Doctoral Dissertation

Compositional Confluence Criteria for Term Rewriting

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Abstract

Term rewriting is a computation model for equational reasoning. This model underlies various fields in computer science, including functional programming, automated theorem proving, and software verification based on equational specifications. In rewriting there are two fundamental properties. One is termination, which ensures finiteness of computation steps. The other is confluence. It guarantees uniqueness of computation results without relying on any specific computational strategy. For a computational system with non-determinism, confluence corresponds to consistency of the system. It plays a key role as well-definedness of function definitions and correctness of specifications.

In this thesis we present a new approach for analyzing confluence of left-linear term rewrite systems based on compositional confluence criteria. A compositional confluence criterion means a sufficient condition that, given a rewrite system and its subsystem, confluence of the subsystem implies confluence of the original system. Since such a subsystem can be analyzed by any other (compositional) confluence criterion, compositional confluence criteria can be applied to subsystems successively. This method enables us to decompose a rewrite system into its subsystem for showing confluence of the original one.

In order to obtain compositional confluence criteria, we develop a variant of decreasing diagrams method. It is known that most of confluence criteria for left-linear rewrite systems can be shown by the method. Exploiting this fact, we demonstrate how those confluence criteria can be recast into compositional criteria by adopting a compositional version of the decreasing diagram method in their proofs. Furthermore we show how existing confluence criteria based on decreasing diagrams are generalized to ones composable with other criteria. We also show how such a criterion can be used as a reduction method to remove rewrite rules unnecessary for confluence analysis. Effectiveness of these approaches is assessed by experimental data based on our confluence tool Hakusan. In addition to these contributions, we prove that Toyama's parallel closedness result based on parallel critical pairs subsumes his almost parallel closedness theorem.

keywords: *term rewriting, confluence, decreasing diagrams, parallel closedness, automation.*

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Chapter 1

Introduction

1.1 Term Rewrite Systems and Confluence

Term rewriting is a computational model for equational reasoning. This model underlies various fields in computer science, including functional programming, automated theorem proving, and software verification based on equational specifications. In term rewriting, terms are rewritten to different terms by using *rewrite rules*. This step on terms is called a *rewrite step*, and computation is represented by a sequence of rewrite steps.

The following system of rewrite rules (*rewrite system*) represents arithmetic on Peano numbers:

$$\begin{array}{rrcl@{\qquad}rrcl} 1: & x + 0 & \to & x & 3: & x \times 0 & \to & 0 \\ 2: & \mathsf{s}(x) + y & \to & \mathsf{s}(x + y) & 4: & \mathsf{s}(x) \times y & \to & (x \times y) + y \end{array}$$

One can compute $(\mathsf{s}(x) + 0) \times 0$ by the following rewrite sequence:

$$\begin{aligned} (\mathsf{s}(x) + 0) \times 0 &\to_{\{1\}} \mathsf{s}(x) \times 0 \\ &\to_{\{3\}} 0 \end{aligned}$$

Here $\to_{\{i\}}$ denotes a rewrite step with the i-th rule. The first step rewrites the subterm $\mathsf{s}(x) + 0$ to the term $\mathsf{s}(x)$ by using the 1st rule, and the next step rewrites $\mathsf{s}(x) \times 0$ to 0 by using the 3rd rule. This rewrite sequence is not the only way to compute $(\mathsf{s}(x) + 0) \times 0$. For instance, the next rewrite sequence is another one:

$$\begin{aligned} (\mathsf{s}(x) + 0) \times 0 &\to_{\{2\}} \mathsf{s}(x + 0) \times 0 \\ &\to_{\{1\}} \mathsf{s}(x) \times 0 \\ &\to_{\{3\}} 0 \end{aligned}$$

We write $(\mathsf{s}(x) + 0) \times 0 \to^* 0$ if there exists a rewrite sequence from $(\mathsf{s}(x) + 0) \times 0$ to 0.

In rewriting there are two important properties. One is *termination* and the other is *confluence*. The former ensures finiteness of rewrite sequences, while the

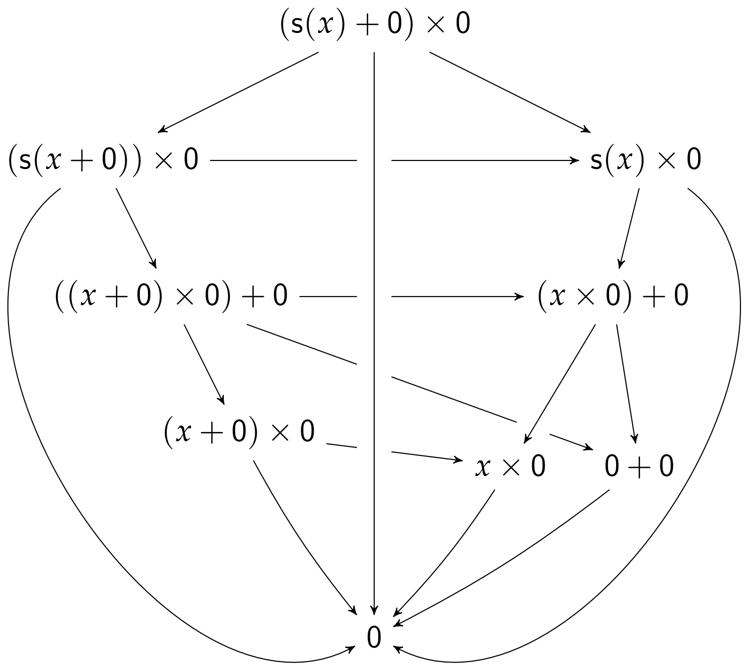
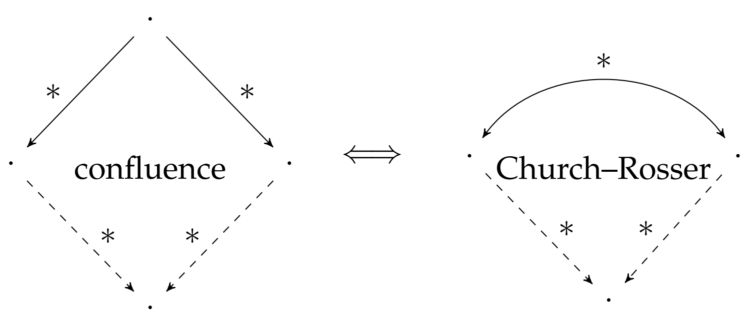


Figure 1.1: All possible rewrite sequences starting from $(\mathsf{s}(x) + 0) \times 0$.

latter guarantees uniqueness of computation results without relying on any specific computation strategy. The introductory example has the confluence property. In fact, any maximal rewrite sequence from $(\mathsf{s}(x) + 0) \times 0$ ends in 0 (see Fig. 1.1).

Rewriting is used for proving equality on terms. Given terms t and u, a question is whether $t \leftrightarrow^* u$ holds. Here $\leftrightarrow^*$ is the reflexive transitive closure of $\leftarrow \cup \rightarrow$. The question is decidable if the rewrite relation $\rightarrow$ is terminating and has *the Church–Rosser property (CR)*. The role of the property guarantees existence of a term that can be reached from both t and u by rewriting. This property is equivalent to confluence. In general, confluence is undecidable.



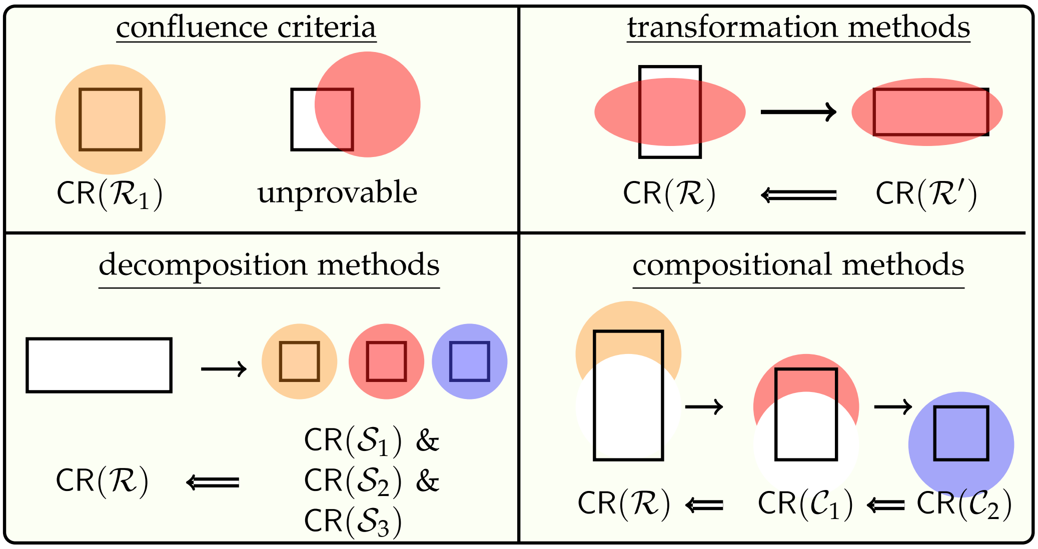


Figure 1.2: Classification of confluence analysis techniques.

1.2 Compositional Approach

Over the last half-century, confluence analysis has been investigated and many proof techniques for confluence analysis have been developed. We recall the existing three techniques: *confluence criteria*, *transformation methods*, and *decomposition methods*; see Fig. 1.2. In the figure, a circle means a confluence criterion, and a box means a rewrite system. If a circle covers a box then the corresponding confluence criterion can prove confluence of the corresponding system.

The most standard method for proving confluence of a system is applying *confluence criteria* to the system *directly*. Today various powerful confluence criteria are known. Many of them rely on one of two properties: termination [KB70] and orthogonality [Ros73]. The confluence of the introductory example can be shown by the criterion based on termination but not orthogonality.

A *transformation method* is a function on rewrite systems. It transforms a rewrite system $\mathcal{R}$ into another rewrite system $\mathcal{R}'$ whose confluence implies confluence of $\mathcal{R}$. The currying transformation by Kahrs [Kah95] is a typical instance of this method. Another instance is redundant rule elimination [SH15, NFM15].

In 1987, Toyama proposed modularity [Toy87]. This research originated a new confluence analysis based on *decomposition methods*. Given a rewrite system $\mathcal{R}$, a decomposition method splits $\mathcal{R}$ into its subsystems $\mathcal{R}_1, \ldots, \mathcal{R}_n$ such that $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_n$. Then under certain condition, $\mathcal{R}$ is confluent if and only if $\mathcal{R}_i$ is confluent for all $1 \leqslant i \leqslant n$. In this situation we can apply different confluence criteria to individual subsystems $\mathcal{R}_1, \ldots, \mathcal{R}_n$.

We give an example of a confluence proof based on a decomposition method.

Let's consider the following rewrite system.

$$\begin{array}{llllll} 1: & x + 0 \to x & 3: & \mathsf{if}(\mathsf{true}, x, y) \to x & 5: & x \wedge y \to y \wedge x \\ 2: & \mathsf{s}(x) + y \to \mathsf{s}(x + y) & 4: & \mathsf{if}(\mathsf{false}, x, y) \to y & 6: & x \vee y \to y \vee x \end{array}$$

This can be naturally separated into two subsystems: the subsystem for addition $\mathcal{R}_1 = \{1, 2\}$ and the subsystem for Boolean functions $\mathcal{R}_2 = \{3, 4, 5, 6\}$. Since these subsystems share no signature, this decomposition satisfies the condition of modularity. Therefore the confluence of $\mathcal{R}$ is equivalent to the confluence of $\mathcal{R}_1$ and $\mathcal{R}_2$. In this case, the confluence of $\mathcal{R}_1$ and $\mathcal{R}_2$ are shown by Knuth–Bendix criterion [KB70] and orthogonality [Ros73], respectively. Moreover, we can apply again the decomposition method to $\mathcal{R}_2$ because $\{3, 4\}$ and $\{5, 6\}$ share no signature. Note that neither of the confluence criteria can be applied to the original system $\mathcal{R}$ due to the lack of termination and orthogonality.

Confluence analysis based on decomposition methods is effective in various rewrite systems that are unions of independent subsystems. However, the method is hard to apply to a more realistic setting: one subsystem strongly depends on another. For instance, consider the introductory system:

$$\begin{array}{llll} 1: & x + 0 \to x & 3: & x \times 0 \to 0 \\ 2: & \mathsf{s}(x) + y \to \mathsf{s}(x + y) & 4: & \mathsf{s}(x) \times y \to (x \times y) + y \end{array}$$

Intuitively, the system can be separated into two subsystems: $\{1, 2\}$ for addition and $\{3, 4\}$ for multiplication. Unlike the last example, the definition of multiplication relies on that of addition. In fact, *hierarchical combination* [Ohl02] in the area of modularity guarantees that this hierarchical decomposition is valid (see [Ohl02, Corollary 8.6.38]). However, the decomposition method strongly depends on signatures. If we add new multiplication rule 5 which uses addition on the left-hand side then the method fails:

$$\begin{array}{llll} 1: & x + 0 \to x & 3: & x \times 0 \to 0 \\ 2: & \mathsf{s}(x) + y \to \mathsf{s}(x + y) & 4: & \mathsf{s}(x) \times y \to (x \times y) + y \\ & & 5: & (0 + y) \times z \to y \times z \end{array}$$

Our idea is to establish a confluence analysis that exploits confluence of subsystems.

1.3 Compositional Confluence Criteria

In this thesis we propose a new confluence analysis based on *compositional confluence criteria*. Given a rewrite system $\mathcal{R}$ and a subsystem $\mathcal{C}$ of $\mathcal{R}$, a compositional confluence criterion $Q(\mathcal{R}, \mathcal{C})$ means a sufficient condition for the following implication:

$$\mathcal{R} \text{ is confluent} \Longleftarrow \mathcal{C} \text{ is confluent}$$

In other words, compositional confluence criteria are confluence criteria that exploit confluence of subsystems. For example, by using the compositional criterion Theorem 4.2.4 (denoted by Q) the confluence of the above rewrite system can be reduced to that of the subsystem addition:

$$\{1,2,3,4,5\} \text{ is confluent} \Longleftarrow \{1,2\} \text{ is confluent} \qquad \text{by } Q(\{1,2,3,4,5\},\{1,2\})$$

Then the confluence of $\{1,2,3,4,5\}$ follows from the confluence of $\{1,2\}$.

This method can be composed to a (compositional) confluence criterion. Notably, confluence of a rewrite system $\mathcal{R}$ can be shown by successive application of compositional confluence criteria $Q_1, \ldots, Q_{n+1}$:

$$\begin{aligned}
\mathcal{R} \text{ is confluent} \Longleftarrow\ & \mathcal{C}_1 \text{ is confluent} && \text{by } Q_1(\mathcal{C}_0, \mathcal{C}_1) \\
\Longleftarrow\ & \mathcal{C}_2 \text{ is confluent} && \text{by } Q_2(\mathcal{C}_1, \mathcal{C}_2) \\
& \vdots && \\
\Longleftarrow\ & \mathcal{C}_n \text{ is confluent} && \text{by } Q_n(\mathcal{C}_{n-1}, \mathcal{C}_n) \\
\Longleftarrow\ & \varnothing \text{ is confluent} && \text{by } Q_{n+1}(\mathcal{C}_n, \varnothing)
\end{aligned}$$

Here $\mathcal{C}_0 = \mathcal{R}$ and each $\mathcal{C}_{i+1}$ is a proper subsystem of $\mathcal{C}_i$ (i.e., $C_{i+1} \supsetneq \mathcal{C}_i$). Because the empty system $\varnothing$ is trivially confluent, the implications indicate that $\mathcal{R}$ is confluent. For any compositional criterion Q, we can recast it as an ordinary confluence criterion by taking the empty subsystem, i.e., $Q(\mathcal{R}, \varnothing)$.

For developing compositional confluence criteria, we introduce a variant of *decreasing diagrams*. The decreasing diagram method [vO94, vO08] is a powerful confluence criterion for abstract rewriting. It is known that a number of confluence criteria [KB70, Hue80, Toy81, Toy88, Gra96, vO97, vO08, HM11, ZFM15] for (left-linear) rewrite systems can be shown by the method. Therefore, a compositional version of a decreasing diagram criterion is used as a uniform framework that derives compositional confluence criteria from existing confluence criteria based on decreasing diagrams.

This thesis studies three compositional confluence criteria that originate from orthogonality, rule labeling [vO08, ZFM15], and critical pair systems [HM11]. When deriving the compositional versions we exploit the notion of *parallel critical pairs*.

We demonstrate how compositional methods work. Consider the following rewrite system:

$$\begin{array}{llllll}
1: & 0 \times y \to 0 & 3: & \mathsf{s}(x) \times y \to (x \times y) + y & 5: & (x + y) + z \to x + (y + z) \\
2: & x \times 0 \to 0 & 4: & 0 + x \to x & 6: & x + (y + z) \to (x + y) + z
\end{array}$$

Let $\mathcal{C}_0 = \{1,2,3,4,5,6\}$. First, we take the subsystem $\mathcal{C}_1 = \{2,4,5,6\}$. The confluence of $\mathcal{R}$ can be shown from that of $\mathcal{C}_1$ by the compositional version of orthogonality (Theorem 4.2.4):

$$\mathcal{R} \text{ is confluent } \Longleftarrow \mathcal{C}_1 \text{ is confluent} \qquad \text{by Theorem 4.2.4 with } \mathcal{C}_0 \text{ and } \mathcal{C}_1$$

Second, in order to show the confluence of $\mathcal{C}_1$ we take the subsystem $\mathcal{C}_2 = \{4,5\}$. This is done by the compositional version of rule labeling (Theorem 4.3.4).

$$\mathcal{C}_1 \text{ is confluent } \Longleftarrow \mathcal{C}_2 \text{ is confluent} \qquad \text{by Theorem 4.3.4 with } \mathcal{C}_1 \text{ and } \mathcal{C}_2$$

Third, the confluence of $\mathcal{C}_2$ follows from the compositional version of critical pair systems (Theorem 4.4.6) with the empty subsystem $\varnothing$.

$$\mathcal{C}_2 \text{ is confluent } \Longleftarrow \varnothing \text{ is confluent} \qquad \text{by Theorem 4.4.6 with } \mathcal{C}_2 \text{ and } \varnothing$$

Because the empty system is confluent, the confluence of $\mathcal{R}$ is concluded.

1.4 Reduction Methods

In order to use a compositional confluence criterion, we need to find a suitable confluence subsystem from exponentially many subsystems. Therefore, confluence analysis based on successive application of compositional confluence criteria tend to suffer from a search space problem. If we naively implement it, search spaces grow super-exponentially with the size of an input system.

Given a rewrite system $\mathcal{R}$ and a subsystem $\mathcal{C}$ of $\mathcal{R}$, a reduction method $Q(\mathcal{R},\mathcal{C})$ means a sufficient condition for the following equivalence:

$$\mathcal{R} \text{ is confluent } \Longleftrightarrow \mathcal{C} \text{ is confluent}$$

This method reduces the confluence problem of $\mathcal{R}$ to that of the subsystem $\mathcal{C}$. Due to the equivalence, we do not need to verify confluence of $\mathcal{C}$ in advance. In other words, if $\mathcal{C}$ is not confluent then $\mathcal{R}$ is not confluent. Consider the following TRS $\mathcal{R}$:

$$\begin{array}{llllll}
1: & 0 \times y \to 0 & 3: & \mathsf{s}(x) \times y \to (x \times y) + y & 5: & (x + y) + z \to x + (y + z) \\
2: & x \times 0 \to 0 & 4: & 0 + x \to x & 6: & x + (y + z) \to (x + y) + z
\end{array}$$

Let $\mathcal{C}_0 = \mathcal{R}$, $\mathcal{C}_1 = \{1,2,4,5,6\}$, and $\mathcal{C}_2 = \{4,5,6\}$. We can reduce the confluence of $\mathcal{R}$ into that of the subsystem $\mathcal{C}_2$ by the reduction method (Corollary 5.1.4):

$$\begin{aligned}
\mathcal{R} \text{ is confluent} &\Longleftrightarrow \mathcal{C}_1 \text{ is confluent} && \text{by Corollary 5.1.4 with } \mathcal{C}_0 \text{ and } \mathcal{C}_1 \\
&\Longleftrightarrow \mathcal{C}_2 \text{ is confluent} && \text{by Corollary 5.1.4 with } \mathcal{C}_1 \text{ and } \mathcal{C}_2
\end{aligned}$$

Thus, the confluence of $\mathcal{R}$ and that of $\mathcal{C}_2$ are equivalent. Let $\mathcal{C}_3 = \{4, 5\}$. The confluence proof of $\mathcal{C}_2$ is similar to the proof presented in the last section.

$$
\begin{aligned}
\mathcal{C}_2 \text{ is confluent} &\Longleftarrow \mathcal{C}_3 \text{ is confluent} && \text{by Theorem 4.3.4 with } \mathcal{C}_2 \text{ and } \mathcal{C}_3 \\
&\Longleftarrow \varnothing \text{ is confluent} && \text{by Theorem 4.4.6 with } \mathcal{C}_3 \text{ and } \varnothing
\end{aligned}
$$

1.5 Contribution

Our contributions are summarized as follows:

- We propose a new confluence analysis based on compositional confluence criteria. In Section 4.1 we present a variant of decreasing diagrams for compositional confluence criteria. By using this, in the subsequent sections we exemplify this approach by deriving compositional confluence criteria from orthogonality [Ros73], rule labeling [vO08, ZFM15], and critical pair systems [HM11]. All of them subsume the original one. Through these section, we show how existing confluence criteria based on decreasing diagrams are generalized to a compositional form in a systematic way. In Chapter 6 we evaluate these criteria by our prototype confluence tool `Hakusan`.

- We present reduction methods for confluence analysis in Chapter 5. First we give a easy sufficient condition for reduction methods based on a signature extension result. This condition ensures the converse of compositional confluence criteria. Using the sufficient condition, we introduce two reduction methods. One is a simple reduction method (Corollary 5.1.4) based on the compositional version of orthogonality introduced from Section 4.2. This method can be regarded as a transformation method. In Chapter 7 we show that this generalizes a transformation method [NFM15] for left-linear rewrite systems. The other reduction method (Corollary 5.1.10) is based on a compositional criterion for non-left-linear TRSs [KH12].

  For automation, we demonstrate how the former reduction method is encoded into linear arithmetic constraints.

- In Chapter 3 we elucidate the hierarchy of Huet's parallel closedness and its variants by showing that Toyama's almost parallel closedness [Toy88] is subsumed by his earlier result based on parallel critical pairs [Toy81]. In summary, the result dated 1981 is more powerful than the more recent results dated in 1988–1996. It is worth noting that Toyama's earlier result is subsumed by rule labeling based on parallel critical pairs (see Section 4.3).

The thesis is based on the papers [SH22, SH]. The research results in Chapters 3 and 4 are taken from [SH22], and the results in Chapter 5 are from [SH].

Chapter 2

Preliminaries

In this chapter we introduce notions and notations for term rewriting. We assume familiarity with set theory. More details about term rewriting are referred to books [BN98, Ohl02, Ter03].

2.1 Abstract Rewriting

Binary relation is a fundamental notion for term rewriting. This notion captures rewrite steps. A pair of a set and a binary relation on the set is called an *abstract rewrite system*, and term rewrite systems are considered as their instances. In this section we introduce properties for abstract rewrite systems, including the confluence property.

We start with binary relations. For emphasizing the direction of a binary relation, we denote it by an arrow notation $\to$.

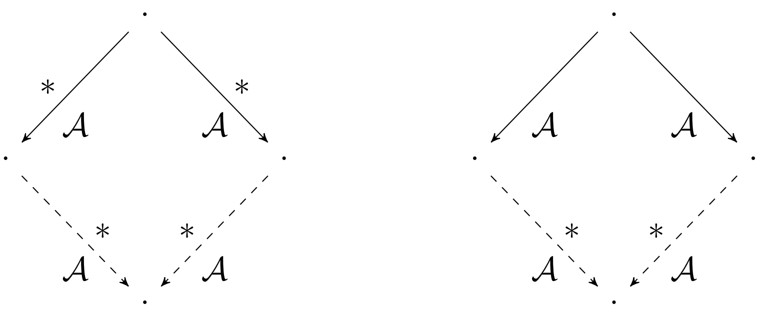
Definition 2.1.1. *Let $\to_1$ and $\to_2$ be binary relations on a set A. The* composition *$\to_1 \cdot \to_2$ of $\to_1$ and $\to_2$ is the binary relation $\{(x,z) \mid x \to_1 y \text{ and } y \to_2 z\}$. Let $\to$ be a relation on A and n a natural number. We inductively define the n*-step relation *$\to^n$ as follows:*

$$\to^n = \begin{cases} id_A & \text{if } n = 0 \\ \to \cdot \to^{n-1} & \text{otherwise} \end{cases}$$

Here id_A is the identity *relation on A, defined as $\{(x,x) \mid x \in A\}$.*

Definition 2.1.2. *We define closure operators for relations. Let $\to$ be a binary relation on a set A.*

- *The* reflexive closure *$\to^=$ of $\to$ is defined as the union of $\to$ and $\to^0$.*
- *The* transitive closure *$\to^+$ of $\to$ is defined as the union of $\to^i$ for all $i \geqslant 1$.*
- *The* reflexive transitive closure *$\to^*$ of $\to$ is defined as the union of $\to^0$ and $\to^+$.*
- *The* inverse *$\leftarrow$ of $\to$ is defined as the relation $\{(y,x) \mid x \to y\}$.*



confluence local confluence

Figure 2.1: Confluence on ARSs

- *The* symmetric closure $\leftrightarrow$ *of* $\rightarrow$ *is defined as the union of* $\rightarrow$ *and* $\leftarrow$.

Note that the reflexive transitive closure $\leftrightarrow^*$ of a symmetric relation $\leftrightarrow$ is an equivalence relation.

Definition 2.1.3. *Let I be a set. An (I-indexed)* abstract rewrite system *(ARS) $\mathcal{A}$ is a pair $(A, \{\rightarrow_\alpha\}_{\alpha \in I})$ consisting of a set A and a family of relations $\rightarrow_\alpha$ on A for all $\alpha \in I$.*

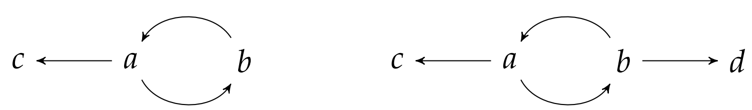
Given a subset J of I, we write $x \rightarrow_J y$ if $x \rightarrow_\alpha y$ for some index $\alpha \in J$. The relation $\rightarrow_I$ is referred to as $\rightarrow_\mathcal{A}$. Let $\mathcal{A}$ and $\mathcal{B}$ be ARSs. A conversion of form $b \;{}_\mathcal{A}\!\!\leftarrow a \rightarrow_\mathcal{B} c$ is called a *local peak* (or simply *peak*) between $\mathcal{A}$ and $\mathcal{B}$. We say that a peak $b \;{}_\mathcal{A}\!\!\leftarrow a \rightarrow_\mathcal{B} c$ is *joinable* if $b \rightarrow_\mathcal{B}^* \cdot {}^*_\mathcal{A}\!\!\leftarrow c$ holds. Then the confluence property is formalized as follows.

Definition 2.1.4. *An ARS $\mathcal{A}$ is* confluent *if ${}^*_\mathcal{A}\!\!\leftarrow \cdot \rightarrow^*_\mathcal{A} \subseteq \rightarrow^*_\mathcal{A} \cdot {}^*_\mathcal{A}\!\!\leftarrow$ holds. It is* locally confluent *if ${}_\mathcal{A}\!\!\leftarrow \cdot \rightarrow_\mathcal{A} \subseteq \rightarrow^*_\mathcal{A} \cdot {}^*_\mathcal{A}\!\!\leftarrow$ holds.*

Example 2.1.5. *Let A be the set $\{a, b, c, d\}$. Consider the two ARSs $\mathcal{A}_1$ and $\mathcal{A}_2$:*

$$\mathcal{A}_1 = (A, \{(a, b), (a, c), (b, a)\})$$
$$\mathcal{A}_2 = (A, \{(a, b), (a, c), (b, a), (b, d)\}).$$

The ARS $\mathcal{A}_1$ is confluent because $x \rightarrow^*_{\mathcal{A}_1} c$ for all $x \in A$. Hence every x and y with $x \;{}^*_{\mathcal{A}_1}\!\!\leftarrow \cdot \rightarrow^*_{\mathcal{A}_1} y$ satisfy $x \rightarrow^*_{\mathcal{A}_1} c \;{}^*_{\mathcal{A}_1}\!\!\leftarrow y$. However, in the case of $\mathcal{A}_2$ there is no element z such that $c \rightarrow^*_{\mathcal{A}_2} z \;{}^*_{\mathcal{A}_2}\!\!\leftarrow d$. Thus the ARS $\mathcal{A}_2$ does not have the confluent property.



(a) The confluent ARS $\mathcal{A}_1$ *(b) The non-confluent ARS $\mathcal{A}_2$*

The confluence property is not decidable but many sufficient conditions for confluence are known. In the remaining part of this section, we introduce two notions *termination* and *commutation*, and indicate confluence criteria based on them.

Termination is another fundamental property for abstract rewriting. It represents finiteness of rewrite steps. A binary relation $\to$ is *terminating* if there exists no infinite sequence $a_0 \to a_1 \to \cdots$.

Definition 2.1.6. *An ARS $\mathcal{A}$ is* terminating *if $\to_{\mathcal{A}}$ is terminating.*

By using termination, confluence can be characterized by local confluence. The next confluence criterion is known as Newman's Lemma.

Theorem 2.1.7 ([New42]). *A terminating ARS is confluent if and only if it is locally confluent.*

This notion can be generalized to a *relative termination* on two ARSs. We define the relative rewrite step $\to_{\mathcal{A}/\mathcal{B}}$ as $\to_{\mathcal{B}}^* \cdot \to_{\mathcal{A}} \cdot \to_{\mathcal{B}}^*$ for ARSs $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.1.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be ARSs. We say that $\mathcal{A}$ is* relatively terminating *with respect to $\mathcal{B}$, or simply $\mathcal{A}/\mathcal{B}$ is* terminating, *if $\to_{\mathcal{A}/\mathcal{B}}$ is terminating.*

Note that if $\to_{\mathcal{B}} \subseteq \to_{\mathcal{A}}^*$ holds then $\to_{\mathcal{A}/\mathcal{B}}^* = \to_{\mathcal{A}}^*$ holds. In the case, termination of $\mathcal{A}$ and that of $\mathcal{A}/\mathcal{B}$ coincide. Therefore relative termination is a generalization of termination. Moreover if $\to_{\mathcal{A}} = \varnothing$ then termination of $\mathcal{A}/\mathcal{B}$ is trivial because $\to_{\mathcal{A}/\mathcal{B}} = \varnothing$. In Section 4.4 we discuss confluence criteria based on relative termination.

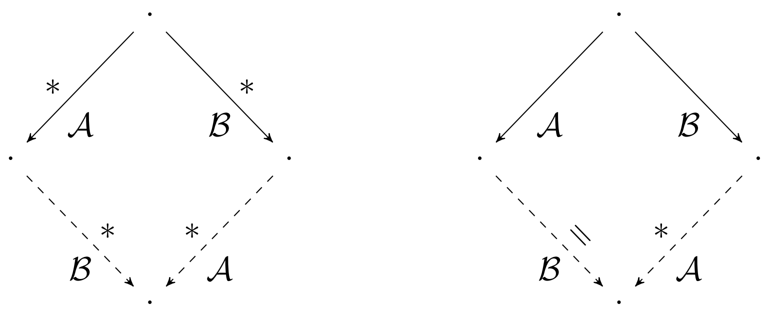
The next property is *commutation*. Commutation is a property that generalizes confluence. While confluence is a property on an ARS, commutation is a property on two ARSs. This property distinguishes the direction of rewriting.

Definition 2.1.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be ARSs.*

- *We say that $\mathcal{A}$ and $\mathcal{B}$* commute *if ${}_{\mathcal{A}}^{*}\!\leftarrow \cdot \to_{\mathcal{B}}^* \subseteq \to_{\mathcal{B}}^* \cdot {}_{\mathcal{A}}^{*}\!\leftarrow$.*
- *We say that $\mathcal{A}$* strongly commutes *with $\mathcal{B}$ if ${}_{\mathcal{A}}\!\leftarrow \cdot \to_{\mathcal{B}} \subseteq \to_{\mathcal{B}}^= \cdot {}_{\mathcal{A}}^{*}\!\leftarrow$.*
- *We say that $\mathcal{A}$ and $\mathcal{B}$* locally commute *if ${}_{\mathcal{A}}\!\leftarrow \cdot \to_{\mathcal{B}} \subseteq \to_{\mathcal{B}}^* \cdot {}_{\mathcal{A}}^{*}\!\leftarrow$.*

It is easy to see that an ARS $\mathcal{A}$ is confluent if and only if it is *self-commuting*, that is, $\mathcal{A}$ and $\mathcal{A}$ commute. This suggests confluence analysis via commutation. The next commutation criterion is referred in Section 3.2.

Theorem 2.1.10 ([Hin64]). *ARSs $\mathcal{A}$ and $\mathcal{B}$ commute if $\mathcal{A}$ strongly commutes with $\mathcal{B}$.*



commutation   strong commutation

Figure 2.2: Commutation on ARSs

2.2 Terms and Substitutions

Term rewriting is abstract rewriting whose elements are *terms* and a relation is the *rewrite relation* induced from *rewrite rules*. In this section, we introduce notions and notations on terms. Others are presented in the next section.

Terms consist of *function symbols* and *variables*. We use symbols $\mathcal{F}$ and $\mathcal{V}$ for denoting sets of function symbols and variables, respectively.

Definition 2.2.1. *A* signature *$\mathcal{F}$ is a set of function symbols, where each function symbol f is associated with non-negative integer n, the* arity *of f. We write $f^{(n)}$ when we need to indicate the arity of f. If $c \in \mathcal{F}$ has 0-arity, we call c a* constant.

Definition 2.2.2. *Let $\mathcal{F}$ be a signature and $\mathcal{V}$ a countable set of variables such that $\mathcal{F} \cap \mathcal{V} = \varnothing$. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of* terms *(over $\mathcal{F}$) is defined as follows:*

- *$x \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ if $x \in \mathcal{V}$, and*
- *$f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ if $f^{(n)} \in \mathcal{F}$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.*

We usually use letters $s, t, u, \ldots$ for terms and $x, y, z, \ldots$ for variables.

Throughout this thesis, we fix a countable set $\mathcal{V}$ of variables. We introduce the notion *position*, a sequence of positive integers, to represent locations of *subterms* on terms. The empty sequence ϵ is called the *root* position.

Definition 2.2.3. *Let p and q be positions.*

- *$p \cdot q$ (or simply pq) is the* concatenation *of p and q.*
- *$p \leqslant q$ if $p \cdot p' = q$ for some position p'.*
- *p and q are* parallel *if $p \nleqslant q$ and $q \nleqslant p$. A set of positions is called* parallel *if all pairs of distinct positions in the set are so.*

Definition 2.2.4. *Let t be a term.*

- *The set of all variables and the set of all function symbols in t are denoted by $\mathcal{V}\mathsf{ar}(t)$ and $\mathcal{F}\mathsf{un}(t)$, respectively.*
- *The set $\mathcal{P}\mathsf{os}(t)$ of all positions in t is defined as follows:*

$$\mathcal{P}\mathsf{os}(t) = \begin{cases} \{\epsilon\} & \text{if } t \in \mathcal{V} \\ \{\epsilon\} \cup \{ip \mid 1 \leqslant i \leqslant n \text{ and } p \in \mathcal{P}\mathsf{os}(t_i)\} & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- *The set $\mathcal{P}\mathsf{os}_{\mathcal{F}}(t)$ of all function positions and the set $\mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$ of variable positions in t are defined as follows:*

$$\mathcal{P}\mathsf{os}_{\mathcal{F}}(t) = \begin{cases} \varnothing & \text{if } t \in \mathcal{V} \\ \{\epsilon\} \cup \{ip \mid 1 \leqslant i \leqslant n \text{ and } p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(t_i)\} & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

$$\mathcal{P}\mathsf{os}_{\mathcal{V}}(t) = \mathcal{P}\mathsf{os}(t) \setminus \mathcal{P}\mathsf{os}_{\mathcal{F}}(t)$$

- *The* subterm *$t|_p$ of t at position p is defined as follows:*

$$t|_p = \begin{cases} t & \text{if } p = \epsilon \\ t_i|_{p'} & \text{if } p = i \cdot p' \text{ and } t = f(t_1, \ldots, t_i, \ldots, t_n) \end{cases}$$

  We say that s is a subterm *of t if there exists a position p such that $s = t|_p$. A subterm $t|_p$ of t is called a* proper *subterm if $p \neq \epsilon$.*

- *The term $t[u]_p$ results from replacing the subterm of t at p by a term u, defined by*

$$t[u]_p = \begin{cases} u & \text{if } p = \epsilon \\ f(t_1, \ldots, t_i[u]_{p'}, \ldots, t_n) & \text{if } p = i \cdot p' \text{ and } t = f(t_1, \ldots, t_i, \ldots, t_n) \end{cases}$$

- *The* size *of t, denoted by $|t|$, is the number of occurrences of functions symbols and variables in t.*
- *The term t is* linear *if every variable in t occurs exactly once.*

In order to compare terms, we introduce notions of substitution and unification on terms.

Definition 2.2.5. *A* substitution *is a mapping $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ whose* domain *$\mathcal{D}\mathsf{om}(\sigma)$ is finite. Here $\mathcal{D}\mathsf{om}(\sigma)$ stands for the set $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$. Let t be a term. The application $t\sigma$ is defined as follows:*

$$t\sigma = \begin{cases} \sigma(t) & \text{if } t \in \mathcal{V} \\ f(t_1\sigma, \ldots, t_n\sigma) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

We write $\sigma\tau$ for the composition *of substitutions σ and τ, defined by $t(\sigma\tau) = (t\sigma)\tau$ for all terms t.*

We say that a term u is an *instance* of a term t if $u = t\sigma$ for some σ. A substitution is called a *renaming* if it is a bijection on variables.

Definition 2.2.6. *A* unifier *σ of an equality $s \approx t$ is a substitution such that $s\sigma = t\sigma$. A unifier σ of s and t is a* most general unifier *if for every unifier σ' of s and t there is some substitution τ such that $\sigma\tau = \sigma'$.*

2.3 Term Rewrite Systems

Rewrite rules are pairs of terms, and they represent transformation steps for rewriting. A set of rewrite rules, a *term rewrite system*, establishes these transformation steps. The relation induced from a rewrite system is called a *rewrite relation*.

Definition 2.3.1. *A* term rewrite system *(TRS) over a signature $\mathcal{F}$ is a set of rewrite rules. Here a pair (ℓ, r) of terms over $\mathcal{F}$ is a* rewrite rule *(or simply a* rule*) if $\ell \notin \mathcal{V}$ and $\mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(\ell)$. We write $\ell \to r$ instead of (ℓ, r).*

We call subsets of $\mathcal{R}$ *subsystems*. We use symbols $\mathcal{R}, \mathcal{S}, \ldots$ to denote TRSs.

Definition 2.3.2. *The* rewrite relation *$\to_{\mathcal{R}}$ of a TRS $\mathcal{R}$ is defined on terms as follows: $s \to_{\mathcal{R}} t$ if $s|_p = \ell\sigma$ and $t = s[r\sigma]_p$ for some rule $\ell \to r \in \mathcal{R}$, position p, and substitution σ. We write $s \xrightarrow{p}_{\mathcal{R}} t$ if the rewrite position p is relevant.*

We write $\mathcal{F}\mathsf{un}(\ell \to r)$ for $\mathcal{F}\mathsf{un}(\ell) \cup \mathcal{F}\mathsf{un}(r)$, and $\mathcal{F}\mathsf{un}(\mathcal{R})$ for the union of $\mathcal{F}\mathsf{un}(\ell \to r)$ for all rules $\ell \to r \in \mathcal{R}$. The set $\{f \mid f(\ell_1, \ldots, \ell_n) \to r \in \mathcal{R}\}$ is the set of *defined symbols* and denoted by $\mathcal{D}_{\mathcal{R}}$. A TRS $\mathcal{R}$ is *left-linear* if ℓ is linear for all $\ell \to r \in \mathcal{R}$.

Example 2.3.3. *Consider the next TRS $\mathcal{R}$ over the signature $\{\mathsf{a}^{(1)}, \mathsf{b}^{(1)}, \mathsf{f}^{(2)}, \mathsf{g}^{(1)}\}$:*

$$\mathsf{a}(x) \to \mathsf{b}(x) \qquad\qquad \mathsf{f}(x, y) \to \mathsf{g}(\mathsf{f}(x, y))$$

The term $\mathsf{f}(\mathsf{a}(x), \mathsf{a}(y))$ can be rewritten to three different terms by $\to_{\mathcal{R}}$ as follows.

- *$\mathsf{f}(\mathsf{a}(x), \mathsf{a}(y)) \to_{\mathcal{R}} \mathsf{g}(\mathsf{f}(\mathsf{a}(x), \mathsf{a}(y)))$ is obtained by the second rule with the root position ϵ, and the substitution σ such that $\sigma(x) = \mathsf{a}(x)$ and $\sigma(y) = \mathsf{a}(y)$.*
- *$\mathsf{f}(\mathsf{a}(x), \mathsf{a}(y)) \to_{\mathcal{R}} \mathsf{f}(\mathsf{b}(x), \mathsf{a}(y))$ is obtained by the first rule with the position* 1*, and the identity substitution.*
- *$\mathsf{f}(\mathsf{a}(x), \mathsf{a}(y)) \to_{\mathcal{R}} \mathsf{f}(\mathsf{a}(x), \mathsf{b}(y))$ is obtained by the first rule with the position* 2 *and the substitution σ such that $\sigma(x) = y$.*

Definition 2.3.4. *Let $\mathcal{F}$ be a signature and $\mathcal{R}$ a TRS. The ARS* induced from *$\mathcal{R}$ is defined by $(\mathcal{T}(\mathcal{F}, \mathcal{V}), \{\to_{\mathcal{R}}\})$.*

We use notions and notations of ARSs for TRSs. For instance, a TRS $\mathcal{R}$ over $\mathcal{F}$ is (locally) confluent if the ARS $(\mathcal{T}(\mathcal{F},\mathcal{V}),\{\to_{\mathcal{R}}\})$ is so. Similarly, two TRSs commute if their corresponding ARSs commute. When $\mathcal{F} = \mathcal{F}\mathsf{un}(\mathcal{R})$, we omit the signature $\mathcal{F}$ from statements.

Local confluence of TRSs is characterized by the notion of *critical pair*. We say that a rule $\ell_1 \to r_1$ is a *variant* of a rule $\ell_2 \to r_2$ if $\ell_1\rho = \ell_2$ and $r_1\rho = r_2$ for some renaming ρ.

Definition 2.3.5. *Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs. Suppose that the following conditions hold:*

- *$\ell_1 \to r_1$ and $\ell_2 \to r_2$ are variants of rules in $\mathcal{R}$ and in $\mathcal{S}$, respectively,*
- *$\ell_1 \to r_1$ and $\ell_2 \to r_2$ have no common variables,*
- *$p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$,*
- *σ is a most general unifier of ℓ_1 and $\ell_2|_p$, and*
- *if $p = \epsilon$ then $\ell_1 \to r_1$ is not a variant of $\ell_2 \to r_2$.*

The triple $(\ell_1 \to r_1, p, \ell_2 \to r_2)$ is called an overlap *between $\mathcal{R}$ and $\mathcal{S}$, and the local peak $(\ell_2\sigma)[r_1\sigma]_p \;{}_{\mathcal{R}}\!\xleftarrow{p} \ell_2\sigma \xrightarrow{\epsilon}_{\mathcal{S}} r_2\sigma$ is called a* critical peak *between $\mathcal{R}$ and $\mathcal{S}$. When $t \;{}_{\mathcal{R}}\!\xleftarrow{p} s \xrightarrow{\epsilon}_{\mathcal{S}} u$ is a critical peak, the pair (t,u) is called a* critical pair*. To clarify the orientation of the pair, we denote it as the binary relation $t \;{}_{\mathcal{R}}\!\xleftarrow{p}\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{S}} u$, see [Der05]. Moreover, we write $t \;{}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{S}} u$ if $t \;{}_{\mathcal{R}}\!\xleftarrow{p}\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{S}} u$ for some position p.*

Theorem 2.3.6 ([Hue80])**.** *A TRS $\mathcal{R}$ is locally confluent if and only if the inclusion ${}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \to_{\mathcal{R}}^{*} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow$ holds.*

Combining it with Newman's Lemma [New42], we obtain Knuth and Bendix' criterion [KB70].

Theorem 2.3.7 ([KB70])**.** *A terminating TRS $\mathcal{R}$ is confluent if and only if the inclusion ${}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \to_{\mathcal{R}}^{*} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow$ holds.*

We define the parallel step relation, which plays a key role in analysis of local peaks.

Definition 2.3.8. *Let $\mathcal{R}$ be a TRS and let P be a set of parallel positions. The* parallel step *$\xrightarrow{P}\!\!\!\!\!\!\!\!\;\,\|\;\;_{\mathcal{R}}$ is inductively defined on terms as follows:*

- *$x \xrightarrow{P}\!\!\!\!\!\!\!\!\;\,\|\;\;_{\mathcal{R}} x$ if x is a variable and $P = \varnothing$.*
- *$\ell\sigma \xrightarrow{P}\!\!\!\!\!\!\!\!\;\,\|\;\;_{\mathcal{R}} r\sigma$ if $\ell \to r$ is an $\mathcal{R}$-rule, σ is a substitution, and $P = \{\epsilon\}$.*
- *$f(s_1,\ldots,s_n) \xrightarrow{P}\!\!\!\!\!\!\!\!\;\,\|\;\;_{\mathcal{R}} f(t_1,\ldots,t_n)$ if f is an n-ary function symbol in $\mathcal{F}$, $s_i \xrightarrow{P_i}\!\!\!\!\!\!\!\!\;\,\|\;\;_{\mathcal{R}} t_i$ for all $1 \leqslant i \leqslant n$, and $P = \{i \cdot p \mid 1 \leqslant i \leqslant n \text{ and } p \in P_i\}$.*

We write $s \mathrel{\shortparallel\!\!\!\!\!\longrightarrow}_{\mathcal{R}} t$ if $s \xrightarrow{P}\!\!\!\!\!\!\!\shortparallel\;\;_{\mathcal{R}} t$ for some set P of parallel positions.

Note that $\mathrel{\shortparallel\!\!\!\!\!\longrightarrow}_{\mathcal{R}}$ is a reflexive relation and $\to_{\mathcal{R}}^{=} \subseteq \mathrel{\shortparallel\!\!\!\!\!\longrightarrow}_{\mathcal{R}} \subseteq \to_{\mathcal{R}}^{*}$ hold. As the latter entails $\to_{\mathcal{R}}^{*} = \mathrel{\shortparallel\!\!\!\!\!\longrightarrow}_{\mathcal{R}}^{*}$, we obtain the following useful characterizations.

Lemma 2.3.9. *A TRS $\mathcal{R}$ is confluent if and only if $\mathrel{\shortparallel\!\!\!\!\!\longrightarrow}_{\mathcal{R}}$ is confluent. Similarly, TRSs $\mathcal{R}$ and $\mathcal{S}$ commute if and only if $\mathrel{\shortparallel\!\!\!\!\!\longrightarrow}_{\mathcal{R}}$ and $\mathrel{\shortparallel\!\!\!\!\!\longrightarrow}_{\mathcal{S}}$ commute.*

Example 2.3.10. *In order to demonstrate parallel steps, we consider the following TRS $\mathcal{R}$ over the signature $\{\mathsf{a}^{(1)}, \mathsf{b}^{(1)}, \mathsf{f}^{(2)}, \mathsf{g}^{(1)}\}$:*

$$\mathsf{a}(x) \to \mathsf{b}(x) \qquad\qquad \mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) \to \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)))$$

By the rewrite step $\to_{\mathcal{R}}$, the term $\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y))$ is rewritten to three different terms $\mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)))$, $\mathsf{f}(\mathsf{b}(x),\mathsf{a}(y))$, and $\mathsf{f}(\mathsf{a}(x),\mathsf{b}(y))$:

$$\begin{tikzcd}
 & \mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) \arrow[dl, "\epsilon"'] \arrow[d, "1"'] \arrow[dr, "2"] & \\
\mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y))) & \mathsf{f}(\mathsf{b}(x),\mathsf{a}(y)) & \mathsf{f}(\mathsf{a}(x),\mathsf{b}(y))
\end{tikzcd}$$

In the case of the parallel step $\mathrel{\shortparallel\!\!\!\!\!\longrightarrow}_{\mathcal{R}}$, we additionally have the term $\mathsf{f}(\mathsf{b}(x),\mathsf{b}(y))$ by rewriting subterms $\mathsf{a}(x)$ and $\mathsf{a}(y)$ at the same time. This step is valid because positions 1 and 2 are parallel.

$$\begin{tikzcd}
 & & \mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) \arrow[dll, "\{\epsilon\}"'] \arrow[dl, "\{1\}"'] \arrow[dr, "\{2\}"] \arrow[drr, "\{1,2\}"] & & \\
\mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y))) & \mathsf{f}(\mathsf{b}(x),\mathsf{a}(y)) & & \mathsf{f}(\mathsf{a}(x),\mathsf{b}(y)) & \mathsf{f}(\mathsf{b}(x),\mathsf{b}(y))
\end{tikzcd}$$

Moreover we can obtain identical step $\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) \xrightarrow{\varnothing}\!\!\!\!\!\!\!\shortparallel\;\;\; \mathsf{f}(\mathsf{a}(x),\mathsf{a}(y))$. Note that the next step is invalid as a parallel step because $\{\epsilon, 1, 2\}$ is not a set of parallel positions.

$$\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) \xrightarrow{\{\epsilon,1,2\}}\!\!\!\!\!\!\!\!\!\shortparallel\;\;\;\;\; \mathsf{g}(\mathsf{f}(\mathsf{b}(x),\mathsf{b}(y)))$$

Chapter 3

Parallel Closedness

In this chapter we discuss relationships among confluence criteria: *parallel closedness* and its variants. The original criterion was introduced by Huet [Hue80], and it tests confluence by relating critical pairs to parallel steps. Today three variants of parallel closedness are known: almost parallel closedness [Toy88] , Gramlich's criterion [Gra96], and Toyama's criterion [Toy81]. The last two criteria are based on *parallel critical pairs*, which capture parallel local peaks. The subsumption relationships between three criteria are known except the one between the first and the last criteria.

In Section 3.1 we recall four existing parallel closedness. For the sake of self-containedness we give a proof for Toyama's criterion in Section 3.2. In Section 3.3 we show that Toyama's criterion subsumes others.

3.1 Parallel Closedness and Variants

Toyama made two variations of Huet's parallel closedness theorem [Hue80] in 1981 [Toy81] and in 1988 [Toy88], but their relation has not been known. In this section we recall his and related results, and then show that Toyama's earlier result subsumes the later one. For brevity we omit the subscript $\mathcal{R}$ from $\to_\mathcal{R}$, $\mathrel{\shortparallel\!\!\!\!\to}_\mathcal{R}$, and ${}_\mathcal{R}{\leftarrow}\rtimes\xrightarrow{\epsilon}_\mathcal{R}$ when it is clear from the contexts.

Definition 3.1.1 ([Hue80])**.** *A TRS is* parallel closed *if* ${\leftarrow}\rtimes\xrightarrow{\epsilon} \subseteq \mathrel{\shortparallel\!\!\!\!\to}$ *holds.*

Theorem 3.1.2 ([Hue80])**.** *A left-linear TRS is confluent if it is parallel closed.*

In 1988, Toyama showed that the closing form for *overlay* critical pairs, originating from root overlaps, can be relaxed. We write $t \xleftarrow{>\epsilon}\rtimes\xrightarrow{\epsilon} u$ if $t \xleftarrow{p}\rtimes\xrightarrow{\epsilon} u$ holds for some $p > \epsilon$.

Definition 3.1.3 ([Toy88])**.** *A TRS is* almost parallel closed *if* $\xleftarrow{\epsilon}\rtimes\xrightarrow{\epsilon} \subseteq \mathrel{\shortparallel\!\!\!\!\to} \cdot {}^{*}\!\!\leftarrow$ *and* $\xleftarrow{>\epsilon}\rtimes\xrightarrow{\epsilon} \subseteq \mathrel{\shortparallel\!\!\!\!\to}$ *hold.*

Theorem 3.1.4 ([Toy88])**.** *A left-linear TRS is confluent if it is almost parallel closed.*

Example 3.1.5. *Consider the following left-linear and non-terminating TRS, which is a variant of the TRS in [Gra96, Example 5.4].*

$$\begin{array}{rlrl} \mathsf{a}(x) & \to \mathsf{b}(x) & \mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) & \to \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y))) \\ \mathsf{f}(\mathsf{b}(x),y) & \to \mathsf{g}(\mathsf{f}(\mathsf{a}(x),y)) & \mathsf{f}(x,\mathsf{b}(y)) & \to \mathsf{g}(\mathsf{f}(x,\mathsf{a}(y))) \end{array}$$

Out of the three critical pairs, two critical pairs including the next diagram (i) are closed by single parallel steps. The remaining pair (ii) joins by performing a single parallel step on each side:

$$\begin{tikzcd} \mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) \arrow[r, "\epsilon"] \arrow[d, "1"'] & \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y))) \\ \mathsf{f}(\mathsf{b}(x),\mathsf{a}(y)) \arrow[ur, dashed, bend right, "\parallel" marking] & \end{tikzcd}$$

(i)

$$\begin{tikzcd} \mathsf{f}(\mathsf{b}(x),\mathsf{b}(y)) \arrow[r, "\epsilon"] \arrow[d, "\epsilon"'] & \mathsf{g}(\mathsf{f}(\mathsf{b}(x),\mathsf{a}(y))) \arrow[d, dashed, "\parallel" marking] \\ \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{b}(y))) \arrow[r, "\parallel" marking] & \mathsf{g}(\mathsf{f}(\mathsf{b}(x),\mathsf{b}(y))) \end{tikzcd}$$

(ii)

Thus, the TRS is almost parallel closed. Hence, the TRS is confluent.

Inspired by almost parallel closedness, Gramlich [Gra96] developed a confluence criterion based on *parallel critical pairs* in 1996. Let t be a term and let P be a set of parallel positions in t. We write $\mathcal{V}\mathsf{ar}(t,P)$ for the union of $\mathcal{V}\mathsf{ar}(t|_p)$ for all $p \in P$. By $t[u_p]_{p\in P}$ we denote the term that results from replacing in t the subterm at p by a term u_p for all $p \in P$.

Definition 3.1.6. *Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs, $\ell \to r$ a variant of an $\mathcal{S}$-rule, and $\{\ell_p \to r_p\}_{p\in P}$ a family of variants of $\mathcal{R}$-rules, where P is a set of positions. A local peak*

$$(\ell\sigma)[r_p\sigma]_{p\in P}\ {}_{\mathcal{R}}\mathrel{\leftarrow\!\!\!\!\|\,}\ \ell\sigma \xrightarrow{\epsilon}_{\mathcal{S}} r\sigma$$

is called a parallel critical peak *between $\mathcal{R}$ and $\mathcal{S}$ if the following conditions hold:*

- *$P \subseteq \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell)$ is a non-empty set of parallel positions in ℓ,*
- *none of rules $\ell \to r$ and $\ell_p \to r_p$ for $p \in P$ shares a variable with other rules,*
- *σ is a most general unifier of $\{\ell_p \approx (\ell|_p)\}_{p\in P}$, and*
- *if $P = \{\epsilon\}$ then $\ell_\epsilon \to r_\epsilon$ is not a variant of $\ell \to r$.*

When $t\ {}_{\mathcal{R}}\overset{P}{\mathrel{\leftarrow\!\!\!\!\|\,}}\ s \xrightarrow{\epsilon}_{\mathcal{S}} u$ is a parallel critical peak, the pair (t,u) is called a parallel critical pair, *and denoted by $t\ {}_{\mathcal{R}}\overset{P}{\mathrel{\leftarrow\!\!\!\!\|\,}}\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{S}} u$. In the case of $P \nsubseteq \{\epsilon\}$ the parallel critical pair is written as $t\ {}_{\mathcal{R}}\overset{>\epsilon}{\mathrel{\leftarrow\!\!\!\!\|\,}}\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{S}} u$. Whenever no confusion arises, we abbreviate ${}_{\mathcal{R}}\mathrel{\leftarrow\!\!\!\!\|\,}\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{R}}$ to $\mathrel{\leftarrow\!\!\!\!\|\,}\!\rtimes\!\xrightarrow{\epsilon}$.*

Consider a local peak $t\ {}_{\mathcal{R}}\!\overset{P}{\leftarrow\!\!\!\!+\!\!\!\!+}\ s \xrightarrow{\epsilon}_{\mathcal{S}} u$ that employs a rule $\ell_p \to r_p$ at $p \in P$ in the left step and a rule $\ell \to r$ in the right step. We say that the peak is *orthogonal* if either $P \cap \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell) = \varnothing$, or $P = \{\epsilon\}$ and $\ell_\epsilon \to r_\epsilon$ is a variant of $\ell \to r$.[1] A local peak $t\ {}_{\mathcal{R}}\!\overset{p}{\leftarrow}\ s \xrightarrow{\epsilon}_{\mathcal{S}} u$ is *orthogonal* if $t\ {}_{\mathcal{R}}\!\overset{\{p\}}{\leftarrow\!\!\!\!+\!\!\!\!+}\ s \xrightarrow{\epsilon}_{\mathcal{S}} u$ is.

Theorem 3.1.7 ([Gra96])**.** *A left-linear TRS is confluent if the inclusions $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{-\!\!\!+\!\!\!+\!\!\!\to} \cdot {}^*\!\!\leftarrow$ and $\overset{>\epsilon}{\leftarrow\!\!\!\!+\!\!\!\!+}\!\rtimes\!\xrightarrow{\epsilon} \subseteq \to^*$ hold.*

Unfortunately, this criterion by Gramlich does not subsume (almost) parallel closedness.

Example 3.1.8 (Continued from Example 3.1.5)**.** *The TRS admits the parallel critical peak* $\mathsf{f}(\mathsf{b}(x),\mathsf{b}(y)) \overset{\{1,2\}}{\leftarrow\!\!\!\!+\!\!\!\!+} \mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) \xrightarrow{\epsilon} \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)))$*. However, the relation* $\mathsf{f}(\mathsf{b}(x),\mathsf{b}(y)) \to^* \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)))$ *does not hold.*

As noted in the paper [Gra96], Toyama [Toy81] had already obtained in 1981 a closedness result that subsumes Theorem 3.1.7. His idea is to impose variable conditions on parallel steps $\mathrel{-\!\!\!+\!\!\!+\!\!\!\to}$.

Theorem 3.1.9 ([Toy81])**.** *A left-linear TRS is confluent if the following conditions hold:*

1. *The inclusion $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{-\!\!\!+\!\!\!+\!\!\!\to} \cdot {}^*\!\!\leftarrow$ holds.*
2. *For every parallel critical peak $t \overset{P}{\leftarrow\!\!\!\!+\!\!\!\!+} s \xrightarrow{\epsilon} u$ there exist a term v and a set P' of parallel positions such that $t \to^* v \overset{P'}{\leftarrow\!\!\!\!+\!\!\!\!+} u$ and $\mathcal{V}\mathrm{ar}(v, P') \subseteq \mathcal{V}\mathrm{ar}(s, P)$.*

In the next section we give a proof of Theorem 3.1.9, introducing a key lemma for parallel critical peak analysis.

Example 3.1.10 (Continued from Example 3.1.8)**.** *The confluence of the TRS in Example 3.1.5 can be shown by Theorem 3.1.9. Since condition (1) of Theorem 3.1.9 follows from the almost parallel closedness, it is enough to verify condition (2). The following parallel critical peak, which Theorem 3.1.7 fails to handle, admits the following diagram:*

$$\begin{tikzcd}
\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) \arrow[r, "\epsilon"] \arrow[d, "\{1,2\}"'] & \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y))) \arrow[d, dashed, "\{1\cdot 2\}"] \\
\mathsf{f}(\mathsf{b}(x),\mathsf{b}(y)) \arrow[r, dashed] & \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{b}(y)))
\end{tikzcd}$$

[1] As the name suggests, every local peak ${}_{\mathcal{R}}\!\overset{P}{\leftarrow\!\!\!\!+\!\!\!\!+} \cdot \xrightarrow{\epsilon}_{\mathcal{R}}$ is orthogonal for *orthogonal TRSs*, see Section 4.2.

$$
\begin{tikzcd}
s \arrow[r, "\epsilon"] \arrow[d, "P"'] & u \arrow[d, dashed, "P'"] \\
t \arrow[r, dashed, "\epsilon\ \ ="] & v
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
s = s_0\sigma \arrow[r, "\epsilon"] \arrow[dd, "P"'] \arrow[d, dashed, "P_0"] & u_0\sigma = u \\
t_0\sigma \arrow[d, dashed, "P \setminus P_0"] & \\
t = t_0\tau &
\end{tikzcd}
$$

(a) Γ is orthogonal (b) Γ is not orthogonal

Figure 3.1: The claims of Lemma 3.2.1.

Because $\mathcal{V}\mathsf{ar}(\mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{b}(y))),\{1\cdot 2\}) = \{y\} \subseteq \{x,y\} = \mathcal{V}\mathsf{ar}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)),\{1,2\})$ *holds, the parallel critical peak satisfies condition (2) in Theorem 3.1.9. Similarly, we can find suitable diagrams for the other parallel critical peaks. Hence, (2) holds for the TRS.*

3.2 Proof of Toyama's Parallel Closedness

For the sake of self-containedness, we give a proof of Theorem 3.1.9 in this section.[2] The first part of the next lemma is a strengthened version of the Parallel Moves Lemma [BN98, Lemma 6.4.4]. Here a variable condition like Theorem 3.1.9 is associated. We write $\sigma \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \tau$ if $x\sigma \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} x\tau$ for all variables x.

Lemma 3.2.1. *Let $\mathcal{R}$ be a TRS and $\ell \to r$ a left-linear rule. Consider a local peak Γ of the form $t\ {}_{\mathcal{R}}\!\overset{P}{\leftarrow\!\!\!\!+\!\!\!+}\ s \xrightarrow{\epsilon}_{\{\ell\to r\}} u$.*

1. *If Γ is orthogonal, $t \xrightarrow{\epsilon}{}^{=}_{\{\ell\to r\}} v\ {}_{\mathcal{R}}\!\overset{P'}{\leftarrow\!\!\!\!+\!\!\!+}\ u$ and $\mathcal{V}\mathsf{ar}(v,P') \subseteq \mathcal{V}\mathsf{ar}(s,P)$ for some v and P'.*

2. *Otherwise, there exist a parallel critical peak $t_0\ {}_{\mathcal{R}}\!\overset{P_0}{\leftarrow\!\!\!\!+\!\!\!+}\ s_0 \xrightarrow{\epsilon}_{\{\ell\to r\}} u_0$ and substitutions σ and τ such that $s = s_0\sigma$, $t = t_0\tau$, $u = u_0\sigma$, $\sigma \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \tau$, $t_0\sigma \xrightarrow{P\setminus P_0}_{\mathcal{R}} t_0\tau$, and $P_0 \subseteq P$.*

See the diagrams in Fig. 3.1.

Proof. 1. Suppose that Γ is orthogonal. If $s \xrightarrow{\{\epsilon\}}_{\{\ell'\to r'\}} t$ holds for some variant $\ell' \to r'$ of $\ell \to r$ then $t = u$. Thus, $t \to^{=} u \overset{\varnothing}{\leftarrow\!\!\!\!+\!\!\!+} u$. Otherwise, $P \cap \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell)$

[2] The report is hardly accessible. In fact I fail to access it.

is $\varnothing$. Since $s \xrightarrow{\epsilon}_{\{\ell \to r\}} u$ holds, there exists a substitution σ with $s = \ell\sigma$ and $u = r\sigma$. As $\ell\sigma \overset{P}{\twoheadrightarrow} t$, ℓ is linear, and $P \cap \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell) = \varnothing$, straightforward induction on ℓ shows existence of τ such that $t = \ell\tau$ and $\sigma \twoheadrightarrow_{\mathcal{R}} \tau$. Take $v = r\tau$ and define P' the set:

$$\{p'_1 \cdot p_2 \mid p_1 \cdot p_2 \in P,\, p'_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(r), \text{ and } \ell|_{p_1} = r|_{p'_1} \text{ for some } p_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(\ell)\}$$

Clearly, $t \xrightarrow{\epsilon}_{\{\ell \to r\}} v$ holds. So it remains to show $u \overset{P'}{\twoheadrightarrow}_{\mathcal{R}} v$ and $\mathcal{V}\mathsf{ar}(v, P') \subseteq \mathcal{V}\mathsf{ar}(s, P)$. Let p' be an arbitrary position in P'. There exist positions $p_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(\ell)$, $p'_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(r)$, and p_2 such that $p' = p'_1 \cdot p_2$, $p_1 \cdot p_2 \in P$, and $\ell|_{p_1} = r|_{p'_1}$. Denoting $p_1 \cdot p_2$ by p, we have the identities:

$$\begin{aligned} u|_{p'} &= (r\sigma)|_{p'_1 \cdot p_2} = (r|_{p'_1}\sigma)|_{p_2} = (\ell|_{p_1}\sigma)|_{p_2} = (\ell\sigma)|_{p_1 \cdot p_2} = s|_p \\ v|_{p'} &= (r\tau)|_{p'_1 \cdot p_2} = (r|_{p'_1}\tau)|_{p_2} = (\ell|_{p_1}\tau)|_{p_2} = (\ell\tau)|_{p_1 \cdot p_2} = t|_p \end{aligned}$$

From $s \overset{P}{\twoheadrightarrow}_{\mathcal{R}} t$ we obtain $s|_p \xrightarrow{\epsilon}_{\mathcal{R}} t|_p$ and thus $u|_{p'} \xrightarrow{\epsilon}_{\mathcal{R}} v|_{p'}$. Therefore, $u \overset{P'}{\twoheadrightarrow}_{\mathcal{R}} v$ is obtained. Moreover, we have $\mathcal{V}\mathsf{ar}(v|_{p'}) = \mathcal{V}\mathsf{ar}(t|_p) \subseteq \mathcal{V}\mathsf{ar}(s|_p) \subseteq \mathcal{V}\mathsf{ar}(s, P)$. As $\mathcal{V}\mathsf{ar}(v, P')$ is the union of $\mathcal{V}\mathsf{ar}(v|_{p'})$ for all $p' \in P'$, the desired inclusion $\mathcal{V}\mathsf{ar}(v, P') \subseteq \mathcal{V}\mathsf{ar}(s, P)$ follows.

2. Suppose that Γ is not orthogonal. By $\ell_p \to r_p$ we denote the rule employed at the rewrite position $p \in P$ in $s \overset{P}{\twoheadrightarrow}_{\mathcal{R}} t$. Let $P_0 = P \cap \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell)$ and $P_1 = P \setminus P_0$. Since P is a set of parallel positions, $s \overset{P}{\twoheadrightarrow} t$ is split into the two steps $s \overset{P_0}{\twoheadrightarrow}_{\mathcal{R}} v \overset{P_1}{\twoheadrightarrow}_{\mathcal{R}} t$, where $v = s[t|_p]_{p \in P_0}$.

   First, we show that $v \overset{P_0}{\twoheadleftarrow} s \xrightarrow{\epsilon}_{\{\ell \to r\}} u$ is an instance of a parallel critical peak. Let p be an arbitrary position in P_0. Because of $s \xrightarrow{\epsilon}_{\{\ell \to r\}} u$, we have $s = \ell\mu$ and $u = r\mu$ for some μ. Suppose that $\ell'_p \to r'_p$ is a renamed variant of $\ell_p \to r_p$ with fresh variables. There exists a substitution μ_p such that $s|_p = \ell'_p\mu_p$ and $t|_p = r'_p\mu_p$. Note that $\mathcal{D}\mathsf{om}(\mu) \cap \mathcal{D}\mathsf{om}(\mu_p) = \varnothing$. We define the substitution ν as follows:

$$\nu(x) = \begin{cases} x\mu_p & \text{if } p \in P_0 \text{ and } x \in \mathcal{V}\mathsf{ar}(\ell'_p) \\ x\mu & \text{otherwise} \end{cases}$$

   Because every ℓ'_p with $p \in P_0$ is linear and do not share variables with each other, ν is well-defined. Since ℓ neither share variables with ℓ'_p, we obtain the identities:

$$\ell'_p\nu = \ell'_p\mu_p = s|_p = \ell|_p\mu = \ell|_p\nu$$

   Thus, ν is a unifier of $E = \{\ell'_p \approx \ell|_p\}_{p \in P_0}$. Let V denote the set of all variables occurring in E. According to [Ede85, Proposition 4.10], there exists a most general unifier ν' of E such that $\mathcal{D}\mathsf{om}(\nu') \subseteq V$. Thus, there is a substitution σ with $\nu = \nu'\sigma$. Let $s_0 = \ell\nu'$, $t_0 = (\ell\nu')[r'_p\nu']_{p \in P_0}$, and $u_0 = r\nu'$. The peak $t_0 \xleftarrow{P_0}\!\!\!\!\!+\!\!\!+ s_0 \xrightarrow{\epsilon} u_0$ is a parallel critical peak, and $v \xleftarrow{P_0}\!\!\!\!\!+\!\!\!+ s \xrightarrow{\epsilon} u$ is an instance of the peak by the substitution σ:

$$
\begin{aligned}
s_0\sigma &= \ell\nu'\sigma = \ell\nu = \ell\mu = s \\
t_0\sigma &= (\ell\nu'\sigma)[r'_p\nu'\sigma]_{p \in P_0} = (\ell\nu)[r'_p\nu]_{p \in P_0} = (\ell\mu)[r'_p\mu_p]_{p \in P_0} = v \\
u_0\sigma &= r\nu'\sigma = r\nu = r\mu = u
\end{aligned}
$$

Next, we construct a substitution τ so that it satisfies $\sigma \twoheadrightarrow_{\mathcal{R}} \tau$ and $t_0\sigma \xrightarrow{P_1}\!\!\!\!\!\!\!+\!\!\!+_{\mathcal{R}} t_0\tau$. Given a variable $x \in \mathcal{V}\mathsf{ar}(\ell)$, we write p_x for a variable occurrence of x in ℓ. Due to linearity of ℓ, the position p_x is uniquely determined. Let $W = \mathcal{V}\mathsf{ar}(\ell) \setminus \mathcal{V}\mathsf{ar}(\ell, P_0)$. Note that $W \cap V = \varnothing$ holds. We define the substitution τ as follows:

$$
\tau(x) = \begin{cases} t|_{p_x} & \text{if } x \in W \\ x\sigma & \text{otherwise} \end{cases}
$$

To verify $\sigma \twoheadrightarrow_{\mathcal{R}} \tau$, consider an arbitrary variable x. We show $x\sigma \twoheadrightarrow_{\mathcal{R}} x\tau$. If $x \notin W$ then $x\sigma = x\tau$, from which the claim follows. Otherwise, the definitions of V and ν' yield the implications:

$$
x \in W \implies x \notin V \implies x \notin \mathcal{D}\mathsf{om}(\nu') \implies x\nu' = x
$$

So $s_0|_{p_x} = x$ follows from the identities:

$$
s_0|_{p_x} = (\ell\nu)|_{p_x} = \ell|_{p_x}\nu = x\nu = x
$$

Let $Q_x = \{q \mid p_x q \in P_1\}$. As $s \xrightarrow{P_0}\!\!\!\!\!\!\!+\!\!\!+_{\mathcal{R}} v \xrightarrow{P_1}\!\!\!\!\!\!\!+\!\!\!+_{\mathcal{R}} t$ implies $s|_{p_x} = v|_{p_x} \xrightarrow{Q_x}\!\!\!\!\!\!\!+\!\!\!+_{\mathcal{R}} t|_{p_x}$, we obtain $x\sigma = s_0|_{p_x}\sigma = (s_0\sigma)|_{p_x} = s|_{p_x} \xrightarrow{Q_x}\!\!\!\!\!\!\!+\!\!\!+_{\mathcal{R}} t|_{p_x} = x\tau$. Therefore, the claim is verified.

The remaining task is to show $t_0\sigma \xrightarrow{P_1}\!\!\!\!\!\!\!+\!\!\!+_{\mathcal{R}} t_0\tau$. Let $p \in P_1$. As $s_0|_{p_x} = x$ and $s_0 \xrightarrow{P_0}\!\!\!\!\!\!\!+\!\!\!+_{\mathcal{R}} t_0$ imply $x = t_0|_{p_x}$, the equation $(s_0\sigma)|_p = (t_0\sigma)|_p$ follows. By the definition of τ we have $(t_0\tau)|_{p_x} = t|_{p_x}$, which leads to $(t_0\tau)|_p = t|_p$. Hence, we obtain the relations

$$
(t_0\sigma)|_p = (s_0\sigma)|_p = s|_p \xrightarrow{\{\epsilon\}}\!\!\!\!\!\!\!+\!\!\!+_{\mathcal{R}} t|_p = (t_0\tau)|_p
$$

which entails the desired parallel step $t_0\sigma \xrightarrow{P_1}\!\!\!\!\!\!\!+\!\!\!+_{\mathcal{R}} t_0\tau$.

∎

The following lemma taken from [ZFM15] is used for composing parallel steps. In order to make the thesis self-contained, we give a proof of it.

Lemma 3.2.2 ([ZFM15, Lemma 51(b)]). *If $s \xrightarrow{P}\!\!\!\!\!\!\!\!\!+\!\!\!+\!\!\!\!\!\;_{\mathcal{R}} t$, $\sigma \twoheadrightarrow_{\mathcal{R}} \tau$, and $x\sigma = x\tau$ for all $x \in \mathcal{V}\mathsf{ar}(t, P)$ then $s\sigma \twoheadrightarrow_{\mathcal{R}} t\tau$.*

Proof. We show the claim by induction on derivation of $\twoheadrightarrow$.

1. If $P = \varnothing$, $s \in \mathcal{V}$, and $s = t$ then $s\sigma \twoheadrightarrow_{\mathcal{R}} s\tau = t\tau$ by assumption.

2. If $P = \{\epsilon\}$, and $s = \ell\nu$ and $t = r\nu$ for some $\ell \to r \in \mathcal{R}$ and ν then $\mathcal{V}\mathsf{ar}(t, \{\epsilon\}) = \mathcal{V}\mathsf{ar}(t)$. By assumption $x\sigma = x\tau$ holds for all $x \in \mathcal{V}\mathsf{ar}(t)$, and thus $t\sigma = t\tau$. Because of $s\sigma = (\ell\nu)\sigma$ and $t\sigma = (r\nu)\sigma$, the relation
$$s\sigma = \ell(\nu\sigma) \xrightarrow{\epsilon}_{\mathcal{R}} r(\nu\sigma) = t\sigma = t\tau$$
holds.

3. If $P \not\subseteq \{\epsilon\}$ then we have $P = \{ip \mid 1 \leqslant i \leqslant n \text{ and } p \in P_i\}$, $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, and $s_i \xrightarrow{P_i} t_i$ for all $1 \leqslant i \leqslant n$. It is enough to show the claim that every s_i satisfies $s_i\sigma \twoheadrightarrow t_i\tau$. Because $\mathcal{V}\mathsf{ar}(t_i, P_i) \subseteq \mathcal{V}\mathsf{ar}(t, P)$ holds, $x\sigma = x\tau$ for all $x \in \mathcal{V}\mathsf{ar}(t_i, P_i)$. According to the induction hypothesis we have $s_i\sigma \twoheadrightarrow t_i\tau$.

∎

Now we give a proof for Theorem 3.1.9. Note that Fig. 3.2 illustrates diagrams of the proof.

Proof (of Theorem 3.1.9). In order to show confluence, we show that $\to$ strongly commutes with $\twoheadrightarrow$. Let $\Gamma\colon t \xleftarrow{P} s \xrightarrow{q} u$ be a local peak. We perform structural induction on s. Depending on the shape of Γ, we distinguish six cases.

1. If P is empty then we are done.

2. If $q = \epsilon$ and Γ is orthogonal then we have $t \to^* v \twoheadleftarrow u$ for some v by Lemma 3.2.1.

3. If $P = \{\epsilon\}$ and Γ is orthogonal then $t \xleftarrow{\epsilon} s \to u$ holds. Since $s \to u$ yields $t \xleftarrow{\epsilon} s \twoheadrightarrow u$, we obtain $t \twoheadrightarrow \cdot\, {}^{=}\!\!\leftarrow u$ by Lemma 3.2.1(1). Hence $t \to^* \cdot \twoheadleftarrow u$ follows from $\to^{=} \subseteq \twoheadrightarrow$.

4. If $q = \epsilon$ and Γ is not orthogonal then there exists a parallel critical peak $t_0 \xleftarrow{P_0} s_0 \xrightarrow{\epsilon} u_0$ and substitutions σ and τ such that $s = s_0\sigma$, $t = t_0\tau$, $u = u_0\sigma$, $\sigma \twoheadrightarrow \tau$, $t_0\sigma \xrightarrow{P \setminus P_0} t_0\tau$, and $\varnothing \neq P_0 \subseteq P$. By assumption $t_0 \to^* v_0 \xleftarrow{P_0'} u_0$

for some v_0 and P_0' with $\mathcal{V}\mathsf{ar}(v_0, P_0') \subseteq \mathcal{V}\mathsf{ar}(s_0, P_0)$. Since the rewrite steps are closed under substitutions, the following relations are obtained:

$$t = t_0\tau \to^* v_0\tau \qquad\qquad v_0\sigma \xleftarrow{P_0'}\!\!\!\!\!\!\!\!\!\!+\!\!\!\!+\; u_0\sigma = u$$

Since $t_0\sigma|_p = t_0\tau|_p$ holds for all $p \in P_0$, the identity $x\sigma = x\tau$ holds for all $x \in \mathcal{V}\mathsf{ar}(s_0, P_0)$. Therefore $x\sigma = x\tau$ for all $x \in \mathcal{V}\mathsf{ar}(s_0, P_0)$. Because $u_0 \xrightarrow{P_0'}\!\!\!\!\!\!\!\!\!+\!\!\!\!+\; v_0$, $\sigma \twoheadrightarrow \tau$, and $x\sigma = x\tau$ for all $x \in \mathcal{V}\mathsf{ar}(v_0, P_0')$ hold, Lemma 3.2.2 yields $u_0\sigma \twoheadrightarrow v_0\tau$. Hence $t = t_0\tau \to^* v_0\tau \twoheadleftarrow u_0\sigma = u$ is obtained.

5. If $P = \{\epsilon\}$ and Γ is not orthogonal then $t \xleftarrow{\epsilon} s \xrightarrow{q} u$ holds. Since the peak can be regarded as $t \xleftarrow{\epsilon} s \xrightarrow{\{q\}}\!\!\!\!\!\!\!\!\!+\!\!\!\!+\; u$, by Lemma 3.2.1(2) there exist a parallel critical peak $t_0 \xleftarrow{\epsilon} s_0 \xrightarrow{P_0}\!\!\!\!\!\!\!\!\!+\!\!\!\!+\; u_0$ and substitutions σ and τ such that $s = s_0\sigma$, $t = t_0\sigma$, $u = u_0\tau$, $\sigma \twoheadrightarrow \tau$, $u_0\sigma \xrightarrow{P \setminus P_0}\!\!\!\!\!\!\!\!\!\!\!\!+\!\!\!\!+\; u_0\tau$, and $\varnothing \neq P_0 \subseteq \{q\}$. Then $P_0 = \{q\}$ and $\sigma = \tau$ hold, and hence $t_0 \xleftarrow{\epsilon} s_0 \xrightarrow{q} u_0$ is a critical peak. By assumption there exists a term v_0 such that $t_0 \to^* v_0 \twoheadleftarrow u_0$. Because rewrite steps are closed under substitutions, the desired relation is obtained:

$$t = t_0\sigma \to^* v_0\sigma \twoheadleftarrow u_0\sigma = u_0\tau = u$$

6. If $P \not\subseteq \{\epsilon\}$ and $q = i \cdot q'$ for some $i \in \mathbb{N}$ then s, t, and u can be written as $f(s_1, \ldots, s_i, \ldots, s_n)$, $f(t_1, \ldots, t_i, \ldots, t_n)$, and $f(s_1, \ldots, u_i, \ldots, u_n)$ respectively. We denote the subset $\{i \cdot p \mid i \cdot p \in P\}$ of P by P_i for each i. Then $t_i \xleftarrow{Q_i}\!\!\!\!\!\!\!\!\!+\!\!\!\!+\; s_i \xrightarrow{q'} u_i$ holds for some $1 \leqslant i \leqslant n$. Here $Q_i = \{p \mid i \cdot p \in P_i\}$. By the induction hypotheses we have $t_i \to^* v_i \xleftarrow{Q_i'}\!\!\!\!\!\!\!\!\!+\!\!\!\!+\; u_i$ for some Q_i' and v_i. Let $P' = (P \setminus P_i) \cup \{i \cdot p' \mid p' \in Q_i'\}$. Since i is parallel for all positions in $P \setminus P_i$, the set P' is a set of parallel positions. Hence $u = f(s_1, \ldots, u_i, \ldots, s_n) \xrightarrow{P'}\!\!\!\!\!\!\!\!\!+\!\!\!\!+\; f(t_1, \ldots, v_i, \ldots, t_n)$ holds. Therefore, we obtain the desired relations:

$$t = f(t_1, \ldots, t_n) \to^* f(t_1, \ldots, v_i, \ldots, t_n) \twoheadleftarrow f(s_1, \ldots, u_i, \ldots, s_n) = u$$

Since $\to$ strongly commutes with $\twoheadrightarrow$, they commute by Theorem 2.1.10. Because of $\to^* = \twoheadrightarrow^*$, we conclude that $\to$ is self-commuting. Thus $\to$ is confluent. ∎

3.3 Comparison

As we have already seen in Section 3.1, Theorem 3.1.4 subsumes Theorem 3.1.2. Now we show that Theorem 3.1.9 even subsumes Theorem 3.1.4. The next lemma is irrelevant here but will be used in the subsequent sections. Note that the second part corresponds to [ZFM15, Lemma 55].

$$
\begin{tikzcd}
s = s_0\sigma \arrow[r, "\epsilon"] \arrow[d, dashed, "P_0 \shortparallel"'] & u_0\sigma = u \arrow[d, dashed, "\shortparallel P'_0"] \arrow[dd, dashed, bend left, "\shortparallel"] \\
t_0\sigma \arrow[r, dashed, "*"] \arrow[d, dashed, "P \setminus P_0 \shortparallel"'] & v_0\sigma \arrow[d, dashed, "\shortparallel"] \\
t = t_0\tau \arrow[r, dashed, "*"] & v_0\tau
\end{tikzcd}
$$

Case (4)

$$
\begin{tikzcd}
s = s_0\sigma \arrow[r, "\{q\}", "\shortparallel"'] \arrow[d, "\epsilon"'] & u_0\sigma = u \arrow[d, dashed, "\shortparallel"] \\
t = t_0\sigma \arrow[r, dashed, "*"] & v_0\sigma
\end{tikzcd}
$$

Case (5)

$$
\begin{tikzcd}
f(s_1, \ldots, s_i, \ldots, s_n) \arrow[r, "i \cdot q"] \arrow[d, "P_i \shortparallel"'] & f(s_1, \ldots, u_i, \ldots, s_n) \arrow[d, dashed, "\shortparallel P'_i"] \arrow[dd, dashed, bend left, "\shortparallel"] \\
f(s_1, \ldots, t_i, \ldots, s_n) \arrow[r, dashed, "*"] \arrow[d, "P \setminus P_i \shortparallel"'] & f(s_1, \ldots, v_i, \ldots, s_n) \arrow[d, dashed, "\shortparallel P \setminus P_i"] \\
f(t_1, \ldots, t_i, \ldots, t_n) \arrow[r, dashed, "*"] & f(t_1, \ldots, v_i, \ldots, t_n)
\end{tikzcd}
$$

Case (6)

Figure 3.2: Proof of Theorem 3.1.9.

For almost parallel closed TRSs the above statement is extended to local peaks $\leftarrow\!\!\shortparallel\!\!- \cdot -\!\!\shortparallel\!\!\rightarrow$ of parallel steps. In its proof we measure parallel steps $s \overset{P}{-\!\!\shortparallel\!\!\rightarrow} t$ in such a local peak by the *total size of contractums* $|t|_P$, namely the sum of $|(t|_p)|$ for all $p \in P$. Note that this measure attributes to [OO97, LJ14].

Lemma 3.3.1. *Consider a left-linear almost parallel closed TRS. If* $t \overset{P_1}{\leftarrow\!\!\shortparallel\!\!-} s \overset{P_2}{-\!\!\shortparallel\!\!\rightarrow} u$ *then*

- $t \to^* v_1 \overset{P_1'}{\leftarrow\!\!\shortparallel\!\!-} u$ *for some* v_1 *and* P_1' *with* $\mathcal{V}\mathsf{ar}(v_1, P_1') \subseteq \mathcal{V}\mathsf{ar}(s, P_1)$*, and*
- $t \overset{P_2'}{-\!\!\shortparallel\!\!\rightarrow} v_2 \; {}^*\!\!\leftarrow u$ *for some* v_2 *and* P_2' *with* $\mathcal{V}\mathsf{ar}(v_2, P_2') \subseteq \mathcal{V}\mathsf{ar}(s, P_2)$.

Proof. Let $\Gamma\colon t \overset{P_1}{\leftarrow\!\!\shortparallel\!\!-} s \overset{P_2}{-\!\!\shortparallel\!\!\rightarrow} u$ be a local peak. We show the claim by well-founded induction on $(|t|_{P_1} + |u|_{P_2}, s)$ with respect to $\succ$. Here $(m, s) \succ (n, t)$ if either $m > n$, or $m = n$ and t is a proper subterm of s. Depending on the shape of Γ, we distinguish six cases.

1. If P_1 or P_2 is empty then the claim follows from the fact: $\mathcal{V}\mathsf{ar}(v, P) \subseteq \mathcal{V}\mathsf{ar}(w, P)$ if $w \overset{P}{-\!\!\shortparallel\!\!\rightarrow} v$.

2. If P_1 or P_2 is $\{\epsilon\}$ and Γ is orthogonal then Lemma 3.2.1(1) applies.

3. If $P_1 = P_2 = \{\epsilon\}$ and Γ is not orthogonal then Γ is an instance of a critical peak. By almost parallel closedness $t \to^* v_1 \overset{Q_1}{\leftarrow\!\!\shortparallel\!\!-} u$ and $t \overset{Q_2}{-\!\!\shortparallel\!\!\rightarrow} v_2 \; {}^*\!\!\leftarrow u$ for some v_1, v_2, Q_1, and Q_2. For each $k \in \{1, 2\}$ we have $s \to^* v_k$, so $\mathcal{V}\mathsf{ar}(v_k) \subseteq \mathcal{V}\mathsf{ar}(s)$ follows. Thus, $\mathcal{V}\mathsf{ar}(v_k, Q_k) \subseteq \mathcal{V}\mathsf{ar}(v_k) \subseteq \mathcal{V}\mathsf{ar}(s) = \mathcal{V}\mathsf{ar}(s, \{\epsilon\})$. The claim holds.

4. If $P_1 \nsubseteq \{\epsilon\}$, $P_2 = \{\epsilon\}$, and Γ is not orthogonal then there is $p \in P_1$ such that $s' \overset{p}{\leftarrow} s \overset{\epsilon}{\to} u$ is an instance of a critical peak and $s' \xrightarrow[\shortparallel]{P_1 \setminus \{p\}} t$ follows by Lemma 3.2.1(2) where $P = \{p\}$. By the almost parallel closedness $s' \overset{P_2'}{-\!\!\shortparallel\!\!\rightarrow} u$ for some P_2'. Since P_2' is a set of parallel positions in u, we have $|u|_{\{\epsilon\}} = |u| \geqslant |u|_{P_2'}$. As $|u|_{\{\epsilon\}} \geqslant |u|_{P_2'}$ and $|t|_{P_1} > |t|_{P_1 \setminus \{p\}}$ yield $|t|_{P_1} + |u|_{\{\epsilon\}} > |t|_{P_1 \setminus \{p\}} + |u|_{P_2'}$, we obtain the inequality:
$$(|t|_{P_1} + |u|_{P_2}, s) \succ (|t|_{P_1 \setminus \{p\}} + |u|_{P_2'}, s')$$
Thus, the claim follows by the induction hypothesis for $t \xleftarrow[\shortparallel]{P_1 \setminus \{p\}} s' \overset{P_2'}{-\!\!\shortparallel\!\!\rightarrow} u$ and $\mathcal{V}\mathsf{ar}(s', P_1 \setminus \{p\}) \subseteq \mathcal{V}\mathsf{ar}(s, P_1)$ and $\mathcal{V}\mathsf{ar}(s', P_2') \subseteq \mathcal{V}\mathsf{ar}(s, \{\epsilon\})$.

5. If $P_1 = \{\epsilon\}$, $P_2 \nsubseteq \{\epsilon\}$, and Γ is not orthogonal then the proof is analogous to the last case.

6. If $P_1 \nsubseteq \{\epsilon\}$ and $P_2 \nsubseteq \{\epsilon\}$ then we may assume $s = f(s_1,\dots,s_n)$, $t = f(t_1,\dots,t_n)$, $u = f(u_1,\dots,u_n)$, and $t_i \xleftarrow{P_1^i}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!+\!\!\!\!+\ \ s_i \xrightarrow{P_2^i}\!\!\!\!\!\!\!\!\!+\!\!\!\!+\ \ u_i$ for all $1 \leqslant i \leqslant n$. Here P_k^i denotes the set $\{p \mid i \cdot p \in P_k\}$. For each $i \in \{1,\dots,n\}$, we have $|t|_{P_1} \geqslant |t_i|_{P_1^i}$ and $|u|_{P_2} \geqslant |u_i|_{P_2^i}$, and therefore $|t|_{P_1} + |u|_{P_2} \geqslant |t_i|_{P_1^i} + |u_i|_{P_2^i}$. So we deduce the following inequality:

$$(|t|_{P_1} + |u|_{P_2}, s) \succ (|t_i|_{P_1^i} + |u_i|_{P_2^i}, s_i)$$

Consider the i-th peak $t_i \xleftarrow{P_1^i}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!+\!\!\!\!+\ \ s_i \xrightarrow{P_2^i}\!\!\!\!\!\!\!\!\!+\!\!\!\!+\ \ u_i$. By the induction hypothesis it admits valleys of the forms $t_i \to^* v_1^i \xleftarrow{Q_1^i}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!+\!\!\!\!+\ \ u_i$ and $t_i \xrightarrow{Q_2^i}\!\!\!\!\!\!\!\!\!+\!\!\!\!+\ \ v_2^i \ {}^*\!\!\leftarrow u_i$ such that $\mathcal{V}\mathsf{ar}(v_k^i, Q_k^i) \subseteq \mathcal{V}\mathsf{ar}(s_i, P_k^i)$ for both $k \in \{1,2\}$. For each k, define $Q_k = \{i \cdot q \mid 1 \leqslant i \leqslant n \text{ and } q \in Q_k^i\}$ and $v_k = f(v_k^1,\dots,v_k^n)$. Then we have $t \to^* v_1 \xleftarrow{Q_1}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!+\!\!\!\!+\ \ u$ and $t \xrightarrow{Q_2}\!\!\!\!\!\!\!\!\!+\!\!\!\!+\ \ v_2 \ {}^*\!\!\leftarrow u$. Moreover,

$$\mathcal{V}\mathsf{ar}(v_k, Q_k) = \bigcup_{i=1}^{n} \mathcal{V}\mathsf{ar}(v_k^i, Q_k^i) \subseteq \bigcup_{i=1}^{n} \mathcal{V}\mathsf{ar}(s_i, P_k^i) = \mathcal{V}\mathsf{ar}(s, P_k)$$

holds. Hence, the claim follows.

∎

Theorem 3.3.2. *Every left-linear and almost parallel closed TRS satisfies conditions (1) and (2) of Theorem 3.1.9. In other words, Theorem 3.1.9 subsumes Theorem 3.1.4.*

Proof. Since (parallel) critical peaks are instances of $\leftarrow\!\!\!\!+\!\!\!\!+ \cdot +\!\!\!\!+\!\!\!\!\rightarrow$, Lemma 3.3.1 entails the claim. ∎

Note that Theorem 3.1.4 does not subsume Theorem 3.1.9 as witnessed by the TRS consisting of the four rules

$$\mathsf{f(a)} \to \mathsf{c} \qquad \mathsf{a} \to \mathsf{b} \qquad \mathsf{f(b)} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{b}$$

Therefore, Theorem 3.1.9 is the most general theorem among parallel closedness theorems. In Section 4.3 we will see that Theorem 3.1.9 is subsumed by a variant of rule labeling.

Finally we compare Theorem 3.1.9 with Huet's another closedness result. The following criterion is known as *strong closedness*. We say that a TRS $\mathcal{R}$ is *linear* if ℓ and r are linear for all $\ell \to r \in \mathcal{R}$.

Theorem 3.3.3 ([Hue80]). *A linear TRS is confluent if $t \to^* \cdot {}^{=}\!\!\leftarrow u$ and $t \to^{=} \cdot {}^{*}\!\!\leftarrow u$ for all $t \leftarrow\!\rtimes\!\xrightarrow{\epsilon} u$.*

We show that Theorem 3.1.9 and this criterion are not comparable even if only linear TRSs are considered. To see it, consider the TRS $\mathcal{R}_1$:

$$\mathsf{f}(\mathsf{a}, y) \to \mathsf{g}(y) \qquad \mathsf{a} \to \mathsf{b} \qquad \mathsf{f}(\mathsf{b}, y) \to \mathsf{h}(y) \qquad \mathsf{g}(y) \to \mathsf{h}(y)$$

The only one critical pair of the TRS is $\mathsf{f}(\mathsf{b}, y) \leftarrow\!\rtimes\!\xrightarrow{\epsilon} \mathsf{g}(y)$, and it is closed by the sequence $\mathsf{f}(\mathsf{b}, y) \xrightarrow{\epsilon} \mathsf{h}(y) \xleftarrow{\epsilon} \mathsf{g}(y)$. Thus $\mathcal{R}_1$ satisfies the condition of Theorem 3.3.3. However, because of $\mathcal{V}\mathsf{ar}(\mathsf{h}(y), \{\epsilon\}) \not\subseteq \mathcal{V}\mathsf{ar}(\mathsf{f}(\mathsf{b}, y), \{1\})$, this does not satisfy the condition of Theorem 3.1.9.

Conversely, Theorem 3.3.3 does not subsume Theorem 3.1.9 for even linear TRSs. Consider the next TRS $\mathcal{R}_2$ that contains no variables:

$$\mathsf{c} \to \mathsf{f}(\mathsf{a}, \mathsf{a}) \qquad \mathsf{c} \to \mathsf{f}(\mathsf{b}, \mathsf{b}) \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

There are two critical pairs: $\mathsf{f}(\mathsf{a}, \mathsf{a}) \leftarrow\!\rtimes\!\xrightarrow{\epsilon} \mathsf{f}(\mathsf{b}, \mathsf{b})$ and $\mathsf{f}(\mathsf{b}, \mathsf{b}) \leftarrow\!\rtimes\!\xrightarrow{\epsilon} \mathsf{f}(\mathsf{a}, \mathsf{a})$. They are joinable by not only $\to^2 \cdot {}^2\!\leftarrow$ but also $\mathrel{+\!\!\!\!\!\!\twoheadrightarrow} \cdot \mathrel{\twoheadleftarrow\!\!\!\!\!\!+}$. The former does not satisfy the condition of Theorem 3.3.3, whereas the latter satisfies all conditions of Theorem 3.1.9. Hence the confluence of $\mathcal{R}_2$ can be shown by Theorem 3.1.9 but not Theorem 3.3.3.

Chapter 4

Compositional Confluence Criteria

In this chapter we discuss our main topic. The first section introduces a variant of decreasing diagrams that takes *commuting* ARSs as parameters. The next section demonstrates how the orthogonality result can be enhanced to a compositional confluence criterion by the variant. Adopting this approach, we derive two powerful compositional confluence criteria from the existing confluence criteria based on rule labeling and critical pair systems. These are presented in the remaining two sections.

4.1 Decreasing Diagrams with Commuting Subsystems

We make a variant of decreasing diagrams [vO94, vO08], which will be used in the subsequent sections for deriving compositional confluence criteria for term rewrite systems. First we recall the commutation version of the technique [vO08]. Let $\mathcal{A} = (A, \{\to_{1,\alpha}\}_{\alpha \in I})$ and $\mathcal{B} = (A, \{\to_{2,\beta}\}_{\beta \in J})$ be I-indexed and J-indexed ARSs on the same domain, respectively. Let $>$ be a well-founded order on $I \cup J$. By $\curlyvee\alpha$ we denote the set $\{\beta \in I \cup J \mid \alpha > \beta\}$, and by $\curlyvee\alpha\beta$ we denote $(\curlyvee\alpha) \cup (\curlyvee\beta)$. We say that a local peak $b \;{}_{1,\alpha}\!\leftarrow a \to_{2,\beta} c$ is *decreasing* if

$$b \xleftrightarrow[\curlyvee\alpha]{*} \cdot \xrightarrow[2,\beta]{=} \cdot \xleftrightarrow[\curlyvee\alpha\beta]{*} \cdot \xleftarrow[1,\alpha]{=} \cdot \xleftrightarrow[\curlyvee\beta]{*} c$$

holds. Here $\leftrightarrow_K$ stands for the union of ${}_{1,\gamma}\!\leftarrow$ and $\to_{2,\gamma}$ for all $\gamma \in K$. The ARSs $\mathcal{A}$ and $\mathcal{B}$ are *decreasing* if every local peak $b \;{}_{1,\alpha}\!\leftarrow a \to_{2,\beta} c$ with $(\alpha, \beta) \in I \times J$ is decreasing. In the case of $\mathcal{A} = \mathcal{B}$, we simply say that $\mathcal{A}$ is decreasing.

Theorem 4.1.1 ([vO08]). *If two ARSs are decreasing then they commute.*

We present the abstract principle of our compositional criteria. The idea of using the least index in the decreasing diagram technique is taken from [JL12, FvO13, DFJL22].

Theorem 4.1.2. *Let $\mathcal{A} = (A, \{\rightarrow_{1,\alpha}\}_{\alpha \in I})$ and $\mathcal{B} = (A, \{\rightarrow_{2,\beta}\}_{\beta \in I})$ be I-indexed ARSs equipped with a well-founded order $>$ on I. Suppose that $\perp$ is the least element in I and $\rightarrow_{1,\perp}$ and $\rightarrow_{2,\perp}$ commute. The ARSs $\mathcal{A}$ and $\mathcal{B}$ commute if every local peak ${}_{1,\alpha}\!\leftarrow \cdot \rightarrow_{2,\beta}$ with $(\alpha, \beta) \in I^2 \setminus \{(\perp, \perp)\}$ is decreasing.*

Proof. We define the two ARSs $\mathcal{A}' = (A, \{\Rightarrow_{1,\alpha}\}_{\alpha \in I})$ and $\mathcal{B}' = (A, \{\Rightarrow_{2,\alpha}\}_{\alpha \in I})$ as follows:

$$\Rightarrow_{i,\alpha} = \begin{cases} \rightarrow^*_{i,\alpha} & \text{if } \alpha = \perp \\ \rightarrow_{i,\alpha} & \text{otherwise} \end{cases}$$

Since $\rightarrow^*_{\mathcal{A}} = \Rightarrow^*_{\mathcal{A}}$ and $\rightarrow^*_{\mathcal{B}} = \Rightarrow^*_{\mathcal{B}}$, the commutation of $\mathcal{A}$ and $\mathcal{B}$ follows from that of $\mathcal{A}'$ and $\mathcal{B}'$. We show the latter by proving decreasingness of $\mathcal{A}'$ and $\mathcal{B}'$ with respect to the given well-founded order $>$. Let Γ be a local peak of form ${}_{1,\alpha}\!\Leftarrow \cdot \Rightarrow_{2,\beta}$. We distinguish four cases.

- If neither α nor β is $\perp$ then decreasingness of Γ follows from the assumption.

- If both α and β are $\perp$ then the commutation of $\rightarrow_{1,\perp}$ and $\rightarrow_{2,\perp}$ yields the inclusion:

  $$\underset{1,\perp}{\Longleftarrow} \cdot \underset{2,\perp}{\Longrightarrow} \subseteq \underset{2,\perp}{\Longrightarrow} \cdot \underset{1,\perp}{\Longleftarrow}$$

  Thus Γ is decreasing.

- If $\beta > \alpha = \perp$ then we have ${}_{1,\alpha}\!\leftarrow \cdot \rightarrow_{2,\beta} \subseteq \rightarrow^{=}_{2,\beta} \cdot \leftrightarrow^*_{\curlyvee\beta}$ Therefore, easy induction on n shows the inclusion ${}^{\;n}_{1,\alpha}\!\leftarrow \cdot \rightarrow_{2,\beta} \subseteq \rightarrow^{=}_{2,\beta} \cdot \leftrightarrow^*_{\curlyvee\beta}$ for all $n \in \mathbb{N}$. Thus,

  $$\underset{1,\alpha}{\Longleftarrow} \cdot \underset{2,\beta}{\Longrightarrow} = \overset{*}{\underset{1,\alpha}{\longleftarrow}} \cdot \underset{2,\beta}{\longrightarrow} \subseteq \overset{=}{\underset{2,\beta}{\longrightarrow}} \cdot \overset{*}{\underset{\curlyvee\beta}{\longleftrightarrow}} = \overset{=}{\underset{2,\beta}{\Longrightarrow}} \cdot \overset{*}{\underset{\curlyvee\beta}{\Longleftrightarrow}}$$

  holds, where $\Leftrightarrow_J$ stands for ${}_{1,J}\!\Leftarrow \cup \Rightarrow_{2,J}$. Hence Γ is decreasing.

- The case that $\alpha > \beta = \perp$ is analogous to the last case.

∎

Because confluence is characterized by self-commutation, we obtain the following corollary from Theorem 4.1.2.

Corollary 4.1.3. *Let $\mathcal{A} = (A, \{\rightarrow_\alpha\}_{\alpha \in I})$ be an I-indexed ARS equipped with a well-founded order $>$ on I. Suppose that $\perp$ is the least element in I and $\rightarrow_\perp$ is confluent. The ARS $\mathcal{A}$ is confluent if every local peak ${}_\alpha\!\leftarrow \cdot \rightarrow_\beta$ with $(\alpha, \beta) \in I^2 \setminus \{(\perp, \perp)\}$ is decreasing.*

4.2 Orthogonality

As a first example of compositional confluence criteria for term rewrite systems, we pick up a compositional version of Rosen's confluence criterion by orthogonality [Ros73]. *Orthogonal* TRSs are left-linear TRSs having no critical pairs. Their confluence property can be shown by decreasingness of parallel steps. We briefly recall its proof. Left-linear TRSs are *mutually orthogonal* if ${}_{\mathcal{R}}{\leftarrow}\rtimes\xrightarrow{\epsilon}_{\mathcal{S}} = \varnothing$ and ${}_{\mathcal{S}}{\leftarrow}\rtimes\xrightarrow{\epsilon}_{\mathcal{R}} = \varnothing$. Note that orthogonality of $\mathcal{R}$ and mutual orthogonality of $\mathcal{R}$ and $\mathcal{R}$ are equivalent.

Lemma 4.2.1 ([BN98, Theorem 9.3.11]). *For mutually orthogonal TRSs $\mathcal{R}$ and $\mathcal{S}$ the inclusion ${}_{\mathcal{R}}{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathcal{S}} \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathcal{S}} \cdot {}_{\mathcal{R}}{\leftarrow\!\!\!\!\!\parallel}$ holds.*

Theorem 4.2.2 ([Ros73]). *Every orthogonal TRS $\mathcal{R}$ is confluent.*

Proof. Let $\mathcal{A} = (\mathcal{T}(\mathcal{F},\mathcal{V}), \{\mathrel{\parallel\!\!\!\!\!\rightarrow}_1\})$ be the ARS equipped with the empty order $>$ on the index set $\{1\}$, where $\mathrel{\parallel\!\!\!\!\!\rightarrow}_1 = \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathcal{R}}$. According to Lemma 2.3.9 and Theorem 4.1.1, it is enough to show that $\mathcal{A}$ is decreasing. Since Lemma 4.2.1 yields ${}_1{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow}_1 \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow}_1 \cdot {}_1{\leftarrow\!\!\!\!\!\parallel}$, the decreasingness of $\mathcal{A}$ follows. ∎

The theorem can be recast as a compositional criterion that uses a confluent subsystem $\mathcal{C}$ of a given TRS $\mathcal{R}$. For this sake we switch the underlying criterion from Theorem 4.1.1 to Theorem 4.1.2 and Corollary 4.1.3, setting the relation of the least index $\perp$ to $\mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathcal{C}}$.

Theorem 4.2.3. *A left-linear TRS $\mathcal{R}$ is confluent if $\mathcal{R}$ and $\mathcal{R} \setminus \mathcal{C}$ are mutually orthogonal for some confluent TRS $\mathcal{C}$ with $\mathcal{C} \subseteq \mathcal{R}$.*

Proof. Suppose that $\mathcal{C} \subseteq \mathcal{R}$ and $\mathcal{C}$ is confluent. Let $\mathcal{A} = (\mathcal{T}(\mathcal{F},\mathcal{V}), \{\mathrel{\parallel\!\!\!\!\!\rightarrow}_0, \mathrel{\parallel\!\!\!\!\!\rightarrow}_1\})$ be the ARS equipped with the well-founded order $1 > 0$, where $\mathrel{\parallel\!\!\!\!\!\rightarrow}_0 = \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathcal{C}}$ and $\mathrel{\parallel\!\!\!\!\!\rightarrow}_1 = \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathcal{R}\setminus\mathcal{C}}$. Since $\mathcal{C}$ is confluent, $\mathcal{C}$ and $\mathcal{C}$ commute. So $\mathrel{\parallel\!\!\!\!\!\rightarrow}_0$ and $\mathrel{\parallel\!\!\!\!\!\rightarrow}_0$ commute too. According to Lemma 2.3.9 and Theorem 4.1.2, it is sufficient to show that all local peak ${}_i{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow}_j$ with $(i,j) \neq (0,0)$ are decreasing. Since $\mathcal{R}$ and $\mathcal{R} \setminus \mathcal{C}$ are mutually orthogonal, $\mathcal{R} \setminus \mathcal{C}$ and $\mathcal{R} \setminus \mathcal{C}$ as well as $\mathcal{C}$ and $\mathcal{R} \setminus \mathcal{C}$ are mutually orthogonal. Therefore, Lemma 4.2.1 yields the following inclusions:

$$ {}_{\mathcal{R}\setminus\mathcal{C}}{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathcal{R}\setminus\mathcal{C}} \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathcal{R}\setminus\mathcal{C}} \cdot {}_{\mathcal{R}\setminus\mathcal{C}}{\leftarrow\!\!\!\!\!\parallel} \qquad {}_{\mathcal{C}}{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathcal{R}\setminus\mathcal{C}} \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathcal{R}\setminus\mathcal{C}} \cdot {}_{\mathcal{C}}{\leftarrow\!\!\!\!\!\parallel} $$

So ${}_k{\leftarrow\!\!\!\!\!\parallel} \cdot \mathrel{\parallel\!\!\!\!\!\rightarrow}_m \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow}_m \cdot {}_k{\leftarrow\!\!\!\!\!\parallel}$ holds for all $(k,m) \in \{0,1\}^2 \setminus \{(0,0)\}$, from which the decreasingness of $\mathcal{A}$ follows. Hence, Theorem 4.1.2 applies. ∎

We can derive a more general criterion by exploiting the flexible valley form of decreasing diagrams. We will adopt parallel critical pairs. It causes no loss of confluence proving power of Theorem 4.2.3 as ${}_{\mathcal{R}}{\leftarrow\!\!\!\!\!\parallel}\rtimes\xrightarrow{\epsilon}_{\mathcal{S}} = \varnothing$ is equivalent to ${}_{\mathcal{R}}{\leftarrow}\rtimes\xrightarrow{\epsilon}_{\mathcal{S}} = \varnothing$.

Theorem 4.2.4. *A left-linear TRS $\mathcal{R}$ is confluent if ${}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+} \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}}$ holds for some confluent TRS $\mathcal{C}$ with $\mathcal{C} \subseteq \mathcal{R}$.*

Proof. Recall the ARS used in the proof of Theorem 4.2.3. According to Lemma 2.3.9 and Theorem 4.1.2, it is sufficient to show that every local peak

$$\Gamma : t \mathrel{\underset{k}{\overset{P}{\leftarrow\!\!\!\!\!+\!\!\!+}}} s \mathrel{\underset{m}{\overset{Q}{+\!\!\!+\!\!\!\!\!\to}}} u$$

with $(k, m) \neq (0, 0)$ is decreasing. To this end, we show $t \mathrel{+\!\!\!+\!\!\!\!\!\to}_m \cdot \mathrel{\leftarrow\!\!\!\!\!+\!\!\!+\!\!\!\!\!\to}^*_0 \cdot {}_k\!\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+} u$ by structural induction on s. Depending on the shape of Γ, we distinguish five cases.

1. If P or Q is empty then the claim is trivial.

2. If P or Q is $\{\epsilon\}$ and Γ is orthogonal then Lemma 3.2.1(1) yields $t \mathrel{+\!\!\!+\!\!\!\!\!\to}_m \cdot {}_k\!\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+} u$.

3. If $P \neq \varnothing$, $Q = \{\epsilon\}$, and Γ is not orthogonal then by Lemma 3.2.1(2) there exist a parallel critical peak $t_0 \; {}_k\!\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+} s_0 \xrightarrow{\epsilon}_m u_0$ and substitutions σ and τ such that $s = s_0\sigma$, $t = t_0\tau$, $u = u_0\sigma$, and $\sigma \mathrel{+\!\!\!+\!\!\!\!\!\to}_k \tau$. The assumption $t_0 \leftrightarrow^*_{\mathcal{C}} u_0$ yields $t_0\tau \mathrel{\leftarrow\!\!\!\!\!+\!\!\!+\!\!\!\!\!\to}^*_0 u_0\tau$ because $\to$ is closed under substitutions and $\to \subseteq \mathrel{+\!\!\!+\!\!\!\!\!\to}$. Therefore, $t = t_0\tau \mathrel{\leftarrow\!\!\!\!\!+\!\!\!+\!\!\!\!\!\to}^*_0 u_0\tau \; {}_k\!\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+} u_0\sigma = u$ follows.

4. If $P = \{\epsilon\}$, $Q \neq \varnothing$, and Γ is not orthogonal then the proof is analogous to the last case.

5. If $P \nsubseteq \{\epsilon\}$ and $Q \nsubseteq \{\epsilon\}$ then s, t, and u can be written as $f(s_1, \ldots, s_n)$, $f(t_1, \ldots, t_n)$, and $f(u_1, \ldots, u_n)$ respectively, and moreover, $t_i \; {}_k\!\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+} s_i \mathrel{+\!\!\!+\!\!\!\!\!\to}_m u_i$ holds for all $1 \leqslant i \leqslant n$. For every i the induction hypothesis yields $t_i \mathrel{+\!\!\!+\!\!\!\!\!\to}_m v_i \mathrel{\leftarrow\!\!\!\!\!+\!\!\!+\!\!\!\!\!\to}^*_0 w_i \; {}_k\!\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+} u_i$ for some v_i and w_i. Therefore, the desired conversion $t \mathrel{+\!\!\!+\!\!\!\!\!\to}_m v \mathrel{\leftarrow\!\!\!\!\!+\!\!\!+\!\!\!\!\!\to}^*_0 w \; {}_k\!\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+} u$ holds for $v = f(v_1, \ldots, v_n)$ and $w = f(w_1, \ldots, w_n)$.

∎

From Takahashi's proposition [Tak93] (see also [Ter03, Proposition 9.3.5]) we can deduce that ${}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+} \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq =$ is equivalent to ${}_{\mathcal{R}}\!\leftarrow \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq =$. Thus, Theorem 4.2.4 subsumes Theorem 4.2.3. Note that when $\mathcal{C} = \varnothing$, Theorem 4.2.4 simulates the weak orthogonality criterion.

$$\begin{tikzcd}
s_0\sigma \arrow[rr, "\{\epsilon\}", "m"'] \arrow[d, "k"] & & u_0\sigma = u \arrow[dd, dashed, "k"] \\
t_0\sigma \arrow[d, "k"] & & \\
t = t_0\tau \arrow[rr, leftrightarrow, dashed, "*", "0"'] & & u_0\tau
\end{tikzcd}$$

Figure 4.1: Proof of Theorem 4.2.4 (3).

Example 4.2.5. *By successive application of Theorem 4.2.4 we show the confluence of the left-linear TRS $\mathcal{R}$ (COPS [HNM18] number 62), taken from [OO03]:*

$$\begin{array}{lrcl@{\qquad}lrcl}
1: & x - 0 & \to & x & 10: & \mathsf{gcd}(x,0) & \to & x \\
2: & 0 - x & \to & 0 & 11: & \mathsf{gcd}(0,x) & \to & x \\
3: & \mathsf{s}(x) - \mathsf{s}(y) & \to & x - y & 12: & \mathsf{gcd}(x,y) & \to & \mathsf{gcd}(y,\mathsf{mod}(x,y)) \\
4: & x < 0 & \to & \mathsf{false} & 13: & \mathsf{if}(\mathsf{true},x,y) & \to & x \\
5: & 0 < \mathsf{s}(y) & \to & \mathsf{true} & 14: & \mathsf{if}(\mathsf{false},x,y) & \to & y \\
6: & \mathsf{s}(x) < \mathsf{s}(y) & \to & x < y & & & & \\
7: & \mathsf{mod}(x,0) & \to & x & & & & \\
8: & \mathsf{mod}(0,y) & \to & 0 & & & & \\
9: & \mathsf{mod}(x,\mathsf{s}(y)) & \to & \multicolumn{5}{l}{\mathsf{if}(x < \mathsf{s}(y), x, \mathsf{mod}(x - \mathsf{s}(y), \mathsf{s}(y)))}
\end{array}$$

Let $\mathcal{C} = \{5,7,8,10,11,13\}$. The six non-trivial parallel critical pairs of $\mathcal{R}$ are

$$\begin{array}{ll}
(x, \mathsf{gcd}(0,\mathsf{mod}(x,0))) & (0, \mathsf{if}(0 < \mathsf{s}(y), 0, \mathsf{mod}(0 - \mathsf{s}(y), \mathsf{s}(y)))) \\
(y, \mathsf{gcd}(y,\mathsf{mod}(0,y))) &
\end{array}$$

and their symmetric versions. All of them are joinable by $\mathcal{C}$. So it remains to show that $\mathcal{C}$ is confluent. Because $\mathcal{C}$ only admits trivial parallel critical pairs, ${}_{\mathcal{C}}{\leftarrow\!\!\shortparallel} \rtimes \xrightarrow{\epsilon}_{\mathcal{C}} \subseteq \leftrightarrow^*_{\varnothing}$ holds. Therefore, the confluence of $\mathcal{C}$ is concluded if we show the confluence of the empty system. The latter claim is trivial. This completes the proof.

It is worth noting that $\mathcal{C}$ is the smallest and confluent suitable subsystem of $\mathcal{R}$. The other suitable subsystems are supersets of $\mathcal{C}$ or non-confluent subsystems. For example, the subsystem $\mathcal{C}' = \{8,9,10,11,12\}$ satisfies ${}_{\mathcal{R}}{\leftarrow\!\!\shortparallel} \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}'}$, because all critical pairs are originated from rules in $\mathcal{C}'$, but $\mathcal{C}'$ is not confluent.

Theorem 4.2.4 is a generalization of Toyama's unpublished result:

Corollary 4.2.6 ([Toy17]). *A left-linear TRS $\mathcal{R}$ is confluent if ${}_{\mathcal{R}}{\leftarrow\!\!\!\!\!\!\!+\!\!\!+} \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}}$ holds for some terminating and confluent TRS $\mathcal{C}$ with $\mathcal{C} \subseteq \mathcal{R}$.*

4.3 Rule Labeling

In this section we recast the *rule labeling* criterion [vO08, ZFM15, DFJL22] in a compositional form. Rule labeling is a direct application of decreasing diagrams to confluence proofs for TRSs. It labels rewrite steps by their employed rewrite rules and compares indexes of them. Among others, we focus on the variant of rule labeling based on parallel critical pairs, introduced by Zankl et al. [ZFM15].

Definition 4.3.1. *Let $\mathcal{R}$ be a TRS. A* labeling function *for $\mathcal{R}$ is a function from $\mathcal{R}$ to $\mathbb{N}$. Given a labeling function ϕ and a number $k \in \mathbb{N}$, we define the TRS $\mathcal{R}_{\phi,k}$ as follows:*

$$\mathcal{R}_{\phi,k} = \{\ell \to r \in \mathcal{R} \mid \phi(\ell \to r) \leqslant k\}$$

*The relations $\to_{\mathcal{R}_{\phi,k}}$ and $\mathrel{+\!\!\!+\!\!\!\!\!\!\!\rightarrow}_{\mathcal{R}_{\phi,k}}$ are abbreviated to $\to_{\phi,k}$ and $\mathrel{+\!\!\!+\!\!\!\!\!\!\!\rightarrow}_{\phi,k}$. Let ϕ and ψ be labeling functions for $\mathcal{R}$. We say that a local peak $t \mathrel{\underset{\phi,k}{\overset{P}{\leftarrow\!\!\!\!\!\!\!+\!\!\!+}}} s \xrightarrow[\psi,m]{\epsilon} u$ is (ψ,ϕ)-*decreasing *if*

$$t \underset{\curlyvee k}{\overset{*}{\longleftrightarrow}} \cdot \underset{\psi,m}{\mathrel{+\!\!\!+\!\!\!\!\!\!\!\longrightarrow}} \cdot \underset{\curlyvee km}{\overset{*}{\longleftrightarrow}} v \mathrel{\underset{\phi,k}{\overset{P'}{\leftarrow\!\!\!\!\!\!\!+\!\!\!+}}} \cdot \underset{\curlyvee m}{\overset{*}{\longleftrightarrow}} u$$

and $\mathcal{V}\mathsf{ar}(v, P') \subseteq \mathcal{V}\mathsf{ar}(s, P)$ for some set P' of parallel positions and term v. Here $\leftrightarrow_K$ stands for the union of ${}_{\phi,k}{\leftarrow}$ and $\to_{\psi,k}$ for all $k \in K$.

The following theorem is a variant of the rule labeling method based on parallel critical pairs.

Theorem 4.3.2 ([ZFM15, Theorem 56]). *Let $\mathcal{R}$ be a left-linear TRS, and ϕ and ψ its labeling functions. The TRS $\mathcal{R}$ is confluent if the following conditions hold for all $k, m \in \mathbb{N}$.*

- *Every parallel critical peak of form $t \mathrel{\underset{\phi,k}{\leftarrow\!\!\!\!\!\!\!+\!\!\!+}} s \xrightarrow[\psi,m]{\epsilon} u$ is (ψ,ϕ)-decreasing.*
- *Every parallel critical peak of form $t \mathrel{\underset{\psi,m}{\leftarrow\!\!\!\!\!\!\!+\!\!\!+}} s \xrightarrow[\phi,k]{\epsilon} u$ is (ϕ,ψ)-decreasing.*

With a small example we illustrate the usage of rule labeling.

Example 4.3.3. *Consider the left-linear TRS $\mathcal{R}$:*

$$(x+y)+z \to x+(y+z) \qquad\qquad x+(y+z) \to (x+y)+z$$

We define the labeling functions ϕ and ψ as follows: $\phi(\ell \to r) = 0$ and $\psi(\ell \to r) = 1$ for all $\ell \to r \in \mathcal{R}$. All parallel critical peaks can be closed by $\to_{\phi,0}$-steps, like the following diagram:

$$
\begin{tikzcd}[column sep=small]
s = ((x+y)+z)+w \arrow[rr, "\epsilon", "{\psi,1}"'] \arrow[d, "{\{1\}\ \shortparallel\ \phi,0}"] & & (x+y)+(z+w) \arrow[d, dashed, "{\varnothing\ \shortparallel\ \phi,0}"] \\
(x+(y+z))+w & ((x+y)+z)+w \arrow[l, dashed, "{\phi,0}"'] & (x+y)+(z+w) = v \arrow[l, dashed, "{\phi,0}"']
\end{tikzcd}
$$

As $\mathcal{V}\mathsf{ar}(v,\varnothing) = \varnothing \subseteq \{x,y,z\} = \mathcal{V}\mathsf{ar}(s,\{1\})$*, this parallel critical peak is* (ψ,ϕ)*-decreasing. In a similar way the other peaks can also be verified. Hence, the TRS* $\mathcal{R}$ *is confluent.*

We make the rule labeling compositional. The next theorem is a compositional version of the rule labeling criterion. Note that by taking $\mathcal{C} := \mathcal{R}_{\phi,0} = \mathcal{R}_{\psi,0}$ it can be used as a compositional confluence criterion parameterized by $\mathcal{C}$.

Theorem 4.3.4. *Let* $\mathcal{R}$ *be a left-linear TRS, and* ϕ *and* ψ *its labeling functions. Suppose that* $\mathcal{R}_{\phi,0}$ *and* $\mathcal{R}_{\psi,0}$ *commute. The TRS* $\mathcal{R}$ *is confluent if the following conditions hold for all* $(k,m) \in \mathbb{N}^2 \setminus \{(0,0)\}$*.*

- *Every parallel critical peak of form* $t \underset{\phi,k}{\mathrel{\mathrlap{\,\,\shortparallel}\longleftarrow}} s \xrightarrow[\psi,m]{\epsilon} u$ *is* (ψ,ϕ)*-decreasing.*
- *Every parallel critical peak of form* $t \underset{\psi,m}{\mathrel{\mathrlap{\,\,\shortparallel}\longleftarrow}} s \xrightarrow[\phi,k]{\epsilon} u$ *is* (ϕ,ψ)*-decreasing.*

Proof. Consider the ARSs $(\mathcal{T}(\mathcal{F},\mathcal{V}), \{\mathrel{\mathrlap{\,\,\shortparallel\!\shortmid}\longrightarrow}_{\phi,k}\}_{k\in\mathbb{N}})$ and $(\mathcal{T}(\mathcal{F},\mathcal{V}), \{\mathrel{\mathrlap{\,\,\shortparallel\!\shortmid}\longrightarrow}_{\psi,m}\}_{m\in\mathbb{N}})$. According to Lemma 2.3.9 and Theorem 4.1.2, it is sufficient to show that every local peak

$$\Gamma\colon t \underset{\phi,k}{\overset{P}{\mathrel{\mathrlap{\,\,\shortparallel\!\shortmid}\longleftarrow}}} s \underset{\psi,m}{\overset{Q}{\mathrel{\mathrlap{\,\,\shortparallel\!\shortmid}\longrightarrow}}} u$$

with $(k,m) \neq (0,0)$ is decreasing. To this end, we perform structural induction on s. Depending on the shape of Γ, we distinguish five cases.

1. If P or Q is empty then the claim is trivial.
2. If P or Q is $\{\epsilon\}$ and Γ is orthogonal then Lemma 3.2.1(1) yields
$$t \underset{\psi,m}{\mathrel{\mathrlap{\,\,\shortparallel\!\shortmid}\longrightarrow}} \cdot \underset{\phi,k}{\mathrel{\mathrlap{\,\,\shortparallel\!\shortmid}\longleftarrow}} u.$$
3. If $P \neq \varnothing$, $Q = \{\epsilon\}$, and Γ is not orthogonal then by Lemma 3.2.1(2) there exist a parallel critical peak $t_0 \underset{\phi,k'}{\overset{P_1}{\mathrel{\mathrlap{\,\,\shortparallel}\longleftarrow}}} s_0 \xrightarrow[\psi,m]{\epsilon} u_0$ and substitutions σ and τ such that $k' \leqslant k$, $t = t_0\tau$, $u = u_0\sigma$, $\sigma \underset{\phi,k}{\mathrel{\mathrlap{\,\,\shortparallel\!\shortmid}\longrightarrow}} \tau$, $t_0\sigma \underset{\phi,k}{\overset{P\setminus P_1}{\mathrel{\mathrlap{\,\,\shortparallel\!\shortmid}\longrightarrow}}} t_0\tau$, and $P_1 \subseteq P$. We distinguish two subcases.[1] If $k' = 0$ and $m = 0$ then $t_0 \underset{0}{\overset{*}{\mathrel{\mathrlap{\,\,\shortparallel\!\shortmid}\longleftrightarrow}}} u_0$. As $\mathrel{\mathrlap{\,\,\shortparallel\!\shortmid}\longrightarrow}$ is

[1] The preliminary version of this thesis [SH22] lacks this case analysis.

closed under substitutions, $t_0\tau \xleftrightarrow[0]{*} u_0\tau$ follows. The step can be written as $t_0\tau \xleftrightarrow[\curlyvee k]{*} u_0\tau$ because $(k,m) \neq (0,0)$ and $m = 0$ imply $k > 0$. Summing them up, we obtain the sequence

$$t = t_0\tau \xleftrightarrow[\curlyvee k]{*} u_0\tau \xleftarrow[\phi,k]{} u_0\sigma = u$$

from which we conclude decreasingness of Γ. Otherwise, $k' > 0$ or $m > 0$ holds. The assumption yields

$$t_0 \xleftrightarrow[\curlyvee k']{*} \cdot \xrightarrow[\psi,m]{} \cdot \xleftrightarrow[\curlyvee k'm]{*} v_0 \xleftarrow[\phi,k']{P_1'} w_0 \xleftrightarrow[\curlyvee m]{*} u_0$$

and $\mathcal{V}\mathsf{ar}(v_0, P_1') \subseteq \mathcal{V}\mathsf{ar}(s_0, P_1)$ for some v_0, w_0, and P_1'. Since $k' \leqslant k$ and the rewrite steps are closed under substitutions, the following relations are obtained:

$$t_0\tau \xleftrightarrow[\curlyvee k]{*} \cdot \xrightarrow[\psi,m]{} \cdot \xleftrightarrow[\curlyvee km]{*} v_0\tau \qquad\qquad w_0\sigma \xleftrightarrow[\curlyvee m]{*} u_0\sigma$$

Since $t_0\sigma|_p = t_0\tau|_p$ holds for all $p \in P_1$, the identity $x\sigma = x\tau$ holds for all $x \in \mathcal{V}\mathsf{ar}(s_0, P_1)$. Therefore, $x\sigma = x\tau$ holds for all $x \in \mathcal{V}\mathsf{ar}(v_0, P_1')$. Because $w_0 \xrightarrow[\phi,k]{P_1'} v_0$, $\sigma \xrightarrow[\phi,k]{} \tau$, and $x\sigma = x\tau$ for all $x \in \mathcal{V}\mathsf{ar}(v_0, P_1')$ hold, Lemma 3.2.2 yields $w_0\sigma \xrightarrow[\phi,k]{} v_0\tau$. Hence, the decreasingness of Γ is witnessed by the following sequence:

$$t = t_0\tau \xleftrightarrow[\curlyvee k]{*} \cdot \xrightarrow[\psi,m]{} \cdot \xleftrightarrow[\curlyvee km]{*} v_0\tau \xleftarrow[\phi,k]{} w_0\sigma \xleftrightarrow[\curlyvee m]{*} u_0\sigma = u$$

Note that the construction is depicted in Fig. 4.2.

4. If $P = \{\epsilon\}$, $Q \neq \varnothing$, and Γ is not orthogonal then the proof is analogous to the last case.

5. If $P \nsubseteq \{\epsilon\}$ and $Q \nsubseteq \{\epsilon\}$ then s, t, and u can be written as $f(s_1,\ldots,s_n)$, $f(t_1,\ldots,t_n)$, and $f(u_1,\ldots,u_n)$ respectively, and moreover, $t_i \xleftarrow[\phi,k]{} s_i \xrightarrow[\psi,m]{} u_i$ holds for all $1 \leqslant i \leqslant n$. By the induction hypotheses we have $t_i \xleftrightarrow[\curlyvee k]{*} \cdot \xrightarrow[\psi,m]{} \cdot \xleftrightarrow[\curlyvee km]{*} \cdot \xleftarrow[\phi,k]{} \cdot \xleftrightarrow[\curlyvee m]{*} u_i$ for all $1 \leqslant i \leqslant n$. Therefore, we obtain the desired relations:

$$t = f(t_1,\ldots,t_n) \xleftrightarrow[\curlyvee k]{*} \cdot \xrightarrow[\psi,m]{} \cdot \xleftrightarrow[\curlyvee km]{*} \cdot \xleftarrow[\phi,k]{} \cdot \xleftrightarrow[\curlyvee m]{*} f(u_1,\ldots,u_n) = u$$

Hence Γ is decreasing.

$$
\begin{tikzcd}[column sep=large, row sep=large]
s_0\sigma \arrow[rrr, "\{\epsilon\}", "{\psi, m}"'] \arrow[dd, "{\phi, k}", "P_1"'] & & & u_0\sigma = u \arrow[d, dashed, leftrightarrow, "\curlyvee m", "*"'] \\
& & & w_0\sigma \arrow[d, dashed, "{\phi, k}", "P_1'"'] \arrow[dd, dashed, bend left=60, "{\phi, k}"] \\
t_0\sigma \arrow[r, dashed, leftrightarrow, "*", "\curlyvee k"'] \arrow[d, "{\phi, k}"] & \cdot \arrow[r, dashed, "{\psi, m}"'] & \cdot \arrow[r, dashed, leftrightarrow, "*", "\curlyvee km"'] & v_0\sigma \arrow[d, dashed, "{\phi, k}"] \\
t = t_0\tau \arrow[r, dashed, leftrightarrow, "*", "\curlyvee k"'] & \cdot \arrow[r, dashed, "{\psi, m}"'] & \cdot \arrow[r, dashed, leftrightarrow, "*", "\curlyvee km"'] & v_0\tau
\end{tikzcd}
$$

Figure 4.2: Proof of Theorem 4.3.4(3).

∎

The original version of rule labeling (Theorem 4.3.2) is a special case of Theorem 4.3.4: Suppose that labeling functions ϕ and ψ for a left-linear TRS $\mathcal{R}$ satisfy the conditions of Theorem 4.3.2. By taking the labeling functions ϕ' and ψ' with

$$\phi'(\ell \to r) = \phi(\ell \to r) + 1 \qquad\qquad \psi'(\ell \to r) = \psi(\ell \to r) + 1$$

Theorem 4.3.4 applies for ϕ', ψ', and the empty TRS $\mathcal{C}$.

The next example shows the combination of our rule labeling variant (Theorem 4.3.4) with Knuth–Bendix' criterion (Theorem 2.3.7).

Example 4.3.5. *Consider the left-linear TRS $\mathcal{R}$:*

$$1\colon\ 0 + x \to x \quad 2\colon\ (x+y)+z \to x+(y+z) \quad 3\colon\ x+(y+z) \to (x+y)+z$$

Let $\mathcal{C} = \{1, 2\}$. We define the labeling functions ϕ and ψ as follows:

$$\phi(\ell \to r) = \psi(\ell \to r) = \begin{cases} 0 & \text{if } \ell \to r \in \mathcal{C} \\ 1 & \text{otherwise} \end{cases}$$

For instance, the parallel critical pairs involving rule 3 admit the following diagrams:

$$
\begin{tikzcd}
x+(0+z) \arrow[r, "\epsilon", "\psi,1"'] \arrow[d, "\phi,0", "\{2\}"', "\#" marking] & (x+0)+z \arrow[d, dashed, "\phi,0"] \\
x+z & x+(0+z) \arrow[l, dashed, "\phi,0"]
\end{tikzcd}
\qquad
\begin{tikzcd}
x+(y+(z+w)) \arrow[r, "\epsilon", "\psi,1"'] \arrow[d, "\phi,1", "\{2\}"', "\#" marking] & (x+y)+(z+w) \arrow[d, dashed, "\phi,0"] \\
x+((y+z)+w) & x+(y+(z+w)) \arrow[l, dashed, "\{2\}", "\phi,1"', "\#" marking]
\end{tikzcd}
$$

They fit for the conditions of Theorem 4.3.4. The other parallel critical pairs also admit suitable diagrams. Therefore, it remains to show that $\mathcal{C}$ is confluent. Since $\mathcal{C}$ is terminating and all its critical pairs are joinable, confluence of $\mathcal{C}$ follows by Knuth and Bendix' criterion (Theorem 2.3.7). Thus, $\mathcal{R}_{\phi,0}$ and $\mathcal{R}_{\psi,0}$ commute because $\mathcal{R}_{\phi,0} = \mathcal{R}_{\psi,0} = \mathcal{C}$. Hence, by Theorem 4.3.4 we conclude that $\mathcal{R}$ is confluent.

While a proof for Theorem 4.2.4 is given in Section 4.2, here we present an alternative proof based on Theorem 4.3.4.

Proof of Theorem 4.2.4. Define the labeling functions ϕ and ψ as in Example 4.3.5. Then Theorem 4.3.4 applies. ∎

We conclude the section by stating that rule labeling based on parallel critical pairs (Theorem 4.3.2) subsumes parallel closedness based on parallel critical pairs (Theorem 3.1.9): Suppose that conditions (a) and (b) of Theorem 3.1.9 hold. We define ϕ and ψ as the constant rule labeling functions $\phi(\ell \to r) = 1$ and $\psi(\ell \to r) = 0$. By using structural induction as well as Lemmata 3.2.1 and 3.2.2 we can prove the implication

$$t \underset{\phi,1}{\overset{P_1}{\twoheadleftarrow\!\!\!\!\!\!+\!\!\!\!+}} s \underset{\psi,0}{\twoheadrightarrow\!\!\!\!\!\!+\!\!\!\!+} u \implies t \xrightarrow[\psi,0]{*} v \underset{\phi,1}{\overset{P_1'}{\twoheadleftarrow\!\!\!\!\!\!+\!\!\!\!+}} u \text{ and } \mathcal{V}\mathsf{ar}(v, P_1') \subseteq \mathcal{V}\mathsf{ar}(s, P_1) \text{ for some } P_1'$$

Thus, the conditions of Theorem 4.3.2 follow. As a consequence, our compositional version (Theorem 4.3.4) is also a generalization of parallel closedness.

4.4 Critical Pair Systems

The last example of compositional criteria in this thesis is a variant of the confluence criterion by critical pair systems [HM11]. It is known that the original criterion is a generalization of the orthogonal criterion (Theorem 4.2.2) and Knuth and Bendix' criterion (Theorem 2.3.7) for left-linear TRSs.

Definition 4.4.1. *The* critical pair system $\mathsf{CPS}(\mathcal{R})$ *of a TRS $\mathcal{R}$ is defined as the TRS:*

$$\{s \to t, s \to u \mid t \;{}_{\mathcal{R}}{\leftarrow}\; s \xrightarrow{\epsilon}_{\mathcal{R}} u \textit{ is a critical peak}\}$$

Theorem 4.4.2 ([HM11]). *A left-linear and locally confluent TRS $\mathcal{R}$ is confluent if* $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$ *is terminating (i.e.,* $\mathsf{CPS}(\mathcal{R})$ *is relatively terminating with respect to $\mathcal{R}$).*

The theorem is shown by using the decreasing diagram technique (Theorem 4.1.1), see [HM11].

Example 4.4.3. *Consider the left-linear and non-terminating TRS $\mathcal{R}$:*

$$\mathsf{s}(\mathsf{p}(x)) \to \mathsf{p}(\mathsf{s}(x)) \qquad \mathsf{p}(\mathsf{s}(x)) \to x \qquad \infty \to \mathsf{s}(\infty)$$

The TRS $\mathcal{R}$ admits two critical pairs and they are joinable:

$$\begin{tikzcd} & \mathsf{s}(\mathsf{p}(\mathsf{s}(x))) \arrow[dl] \arrow[dr, "\epsilon"] & \\ \mathsf{s}(x) & & \mathsf{p}(\mathsf{s}(\mathsf{s}(x))) \arrow[ll, dashed, bend left] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{p}(\mathsf{s}(\mathsf{p}(x))) \arrow[dl] \arrow[dr, "\epsilon"] & \\ \mathsf{p}(\mathsf{p}(\mathsf{s}(x))) \arrow[rr, dashed, bend right] & & \mathsf{p}(x) \end{tikzcd}$$

The critical pair system $\mathsf{CPS}(\mathcal{R})$ *consists of the four rules:*

$$\begin{array}{ll} \mathsf{s}(\mathsf{p}(\mathsf{s}(x))) \to \mathsf{s}(x) & \mathsf{p}(\mathsf{s}(\mathsf{p}(x))) \to \mathsf{p}(\mathsf{p}(\mathsf{s}(x))) \\ \mathsf{s}(\mathsf{p}(\mathsf{s}(x))) \to \mathsf{p}(\mathsf{s}(\mathsf{s}(x))) & \mathsf{p}(\mathsf{s}(\mathsf{p}(x))) \to \mathsf{p}(x) \end{array}$$

The termination of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$ *can be shown by, e.g., the termination tool* NaTT *(cf. Section 6.3). Hence the confluence of $\mathcal{R}$ follows by Theorem 4.4.2.*

We argue about the parallel critical pair version of $\mathsf{CPS}(\mathcal{R})$:

$$\mathsf{PCPS}(\mathcal{R}) = \{s \to t, s \to u \mid t \;{}_{\mathcal{R}}{\leftarrow\!\!\!\!\!\!\!+\!\!\!\!+}\; s \xrightarrow{\epsilon}_{\mathcal{R}} u \text{ is a parallel critical peak}\}$$

Interestingly, replacing $\mathsf{CPS}(\mathcal{R})$ by $\mathsf{PCPS}(\mathcal{R})$ in Theorem 4.4.2 results in the same criterion (see [ZFM15]). Since $\to_{\mathsf{CPS}(\mathcal{R})} \subseteq \to_{\mathsf{PCPS}(\mathcal{R})} \subseteq \to_{\mathsf{CPS}(\mathcal{R})} \cdot \mathrel{+\!\!\!\!+\!\!\!\!\!\!\!\to}_{\mathcal{R}}$ holds, $\to_{\mathsf{CPS}(\mathcal{R})/\mathcal{R}} = \to_{\mathsf{PCPS}(\mathcal{R})/\mathcal{R}}$ follows. So the termination of $\mathsf{PCPS}(\mathcal{R})/\mathcal{R}$ is equivalent to that of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$. However, a compositional form of Theorem 4.4.2 may benefit from the use of parallel critical pairs, as seen in Section 4.2.

Definition 4.4.4. *Let $\mathcal{R}$ and $\mathcal{C}$ be TRSs. The* parallel critical pair system *of $\mathcal{R}$ modulo $\mathcal{C}$, written as* $\mathsf{PCPS}(\mathcal{R},\mathcal{C})$*, is defined as the TRS:*

$$\{s \to t, s \to u \mid t \;{}_{\mathcal{R}}{\leftarrow\!\!\!\!\!\!\!+\!\!\!\!+}\; s \xrightarrow{\epsilon}_{\mathcal{R}} u \textit{ is a parallel critical peak but not } t \leftrightarrow^*_{\mathcal{C}} u\}$$

Note that $\mathsf{PCPS}(\mathcal{R},\varnothing) \subseteq \mathsf{PCPS}(\mathcal{R})$ holds in general, and $\mathsf{PCPS}(\mathcal{R},\varnothing) \subsetneq \mathsf{PCPS}(\mathcal{R})$ when $\mathcal{R}$ admits a trivial critical pair.

The next lemma relates $\mathsf{PCPS}(\mathcal{R},\mathcal{C})$ to closing forms of parallel critical peaks.

Lemma 4.4.5. *Let $\mathcal{R}$ be a left-linear TRS and $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{C}$ subsets of $\mathcal{R}$, and let* $\mathcal{P} = \mathsf{PCPS}(\mathcal{R},\mathcal{C})$*. Suppose that* ${}_{\mathcal{R}}{\leftarrow\!\!\!\!\!\!\!+\!\!\!\!+} \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \to^*_{\mathcal{R}} \cdot {}^{*}_{\mathcal{R}}\!\leftarrow$ *holds. If $t \;{}_{\mathcal{R}_1}{\leftarrow\!\!\!\!\!\!\!+\!\!\!\!+}\; s \mathrel{+\!\!\!\!+\!\!\!\!\!\!\!\to}_{\mathcal{R}_2} u$ then* *(i)* $t \mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\mathcal{R}_2} \cdot \leftrightarrow^*_{\mathcal{C}} \cdot {}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\Vert\!\!-} u$*, or*

(ii) $t \; {}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\Vert\!\!-} t' \; {}_{\mathcal{P}}\!\leftarrow s \rightarrow_{\mathcal{P}} u' \mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\mathcal{R}_2} u$ *and* $t' \rightarrow^*_{\mathcal{R}} \cdot {}^*_{\mathcal{R}}\!\leftarrow u'$ *for some* t' *and* u'*.*

Proof. Let $\Gamma\colon t \; {}_{\mathcal{R}_1}\!\overset{P}{\mathrel{\leftarrow\!\!\!\Vert\!\!-}} s \overset{Q}{\mathrel{-\!\!\Vert\!\!\!\rightarrow}}_{\mathcal{R}_2} u$ be a local peak. We use structural induction on s. Depending on the form of Γ, we distinguish five cases.

1. If P or Q is the empty set then (i) holds trivially.

2. If P or Q is $\{\epsilon\}$ and Γ is orthogonal then (i) follows by Lemma 3.2.1(1).

3. If $P \neq \varnothing$, $Q = \{\epsilon\}$, and Γ is not orthogonal then we distinguish two cases.
   - If there exist P_0, t_0, u_0, and σ such that "$P_0 \subseteq P$, $t \; {}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\Vert\!\!-} t_0\sigma \; {}_{\mathcal{R}_1}\!\overset{P_0}{\mathrel{\leftarrow\!\!\!\Vert\!\!-}} s \xrightarrow{\epsilon}_{\mathcal{R}_2} u_0\sigma = u$, and $t_0 \; {}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\Vert\!\!-} \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} u_0$" but not $t_0 \leftrightarrow^*_{\mathcal{C}} u_0$. Take $t' = t_0\sigma$ and $u' = u_0\sigma$. Then $t_0\tau \; {}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\Vert\!\!-} t_0\sigma \; {}_{\mathcal{P}}\!\leftarrow s \rightarrow_{\mathcal{P}} u_0\sigma = u$ holds and by the assumption $t' \rightarrow^*_{\mathcal{R}} \cdot {}^*_{\mathcal{R}}\!\leftarrow u'$ also holds. Hence (ii) follows.
   - Otherwise, whenever P_0, t_0, u_0, and σ satisfy the conditions quoted in the last item, $t_0 \leftrightarrow^*_{\mathcal{C}} u_0$ holds. Because Γ is not orthogonal, by Lemma 3.2.1(2) there exist P_0, t_0, u_0, σ, and τ such that $P_0 \subseteq P$, $t = t_0\tau \; {}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\Vert\!\!-} t_0\sigma \; {}_{\mathcal{R}_1}\!\overset{P_0}{\mathrel{\leftarrow\!\!\!\Vert\!\!-}} s \xrightarrow{\epsilon}_{\mathcal{R}_2} u_0\sigma = u$, and $\sigma \mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\mathcal{R}_1} \tau$. Thus $t_0 \leftrightarrow^*_{\mathcal{C}} u_0$ follows. Therefore, $t = t_0\tau \leftrightarrow^*_{\mathcal{C}} u_0\tau \; {}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\Vert\!\!-} u_0\sigma = u$, and hence (i) holds.

4. If $P = \{\epsilon\}$, $Q \nsubseteq \{\epsilon\}$, and Γ is not orthogonal then the proof is analogous to the last case.

5. If $P \nsubseteq \{\epsilon\}$ and $Q \nsubseteq \{\epsilon\}$ then s, t, and u can be written as $f(s_1,\ldots,s_n)$, $f(t_1,\ldots,t_n)$, and $f(u_1,\ldots,u_n)$ respectively, and $\Gamma_i\colon t_i \; {}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\Vert\!\!-} s_i \mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\mathcal{R}_2} u_i$ holds for all $1 \leqslant i \leqslant n$. For every peak Γ_i the induction hypothesis yields (i) or (ii). If (i) holds for all Γ_i then (i) is concluded for Γ. Otherwise, some Γ_i satisfies (ii). By taking $t' = f(s_1,\ldots,t_i,\ldots,s_n)$ and $u' = f(s_1,\ldots,u_i,\ldots,s_n)$ we have $t \; {}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\Vert\!\!-} t' \; {}_{\mathcal{P}}\!\leftarrow s \rightarrow_{\mathcal{P}} u' \mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\mathcal{P}} u$. From $t_i \rightarrow^*_{\mathcal{R}} \cdot {}^*_{\mathcal{R}}\!\leftarrow u_i$ we obtain $t' \rightarrow^*_{\mathcal{R}} \cdot {}^*_{\mathcal{R}}\!\leftarrow u'$. Hence Γ satisfies (ii).

∎

The next theorem is a compositional confluence criterion based on parallel critical pair systems.

Theorem 4.4.6. *Let* $\mathcal{R}$ *be a left-linear TRS and* $\mathcal{C}$ *a confluent TRS with* $\mathcal{C} \subseteq \mathcal{R}$*. The TRS* $\mathcal{R}$ *is confluent if* ${}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\Vert\!\!-} \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \rightarrow^*_{\mathcal{R}} \cdot {}^*_{\mathcal{R}}\!\leftarrow$ *and* $\mathcal{P}/\mathcal{R}$ *is terminating, where* $\mathcal{P} = \mathsf{PCPS}(\mathcal{R},\mathcal{C})$*.*

Proof. Let $\bot$ be a fresh symbol and let $I = \mathcal{T}(\mathcal{F},\mathcal{V}) \cup \{\bot\}$. We define the relation $>$ on I as follows: $\alpha > \beta$ if $\alpha \neq \bot = \beta$ or $\alpha \to^{+}_{\mathcal{P}/\mathcal{R}} \beta$. Since $\mathcal{P}/\mathcal{R}$ is terminating, $>$ is a well-founded order. Let $\mathcal{A} = (\mathcal{T}(\mathcal{F},\mathcal{V}), \{\mathrel{+\!\!\!\!\!\!\to}_\alpha\}_{\alpha \in I})$ be the ARS where $\mathrel{+\!\!\!\!\!\!\to}_\alpha$ is defined as follows: $s \mathrel{+\!\!\!\!\!\!\to}_\alpha t$ if either $\alpha = \bot$ and $s \mathrel{+\!\!\!\!\!\!\to}_\mathcal{C} t$, or $\alpha \neq \bot$ and $\alpha \to^*_\mathcal{R} s \mathrel{+\!\!\!\!\!\!\to}_{\mathcal{R}\setminus\mathcal{C}} t$. Since the commutation of $\mathcal{C}$ and $\mathcal{C}$ follows from confluence of $\mathcal{C}$, Lemma 2.3.9 yields the commutation of $\to_\bot$ and $\to_\bot$. According to Lemma 2.3.9 and Theorem 4.1.2, it is sufficient to show that every local peak

$$\Gamma \colon t \underset{\alpha}{\mathrel{\leftarrow\!\!\!\!\!\!+}} s \underset{\beta}{\mathrel{+\!\!\!\!\!\!\to}} u$$

with $(\alpha,\beta) \in I^2 \setminus \{(\bot,\bot)\}$ is decreasing. By the definition of $\mathcal{A}$ we have $s \mathrel{+\!\!\!\!\!\!\to}_{\mathcal{R}_1} t$ and $s \mathrel{+\!\!\!\!\!\!\to}_{\mathcal{R}_2} u$ for some TRSs $\mathcal{R}_1, \mathcal{R}_2 \in \{\mathcal{R}\setminus\mathcal{C}, \mathcal{C}\}$. Using Lemma 4.4.5, we distinguish two cases.

1. Suppose that Lemma 4.4.5((i)) holds for Γ. Then $t \mathrel{+\!\!\!\!\!\!\to}_{\mathcal{R}_2} t' \leftrightarrow^*_\mathcal{C} u' \;{}_{\mathcal{R}_1}\!\!\mathrel{\leftarrow\!\!\!\!\!\!+} u$ holds for some t' and u'. If $\mathcal{R}_2 = \mathcal{R}\setminus\mathcal{C}$ then $t \mathrel{+\!\!\!\!\!\!\to}_\beta t'$ follows from $\beta \to^*_\mathcal{R} s \to^*_\mathcal{R} t \mathrel{+\!\!\!\!\!\!\to}_{\mathcal{R}\setminus\mathcal{C}} t'$. Otherwise, $\mathcal{R}_2 = \mathcal{C}$ yields $t \mathrel{+\!\!\!\!\!\!\to}_\bot t'$. In either case $t \mathrel{+\!\!\!\!\!\!\to}_{\{\beta,\bot\}} t'$ is obtained. Similarly, $u \mathrel{+\!\!\!\!\!\!\to}_{\{\alpha,\bot\}} u'$ is obtained. Moreover, $t' \mathrel{\leftarrow\!\!\!\!\!\!+\!\!\!\!\!\!\to}^*_\bot u'$ follows from $t' \leftrightarrow^*_\mathcal{C} u'$. Since $(\alpha,\beta) \neq (\bot,\bot)$ yields $\bot \in {\curlyvee}\alpha\beta$ and the reflexivity of $\mathrel{+\!\!\!\!\!\!\to}_\bot$ yields $\mathrel{+\!\!\!\!\!\!\to}_{\{\delta,\bot\}} \subseteq \mathrel{+\!\!\!\!\!\!\to}^=_\delta \cdot \mathrel{+\!\!\!\!\!\!\to}_\bot$ for any δ, we obtain the desirable conversion $t \overset{=}{\underset{\beta}{\mathrel{+\!\!\!\!\!\!\to}}} t' \overset{*}{\underset{{\curlyvee}\alpha\beta}{\mathrel{\leftarrow\!\!\!\!\!\!+\!\!\!\!\!\!\to}}} u' \overset{=}{\underset{\alpha}{\mathrel{\leftarrow\!\!\!\!\!\!+}}} u$. Hence, Γ is decreasing.

2. Suppose that Lemma 4.4.5((ii)) holds for Γ. We have $t \;{}_{\mathcal{R}_1}\!\!\mathrel{\leftarrow\!\!\!\!\!\!+} t' \;{}_\mathcal{P}\!\!\leftarrow s \to_\mathcal{P} u' \mathrel{+\!\!\!\!\!\!\to}_{\mathcal{R}_2} u$ and $t' \to^*_\mathcal{R} v \;{}^*_\mathcal{R}\!\!\leftarrow u'$ for some t', u', and v. As $(\alpha,\beta) \neq (\bot,\bot)$, we have $\alpha \to^*_\mathcal{R} s \to_\mathcal{P} t'$ or $\beta \to^*_\mathcal{R} s \to_\mathcal{P} t'$, from which $\alpha > t'$ or $\beta > t'$ follows. Thus, $t' \in {\curlyvee}\alpha\beta$. If $\mathcal{R}_2 = \mathcal{R}\setminus\mathcal{C}$ then $t' \mathrel{+\!\!\!\!\!\!\to}_{t'} t$. Otherwise, $\mathcal{R}_2 = \mathcal{C}$ yields $t' \mathrel{+\!\!\!\!\!\!\to}_\bot t$. So in either case $t' \mathrel{+\!\!\!\!\!\!\to}_{{\curlyvee}\alpha\beta} t$ holds. Next, we show $t' \mathrel{\leftarrow\!\!\!\!\!\!+\!\!\!\!\!\!\to}^*_{{\curlyvee}\alpha\beta} v$. Consider terms w and w' with $t' \to^*_\mathcal{R} w \to_\mathcal{R} w' \to^*_\mathcal{R} v$. We have $w \mathrel{+\!\!\!\!\!\!\to}_{t'} w'$ or $w \mathrel{+\!\!\!\!\!\!\to}_\bot w'$. So $w \mathrel{+\!\!\!\!\!\!\to}_{{\curlyvee}\alpha\beta} w'$ follows by $\{t',\bot\} \subseteq {\curlyvee}\alpha\beta$. Summing up, we obtain $t \mathrel{\leftarrow\!\!\!\!\!\!+}_{{\curlyvee}\alpha\beta} t' \mathrel{\leftarrow\!\!\!\!\!\!+\!\!\!\!\!\!\to}^*_{{\curlyvee}\alpha\beta} v$. In a similar way $u \mathrel{\leftarrow\!\!\!\!\!\!+}_{{\curlyvee}\alpha\beta} u' \mathrel{\leftarrow\!\!\!\!\!\!+\!\!\!\!\!\!\to}^*_{{\curlyvee}\alpha\beta} v$ is obtained. Therefore $t \underset{{\curlyvee}\alpha\beta}{\mathrel{\leftarrow\!\!\!\!\!\!+}} t' \overset{*}{\underset{{\curlyvee}\alpha\beta}{\mathrel{+\!\!\!\!\!\!\to}}} v \overset{*}{\underset{{\curlyvee}\alpha\beta}{\mathrel{\leftarrow\!\!\!\!\!\!+}}} u' \underset{{\curlyvee}\alpha\beta}{\mathrel{+\!\!\!\!\!\!\to}} u$, and hence Γ is decreasing.

∎

We claim that Theorem 4.4.2 is subsumed by Theorem 4.4.6. Suppose that $\mathcal{C}$ is the empty TRS. Trivially $\mathcal{C}$ is confluent. Because $\mathsf{PCPS}(\mathcal{R},\mathcal{C})$ is a subset of $\mathsf{PCPS}(\mathcal{R})$, termination of $\mathsf{PCPS}(\mathcal{R},\mathcal{C})/\mathcal{R}$ follows from that of $\mathsf{PCPS}(\mathcal{R})/\mathcal{R}$, which is equivalent to termination of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$. Finally, any confluent TRS satisfies the inclusion ${}_\mathcal{R}\!\!\mathrel{\leftarrow\!\!\!\!\!\!+} \rtimes \overset{\epsilon}{\to}_\mathcal{R} \subseteq \to^*_\mathcal{R} \cdot {}^*_\mathcal{R}\!\!\leftarrow$. Thus, whenever Theorem 4.4.2 applies, Theorem 4.4.6 applies.

Theorem 4.4.6 also subsumes Theorem 4.2.4. Suppose that $\mathcal{C}$ is a confluent subsystem of $\mathcal{R}$. If ${}_{\mathcal{R}}{\leftarrow\!\!\!+\!\!\!\!+} \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}}$ then $\mathsf{PCPS}(\mathcal{R},\mathcal{C}) = \varnothing$, which leads to termination of $\mathsf{PCPS}(\mathcal{R},\mathcal{C})/\mathcal{R}$. Hence, Theorem 4.4.6 applies. Note that if $\mathcal{C} = \mathcal{R}$ then $\mathsf{PCPS}(\mathcal{R},\mathcal{C}) = \varnothing$.

Example 4.4.7. *Consider the left-linear TRS $\mathcal{R}$:*

$$\begin{array}{lll}
1\colon\ \mathsf{s}(\mathsf{p}(x)) \to x & 3\colon\ x + \mathsf{0} \to x & 5\colon\ x + \mathsf{s}(y) \to \mathsf{s}(x+y) \\
2\colon\ \mathsf{p}(\mathsf{s}(x)) \to x & 4\colon\ \mathsf{0} + x \to x + \mathsf{0} & 6\colon\ x + \mathsf{p}(y) \to \mathsf{p}(x+y)
\end{array}$$

We show the confluence of $\mathcal{R}$ by the combination of Theorem 4.4.6 and orthogonality. Let $\mathcal{C} = \{3\}$. The TRS $\mathsf{PCPS}(\mathcal{R},\mathcal{C})$ consists of the eight rules:

$$\begin{array}{ll}
\mathsf{0} + \mathsf{s}(x) \to \mathsf{s}(\mathsf{0} + x) & x + \mathsf{s}(\mathsf{p}(y)) \to \mathsf{s}(x + \mathsf{p}(y)) \\
\mathsf{0} + \mathsf{s}(x) \to \mathsf{s}(x) + \mathsf{0} & x + \mathsf{s}(\mathsf{p}(y)) \to x + y \\
\mathsf{0} + \mathsf{p}(x) \to \mathsf{p}(\mathsf{0} + x) & x + \mathsf{p}(\mathsf{s}(y)) \to \mathsf{p}(x + \mathsf{s}(y)) \\
\mathsf{0} + \mathsf{p}(x) \to \mathsf{p}(x) + \mathsf{0} & x + \mathsf{p}(\mathsf{s}(y)) \to x + y
\end{array}$$

The termination of $\mathsf{PCPS}(\mathcal{R},\mathcal{C})/\mathcal{R}$ can be shown by, e.g., the termination tool NaTT*. Since $\mathcal{C}$ is orthogonal and all parallel critical pairs of $\mathcal{R}$ are joinable by $\mathcal{R}$, Theorem 4.4.6 applies. Note that the confluence of $\mathcal{R}$ can neither be shown by Theorem 4.3.2 nor Theorem 4.4.2. The former theorem fails due to the lack of suitable labeling functions for the following diagrams:*

$$\begin{tikzcd}
x + \mathsf{s}(\mathsf{p}(y)) \arrow[r, "\epsilon", "5"'] \arrow[d, "\{2\}"', "1"] & \mathsf{s}(x + \mathsf{p}(y)) \arrow[d, dashed, "6"] \\
x + y & \mathsf{s}(\mathsf{p}(x+y)) \arrow[l, dashed, "1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
x + \mathsf{p}(\mathsf{s}(y)) \arrow[r, "\epsilon", "6"'] \arrow[d, "\{2\}"', "2"] & \mathsf{p}(x + \mathsf{s}(y)) \arrow[d, dashed, "5"] \\
x + y & \mathsf{p}(\mathsf{s}(x+y)) \arrow[l, dashed, "2"]
\end{tikzcd}$$

The latter fails due to the non-termination of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$. The culprit is the idempotent rule $\mathsf{0} + \mathsf{0} \to \mathsf{0} + \mathsf{0}$ in $\mathsf{CPS}(\mathcal{R})$, originating from the critical peak $\mathsf{0} \leftarrow \mathsf{0} + \mathsf{0} \to \mathsf{0} + \mathsf{0}$. In contrast, the rule does not belong to $\mathsf{PCPS}(\mathcal{R},\mathcal{C})$ because the conversion $\mathsf{0} \leftrightarrow^*_{\mathcal{C}} \mathsf{0} + \mathsf{0}$ holds.

Chapter 5

Reduction Method

We present *reduction methods* for confluence problems. A reduction method is a compositional confluence criterion that, given a TRS $\mathcal{R}$ and its subsystem $\mathcal{C}$, confluence of $\mathcal{R}$ implies that of $\mathcal{C}$. In this situation confluence of $\mathcal{R}$ is equivalent to that of $\mathcal{C}$. In other words, reduction methods can remove rewrite rules from TRSs, that are redundant for confluence. We introduce two reduction methods and an automation technique based on SMT solvers.

5.1 Reduction Methods

Compositional confluence criteria address the 'if' direction. Our question here is how to guarantee the reverse direction. First we develop a simple criterion that exploits the fact that confluence is preserved under signature extensions. In order to establish the 'only-if' direction, we show that if TRSs $\mathcal{R}$ and $\mathcal{C}$ satisfy $\mathcal{R}{\restriction_{\mathcal{C}}} \subseteq \to_{\mathcal{C}}^*$ then confluence of $\mathcal{R}$ implies confluence of $\mathcal{C}$. Here $\mathcal{R}{\restriction_{\mathcal{C}}}$ stands for the following subsystem of $\mathcal{R}$:

$$\mathcal{R}{\restriction_{\mathcal{C}}} = \{\ell \to r \in \mathcal{R} \mid \mathcal{F}\mathsf{un}(\ell) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})\}$$

The following auxiliary lemma explains the role of the condition $\mathcal{R}{\restriction_{\mathcal{C}}} \subseteq \to_{\mathcal{C}}^*$. Note that we do not assume $\mathcal{C} \subseteq \mathcal{R}$.

Lemma 5.1.1. *Suppose* $\mathcal{R}{\restriction_{\mathcal{C}}} \subseteq \to_{\mathcal{C}}^*$.

1. *If* $s \to_{\mathcal{R}} t$ *and* $\mathcal{F}\mathsf{un}(s) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})$ *then* $s \to_{\mathcal{C}}^* t$ *and* $\mathcal{F}\mathsf{un}(t) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})$.

2. *If* $s \to_{\mathcal{R}}^* t$ *and* $s \in \mathcal{T}(\mathcal{F}\mathsf{un}(\mathcal{C}), \mathcal{V})$ *then* $s \to_{\mathcal{C}}^* t$.

Proof. We only show the first claim, because then the second claim is shown by straightforward induction. Suppose $s \in \mathcal{T}(\mathcal{F}\mathsf{un}(\mathcal{C}), \mathcal{V})$ and $s \to_{\mathcal{R}} t$. There exist a rule $\ell \to r \in \mathcal{R}$, a position $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(s)$, and a substitution σ such that $s|_p = \ell\sigma$, $t = s[r\sigma]_p$, and $\mathcal{F}\mathsf{un}(\ell\sigma) \subseteq \mathcal{F}\mathsf{un}(s) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})$. Then $\mathcal{F}\mathsf{un}(x\sigma) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})$ for all $x \in \mathcal{V}\mathsf{ar}(\ell)$. Moreover $\mathcal{F}\mathsf{un}(x\sigma) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})$ holds for all $x \in \mathcal{V}\mathsf{ar}(r)$ because $\ell \to r$ is a rewrite rule. We distinguish two cases:

- If $\ell \to r \in \mathcal{R}{\restriction_{\mathcal{C}}}$ then $\mathcal{F}\mathsf{un}(\ell) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})$. Since $\mathcal{R}{\restriction_{\mathcal{C}}} \subseteq \to_{\mathcal{C}}^{*}$ holds, we have $s \to_{\mathcal{C}}^{*} t$ and $\mathcal{F}\mathsf{un}(r) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})$. From the latter $\mathcal{F}\mathsf{un}(r\sigma) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})$ holds. Thus the inclusion $\mathcal{F}\mathsf{un}(t) = \mathcal{F}\mathsf{un}(s[r\sigma]_p) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})$ is obtained.
- Otherwise, $\mathcal{F}\mathsf{un}(\ell) \nsubseteq \mathcal{F}\mathsf{un}(\mathcal{C})$. However, we have $\mathcal{F}\mathsf{un}(\ell) \subseteq \mathcal{F}\mathsf{un}(s) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})$, so this case does not happen.

∎

As a consequence of Lemma 5.1.1(2), confluence of $\mathcal{R}$ carries over to confluence of $\mathcal{C}$, when the inclusion $\mathcal{R}{\restriction_{\mathcal{C}}} \subseteq \to_{\mathcal{C}}^{*}$ holds and the signature of $\mathcal{C}$ is $\mathcal{F}\mathsf{un}(\mathcal{C})$. The restriction against the signature of $\mathcal{C}$ can be lifted by the fact that confluence is preserved under signature extensions:

Proposition 5.1.2. *A TRS $\mathcal{C}$ is confluent if and only if the implication*

$$t \;{}_{\mathcal{C}}^{*}\!\leftarrow s \to_{\mathcal{C}}^{*} u \implies t \to_{\mathcal{C}}^{*} \cdot {}_{\mathcal{C}}^{*}\!\leftarrow u$$

holds for all terms $s, t, u \in \mathcal{T}(\mathcal{F}\mathsf{un}(\mathcal{C}), \mathcal{V})$.

Proof. Toyama [Toy87] showed that the confluence property is *modular*, i.e., the union of two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over signatures $\mathcal{F}_1$ and $\mathcal{F}_2$ with $\mathcal{F}_1 \cap \mathcal{F}_2 = \varnothing$ is confluent if and only if both $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent. Let $\mathcal{C}$ be a TRS over a signature $\mathcal{F}$. The claim follows by taking $\mathcal{R}_1 = \mathcal{C}$, $\mathcal{R}_2 = \varnothing$, $\mathcal{F}_1 = \mathcal{F}\mathsf{un}(\mathcal{C})$, and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. ∎

Now we are ready to show the main claim.

Theorem 5.1.3. *Suppose $\mathcal{R}{\restriction_{\mathcal{C}}} \subseteq \to_{\mathcal{C}}^{*}$. If $\mathcal{R}$ is confluent then $\mathcal{C}$ is confluent.*

Proof. Suppose that $\mathcal{R}$ is confluent. It is enough to show the implication in Proposition 5.1.2 for all $s, t, u \in \mathcal{T}(\mathcal{F}\mathsf{un}(\mathcal{C}), \mathcal{V})$. Suppose $t \;{}_{\mathcal{C}}^{*}\!\leftarrow s \to_{\mathcal{C}}^{*} u$. By confluence of $\mathcal{R}$ we have $t \to_{\mathcal{R}}^{*} v \;{}_{\mathcal{R}}^{*}\!\leftarrow u$ for some v. By assumption Lemma 5.1.1(2) yields $t \to_{\mathcal{C}}^{*} v \;{}_{\mathcal{C}}^{*}\!\leftarrow u$. ∎

A reduction method can be obtained by combining a compositional confluence criterion with Theorem 5.1.3. Here we present the combination of Theorem 4.2.4 with Theorem 5.1.3 and its automation technique.

Corollary 5.1.4. *Let $\mathcal{C}$ be a subsystem of a left-linear TRS $\mathcal{R}$ such that ${}_{\mathcal{R}}\!\leftarrow\!\!\!\!+\!\!\!\!\rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow_{\mathcal{C}}^{*}$ and $\mathcal{R}{\restriction_{\mathcal{C}}} \subseteq \to_{\mathcal{C}}^{*}$. The TRS $\mathcal{R}$ is confluent if and only if $\mathcal{C}$ is confluent.*

The following example illustrates how Corollary 5.1.4 is used for automating confluence analysis.

Example 5.1.5. *We show the confluence of the following left-linear TRS $\mathcal{R}$:*

$$\begin{array}{llllll} 1: & x + 0 \to x & 3: & 0 + y \to y & 5: & \mathsf{s}(x) + y \to \mathsf{s}(x + y) \\ 2: & x \times 0 \to 0 & 4: & \mathsf{s}(x) \times 0 \to 0 & 6: & \mathsf{s}(x) \times y \to (x \times y) + y \end{array}$$

Applying the reduction method of Corollary 5.1.4 repeatedly, we remove rules unnecessary for confluence analysis.

1. *The TRS $\mathcal{R}$ has four non-trivial parallel critical pairs and they admit the following diagrams:*

$$\begin{tikzcd} & \mathsf{s}(x) + 0 \arrow[dl, "\shortparallel"' marking] \arrow[dr, "\epsilon"] & \\ \mathsf{s}(x) & & \mathsf{s}(x + 0) \arrow[ll, bend left, "1"] \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{s}(x) + 0 \arrow[dl, "\shortparallel"' marking] \arrow[dr, "\epsilon"] & \\ \mathsf{s}(x + 0) \arrow[rr, bend right, "1"'] & & \mathsf{s}(x) \end{tikzcd}$$

$$\begin{tikzcd} & \mathsf{s}(x) \times 0 \arrow[dl, "\shortparallel"' marking] \arrow[dr, "\epsilon"] & \\ 0 & & (x \times 0) + 0 \arrow[dl, "1"] \\ & x \times 0 \arrow[ul, "2"] & \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{s}(x) \times 0 \arrow[dl, "\shortparallel"' marking] \arrow[dr, "\epsilon"] & \\ (x \times 0) + 0 \arrow[dr, "1"'] & & 0 \\ & x \times 0 \arrow[ur, "2"'] & \end{tikzcd}$$

   Therefore, ${}_{\mathcal{R}}\!\leftarrow\!\shortparallel\!\!- \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}_0}$ holds for $\mathcal{C}_0 = \{1,2\}$. As $\mathcal{F}\mathsf{un}(\mathcal{C}_0) = \{0, +, \times\}$, we have $\mathcal{R}\restriction_{\mathcal{C}_0} = \{1,2,3\}$. However, $\mathcal{R}\restriction_{\mathcal{C}_0} \subseteq \to^*_{\mathcal{C}_0}$ does not hold due to $0 + y \not\to^*_{\mathcal{C}_0} y$. So we extend $\mathcal{C}_0$ to $\mathcal{C} = \mathcal{C}_0 \cup \{3\}$. Then $\mathcal{R}\restriction_{\mathcal{C}} = \{1,2,3\} \subseteq \to^*_{\mathcal{C}}$ holds. Because $\mathcal{C}$ is a superset of $\mathcal{C}_0$, the inclusion ${}_{\mathcal{R}}\!\leftarrow\!\shortparallel\!\!- \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}}$ holds too. According to Corollary 5.1.4, the confluence problem of $\mathcal{R}$ is reduced to that of $\mathcal{C}$.

2. *Since $\mathcal{C}$ only admits a trivial parallel critical pair, it is closed by the empty system $\varnothing$. Moreover, the inclusion $\mathcal{C}\restriction_{\varnothing} = \varnothing \subseteq \to^*_{\varnothing}$ holds. Hence, by Corollary 5.1.4 the confluence of $\mathcal{C}$ is reduced to the confluence of the empty system $\varnothing$.*

3. *The confluence of the empty system $\varnothing$ is trivial.*

Hence we conclude that $\mathcal{R}$ is confluent. Note that in the first step all subsystems $\mathcal{C}'$ including $\mathcal{C}_0$ or $\{1,4,6\}$ satisfy the inclusion ${}_{\mathcal{R}}\!\leftarrow\!\shortparallel\!\!- \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}'}$ but some of them (e.g., $\{1,4,6\}$) are non-confluent. The additional requirement $\mathcal{R}\restriction_{\mathcal{C}'} \subseteq \to^*_{\mathcal{C}'}$ excludes such subsystems.

We give another application of Theorem 5.1.3. Because the theorem has no linearity restriction, we can obtain a reduction method for non-left-linear TRSs by employing a compositional criterion that supports non-left-linear TRSs. Here we use the result of [KH12].

Definition 5.1.6. *Given a term t, we write* $\mathsf{REN}(t)$ *for the linear term results from replacing each variable occurrence by a fresh variable. Given a TRS $\mathcal{R}$,* $\mathsf{REN}(\mathcal{R})$ *denotes the set* $\{\mathsf{REN}(\ell) \to r \mid \ell \to r \in \mathcal{R}\}$.

In general $\mathsf{REN}(\mathcal{R})$ is not a TRS. In fact, if $\mathcal{R}$ contains a rule $\ell \to r$ that has at least one variable in r then $\mathsf{REN}(\ell) \to r$ is not a rewrite rule, because the inclusion $\mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(\mathsf{REN}(\ell))$ does not hold. We call such a rule an *extended rewrite rule*. Formally, a pair (ℓ, r) of terms is an extended rewrite rule if $\ell \notin \mathcal{V}$. A set of extended rewrite rule is called an *extended TRS (eTRS)*.

Definition 5.1.7. *Let $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{E}$ be eTRSs. Suppose that the following conditions hold:*

- *$\ell_1 \to r_1$ and $\ell_2 \to r_2$ are variants of rules in $\mathcal{R}_1$ and in $\mathcal{R}_2$, respectively,*
- *$\ell_1 \to r_1$ and $\ell_2 \to r_2$ have no common variables,*
- *$p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$,*
- *σ is a substitution that satisfies $\ell_1\sigma \leftrightarrow^*_{\mathcal{E}} \ell_2|_p\sigma$, and*
- *if $p = \epsilon$ then $\ell_1 \to r_1$ is not a variant of $\ell_2 \to r_2$.*

*The triple $(\ell_1 \to r_1, p, \ell_2 \to r)$ is called an $\mathcal{E}$-*overlap *between $\mathcal{R}_1$ and $\mathcal{R}_2$, and the pair $((\ell_2\sigma)[r_1\sigma]_p, r_2\sigma)$ is called an $\mathcal{E}$-*extended critical pair *between $\mathcal{R}_1$ and $\mathcal{R}_2$. As in the case of ordinary critical pairs, we denote an extended critical pair (t, u) by $t \ {}_{\mathcal{R}}{\leftarrow}_{\mathcal{E}}{\rtimes}\xrightarrow{\epsilon}_{\mathcal{R}} u$.*

Note that a number of $\mathcal{E}$-extended critical pairs may exist for an $\mathcal{E}$-overlap in this definition. Moreover we cannot assume finiteness of $\mathcal{E}$-critical pairs. If $\mathcal{E} = \varnothing$ then σ is a most general unifier of ℓ and $\ell_2|_p$.

Definition 5.1.8. *Let $\mathcal{R}$ and $\mathcal{C}$ be a TRSs. We say that $\mathcal{R}$ and $\mathcal{C}$ are* strongly non-overlapping *if there is no $\varnothing$-overlap between* $\mathsf{REN}(\mathcal{R})$ *and* $\mathsf{REN}(\mathcal{C})$, *and* $\mathsf{REN}(\mathcal{C})$ *and* $\mathsf{REN}(\mathcal{R})$, *respectively.*

Theorem 5.1.9 ([KH12, Theorem 2])**.** *Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs. Suppose that $\mathcal{S}$ is confluent, $\mathcal{R}/\mathcal{S}$ is terminating, and $\mathcal{R}$ and $\mathcal{S}$ are strongly non-overlapping. The TRS $\mathcal{R} \cup \mathcal{S}$ is confluent if and only if ${}_{\mathcal{R}}{\leftarrow}_{\mathcal{S}}{\rtimes}\xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \to^*_{\mathcal{R}\cup\mathcal{S}} \cdot {}_{\mathcal{R}\cup\mathcal{S}}^{*}{\leftarrow}$.*

Note that the criterion subsumes Theorem 4.2.2. If $\mathcal{R}$ is orthogonal then $\mathcal{R}$ trivially satisfies all conditions by taking $\mathcal{S} = \varnothing$. Now we give a reduction method for non-left-linear TRSs.

Corollary 5.1.10. *Let $\mathcal{C}$ be a subsystem of a TRS $\mathcal{R}$ and $\mathcal{P} = \mathcal{R} \setminus \mathcal{C}$. Suppose that $\mathcal{P}/\mathcal{C}$ is terminating, $\mathcal{P}$ and $\mathcal{C}$ are strongly non-overlapping, ${}_{\mathcal{P}}{\leftarrow}_{\mathcal{C}}{\rtimes}\xrightarrow{\epsilon}_{\mathcal{P}} \subseteq \to^*_{\mathcal{R}} \cdot {}_{\mathcal{R}}^{*}{\leftarrow}$, and $\mathcal{R}\restriction_{\mathcal{C}} = \mathcal{C}$. The TRS $\mathcal{R}$ is confluent if and only if $\mathcal{C}$ is confluent.*

Under the assumption, the conditions $\mathcal{R}{\restriction_\mathcal{C}} = \mathcal{C}$ and $\mathcal{R}{\restriction_\mathcal{C}} \subseteq \to_\mathcal{C}^*$ are equivalent. We briefly describe the reason. Assume to contrary there is $\ell \to r \in \mathcal{R}{\restriction_\mathcal{C}} \setminus \mathcal{C}$ such that $\ell \to_\mathcal{C}^* r$. If $\ell = r$ then $\mathcal{P}/\mathcal{C}$ is not terminating. Otherwise, $\mathcal{P}$ and $\mathcal{C}$ are strongly non-overlapping because $\ell \to r \in \mathcal{P}$ and $\ell \to_{\{\ell' \to r'\}} \cdot \to_\mathcal{C}^* r$ for some $\ell' \to r' \in \mathcal{C}$. Hence $\mathcal{R}{\restriction_\mathcal{C}} = \mathcal{C}$.

This reduction method is useful to remove redundant non-left-linear rules from TRSs. The following proposition helps us to check joinability of $\mathcal{E}$-critical pairs.

Proposition 5.1.11 ([KH12, Lemma 12 and Theorem 15])**.** *Let $\mathcal{R}$ and $\mathcal{E}$ be TRSs. If $\mathcal{E}$ is confluent, $\mathcal{R}$ and $\mathcal{E}$ are strongly non-overlapping, and ${}_\mathcal{R}{\leftarrow} \rtimes \xrightarrow{\epsilon}_\mathcal{R} \subseteq \to_\mathcal{R}^* \cdot {}_\mathcal{R}^*{\leftarrow}$ then ${}_\mathcal{R}{\leftarrow}_\mathcal{C} \rtimes \xrightarrow{\epsilon}_\mathcal{R} \subseteq \to_\mathcal{R}^* \cdot {}_\mathcal{R}^*{\leftarrow}$.*

Example 5.1.12 ([KH12, Example 10])**.** *Consider the following non-left-linear TRS $\mathcal{R}$:*

$$\begin{array}{lrcllrcl} 1: & \mathsf{eq}(n, xs, xs) & \to & \mathsf{true} & 3: & \mathsf{nats} & \to & \mathsf{0} : \mathsf{inc}(\mathsf{nats}) \\ 2: & \mathsf{eq}(\mathsf{s}(n), x : xs, x : ys) & \to & \mathsf{eq}(n, xs, ys) & 4: & \mathsf{inc}(x : xs) & \to & \mathsf{s}(x) : \mathsf{inc}(xs) \end{array}$$

We apply Corollary 5.1.10 to $\{1, 2, 3, 4\}$ by taking $\mathcal{C} = \{3, 4\}$. Let $\mathcal{P} = \mathcal{R} \setminus \mathcal{C}$. Suppose that $\mathsf{REN}(\mathcal{R})$ is the following eTRS:

$$\begin{array}{lrcllrcl} 1: & \mathsf{eq}(x_1, x_2, x_3) & \to & \mathsf{true} & 3: & \mathsf{nats} & \to & \mathsf{0} : \mathsf{inc}(\mathsf{nats}) \\ 2: & \mathsf{eq}(\mathsf{s}(x_1), x_2 : x_3, x_4 : x_5) & \to & \mathsf{eq}(x, xs, ys) & 4: & \mathsf{inc}(x_1 : x_2) & \to & \mathsf{s}(x) : \mathsf{inc}(xs) \end{array}$$

If the confluence of $\mathcal{C}$ holds then we can show that $\mathcal{R}$ and $\mathcal{C}$ satisfy conditions of Theorem 5.1.9 as follows:

- *The termination of $\mathcal{P}/\mathcal{C}$ can be shown by, e.g., the termination tool* NaTT.
- *The TRSs $\mathcal{P}$ and $\mathcal{C}$ are strongly non-overlapping because $\mathcal{C}$-overlaps $(1, \epsilon, 2)$ and $(2, \epsilon, 1)$ on $\mathsf{REN}(\mathcal{R})$ are overlaps on $\mathcal{P}$.*
- *Because there are only two overlaps $(1, \epsilon, 2)$ and $(2, \epsilon, 1)$ on $\mathcal{P}$, critical pairs on $\mathcal{P}$ are:*

  $$\mathsf{true} \;{}_\mathcal{P}{\leftarrow} \rtimes \xrightarrow{\epsilon}_\mathcal{P} \mathsf{eq}(n, xs, xs) \qquad \mathsf{eq}(n, xs, xs) \;{}_\mathcal{P}{\leftarrow} \rtimes \xrightarrow{\epsilon}_\mathcal{P} \mathsf{true}$$

  Both are joinable by rule 1. Since the confluence $\mathcal{C}$ is assumed, and $\mathcal{P}$ and $\mathcal{C}$ strongly non-overlapping, by Proposition 5.1.11 all $\mathcal{C}$-extended critical pairs on $\mathcal{P}$ are joinable.

On the other hand it is easy to show that $\mathcal{R}$ and $\mathcal{C}$ satisfy Theorem 5.1.3. In fact we have $\mathcal{R}{\restriction_\mathcal{C}} = \mathcal{C}$ because $\mathsf{eq} \notin \mathcal{F}\mathsf{un}(\mathcal{C})$ holds. Hence $\mathcal{R}$ is confluent if and only if $\mathcal{C}$ is so.

We again apply Corollary 5.1.10 to $\mathcal{C}$ with $\varnothing$. Since $\mathcal{C}$ is orthogonal, $\mathcal{C}$ and $\varnothing$ satisfies Theorem 5.1.9. Because of $\mathcal{C}{\restriction_\varnothing} = \varnothing$, by Corollary 5.1.10 the confluence of $\mathcal{C}$ and $\varnothing$ coincide.

Therefore we conclude that $\mathcal{R}$ is confluent.

5.2 Automation

Corollary 5.1.4 can be automated as follows. Suppose that we have found a subsystem $\mathcal{C}_0$ of a given left-linear TRS $\mathcal{R}$ such that ${}_{\mathcal{R}}{\leftarrow\!\!\!+\!\!\!\rtimes\!\!\xrightarrow{\epsilon}}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}_0}$. We extend $\mathcal{C}_0$ to $\mathcal{C}$ so that (i) $\mathcal{C}_0 \subseteq \mathcal{C} \subsetneq \mathcal{R}$ and (ii) $\mathcal{R}\restriction_{\mathcal{C}} \subseteq \to^{\leqslant k}_{\mathcal{C}}$ for a designated number $k \in \mathbb{N}$. This search problem can be reduced to a SAT problem. Let $\mathsf{S}_k(\ell \to r)$ be the following set of subsystems:

$$\mathsf{S}_k(\ell \to r) = \{\{\beta_1, \ldots, \beta_n\} \mid \ell \to_{\beta_1} \cdots \to_{\beta_n} r \text{ and } n \leqslant k\}$$

In our SAT encoding we use two kinds of propositional variables: $x_{\ell\to r}$ and y_f. The former represents $\ell \to r \in \mathcal{C}$, and the latter represents $f \in \mathcal{F}\mathsf{un}(\mathcal{C})$. With these variables the search problem for $\mathcal{C}$ is encoded as follows:

$$\bigwedge_{\alpha\in\mathcal{C}_0} x_\alpha \wedge \bigvee_{\alpha\in\mathcal{R}} \neg x_\alpha \wedge \bigwedge_{\alpha\in\mathcal{R}} \Bigl(\neg x_\alpha \vee \bigwedge_{f\in\mathcal{F}\mathsf{un}(\alpha)} y_f\Bigr)$$
$$\wedge \bigwedge_{\alpha\in\mathcal{R}\setminus\mathcal{C}_0} \Bigl(\bigl(\bigvee_{\mathcal{S}\in\mathsf{S}_k(\alpha)} x_{\mathcal{S}}\bigr) \vee \bigl(\neg \bigwedge_{f\in\mathcal{F}\mathsf{un}(\ell)} y_f\bigr)\Bigr)$$

Here $x_{\mathcal{S}} = x_{\beta_1} \wedge \cdots \wedge x_{\beta_n}$ for $\mathcal{S} = \{\beta_1, \ldots, \beta_n\}$. It is easy to see that the first two clauses encode condition (i) and the third clause characterizes $\mathcal{F}\mathsf{un}(\mathcal{C})$. The last clause encodes condition (ii).

Example 5.2.1 (Continued from Example 5.1.5). *Recall that* ${}_{\mathcal{R}}{\leftarrow\!\!\!+\!\!\!\rtimes\!\!\xrightarrow{\epsilon}}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}_0}$ *holds for* $\mathcal{C}_0 = \{1,2\}$*. Setting* $k = 5$*, we compute* $\mathsf{S}_k(\alpha)$ *for each rule* $\alpha \in \mathcal{R} \setminus \mathcal{C}_0 = \{3,4,5,6\}$*:*

$$\mathsf{S}_k(3) = \{\{3\}\} \quad \mathsf{S}_k(4) = \{\{2\},\{1,2,6\},\{2,3,6\}\} \quad \mathsf{S}_k(5) = \{\{5\}\} \quad \mathsf{S}_k(6) = \{\{6\}\}$$

The SAT encoding explained above results in the following formula

$$\begin{aligned}
(x_1 \wedge x_2) \wedge{} & (\neg x_1 \vee \cdots \vee \neg x_6) \\
\wedge{} & (\neg x_1 \vee (y_0 \wedge y_+)) \\
\wedge{} & (\neg x_2 \vee (y_0 \wedge y_\times)) \\
\wedge{} & (\neg x_3 \vee (y_0 \wedge y_+)) && \wedge (x_3 \vee \neg(y_0 \wedge y_+)) \\
\wedge{} & (\neg x_4 \vee (y_0 \wedge y_\mathsf{s} \wedge y_\times)) && \wedge (X \vee \neg(y_\mathsf{s} \wedge y_0 \wedge y_\times)) \\
\wedge{} & (\neg x_5 \vee (y_\mathsf{s} \wedge y_+)) && \wedge (x_5 \vee \neg(y_\mathsf{s} \wedge y_+)) \\
\wedge{} & (\neg x_6 \vee (y_\mathsf{s} \wedge y_+ \wedge y_\times)) && \wedge (x_6 \vee \neg(y_\mathsf{s} \wedge y_\times))
\end{aligned}$$

with $X = x_2 \vee (x_1 \wedge x_2 \wedge x_6) \vee (x_2 \wedge x_3 \wedge x_6)$*. The formula is satisfied if we assign true to* x_1*,* x_2*,* x_3*,* y_0*,* y_+*, and* $y_\times$*, and false to the other variables. This assignment corresponds to* $\mathcal{C} = \{1,2,3\}$*. Note that for this formula there is no other solution.*

Chapter 6

Confluence Tool — Hakusan

In order to evaluate presented approaches in Chapters 4 and 5, we developed a confluence tool Hakusan. The tool supports the main three compositional confluence criteria (Theorems 4.2.4, 4.3.4, and 4.4.6) and their original versions (Theorems 4.2.2, 4.3.2, and 4.4.2). Moreover the reduction method (Corollary 5.1.4) is available.

In the subsequent sections we describe the usage and features of the tool, and then we report experimental data for these implemented criteria. Note that the descriptions about the tool of experimental data below are based on version 0.8 of Hakusan.

6.1 Usage

Hakusan is a confluence tool written in Haskell. It automatically solves a confluence problem for a given TRS file. The source code of the tool is available at:

```
https://www.jaist.ac.jp/project/saigawa/
```

The input file is a text file written in the TRS format [MNS21] and the output is a (non-)confluence proof of the input TRS. The tool runs on a command line interface with options and an input TRS.

```
hakusan [configurations] [criterion] <input.trs>
```

The optional argument `[criterion]` specifies the (compositional) confluence criterion to be used. Some criteria require external tools, which can be specified by preceding optional arguments `[configurations]`.

The basic usage of Hakusan is just typing the command:

```
hakusan <input.trs>
```

Then Hakusan starts confluence analysis. There are three patterns of outputs.

- YES means that confluence of an input TRS was proved.

```
(VAR x y z)
(RULES
  +(+(x,y),z) -> +(x,+(y,z))
  +(x,+(y,z)) -> +(+(x,y),z)
  +(0,x) -> x
)
```

Figure 6.1: The TRS in Example 4.3.5.

- `NO` means that non-confluence of an input TRS was proved.
- `MAYBE` means that nothing was concluded.

In addition, the first two statuses are followed by the corresponding proof scripts. Moreover, `MAYBE` contains two cases: one is not provable theoretically, and the other is an inadequacy of the specified method and parameters (see the first example below).

A short usage is available in help messages by running the tool with no input.

```
hakusan
```

More details can be found in the next section. In the subsequent parts, we briefly show several usages of `Hakusan` with a few examples of input and output.

Confluence checking by a specified criterion. In order to demonstrate this feature, we recall the confluence proof of the TRS $\mathcal{R}$ in Example 4.3.5. The input is a text file listed in Fig. 6.1.

Since the confluence of the TRS is shown by the combination of parallel rule labeling (Theorem 4.3.4 with Knuth–Bendix's criterion (Theorem 2.3.7), we can obtain a confluence proof by the following command:

```
hakusan -prl-pcps 5 <input.trs>
```

The option `-prl-pcps k` stands for *parallel rule labeling with parallel critical pair systems*. Here the number `k` is the maximum length of rewrite steps used in each criterion. When this option is specified, `Hakusan` tries to find a subsystem $\mathcal{C}$ such that $\mathcal{R}$ with $\mathcal{C}$ satisfies the condition of parallel rule labeling, and $\mathcal{C}$ with $\varnothing$ satisfies the condition of parallel critical pair systems.

The result of the command is the status `YES` and a proof text, see Fig. 6.2. As mentioned above, the status means that confluence of the input TRS was successfully proved, and the proof text is a sequence of individual proofs corresponding to each applications of compositional confluence criteria. In this case, the proof text consists of two applications of the compositional confluence criteria mentioned above.

Note that the command `hakusan -prl-pcps 1 <input.trs>` outputs `MAYBE`. The cause is that k-steps length `1` is too short to find join sequences of (parallel) critical pairs on the problem.

```
YES

# Compositional parallel rule labeling (Shintani and Hirokawa 2022).

Consider the left-linear TRS R:

  +(+(x,y),z) -> +(x,+(y,z))
  +(x,+(y,z)) -> +(+(x,y),z)
  +(0(),x) -> x

Let C be the following subset of R:

  +(+(x,y),z) -> +(x,+(y,z))
  +(0(),x) -> x

All parallel critical peaks (except C's) are decreasing wrt rule labeling:

  phi(+(+(x,y),z) -> +(x,+(y,z))) = 0
  phi(+(x,+(y,z)) -> +(+(x,y),z)) = 1
  phi(+(0(),x) -> x) = 0

  psi(+(+(x,y),z) -> +(x,+(y,z))) = 0
  psi(+(x,+(y,z)) -> +(+(x,y),z)) = 1
  psi(+(0(),x) -> x) = 0


Therefore, the confluence of R follows from that of C.

# Compositional parallel critical pair system (Shintani and Hirokawa 2022).

Consider the left-linear TRS R:

  +(+(x,y),z) -> +(x,+(y,z))
  +(0(),x) -> x

Let C be the following subset of R:

  (empty)

The parallel critical pair system PCPS(R,C) is:

  +(+(+(x1_1,x1_2),y2),y3) -> +(+(x1_1,+(x1_2,y2)),y3)
  +(+(+(x1_1,x1_2),y2),y3) -> +(+(x1_1,x1_2),+(y2,y3))
  +(+(0(),y2),y3) -> +(y2,y3)
  +(+(0(),y2),y3) -> +(0(),+(y2,y3))

All pairs in PCP(R) are joinable and PCPS(R,C)/R is terminating.
Therefore, the confluence of R follows from that of C.

# emptiness

The empty TRS is confluent.
```

Figure 6.2: The proof of Fig. 6.1.

```
(VAR x y z)
(RULES
  +(x,0) -> x
  *(x,0) -> 0
  +(0,x) -> x
  *(s(x),0) -> 0
  +(s(x),y) -> s(+(x,y))
  *(s(x),y) -> +(*(x,y),y)
)
```

Figure 6.3: The TRS in Example 5.1.5.

Confluence checking with the reduction method. We use Example 5.1.5 to demonstrate the reduction method feature. The TRS in the example is formalized as a text file (Fig. 6.3). Identically, we use Corollary 5.1.4 as a reduction method to prove the confluence of the input TRS. The next command consists of the configuration for the reduction method and a criterion option:

```
hakusan -reduce 5 -orthogonal+ 5 <input.trs>
```

By activating the reduction method with the option `-reduce k`, the method is inserted before an application of each (compositional) confluence criterion. The criterion option `-orthogonal+ k` orders `Hakusan` to use successive application of Theorem 4.2.4 as confluence criterion. Then the reduction method is tested before each application of Theorem 4.2.4. The common parameter `k` is the same as the previous example.

The output proof is available at Fig. 6.4. The proof structure is same as the last one.

Non-confluence checking. `Hakusan` also supports a feature for non-confluence checking. Consider the following TRS:

$$x \cdot \mathsf{e} \to x \qquad x \cdot x^{-1} \to \mathsf{e}$$
$$\mathsf{e} \cdot x \to x \qquad x^{-1} \cdot x \to \mathsf{e}$$

The TRS file Fig. 6.5 is a representation of it. Since the TRS has a non-joinable peak $\mathsf{e} \leftarrow \mathsf{e} \cdot \mathsf{e}^{-1} \to \mathsf{e}^{-1}$, it is not confluent.

For non-confluence TRSs, `Hakusan` may output the status `NO` with a witness of non-confluence if the non-confluence feature is activated by the configuration option `-noncr`. Here is an example command:

```
hakusan -noncr <input.trs>
```

In this case, `Hakusan` outputs `NO`, and the non-joinable peak is displayed as a witness of the non-confluence (Fig. 6.6).

```
YES

# parallel critical pair closing system (Shintani and Hirokawa 2022, Section 8
in LMCS 2023)

Consider the left-linear TRS R:

  +(x,0()) -> x
  *(x,0()) -> 0()
  +(0(),x) -> x
  *(s(x),0()) -> 0()
  +(s(x),y) -> s(+(x,y))
  *(s(x),y) -> +(*(x,y),y)

Let C be the following subset of R:

  *(x,0()) -> 0()
  +(x,0()) -> x
  +(0(),x) -> x

The TRS R is left-linear and all parallel critical pairs are joinable by C.
Therefore, the confluence of R is equivalent to that of C.

# parallel critical pair closing system (Shintani and Hirokawa 2022, Section 8
in LMCS 2023)

Consider the left-linear TRS R:

  *(x,0()) -> 0()
  +(x,0()) -> x
  +(0(),x) -> x

Let C be the following subset of R:

  *(x,0()) -> 0()

The TRS R is left-linear and all parallel critical pairs are joinable by C.
Therefore, the confluence of R is equivalent to that of C.

# parallel critical pair closing system (Shintani and Hirokawa 2022, Section 8
in LMCS 2023)

Consider the left-linear TRS R:

  *(x,0()) -> 0()

Let C be the following subset of R:

  (empty)

The TRS R is left-linear and all parallel critical pairs are joinable by C.
Therefore, the confluence of R is equivalent to that of C.

# emptiness

The empty TRS is confluent.
```

Figure 6.4: The proof of Fig. 6.3.

```
(VAR x)
(RULES
  .(x,e) -> x
  .(e,x) -> x
  .(x,-(x)) -> e
  .(-(x),x) -> e
)
```

Figure 6.5: The non-confluent TRS.

```
NO

peak

e()
*<-
.(-(e()),e())
->*
-(e())

is not joinable
```

Figure 6.6: The proof of Fig. 6.5

6.2 Features

We explain more detailed features of `Hakusan`. First, we explain how to specify external tools and how to enable the reduction method. Next, we describe detailed descriptions of all criterion options.

Configurations must be placed before criterion options. Here is a configuration example of `Hakusan`.

```
hakusan -ttx /path/to/tool -noncr -reduce 5 <input.trs>
```

External tools. `Hakusan` requires two external tools for the following criterion options: `-cps`, `-pcps`, `-prl`, `-pcps-`X, and `-prl-`X. The parameter X is explained later. The following list is instructions on external tools and corresponding criterion options.

- `-ttx <path>`: This option specifies the path of an external termination tool for the XML format.[1] A passed termination tool is used for testing relative termination for Theorem 4.4.2 and Theorem 4.4.6 (`-cps`, `-pcps`, `-pcps-`X, and `-prl-pcps` options). The default termination tool is `NaTT` [YKT14], the default path is `NaTT.exe`.

[1]See the web site `https://termination-portal.org/wiki/TPDB`

Table 6.1: Main options for configurations.

`-ttx <path>`	use external termination tool `<path>` with the XML format.
`-tt <path>`	use external termination tool `<path>` with the WST format.
`-smt <path>`	use external SMT solver `<path>`.
`-reduce <k>`	enable reduction method with at most `k`-rewrite steps.

- `-tt <path>`: This option is same as `-ttx`, but it specifies the path of an external termination tool for the WST format. [1] Other descriptions are equivalent to the option `-ttx`.
- `-smt <path>`: This option specifies the path of an external SMT solver for the SMT-LIB 2 format [BFT17]. This solver is used for solving linear arithmetic constraints of Theorem 4.3.4 (`-prl`, `-prl-X`, and `-pcps-prl` options). The default SMT solver is `Z3` [dMB08], the default path is `z3`.

Non-confluence checking. The non-confluence checking feature of `Hakusan` is based on the TCAP approximation [ZFM11]. The feature is activated by the option `-noncr`. Non-confluence of input TRSs are tested before applications of a (compositional) confluence criterion. If non-confluence of an input TRS is proved then the status `NO` is output with a witness.

Reduction Method. The option `-reduce <k>` activates the reduction method (Corollary 5.1.4). If this option is specified then the reduction method is inserted before each application of confluence criteria. Note that the reduction method is successively applied similarly to Example 5.1.5. Due to the performance, suitable subsystems for Theorem 4.2.4 are calculated from rewrite rules that are used in join sequences for each parallel critical pair on the original system. Joinability of each parallel critical pair (t, u) is tested by the relation:

$$t \xrightarrow{\leqslant 5} \cdot \xleftarrow{\leqslant 5} u$$

Contrary to compositional confluence criteria described later, this feature only tests a first candidate of suitable subsystems.

Criterion options. These options enable corresponding (compositional) confluence criteria. If several criterion options are specified at once then the one of criterion options is used with the internal priority of options. Main options for (compositional) confluence criteria are listed in Table 6.2.

Table 6.2: Main options for confluence criteria.

-orthogonal	apply Theorem 4.2.2.
-orthogonal+ <k>	successively apply Theorem 4.2.4.
-prl <k>	apply Theorem 4.3.2
-prl-<X> <k>	consecutively apply Theorem 4.3.4 with <X>.
-cps <k>	apply Theorem 4.4.2.
-pcps-<X> <k>	consecutively apply Theorem 4.4.6 with <X>.

Here k is the maximal length of rewrite steps and X is one of orthogonal, prl, pcps, or empty.

We list individual options for the aforementioned (compositional) confluence criteria in Chapter 4.

- -orthogonal: If the option is specified then Hakusan checks orthogonality of an input TRS (see Theorem 4.2.2).
- -orthogonal+ <k>: Hakusan applies the confluence criterion that is obtained by successive application of Theorem 4.2.4 (see Example 4.2.5). Joinability of each parallel critical pair (t, u) is tested by the relation:

  $$t \xrightarrow{\leqslant k} \cdot \xleftarrow{\leqslant k} u$$

  Every subsystem $\mathcal{C}$ is searched by enumeration.
- -prl <k> and -prl-<X> <k>: For the former option, Theorem 4.3.2 is applied. For the latter option, Hakusan applies Theorem 4.3.4 with a corresponding criterion <X>. The decreasingness of each parallel critical peak of the form $t \;{}_{\phi,k}\overset{P}{\leftarrow\!\!\!\!\shortparallel}\; s \overset{\epsilon}{\rightarrow}_{\psi,m} u$ is tested by existence of a conversion of the form

  $$t \underset{\Upsilon k}{\longrightarrow}^{i_1} \cdot \underset{\psi,m}{\shortparallel\!\!\!\!\longrightarrow}^{i_2} \cdot \underset{\Upsilon km}{\longrightarrow}^{i_3} \cdot {}^{j_3}\underset{\Upsilon km}{\longleftarrow} v \;{}^{j_2}\underset{\phi,k}{\overset{P'}{\longleftarrow\!\!\!\!\shortparallel}} \cdot {}^{j_1}\underset{\Upsilon m}{\longleftarrow} u$$

  such that $i_1, i_3, j_1, j_3 \in \mathbb{N}$, $i_2, j_2 \in \{0, 1\}$, $i_1 + i_2 + i_3 \leqslant k$, $j_1 + j_2 + j_3 \leqslant k$, and $\mathcal{V}\mathsf{ar}(v, P') \subseteq \mathcal{V}\mathsf{ar}(s, P)$ holds. This is encoded into linear arithmetic constraints [HM11], and they are solved by a configured SMT solver.

  For the option -prl-<X>, every subsystem $\mathcal{C}$ is searched by enumeration, and confluence of $\mathcal{C}$ is proved by the corresponding criterion <X>.
- -cps <k>: If the option is given then Hakusan applies Theorem 4.4.2. Joinability of critical pairs is tested by the same relation of -orthogonal+.

- `-pcps <k>` and `-pcps-<X> <k>`: If these options are given then Hakusan applies Theorem 4.4.6. For the first option, the criterion is used as an ordinary confluence criterion, that is $\mathcal{C} = \varnothing$. In the latter case, confluence of subsystem $\mathcal{C}$ is proved by the corresponding criterion `<X>`. Joinability of each parallel critical pair (t, u) is tested by the relation:

$$t \xrightarrow{\leqslant k} \cdot \xleftarrow{\leqslant k} u$$

  In order to determine termination of $\mathcal{R}/\mathcal{C}$, a configured termination tool is used.

  Every subsystem $\mathcal{C}$ is searched by enumeration.

In both of `-prl-<X>` and `-pcps-<X>`, a confluence criterion X is the one of `prl`, `pcps`, `orthogonal`, and `empty`. The first three are equivalent to the corresponding criterion options. Here `empty` is a confluence criterion that checks only emptiness of TRSs.

When no criterion option is specified, Hakusan automatically enables configuration options `-noncr` and `-reduce 5`, and the criterion options `-prl-pcps 5` and `-pcps-prl 5`. These specified criteria are applied sequentially, i.e., `-pcps-prl` is enabled when `-prl-pcps` outputs `MAYBE`. Hence the result of the command `hakusan <input.trs>` is identical to the union of the following results of two commands:

```
hakusan -noncr -reduce 5 -prl-pcps 5 <input.trs>
```

and

```
hakusan -noncr -reduce 5 -pcps-prl 5 <input.trs>
```

The comparison of the default parameter `k` can be found in Fig. 6.8 in Section 6.3.

6.3 Experiments

In order to evaluate the aforementioned techniques, we compare Hakusan (version 0.8) with existing methods and tools by experimental results. The experimental data are available at the project page:

`https://www.jaist.ac.jp/project/saigawa/`

The problem set used in experiments consists of 462 left-linear TRSs taken from the confluence problems database COPS [HNM18]. Out of the 462 TRSs, at least 191 are known to be non-confluent. The tests were run on a PC with Intel Core i7-1165G7 CPU (2.80 GHz) and 16 GB memory of RAM. Table 6.3 summarizes the results. The columns in the table stand for the following confluence criteria:

Table 6.3: Comparison for individual methods.

	O	**R**	**C**	**OO**	**RC**	**CR**	**rOO**	**rRC**	**rCR**
# of proved TRSs	20	135	59	88	152	143	91	153	146
timeouts	0	20	9	12	84	34	9	82	43
errors	0	1	1	3	2	2	3	2	2

- **O**: Orthogonality (Theorem 4.2.2).
- **R**: Rule labeling (Theorem 4.3.2).
- **C**: The criterion by critical pair systems (Theorem 4.4.2).
- **OO**: Successive application of Theorem 4.2.4 (see Example 4.2.5).
- **RC**: Theorem 4.3.4, where confluence of a subsystem $\mathcal{C}$ is shown by Theorem 4.4.6 with the empty subsystem.
- **CR**: Theorem 4.4.6, where confluence of a subsystem $\mathcal{C}$ is shown by Theorem 4.3.4 with the empty subsystem.
- **rOO**, **rRC**, and **rCR**: They are same as **OO**, **RC**, and **CR** but Corollary 5.1.4 is always used repeatedly before a (compositional) confluence criterion is applied.

In the table, the row "# of proved TRSs" is the number of `YES` problems. The row "timeouts" is the number of problems with neither `YES`, `NO`, nor `MAYBE`. The last row is the number of run-time errors.

We briefly explain configurations of this run. Timeouts are 150 seconds and the maximal length k of rewrite steps is 5. These numbers are intended to maximize the number of proved TRSs (see Figure 6.8 and 6.7). In both charts x-axis indicates the number of proved TRSs and y-axis indicates timeouts in seconds and length of rewrite steps, respectively. Relative termination, required by Theorems 4.4.2 and 4.4.6, is checked by employing the termination tool `NaTT` (version 2.3) [YKT14]. The SMT solver `z3` (version 4.8.7) is used for solving SAT problems for Theorem 4.3.4 and the reduction method (Corollary 5.1.4).

As theoretically expected, in the experiments **O**, **R**, and **C** are subsumed by their compositional versions **OO**, **RC**, and **CR**, respectively. Moreover, **OO** is subsumed by **R**, **RC**, and **CR**. Due to timeouts, **CR** misses three systems of which **R** can prove confluence. While the union of **R** and **C** amounts to 145, the union of **RC** and **CR** amounts to 153. Differences between **RC** and **CR** are summarized as follows:

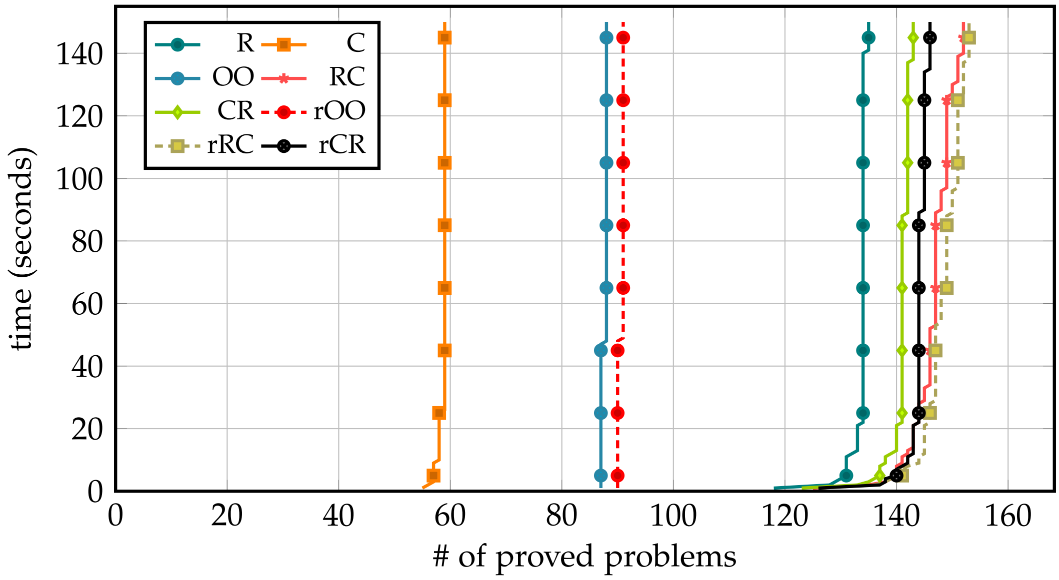


Figure 6.7: The numbers of proved problems in run time.

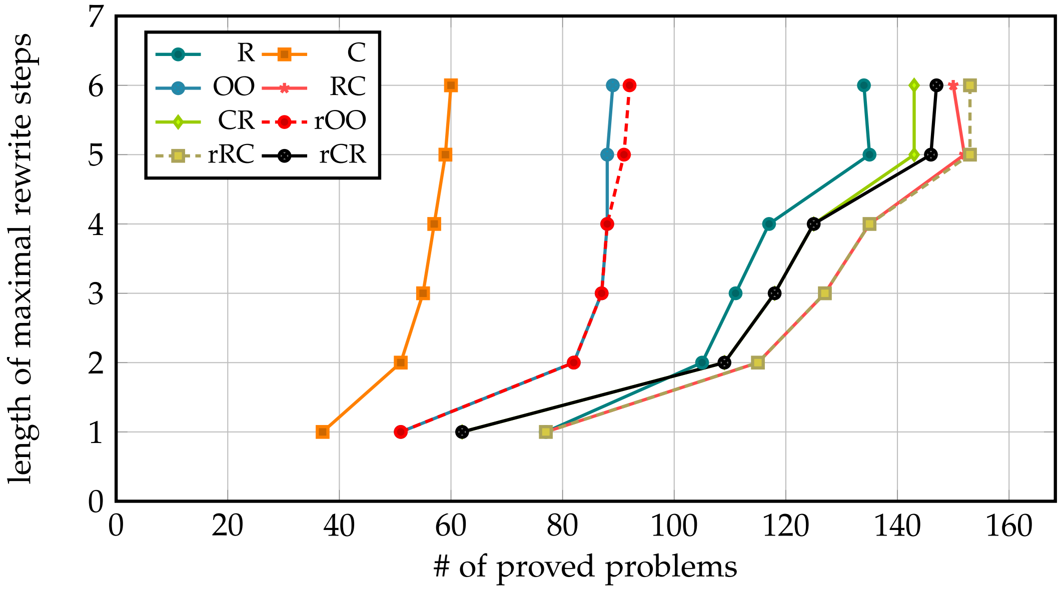


Figure 6.8: The number of proved problems in maximal length of rewrite steps.

- Three systems (COPS numbers `994`, `1001`, and `1029`) are proved by **RC** but not by **CR** nor **R**. One of them is the next TRS (COPS number `994`). **RC** uses the subsystem $\{2, 4, 6\}$ whose confluence is shown by **C**.

$$\begin{array}{lll} 1\colon\ \mathsf{a}(\mathsf{b}(x)) \to \mathsf{a}(\mathsf{c}(x)) & 3\colon\ \mathsf{c}(\mathsf{b}(x)) \to \mathsf{a}(\mathsf{b}(x)) & 5\colon\ \mathsf{c}(\mathsf{c}(x)) \to \mathsf{c}(\mathsf{c}(x)) \\ 2\colon\ \mathsf{a}(\mathsf{c}(x)) \to \mathsf{c}(\mathsf{b}(x)) & 4\colon\ \mathsf{b}(\mathsf{c}(x)) \to \mathsf{a}(\mathsf{c}(x)) & 6\colon\ \mathsf{c}(\mathsf{c}(x)) \to \mathsf{c}(\mathsf{b}(x)) \\ & & 7\colon\ \mathsf{c}(\mathsf{b}(x)) \to \mathsf{a}(\mathsf{b}(x)) \end{array}$$

- The only TRS where **CR** is advantageous to **RC** is COPS number `132`:

$$\begin{array}{ll} 1\colon\ -(x+y) \to (-x)+(-y) & 3\colon\ -(-x) \to x \\ 2\colon\ (x+y)+z \to x+(y+z) & 4\colon\ x+y \to y+x \end{array}$$

  Its confluence is shown by the composition of Theorem 4.4.6 and Theorem 4.3.2, the latter of which proves the subsystem $\{1, 2, 4\}$ confluent.

The columns **rOO**, **rRC**, and **rCR** in Table 6.3 show that the use of the reduction method (Corollary 5.1.4) basically improves the power and efficiency of the underlying compositional confluence criteria. Close inspection of experimental data reveals the following facts.

- The confluence proving powers of **rOO** and **OO** are theoretically equivalent, because the reduction method as a compositional confluence criterion is an instance of **OO**. In the experiments **rOO** handled three more systems. This is due to the improvement of efficiency. The same argument holds for the relation between **rRC** and **RC**.
- While the use of the reduction method improves the efficiency in most of cases, there are a few exceptions (e.g., COPS number `689`). The bottleneck is the reachability test by $\to_{\mathcal{C}^{\leqslant k}}$.
- The reduction method and **C** are incomparable with each other. Hence **rCR** is more powerful than **CR**. In the experiments, **rCR** subsumes **CR** and it includes three more systems. As a drawback, **rCR** has four more timeouts.
- Among **rOO**, **rRC**, and **rCR**, the second criterion is the most powerful. As in the cases of their underlying criteria, the results of **rOO** are subsumed by both **rRC** and **rCR**, and COPS number `132` is the only problem where **rCR** outperforms **rRC**.

For the sake of comparison the results of the following confluence tools are included in Table 6.4. All tools are taken from the TRS category of the annual confluence competition (CoCo) [MNS21] held in 2022.

Table 6.4: Comparison for confluence tools.

	Hakusan	ACP	CoLL	CONFident	CSI
# of proved TRSs	154	198	177	97	215
timeouts	77	49	167	43	3
errors	3	5	4	0	0

- ACP [AYT09] is the first modern automated confluence prover. The main feature is supporting several divide-and-conquer methods, including modularity and commutative decompositions. We use version 0.62 which is participated in CoCo 2022.
- CONFident [GL20] is a confluence prover for context sensitive and conditional term rewrite systems. The tool boils down confluence problems to corresponding logical problems such as joinability and feasibility. Exported logical problems are solved by external tools. We use CoCo 2022 version of CONFident. Note that the main targets of CONFident are extensions of ordinary TRSs. So this experimentation unfits the tool.
- CSI is an automated confluence prover employing decomposition methods [ZFM11] and transformation methods [NFM17]. This tool won the TRS category of CoCo 2022. We use version 1.25 which is participated in CoCo 2022.
- CoLL-Saigawa is the predecessor of Hakusan. This tool consists of two different confluence tools: CoLL [SH15] and Saigawa [Hir14]. The former is used for left-linear problems and the latter is used for other problems. We abbreviate CoLL-Saigawa to CoLL. We use version 1.6 which is participated in CoCo 2022.

The result of Hakusan is the union of the results of **rRC** and **rCR**. The former proves 153 problems out of 154 problems. Among them, the three systems COPS numbers `994`, `1001`, and `1029` are newly proved by Hakusan. All results of **rCR** are subsumed by other tools.

Chapter 7

Conclusion

We studied how compositional confluence criteria can be derived from confluence criteria based on the decreasing diagrams technique, and showed that Toyama's almost parallel closedness theorem is subsumed by his earlier theorem based on parallel critical pairs. We conclude this thesis by mentioning related work and future work.

Commutation version of compositional criteria. In this thesis we presented several compositional confluence criteria. However, the underlying abstract theorem Theorem 4.1.2 of them is a decreasing diagram method based on commutation. So compositional commutation criteria are naturally considered. Recasting compositional commutation is straightforward for Theorem 4.2.4 and Theorem 4.3.4. Other criteria, including reduction methods, need to be investigated.

Simultaneous critical pairs. van Oostrom [vO97] showed the almost development closedness theorem: A left-linear TRS is confluent if the inclusions

$$\xleftarrow{\epsilon} \rtimes \xrightarrow{\epsilon} \subseteq \xrightarrow{*} \cdot \mathrel{\leftarrow\!\!\circ\!\!-} \qquad\qquad \xleftarrow{>\epsilon} \rtimes \xrightarrow{\epsilon} \subseteq \mathrel{-\!\!\circ\!\!\rightarrow}$$

hold, where $\mathrel{-\!\!\circ\!\!\rightarrow}$ stands for the multi-step [Ter03, Section 4.7.2]. Okui [Oku98] showed the simultaneous closedness theorem: A left-linear TRS is confluent if the inclusion

$$\mathrel{\leftarrow\!\!\circ\!\!-} \rtimes \rightarrow \subseteq \xrightarrow{*} \cdot \mathrel{\leftarrow\!\!\circ\!\!-}$$

holds, where $\mathrel{\leftarrow\!\!\circ\!\!-} \rtimes \rightarrow$ stands for the set of simultaneous critical pairs [Oku98]. As this inclusion characterizes the inclusion $\mathrel{\leftarrow\!\!\circ\!\!-} \cdot \rightarrow \subseteq \rightarrow^* \cdot \mathrel{\leftarrow\!\!\circ\!\!-}$, simultaneous closedness subsumes almost development closedness. The main result in Section 3.1 is considered as a counterpart of this relationship in the setting of parallel critical pairs.

Critical-pair-closing systems. A TRS $\mathcal{C}$ is called *critical-pair-closing* for a TRS $\mathcal{R}$ if

$${}_{\mathcal{R}}\!\leftarrow \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow_{\mathcal{C}}^*$$

holds. It is known that a left-linear TRS $\mathcal{R}$ is confluent if $\mathcal{C}_{\mathsf{d}}/\mathcal{R}$ is terminating for some confluent critical-pair-closing TRS $\mathcal{C}$ with $\mathcal{C} \subseteq \mathcal{R}$, see [HNvOO19]. Here $\mathcal{C}_{\mathsf{d}}$ denotes the set of all duplicating rules in $\mathcal{C}$. Theorem 4.2.4 imposes closedness by $\mathcal{C}$ on all *parallel* critical pairs in return to removal of the relative termination condition. Investigating whether the latter subsumes the former is our future work.

Rule labeling. Dowek et al. [DFJL22, Theorem 38] extended rule labeling based on parallel critical pairs [ZFM15] to take higher-order rewrite systems. If we restrict their method to a first-order setting, it corresponds to the case that a complete TRS is employed for $\mathcal{C}$ in Theorem 4.3.4, and thus, it can be seen as a generalization of Corollary 4.2.6 by Toyama [Toy17].

Critical pair systems. Hirokawa and Middeldorp [HM13] generalized Theorem 4.4.2 by replacing $\mathsf{CPS}(\mathcal{R})$ by the following subset:

$$\mathsf{CPS}'(\mathcal{R}) = \{s \to t, s \to u \mid t \;{}_{\mathcal{R}}\!\leftarrow s \xrightarrow{\epsilon}_{\mathcal{R}} u \text{ is a critical peak but not } t \mathrel{\circ\!\!\!\!\to}_{\mathcal{R}} u\}$$

This variant subsumes van Oostrom's development closedness theorem [vO97]. We anticipate that in a similar way our compositional variant (Theorem 4.4.6) is extended to subsume the parallel closedness theorem based on parallel critical pairs (Theorem 3.1.9).

Redundant rules. *Redundant rule elimination* by Nagele et al. [NFM15, Corollary 9] can be regarded as a compositional confluence criterion. It states that a TRS $\mathcal{R}$ is confluent if there exists a confluent subsystem $\mathcal{C}$ such that $\mathcal{R} \setminus \mathcal{C} \subseteq \leftrightarrow_{\mathcal{C}}^*$ holds. When $\mathcal{R}$ is left-linear, the criterion is subsumed by Theorem 4.2.4. This is verified by the following trivial fact:

Fact 7.0.1. *Let $\mathcal{C}$ be a subsystem of a TRS $\mathcal{R}$. If $\mathcal{R} \setminus \mathcal{C} \subseteq \leftrightarrow_{\mathcal{C}}^*$ then ${}_{\mathcal{R}}\!\leftarrow\!\!\!\!\!+\!\!\!\!\!\!\!\!\!\!\!\rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow_{\mathcal{C}}^*$.*

The converse does not hold in general. To see it, consider the one-rule TRS $\mathcal{R}$ consisting of $\mathsf{a} \to \mathsf{b}$. The empty TRS $\mathcal{C} = \varnothing$ satisfies ${}_{\mathcal{R}}\!\leftarrow\!\!\!\!\!+\!\!\!\!\!\!\!\!\!\!\!\rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow_{\mathcal{C}}^*$ but $\mathcal{R} \setminus \mathcal{C} \subseteq \leftrightarrow_{\mathcal{C}}^*$ does not hold as $\mathsf{a} \not\leftrightarrow_{\mathcal{C}}^* \mathsf{b}$. There is another form of redundant rule elimination ([NFM15, Corollary 6] and [SH15]). It states that a TRS $\mathcal{R}$ is confluent if and only if $\mathcal{R} \subseteq \to_{\mathcal{C}}^*$ for some confluent $\mathcal{C} \subseteq \mathcal{R}$. This criterion is regarded as a reduction method for confluence analysis. In fact, it is an instance of Corollary 5.1.4 for left-linear TRSs, since $\mathcal{R}\!\upharpoonright_{\mathcal{C}} \subseteq \mathcal{R}$ and ${}_{\mathcal{R}}\!\leftarrow\!\!\!\!\!+\!\!\!\!\!\!\!\!\!\!\!\rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow_{\mathcal{C}}^*$ hold. We want to stress that a reduction method is obtained by any combination of a compositional confluence criterion with Theorem 5.1.3.

Certification. Bugs of implementation make tools unreliable. How to verify the correctness of a proof generated by a tool? One solution is a certified proof. CeTA [TS09] is a certifier for rewriting, which is based on an Isabelle/HOL library IsaFoR. It provides a variety of certification interfaces for termination, complexity, confluence, etc. We plan to support CeTA with Hakusan.

Modularity and automation. Last but not least, we discuss relations between modularity and reduction methods. Organizing compositional criteria as a reduction method is a key for effective automation. Therefore, developing a generalization of Theorem 5.1.3 is our primary future work. Ohlebusch [Ohl02] showed that if the union of *composable* TRSs $\mathcal{R}$ and $\mathcal{C}$ is confluent then both $\mathcal{R}$ and $\mathcal{C}$ are confluent. When $\mathcal{C}$ is a subsystem of $\mathcal{R}$, this result is rephrased as follows: If $\mathcal{D}_{\mathcal{R}\setminus\mathcal{C}} \cap \mathcal{F}\mathsf{un}(\mathcal{C}) = \varnothing$ then confluence of $\mathcal{R}$ implies that of $\mathcal{C}$. Therefore, this can be used as an alternative of Theorem 5.1.3. Unfortunately, $\mathcal{R}\restriction_{\mathcal{C}} \subseteq \mathcal{C}$ follows from $\mathcal{D}_{\mathcal{R}\setminus\mathcal{C}} \cap \mathcal{F}\mathsf{un}(\mathcal{C}) = \varnothing$. So composability as a reduction method is still in the realm of our criterion (Theorem 5.1.3). Similarly, we can argue that the theorem also subsumes the persistency result [AT97] as a base criterion for reduction methods. Yet, we anticipate that this work benefits from studies of more advanced modularity results such as *layer systems* [FMZvO15].

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