A) \tag{Lin}$$

Let V be infinite. Baaz showed the following weak completeness for **LGP**.

Theorem 6.5.1 (Baaz)**.** For all $A \in \mathsf{Form}$, $\mathbf{LGP} \vdash A$ iff $A \in \mathbf{GP}(V)$.

As is well-known (e.g. [45, Theorem 19, Chapter 4]), Kripke-semantically (Lin) corresponds to lineally ordered Kripke frames. Thus as an improvement, we obtain a *strong* completeness proof for **LGP**, in view of Theorem 6.3.2. More specifically, let us denote $\vdash_l$ and $\vDash_l$ for the derivability in **LGP** and semantic consequence with respect to the class of linearly ordered models, respectively.

Proposition 6.5.1. $\Gamma \vdash_l A$ iff $\Gamma \vDash_l A$.

Proof. For soundness, we have to check that (Lin) holds in any linearly ordered model. Given a linearly ordered model $(W, r, \leq, \mathcal{V})$ and formulae A and B, let us denote $V(A) = \{w : w \Vdash_l A\}$ and $V(B) = \{w : w \Vdash_l B\}$. Then we have $V(A) \subseteq V(B)$ or $V(B) \subseteq V(A)$. Hence $r \Vdash_l A \to B \vee B \to A$.

For completeness, we have to check that the counter-model construction of Theorem 6.3.2 creates a linearly ordered model. Suppose otherwise. Then there are states Σ_1 and Σ_2 such that neither $\Sigma_1 \subseteq \Sigma_2$ nor $\Sigma_2 \subseteq \Sigma_1$. Then we can find a formula A_1 in Σ_1 not in Σ_2, and A_2 in Σ_2 not in Σ_1. Now as the base state Π is a prime Π-theory, $A_1 \to A_2 \vee A_2 \to A_1 \in \Pi$, and so $A_1 \to A_2 \in \Pi$ or $A_2 \to A_1 \in \Pi$. Without loss of generality, assume the former. Then because Σ_1 is a Π-theory, $A_1 \wedge (A_1 \to A_2) \in \Sigma_1$; thus $A_2 \in \Sigma_1$, a contradiction. Therefore the counter-model has to be linearly ordered. This completes the proof. □

remark 6.5.1. The above result clarifies that $\mathbf{IPC}^{@}$ is a generalisation of **LGP** to include non-linearly ordered models. To give a further comparison, for **LGP** it is observed in [2] that $\neg\neg A$ is a dual projection operator of $@A$, attaining 1 if $A \neq 0$ and 0 otherwise. In the setting of $\mathbf{IPC}^{@}$, this true-if-not-false type of operator is perhaps better captured by $\neg@\neg A$ (i.e. $\sim\neg A$). $w \Vdash_a \neg@\neg A$ iff for some $u \in W, u \Vdash_a A$; so while $\neg\neg A \to \neg@\neg A$ holds in general, $\neg@\neg A \to \neg\neg A$ does not. One may readily check that this latter implication is equivalent to *the weak excluded middle* $\neg A \vee \neg\neg A$ as an axiom; in particular $\neg@\neg A$ and $\neg\neg A$ becomes equivalent in **LGP**, because (Lin) implies the weak excluded middle.

6.5.2 A reformulation of global intuitionistic logic

Next we shall consider propositional global intuitionistic logic (to be called **GIPC**). Let us first look at the formulation of the logic in sequent calculus as given in [118, 1]. The system will be described in the language $\mathcal{L}_{\perp}^{@}$. Originally, however, $\Box$ was used in place of @, and $\neg$ was taken as primitive, rather than $\perp$. We shall call the calculus **LGJ** and the derivability by $\vdash_{gGI}$.

Definition 6.5.3 (Titani & Aoyama)**.** The rule of the calculus **LGJ** are as follows.

$$A \Rightarrow A \ (\mathrm{Ax}) \qquad\qquad \perp \Rightarrow \ (\mathrm{L}\perp)$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (\mathrm{LW}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (\mathrm{RW})$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (\mathrm{LC}) \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ (\mathrm{RC})$$

$$\frac{\Gamma, A, B, \Pi \Rightarrow \Delta}{\Gamma, B, A, \Pi \Rightarrow \Delta}\ (\mathrm{LE}) \qquad \frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda}\ (\mathrm{RE})$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ (\mathrm{Cut})$$

$$\frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ (\mathrm{L}\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\mathrm{R}\wedge)$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\mathrm{L}\vee) \qquad \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ (\mathrm{R}\vee)$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Pi \Rightarrow \Lambda}{A \to B, \Gamma, \Pi \Rightarrow \Delta, \Lambda}\ (\mathrm{L}\!\to) \qquad \frac{A, \Gamma \Rightarrow \bar{\Delta}, B}{\Gamma \Rightarrow \bar{\Delta}, A \to B}\ (\mathrm{R}\!\to)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{@A, \Gamma \Rightarrow \Delta}\ (\mathrm{L}@) \qquad \frac{\bar{\Gamma} \Rightarrow \bar{\Delta}, A}{\bar{\Gamma} \Rightarrow \bar{\Delta}, @A}\ (\mathrm{R}@)$$

In the above, $i \in \{1, 2\}$ and $\bar{\Gamma}$ and $\bar{\Delta}$ are finite sequences of @-*closed formulae*, which are formulae built from $\perp$ and formulae of the form $@A$, by the connectives $\wedge, \vee, \to$.

For example, $@@A, @A \wedge @(\bot \to C), \neg @(\neg A \vee B)$ are all @-closed formulae. We shall denote @-closed formulae by $\bar{A}, \bar{B}$ and so on.

We wish to compare **GIPC** with **IPC**$^@$. For this purpose it is preferable to have at hand a Hilbert-style axiomatisation. This we claim to be the following.

Definition 6.5.4. The system **GIPC** consists of intuitionistic axioms, (K)-(aLEM), (MP),(RN) and the following axiom scheme:

$$(@A \to @B) \to @(@A \to B) \tag{aSFT}$$

The derivability in **GIPC** will be denoted by $\vdash_{GI}$.

remark 6.5.2. Note that the deduction theorem, in the form of Theorem 6.2.1, holds for **GIPC** as well, by the same argument.

We now show a lemma before proving that **LGJ** and **GIPC** are equivalent.

Lemma 6.5.1. Let $\bar{A}$ be @-closed. Then, (i) $\vdash_{GI} \bar{A} \vee \bar{A} \to B$, and (ii) $\vdash_{GI} \bar{A} \to @\bar{A}$.

Proof. For (i), we argue by induction on the complexity of A.

- If $\bar{A} \equiv \bot$, then $\vdash_{GI} \bot \vee \bot \to B$.
- If $\bar{A} \equiv @A$, then $@A \vee @A \to B$ is an instance of (aLEM).
- If $\bar{A} \equiv \bar{C} \wedge \bar{D}$, then by IH $\vdash_{GI} \bar{C} \vee \bar{C} \to B$ and $\vdash_{GI} \bar{D} \vee \bar{D} \to B$. So $\vdash_{GI} (\bar{C} \wedge \bar{D}) \vee (\bar{C} \wedge \bar{D}) \to B$.
- If $\bar{A} \equiv \bar{C} \vee \bar{D}$, similarly $\vdash_{GI} (\bar{C} \vee \bar{D}) \vee (\bar{C} \vee \bar{D}) \to B$.
- If $\bar{A} \equiv \bar{C} \to \bar{D}$, by IH $\vdash_{GI} \bar{C} \vee \bar{C} \to \bar{D}$ and $\vdash_{GI} \bar{D} \vee \bar{D} \to B$. So $\vdash_{GI} (\bar{C} \to \bar{D}) \vee (\bar{C} \to \bar{D}) \to B$.

For (ii), we similarly argue by induction on A.

- If $\bar{A} \equiv \bot$, then $\bot \to @\bot$ is an instance of intuitionistic axioms.
- If $\bar{A} \equiv @A$, then $@A \to @@A$ is an instance of (4).
- If $\bar{A} \equiv \bar{B} \wedge \bar{C}$, then by IH $\vdash_{GI} \bar{B} \to @\bar{B}$ and $\vdash_{GI} \bar{C} \to @\bar{C}$. Thus $\vdash_{GI} \bar{B} \wedge \bar{C} \to @\bar{B} \wedge @\bar{C}$. Now it is easy to check via the deduction theorem that $\vdash_{GI} @\bar{B} \wedge @\bar{C} \to @(\bar{B} \wedge \bar{C})$. Hence $\vdash_{GI} \bar{B} \wedge \bar{C} \to @(\bar{B} \wedge \bar{C})$.
- If $\bar{A} \equiv \bar{B} \vee \bar{C}$, then using the same IH as above, we see $\vdash_{GI} \bar{B} \vee \bar{C} \to @\bar{B} \vee @\bar{C}$. Again it is an easy consequence of the deduction theorem that $\vdash_{GI} @\bar{B} \to @(\bar{B} \vee \bar{C})$ and $\vdash_{GI} @\bar{C} \to @(\bar{B} \vee \bar{C})$. Hence $\vdash_{GI} \bar{B} \vee \bar{C} \to @(\bar{B} \vee \bar{C})$.
- If $\bar{A} \equiv \bar{B} \to \bar{C}$, then using (T) and the IH that $\vdash_{GI} \bar{C} \to @\bar{C}$ we infer $\vdash_{GI} (\bar{B} \to \bar{C}) \to (@\bar{B} \to @\bar{C})$. Thus by (aSFT) $\vdash_{GI} (\bar{B} \to \bar{C}) \to @(@\bar{B} \to \bar{C})$. Also by the IH that $\vdash_{GI} \bar{B} \to @\bar{B}$ we have $\vdash_{GI} (@\bar{B} \to \bar{C}) \to (\bar{B} \to \bar{C})$. So by (RN) and (K), $\vdash_{GI} @(@\bar{B} \to \bar{C}) \to @(\bar{B} \to \bar{C})$. Combining the above two observations, we conclude $\vdash_{GI} (\bar{B} \to \bar{C}) \to @(\bar{B} \to \bar{C})$.

This completes the proof. □

Proposition 6.5.2. The following equivalence hold between **LGJ** and **GIPC**.
(i) If $\vdash_{GI} A$ then $\vdash_{gGI} \Rightarrow A$.
(ii) If $\vdash_{gGI} \Gamma \Rightarrow \Delta$ then $\vdash_{GI} \bigwedge \Gamma \to \bigvee \Delta$.

Proof. For (i), given the correspondence in intuitionistic logic, it suffices to consider axioms involving @ and (RN). Here we show cases for (aLEM) and (aSFT), which are stated but not shown in [1, Proposition 2.1]; other cases are immediate.

<u>(aLEM)</u>

$$\dfrac{\dfrac{\dfrac{@A \Rightarrow @A}{@A \Rightarrow @A, B}\,\text{(RW)}}{\Rightarrow @A, @A \to B}\,(\text{R}{\to})}{\Rightarrow @A \vee @A \to B}\,(\text{R}{\vee}),(\text{RC})$$

<u>(aSFT)</u>

$$\dfrac{\dfrac{\dfrac{\dfrac{@A \Rightarrow @A \qquad \dfrac{B \Rightarrow B}{@B \Rightarrow B}\,(\text{L}@)}{@A \to @B, @A \Rightarrow B}\,(\text{L}{\to})}{@A \to @B \Rightarrow @A \to B}\,(\text{R}{\to})}{@A \to @B \Rightarrow @(@A \to B)}\,(\text{R}@)}{\Rightarrow (@A \to @B) \to @(@A \to B)}\,(\text{R}{\to})$$

For (ii), we treat here the cases for (R→), (L@) and (R@).

- For (R→), by IH $\vdash_{GI} (\bigwedge \Gamma \wedge A) \to (\bigvee \bar{\Delta} \vee B)$. So $\vdash_{GI} \bigwedge \Gamma \to (A \to (\bigvee \bar{\Delta} \vee B))$. Now by Lemma 6.5.1 (i), $\vdash_{GI} \bigvee \bar{\Delta} \vee \bigvee \bar{\Delta} \to B$. Thus $\vdash_{GI} \bigwedge \Gamma \to (\bigvee \bar{\Delta} \vee A \to B)$.
- For (L@), by IH $\vdash_{GI} (A \wedge \bigwedge \Gamma) \to \bigvee \Delta$. Then $\vdash_{GI} A \to (\bigwedge \Gamma \to \bigvee \Delta)$. So by (T) $\vdash_{GI} @A \to (\bigwedge \Gamma \to \bigvee \Delta)$. Hence $\vdash_{GI} (@A \wedge \bigwedge \Gamma) \to \bigvee \Delta$.
- For (R@), by IH $\vdash_{GI} \bigwedge \bar{\Gamma} \to (\bigvee \bar{\Delta} \vee A)$. Then $\vdash_{GI} (\bigwedge \bar{\Gamma} \wedge (\bigvee \bar{\Delta} \to @A)) \to A$. Thus by (RN) and (K), $\vdash_{GI} @(\bigwedge \bar{\Gamma} \wedge (\bigvee \bar{\Delta} \to @A)) \to @A$. Here we note $@(\bigwedge \bar{\Gamma} \wedge (\bigvee \bar{\Delta} \to @A))$ is @-closed. So by Lemma 6.5.1 (ii), $\vdash_{GI} (\bigwedge \bar{\Gamma} \wedge (\bigvee \bar{\Delta} \to @A)) \to @A$. Also by Lemma 6.5.1 (i), $\vdash_{GI} \bigvee \bar{\Delta} \vee \bigvee \bar{\Delta} \to @A$. From these we deduce $\vdash_{GI} \bigwedge \bar{\Gamma} \to (\bigvee \bar{\Delta} \vee @A)$.

This completes the proof. □

6.5.3 Globalization and actuality

We are now ready to compare $\mathbf{IPC}^{@}$ and **GIPC**. We first observe that the former logic contains the latter.

Proposition 6.5.3. $\mathbf{IPC}^{@} \supseteq$ **GIPC**.

Proof. It suffices to observe that (aSFT) is derivable in $\mathbf{IPC}^{@}$. Applying (RN) and (aDIS) to (aLEM), we obtain $\vdash_a @A \vee @(@A \to B)$. Then on one hand, since $\vdash_a @A \to ((@A \to @B) \to @B)$ and $\vdash_a @B \to @(@A \to B)$ (the derivation of the latter uses (RN) and (K)), we have $\vdash_a @A \to ((@A \to @B) \to @(@A \to B))$. On the other hand, it is immediate that $\vdash_a @(@A \to B) \to ((@A \to @B) \to @(@A \to B))$. Therefore $\vdash_a (@A \to @B) \to @(@A \to B)$. □

remark 6.5.3. Baaz, in [2], states sequent rules for $\triangle$ of **LGP**. It turns out that the same rules can be used to formulate a calculus for $\mathbf{IPC}^{@}$. It is obtained from **LGJ** by relaxing (R@) to

$$\dfrac{\bar{\Gamma} \Rightarrow \Delta, A}{\bar{\Gamma} \Rightarrow \Delta, @A}\,(\text{R}@)$$

By Proposition 6.5.3, we can use Lemma 6.5.1 for $\mathbf{IPC}^{@}$ as well. Then we can argue analogously to Proposition 6.5.2; the treatments of cases for the new (R@) and (aDIS) are straightforward.

To show that the inclusion of the above proposition is strict, we shall turn to $\mathbf{TCC}_\omega$. We shall make use of the fact that the following formulae and rule are derivable in $\mathbf{TCC}_\omega$.

$$(\sim A \to A) \to A \quad \text{(t1)} \qquad \sim\sim A \to A \quad \text{(t3)}$$

$$\sim(A \to A) \to B \quad \text{(t2)} \qquad \frac{A}{\sim\sim A} \quad \text{(t4)}$$

Moreover, the same form of the deduction theorem as $\mathbf{IPC}^\sim$ holds in $\mathbf{TCC}_\omega$.

Quite similarly to the situation with $\mathbf{IPC}^@$ and $\mathbf{IPC}^\sim$, we have the following translations between **GIPC** and $\mathbf{TCC}_\omega$.

Definition 6.5.5. Let $()^\sim$ and $()^@$ be translations between $\mathcal{L}^@_\bot$ and $\mathcal{L}^\sim$ such that:

$$\begin{aligned} p^\sim &= p & p^@ &= p \\ (A \circ B)^\sim &= A^\sim \circ B^\sim & (A \circ B)^@ &= A^@ \circ B^@ \\ (@A)^\sim &= \sim\sim A^\sim & (\sim A)^@ &= \neg @ A^@ \\ \bot^\sim &= \sim(p_0 \to p_0) \end{aligned}$$

where p_0 is a fixed propositional variable, and $\circ \in \{\wedge, \vee \to\}$.

Lemma 6.5.2. $\vdash_{GI} A \leftrightarrow (A^\sim)^@$ and $\vdash_t A \leftrightarrow (A^@)^\sim$.

Proof. By induction on A. Here we look at the cases $A \equiv @B$ and $A \equiv \sim B$.

For the former, we need to show $\vdash_{GI} @B \leftrightarrow \neg@\neg@(B^\sim)^@$. By IH $\vdash_{GI} B \leftrightarrow (B^\sim)^@$, so its suffices to show $\vdash_{GI} @B \leftrightarrow \neg@\neg@B$. We first note $\neg@B$ is @-closed, thus $\vdash_{GI} \neg@\neg@B \leftrightarrow \neg\neg@B$. Also $\vdash_{GI} \neg\neg@B \leftrightarrow @B$ from (aLEM). Therefore we conclude $\vdash_{GI} @B \leftrightarrow \neg@\neg@B$ as desired.

For the latter, we need $\vdash_t \sim B \leftrightarrow (\sim\sim(B^@)^\sim \to \sim(p_0 \to p_0))$. Again by IH $\vdash_t B \leftrightarrow (B^@)^\sim$. Then the equivalence follows by $\sim B \to \neg\neg B \to \sim(p_0 \to p_0)$, (t1) and (t2). ☐

Proposition 6.5.4. We have that (i) for all $A \in \mathsf{Form}$, $\vdash_{GI} A$ iff $\vdash_t A^\sim$, and (ii) for all $A \in \mathsf{Form}^\sim$, $\vdash_t A$ iff $\vdash_{GI} A^@$.

Proof. By Lemma 6.5.2, it suffices to show the left-to-right direction.

For (i), we need to check the translations of (K)-(aLEM), (aSFT) and (RN) hold in $\mathbf{TCC}_\omega$.

- (K) is translated as $\sim\sim(A^\sim \to B^\sim) \to (\sim\sim A^\sim \to \sim\sim B^\sim)$, the derivability of which is immediate from the deduction theorem and (RC).
- (T) is translated as $\sim\sim A^\sim \to A^\sim$, which is an instance of (t3).
- (4) is translated as $\sim\sim A^\sim \to \sim\sim\sim\sim A^\sim$. This follows from $\sim\sim\sim A^\sim \to (\sim\sim A^\sim \to \sim A^\sim)$ and (t1), which imply $\sim\sim\sim A^\sim \to \sim A^\sim$; then use (RC).
- (aLEM) becomes $\sim\sim A^\sim \vee \sim\sim A^\sim \to B^\sim$, a consequence of $\sim\sim A \vee \sim A^\sim$ and $\sim A^\sim \to (\sim\sim A^\sim \to B^\sim)$.
- For (aSFT), we need to show $\vdash_t (\sim\sim A^\sim \to \sim\sim B^\sim) \to \sim\sim(\sim\sim A^\sim \to B^\sim)$. First $\vdash_t \sim A^\sim \vee \sim\sim A^\sim$ and $\vdash_t \sim\sim\sim A^\sim \to \sim A^\sim$ as seen above. So $\vdash_t (\sim\sim A^\sim \to \sim\sim B^\sim) \to (\sim A^\sim \vee \sim\sim B^\sim)$. We shall show $\vdash_t (\sim A^\sim \vee \sim\sim B^\sim) \to \sim\sim(\sim\sim A^\sim \to B^\sim)$. On one hand, $\vdash_t \sim A^\sim \to \sim\sim(\sim\sim A^\sim \to B^\sim)$ from $\vdash_t \sim A^\sim \to (\sim\sim A^\sim \to B^\sim)$, (t3) and (RC). On the other hand, $\vdash_t \sim\sim B^\sim \to \sim\sim(\sim\sim A^\sim \to B^\sim)$ from (RC). Thus $\vdash_t (\sim A^\sim \vee \sim\sim B^\sim) \to \sim\sim(\sim\sim A^\sim \to B^\sim)$ as required.

- Finally, (RN) is replicable by (t4).

For (ii), we need to check $A \vee \sim A$,$\sim A \to (\sim\sim A \to B)$ and (RC).

- $A \vee \sim A$ is translated into $A^@ \vee \neg @A^@$, which is an instance of (aLEM).
- $\sim A \to (\sim\sim A \to B)$ is translated into $\neg @A^@ \to (\neg @\neg @A^@ \to B^@)$. As we observed in Lemma 6.5.2, $\neg @\neg @A^@$ is equivalent to $\neg\neg @A^@$; so it follows from *Ex falso*.
- For (RC), we need to derive $\neg @B \to \neg @A$ from $A \to B$. This is possible with (RN),(K) and by contraposition.

This completes the proof. □

The translation allows us to use the Kripke semantics for $\mathbf{TCC}_\omega$.
We are now ready to separate the two systems.

Corollary 6.5.1. IPC$^@ \supsetneq$ GIPC.

Proof. First, recall that we have the following valuation for $\sim\sim A$.

$$w \Vdash_{\mathcal{K}t} \sim\sim A \text{ iff } w' \Vdash_{\mathcal{K}t} A \text{ for all } w'.$$

Now, if **GIPC** proves (aDIS), then by Proposition 6.5.4 $\sim\sim(p \vee q) \to \sim\sim p \vee \sim\sim q$ is provable in $\mathbf{TCC}_\omega$. On the other hand, if we consider a model where $W = \{w, w'\}$, $\leq = \{(w,w),(w',w')\}$, $\mathcal{V}(p)=\{w\}$ and $\mathcal{V}(q)=\{w'\}$, then $w \Vdash_{\mathcal{K}t} \sim\sim(p \vee q)$, but $w \nVdash_{\mathcal{K}t} \sim\sim p$ and $w \nVdash_{\mathcal{K}t} \sim\sim q$. Hence this is a countermodel for $\sim\sim(p \vee q) \to \sim\sim p \vee \sim\sim q$. So by the previous theorem, $\nvdash_t \sim\sim(p \vee q) \to \sim\sim p \vee \sim\sim q$. A contradiction. Therefore **GIPC** does not prove (aDIS). □

remark 6.5.4. Note that given a model of $\mathbf{TCC}_\omega$, we can define a model for $\mathcal{L}_\bot^@$ such that

$$w \Vdash @A \text{ iff } w' \Vdash A \text{ for all } w'.$$

Then, it is not difficult to see that each such model corresponds to the original model similarly to Lemma 6.5.2 and Proposition 6.5.4. Therefore, it is an immediate consequence of Theorem 5.3.2 that this gives a sound and weakly complete Kripke semantics for **GIPC**. (This semantics can be also obtained from Ono's semantics via Gordienko's technique; see below.)

We offer a few more words about **GIPC**. In [95], Hiroakira Ono extensively discussed intuitionistic modal systems which are defined by axioms that classically define **S5** when added to **S4**. Aoyama [1] compared some of these systems with **GIPC**,[4] but he did not compare with the strongest of Ono's systems, $\mathbf{L}_4$. It is defined by intuitionistic axioms plus (K)-(4), $@A \vee @\neg @A$, (MP) and (RP). The Kripke semantics for $\mathbf{L}_4$ in [95] is characterised by modal relation R that is an equivalence relation; this corresponds to the original semantics of $\mathbf{TCC}_\omega$, from which Gordienko derived [54, Lemma 4.4] the semantics of Definition 5.3.3. This observation and Proposition 6.5.4 suggest a close relationship between **GIPC** and $\mathbf{L}_4$. In fact, the two systems turn out to coincide.

Proposition 6.5.5. GIPC = $\mathbf{L}_4$

[4]Some of the comparisons offered in [1] are also observed by Hidenori Kurokawa in [77].

Proof. On one hand, $\neg @A$ is @-closed, so by Lemma 6.5.1 (ii) $\neg @A \to @\neg @A$ is derivable in **GIPC**. Thus with (aLEM), $@A \vee @\neg @A$ is derivable in **GIPC**. Consequently **GIPC** contains $\mathbf{L}_4$. On the other hand, $@A \vee @\neg @A$ implies (aLEM) with *Ex falso* and (T). Moreover, $(@A \to @B) \to @(@A \to @B)$ is known to be derivable in $\mathbf{L}_4$ (cf. [95, Figure 2.1]), and it is a consequence of (T), (RN) and (K) that $@(@A \to @B) \to @(@A \to B)$ holds, so (aSFT) is also derivable in $\mathbf{L}_4$. Thus $\mathbf{L}_4$ contains **GIPC** as well. □

6.5.4 Sequent calculi for logics with empirical negation

Finally, we shall use the results obtained so far to formulate sequent calculi for $\mathbf{TCC}_\omega$ and $\mathbf{IPC}^\sim$. We begin with introducing an analogue of @-closed for formulae in $\mathcal{L}^\sim$.

Definition 6.5.6. We define the class of *$\sim$-closed formulae* by the next clauses.
(i) $\bot$, $\sim A$ are $\sim$-closed.
(ii) If $\bar{B}$ and $\bar{C}$ are $\sim$-closed, then $\bar{B} \circ \bar{C}$ is $\sim$-closed, where $\circ \in \{\wedge, \vee, \to\}$.

It is straightforward to check that if $\bar{A}$ is $\sim$-closed, then $\bar{A}^@$ is @-closed.

Lemma 6.5.3. $\vdash_t \bar{A} \to \sim\sim\bar{A}$.

Proof. By the above observation and Lemma 6.5.1 (ii), we have $\vdash_{GI} \bar{A}^@ \to @\bar{A}^@$. Thus by Proposition 6.5.4 (i) and Lemma 6.5.2, $\vdash_t \bar{A} \to \sim\sim\bar{A}$. □

The sequent rules for $\sim$ corresponding to $\mathbf{TCC}_\omega$ is obtained by the following

$$\frac{\bar{\Gamma} \Rightarrow \bar{\Delta}, A}{\sim A, \bar{\Gamma} \Rightarrow \bar{\Delta}}\ (\mathrm{L}\sim) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim A}\ (\mathrm{R}\sim)$$

where $\bar{\Gamma}, \bar{\Delta}$ are $\sim$-closed. The sequent calculus **LT** for $\mathbf{TCC}_\omega$ is obtained by adding the above rules to the positive and non-modal fragment of **LGJ** (derivability denoted by $\vdash_{gT}$).

Theorem 6.5.2. $\vdash_{gT} \Gamma \Rightarrow \Delta$ iff $\vdash_t \bigwedge \Gamma \to \bigvee \Delta$.

Proof. For the right-to-left direction, we need to check the cases for $A \vee \sim A$, $\sim A \to (\sim\sim A \to B)$ and (RC). Each case is straightforward. For the right-to-left direction, we must check the cases for (L$\sim$) and (R$\sim$). The latter case is simple; for the former case, $\vdash_t \bar{\Gamma} \to (\bigvee \bar{\Delta} \vee A)$ by IH. Then by (MP) and Lemma 6.5.3, $\bar{\Gamma} \vdash_t \sim\sim \bigvee \bar{\Delta} \vee A$. So $\bar{\Gamma} \vdash_t \sim \bigvee \bar{\Delta} \to A$ and thus by (RC) and (t3), we obtain $\bar{\Gamma} \vdash_t \sim A \to \bigvee \bar{\Delta}$. Hence by deduction theorem and Lemma 6.5.3 again, we conclude $\vdash_t (\bar{\Gamma} \wedge \sim A) \to \bigvee \bar{\Delta}$. □

A sequent calculus for $\mathbf{IPC}^\sim$ has not been considered before. We can now obtain one by removing the condition that $\bar{\Delta}$ is $\sim$-closed in (L$\sim$). The correspondence with the Hilbert-style system is straightforwardly demonstrable.

6.6 Discussion

In this chapter, we introduced $\mathbf{IPC}^{@}$, an expansion of **IPC**, obtained by adding actuality operator, and compared with systems including **LGP** of Baaz, **GIPC** of Titani and $\mathbf{IPC}^{\sim}$ of De, obtained by adding projection operator, globalization operator and empirical negation respectively. What emerged is the following hierarchy of systems in $\mathcal{L}^{@}_{\bot}$, each corresponding to a system in $\mathcal{L}^{\sim}$.

(In $\mathcal{L}^{@}_{\bot}$)		(In $\mathcal{L}^{\sim}$)
LGP	$\Longleftrightarrow$	$\mathbf{IPC}^{\sim}$ + (Lin)
$\mathbf{IPC}^{@}$	$\Longleftrightarrow$	$\mathbf{IPC}^{\sim}$
$\mathbf{GIPC} = \mathbf{L}_4$	$\Longleftrightarrow$	$\mathbf{TCC}_{\omega}$

With respect to these systems, we make some additional observations and mention a few future directions.

Hybrid logic Since there are clear connections between hybrid logics and logics with actuality operator, and in particular there are some results on hybrid logics based on intuitionistic logic (cf. [12, 13]), a comparison of $\mathbf{IPC}^{@}$ to these systems will be of great interest.

Kripke semantics vs. Beth semantics We observed that @ in $\mathbf{IPC}^{@}$ and $\sim$ in $\mathbf{IPC}^{\sim}$ are inter-definable (in the presence of $\bot$ in the language), and similarly for **GIPC** and $\mathbf{TCC}_{\omega}$. As we have noted, a crucial difference between the semantics of $\mathbf{IPC}^{\sim}$ and $\mathbf{TCC}_{\omega}$ (hence the interpretation of @) is that models in the former always has a base state, while the latter in general does not. As a result, Kripke-semantically, even though both @ can be understood as a globalization operator (i.e. true iff true everywhere), only the former can be interpreted as an actuality operator. Yet one may wonder whether one could view @ in **GIPC** as a sort of actuality operator. To this question, one may again answer that Beth semantics allows such an interpretation, in the same way the negation of $\mathbf{TCC}_{\omega}$ can be viewed as an empirical negation. Thus there are two types of actuality operator/empirical negation in intuitionistic logic, Kripke-type and Beth-type.

With this kind of perspective, we can connect results related to **GIPC** with empirical negation. For instance, Titani's global intuitionistic set theory can be seen as a mathematical theory with Beth-type empirical negation, by reading $\neg\Box$ as $\sim$. This could then encourage the investigation of intuitionistic set theory with Kripke-type empirical negation, as a possible future direction.

Quantifiers Global intuitionistic logic was originally formulated in a first-order language. Moreover, quantification for **LGP** has been investigated in [2, 3]. From this perspective, it seems to be a natural direction to consider first-order systems for $\mathbf{IPC}^{@}$. This can be particularly interesting because like disjunction, existential quantifier has differing interpretations in Kripke and Beth models. Therefore we might be able to find an interesting interaction between quantifiers and modal operators. Moreover, for the purpose of comparing $\mathbf{IPC}^{@}$ to **S5A** of Crossley and

Humberstone, we also need quantifiers, and this will be yet another motivation for adding quantifiers.

Hypersequent calculi The sequent calculus for global intuitionistic logic **GI** defined by Titani and Aoyama is not cut-eliminatable, as observed by Agata Ciabattoni in [23, p.437]. She instead formulated a cut-free hypersequent calculus for **GI** and for **GIF**. We may then expect a similar approach to be quite beneficial in pursuing cut-free sequent calculi for the systems we have considered, namely $\mathbf{IPC}^{@}$, $\mathbf{IPC}^{\sim}$ and $\mathbf{TCC}_{\omega}$.

Chapter 7

Concluding remarks

7.1 Summary of the contents

In this thesis, we have investigated various types of negation; in Chapter 3, we saw that the extension of Ishii's class for Ishihara's problem of decidable variables, by means of the classes of weak excluded middle and double negation elimination. This also allowed the extension of the result to Glivenko's logic. Moreover, we observed the extension of the problem into minimal logic by considering classes of the avoidability of Q.

In Chapter 4, we have seen how the logic of Vakarelov relates to the framework of subminimal negation, with the correspondence between the semantics established. This further enabled us to formulate a cut-free sequent calculus for the logic. In addition, we formulated a new countably infinite class of logics with subminimal negation.

In Chapter 5, we looked at how different semantics capture empirical and co-negation with different logics. For these systems, we formulated a uniform axiomatisation in terms of the rules (RC), and obtained the corresponding frame conditions. They were then utilised to create cut-free labelled sequent calculi for the systems.

In Chapter 6, we explored actuality operator as the dual notion of empirical negation in the intuitionistic setting. We formulated the proof theory for the semantics of Humberstone, and showed the strong completeness between them. We then made comparisons, with adjacent systems, including classical logic with actuality operator, Gödel-Dummett logic with projection operator and global intuitionistic logic.

These enquiries offer us many crucial insights into the nature of alternative negations in intuitionistic setting, the interaction between different proof systems and semantics, as well as the relationship between various logics from the viewpoint of negation.

7.2 Future directions

Although we have already discussed some possible future directions for each topic at the end of respective chapter, we would like to add a few words regarding a wider picture emerging from the interactions among the topics, including its potential influences. Whilst the enquiries of this thesis are mostly independent of each other, it can still be expected that one broadens the perspective for negation in intuitionistic setting further by combining the interests of the topics. For instance, one may investigate a possible generalisation of the framework of subminimal negation which is able to incorporate empirical negation. Another prospect would be to extend the range of Ishihara's problem to include non-standard negations.

For a more long-term perspectives, I should suggest the following future influences to various academic disciplines.

With respect to mathematics, it can be expected that the types of negation we explored to be incorporated into the formal and informal discourse of mathematics. Different notions of negation, such as the ones we treated in this thesis, are made precise by the use of the formal-mathematical frameworks. This shall mean, from a slightly different viewpoint, that the negations have developed a closer affinity with mathematical discourse. Although it has been generally accepted that informal mathematics can be formalised with the connectives $\wedge, \vee, \rightarrow$ (with quantifiers), it is possible that the alternative negations, made mathematics-friendly via formalisation, contribute to enrich new mathematical perspectives and enquiries. For instance, subminimal negation should bring more paraconsistency into mathematics. Furthermore, empirical negation and co-negation may allow to introduce a more empirical perspective to mathematics, to allow to talk about statements that are initially not the case. If the mathematical value of the negations is confirmed in the formal setting, then the use of the concepts can potentially be extended even to more informal mathematical discourses. This prospective is also of significant value to intuitionistic logic, because many of the negations crucially depending on the intuitionistic setting in their formalisations.

In this respect, we may also recall the fact that a proof of a proposition in intuitionistic logic corresponds to a construction of a program, a principle known as *Curry-Howard correspondence* [27, 64, 112]. It fundamentally connects the proof theory of intuitionistic logic and type theory. Given such connections, alternative negations in intuitionistic logic can also be expected to find applications in the aforementioned areas, e.g. by serving as a foundation for a weak theory of constructive mathematics, or as a guide to find their type-theoretic equivalents.

Finally, with respect to philosophy, one of the core significance of the present enquiry lies in the potential to uncover the aspects of Brouwerian philosophy. From its origin, intuitionistic logic has had a strong philosophical flavour due to the influence of Brouwer. As is well known, Brouwer's mathematical philosophy is closely related to his solipsistic mysticism [18]. Mathematically, this point of view is mirrored in the notion of *creative subject* argument [121, p.236], which incorporates the growth of knowledge of an *idealised mathematician* into the arguments in intuitionistic mathematics. Kripke and Beth semantics can be interpreted [133, pp.166-167] as the visualisation of the intellectual journey of the creative subject. Hence Brouwerian philosophy is well-reflected in the semantics of intuitionistic logic. This in turn suggests that the ideas of Brouwer can be analysed with the aid of the semantics. Therefore enriching of intuitionistic semantics with new types of negation should contribute to expand and enrich the philosophical system of Brouwer.

Publications

1. Niki, S. (2020): Empirical Negation, Co-negation and Contraposition Rule I: Semantical Investigations. Bulletin of the Section of Logic. DOI: https://doi.org/10.18778/0138-0680.2020.12

2. Niki, S. (2020): Empirical Negation, Co-negation and Contraposition Rule II: Proof-Theoretical Investigations. Bulletin of the Section of Logic. DOI: https://doi.org/10.18778/0138-0680.2020.13

3. Niki, S. (2020): Subminimal Logics in Light of Vakarelov's Logic. Studia Logica 108(5):967-987.

4. Niki, S. and Omori, H. (2020): Actuality in Intuitionistic Logic. in Olivetti, N., Verbrugge, R., Negri, S. and Sandu, G. (eds.), Advances in Modal Logic 13. pp.459-479.

5. Niki, S. and Schuster, P. (2020): On Scott's semantics for many-valued logic. Journal of Logic and Computation 30(6):1291-1302.

6. Niki, S. Decidable variables for constructive logics. Mathematical Logic Quarterly. accepted.

Bibliography

[1] Hiroshi Aoyama. The semantic completeness of a global intuitionistic logic. *Mathematical Logic Quarterly*, 44(2):167–175, 1998.

[2] Matthias Baaz. Infinite-valued Gödel logics with 0-1-projections and relativizations. In Petr Hájek, editor, *Gödel '96*, pages 23–33. Springer, 1996.

[3] Matthias Baaz, Norbert Preining, and Richard Zach. Completeness of a hypersequent calculus for some first-order godel logics with delta. In *36th International Symposium on Multiple-Valued Logic (ISMVL'06)*, pages 9–9. IEEE, 2006.

[4] Francesco Berto. A modality called 'negation'. *Mind*, 124(495):761–793, 2015.

[5] Francesco Berto and Greg Restall. Negation on the Australian Plan. *Journal of Philosophical Logic*, 48:1119–1144, 2019.

[6] Evert Willem Beth. semantic construction of intuitionistic logic. *(Mededelingen van de) Koninklijke Nederlandse Akademie van Wetenschappen (Afdeling Letteren) (Nieuwe Reeks)*, 19(11):357–388, 1956.

[7] Guram Bezhanishvili. Varieties of monadic Heyting algebras. Part I. *Studia Logica*, 61(3):367–402, 1998.

[8] Nick Bezhanishvili, Almudena Colacito, and Dick de Jongh. A study of subminimal logics of negation and their modal companions. In *International Tbilisi Symposium on Logic, Language, and Computation*, pages 21–41. Springer, 2018.

[9] Gavin M. Bierman and Valeria CV de Paiva. On an intuitionistic modal logic. *Studia Logica*, 65(3):383–416, 2000.

[10] Marta Bílková and Almudena Colacito. Proof theory for positive logic with weak negation. *Studia Logica*, 108(4):649–686, 2020.

[11] Milan Božić and Kosta Došen. Models for normal intuitionistic modal logics. *Studia Logica*, 43(3):217–245, 1984.

[12] Torben Braüner. Axioms for classical, intuitionistic, and paraconsistent hybrid logic. *Journal of Logic, Language and Information*, 15(3):179–194, 2006.

[13] Torben Braüner. Intuitionistic hybrid logic: Introduction and survey. *Information and Computation*, 209(12):1437–1446, 2011.

[14] L.E.J. Brouwer. Der onbetrouwbaarheid der logische principes [the unreliability of logical principles]. In Arend Heyting, editor, *L.E.J.Brouwer collected papers*, volume 1, pages 107–111. North-Holland, 1975.

[15] L.E.J. Brouwer. Intuitionism and formalism. In Arend Heyting, editor, *L.E.J.Brouwer collected papers*, volume 1, pages 123–138. North-Holland, 1975.

[16] L.E.J. Brouwer. Over de grondslagen der wisukunde [on the foundations of mathematics]. In Arend Heyting, editor, *L.E.J.Brouwer collected papers*, volume 1, pages 11–101. North-Holland, 1975.

[17] Luitzen Egbertus Jan Brouwer. Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Zweiter teil. In Arend Heyring, editor, *L.E.J. Brouwer Collected Works 1: Philosophy and Foundations of Mathematics*, pages 191–221. North-Holland, 1975.

[18] Luitzen Egbertus Jan Brouwer. Life, art, and mysticism. *Notre Dame Journal of Formal Logic*, 37(3):389–429, 1996.

[19] R. A. Bull. A modal extension of intuitionist logic. *Notre Dame Journal of Formal Logic*, 6(2):142–146, 1965.

[20] R. A. Bull. MIPC as the formalisation of an intuitionist concept of modality. *The Journal of Symbolic Logic*, 31(4):609–616, 1966.

[21] José Luis Castiglioni and Rodolfo Cristian Ertola Biraben. Strict paraconsistency of truth-degree preserving intuitionistic logic with dual negation. *Logic Journal of the IGPL*, 22(2):268–273, 2014.

[22] Alexender Chagrov and Michael Zakharyaschev. *Modal Logic*. Oxford University Press, 1997.

[23] Agata Ciabattoni. A proof-theoretical investigation of global intuitionistic (fuzzy) logic. *Archive for Mathematical Logic*, 44(4):435–457, 2005.

[24] Almudena Colacito. Minimal and subminimal logic of negation. Master's thesis, University of Amsterdam, 2016.

[25] Almudena Colacito, Dick de Jongh, and Ana Lucia Vargas. Subminimal negation. *Soft Computing*, 21:165–174, 2017.

[26] John N Crossley and Lloyd Humberstone. The logic of "actually". *Reports on Mathematical Logic*, 8(1):1–29, 1977.

[27] Haskell Brooks Curry. The combinatory foundations of mathematical logic. *The Journal of Symbolic Logic*, 7(2):49–64, 1942.

[28] Martin Davies and Lloyd Humberstone. Two notions of necessity. *Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition*, 38(1):1–30, 1980.

[29] Michael De. Empirical negation. *Acta Analytica*, 28:49–69, 2013.

[30] Michael De and Hitoshi Omori. More on empirical negation. In Rajeev Goreé, Barteld Kooi, and Agi Kurucz, editors, *Advances in Modal Logic*, volume 10, pages 114–133. College Publications, 2014.

[31] Michael De and Hitoshi Omori. Classical and empirical negation in subintuitionistic logic. In *Advances in Modal Logic*, pages 217–235, 2016.

[32] Michael De and Hitoshi Omori. There is more to negation than modality. *Journal of Philosophical logic*, 47(2):281–299, 2018.

[33] Kosta Došen. Negation as a modal operator. *Reports on Mathematical Logic*, 20:15–27, 1986.

[34] Kosta Došen. Negation in the light of modal logic. In Dov M. Gabbay and Heinrich Wansing, editors, *What is Negation?* Kluwer Academic Publishing, 1999.

[35] Kosta Došen. Negative modal operators in intuitionistic logic. *Publication de lInstutute Mathematique, Nouv. Ser*, 35:3–14, 1984.

[36] Kosta Došen. Models for stronger normal intuitionistic modal logics. *Studia Logica*, 44(1):39–70, 1985.

[37] Kosta Došen and Peter Schroeder-Heister. *Substructural Logics*. Oxford University Press, Inc., 1994.

[38] Michael Dummett. A propositional calculus with denumerable matrix. *the journal of symbolic logic*, 24:97–106, 1959.

[39] Christian Espíndola. A short proof of Glivenko theorems for intermediate predicate logics. *Archive for mathematical logic*, 52(7-8):823–826, 2013.

[40] Thomas Macaulay Ferguson. Extensions of Priest-da Costa logic. *Studia Logica*, 102:145–174, 2013.

[41] Frederic B Fitch. Intuitionistic modal logic with quantifiers. *Portugaliae mathematica*, 7(2):113–118, 1948.

[42] Hans Freudenthal. Zur intuitionistischen deutung logischer formeln. *Compositio Mathematica*, 4:112–116, 1937.

[43] Harvey Friedman. Intuitionistic completeness of Heyting's predicate calculus. In *Notices of the American Mathematical Society*, volume 22(6), pages A648–A648, 1975.

[44] Peter Fritz. What is the correct logic of necessity, actuality and apriority? *The Review of Symbolic Logic*, 7(3):385–414, 2014.

[45] Dov M Gabbay. *Semantical investigations in Heyting's intuitionistic logic*, volume 148 of *Synthese Library*. Springer, 1981.

[46] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated lattices: an algebraic glimpse at substructural logics*. Elsevier, 2007.

[47] Gerhard Gentzen. Investigations into logical deduction. In Manfred Egon Szabo, editor, *The collected papers of Gerhard Gentzen*, pages 68–131. North-Holland, 1969.

[48] Gerhard Gentzen. On the relation between intuitionist and classical arithmetic. In Manfred Egon Szabo, editor, *The collected papers of Gerhard Gentzen*, pages 53–67. North-Holland, 1969.

[49] Paul Carl Gilmore. *The effect of Griss' criticism of the intuitionistic logic on deductive theories formalized within the intuitionistic logic.* Drukkerij Holland, 1953.

[50] Valerii Glivenko. On some points of the logic of Mr. Brouwer. In Paolo Mancosu, editor, *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s*, pages 301–305. Oxford University Press, 1998.

[51] Kurt Gödel. Eine Interpretation des intuitionistichen Aussagenkalküls. In Solomon Feferman, John W. Dawson Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, and Jean van Heienoort, editors, *Kurt Gödel: collected works*, volume I, page 301. Oxford University Press, 1986.

[52] Kurt Gödel. Zur intuitionistischen Arithmetik and Zahlenthorie. In Solomon Feferman, John W. Dawson Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, and Jean van Heienoort, editors, *Kurt Gödel: collected works*, volume I, pages 286–295. Oxford University Press, 1986.

[53] Kurt Gödel. Zur intuitionistischen Aussagenkalkül. In Solomon Feferman, John W. Dawson Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, and Jean van Heienoort, editors, *Kurt Gödel: collected works*, volume I, pages 222–225. Oxford University Press, 1986.

[54] Andrei Borisovich Gordienko. A paraconsistent extension of Sylvan's logic. *Algebra and Logic*, 46(5):289–296, 2007.

[55] Dominic Gregory. Completeness and decidability results for some propositional modal logics containing actually operators. *Journal of Philosophical Logic*, 30(1):57–78, 2001.

[56] George Franois Cornelis Griss. Negationless intuitionistic nathematic I. *Indagationes Mathematicae*, 8:675–681, 1946.

[57] Arend Heyting. Bemerkungen zu dem Aufsatz von Herrn Freudenthal "zur intuitionistischen deutung logischer formeln". *Compositio Mathematica*, 4:117–118, 1937.

[58] Arend Heyting. GFC Griss and his negationless intuitionistic mathematics. *Synthese*, pages 91–96, 1953.

[59] Arend Heyting. *Intuitionism: An Introduction.* North Holland, third revised edition, 1976.

[60] Arend Heyting. The formal rules of intuitionistic logic. In Paolo Mancosu, editor, *From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s*, pages 311–327. Oxford University Press, 1998.

[61] Arend Heyting. On intuitionistic logic. In Paolo Mancosu, editor, *From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s*, pages 306–310. Oxford University Press, 1998.

[62] Laurence Horn and Heinrich Wansing. Negation. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Spring 2020 edition, 2020.

[63] Tsutomu Hosoi and Hiroakira Ono. Intermediate propositional logics (a survey). *Journal of Tsuda College*, 5:67–82, 1973.

[64] William Alvin Howard. The formulae-as-types notion of construction. *To HB Curry: essays on combinatory logic, lambda calculus and formalism*, 44:479–490, 1980.

[65] Lloyd Humberstone. Two-dimensional adventures. *Philosophical Studies*, 118(1-2):17–65, 2004.

[66] Lloyd Humberstone. Extensions of intuitionistic logic without the deduction theorem: Some simple examples. *Reports of Mathematical Logic*, 40:45–82, 2006.

[67] Rosalie Iemhoff. Intuitionism in the philosophy of mathematics. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, fall 2020 edition, 2020.

[68] Hajime Ishihara. Classical propositional logic and decidability of variables in intuitionistic propositional logic. *Logical Methods in Computer Science (LMCS)*, 10(3), 2014.

[69] Katsumasa Ishii. A note on decidability of variables in intuitionistic propositional logic. *Mathematical Logic Quarterly*, 64(3):183–184, 2018.

[70] Ingebrigt Johansson. Der minimalkalkül, ein reduzierter intuitionistischer formalismus. *Compositio Mathematica*, 4:119–136, 1937.

[71] Stephen Cole Kleene. On the interpretation of intuitionistic number theory. *The journal of symbolic logic*, 10(4):109–124, 1945.

[72] Andrei Nikolaevich Kolmogorov. On the interpretation of intuitionistic logic. In Vladimir Mikhailovich Tikhomirov, editor, *Selected Works of A.N. Kolmogorov*, volume 1: Mathematics and Mechanics, pages 151–158. Kluwer Academic Publishers, 1991.

[73] Andrei Nikolaevich Kolmogorov. On the tertium non datur principle. In Vladimir Mikhailovich Tikhomirov, editor, *Selected Works of A.N. Kolmogorov*, volume 1: Mathematics and Mechanics, pages 40–68. Kluwer Academic Publishers, 1991.

[74] Saul Aaron Kripke. Semantical analysis of intuitionistic logic I. In *Studies in Logic and the Foundations of Mathematics*, volume 40, pages 92–130. Elsevier, 1965.

[75] Victor Nikolaevich Krivtsov. An intuitionistic completeness theorem for classical predicate logic. *Studia Logica*, 96(1):109–115, 2010.

[76] Sigekatu Kuroda. Intuitionistische untersuchungen der formalistischen logik. *Nagoya Mathematical Journal*, 2:35–47, 1951.

[77] Hidenori Kurokawa. Hypersequent calculi for intuitionistic logic with classical atoms. *Annals of Pure and Applied Logic*, 161(3):427–446, 2009.

[78] Daniel Leivant. Syntactic translations and provably recursive functions. *The Journal of Symbolic Logic*, 50(3):682–688, 1985.

[79] Edward John Lemmon. *Beginning logic*. Hackett Publishing, 1978.

[80] Edgar George Kenneth López-Escobar. Constructions and negationless logic. *Studia Logica*, 30(1):7–19, 1972.

[81] John CC McKinsey and Alfred Tarski. Some theorems about the sentential calculi of lewis and heyting. *The journal of symbolic logic*, 13(1):1–15, 1948.

[82] Sara Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5-6):507–544, 2005.

[83] Sara Negri. Proof analysis in non-classical logics. In C. Dimitracopoulos, L. Newelski, D. Normann, and J. Steel, editors, *ASL Lecture Notes in Logic*, volume 28, pages 107–128. Cambridge University Press, 2007.

[84] Sara Negri and Jan Von Plato. *Proof analysis: a contribution to Hilbert's last problem*. Cambridge University Press, 2011.

[85] David Nelson. Recursive functions and intuitionistic number theory. *Transactions of the American Mathematical Society*, 61(2):307–368, 1947.

[86] David Nelson. Non-null implication. *The Journal of Symbolic Logic*, 31(4):562–572, 1966.

[87] David Nelson. A complete negationless system. *Studia Logica*, 32(1):41–47, 1973.

[88] Satoru Niki. Decidable variables for constructive logics. Submitted.

[89] Satoru Niki. Empirical negation, co-negation and contraposition rule I: Semantical investigations. *Bulletin of the Section of Logic*, 49(3):231–253, 2020.

[90] Satoru Niki. Empirical negation, co-negation and the contraposition rule II: Proof-theoretical investigations. *Bulletin of the Section of Logic*, 2020. DOI: https://doi.org/10.18778/0138-0680.2020.13.

[91] Satoru Niki. Subminimal logics in light of Vakarelovs logic. *Studia Logica*, 108(5):967–987, 2020.

[92] Satoru Niki and Hitoshi Omori. Actuality in intuitionistic logic. In Nicola Olivetti, Rineke Verbrugge, Sara Negri, and Gabriel Sandu, editors, *Advances in Modal Logic*, volume 13, pages 459–479. College Publications, 2020.

[93] Iwao Nishimura. On formulas of one variable in intuitionistic propositional calculus. *The Journal of Symbolic Logic*, 25(4):327–331, 1960.

[94] Sergei Odintsov. *Constructive Negations and Paraconsistency: A Categorical Approach to L-fuzzy Relations*. Springer, 2008.

[95] Hiroakira Ono. On some intuitionistic modal logics. *Publications of the Research Institute for Mathematical Sciences*, 13(3):687–722, 1977.

[96] Hiroakira Ono. *Proof Theory and Algebra in Logic*. Springer, 2019.

[97] Ivan Efremovich Orlov. Ischislenie sovmestnosti predlozheniy. *Matematicheskiy Sbornik*, 35(4):263–286, 1928.

[98] Mauricio Osorio, José Luis Carballido, Claudia Zepeda, and José Abel Castellanos. Weakening and extending $\mathbb{Z}$. *Logica Universalis*, 9(3):383–409, 2015.

[99] Mauricio Osorio and José Abel Castellanos Joo. Equivalence among RC-type paraconsistent logics. *Logic Journal of the IGPL*, 25(2):239–252, 2017.

[100] Graham Priest. Dualising intuitionistic negation. *Principia*, 13(2):165–184, 2009.

[101] Arthur N Prior. *Time and modality*. Oxford University Press, 1957.

[102] Cecylia Rauszer. A formalization of the propositional calculus of H-B logic. *Studia Logica*, 33(1):23–34, 1974.

[103] Cecylia Rauszer. Applications of Kripke models to Heyting-Brouwer logic. *Studia Logica*, 36(1):61–71, 1977.

[104] Cecylia Rauszer. *An algebraic and Kripke-style approach to a certain extension of intuitionistic logic*. Państwowe Wydawnictwo Naukowe, 1980.

[105] Greg Restall. Subintuitionistic Logics. *Notre Dame Journal of Formal Logic*, 35(1):116–126, 1994.

[106] Greg Restall. Extending intuitionistic logic with subtraction. unpublished, 1997.

[107] Greg Restall. *An introduction to substructural logics*. Routledge, 2002.

[108] Ladislav Rieger. On the lattice theory of Brouwerian propositional logic. *Acta Faclutatis Rerum Naturalium Universitatis Carolinae*, 189, 1949.

[109] Krister Segerberg. Propositional logics related to Heyting's and Johansson's. *Theoria*, 34(1):26–61, 1968.

[110] Gisèle Fischer Servi. On modal logic with an intuitionistic base. *Studia Logica*, 36(3):141–149, 1977.

[111] Alex K Simpson. *The proof theory and semantics of intuitionistic modal logic.* PhD thesis, University of Edinburgh, 1994.

[112] Morten Heine Sørensen and Pawel Urzyczyn. *Lectures on the Curry-Howard isomorphism.* Elsevier, 2006.

[113] Vladimir H Sotirov. Modal theories with intuitionistic logic. In *Proceedings of the Conference on Mathematical Logic, Sofia*, pages 139–171, 1980.

[114] Shawn Standefer. Actual issues for relevant logics. *Ergo, an Open Access Journal of Philosophy*, 7, 2020.

[115] Yannis Stephanou. The meaning of actually. *dialectica*, 64(2):153–185, 2010.

[116] Richard Sylvan. Variations on da Costa C systems and dual-intuitionistic logics I. analyses of C_ω and CC_ω. *Studia Logica*, 49(1):47–65, 1990.

[117] Gaisi Takeuti and Satoko Titani. Global intuitionistic fuzzy set theory. In Petr Hájek, editor, *The Mathematics of Fuzzy Systems*, pages 291–301. TUV-Verlag, 1986.

[118] Satoko Titani. Completeness of global intuitionistic set theory. *The Journal of Symbolic Logic*, 62(2):506–528, 1997.

[119] Anne Sjerp Troelstra and Joan Rand Moschovakis. A.S. troelstra, D. van Dalen, constructivism in mathematics corrections. [accessed 10/Jan/2019].

[120] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, second edition, 2000.

[121] Anne Sjerp Troelstra and Dirk van Dalen. *Constructivism in Mathematics: An Introduction*, volume I. Elsevier, 1988.

[122] Anne Sjerp Troelstra and Dirk van Dalen. *Constructivism in Mathematics: An Introduction*, volume II. Elsevier, 1988.

[123] Toshio Umezawa. Über die zwischensysteme der Aussagenlogik. *Nagoya mathematical journal*, 9:181–189, 1955.

[124] Toshio Umezawa. On intermediate propositional logics. *The Journal of Symbolic Logic*, 24(1):20–36, 1959.

[125] Aldo Ursini. A modal calculus analogous to K4W, based on intuitionistic propositional logic, I^o. *Studia Logica*, 38(3):297–311, 1979.

[126] Dimiter Vakarelov. Consistency, completeness and negation. In Graham Priest, Richard Routley, and Jean Norma, editors, *Paraconsistent Logic: Essays on the Inconsistent*, Analytica, pages 328–363. Philosophia Verlag, 1989.

[127] Dimiter Vakarelov. Nelson's negation on the base of weaker versions of intuitionistic negation. *Studia Logica*, 80:393–430, 2005.

[128] V. Valpola. Ein system der negationslosen Logik mit ausschliesslich realisierbaren prädikaten. *Acta Philosophica Fennica*, 9:1–247, 1953.

[129] Mark van Atten. The development of intuitionistic logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2017 edition, 2017.

[130] Mark van Atten. Luitzen Egbertus Jan Brouwer. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2020 edition, 2020.

[131] Dirk van Dalen. *L.E.J. Brouwer–topologist, intuitionist, philosopher: How mathematics is rooted in life*. Springer Science & Business Media, 2012.

[132] Dirk van Dalen. *L.E.J. Brouwer: Topologist, Intuitionist, Philosopher*. Springer, 2013.

[133] Dirk van Dalen. *Logic and Structure*. Springer, fifth edition, 2014.

[134] Tim van der Molen. The Johansson/Heyting letters and the birth of minimal logic. *ILLC Technical Notes X-2016-03, Universiteit van Amsterdam*, 2016.

[135] Wim Veldman. An intuitionistic completeness theorem for intuitionistic predicate logic. *The Journal of Symbolic Logic*, 41(1):159–166, 1976.

[136] Jan von Plato. Skolem's discovery of Gödel-Dummett logic. *Studia Logica*, 73(1):153–157, 2003.

[137] Pieter Gaele Johannes Vredenduin. The logic of negationless mathematics. *Compositio mathematica*, 11:204–270, 1953.

[138] Peter W Woodruff. A note on JP'. *Theoria*, 36(2):183–184, 1970.