Title	量子動的代数:オーソモジュラー束から量子プログラムの代数へ
Author(s)	高木, 翼
Citation	
Issue Date	2023-09
Type	Thesis or Dissertation
Text version	ETD
URL	http://hdl.handle.net/10119/18773
Rights	
Description	Supervisor: 緒方 和博, 先端科学技術研究科, 博士

Quantum Logic is the logic of quantum mechanics, originated by von Neumann and Birkhoff in 1936. Half a century later, computation based on quantum mechanics, namely quantum computation, was invented and has developed dramatically up to the present day. Therefore, it is expected to incorporate the viewpoint of quantum computation to reformulate Quantum Logic into a modern logic.

In this dissertation, we study algebraic structures. The algebraic structure of Quantum Logic is known as orthomodular lattices, which are abstractions of the sets of all closed subspaces of Hilbert spaces. Orthomodular lattices are challenging to deal with because the distributive law does not hold. Besides that, orthomodularity is not determined by any first-order properties of the accessibility relation of Kripke frames for Quantum Logic. These features are not found in other well-known algebraic structures of logics, such as Boolean lattices (Classical Logic), Heyting lattices (Intuitionistic Logic), or Modal algebras (Modal Logic).

Interestingly, orthomodular lattices, with these hard-to-handle properties, can be made easier to deal with by adding the notion of quantum programs. This is another motivation for extending orthomodular lattices to algebra of quantum programs. Incorporating a quantum computation perspective into orthomodular lattices is not only beneficial for its reformulating as modern algebras, but also from a technical point of view.

We name the algebra of quantum programs Quantum Dynamic Algebra (QDA). QDA is constructed by combing the algebra of quantum mechanics (orthomodular lattices), Regular Program Algebra, and Modal Algebra. The quantum programs that QDA can express are limited to regular programs. However, regular programs are expressive enough to describe various programs, such as conditional programs, guarded commands, while programs, and until programs. The interpretation of tests in QDA differs from that in Dynamic Algebra (the algebra of classical programs): tests are interpreted as the execution of projective measurement in quantum mechanics. Because the execution of projective measurement may change the current state, there is no corresponding notion in Dynamic Algebra.

QDA provides the algebraic foundation for quantum program verification. We show that the inference rules in Hoare Logic are valid in QDA if the usual conjunction is replaced by the Sasaki conjunction. The validity of the Hoare-like inference rules means that the inference rules in Hoare Logic also work in the quantum setting as long as the appropriate logical connective(s) are chosen. Behind this is the fact that the more fundamental law called the law of residuation holds if the usual conjunction is replaced by the Sasaki conjunction in Quantum Logic. The law of residuation is significant because the law corresponds to one of the most significant theorems called the deduction theorem in logics, such as Classical Logic, Intuitionistic Logic, Modal Logic. More generally, algebras that satisfy the law of residuation give algebraic semantics for various substructural logics. There

has been no discussion of this kind of relevance to the field of logic (not only Hoare Logic) in existing studies.

Another achievement of this study is to show the Stone-type representation theorem for QDA at the cost of simplifying QDA to star-free (iteration-free) QDA. It is traditionally known that the iteration operator is challenging to deal with. For example, the Stone-type representation theorem has been proved only for star-free (Classical) Dynamic Algebra. The difficulty arises because the iterative operator allows the existence of non-standard Kripke models. Even for star-free QDAs, the proof of the Stone-type representation theorem is not straightforward. (Star-free) QDA is made up of a complex combination of multiple algebras, namely an orthomodular lattice, a regular program algebra, and a modal algebra. It is not apparent to prove the Stone-type representation theorem consistent with all these algebras.

Proving Stone-type representation theorems is significant because it reveals the relation between algebraic semantics and Kripke semantics. The most well-known Stone-type representation theorem is Stone's representation theorem for Boolean lattices. The theorem is extended to Jónsson–Tarski theorem, also known as the Stone-type representation theorem for modal algebras. However, although the algebraic structure of DQL is an extension of the modal algebra, its Stone-type representation theorem has not been known so far.

In summary, we formulate QDA for reformulating the algebraic structure of quantum mechanics into a modern algebra from the perspective of quantum computation. We also show the Stone-type representation theorem for its star-free fragment, which ensures the theoretical adequacy of QDA. Furthermore, it is expected to apply QDA to quantum program verification because Hoare-like inference rules are valid in QDA.

Key Words: Orthomodular Lattice, Dynamic Algebra, Sasaki Conjunction, Quantum Program Verification, Stone-type Representation Theorem