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Master's Thesis

Awareness Logic of Abstraction

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Abstract

We aim to formalize abstraction as a model transformation tailored to the reasoning abilities of agents. The characteristic of our work is to capture the epistemological aspect of the abstraction of a model. It is motivated not to develop the formal theory of knowledge but to develop the formal theory of modeling because the motivation is an issue of how to share models representing the same situation among people with different reasoning abilities. For this motivation, Awareness Logic as the Logic of Abstraction is given. Since traditional research for awareness logic does not aim at developing a formal theory of modeling, we aim to get a good foundation of the basics of a formal theory of modeling in awareness logic.

This thesis consists of two parts. The first part introduces **Awareness Logic with Global Propositional Awareness(ALGP)**. **ALGP** is the logic with global awareness(an agent's awareness is the same in all possible worlds) and propositional awareness(an agent is only aware of formulas containing occurrences of a subset of all atomic propositions). In addition, a sound and complete axiomatization of **ALGP** is shown.

The second part investigates Awareness Logic as the Logic of Abstraction. We compare among three abstractions: "*atoms-based abstraction*", "*filtration-based abstraction*", and "*bisimulation-based abstraction*".

The second part also introduces **Awareness Logic of Filtration(ALF)**. **ALF** is given by adding an implicit abstraction operator to **ALGP**. Non-compactness in the semantics of **ALF** is shown.

Common awareness and distributed awareness are given to **ALGP** as a macro. A quotient model with common awareness and a nested quotient model are considered different approaches for obtaining a comprehensible model among agents with different reasoning abilities. This thesis obtains semantics of nested abstraction introducing common awareness and nested quotient model.

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Kosuke Udatsu

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Chapter 1

Introduction

1.1 Background and Motivation

Epistemic Logic(EL) is a subfield of Modal Logic that is intended reasoning about knowledge, belief, and related notions. **EL** what we know today was established by Hintikka(1962). In **EL**, the necessity operator $\Box_i$ is interpreted as the knowledge operator such that $\Box_i\varphi$ stands for "*agent i knows φ holds*". The knowledge operator satisfies the following rules:

- (Nec): $\vdash \varphi$ implies $\vdash \Box_i\varphi$,
- (K): $\vdash \Box_i\varphi \land \Box_i(\varphi \to \psi) \to \Box_i\psi$.

The first one represents that if a formula φ is a tautology, then agent i knows φ is true. The second one makes the knowledge closed under logical consequence, meaning that if agent i knows φ and $\varphi \to \psi$ are true, then the agent i automatically knows ψ is true. Then, the agent knows all the tautologies. The abilities are not actual human reasoning abilities. The problem is called **the Problem of Logical Omniscience**.

Awareness Logic(AL) is a subfield of **EL** that limits the reasoning abilities of agents for solving the problem of logical omniscience. **AL** was established by Fagin and Halpern in 1987 [3]. They introduced the notion called **awareness** as a filter for knowledge. Awareness splits knowledge into implicit knowledge and explicit knowledge. **Implicit Knowledge** represents the ideal knowledge presented in traditional epistemic logic. On the other hand, **explicit knowledge** represents the knowledge of an agent whose reasoning abilities are limited by awareness. In other words, an agent's explicit knowledge is an implicit knowledge of which the agent is aware.

In this way, we can syntactically consider limited reasoning by introducing an awareness operator to EL. On the other hand, we cannot syntactically consider limited modeling in **AL**. There are two cases where we need limited modeling. First, we will need limited modeling when we want to share a model with others. There are many possibilities hidden in the real world. But, it is difficult to model all of them because we are not aware of all of them. We have no choice but to disregard what we are unaware of when modeling them. Further, we have to impose different limitations on modeling because different agents have different reasoning abilities. Then, we may not share the same model. We need to give a limited model tailored to others' reasoning abilities if we want to share a model with them. Second, we will need limited modeling when we cannot understand the requirements of a program. For example, it may be impossible to debug a large-scale program because it is complex and may be difficult to understand. We will rely on the automatic detection of specifications because computers can perform calculations on a larger scale than humans. However, even if the specifications of the entire program were given, humans would not be able to understand it. We want a limited model tailored to our abilities and representing a part of a specification of a large-scale program.

With such motivations, we would like to develop Awareness Logic of *Abstraction*. In this thesis, *Abstraction* is a non-technical term considering the above motivations and the following two requirements. First, abstraction is required to be a model transformation that gives a simpler model. Second, it is required that abstraction preserves the truth of formulas attracting attention. **Logic of Abstraction** [1] is a prior research on an abstraction of a model. Logic of Abstraction is propositional dynamic logic including a dynamic abstraction operator as a transformation to tailor a model to the relevant issues under discussion.

Semantical approaches to modeling unawareness(e.g. [7]) give abstraction another point of view. Comparison with modeling unawareness would be a future work.

There are three main ways to formalize abstraction. In this thesis, each formalized abstraction is called as follows.:

1. filtration-based abstraction
2. bisimulation-based abstraction
3. atoms-based abstraction

Our choice is a filtration-based abstraction. Filtration-based abstraction is a formalized abstraction by logical equivalence. There are several possibilities for filtration-based abstractions because there are several filtrations of a Kripke model ([8] p.78-p.81). Bisimulation-based abstraction is a formalized abstraction by observational equivalence. Research on **speculative knowledge**[11] gives us a point of view for semantics applying bisimulation. Atoms-based abstraction is a formalized abstraction preserving the truth of atomic propositions attracting attention. **Awareness Logic with Partition(ALP)** [6] gives us a point of view for an atoms-based abstraction operator.

1.2 Our Proposals

There are several restrictions on awareness [3] [6]. First, we consider awareness that an agent is only aware of formulas containing occurrences of a subset of all atomic propositions. Such awareness and the restriction on awareness are called **propositional awareness** and **awareness generated by primitive propositions(gpp)**, respectively. gpp is a restriction on awareness considered in [3]. Second, similar to [6], we also consider awareness that an agent's awareness is the same in all possible worlds. Such awareness is called **global awareness** in this thesis. Global and propositional awareness is called **global propositional awareness**.

This thesis consisting of two parts introduces two logics. Part I and II introduce **Awareness Logic with Global Propositional Awareness(ALGP)** and **Awareness Logic of Filtration(ALF)**, respectively. **ALGP** is given adding restriction of global propositional awareness to **AL**. Note **ALGP** is not Awareness Logic of Abstraction. **ALF** is given by extending **ALGP** to Awareness Logic of Abstraction. Awareness of **ALF** is also global propositional awareness.

As described in Section 1.1, it is motivated to give a model regarding what agents are commonly aware of. With such motivations, we introduce **common awareness**. Common awareness is formalized by introducing a finite sequence of symbols called **agent expressions**. On the other hand, abstraction by chaining abstractions with different awareness can also be considered another approach to obtaining mutually comprehensible models among agents with different reasoning abilities. A chain of abstractions with different awareness is called **nested abstraction** through this thesis. Now, we explain an example showing how nested abstractions are a natural way to obtain mutually comprehensible models with agents with different reasoning abilities.

First, we give a Kripke model shaped perspective view of a cube as Figure 1.1(Now, some accessibility relations are abbreviated, but any possible world can be reachable to all possible worlds.). Suppose there are two agents i and j. The Kripke model represents "*agent i do not know the truth of p_1, p_2 and p_3, respectively*". Suppose agent i is only aware of p_1 and p_2 and agent j is only aware of p_1 and p_3. Then, Figure 1.2 shaped like a projection of a cube is a model tailored to our awareness. At this time, the only atomic proposition agent j is aware of in Figure 1.2 is p_1. Hence, abstraction tailored to j's awareness that agent i can understand is abstraction preserving the truth of all formulas only containing p_1. Agent i can obtain Figure 1.3 by such an abstraction. Since Figure 1.3 is a mutually comprehensible model for agents i and j, the nested abstraction in the above example can provide a mutually comprehensible model.

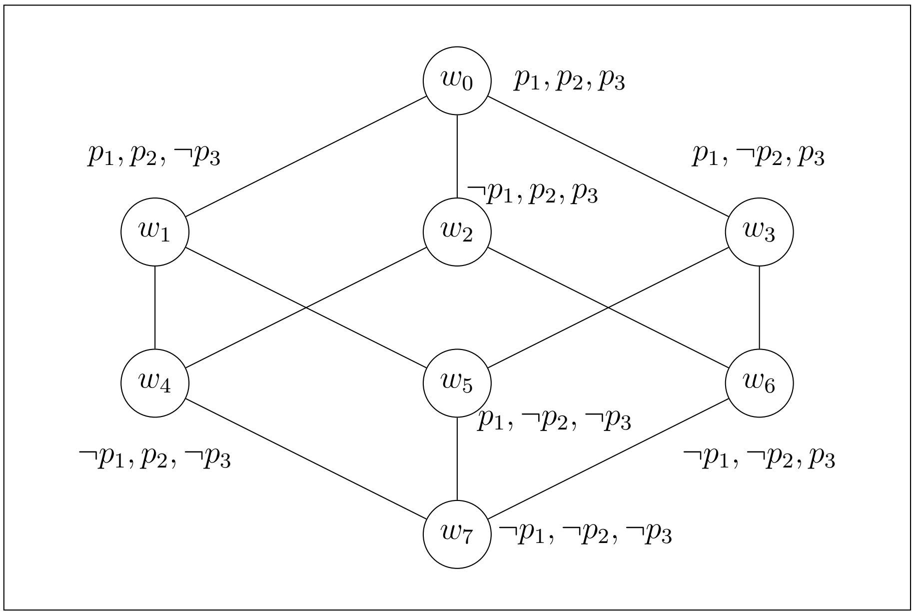

Figure 1.1: Example of Kripke Model(A Perspective View of A Cube)

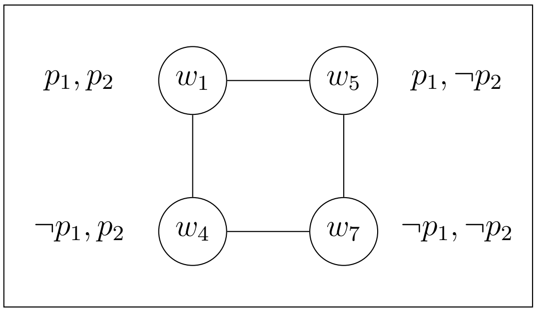

Figure 1.2: Example of Kripke Model(A Projection of Figure 1.1)

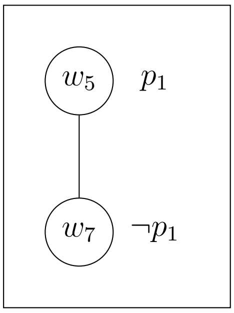

Figure 1.3: Example of Kripke Model Shared by The Others

Programmers are not always able to observe the logical control of a program because there are programs with unobservable behavior from outside. Assume that two programmers i and j know observable logical control of a program on each awareness. i and j have to check the correctness of a specification of a program. When i and j say "as far as I can observe, the program works as a specification", how validated was the program? At the least, it will be possible to verify more specifications than i could verify. We investigate the composition of an observation as a product of two models based on our graphical intention.: We represent "*agent i don't know the truth of p_1 and p_2, respectively*" like the Figure 1.4, which looks like a square (Now, some accessibility relations are abbreviated, but any possible world can be reachable to all possible worlds.).:

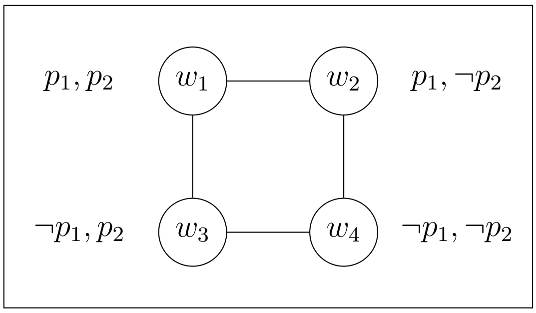

Figure 1.4: Example of Kripke Model(i's Perspective)

Suppose that there is a fact "*agent i don't know the truth of p_3*". We represent it like the Figure 1.5, which looks like a line (Now, some accessibility relations are abbreviated, but any possible world can be reachable to itself, too.).:

Figure 1.6, which looks like a perspective view of a cube captures two perspectives "*We don't know the truth of p_1 and p_2, respectively*" and "*We don't know the truth of p_3*" at the same time (Now, some accessibility relations are abbreviated, but any possible world can be reachable to all possible worlds.).:

In this way, the product of models represents a composition of individual perspectives. We also introduce **distributed awareness** as a mutual complement among different reasoning abilities. We will show the differences between a mutual complement and a composition of awareness by a product of models.

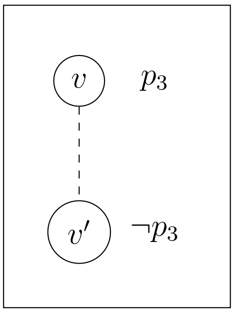

Figure 1.5: Example of Kripke Model(j's Perspective)

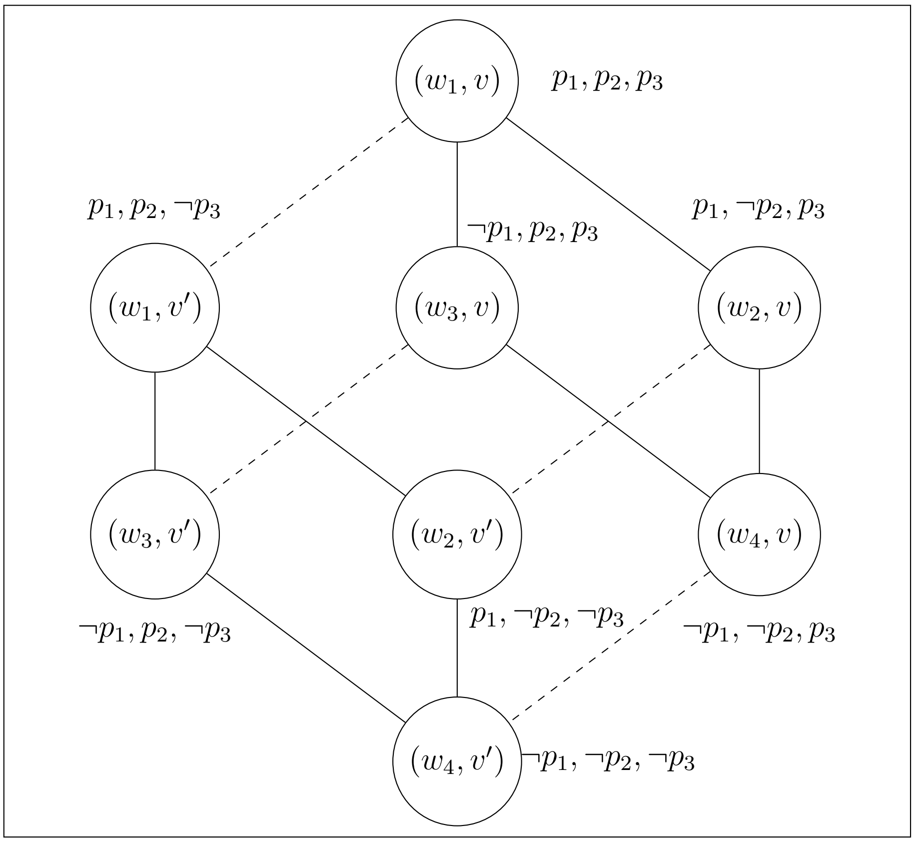

Figure 1.6: Example of Kripke Model Capturing Two Perspectives

Global propositional awareness gives **ALGP** and **ALF** three main limitations. First, there is a set of formulas without compactness in the semantics of **ALF**. Then, it is difficult to prove the completeness theorem. Second, **ALF** cannot represent "*the agent is aware of* p *and does not know whether others are aware of* p *or not*" by restriction to global awareness. Third, $K : K_i\varphi \wedge K_i(\varphi \to \psi) \to K_i\psi$ is valid for explicit knowledge by restriction to propositional awareness. Instead of introducing K for explicit knowledge, a condition in which an agent is aware of a formula and unaware of its subformula, and a neighborhood model[4] are used as substitutes.

At the cost of these limitations, we obtain several results. **ALF** can be introduced in the standard Kripke semantics without ternary or more relations by restriction to global and propositional awareness. Although traditional filtration gives a contracted model from a given model by globally applying the equivalence of truth of formulas, we can define an abstraction operator from traditional filtration by restriction to global awareness. Further, the compatibility of filtration-based abstraction to bisimulation-based abstraction on image-finite models by restriction to propositional awareness is shown.

A quotient model with agent expressions is introduced as filtration of a Kripke model. Further, **a nested quotient model** and **a product quotient model** are also introduced. The result for an image-finite model that a nested quotient model corresponds to a quotient model with common awareness is obtained. The result for an image-finite model that a product quotient model does not necessarily correspond to a quotient model with distributed awareness is obtained.

1.3 Thesis Outline

The rest of this thesis is organized as follows. Part I shows a sound and complete axiomatization for existing logic and Awareness Logic with Global Propositional Awareness, respectively. Part II investigates Awareness Logic of Abstraction.

Part I consists of chapters 2, 3, and 4. Chapters 2 and 3 introduce the background needed in the rest of the thesis. Note that these chapters do not contain an original work. Chapter 4 adds global awareness to Awareness Logic with gpp and shows a sound and complete axiomatization in **ALGP**.

Part II consists of chapters 5, 6, 7, 8, and 9. Chapter 5 introduces the syntax and semantics of Awareness Logic of Filtration and shows the properties of the filtration-based abstraction presented in this thesis. Chapter 6 compares three abstractions called filtration-based, bisimulation-based, and atoms-based abstraction. Chapter 7 investigates the non-compactness in **ALF** and the condition of a reduction from **ALF** to **ALGP**. Chapter 8 introduces a quotient model with an agent expression, a nested quotient model, and a product quotient model. Chapter 9 aims at reductions from a nested abstraction and concretization to an abstraction with an agent expression, respectively.

Part I

Soundness and Completeness

Notational Convention

In this thesis, the following symbols and brackets () are used to construct formulas:

- (**The propositional constants**) : $\top, \bot$;
- (**Atomic propositions**): p, q, r;
- (**Boolean connectives**): $\wedge, \vee, \rightarrow, \neg$;
- (**Agents**): i, j, k;
- (**Agent expressions**): a, b, c;
- (**Connectives for agent expressions**): $+, \cdot$;
- (**Modal operators**): $\Box, \Diamond, \Box_i, \Diamond_i, A_a, [\approx]_a, K_i, A_a^\alpha, [\approx]_a^\alpha, K_i^\alpha$ (where α is a finite sequence of agent expressions).

Also, we use the following letters to represent possible worlds.

- (**Possible worlds**): w, v, u, s, t, x, y;

$\varphi \equiv \psi$ represents that φ is a formula that has the same form as ψ. $\varphi := \psi$ represents that ψ is an abbreviations for φ. $M \cong M'$ represents that there is an isomorphism between two models M and M'.

Chapter 2

Modal Logic

This section reviews the basic modal logic.

2.1 Language and Semantics

Definition 2.1.1. (**Language $\mathcal{L}_{ML}$**)
Let P be countable set of atomic propositions. The language of the basic modal logic $\mathcal{L}_{ML}$ is defined by the following:

$$\mathcal{L}_{ML} \ni \varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

where $p \in P$.

In addition, we introduce abbreviations for the truth $\top$, the falsity $\bot$, the disjunction $\vee$, the implication $\rightarrow$, the logical equivalence $\leftrightarrow$, and the dual operator $\Diamond$ of $\Box$ as follows:

- $\top \ := \ p \rightarrow p$
- $\bot \ := \ \neg\top$;
- $\varphi \vee \psi \ := \ \neg(\neg\varphi \wedge \neg\psi)$;
- $\varphi \rightarrow \psi \ := \ \neg(\varphi \wedge \neg\psi)$;
- $\varphi \leftrightarrow \psi \ := \ (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$;
- $\Diamond\varphi \ := \ \neg \ \Box\neg\varphi$;

The following are the informal senses of $\Box$ and $\Diamond$:

- $\Box\varphi$ stands for 'it is necessary that φ holds'.

- $\Diamond\varphi$ stands for 'it is possible that φ holds'.

Next, we explain the semantics of Modal Logic.

Definition 2.1.2. (**Kripke frame**)
A pair $F = \langle W, \sim \rangle$ is called a **Kripke frame** consist of

1. W is a non-empty set;

2. $\sim$ is a relation on W.

W and $\sim$ are called **a set of possible worlds** and **an accessibility relation** of Kripke frame $\langle W, \sim \rangle$, respectively.

Definition 2.1.3. (**Kripke model**)
Let P be a countable set of atomic propositions. Let $V : P \longrightarrow 2^W$ be a function. Suppose that $F = \langle W, \sim \rangle$ is a Kripke frame. A Kripke model based on F is a tuple $M = \langle F, V \rangle$. V is called a **valuation function** of $\langle W, \sim, V \rangle$.

Definition 2.1.4. (**Satisfaction Relation of $\mathcal{L}_{ML}$**)
Let P be a countable set of atomic propositions. Given any Kripke model $M = \langle W, \sim, V \rangle$ and any possible world $w \in W$. Then, a binary relation $\models$ is defined as follows:

$$\begin{array}{ll} (M,w) \models p & \iff w \in V(p) \\ (M,w) \models \neg\varphi & \iff (M,w) \not\models \varphi \\ (M,w) \models (\varphi \wedge \psi) & \iff (M,w) \models \varphi \text{ and } (M,w) \models \psi; \\ (M,w) \models \Box\varphi & \iff \text{for all } v \in W, w \sim v \text{ implies } (M,v) \models \varphi; \end{array}$$

Then, $\models$ is called a **satisfaction relation** of $\mathcal{L}_{ML}$.

A set of formulas $\Gamma \subseteq \mathcal{L}_{ML}$ is **satisfiable** if there is some model $M = \langle W, \sim, V \rangle$ and a possible world $w \in W$ such that $(M, w) \models \varphi$ for all $\varphi \in \Gamma$. A formula $\varphi \in \mathcal{L}_{ML}$ is satisfiable when $\{\varphi\}$ is satisfiable.

By the definition, the satisfaction relation for $\top, \bot, \vee, \rightarrow, \Diamond$ is derived as follows: Given any (M, w) with $M = \langle W, \sim, V \rangle$ and $w \in W$.

$$\begin{array}{ll} (M,w) \models \top \; ; & \\ (M,w) \not\models \bot \; ; & \\ (M,w) \models \varphi \vee \psi & \iff (M,w) \models \varphi \text{ or } (M,w) \models \psi; \\ (M,w) \models \varphi \rightarrow \psi & \iff (M,w) \models \varphi \text{ implies } (M,w) \models \psi; \\ (M,w) \models \Diamond\varphi & \iff \text{for some } v \in W, w \sim v \text{ and } (M,v) \models \varphi; \end{array}$$

Definition 2.1.5. (**Validity**)
The notion of **validity** is defined over the various levels of semantical structure as follows:

- A formula $\varphi \in \mathcal{L}_{ML}$ is **valid on** a Kripke model M if $(M, w) \models \varphi$ for all $w \in W$. It is denoted $M \models \varphi$.
- A formula $\varphi \in \mathcal{L}_{ML}$ is **valid on** a Kripke frame F if $(F, V) \models \varphi$ for all valuation V for F. It is denoted $F \models \varphi$.
- A formula $\varphi \in \mathcal{L}_{ML}$ is **valid on** a class $\mathbb{F}$ of Kripke frames if $F \models \varphi$ for all $F \in \mathbb{F}$. It is denoted $\mathbb{F} \models \varphi$.

In modal logic, we can consider the correspondences between formulas and conditions on an accessibility relation for a given frame (See Table 2.1). Given a frame $F = \langle W, \sim \rangle$. For example, $\sim$ is reflexive if and only if $F \models \Box\varphi \to \varphi$ for all $\varphi \in \mathcal{L}_{ML}$. $\sim$ is called **serial** if for all $w \in W$ there is a $v \in W$ such that $w \sim v$, i.e., a given frame does not have a possible world that is not reachable to any possible world. If $\sim$ is reflexive, then $\sim$ is serial. $\sim$ is called **euclidean** if $w \sim v$ and $w \sim u$ imply $v \sim u$ for all $w, v, u \in W$, i.e., for all $w \in W$, every reachable possible world from w is reachable to all reachable possible world from w. $\sim$ is reflexive and euclidean if and only if $\sim$ is equivalence.

Formula		Name	Frame Condition
T	$\vdash \Box\varphi \to \varphi$	Reflexive	$\forall w \in W(w \sim w)$
B	$\vdash \varphi \to \Box\Diamond\varphi$	Symmetric	$\forall w, v \in W(w \sim v \to v \sim w)$
4	$\vdash \Box\varphi \to \Box\Box\varphi$	Transitive	$\forall w, v, u \in W(w \sim v \land v \sim u \to w \sim u)$
D	$\vdash \Box\varphi \to \Diamond\varphi$	Serial	$\forall w \in W \exists v \in W(w \sim v)$
5	$\vdash \Diamond\varphi \to \Box\Diamond\varphi$	Euclidean	$\forall w, v, u \in W(w \sim v \land w \sim u \to v \sim u)$

Table 2.1: Correspondence between Frame Conditions and Formulas

Definition 2.1.6. (**The Class of Frames**)
Given a Kripke frame $F = \langle W, \sim \rangle$.

- Every Kripke frames is called **K frame**. The class of K frames is denoted $\mathbb{F}_K$.
- If $\sim$ is an equivalence relation, then F is called **S5 frame**. The class of S5 frames is denoted $\mathbb{F}_{S5}$.

2.2 Axiomatization and Soundness

This section introduces the complete axiomatization for $\mathbb{F}_{S5}$ because the semantics of the classical logic of knowledge is given by the S5 frames. The proof system associated with a class of K frames is called **the system K**, and with S5 frames is called **the system S5**, respectively. The axiom K of $\Box$ is valid on all Kripke models and is also called "*distribution axiom*". The system S5 includes the axioms T and 5. However, We may not admit the axioms B and 4 because $\varphi \to \Box\Diamond\varphi$ and $\Box\varphi \to \Box\Box\varphi$ are deduced by using the following axioms of S5 and inference rules of modal logic. The K system is obtained by excluding the axioms T and 5 from the S5 system.

axiom	
TAUT	All propositional tautologies
K	$\vdash_{S5} \Box\varphi \wedge \Box(\varphi \to \psi) \to \Box\psi$
T	$\vdash_{S5} \Box\varphi \to \varphi$
5	$\vdash_{S5} \neg\Box\varphi \to \Box\neg\Box\varphi$

Table 2.2: Axioms of S5

System S5 has two inference rules called modus ponens(MP) and necessitation(Nec), respectively. Note that φ and $\varphi \to \psi$ in the inference rules (MP) and (Nec) must be formulas deduced without assumption.

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (MP) \qquad \frac{\varphi}{\Box\varphi}\ (Nec)$$

Now, we define the formal proof in the system S5.

Definition 2.2.1. (**Proof of The System S5**)
A **proof of the system S5** from Γ to φ is a finite sequence of formulas $\psi_1, \psi_2, ...\psi_n$ in $\mathcal{L}_{ML}$ such that $\varphi \equiv \psi_n$ and every formula ψ_i in the sequence is the axiom of S5, a formula in Γ, or formula deduced from $\psi_j (j < i)$ by applying an inference rule.

Further, φ is **provable** from Γ in the system S5 if there is a proof of the system S5 from Γ to φ. It is denoted $\Gamma \vdash_{S5} \varphi$. If $\Gamma = \varnothing$, then it is denoted $\vdash_{S5} \varphi$.

Theorem 2.2.2. (**The Soundness Theorem**)
Let $\mathbb{F}_{S5}$ be the class S5 of Kripke frames. For every $\varphi \in \mathcal{L}_{ML}$,

$$\vdash_{S5} \varphi \text{ implies } \mathbb{F}_{S5} \models \varphi.$$

2.3 Completeness

The proof via the canonical model for the completeness theorem is frequently cited in text [2]. Also, text [10] focuses on the completeness theorem of S5. In this section, the procedure for the proof follows the steps outlined in text [10].

Definition 2.3.1. (**Consistent**)
A set Γ of formulas is called **consistent** if $\Gamma \nvdash \bot$.

Definition 2.3.2. (**Maximally Consistent**)
A set Γ is called **maximally consistent** if

1. Γ is consistent: $\Gamma \nvdash \bot$;
2. Γ is maximal: there is no Γ' such that $\Gamma \subsetneq \Gamma'$ and $\Gamma' \nvdash \bot$.

Every maximally consistent set is closed under implication and determines the truth of all formulas. Then, we can consider $\Gamma \vdash \varphi$ if and only if $\varphi \in \Gamma$ for any formulas φ and any maximally consistent set Γ. In particular, φ is in all maximally consistent sets if and only if φ is derivable without assumption formulas. So, we will show the truth lemma(See Lemma 2.3.10) to prove the completeness theorem.

Lemma 2.3.3. Let Γ and φ be a finite consistent set and a formula, respectively. If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\neg\varphi\}$ is a finite consistent set.

Lemma 2.3.4. Γ is consistent if and only if any finite subset $\Delta \subseteq \Gamma$ is consistent.

Without considering compactness, we cannot establish the proof that Γ is satisfiable if and only if every finite subset $\Delta \subseteq \Gamma$ is also satisfiable. However, we can prove the lemma 2.3.4 without consideration of the compactness because $\Gamma \vdash \bot$ means that $\bot$ is deduced by using finite formulas in Γ as assumption formulas and applying the axioms and the inference rules at most finite(See Definition 2.2.1).

Lemma 2.3.5. The set of all finite sequences of members of a countable set is also countable.

The proof of the lemma 2.3.5 is referenced from the text [13] p.4-5. This lemma 2.3.5 shows the language $\mathcal{L}_{ML}$ is a countably infinite set of formulas because every formula is a finite sequence of symbols. If the language $\mathcal{L}_{ML}$ is countable, then we can consider an enumeration of all formulas in $\mathcal{L}_{ML}$.

Lemma 2.3.6. (**Lindenbaum's Lemma**)
If Γ is consistent, there is a maximally consistent set Δ such as $\Gamma \subseteq \Delta$.

Lindenbaum's lemma demonstrates that any consistent set can be extended to form a maximally consistent set. There are two proofs for Lindenbaum's lemma. The proof by an enumeration of all formulas is frequently cited. We refer to the proof in [10] p.179. On the other hand, we can prove Lindenbaum's lemma by an application of Zorn's Lemma without an enumeration of all formulas.

Now, we review the notion of "*validity*". The truth of every formula is determined in any possible world and any Kripke model. In particular, all valid formulas always hold in all possible worlds and all Kripke models. Although validity and a maximally consistent set are defined by satisfiability and provability, respectively, we can make a correspondence between all possible worlds and all maximally consistent sets. Hence, we prove "every formula in all maximally consistent sets is valid" and "every valid formula is in all maximally consistent sets" by introducing the Kripke model covering all maximally consistent sets called "*the canonical model*".

Definition 2.3.7. (**Canonical Model**)
The canonical model $M^C = \langle W^C, \sim^C, V^C \rangle$ is defined as follows:

1. $W^C = \{\Gamma \mid \Gamma \text{ is maximal consistent}\}$;
2. $\Gamma \sim^C \Delta$ iff $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Delta$;
3. $V^C(p) = \{\Gamma \in W^C \mid p \in \Gamma\}$.

The definition of the canonical model above is the same as the definition of the canonical model for proof system K. Since maximally consistent sets of the system S5 do not contain $\neg(\Box_i\varphi \to \varphi)$ and $\neg(\neg\Box_i\varphi \to \Box_i\neg\Box_i\varphi)$ for every formula $\varphi \in \mathcal{L}_{ML}$, we may define $\Gamma \sim^C \Delta$ iff $\{\varphi \mid \Box\varphi \in \Gamma\} = \{\varphi \mid \Box\varphi \in \Delta\}$, too.

Lemma 2.3.8.
If Γ and Δ are maximal consistent sets, then

1. If $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$, then $\psi \in \Gamma$;
2. $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$;
3. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$;
4. $\Gamma \sim^C \Delta$ iff $\{\varphi \mid \Box\varphi \in \Gamma\} = \{\varphi \mid \Box\varphi \in \Delta\}$;

5. $\{\Box\varphi \mid \Box\varphi \in \Gamma\} \vdash \psi$ iff $\{\Box\varphi \mid \Box\varphi \in \Gamma\} \vdash \Box\psi$;

The proof of the lemma 2.3.8 is referenced from the text [10] p179-180. The proof of item 4 needs the axioms (T) and (5). The proof of item 5 needs the axiom (K) and the inference rule (Nec).

Lemma 2.3.9. (**Canonicity**)
$\sim^C$ is equivalence relations for all maximally consistent sets $\Gamma, \Delta \in W^C$.

It is clear by item 4 of the lemma 2.3.8 because = is an equivalence relation. The lemma 2.3.9 shows the canonical model is S5 frame. It is needed to prove the following lemma called Truth Lemma.

Lemma 2.3.10. (**Truth Lemma**)
For every $\varphi \in \mathcal{L}_{ML}$ and every maximally consistent set $\Gamma \in W^C$:

$$\varphi \in \Gamma \text{ iff } (M^C, \Gamma) \models \varphi.$$

Truth Lemma shows "every formula in all maximally consistent sets is valid" and "every valid formula is in all maximally consistent sets". The proof by the induction on the structure of $\varphi \in \mathcal{L}_{ML}$ is referenced from text [10] p.180-p.181.

Theorem 2.3.11. (**The Completeness Theorem**)
For every $\varphi \in \mathcal{L}_{ML}$,

$$\mathbb{F}_{S5} \models \varphi \text{ implies } \vdash_{S5} \varphi.$$

By the contraposition, Lindenbaum's Lemma, and Truth Lemma. The proof is referenced from text [10] p.181.

Chapter 3

Awareness Logic

This section reviews Awareness Logic.

3.1 Language and Semantics

Definition 3.1.1. (**Language $\mathcal{L}_{AL}$**)
Let P be a countable set of atomic formulas and $\mathcal{G}$ be a finite set of agents, where these sets are mutually disjoint. The language $\mathcal{L}_{AL}$ is defined by the following:

$$\mathcal{L}_{AL} \ni \varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i\varphi \mid A_i\varphi$$

where $p \in P$ and $i \in \mathcal{G}$.

In addition, we introduce abbreviations for the truth $\top$, the falsity $\bot$, the disjunction $\vee$, the implication $\rightarrow$, the logical equivalence $\leftrightarrow$, the dual operator $\Diamond_i$ of $\Box_i$ for each $i \in \mathcal{G}$, and the explicit knowledge operator K_i for each $i \in \mathcal{G}$ as follows:

- $\top \ := \ p \rightarrow p$;
- $\bot \ := \ \neg\top$;
- $\varphi \vee \psi \ := \ \neg(\neg\varphi \wedge \neg\psi)$;
- $\varphi \rightarrow \psi \ := \ \neg(\varphi \wedge \neg\psi)$;
- $\varphi \leftrightarrow \psi \ := \ (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$;
- $\Diamond_i\varphi \ := \ \neg\Box_i\neg\varphi$;
- $K_i\varphi \ := \ A_i\varphi \wedge \Box_i\varphi$;

The following are the informal senses of $\Box_i$, A_i and K_i:

- $\Box_i\varphi$ stands for 'agent i knows φ holds implicitly'.
- $A_i\varphi$ stands for 'agent i is aware of φ'.
- $K_i\varphi$ stands for 'agent i knows φ holds explicitly'.

Next, we explain the semantics of Awareness Logic.

Now, we assume the following conditions of awareness for all agents: "agent i is aware of φ if and only if agent i is aware of all atomic propositions contained in φ". This condition has been considered in [3] and is called "*awareness is generated from primitive propositions*, as **gpp** for abbreviation.

Definition 3.1.2. (**Epistemic Awareness Model**)
Given a countable set of atomic propositions P and a finite set of agents $\mathcal{G}$, where these sets are mutually disjoint. An **epistemic awareness model** is a tuple $M = \langle W, \sim, \mathscr{A}, V \rangle$ consisting of:

1. W is a non-empty set of possible worlds;
2. $\sim: \mathcal{G} \longrightarrow 2^{W \times W}$ is a function assigning to each agent $i \in \mathcal{G}$ an equivalence relation $\sim_i$;
3. $\mathscr{A} : \mathcal{G} \times W \longrightarrow 2^P$ is a function.
4. $V : P \longrightarrow 2^W$ is a function.

Then, $\sim, \mathscr{A}, V$ is called **accessibility function**, **awareness function** and **valuation function** of M, respectively. Also, $\sim_i$ returned by the accessibility function is called **accessibility relation** of agent i.

Definition 3.1.3. $At : \mathcal{L}_{AL} \longrightarrow 2^P$ is defined inductively in the followings:

1. $At(\top) = \varnothing$;
2. for all $p \in P$, $At(p) = \{p\}$;
3. for all $\varphi \in \mathcal{L}_{AL}$, $At(\neg\varphi) = At(\varphi)$;
4. for all $\varphi, \psi \in \mathcal{L}_{AL}$, $At(\varphi \wedge \psi) = At(\varphi) \cup At(\psi)$;
5. for all $\varphi \in \mathcal{L}_{AL}$, $At(\Box_i\varphi) = At(\varphi)$;
6. for all $\varphi \in \mathcal{L}_{AL}$, $At(A_i\varphi) = At(\varphi)$;

Definition 3.1.4. (**Satisfaction Relation of $\mathcal{L}_{AL}$**)
Let P be a set of atomic propositions. Given any Kripke model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and any possible world $w \in W$. Then, a binary relation $\models$ is defined as follows:

$$
\begin{aligned}
(M,w) &\models_{AL} p && \iff w \in V(p) \\
(M,w) &\models_{AL} \neg\varphi && \iff (M,w) \not\models_{AL} \varphi \\
(M,w) &\models_{AL} (\varphi \wedge \psi) && \iff (M,w) \models_{AL} \varphi \text{ and } (M,w) \models_{AL} \psi; \\
(M,w) &\models_{AL} A_i\varphi && \iff At(\varphi) \subseteq \mathscr{A}(i,w); \\
(M,w) &\models_{AL} \Box_i\varphi && \iff \text{for all } v \in W, w \sim_i v \text{ implies } (M,v) \models_{AL} \varphi;
\end{aligned}
$$

Then, $\models_{AL}$ is called a **satisfaction relation** of $\mathcal{L}_{AL}$.

A set of formulas $\Gamma \subseteq \mathcal{L}_{AL}$ is **satisfiable** if there is some model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and a possible world $w \in W$ such that $(M,w) \models_{AL} \varphi$ for all $\varphi \in \Gamma$. A formula $\varphi \in \mathcal{L}_{AL}$ is satisfiable when $\{\varphi\}$ is satisfiable.

By the definition, the satisfaction relation for $\top, \bot, \vee, \rightarrow, \Diamond_i, K_i$ is derived as follows: Given any (M,w) with $M = \langle W, \sim, \mathscr{A}, V\rangle$ and $w \in W$.

$$
\begin{aligned}
(M,w) &\models_{AL} \top ; \\
(M,w) &\not\models_{AL} \bot ; \\
(M,w) &\models_{AL} \varphi \vee \psi && \iff (M,w) \models_{AL} \varphi \text{ or } (M,w) \models_{AL} \psi; \\
(M,w) &\models_{AL} \varphi \rightarrow \psi && \iff (M,w) \models_{AL} \varphi \text{ implies } (M,w) \models_{AL} \psi; \\
(M,w) &\models_{AL} \Diamond_i\varphi && \iff \text{for some } v \in W, w \sim_i v \text{ and } (M,v) \models_{AL} \varphi; \\
(M,w) &\models_{AL} K_i\varphi && \iff (M,w) \models_{AL} \Box_i\varphi \text{ and } (M,w) \models_{AL} A_i\varphi;
\end{aligned}
$$

Definition 3.1.5. (**Validity**)
The notion of **validity** is defined over the various levels of semantical structure as follows:

- A formula $\varphi \in \mathcal{L}_{AL}$ is **valid on** an epistemic awareness model M if $(M,w) \models_{AL} \varphi$ for all $w \in W$. It is denoted $M \models_{AL} \varphi$.

- A formula $\varphi \in \mathcal{L}_{AL}$ is **valid** if $(M,w) \models_{AL} \varphi$ for all $w \in W$ and all epistemic awareness models M. It is denoted $\models_{AL} \varphi$.

3.2 Axiomatization and Soundness

This section introduces the complete axiomatization for Awareness Logic with gpp. In the thesis, the axioms A1, A2, and A5 in [3] p.43 are called

TAUT, K, and 5, respectively. Further, the inference rules R1 and R2 in [3] p.43 are called MP and Nec, respectively. It is the same as that of the system S5.

The axioms regarding the condition gpp are referenced from [3] p.56. In this thesis, the axioms $A_i\varphi \leftrightarrow A_i\neg\varphi$, $A_i(\varphi \wedge \psi) \leftrightarrow A_i\varphi \wedge A_i\psi$, $A_i\varphi \leftrightarrow A_iA_j\varphi$, and $A_i\varphi \leftrightarrow A_i\Box_j\varphi$ are called AN, ACN, AA, and A□, respectively.

The axiom K stands for "*Implicit knowledge is closed under implication*". The axiom T stands for "*If an agent implicitly knows φ holds, then φ holds*", i.e., implicit knowledge is always correct. Doxastic Logic regarding reasoning about false information does not admit the axiom T. The axiom 5 stands for "*If an agent does not implicitly know φ holds, then he implicitly knows that he does not implicitly know φ holds*". If we admit the axioms of the system S5, then $4 : \Box_i\varphi \to \Box_i\Box_i\varphi$ must be admitted as the theorem. The theorem 4 stands for "*If an agent implicitly knows φ holds, then he implicitly knows that he implicitly knows φ holds*". Then, each agent knows what he implicitly knows and does not know by axiom 5 and theorem 4.

axiom		
TAUT	All propositional tautologies	
K	$\vdash_{AL} \Box_i\varphi \wedge \Box_i(\varphi \to \psi) \to \Box_i\psi$	Closed under Implication
T	$\vdash_{AL} \Box_i\varphi \to \varphi$	Knowledge implies truth
5	$\vdash_{AL} \neg\Box_i\varphi \to \Box_i\neg\Box_i\varphi$	Negative Introspection
AN	$\vdash_{AL} A_i\varphi \leftrightarrow A_i\neg\varphi$	gpp
ACN	$\vdash_{AL} A_i(\varphi \wedge \psi) \leftrightarrow A_i\varphi \wedge A_i\psi$	gpp
AA	$\vdash_{AL} A_i\varphi \leftrightarrow A_iA_j\varphi$	gpp
A□	$\vdash_{AL} A_i\varphi \leftrightarrow A_i\Box_j\varphi$	gpp

Table 3.1: Axioms of AL with gpp

The following are the inference rules of AL. Note that φ and $\varphi \to \psi$ in the inference rules (MP) and (Nec) must be formulas deduced without assumption.

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (MP) \qquad \frac{\varphi}{\Box_i\varphi}\ (Nec)$$

Now, we call a proof system S5 is extended by adding the axioms (AN), (ACN), (AA), and ($A\Box$) **the system AL**. Next, we define the formal proof in the system AL.

Definition 3.2.1. (**Proof of The System AL**)
A **proof of the system AL** from Γ to φ is a finite sequence of formulas $\psi_1, \psi_2, ...\psi_n$ in $\mathcal{L}_{AL}$ such that $\varphi \equiv \psi_n$ and every formula ψ_i in the sequence is the axiom of AL, a formula in Γ, or formula deduced from $\psi_j (j < i)$ by applying an inference rule.

Further, φ is **provable** from Γ in the system AL if there is a proof of the system AL from Γ to φ. It is denoted $\Gamma \vdash_{AL} \varphi$. If $\Gamma = \varnothing$, then it is denoted $\vdash_{AL} \varphi$.

Theorem 3.2.2. (**The Soundness Theorem**)
For every $\varphi \in \mathcal{L}_{AL}$,

$$\vdash_{AL} \varphi \text{ implies } \models_{AL} \varphi.$$

3.3 Completeness

The definition of maximally consistent sets and the proof of Lindenbaum's Lemma are similar to them in Section 2.3.

The definition of the canonical model of the system S5 (See Definition 2.3.7) is extended to that of the canonical model of the system AL by adding $\mathscr{A}^C(i, \Gamma) = \{p \mid A_i p \in \Gamma\}$. The definition of the canonical model of the system AL containing the axioms regarding the condition gpp is referenced from Definition 67 in [11] p.59.

Definition 3.3.1. (**Canonical Model**)
The canonical model $M^C = \langle W^C, \sim^C, \mathscr{A}^C, V^C \rangle$ is defined as follows:

1. $W^C = \{\Gamma \mid \Gamma \text{ is maximal consistent}\}$;
2. $\Gamma \sim_i^C \Delta$ iff $\{\varphi \mid \Box_i \varphi \in \Gamma\} \subseteq \Delta$;
3. $\mathscr{A}^C(i, \Gamma) = \{p \mid A_i p \in \Gamma\}$;
4. $V^C(p) = \{\Gamma \in W^C \mid p \in \Gamma\}$.

The following lemma 3.3.2 is the same as the lemma 2.3.8. Similarly, the proof is referenced from Lemma 7.4 in text [10] p.179-p.180.

Lemma 3.3.2. If Γ and Δ are maximal consistent sets, then

1. If $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$, then $\psi \in \Gamma$;
2. $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$,
3. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$,

4. $\Gamma \sim_i^C \Delta$ iff $\{\varphi \mid \Box_i \varphi \in \Gamma\} = \{\varphi \mid \Box_i \varphi \in \Delta\}$,

5. $\{\Box_i \varphi \mid \Box_i \varphi \in \Gamma\} \vdash_{AL} \psi$ iff $\{\Box_i \varphi \mid \Box_i \varphi \in \Gamma\} \vdash_{AL} \Box_i \psi$.

Item 1 of the following lemma is the same as the lemma 2.3.7. Similarly, it is clear because $=$ is an equivalence relation. The proof of item 2 of the following lemma is referenced from Lemma 68 in [11] p.59.

Lemma 3.3.3. (**Canonicity**)
For all maximally consistent sets Γ, Δ,

1. $\sim_i^C$ is equivalence relations for all $i \in \mathcal{G}$.

2. gpp: $A_i \varphi \in \Gamma$ iff $At(\varphi) \subseteq \mathscr{A}^C(i, \Gamma)$ for all $i \in \mathcal{G}$.

In the following lemma, the proof of the case for $A_i \varphi$ follows directly from lemma 3.3.3. The proofs of other cases are similar to lemma 2.3.10.

Lemma 3.3.4. (**Truth Lemma**)
For every $\varphi \in \mathcal{L}_{AL}$ and every maximally consistent set $\Gamma \in W^C$:

$$\varphi \in \Gamma \text{ iff } (M^C, \Gamma) \models_{AL} \varphi.$$

The proof of the completeness theorem is similar to that in Section 2.3.

Theorem 3.3.5. (**The Completeness Theorem**)
For every $\varphi \in \mathcal{L}_{AL}$,

$$\models_{AL} \varphi \text{ implies } \vdash_{AL} \varphi.$$

Chapter 4

Awareness Logic with Global Propositional Awareness

This chapter introduces Awareness Logic with Global Propositional Awareness in preparation for introducing Awareness Logic of Abstraction.

4.1 Restrictions on Awareness

Several restrictions on awareness are considered in [3] p.54. First, we list the names of these conditions below.:

- Awareness is generated by primitive propositions;
- Awareness is closed under subformulas;
- An agent knows which formulas he is aware of.

Further, we consider two restrictions of awareness set called **the global definition** and **the local definition**. These definitions of an awareness set are referenced from [6] p.210.

Awareness is generated by primitive propositions

This condition is often abbreviated as **gpp**. Definition 3.1.2 imposes the condition **gpp**. On the other hand, a traditional awareness set $\mathscr{A}$ is a set of formulas, and the satisfiability for awareness operator A_i is defined as $(M, w) \models A_i\varphi \Leftrightarrow \varphi \in \mathscr{A}(i, w)$. gpp is described as the following.:

For all $i \in \mathcal{G}$ and $w \in W, \varphi \in \mathscr{A}(i, w)$ if and only if $At(\varphi) \subseteq \mathscr{A}(i, w)$.

Traditionally, the case of $\varphi \wedge \psi \in \mathscr{A}(i,w)$ and $\varphi \notin \mathscr{A}(i,w)$ is possible. Further, the case of $\neg\varphi \in \mathscr{A}(i,w)$ and $\varphi \notin \mathscr{A}(i,w)$ is possible. However, some people consider that their cases are not natural. For example, if we are aware of "*It is Monday and sunny today*", then we are also aware of "*It is sunny and Monday today*".

If gpp holds, then the following restriction called "*awareness is closed under subformulas*" holds.

Awareness is closed under subformulas

"*Awareness is closed under subformulas*" is described as the following.:

For all $i \in \mathcal{G}$, If $\varphi \in \mathscr{A}(i,w)$ and ψ is a subformula of φ, then $\psi \in \mathscr{A}(i,w)$.

However, if we explicitly know the truth of φ, then we may explicitly know the truth of $\varphi \vee \psi$ even if we are not aware of ψ. When we are reasoning about a computer program, we may know $\varphi \vee \psi$ is true without needing to compute the truth of ψ. In this regard, this restriction is considered inappropriate as a computational notion of awareness. On the other hand, we consider that this restriction is compatible with filtration because Theorem 2.39 in text [8] p.79 holds for every set of formulas closed under subformulas.

The restriction allows **the axiom K** for explicit knowledge ($\vdash K_i\varphi \wedge K_i(\varphi \to \psi) \to K_i\psi$). So, gpp also allows **the axiom K** for explicit knowledge.

An agent knows which formulas he is aware of

This condition is often abbreviated as **ka**.

For all $i \in \mathcal{G}$ and $w, v \in W$, if $w \sim_i v$ implies $\mathscr{A}(i,w) = \mathscr{A}(i,v)$.

By [3] p.54, **ka** corresponds to "$A_i\varphi \to \Box_i A_i\varphi$ *and* $\neg A_i\varphi \to \Box_i \neg A_i\varphi$ *are valid*" for all $i \in \mathcal{G}$. If we admit ka and "*awareness is closed under subformulas*", then an agent explicitly knows he is aware of.

Global Definition of An Awareness Set

Traditionally, the agent's awareness in some possible world may be different from that in different possible worlds. This traditional definition is called the **local definition** of awareness set. On the other hand, The restriction that each agent's awareness is the same in all worlds is called the global definition of awareness set. if we allow the global definition of awareness set, then $\Diamond_i A_j\varphi \wedge \Diamond_i \neg A_j\varphi$ is always false for all $i, j \in \mathcal{G}$. Further, if we allow the global definition of awareness set, then we must allow ka, too.

ALGP allows gpp and the global definition of an awareness set, i.e., ALGP allows all of the above restrictions without the local definition of an awareness set.

4.2 Language and Semantics

Definition 4.2.1. (**Language** $\mathcal{L}_{ALGP}$)
Let P be a countable set of atomic propositions and $\mathcal{G}$ be a finite set of agents. Suppose $e \in \mathcal{G}$. We assume they are mutually disjoint. The language $\mathcal{L}_{ALGP}$ is defined by the following:

$$\mathcal{L}_{ALGP} \ni \varphi \ ::= \ \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i\varphi \mid A_i\varphi$$

where $p \in P$ and $i \in \mathcal{G}$.

We introduce the constant symbol $\top$ as a primitive for consideration of the model tailored to the awareness of those who are not aware of anything. If $\top$ is an abbreviation for $p \to p$, then the satisfiability of $\top$ will not be defined in the model.

In addition, we introduce abbreviations for the falsity $\bot$, the disjunction $\vee$, the implication $\to$, the logical equivalence $\leftrightarrow$, the dual operator $\Diamond_i$ of $\Box_i$ for each $i \in \mathcal{G}$, and the explicit knowledge operator K_i for each $i \in \mathcal{G}$ as follows:

- $\bot \ := \ \neg\top;$
- $\varphi \vee \psi \ := \ \neg(\neg\varphi \wedge \neg\psi);$
- $\varphi \to \psi \ := \ \neg(\varphi \wedge \neg\psi);$
- $\varphi \leftrightarrow \psi \ := \ (\varphi \to \psi) \wedge (\psi \to \varphi);$
- $\Diamond_i\varphi \ := \ \neg\Box_i\neg\varphi;$
- $K_i\varphi \ := \ A_i\varphi \wedge \Box_i\varphi;$

The following are the informal senses of $\Box_i$, A_i, and K_i.

- $\Box_i\varphi$ stands for 'agent i knows φ holds, implicitly'.
- $A_i\varphi$ stands for 'agent i is aware of φ'.

- $K_i\varphi$ stands for 'agent i knows φ holds, explicitly'.

The modalities $\Box_e$ and $\Diamond_e$ are called **the global box** and **the global diamond**, respectively in text [8] p.54.[1] The global diamond is needed to give the reduction axiom for abstraction operators introduced in the paper [1]. In this thesis, the global definition of awareness needs global modalities. But, the modalities $\Box_e$ and $\Diamond_e$ cannot be expressed in the basic modal logic [8]. So, we have to introduce $\Box_e$ or $\Diamond_e$ as a primitive in this thesis.

Next, we introduce the semantics ALGP.

The following model is the same as the Kripke model with awareness function and restriction that $\sim_i$ is an equivalence relation for each $i \in \mathcal{G}$.

Definition 4.2.2. (**Epistemic Awareness Model**)
Let P be a countable set of atomic propositions and $\mathcal{G}$ be a finite set of agents. Suppose $e \in \mathcal{G}$. We assume they are mutually disjoint. M is called **Epistemic awareness model** if it is a tuple $\langle W, \sim, \mathscr{A}, V \rangle$ consisting of:

1. W is a non-empty set of possible worlds ;
2. $\sim: \mathcal{G} \longrightarrow 2^{W \times W}$ is a function assigning to each agent $i \in \mathcal{G}$ an equivalence relation $\sim_i$,and $w \sim_e v$ for all $w, v \in W$;
3. $\mathscr{A} : \mathcal{G} \longrightarrow 2^P$, where $\mathscr{A}(e) = P$;
4. $V : P \longrightarrow 2^W$.

Then, $\sim, \mathscr{A}$, and V are called **accessibility function**, **awareness function** and **valuation function** of M respectively. Also, $\sim_i$ returned by the accessibility function is called **accessibility relation** of agent i.

Definition 4.2.3. $At : \mathcal{L}_{ALGP} \longrightarrow 2^P$ is defined inductively in the followings:

1. $At(\top) = \varnothing$;
2. for all $p \in P$, $At(p) = \{p\}$;
3. for all $\varphi \in \mathcal{L}_{ALGP}$, $At(\neg\varphi) = At(\varphi)$;
4. for all $\varphi, \psi \in \mathcal{L}_{ALGP}$, $At(\varphi \wedge \psi) = At(\varphi) \cup At(\psi)$;
5. for all $\varphi \in \mathcal{L}_{ALGP}$, $At(\Box_i\varphi) = At(\varphi)$;

[1] In the text [8], the symbols the global box and the global diamond are used A and E, respectively. But, in this thesis, the symbol of the global box is $\Box_e$ because it is easily misunderstood that A is the awareness operator.

6. for all $\varphi \in \mathcal{L}_{ALGP}$, $At(A_i\varphi) = At(\varphi)$;

Definition 4.2.4. (**Satisfaction Relation of** $\mathcal{L}_{ALGP}$)
Given (M, w) with $M = \langle W, \sim, \mathscr{A}, V\rangle$ and $w \in W$.

$$
\begin{array}{ll}
(M,w) \models_{ALGP} \top ; & \\
(M,w) \models_{ALGP} p & \iff w \in V(p); \\
(M,w) \models_{ALGP} \neg\varphi & \iff (M,w) \not\models_{ALGP} \varphi; \\
(M,w) \models_{ALGP} \varphi \wedge \psi & \iff (M,w) \models_{ALGP} \varphi \text{ and } (M,w) \models_{ALGP} \psi; \\
(M,w) \models_{ALGP} A_i\varphi & \iff At(\varphi) \subseteq \mathscr{A}(i); \\
(M,w) \models_{ALGP} \Box_i\varphi & \iff \text{for all } v \in W, w \sim_i v \text{ implies } (M,v) \models_{ALGP} \varphi;
\end{array}
$$

Then, $\models_{ALGP}$ is called a **satisfaction relation** of $\mathcal{L}_{ALGP}$.

A set of formulas $\Gamma \subseteq \mathcal{L}_{ALGP}$ is **satisfiable** if there is some model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and a possible world $w \in W$ such that $(M, w) \models \varphi$ for all $\varphi \in \Gamma$. A formula $\varphi \in \mathcal{L}_{ALGP}$ is satisfiable when $\{\varphi\}$ is satisfiable.

By the definition, the satisfaction relation for $\bot, \vee, \rightarrow, \Diamond_i, K_i$ are derived as follows: Given (M, w) with $M = \langle W, \sim, \mathscr{A}, V\rangle$ and $w \in W$.

$$
\begin{array}{ll}
(M,w) \not\models_{ALGP} \bot ; & \\
(M,w) \models_{ALGP} \varphi \vee \psi & \iff (M,w) \models_{ALGP} \varphi \text{ or } (M,w) \models_{ALGP} \psi; \\
(M,w) \models_{ALGP} \varphi \rightarrow \psi & \iff (M,w) \models_{ALGP} \varphi \text{ implies } (M,w) \models_{ALGP} \psi; \\
(M,w) \models_{ALGP} \Diamond_i\varphi & \iff \text{for some } v \in W, w \sim_i v \text{ and } (M,v) \models_{ALGP} \varphi; \\
(M,w) \models_{ALGP} K_i\varphi & \iff (M,w) \models_{ALGP} \Box_i\varphi \text{ and } (M,w) \models_{ALGP} A_i\varphi;
\end{array}
$$

Definition 4.2.5. (**Validity**)
The notion of **validity** is defined over the various levels of semantical structure as follows:

- A formula $\varphi \in \mathcal{L}_{ALGP}$ is **valid on** an epistemic awareness model M if $(M, w) \models \varphi$ for all $w \in W$. It is denoted $M \models_{ALGP} \varphi$.
- A formula $\varphi \in \mathcal{L}_{ALGP}$ is **valid** if $(M, w) \models \varphi$ for all $w \in W$ and all epistemic awareness models M. It is denoted $\models_{ALGP} \varphi$.

4.3 Axiomatization and Soundness

This section introduces the complete axiomatization for ALGP. In the thesis, the axiom inc is referenced from the axiom *inclusion* in [8] p.417. By $e \in \mathcal{G}$, the axioms K, T, and 5 hold for the global modalities. The axioms from

TAUT to A$\Box$ in Table 4.1 are the same as those in Section 3.2. The axioms eA and eUA are for the global definition of awareness sets. Without global modalities, we cannot represent that the same awareness holds for all possible worlds of each model. Although these axioms are not introduced in [6], they are natural as axioms of the global definition of awareness sets. The axiom Ae represents that the global modalities $\Box_e$ and $\Diamond_e$ are the same as the global modalities of modal logic in [8] p.54. ALGP needs axiom A$\top$ because we introduced the constant symbol $\top$ as a primitive for consideration of the model tailored to the awareness of those who are not aware of anything.

axiom		
TAUT	All propositional tautologies	
K	$\vdash \Box_i\varphi \wedge \Box_i(\varphi \to \psi) \to \Box_i\psi$	Closed under Implication
T	$\vdash \Box_i\varphi \to \varphi$	Knowledge implies truth
5	$\vdash \neg\Box_i\varphi \to \Box_i\neg\Box_i\varphi$	Negative Introspection
AN	$\vdash A_i\varphi \leftrightarrow A_i\neg\varphi$	gpp
ACN	$\vdash A_i(\varphi \wedge \psi) \leftrightarrow A_i\varphi \wedge A_i\psi$	gpp
AA	$\vdash A_i\varphi \leftrightarrow A_iA_j\varphi$	gpp
A$\Box$	$\vdash A_i\varphi \leftrightarrow A_i\Box_j\varphi$	gpp
inc	$\vdash \Box_e\varphi \to \Box_i\varphi$	$\sim_e$ includes $\sim_i$
eA	$\vdash A_i\varphi \to \Box_eA_i\varphi$	Global Definition
eUA	$\vdash \neg A_i\varphi \to \Box_e\neg A_i\varphi$	Global Definition
A$\top$	$\vdash A_i\top$	All agents aware of $\top$
Ae	$\vdash A_e\varphi$	The agent e aware of all

Table 4.1: Axioms of ALGP

The inference rules are the same as that of ALGP. Note that φ and $\varphi \to \psi$ in the inference rules (MP) and (Nec) must be formulas deduced without assumption.

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (MP) \qquad \frac{\varphi}{\Box_i\varphi}\ (Nec)$$

Now, we call a proof system AL is extended by adding the axioms (inc), (eA), (eUA), $(A\top)$, and (Ae) **the system ALGP**. Next, we define the formal proof in the system

ALGP.

Definition 4.3.1. (**Proof of The System ALGP**)
A **proof of the system ALGP** from Γ to φ is a finite sequence of formulas $\psi_1, \psi_2, ...\psi_n$ in $\mathcal{L}_{ALGP}$ such that $\varphi \equiv \psi_n$ and every formula ψ_i in the sequence is the axiom of AL, a formula in Γ, or formula deduced from $\psi_j (j < i)$ by applying an inference rule.

Further, φ is **provable** from Γ in the system AL if there is a proof of the system ALGP from Γ to φ. It is denoted $\Gamma \vdash_{AL} \varphi$. If $\Gamma = \varnothing$, then it is denoted $\vdash_{ALGP} \varphi$.

Theorem 4.3.2. (**The Soundness Theorem**)
For every $\varphi \in \mathcal{L}_{ALGP}$,

$$\vdash_{ALGP} \varphi \text{ implies } \models_{ALGP} \varphi.$$

Proof. Take an arbitrary epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ and $w \in W$.

- (**A**$\top$): Take an arbitrary $i \in \mathcal{G}$. By the definition of the function At, $At(\top) \subseteq \mathscr{A}(i)$. By the semantics, $(M, w) \models_{ALGP} A_i\top$. Since $i \in \mathcal{G}$ is an arbitrary agent, $(M, w) \models_{ALGP} A_i\top$ for all $i \in \mathcal{G}$.
- (**Ae**): By the definition of $\mathscr{A}$, $\mathscr{A}(e) = P$. Then, $At(\varphi) \subseteq \mathscr{A}(e)$ for all $\varphi \in \mathcal{L}_{ALGP}$. Therefore, $(M, w) \models_{ALGP} A_e\varphi$.
- (**eA**), (**eUA**): By the definition of the awareness set, $(M, w) \models_{ALGP} A_i\varphi$ is equivalent to $(M, v) \models_{ALGP} A_i\varphi$ for all $v \in W$ and all $i \in \mathcal{G}$. By the definition of $\sim_e$, $(M, w) \models_{ALGP} A_i\varphi$ is equivalent to $(M, w) \models_{ALGP} \Box_e A_i\varphi$.
- (inc): Suppose that $(M, w) \models_{ALGP} \Box_e\varphi$. Then, $(M, v) \models_{ALGP} \varphi$ for all $v \in W$. Since $\sim_i$ is the relation on W for all $i \in \mathcal{G}$, $(M, v') \models_{ALGP} \varphi$ for all $v' \in W$ such that $w \sim_i v'$ for all $i \in \mathcal{G}$. Therefore, $(M, w) \models_{ALGP} \Box_i\varphi$.

$\Box$

4.4 Completeness

The definition of maximally consistent sets, the proof of Lindenbaum's Lemma, and the definition of the canonical model are similar to them in Section 3.3. The following lemma 4.4.1 shows that a model of ALGP can be constructed from the canonical model.

Lemma 4.4.1. Given the canonical model $M^C = \langle W^C, \sim^C, \mathscr{A}^C, V^C \rangle$. For all $i \in \mathcal{G}$ and all $\Gamma, \Delta \in W^C$,

- **inclusion**: if $\Gamma \sim_i^C \Delta$, then $\Gamma \sim_e^C \Delta$.
- **the global definition**: if $\Gamma \sim_e^C \Delta$, then $\mathscr{A}^C(i, \Gamma) = \mathscr{A}^C(i, \Delta)$.

Proof. Given the canonical model $M^C = \langle W^C, \sim^C, \mathscr{A}^C, V^C \rangle$. Take an arbitrary agent $i \in \mathcal{G}$ and two arbitrary maximally consistent sets $\Gamma, \Delta \in W^C$.

- **inclusion**: Suppose $\Gamma \sim_i^C \Delta$. By the lemma 3.3.2, $\{\varphi \mid \Box_i \varphi \in \Gamma\} = \{\varphi \mid \Box_i \varphi \in \Delta\}$. Take an arbitrary formula $\psi \in \{\varphi \mid \Box_i \varphi \in \Gamma\}$. Then, $\Box_i \psi \in \Gamma$. By the axiom (inc), $\{\varphi \mid \Box_e \varphi \in \Gamma\} \subseteq \{\varphi \mid \Box_i \varphi \in \Gamma\}$. Then, $\{\varphi \mid \Box_e \varphi \in \Gamma\} \subseteq \Delta$. Therefore, $\Gamma \sim_e^C \Delta$.
- **the global definition**: Suppose $\Gamma \sim_e^C \Delta$. Take an arbitrary $p \in \mathscr{A}^C(i, \Gamma)$. Then, $A_i p \in \Gamma$. By applying (eA), $\Box_e A_i p \in \Gamma$. By the assumption, $A_i p \in \Delta$. Since $p \in \mathscr{A}^C(i, \Gamma)$ is an arbitrary atomic proposition, $\mathscr{A}^C(i, \Gamma) \subseteq \mathscr{A}^C(i, \Delta)$. Take an arbitrary $q \notin \mathscr{A}^C(i, \Gamma)$. Then, $A_i q \notin \Gamma$, i.e., $\neg A_i q \in \Gamma$. By applying (eUA), $\Box_e \neg A_i q \in \Gamma$. By the assumption, $\neg A_i q \in \Delta$, i.e., $A_i q \notin \Delta$. Since $q \notin \mathscr{A}^C(i, \Gamma)$ is an arbitrary atomic proposition, $\mathscr{A}^C(i, \Delta) \subseteq \mathscr{A}^C(i, \Gamma)$. Therefore, $\mathscr{A}^C(i, \Gamma) = \mathscr{A}^C(i, \Delta)$.

$\Box$

By the properties of maximally consistent sets, φ is in all maximally consistent sets if and only if φ is derivable without assumption formulas. So, we may show that "every formula in all maximally consistent sets is valid" and "every valid formula is in all maximally consistent sets"(Truth Lemma). By proving Truth Lemma, we introduce models such that the disjoint union of these models covers all maximally consistent sets.

Definition 4.4.2. (**Submodel of Canonical Model**)
Let $\mathcal{W}_e$ be the set of the equivalence classes of maximally consistent sets with respect to $\sim_e^C$. A **submodel of canonical model** $M^{C'} = \langle W^{C'}, \sim^C, \mathscr{A}^{C'}, V^{C'} \rangle$ is given for each the equivalence class of a maximally consistent set with respect to $\sim_e^C$ as follows:

1. $W^{C'} \in \mathcal{W}_e$;
2. $\Gamma \sim_i^{C'} \Delta$ iff $\Gamma \sim_i^C \Delta$ for all maximally consistent set $\Gamma, \Delta \in W^{C'}$;
3. $\mathscr{A}^{C'}(i) = \mathscr{A}^C(i, \Gamma)$ for all maximally consistent set $\Gamma \in W^{C'}$;

4. $V^{C'}(p) = V^{C}(p)$ for all $p \in P$.

Lemma 4.4.3. Given a submodel of the canonical model $M^{C'} = \langle W^{C'}, \sim^{C'}, \mathscr{A}^{C'}, V^{C'} \rangle$. For all maximally consistent sets $\Gamma, \Delta \in W^{C'}$,

1. $\top \in \Gamma$,
2. for the agent $e \in \mathcal{G}$, $A_e\varphi \in \Gamma$,
3. gpp: $A_i\varphi \in \Gamma$ iff $At(\varphi) \subseteq \mathscr{A}^{C'}(i)$ for all $i \in \mathcal{G}$.

Proof. Take two arbitrary maximally consistent sets Γ, Δ.

1. Since Γ is consistent, $\Gamma \cup \{\top\}$ is also consistent. Since Γ is maximally consistent, $\top \in \Gamma$.
2. By the axiom (Ae), $A_e\varphi \in \Gamma$ for all $\varphi \in \mathcal{L}_{ALGP}$ because $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ implies $\psi \in \Gamma$ for all $\varphi, \psi \in \mathcal{L}_{ALGP}$.
3. The proof is similar to the lemma 3.3.3.

□

Every submodel is mutually disjoint to the other submodels, and the union of all submodels covers all maximally consistent sets. Therefore, if the truth lemma is valid on every submodel, then all valid formulas are in all maximally consistent sets and every formula is in all maximally consistent sets is valid.

Lemma 4.4.4. (**Truth Lemma**)
For every $\varphi \in \mathcal{L}_{ALGP}$, every submodel of the canonical model $M^{C'}$, and every maximally consistent set $\Gamma \in W^{C'}$:

$$\varphi \in \Gamma \text{ iff } (M^{C'}, \Gamma) \models \varphi.$$

Proof. By the induction on the structure of $\varphi \in \mathcal{L}_{ALGP}$.

- (**Base Case**):
  - (**the case for** $\top$): By the lemma 4.4.3, $\top \in \Gamma$. By the semantics, $(M^{C'}, \Gamma) \models \top$. Therefore, $\top \in \Gamma$ is equialent to $(M^{C'}, \Gamma) \models \top$.
  - (**the case for** $p \in P$): The proof of the case is the same as the lemma 3.3.4.
- (**Induction Hypothesis**): For every maximally consistent set Γ ad for given φ and ψ it is the case that $\varphi \in \Gamma$ iff $(M^{C'}, \Gamma) \models \varphi$ and $\psi \in \Gamma$ iff $(M^{C'}, \Gamma) \models \psi$.

- (**Induction Step**):
  - (**the case for** $\neg\varphi$, $\varphi \wedge \psi$ and $\Box_i\varphi$): The proofs of the cases are the same as the lemma 3.3.4.
  - (**the case for** $A_i\varphi$):
    - $A_i\varphi \equiv A_e\varphi$: By the lemma 4.4.3, $A_e\varphi \in \Gamma$ for all formulas $\varphi \in \mathcal{L}_{ALGP}$. By the semantics, $(M^{C'}, \Gamma) \models A_e\varphi$ for all formulas $\varphi \in \mathcal{L}_{ALGP}$. Therefore, $A_e\varphi \in \Gamma$ is equivalent to $(M^{C'}, \Gamma) \models A_e\varphi$.
    - $A_i\varphi \not\equiv A_e\varphi$: The proof is similar to the lemma 3.3.4.

□

Theorem 4.4.5. (**The Completeness Theorem**)
For every $\varphi \in \mathcal{L}_{ALGP}$,

$$\models \varphi \quad \text{implies} \quad \vdash \varphi.$$

Proof. By the contraposition. Take an arbitrary $\varphi \in \mathcal{L}_{ALGP}$. Suppose $\nvdash \varphi$. Then, $\{\neg\varphi\}$ is a consistent. By Lindenbaum's Lemma, there is a maximally consistent set Γ such that $\{\neg\varphi\} \subseteq \Gamma$. By the Truth Lemma, $(M^{C'}, \Gamma) \models \neg\varphi$. Therefore, $\not\models \varphi$. □

Part II

Awareness Logic of Abstraction

Chapter 5

Awareness Logic of Filtration

In part II, we assume P is a finite set of atomic propositions. Further, we introduce "*Awareness Logic of Abstraction*" by adding the filtration-based equivalence relation to epistemic awareness models.

There are two reasons why filtration is a natural concept as an abstraction. First, a model obtained through filtration is simpler than the original model. Second, a model obtained through filtration preserves the truth of some formulas.

5.1 Agent Expression

We consider awareness of the group of agents for common awareness and distributed awareness. Common awareness and distributed awareness correspond to nested abstraction and mutually complementary concretization, respectively. In this thesis, **expression** means a finite sequence of symbols.

Definition 5.1.1. (**Agent expression** $\mathcal{A}$)
Let $\mathcal{G}$ be a finite set of agents with the special agent e.

$$\mathcal{A} \ni a \ ::= \ i \mid a + a \mid a \cdot a$$

where $i \in \mathcal{G}$. Then, a is called an **agent expression** of $\mathcal{G}$.

$a + b$'s awareness means agent a's awareness or agent b's awareness. In other words, $a + b$'s awareness is the distributed awareness of a and b. $a \cdot b$'s awareness means agent a's awareness and agent b's awareness. In other words, $a \cdot b$'s awareness is the common awareness of a and b.

We introduce A_{a+b} and $A_{a \cdot b}$ as macro to ALGP.

- **Distributed Awareness**: $A_{a+b}\varphi \ := \ \bigwedge_{p\in At(\varphi)}(A_a p \vee A_b p)$;
- **Common Awareness**: $A_{a\cdot b}\varphi \ := \ A_a\varphi \wedge A_b\varphi$,

Lemma 5.1.2. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. For all $\varphi \in \mathcal{L}_{ALGP}$ and all $a \in \mathcal{A}$, $At(A_a\varphi) = At(\varphi)$.

Proof. By the induction of the structure of an agent expression. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Take an arbitrary $\varphi \in \mathcal{L}_{ALGP}$.

- (**Base Case**): By the definition of $At : \mathcal{L}_{ALGP} \longrightarrow 2^P$, $At(A_i\varphi) = At(\varphi)$ for all $\varphi \in \mathcal{L}_{ALGP}$ and $i \in \mathcal{G}$.
- (**Induction Hypothesis**): $At(A_a\varphi) = At(\varphi)$ and $At(A_b\varphi) = At(\varphi)$ for all $\varphi \in \mathcal{L}_{ALGP}$.
- (**Induction Step**):
  - (**The case for** $A_{a+b}\varphi$):
    Since $A_{a+b}\varphi := \bigwedge_{p\in At(\varphi)}(A_a p\vee A_b p)$, $At(A_{a+b}\varphi) = At(\bigwedge_{p\in At(\varphi)}(A_a p\vee A_b p))$. By the definition of $At : \mathcal{L}_{ALGP} \longrightarrow 2^P$, $At(\bigwedge_{p\in At(\varphi)}(A_a p \vee A_b p)) = \bigcup_{p\in At(\varphi)}(At(A_a p) \cup At(A_b p))$. By the induction hypothesis, $\bigcup_{p\in At(\varphi)}(At(A_a p)\cup At(A_b p)) = At(\varphi)$. Therefore, $At(A_{a+b}\varphi) = At(\varphi)$.
  - (**The case for** $A_{a\cdot b}\varphi$):
    Since $A_{a\cdot b}\varphi := A_a\varphi \wedge A_b\varphi$, $At(A_{a\cdot b}\varphi) = At(A_a\varphi \wedge A_b\varphi)$. By the definition of $At : \mathcal{L}_{ALGP} \longrightarrow 2^P$, $At(A_a\varphi \wedge A_b\varphi) = At(A_a\varphi) \cup At(A_b\varphi)$. By the induction hypothesis, $At(A_a\varphi) \cup At(A_b\varphi) = At(\varphi)$. Therefore, $At(A_{a\cdot b}\varphi) = At(\varphi)$.

Therefore, $At(A_a\varphi) = At(\varphi)$ for all $\varphi \in \mathcal{L}_{ALGP}$ and all $a \in \mathcal{A}$. □

Also, We extend the awareness function to that with agent expressions.

Definition 5.1.3. (**Extension of Awareness Function**)
Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, a finite set of agents $\mathcal{G}$ and the set of agent expression $\mathcal{A}$ of $\mathcal{G}$. Then, $\mathscr{A} : \mathcal{A} \longrightarrow 2^P$ is defined inductively:

1. for all $a_1, a_2 \in \mathcal{A}$,
   if $a = (a_1 + a_2)$, then $\mathscr{A}(a) = \mathscr{A}(a_1) \cup \mathscr{A}(a_2)$;

2. for all $a_1, a_2 \in \mathcal{A}$,
   if $a = (a_1 \cdot a_2)$, then $\mathscr{A}(a) = \mathscr{A}(a_1) \cap \mathscr{A}(a_2)$;

Theorem 5.1.4. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. For all $\varphi \in \mathcal{L}_{ALGP}$ and all $a \in \mathcal{A}$,

$$(M, w) \models_{ALGP} A_a\varphi \iff At(\varphi) \subseteq \mathscr{A}(a).$$

Proof. By the induction of the structure of an agent expression. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ and an $w \in W$. Take an arbitrary $\varphi \in \mathcal{L}_{ALGP}$ and two arbitrary agent expressions $a, b \in \mathcal{A}$.

- (**Base Case**): By the semantics, $(M, w) \models_{ALGP} A_i\varphi$ is equivalent to $At(\varphi) \subseteq \mathscr{A}(i)$ for all $i \in \mathcal{G}$.
- (**Induction Hypothesis**): $(M, w) \models_{ALGP} A_a\varphi$ is equivalent to $At(\varphi) \subseteq \mathscr{A}(a)$ and $(M, w) \models_{ALGP} A_b\varphi$ is equivalent to $At(\varphi) \subseteq \mathscr{A}(b)$ for all $\varphi \in \mathcal{L}_{ALGP}$.
- (**Induction Step**):
  - (**The case for** $A_{a+b}\varphi$):
    Since $A_{a+b}\varphi := \bigwedge_{p \in At(\varphi)}(A_a p \vee A_b p)$, $(M, w) \models_{ALGP} A_{a+b}\varphi$ is equivalent to $(M, w) \models_{ALGP} \bigwedge_{p \in At(\varphi)}(A_a p \vee A_b p)$.
    By the semantics, this is equivalent to $(M, w) \models_{ALGP} A_a p$ or $(M, w) \models_{ALGP} A_b p$ for all $p \in At(\varphi)$. By the induction hypothesis, this is equivalent to $At(p) \subseteq \mathscr{A}(a)$ or $At(p) \subseteq \mathscr{A}(b)$ for all $p \in At(\varphi)$. Since $At(p) \subseteq \mathscr{A}(a)$ or $At(p) \subseteq \mathscr{A}(b)$ for all $p \in At(\varphi)$ is equivalent to $At(\varphi) \subseteq \mathscr{A}(a) \cup \mathscr{A}(b)$, $(M, w) \models_{ALGP} A_{a+b}\varphi$ is equivalent to $At(\varphi) \subseteq \mathscr{A}(a + b)$ by the lemma 5.1.2.
  - (**The case for** $A_{a \cdot b}\varphi$):
    Since $A_{a \cdot b}\varphi := A_a\varphi \wedge A_b\varphi$, $(M, w) \models_{ALGP} A_{a \cdot b}\varphi$ is equivalent to $(M, w) \models_{ALGP} A_a\varphi$ and $(M, w) \models_{ALGP} A_b\varphi$. By the induction hypothesis, $(M, w) \models_{ALGP} A_a\varphi$ and $(M, w) \models_{ALGP} A_b\varphi$ is equivalent to $At(\varphi) \subseteq \mathscr{A}(a)$ and $At(\varphi) \subseteq \mathscr{A}(b)$. Since $At(\varphi) \subseteq \mathscr{A}(a)$ and $At(\varphi) \subseteq \mathscr{A}(b)$ is equivalent to $At(\varphi) \subseteq \mathscr{A}(a) \cap \mathscr{A}(b)$, $(M, w) \models_{ALGP} A_{a \cdot b}\varphi$ is equivalent to $At(\varphi) \subseteq \mathscr{A}(a \cdot b)$ by the lemma 5.1.2.

Therefore, $(M, w) \models_{ALGP} A_a\varphi \iff At(\varphi) \subseteq \mathscr{A}(a)$ for all $\varphi \in \mathcal{L}_{ALGP}$ and all $a \in \mathcal{A}$. ∎

5.2 Language and Semantics

Definition 5.2.1. (**Language $\mathcal{L}_{ALF}$**)
Let P and $\mathcal{G}$ be finite sets of atomic formulas and agents, respectively. Suppose $e \in \mathcal{G}$. We assume they are mutually disjoint. Let $\mathcal{A}$ be the set of agent expressions of $\mathcal{G}$. The language $\mathcal{L}_{ALF}$ is defined by the following:

$$\mathcal{L}_{ALF} \ni \varphi \ ::= \ \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i\varphi \mid A_i\varphi \mid [\approx]_a\varphi$$

where $p \in P$, $i \in \mathcal{G}$, and $a \in \mathcal{A}$.

Clearly, $\mathcal{L}_{ALGP} \subset \mathcal{L}_{ALF}$ if P and $\mathcal{G}$ is fixed.

In addition, we introduce abbreviations for the falsity $\bot$, the disjunction $\vee$, the implication $\rightarrow$, the logical equivalence $\leftrightarrow$, the dual operator $\Diamond_i$ of $\Box_i$ for each $i \in \mathcal{G}$, the explicit knowledge operator K_i for each $i \in \mathcal{G}$, the distributed awareness operator A_{a+b}, the common awareness operator $A_{a \cdot b}$, and the dual operator $\langle\approx\rangle_a$ of $[\approx]_a$ for each $a \in \mathcal{A}$ as follows:

- $\bot \ := \ \neg\top$;
- $\varphi \vee \psi \ := \ \neg(\neg\varphi \wedge \neg\psi)$;
- $\varphi \rightarrow \psi \ := \ \neg(\varphi \wedge \neg\psi)$;
- $\varphi \leftrightarrow \psi \ := \ (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$;
- $\Diamond_i\varphi \ := \ \neg\Box_i\neg\varphi$;
- $K_i\varphi \ := \ A_i\varphi \wedge \Box_i\varphi$;
- $A_{a+b}\varphi \ := \ \bigwedge_{p \in At(\varphi)}(A_a p \vee A_b p)$;
- $A_{a \cdot b}\varphi \ := \ A_a\varphi \wedge A_b\varphi$;
- $\langle\approx\rangle_a\varphi \ := \ \neg[\approx]_a\neg\varphi$;

$[\approx]_a$ is called **the implicit abstraction modality** of a. The following are the informal senses of $\Box_i$, $[\approx]_a$, and K_i.

- $\Box_i\varphi$ stands for "*agent i knows φ holds, implicitly*".
- $[\approx]_a\varphi$ stands for "*φ holds after the abstraction tailored to a's awareness*".
- $K_i\varphi$ stands for "*agent i knows φ holds, explicitly*".

An abstracted model called a quotient model is specified for each agent expression when an epistemic awareness model is given. We do not need to define a model regarding dynamic operators like an action model before the definition of the language. Next, we introduce the semantics of ALF.

Definition 5.2.2. $At : \mathcal{L}_{ALF} \longrightarrow 2^P$ is defined inductively in the followings:

1. $At(\top) = \varnothing$;
2. for all $p \in P$, $At(p) = \{p\}$;
3. for all $\varphi \in \mathcal{L}_{ALF}$, $At(\neg\varphi) = At(\varphi)$;
4. for all $\varphi, \psi \in \mathcal{L}_{ALF}$, $At(\varphi \wedge \psi) = At(\varphi) \cup At(\psi)$;
5. for all $\varphi \in \mathcal{L}_{ALF}$, $At(\Box_i\varphi) = At(\varphi)$;
6. for all $\varphi \in \mathcal{L}_{ALF}$, $At(A_a\varphi) = At(\varphi)$;
7. for all $\varphi \in \mathcal{L}_{ALF}$, $At([\approx]_a\varphi) = At(\varphi)$.

Next, we define a filtration-based equivalence relation on W by modal equivalence. Let $\mathcal{L}_{ALGP}|Q$ mean the language $\mathcal{L}_{ALGP}$ with the set of atomic propositions restricted to $Q \subseteq P$.

Definition 5.2.3. (**a-Equivalent**)[1]
Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. The agent expression a induces an equivalence relation $\approx_a$ on W: for all $w, v \in W$,

$$w \approx_a v \text{ iff } \text{ for all } \varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a) \ ((M, w) \models_{ALGP} \varphi \text{ iff } (M, v) \models_{ALGP} \varphi).$$

In other words, two worlds are a-equivalent if and only if they satisfy the same formulas $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a)$.

Definition 5.2.4. (**Satisfaction Relation of $\mathcal{L}_{ALF}$**)
Given epistemic state (M, w) with $M = \langle W, \sim, \mathscr{A}, V\rangle$ and $w \in W$.

$$
\begin{array}{lll}
(M,w) \models_{ALF} \top \ ; & & \\
(M,w) \models_{ALF} p & \iff & w \in V(p); \\
(M,w) \models_{ALF} \neg\varphi & \iff & (M,w) \not\models_{ALF} \varphi; \\
(M,w) \models_{ALF} \varphi \wedge \psi & \iff & (M,w) \models_{ALF} \varphi \text{ and } (M,w) \models_{ALF} \psi; \\
(M,w) \models_{ALF} A_i\varphi & \iff & At(\varphi) \subseteq \mathscr{A}(i); \\
(M,w) \models_{ALF} \Box_i\varphi & \iff & \text{for all } v \in W, w \sim_i v \text{ implies } (M,v) \models_{ALF} \varphi; \\
(M,w) \models_{ALF} [\approx]_a\varphi & \iff & \text{for all } v \in W, w \approx_a v \text{ implies } (M,v) \models_{ALF} \varphi;
\end{array}
$$

[1]This term is given according to [1].

Then, $\models_{ALF}$ is called a **satisfaction relation** of $\mathcal{L}_{ALF}$.

A set of formulas $\Gamma \subseteq \mathcal{L}_{ALF}$ is **satisfiable** if there is some model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and a possible world $w \in W$ such that $(M, w) \models \varphi$ for all $\varphi \in \Gamma$. A formula $\varphi \in \mathcal{L}_{ALF}$ is satisfiable when $\{\varphi\}$ is satisfiable.

By the definition, the satisfaction relation for $\bot, \vee, \rightarrow, \Diamond_i, K_i, A_a, \langle\approx\rangle_a$ are derived as follows: Given epistemic state (M, w) with $M = \langle W, \sim, \mathscr{A}, V\rangle$ and $w \in W$.

$(M, w) \not\models_{ALF} \bot$;
$(M, w) \models_{ALF} \varphi \vee \psi \iff (M, w) \models_{ALF} \varphi$ or $(M, w) \models_{ALF} \psi$;
$(M, w) \models_{ALF} \varphi \rightarrow \psi \iff (M, w) \models_{ALF} \varphi$ implies $(M, w) \models_{ALF} \psi$;
$(M, w) \models_{ALF} \Diamond_i\varphi \iff$ for some $v \in W, w \sim_i v$ and $(M, v) \models_{ALF} \varphi$;
$(M, w) \models_{ALF} K_i\varphi \iff (M, w) \models_{ALF} \Box_i\varphi$ and $(M, w) \models_{ALF} A_i\varphi$;
$(M, w) \models_{ALF} A_a\varphi \iff At(\varphi) \subseteq \mathscr{A}(a)$;
$(M, w) \models_{ALF} \langle\approx\rangle_a\varphi \iff$ for some $v \in W, w \approx_a v$ and $(M, v) \models_{ALF} \varphi$;

5.3 Implicit Abstraction as Filtration

Lemma 5.3.1.
Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. For all $a, b \in \mathcal{A}$ such that $\mathscr{A}(a) \subseteq \mathscr{A}(b)$. and all $w, v \in W$,

$$w \approx_b v \;\Rightarrow\; w \approx_a v.$$

$$w \approx_a v \;\text{ or }\; w \approx_b v \quad\Leftrightarrow\quad w \approx_{a \cdot b} v.$$

$$w \approx_{a+b} v \;\Leftrightarrow\; w \approx_a v \;\text{ and }\; w \approx_b v.$$

Lemma 5.3.2.
Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. For all $a, b \in \mathcal{A}$ and all $w, v \in W$,

$$w \approx_a v \;\text{ or }\; w \approx_b v \quad\Rightarrow\quad w \approx_{a \cdot b} v.$$

Example 5.3.3. There is an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ such that for some $w, v \in W$ and some $a, b \in \mathcal{A}$,

$$w \approx_{a \cdot b} v \not\Rightarrow\; w \approx_a v \;\text{ or }\; w \approx_b v.$$

Let $\mathcal{G} = \{e, i, j\}$ and $P = \{p, q\}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ consisting of:

1. $W = \{w, v\}$;
2. $\sim_i = \{(u, u) \mid u \in W\}$ for all $i \in \mathcal{G} \backslash \{e\}$ and $\sim_e = \{(u, x) \mid u, x \in W\}$.
3. $\mathscr{A}(i) = \{p\}$ and $\mathscr{A}(j) = \{q\}$.
4. $V(p) = \{w\}$ and $V(q) = \{v\}$.

Then, $w \not\approx_i v$, $w \not\approx_j v$, and $w \approx_{i \cdot j} v$ because $\mathscr{A}(i \cdot j) = \varnothing$ (See Figure 5.1).

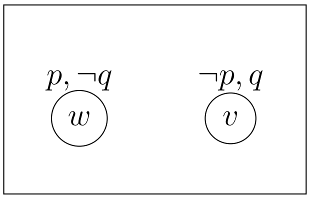

Figure 5.1: $w \approx_{a \cdot b} v \not\Rightarrow w \approx_a v$ or $w \approx_b v$

Lemma 5.3.4.
Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. For all $a, b \in \mathcal{A}$ and all $w, v \in W$,

$$w \approx_{a+b} v \;\Rightarrow\; w \approx_a v \;\text{ and }\; w \approx_b v.$$

Example 5.3.5. There is an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ such that for some $w, v \in W$ and some $a, b \in \mathcal{A}$,

$$w \approx_a v \;\text{ and }\; w \approx_b v \not\Rightarrow\; w \approx_{a+b} v.$$

Let $\mathcal{G} = \{e, i, j\}$ and $P = \{p, q\}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ consisting of:

1. $W = \{w, v, x, y, z\}$;
2. $\sim_i = \{(u, u') \mid u, u' \in \{w, v, x\}\} \cup \{(u, u') \mid u, u' \in \{y, z\}\}$,
   $\sim_j = \{(u, u) \mid u \in W\}$, and $\sim_e = \{(u, u') \mid u, u' \in W\}$.
3. $\mathscr{A}(i) = \{p\}$ and $\mathscr{A}(j) = \{q\}$.
4. $V(p) = \{v, z\}$ and $V(q) = \{w, v, y\}$.

Then, $w \approx_i y$, $w \approx_j y$, and $w \not\approx_{i+j} y$ because $(M, w) \models \Diamond_i(p \wedge q)$ and $(M, y) \models \neg\Diamond_i(p \wedge q)$ (See Figure 5.2).

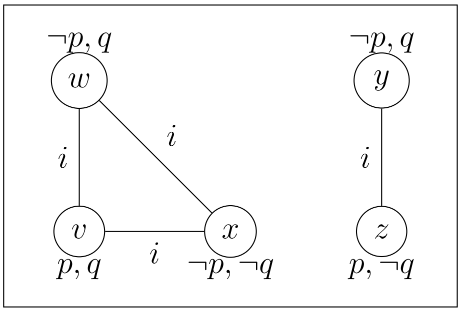

Figure 5.2: $w \approx_a v$ and $w \approx_b v \not\Rightarrow w \approx_{a+b} v$

Proposition 5.3.6.
Given any epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and any $w \in W$. For all $a, b \in \mathcal{A}$ such that $\mathscr{A}(a) \subseteq \mathscr{A}(b)$,

$$\models \langle\approx\rangle_b\varphi \to \langle\approx\rangle_a\varphi$$

$$(M, w) \models \langle\approx\rangle_a\langle\approx\rangle_b\varphi \;\Leftrightarrow\; (M, w) \models \langle\approx\rangle_a\varphi.$$

$$(M, w) \models \langle\approx\rangle_a\langle\approx\rangle_b\varphi \;\Leftrightarrow\; (M, w) \models \langle\approx\rangle_b\langle\approx\rangle_a\varphi \;\Leftrightarrow\; (M, w) \models \langle\approx\rangle_{a\cdot b}\varphi$$

Proposition 5.3.7.
For all $a, b \in \mathcal{A}$ and all $\varphi \in \mathcal{L}_{ALF}$,

$$\models \langle\approx\rangle_{a+b}\varphi \to \langle\approx\rangle_a\varphi \wedge \langle\approx\rangle_b\varphi$$

$$\models \langle\approx\rangle_a\varphi \vee \langle\approx\rangle_b\varphi \to \langle\approx\rangle_{a\cdot b}\varphi$$

Proposition 5.3.8.
For all $a, b \in \mathcal{A}$ and all $\varphi \in \mathcal{L}_{ALF}$,

$$\models [\approx]_{a\cdot b}\varphi \to [\approx]_a[\approx]_b\varphi.$$

Example 5.3.9.
There is an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ such that for some $w \in W$, some $\varphi \in \mathcal{L}_{ALF}$ and some $a, b \in \mathcal{A}$,

$$(M, w) \not\models \langle\approx\rangle_a\langle\approx\rangle_b\varphi \to \langle\approx\rangle_b\langle\approx\rangle_a\varphi.$$

Let $\mathcal{G} = \{e, i, j\}$ and $P = \{p, q\}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ consisting of:

1. $W = \{w, v, x\}$;

2. $\sim_i = \sim_j = \{(u, u) \mid u \in W\}$ and $\sim_e = \{(u, u') \mid u, u' \in W\}$.

3. $\mathscr{A}(i) = \{p\}$ and $\mathscr{A}(j) = \{q\}$.

4. $V(p) = \{v, z\}$ and $V(q) = \{w, v, y\}$.

Then, $(M, w) \models \langle\approx\rangle_i \langle\approx\rangle_j (\neg p \wedge \neg q)$ because $w \approx_i v$, $v \approx_j x$, and $(M, x) \models \neg p \wedge \neg q$. But, $(M, w) \not\models \langle\approx\rangle_j \langle\approx\rangle_i (\neg p \wedge \neg q)$ because there is no $u \in W$ such that $w \approx_j u$ and $u \approx_i x$. Therefore, $(M, w) \not\models \langle\approx\rangle_i \langle\approx\rangle_j (\neg p \wedge \neg q) \rightarrow \langle\approx\rangle_j \langle\approx\rangle_i (\neg p \wedge \neg q)$ (See Figure 5.3).

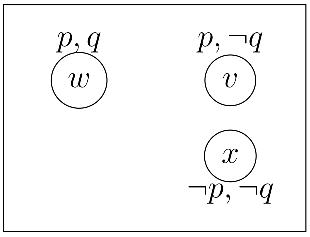

Figure 5.3: $(M, w) \not\models \langle\approx\rangle_a \langle\approx\rangle_b \varphi \rightarrow \langle\approx\rangle_b \langle\approx\rangle_a \varphi$

Chapter 6

Comparison among Abstractions

Chapter 5 introduced a logical equivalence relation called a-equivalent for the semantics of abstraction. However, we can consider other semantics of abstraction by atoms-based equivalence relations or observational equivalence relations. This chapter compares a logical equivalence relation to an atoms-based equivalence relation and a observational equivalence relation.

6.1 Awareness Logic with Partition

First, a filtration-based equivalence relation of ALF is compared to an atoms-based equivalence relation of Awareness Logic with Partition(ALP)[6] called "*indistinguishable relation*". ALP could not be intended for Awareness Logic of Abstraction, but ALF is similar to ALP.

ALP is introduced for the formalization of the knowledge of others in thought. In ALP, an agent's awareness is different for each agent's thought. For example, i's awareness in j_1's thought are i's awareness in j_2's thought are different. However, it is not possible to say which one is the actual i's awareness. An agent's awareness in the other agent's thoughts is similar to an agent's awareness believed by the other agent.

Definition 6.1.1. (**Language** $\mathcal{L}_{ALP}$ [6])
Let P be a countable set of atomic propositions and $\mathcal{G}$ be a finite set of agents. We assume they are mutually disjoint. The language $\mathcal{L}_{ALP}$ is defined by the following:

$$\mathcal{L}_{ALP} \ni \varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid A^i_j\varphi \mid \Box_i\varphi \mid [\approx]^i_j\varphi \mid C^i_j\varphi$$

where $p \in P$ and $i, j \in \mathcal{G}$.

In addition, we introduce abbreviations for the truth $\top$, the falsity $\bot$, the disjunction $\vee$, the implication $\rightarrow$, the logical equivalence $\leftrightarrow$, the dual operator $\Diamond_i$ of $\Box_i$ for each $i \in \mathcal{G}$, the explicit knowledge operator K^i_j for each $i \in \mathcal{G}$, and the dual operator $\langle\approx\rangle^i_j$ of $[\approx]^i_j$ for each $i, j \in \mathcal{G}$ as follows:

- $\top := \varphi \rightarrow \varphi$;
- $\bot := \neg\top$;
- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$;
- $\varphi \rightarrow \psi := \neg(\varphi \wedge \neg\psi)$;
- $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$;
- $\Diamond_i\varphi := \neg\Box_i\neg\varphi$;
- $K^i_j\varphi := A^i_j\varphi \wedge C^i_j\varphi$;
- $\langle\approx\rangle^i_j\varphi := \neg[\approx]^i_j\neg\varphi$;

The following are the informal senses of $\Box_i$, A^i_j, $[\approx]^i_j$ C^i_j ,and K^i_j.[6]:

- $\Box_i\varphi$ stands for "*φ is j's implicit knowledge*".
- $A^i_j\varphi$ stands for "*φ is information that j is aware of from i's viewpoint*"
- $[\approx]^i_j\varphi$ stands for "*φ is information that is true at j's state of awareness from i's viewpoint*".
- $C^i_j\varphi$ stands for "*φ is a kind of j's implicit knowledge from agent i's viewpoint*".
- $K^i_j\varphi$ stands for "*φ is j's explicit knowledge from agent i's viewpoint*".

Definition 6.1.2. (**Epistemic Model with Awareness**[6])
Let P be a countable set of atomic propositions and $\mathcal{G}$ be a finite set of agents. We assume they are mutually disjoint. An epistemic model with awareness M is a tuple $\langle W, \{\sim_i\}_{i\in\mathcal{G}}, \{\mathscr{A}^i_j\}_{i,j\in\mathcal{G}}, V, \{\approx^i_j\}_{i,j\in\mathcal{G}}\rangle$ consisting of:

1. W is a non-empty finite set of possible worlds;
2. $\sim_i \subseteq W \times W$ is an equivalence relation on W for each $i \in \mathcal{G}$;
3. $\mathscr{A}^i_j$ is a non-empty set of atomic propositions satisfying $\mathscr{A}^i_j \subseteq \mathscr{A}^i_i$ for each $i, j \in \mathcal{G}$;

4. $V : P \longrightarrow 2^W$;

5. $w \approx^i_j v$ iff ($w \in V(p)$ iff $v \in V(p)$ for all $p \in \mathscr{A}^i_j$).

Then, $\sim_i, \mathscr{A}^i_j, V$, and $\approx^i_j$ are called **accessibility relation**, **awareness set**, **valuation function**, and **indistinguishable relation** of M, respectively.

Definition 6.1.3. $At : \mathcal{L}_{ALP} \longrightarrow 2^P$ is defined inductively in the followings [6]:

1. $At(\top) = \varnothing$;
2. for all $p \in P$, $At(p) = \{p\}$;
3. for all $\varphi \in \mathcal{L}_{ALP}$, $At(\neg\varphi) = At(\varphi)$;
4. for all $\varphi, \psi \in \mathcal{L}_{ALP}$, $At(\varphi \wedge \psi) = At(\varphi) \cup At(\psi)$;
5. for all $\varphi \in \mathcal{L}_{ALP}$, $At(\Box_i\varphi) = At(\varphi)$;
6. for all $\varphi \in \mathcal{L}_{ALP}$, $At(A^i_j\varphi) = At(\varphi)$;
7. for all $\varphi \in \mathcal{L}_{ALP}$, $At([\approx]^i_j\varphi) = At(\varphi)$;
8. for all $\varphi \in \mathcal{L}_{ALP}$, $At(C^i_j\varphi) = At(\varphi)$;

Let R be a relation on W. Then, R^+ represents the transitive closure of R, i.e. R^+ is the smallest set such that $R \subseteq R^+$ and for all $x, y, z \in W$, if $(x, y) \in R^+$ and $(y, z) \in R^+$, then $(x, z) \in R^+$.

Definition 6.1.4. (**Satisfaction Relation of $\mathcal{L}_{ALP}$** [6])
Given (M, w) with $M = \langle W, \{\sim_i\}_{i\in\mathcal{G}}, \{\mathscr{A}^i_j\}_{i,j\in\mathcal{G}}, V, \{\approx^i_j\}_{i,j\in\mathcal{G}}\rangle$ and $w \in W$.

$(M, w) \models p \iff w \in V(p)$;
$(M, w) \models \neg\varphi \iff (M, w) \not\models \varphi$;
$(M, w) \models \varphi \wedge \psi \iff (M, w) \models \varphi$ and $(M, w) \models \psi$;
$(M, w) \models A^i_j\varphi \iff At(\varphi) \subseteq \mathscr{A}^i_j$;
$(M, w) \models \Box_i\varphi \iff$ for all $v \in W, w \sim_i v$ implies $(M, v) \models \varphi$;
$(M, w) \models [\approx]^i_j\varphi \iff$ for all $v \in W, w \approx^i_j v$ implies $(M, v) \models \varphi$;
$(M, w) \models C^i_j\varphi \iff$ for all $v \in W, (w, v) \in (\sim_j \circ \approx^i_j)^+$ implies $(M, v) \models \varphi$;

Then, $\models$ is called a **satisfaction relation** of $\mathcal{L}_{ALP}$.

A set of formulas $\Gamma \subseteq \mathcal{L}_{ALP}$ is **satisfiable** if there is some model $M = \langle W, \{\sim_i\}_{i\in\mathcal{G}}, \{\mathscr{A}_j^i\}_{i,j\in\mathcal{G}}, V, \{\approx_j^i\}_{i,j\in\mathcal{G}}\rangle$ and a possible world $w \in W$ such that $(M, w) \models \varphi$ for all $\varphi \in \Gamma$. A formula $\varphi \in \mathcal{L}_{ALP}$ is satisfiable when $\{\varphi\}$ is satisfiable.

By the definition, the satisfaction relation for $\top, \bot, \vee, \rightarrow, \Diamond_i, \langle\approx\rangle_j^i$, and K_j^i are derived as follows: Given (M, w) with $M = \langle W, \{\sim_i\}_{i\in\mathcal{G}}, \{\mathscr{A}_j^i\}_{i,j\in\mathcal{G}}, V, \{\approx_j^i\}_{i,j\in\mathcal{G}}\rangle$ and $w \in W$.

$(M, w) \models \top$;

$(M, w) \not\models \bot$;

$(M, w) \models \varphi \vee \psi \iff (M, w) \models \varphi$ or $(M, w) \models \psi$;

$(M, w) \models \varphi \rightarrow \psi \iff (M, w) \models \varphi$ implies $(M, w) \models \psi$;

$(M, w) \models \Diamond_i\varphi \iff$ for some $v \in W, w \sim_i v$ and $(M, v) \models \varphi$;

$(M, w) \models \langle\approx\rangle_j^i\varphi \iff$ for some $v \in W, w \approx_j^i v$ and $(M, v) \models \varphi$;

$(M, w) \models K_j^i\varphi \iff (M, w) \models A_j^i\varphi$ and $(M, w) \models C_j^i\varphi$;

Example 6.1.5. Given finite sets of atomic propositions $P = \{p, q\}$ and agents $\mathcal{G} = \{i, e\}$, respectively. Assume they are mutually disjoint. Given an epistemic awareness model of ALF $M = \langle W, \sim, \mathscr{A}, V\rangle$ consisting of (See Figure 6.1. Shapes drawn with dashed lines represent equivalence classes.):

- $W = \{w, w', v, v'\}$;
- $\sim_i = \{(x, y) \mid x, y \in \{w, w'\}\} \cup \{(x, y) \mid x, y \in \{v, v'\}\}$ and $\sim_e = W \times W$;
- $\mathscr{A}(i) = \{p\}$ and $\mathscr{A}(e) = P$;
- $V(p) = \{w, w', v\}$ and $V(q) = \{w, v'\}$;

Then, $\approx_i = \{(w, w'), (w', w)\} \cup \{(x, x) \mid x \in W\}$ and $\approx_e = \{(x, x) \mid x \in W\}$.

Given an epistemic model with awareness of ALP
$M' = \langle W', \{\sim_i'\}_{i\in\mathcal{G}}, \{\mathscr{A}_j^i\}_{i,j\in\mathcal{G}}, V', \{\approx_j^i\}_{i,j\in\mathcal{G}}\rangle$ consisting of (See Figure 6.2. Shapes drawn with dashed lines represent equivalence classes.):

1. $W' = \{w, w', v, v'\}$;
2. $\sim_i' = \{(x, y) \mid x, y \in \{w, w'\}\} \cup \{(x, y) \mid x, y \in \{v, v'\}\}$ and $\sim_e' = W \times W$;
3. $\mathscr{A}_i^i = \mathscr{A}_e^i = \mathscr{A}_i^e = \{p\}$ and $\mathscr{A}_e^e = P$;

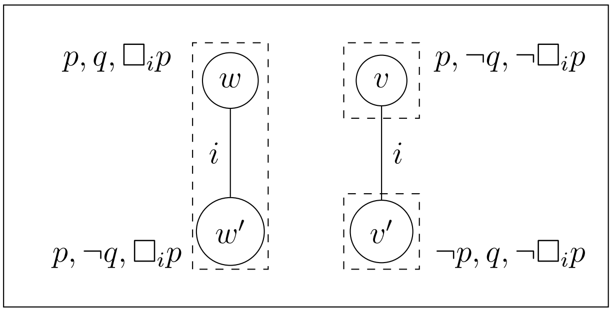

Figure 6.1: Equivalence Class in ALF

4. $V(p) = \{w, w', v\}$ and $V(q) = \{w, v'\}$;
5. $\approx^i_i = \approx^i_e = \approx^e_i = \{(x, y) \mid x, y \in \{w, w', v\}\} \cup \{(v', v')\}$ and $\approx^e_e = \{(x, x) \mid x \in W\}$.

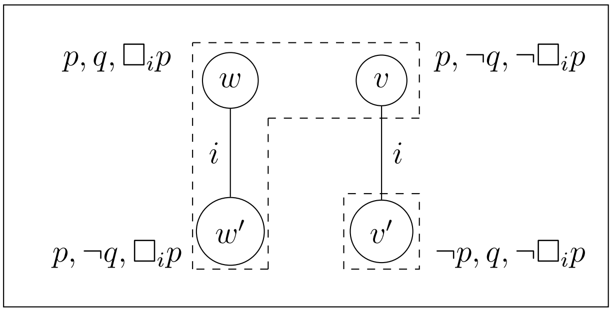

Figure 6.2: Equivalence Class in ALP

In ALP, $p \wedge A^i_j p \to [\approx]^i_j p$ and $\neg p \wedge A^i_j \neg p \to [\approx]^i_j \neg p$ is valid for all $p \in P$, but $\Box_j p \wedge A^i_j \Box_j p \to [\approx]^i_j \Box_j p$ is not valid. In ALF, $\varphi \wedge A_i \varphi \to [\approx]_i \varphi$ is valid for all $\varphi \in \mathcal{L}_{ALGP}$.

Now, we redefine a-equivalent as an atoms-based equivalence relation like an indistinguishable relation of ALP.

Definition 6.1.6. (**Atoms-based a-Equivalent**)
Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. The agent expression $a \in \mathcal{A}$ induces an equivalence relation $\approx_a$ on W: for all $w, v \in W$ and all $p \in \mathscr{A}(a)$,

$$w \approx_a v \text{ iff } (w \in V(p) \text{ iff } v \in V(p)).$$

In other words, two worlds are atoms-based a-equivalent if and only if they satisfy the same formulas p if $p \in \mathscr{A}(a)$.

We consider atoms-based a-equivalent in this section, only. Recall that P is a finite set of atomic propositions. Then, $\Sigma^a = \bigcup_{P' \subseteq P}\{\bigwedge A_a P' \wedge \bigwedge \neg A_a(P \backslash P') \wedge \bigwedge Q \wedge \bigwedge \neg(P' \backslash Q) \mid Q \subseteq P'\}$ is finite and $\bigvee_{\psi \in \Sigma^a}(\psi \wedge \Diamond_e(\psi \wedge \varphi))$ is a formula.

Theorem 6.1.7. Fix $a \in \mathcal{A}$. Let $\approx_a$ be atoms-based a-equivalent. Then, for all $\varphi \in \mathcal{L}_{ALF}$,

$$\models \langle \approx \rangle_a \varphi \leftrightarrow \bigvee_{\psi \in \Sigma^a} (\psi \wedge \Diamond_e(\psi \wedge \varphi)).$$

where $\Sigma^a = \bigcup_{P' \subseteq P}\{\bigwedge A_a P' \wedge \bigwedge \neg A_a(P \backslash P') \wedge \bigwedge Q \wedge \bigwedge \neg(P' \backslash Q) \mid Q \subseteq P'\}$.

Proof. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ for ALGP and $w \in W$. Take an arbitrary $\varphi \in \mathcal{L}_{ALF}$. By the semantics, $(M, w) \models_{ALF} \bigwedge A_a \mathscr{A}(a) \wedge \bigwedge \neg A_a(P \backslash \mathscr{A}(a)) \wedge \bigwedge Q \wedge \bigwedge \neg(\mathscr{A}(a) \backslash Q)$ where $Q = \{p \in \mathscr{A}(a) \mid w \in V(p)\}$. Let $\psi \equiv \bigwedge A_a \mathscr{A}(a) \wedge \bigwedge \neg A_a(P \backslash \mathscr{A}(a)) \wedge \bigwedge Q \wedge \bigwedge \neg(\mathscr{A}(a) \backslash Q)$. Then, $\psi \in \Sigma^a$.

- ($\models_{ALF} \langle \approx \rangle_a \varphi \rightarrow \bigvee_{\psi \in \Sigma^a}(\psi \wedge \Diamond_e(\psi \wedge \varphi)).$):

  Suppose $(M, w) \models_{ALF} \langle \approx \rangle_a \varphi$. By the semantics, $(M, v) \models_{ALF} \varphi$ for some $v \in W$ such that $w \approx_a v$. By the definition of $\approx_a$, $(M, v) \models_{ALF} \psi$. Then, $(M, w) \models_{ALF} \Diamond_e(\psi \wedge \varphi)$.

  By $\psi \in \Sigma^a$, $(M, w) \models_{ALF} \bigvee_{\psi \in \Sigma^a}(\psi \wedge \Diamond_e(\psi \wedge \varphi))$.

- ($\models_{ALF} \bigvee_{\psi \in \Sigma^a}(\psi \wedge \Diamond_e(\psi \wedge \varphi)) \rightarrow \langle \approx \rangle_a \varphi.$):

  By the contraposition. Suppose $(M, w) \not\models_{ALF} \langle \approx \rangle_a \varphi$. By the semantics, $(M, v) \not\models_{ALF} \varphi$ for all $v \in W$ such that $w \approx_a v$.

  Take an arbitrary $\psi' \in \Sigma^a$ such that $\psi \not\equiv \psi'$. If there is a $p \in P$ such that $\models_{ALF} \psi \rightarrow A_a p \not\Leftrightarrow \models_{ALF} \psi' \rightarrow A_a p$, then $(M, w) \not\models_{ALF} \psi'$ by the global definition of awareness. Otherwise, $\models_{ALF} \psi \rightarrow p \not\Leftrightarrow \psi' \rightarrow p$ for some $p \in \mathscr{A}(a)$. Then, $\models_{ALF} \psi \leftrightarrow \neg \bigvee \Sigma^a \backslash \{\psi\}$.

  Since $\models_{ALF} \psi \leftrightarrow \neg \bigvee \Sigma^a \backslash \{\psi\}$, there is no $u \in W$ such that $(M, u) \models_{ALF} \psi$ and $w \not\approx_a u$. Since $(M, w) \models_{ALF} \psi$ and $(M, v) \not\models_{ALF} \varphi$ for all $v \in W$ such that $w \approx_a v$, $(M, w) \not\models_{ALF} \Diamond_e(\varphi \wedge \psi)$.

  Therefore, $(M, w) \not\models_{ALF} \bigvee_{\psi \in \Sigma^a}(\psi \wedge \Diamond_e(\psi \wedge \varphi))$.

Thus, the theorem is valid. □

By atoms-based equivalence relations, any formula of ALF can be transformed into some formula of ALGP.

6.2 Filtration and Bisimulation

This thesis investigates the abstraction of filtration to deal directly with modal equivalence. Although filtration is given by the logical equivalence of some formulas, it is difficult to show the logical equivalence. Generally, we need to explore the whole model to show an equivalence class concerning $\approx_a$. On the other hand, bisimulation is a method that shows logical equivalence by exploring only the worlds that are reachable from a certain possible world. In computer science, bisimulation shows equivalent behavior between two labeled transition systems. When an automaton evaluates logical equivalence, it can only work at the current and neighbor states. Bisimulation shows logical equivalence while exploring like an automaton that evaluates the truth of formulas. Although bisimulation does not correspond unconditionally to filtration, it is a natural concept in computer science.

Definition 6.2.1. (a-**Restricted Bisimulation**)
Let P, Q and $\mathcal{G}$ be a finite set of atomic propositions, a subset of P, and agents, respectively. Let $\mathcal{A}$ be the set of agent expressions of $\mathcal{G}$.

Fix $a \in \mathcal{A}$. Given two awareness epistemic models $M = \langle W, \sim, \mathscr{A}, V\rangle$ and $M' = \langle W', \sim', \mathscr{A}', V'\rangle$. A a-**restricted bisimulation** between M and M' is a relation $\mathfrak{R}_a \subseteq W \times W$ such that for every $(w, w') \in \mathfrak{R}_a$ and for every agent $i \in \mathcal{G}$:

- **atoms:** $w \in V(p)$ iff $w' \in V(p)$ for every $p \in \mathscr{A}(a) \cap \mathscr{A}'(a)$;
- **aware:** $\mathscr{A}(a) = \mathscr{A}'(a)$ and $\mathscr{A}(i) \cap \mathscr{A}(a) = \mathscr{A}'(i) \cap \mathscr{A}(a)$;
- **forth:** if $v \in W$ and $w \sim_i v$, then there is a $v' \in W'$ such that $w' \sim'_i v'$ and $(v, v') \in \mathfrak{R}_a$;
- **back:** if $v' \in W'$ and $w' \sim'_i v'$, then there is a $v \in W$ such that $w \sim_i v$ and $(v, v') \in \mathfrak{R}_a$.

We say that (M, w) and (M', w') are a-restricted bisimilar, notation $(M, w) \simeq_a (M', w')$, if there is a a-restricted bisimulation between M and M' that contains (w, w').

Definition 6.2.2. (**Image Finite**)
Given an epistemic awareness model M $\langle W, \sim, \mathscr{A}, V\rangle$. M is **image-finite** if $\{v \in W \mid w \sim_i v\}$ is finite for all agents $i \in \mathcal{G}$ and all worlds $w \in W$.

Definition 6.2.3. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ and $w, w' \in W$. Suppose M is image-finite and $w \approx_a w'$. Define the following relation Z_a on W:

$$(w, w') \in Z_a \text{ iff } w \approx_a w'.$$

We prove Z is a bisimulation.

Lemma 6.2.4. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ and two arbitrary worlds $w, w' \in W$. Suppose M is image-finite. Then,

$$(M, w) \simeq_{\mathscr{A}(a)} (M, w') \text{ implies } (w, w') \in Z_a.$$

Proof. Given an arbitrary epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$, two arbitrary worlds $w, w' \in W$ and an agent expression $a \in \mathcal{A}$. It is shown by the induction on the structure of $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a)$.

- **Base Case** : Suppose $(M, w) \simeq_{\mathscr{A}(a)} (M, w')$. By **atoms** of the definition 6.2.1, it must be $(M, w) \models p$ if and only if $(M, w') \models p$ for all $p \in \mathscr{A}(a)$.
- **Induction Hypothesis** : $(M, w) \simeq_{\mathscr{A}(a)} (M, w')$ implies $(M, w) \models \varphi \Leftrightarrow (M, w') \models \varphi$ and $At(\varphi) \subseteq \mathscr{A}(a)$.
- **Induction Step** : Suppose $(M, w) \simeq_{\mathscr{A}(a)} (M, w')$. Take two arbitrary formulas $\varphi, \psi \in \mathcal{L}_{ALGP}$ such that $(M, w) \models \varphi \Leftrightarrow (M, w') \models \varphi$, $(M, w) \models \psi \Leftrightarrow (M, w') \models \psi$, and $At(\varphi), At(\psi) \subseteq \mathscr{A}(a)$.
  - **The case for** $\varphi \wedge \psi$:
    By the induction hypothesis, $(M, w) \models \varphi \Leftrightarrow (M, w') \models \varphi$, $(M, w) \models \psi \Leftrightarrow (M, w') \models \psi$, and $At(\varphi), At(\psi) \subseteq \mathscr{A}(a)$. By the semantics, $(M, w) \models \varphi \wedge \psi \Leftrightarrow (M, w') \models \varphi \wedge \psi$ and $At(\varphi \wedge \psi) \subseteq \mathscr{A}(a)$.
  - **The case for** $\neg\varphi$:
    By the induction hypothesis, $(M, w) \models \varphi \Leftrightarrow (M, w') \models \varphi$ and $At(\varphi) \subseteq \mathscr{A}(a)$. By the semantics, $(M, w) \models \neg\varphi \Leftrightarrow (M, w') \models \neg\varphi$ and $At(\neg\varphi) \subseteq \mathscr{A}(a)$.
  - **The case for** $A_b\varphi$:

By the induction hypothesis, $At(\varphi) \subseteq \mathscr{A}(a)$. By the semantics, $At(A_b\varphi) \subseteq \mathscr{A}(a)$ for all $b \in \mathcal{A}$. By the global definition of awareness, $(M, w) \models A_b\varphi \Leftrightarrow (M, w') \models A_b\varphi$. Then, $(M, w) \models A_b\varphi \Leftrightarrow (M, w') \models A_b\varphi$ and $At(A_b\varphi) \subseteq \mathscr{A}(a)$ for all $b \in \mathcal{A}$.

– **The case for** $\Box_i\varphi$:

Take an arbitrary agent $i \in \mathcal{G}$.

  * **back**: Suppose $(M, w) \models \Box_i\varphi$. Take an arbitrary $v' \in W$ such that $w' \sim_i v'$. By **forth** of the definition 6.2.1, there is a $v \in W$ such that $w \sim_i v$ and $(M, v) \simeq_{\mathscr{A}(a)} (M, v')$. By the induction hypothesis, $(M, v) \models \varphi \Leftrightarrow (M, v') \models \varphi$. Since v' is an arbitrary worlds, $(M, w') \models \Box_i\varphi$.
  * **forth**: The proof of **forth** is similar to that of **back**.

Then, $(M, w) \models \Box_i\varphi \Leftrightarrow (M, w') \models \Box_i\varphi$. Also, $At(\Box_i\varphi) \subseteq \mathscr{A}(a)$ by the semantics and the induction hypothesis.

Therefore, $(M, w) \simeq_{\mathscr{A}(a)} (M, w')$ implies $(M, w) \models \varphi \Leftrightarrow (M, w') \models \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}$ such that $At(\varphi) \subseteq \mathscr{A}(a)$. Thus, $w \approx_a v$. Since M is image-finite, $(w, w') \in Z_a$.

□

Lemma 6.2.5. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and two arbitrary worlds $w, w' \in W$. Suppose M is image-finite. Then,

$$(w, w') \in Z_a \text{ implies } (M, w) \simeq_{\mathscr{A}(a)} (M, w').$$

Proof. Given an arbitrary epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, two arbitrary worlds $w, w' \in W$ and an agent expression $a \in \mathcal{A}$. Suppose that M is image-finite and $(w, w') \in Z_a$. By the definition, $w \approx_a w'$.

- **atoms**:

  By $w \approx_a w'$, $w \in V(p)$ if and only if $w' \in V(p)$ for all atomic propositions $p \in \mathscr{A}(a)$.

- **aware**:

  It holds by the global definition of awareness.

- **forth**:

  By the Hennessy–Milner style contradiction. Take an arbitrary agent $i \in \mathcal{G}$ and an arbitrary $v \in W$ such that $w \sim_i v$. Suppose there is no $v' \in W$ such that $w' \sim_i v'$ and $v \approx_a v'$.

  Let n be a positive integer such that $n = \#\{v' \mid w' \sim_i v'\}$ and $v'_k \in W$ be a possible world such that $w' \sim_i v'_k$ for each positive integer $k \leq n$. Since $v \not\approx_a v'_k$ for all $v'_k \in W$, there is a $\varphi_k \in \mathcal{L}_{ALGP}$ such that $At(\varphi) \subseteq \mathscr{A}(a)$, $(M, v) \models \varphi_k$ and $(M, v'_k) \not\models \varphi_k$ for every positive integer $k \leq n$. Since M is image-finite, $\Diamond_i(\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n)$ is a formula. Then, $(M, w) \models \Diamond_i(\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n)$ and $(M, w') \not\models \Diamond_i(\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n)$. This is a contradiction. Therefore, there is a $v' \in W$ such that $w' \sim_i v'$ and $v \approx_a v'$.

  back:

  The proof of **back** is similar to that of **forth**.

Therefore, there is a $\mathscr{A}(a)$-standard bisimular for ALF between (M, w) and (M, w'), i.e., $(M, w) \simeq_{\mathscr{A}(a)} (M, w')$. □

Theorem 6.2.6. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ and two arbitrary worlds $w, w' \in W$. On image-finite models:

$$(M, w) \simeq_{\mathscr{A}(a)} (M, w') \text{ iff } w \approx_a w'.$$

Proof. Given an arbitrary epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$, two arbitrary worlds $w, w' \in W$ and an arbitrary agent expression $a \in \mathcal{A}$. Suppose that M is image-finite.

This theorem is valid by the lemma 6.2.4 and the lemma 6.2.5. □

Chapter 7

Non-Compactness

This chapter shows an example of non-compactness in the semantics of ALF and investigates a condition of a reduction from ALF to ALGP. The sections regarding non-compactness are the section 7.1 and the section 7.2. The section regarding the reduction from ALF to ALGP is the section 7.3. However, there are conditions for the reduction from ALF to ALGP, and this thesis does not show the completeness theorem of ALF.

7.1 Example of Model with Infinite Equivalence Classes

This example is given in [10] p.32. $\varphi \equiv \psi$ means that φ is a formula that has the same form as ψ. $(K_i K_j)^n$ is the abbreviation for $2n$ knowledge operator K_i and K_j in alternation, starting with K_i.

Example 7.1.1. (**Byzantine generals**)
Imagine two allied generals, i and j, standing on two mountain summits, with their enemy in the valley between them. It is generally known that i and j together can easily defeat the enemy, but if only one of them attacks, he will certainly lose the battle.

General i sends a messenger to j with the message p (= "*I propose that we attack the first day of the next month at 8 PM sharp*"). It is not guaranteed, however, that the messenger will arrive.

First, $p \wedge K_i p$ holds because p is general i's proposal. But, $\neg K_j p$ holds.

Suppose that the messenger does reach the other summit and delivers the message to j. Then $K_j p$ and $K_j K_i p$ hold.

But i wants to know that j will attack as well because $\neg K_i K_j p$ holds. Thus, j sends the messenger back with an 'okay'. Suppose the messenger survives again. Then, K_iK_jp and $K_iK_jK_ip$ hold.

But j wants to know that i will attack as well because $\neg K_jK_iK_jp$ holds. Thus, i sends the messenger back with an 'okay'. Suppose the messenger survives again. Then, $K_jK_iK_jp$ and $K_jK_iK_jK_ip$ hold.

But i wants to know that j will attack as well because $\neg K_iK_jK_iK_jp$ holds. Thus, i sends the messenger back with an 'okay'. Suppose the messenger survives again. Then, $K_iK_jK_iK_jp$ and $K_iK_jK_iK_jK_ip$ hold.

Number of times the message was sent	Proposition that holds then.	
0(initial state)	$\psi_0 \wedge \neg K_jp$	$\psi_0 \equiv p \wedge K_ip$
1(From i to j)	$\psi_1 \wedge \neg K_iK_jp$	$\psi_1 \equiv \psi_0 \wedge K_j\psi_0$
2(From j to i)	$\psi_2 \wedge \neg K_jK_iK_jp$	$\psi_2 \equiv \psi_1 \wedge K_iK_j\psi_0$
3(From i to j)	$\psi_3 \wedge \neg K_iK_jK_iK_jp$	$\psi_3 \equiv \psi_2 \wedge K_jK_iK_j\psi_0$
$2n$(From j to i)	$\psi_{2n} \wedge \neg K_j(K_iK_j)^np$	$\psi_{2n} \equiv \psi_{2n-1} \wedge (K_iK_j)^n\psi_0$
$2n+1$(From i to j)	$\psi_{2n+1} \wedge \neg (K_iK_j)^{n+1}p$	$\psi_{2n+1} \equiv \psi_{2n} \wedge K_j(K_iK_j)^n\psi_0$

Table 7.1: Byzantine generals

Suppose that general i and j are aware of p. By this example(Byzantine generals), the epistemic awareness model such that the set of equivalence classes for $\approx_i$ is infinite is shown. Let Φ be the set of formulas such that

$$\begin{aligned}
&\Phi := \{\neg K_jp \wedge p \wedge K_ip,\\
&\neg K_j(K_iK_j)^np \wedge \bigwedge_{0\leq k\leq n}(K_iK_j)^k(p \wedge K_ip) \wedge \bigwedge_{0\leq k\leq n-1} K_j(K_iK_j)^k(p \wedge K_ip),\\
&\neg (K_iK_j)^np \wedge \bigwedge_{0\leq k\leq n-1}(K_iK_j)^k(p \wedge K_ip) \wedge \bigwedge_{0\leq k\leq n-1} K_j(K_iK_j)^k(p \wedge K_ip),\\
&\mid n \in \mathbb{N} \text{ and } 1 \leq n\}.
\end{aligned}$$

Lemma 7.1.2. $\not\models \varphi \wedge \varphi'$ for all $\varphi, \varphi' \in \Phi$ such that $\varphi \not\equiv \varphi'$.

Proof. Take two arbitrary formulas $\varphi, \varphi' \in \Phi$ such that $\varphi \not\equiv \varphi'$.

- Suppose that $\varphi \equiv \neg K_jp \wedge p \wedge K_ip$. Then, $\models \varphi' \rightarrow K_jp$ because φ and φ' are different formulas from each other. Since $\models \varphi \rightarrow \neg K_jp$, $\not\models \varphi \wedge \varphi'$.

- Take two arbitrary natural numbers $n, m \in \mathbb{N}$ with $n, m \geq 1$.
  Suppose that $\varphi \equiv$

  $$\neg(K_iK_j)^n p \wedge \bigwedge_{0 \leq k \leq n-1} (K_iK_j)^k(p \wedge K_ip) \wedge \bigwedge_{0 \leq k \leq n-1} K_j(K_iK_j)^k(p \wedge K_ip).$$

  If $n \leq m$, then $\models \varphi \to \neg(K_iK_j)^n p$ and $\models \varphi' \to (K_iK_j)^n p$. Therefore, $\not\models \varphi \wedge \varphi'$.
  If $m < n$ and $\varphi' \equiv$

  $$\neg(K_iK_j)^m p \wedge \bigwedge_{0 \leq k \leq m-1} (K_iK_j)^k(p \wedge K_ip) \wedge \bigwedge_{0 \leq k \leq m-1} K_j(K_iK_j)^k(p \wedge K_ip),$$

  then $\models \varphi \to (K_iK_j)^m p$ and $\models \varphi' \to \neg(K_iK_j)^m p$. Therefore, $\not\models \varphi \wedge \varphi'$.
  If $m < n$ and $\varphi' \equiv$

  $$\neg K_j(K_iK_j)^m p \wedge \bigwedge_{0 \leq k \leq m} (K_iK_j)^k(p \wedge K_ip) \wedge \bigwedge_{0 \leq k \leq m-1} K_j(K_iK_j)^k(p \wedge K_ip),$$

  then $\models \varphi \to K_j(K_iK_j)^m p$ and $\models \varphi' \to \neg K_j(K_iK_j)^m p$. Therefore, $\not\models \varphi \wedge \varphi'$.

- Take two arbitrary natural numbers $n, m \in \mathbb{N}$ with $n > m \geq 1$. Suppose that $\varphi \equiv$

  $$\neg K_j(K_iK_j)^n p \wedge \bigwedge_{0 \leq k \leq n} (K_iK_j)^k(p \wedge K_ip) \wedge \bigwedge_{0 \leq k \leq n-1} K_j(K_iK_j)^k(p \wedge K_ip),$$

  and $\varphi' \equiv$

  $$\neg K_j(K_iK_j)^m p \wedge \bigwedge_{0 \leq k \leq m} (K_iK_j)^k(p \wedge K_ip) \wedge \bigwedge_{0 \leq k \leq m-1} K_j(K_iK_j)^k(p \wedge K_ip).$$

  Then, $\models \varphi \to K_j(K_iK_j)^m p$ and $\models \varphi' \to \neg K_j(K_iK_j)^m p$. Therefore, $\not\models \varphi \wedge \varphi'$.

Thus, $\not\models \varphi \wedge \varphi'$ for all $\varphi, \varphi' \in \Phi$ such that $\varphi \not\equiv \varphi'$. □

Lemma 7.1.3. For all $\varphi \in \Phi$, there is a (M, w) such that $(M, w) \models \varphi$.

Proof. Let $P = \{p\}$ and $\mathcal{G} = \{e, i, j\}$ be the finite sets of propositions and agents, respectively. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ consisting of (See Figure 7.1):

1. $W = \{w_k \mid k \in \mathbb{N} \backslash \{0\}\}$;

2. $\sim_i = \{(w,w) \mid w \in W\} \cup \{(w_{2k}, w_{2k+1}), (w_{2k+1}, w_{2k}) \mid k \in \mathbb{N}\backslash\{0\}\}$,
   $\sim_j = \{(w,w) \mid w \in W\} \cup \{(w_{2k}, w_{2k-1}), (w_{2k-1}, w_{2k}) \mid k \in \mathbb{N}\backslash\{0\}\}$,
   $\sim_e = \{(w,v) \mid w, v \in W\}$,

3. $\mathscr{A}(i) = \mathscr{A}(j) = \mathscr{A}(e) = P$.

4. $V(p) = W\backslash\{w_1\}$.

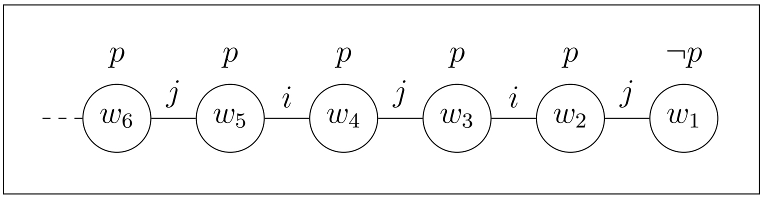

Figure 7.1: Kripke Model Represented Byzantine Generals

Then,

- $(M, w_2) \models \neg K_j p \wedge p \wedge K_i p$.
- for each $n \in \mathbb{N}$,
  $(M, w_{2n+2}) \models \neg K_j (K_i K_j)^n p \wedge \bigwedge_{0 \leq k \leq n} (K_i K_j)^k (p \wedge K_i p)$
  $\wedge \bigwedge_{0 \leq k \leq n-1} K_j (K_i K_j)^k (p \wedge K_i p)$
- for each $n \in \mathbb{N}$,
  $(M, w_{2n+1}) \models \neg (K_i K_j)^n p \wedge \bigwedge_{0 \leq k \leq n-1} (K_i K_j)^k (p \wedge K_i p)$
  $\wedge \bigwedge_{0 \leq k \leq n-1} K_j (K_i K_j)^k (p \wedge K_i p)$

□

Therefore, if $(M, w) \models \varphi$ and $(M, v) \models \varphi'$ for any $\varphi, \varphi' \in \Phi$ such that $\varphi \not\equiv \varphi'$ and $At(\varphi) \cup At(\varphi') \subseteq \mathscr{A}(i)$, then $w \not\approx_i v$. Since Φ is infinite, there is a model such that the set of equivalence classes for $\approx_i$ is infinite.

7.2 Example of Non-Compactness

Let $P = \{p, q\}$ and $\mathcal{G} = \{i, j, e\}$ be the finite set of atomic propositions and agents, respectively. Suppose P and $\mathcal{G}$ are mutually disjoint. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ consisting of (See Figure 7.2):

1. $W = \{w_k \mid k \in \mathbb{N}\backslash\{0\}\}$,

2. $\sim_i = \{(w,w) \mid w \in W\} \cup \{(w_{2k}, w_{2k+1}), (w_{2k+1}, w_{2k}) \mid k \in \mathbb{N}\backslash\{0\}\}$,
   $\sim_j = \{(w,w) \mid w \in W\} \cup \{(w_{2k}, w_{2k-1}), (w_{2k-1}, w_{2k}) \mid k \in \mathbb{N}\backslash\{0\}\}$,
   $\sim_e = \{(w,w') \mid w, w' \in W\}$,

3. $\mathscr{A}(i) = \mathscr{A}(j) = \{p\},\ \mathscr{A}(e) = \{p, q\}$,

4. $V(p) = W\backslash\{w_1\},\ V(q) = W$.

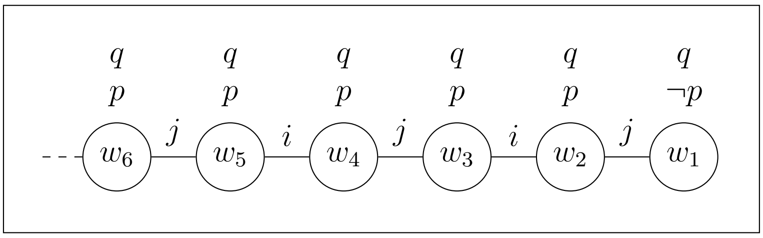

Figure 7.2: φ_k is satisfiable for each $k \in \mathbb{N}\backslash\{0,1\}$.

For each $k \in \mathbb{N}\backslash\{0,1\}$, let φ_k be a formula such that

- $\varphi_2 \equiv p \wedge K_i p$.
- for each $n \in \mathbb{N}\backslash\{0\}$,
  $\varphi_{2n+1} \equiv \bigwedge_{0 \leq k \leq n-1} (K_i K_j)^k (p \wedge K_i p) \wedge \bigwedge_{0 \leq k \leq n-1} K_j (K_i K_j)^k (p \wedge K_i p)$
- for each $n \in \mathbb{N}\backslash\{0\}$,
  $\varphi_{2n+2} \equiv \bigwedge_{0 \leq k \leq n} (K_i K_j)^k (p \wedge K_i p) \wedge \bigwedge_{0 \leq k \leq n-1} K_j (K_i K_j)^k (p \wedge K_i p)$.

Then, $(M, w_{k'}) \models \varphi_k$ for all $k' \in \mathbb{N}$ such that $k \leq k'$ for each $k \in \mathbb{N}\backslash\{0,1\}$ (e.g., Figure 7.3).

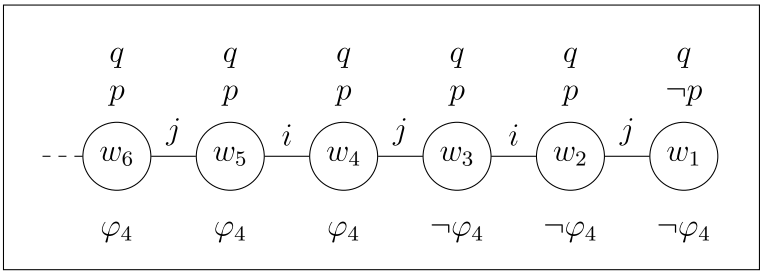

Figure 7.3: φ_4 is true a certain point onwards.

Let Ψ_n be the finite set of formulas such that for each $n \in \mathbb{N}\backslash\{0,1\}$,

$$\Psi_n = \{A_i p \wedge \neg A_i q \wedge [\approx]_i q \wedge \varphi_n \wedge \Diamond_e(\neg[\approx]_i q \wedge \varphi_n)\}.$$

Lemma 7.2.1. Let $n \in \mathbb{N}\backslash\{0,1\}$. Then, Ψ_n is finite satisfiable.

Proof.

- (**The case for** n is an even number): Let n be an even number such that $2 \leq n$. Given an epistemic awareness model $M = \langle W, \sim, V, \mathscr{A} \rangle$ consisting of (See Figure 7.4):

  1. $W = \{w_k \mid k \in \mathbb{N}, 1 \leq k \leq n+1\}$.
  2. $\sim_i = \{(w,w) \mid w \in W\} \cup \{(w_{2k}, w_{2k+1}), (w_{2k+1}, w_{2k}) \mid k \in \mathbb{N}, 1 \leq k \leq n/2\}$,
     $\sim_j = \{(w,w) \mid w \in W\} \cup \{(w_{2k}, w_{2k-1}), (w_{2k-1}, w_{2k}) \mid k \in \mathbb{N}, , 1 \leq k \leq n/2\}$,
     $\sim_e = \{(w,w') \mid w, w' \in W\}$,
  3. $\mathscr{A}(i) = \mathscr{A}(j) = \{p\}$ and $\mathscr{A}(e) = P$.
  4. $V(p) = W\backslash\{w_1\}$ and $V(q) = W\backslash\{w_{n+1}\}$.

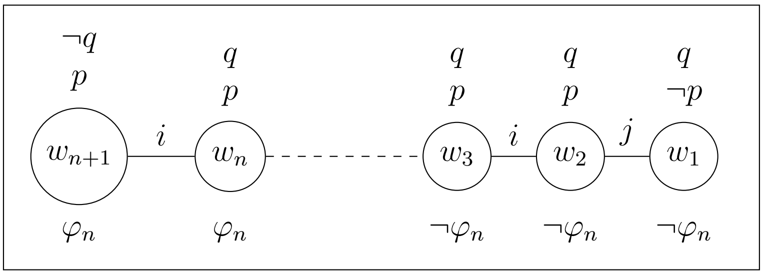

Figure 7.4: Ψ_n is finite satisfiable(n is even).

Since $w \not\approx_i w'$ for all $w, w' \in W$ such that $w \neq w'$ by the lemma 7.1.2, $(M, w_n) \models [\approx]_i q \wedge \varphi_n$ and $(M, w_n) \models \neg[\approx]_i q \wedge \varphi_n$. Then, $(M, w_n) \models \Psi_n$.

- (**The case for** n is an odd number): Let n be an odd number such that $3 \leq n$. Given an epistemic awareness model $M = \langle W, \sim, V, \mathscr{A} \rangle$ consisting of (See Figure 7.5):

  1. $W = \{w_k \mid k \in \mathbb{N}, 1 \leq k \leq n+1\}$.
  2. $\sim_i = \{(w,w) \mid w \in W\} \cup \{(w_{2k}, w_{2k+1}), (w_{2k+1}, w_{2k}) \mid k \in \mathbb{N}, 1 \leq k \leq (n-1)/2\}$,
     $\sim_j = \{(w,w) \mid w \in W\} \cup \{(w_{2k}, w_{2k-1}), (w_{2k-1}, w_{2k}) \mid k \in \mathbb{N}, , 1 \leq k \leq (n+1)/2\}$,
     $\sim_e = \{(w,w') \mid w, w' \in W\}$,

3. $\mathscr{A}(i) = \mathscr{A}(j) = \{p\}$ and $\mathscr{A}(e) = P$.
4. $V(p) = W\backslash\{w_1\}$ and $V(q) = W\backslash\{w_{n+1}\}$.

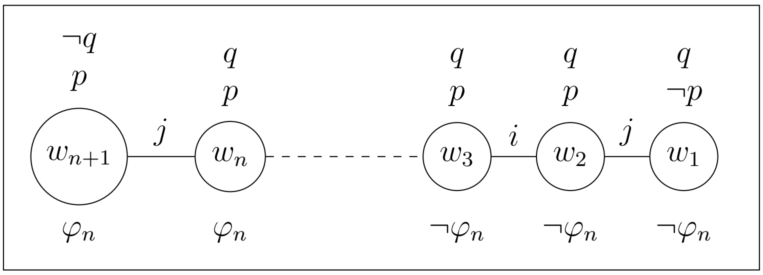

Figure 7.5: Ψ_n is finite satisfiable(n is odd).

Since $w \not\approx_i w'$ for all $w, w' \in W$ such that $w \neq w'$ by the lemma 7.1.2, $(M, w_n) \models [\approx]_i q \wedge \varphi_n$ and $(M, w_n) \models \neg[\approx]_i q \wedge \varphi_n$. Then, $(M, w_n) \models \Psi_n$.

Therefore, Ψ_n is finite satisfiable for each $n \in \mathbb{N}\backslash\{0, 1\}$. $\square$

Lemma 7.2.2. Let $n, m \in \mathbb{N}\backslash\{0, 1\}$ such that $n \leq m$. Given any epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and any $w \in W$. Then,

$$(M, w) \models \Psi_m \text{ implies } (M, w) \models \Psi_n.$$

Proof. Given any epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and any $w \in W$. Since $\models \varphi_m \rightarrow \varphi_n$, if $(M, w) \models \varphi_m$, then $(M, w) \models \varphi_n$. Therefore, $(M, w) \models \Psi_m$ implies $(M, w) \models \Psi_n$. $\square$

Lemma 7.2.3. Let $N \subseteq \mathbb{N}\backslash\{0, 1\}$ be a finite set of natural numbers. Then, $\bigcup_{k \in N} \Psi_k$ is finite satisfiable.

Proof. Let $N \subseteq \mathbb{N}\backslash\{0, 1\}$ be a finite set of natural numbers. Given any epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and any $w \in W$.

Take the natural number $k \in N$ such that $k' \leq k$ for all $k' \in N$. Then, $(M, w) \models \Psi_k$ implies $(M, w) \models Psi_{k'}$ for all $k' \in N$. Therefore, $(M, w) \models \Psi_k$ implies $(M, w) \models \bigcup_{k \in N} \Psi_k$. By the lemma 7.2.1, Ψ_k is finite satisfiable. Therefore, $\bigcup_{k \in N} \Psi_k$ is finite satisfiable, too. $\square$

Now, Ψ_n be extended to Ψ'_n such that

$$\Psi'_n = \Psi_n \cup \{\bigwedge_{r \in P\backslash\{p,q\}} (\Box_e r \wedge \neg A_i r)\} \cup \{\Box_i \varphi \leftrightarrow \Box_e \varphi \mid \varphi \in \mathcal{L}_{ALGP}, i' \in \mathcal{G}\backslash\{i, j\}\}$$

Ψ'_n is satisfiable, too.

Lemma 7.2.4. $\bigcup_{n\in\mathbb{N}\setminus\{0,1\}} \Psi'_n$ is unsatisfiable.

Proof. Suppose that $\bigcup_{n\in\mathbb{N}\setminus\{0,1\}} \Psi'_n$ is satisfiable. Then, there is a (M, w) such that $(M, w) \models \psi$ for all $\psi \in \bigcup_{n\in\mathbb{N}\setminus\{0,1\}} \Psi'_n$. Then, $(M, w) \models [\approx]_i q \wedge (K_i K_j)^k p, [\approx]_i q \wedge (K_j K_i)^k p, [\approx]_i q \wedge K_j(K_i K_j)^k p, [\approx]_i q \wedge K_i(K_j K_i)^k p$ for all $k \in \mathbb{N}$.

Also, there is a (M, v) such that $(M, v) \models \neg[\approx]_i q \wedge (K_i K_j)^k p), \neg[\approx]_i q \wedge (K_j K_i)^k p, \neg[\approx]_i q \wedge K_j(K_i K_j)^k p, \neg[\approx]_i q \wedge K_i(K_j K_i)^k p$ for all $k \in \mathbb{N}$.

Since $w \approx_i v$, $(M, w) \models \neg[\approx]_i q$. This is a contradiction. Therefore, $\bigcup_{k\in\mathbb{N}\setminus\{0\}} \Psi'_k$ is unsatisfiable. □

Theorem 7.2.5. $\bigcup_{n\in\mathbb{N}\setminus\{0,1\}} \Psi'_n$ is not compact.

Proof.
By the lemma 7.2.3, Ψ is satisfiable for all finite subset $\Psi \subseteq \bigcup_{n\in\mathbb{N}\setminus\{0,1\}} \Psi'_n$.
By the lemma 7.2.4, $\bigcup_{n\in\mathbb{N}\setminus\{0,1\}} \Psi'_n$ is unsatisfiable.
Therefore, $\bigcup_{n\in\mathbb{N}\setminus\{0,1\}} \Psi'_n$ is not compact. □

7.3 When Abstraction Operator Can Be Reduced?

The previous section shows the non-compactness of Awareness Logic of Abstraction. This section investigates a condition of reduction from ALF to ALGP. ALF is the logic extended from ALGP by introducing the implicit abstraction operators $[\approx]_a$ for each $a \in \mathcal{A}$.

Definition 7.3.1. (**Paths**)
Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Let $n \in \mathbb{N}$. Suppose $w_1, w_2, ... \in W$. A **path** from w_1 is a sequence $w_1, w_2, ...$ such that $w_k \sim_i w_{k+1}$ for some $i \in \mathcal{G}\setminus\{e\}$ for all $k \in \mathbb{N}$ with $1 \leq k$.

The number of terms in a path is called the **path length**.

Also, if a path from w_1 and a path from v_1 satisfy $w_k \sim_i w_{k+1} \Leftrightarrow v_k \sim_i v_{k+1}$ for all $i \in \mathcal{G}$ and all $k \in \mathbb{N}$ with $1 \leq k$, then both paths are called **equivalent** to each other.

Definition 7.3.2. (**Simple Paths**)
Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Suppose $w_1, w_2, ... \in W$. A **simple path** from w_1 is a sequence $w_1, w_2, ...$ such that $w_k \sim_i w_{k+1}$ for some $i \in \mathcal{G}\setminus\{e\}$ for all $k \in \mathbb{N}$ with $1 \leq k$ and $w_k \neq w_{k'}$ for all $k, k' \in \mathbb{N}$ with $1 \leq k < k'$.

Definition 7.3.3. (**Simple-Path-Length-Bounded**)
An epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ is **simple-path-length-bounded** by $n \in \mathbb{N}\backslash\{0\}$ if there is a positive number n such that n is larger than any simple path length.

M is called **simple-path-length-bounded model** if M is simple-path-length-bounded.

Lemma 7.3.4. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. If M is simple-path-length-bounded, then M is image-finite.

Proof. By the contraposition. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Suppose M is not image-finite. Then, there is a possible world $w \in W$ and an agent $i \in \mathcal{G}$ such that $\{v \mid w \sim_i v\}$ is infinite. Since $\sim_i$ is an equivalence relation, there is an infinite simple path from w via $\sim_i$. Therefore, M is not simple-path-length-bounded. □

Lemma 7.3.5. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Let n be a positive integer. If M is simple-path-length-bounded by n and there is a path from w to w' in M, then there is a path from w to w' such that the path length is n.

Proof. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Let $n \in \mathbb{N}$. Suppose that M is simple-path-length-bounded by n. Take an arbitrary $w_1 \in W$.

Take an arbitrary $w_k \in W$ such that there is a path from w_1 to w_k and the path length is $k \leq n$. Since $\sim_i$ is reflexive for all $i \in \mathcal{G}$, there is a path from w_1 to w_k and the path length is n.

Take an arbitrary $w_k \in W$ such that there is a path from w_1 to w_k and the length is $n < k$. This path is not simple because M is simple-path-length-bounded by n. Take any $w_{k'}, w_{k''}$ in the path such that $w_{k'} = w_{k''}$ and $1 \leq k' < k'' \leq k$. Then, We can remove the path from $w_{k'}$ to $w_{k''}$ and connect the path from w_1 to $w_{k'}$ and the path from $w_{k''}$ to w_k. By such transformations, there is a path from w_1 to w_k such that the path length is at most n. □

Let $(\Diamond_{i_n})^n$ be the abbreviation for n diamond operators $\Diamond_{i_1}\Diamond_{i_2}...\Diamond_{i_n}$ where $i_k \in \mathcal{G}\backslash\{e\}$ for all $k \in \mathbb{N}$ such that $1 \leq k \leq n$.

Lemma 7.3.6. Fix $a \in \mathcal{A}$. Let n be a positive integer. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Suppose that M is simple-path-length-bounded by n. They are equivalent for all $w_n, v_n \in W$.

1. $w_n \approx_a v_n$

2. For all $w_1 \in W$ such that there is a path from w_n to w_1 and the path length is n, there is a $v_1 \in W$ such that there is an equivalent path from v_n to v_1 and for all $p \in \mathscr{A}(a)$, $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$.

Proof. Fix $a \in \mathcal{A}$. Take an arbitrary positive integer n. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ where $w_1, ..., w_n, v_1, ..., v_n \in W$. Suppose that M is simple-path-length-bounded by n. Then, M is image-finite by the lemma 7.3.4.

- $(1 \Rightarrow 2)$: Take an arbitrary $w_1, w_n \in W$ such that there is a path from w_n to w_1 and the path length is n. Take an arbitrary $v_n \in W$ such that $w_n \approx_a v_n$. Suppose that there is no $v_1 \in W$ such that there is an equivalent path from v_n to v_1 and $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$ for all $p \in \mathscr{A}(a)$.

  Let $i_{n-1}, i_{n-2}, ...i_1 \in \mathcal{G}$ be agents such that $w_{k+1} \sim_{i_k} w_k$ and $w_k, w_{k+1} \in W$ is terms of the path from w_n to w_1 for all $k \in \mathbb{N}$ with $1 \leq k < n$. Then, $(M, w_n) \models (\Diamond_{i_n})^n p \not\Leftrightarrow (M, v_n) \models (\Diamond_{i_n})^n p$ and $At((\Diamond_{i_n})^n p) \subseteq \mathscr{A}(a)$ for all $p \in \mathscr{A}(a)$. Therefore, $w_n \not\approx_a v_n$. This is a contradiction. Then, there is $v_1 \in W$ such that there is an equivalent path from v_n to v_1 and $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$ for all $p \in \mathscr{A}(a)$.

- $(2 \Rightarrow 1)$: Take an arbitrary $w_n, v_n \in W$. Suppose that for all $w_1 \in W$ such that there is a path from w_n to w_1 and the path length is n, there is a $v_1 \in W$ such that there is an equivalent path from v_n to v_1 and $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$ for all $p \in \mathscr{A}(a)$. By the lemma 7.3.5, for all $w_1 \in W$ such that there is a path from w_n to w_1, there is a $v_1 \in W$ such that there is an equivalent path from v_n to v_1 and $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$ for all $p \in \mathscr{A}(a)$. Take an arbitrary world $w_1 \in W$. Then, there is $v_1 \in W$ such that there is an equivalent path from v_n to v_1 for some path from w_n to w_1 and $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$ for all $p \in \mathscr{A}(a)$. Then, the following holds: for all positive numbers k with $1 < k \leq n$ and for all $(w_k, v_k) \in W \times W$ such that w_k and v_k are k-th terms of the path from w_n and v_n to w_1 and v_1, respectively,

  - (**atoms**): for all $p \in \mathscr{A}(a)$, $w_k \in V(p) \Leftrightarrow v_k \in V(p)$,
  - (**forth**): for all $i \in \mathcal{G}$, if w_{k-1} is a term of the path from w_n to w_1 and $w_k \sim_i w_{k-1}$, then there is a $v_k \in W$ such that v_{k-1} is a term of the path from v_n to v_1 and $v_k \sim_i v_{k-1}$,
  - (**back**): for all $i \in \mathcal{G}$, if v_{k-1} is a term of the path from v_n to v_1 and $v_k \sim_i v_{k-1}$, then there is a $w_k \in W$ such that w_{k-1} is a term of the path from w_n to w_1 and $w_k \sim_i w_{k-1}$.

Since $w_1 \in W$ are arbitrary, (M, w_n) and (M, v_n) are $\mathscr{A}(a)$ standard bisimilar for ALF. By the theorem 6.2.6, $w_n \approx_a v_n$.

□

Definition 7.3.7. Let P and Q be finite sets of atomic propositions and agents, respectively. Suppose they are mutually disjoint. Fix $a \in \mathcal{A}$. Let Σ_n^a be a set of formulas defined inductively for each positive integer n:

$$\Sigma_1^a := \bigcup_{P' \subseteq P} \{\bigwedge A_a P' \wedge \bigwedge \neg A_a (P \backslash P') \wedge \bigwedge Q \wedge \bigwedge \neg (P' \backslash Q) \mid Q \subseteq P'\}.$$

$$\Sigma_n^a := \{\psi \wedge \bigwedge_{i \in \mathcal{G}} (\bigwedge_{\psi' \in \Sigma_i'} \Diamond_i \psi' \wedge \bigwedge_{\psi'' \in \Sigma_i''} \neg \Diamond_i \psi'') \mid \psi \in \Sigma_1^a,$$
$$\text{for each } i \in \mathcal{G} \text{ there is a subset } \Sigma_i' \subseteq \Sigma_{n-1}^a, \Sigma_i'' = \Sigma_{n-1}^a \backslash \Sigma_i'\}$$

$\varphi \equiv \psi$ means that φ is a formula that has the same form as ψ.

Lemma 7.3.8. Fix $a \in \mathcal{A}$. Then, $\models \varphi \wedge \varphi' \leftrightarrow \bot$ for all $\varphi, \varphi' \in \Sigma_n^a$ such that $\varphi \not\equiv \varphi'$ for all positive integer $n \in \mathbb{N} \backslash \{0\}$.

Proof. Fix $a \in \mathcal{A}$.

- (**The case for** $n = 1$): Take two arbitrary $P', P'' \subseteq P$ such that $P' \neq P''$. Take two arbitrary formulas $\varphi' \in \{\bigwedge A_a P' \wedge \bigwedge \neg A_a (P \backslash P') \wedge \bigwedge Q \wedge \bigwedge \neg (P' \backslash Q) \mid Q \subseteq P'\}$ and $\varphi'' \in \{\bigwedge A_a P'' \wedge \bigwedge \neg A_a (P \backslash P'') \wedge \bigwedge Q \wedge \bigwedge \neg (P'' \backslash Q) \mid Q \subseteq P''\}$. Then, $\models \varphi' \to A_a p \not\Leftrightarrow \models \varphi'' \to A_a p$ for some $p \in P$. Therefore, $\models \varphi' \wedge \varphi'' \to \bot$.

  Take an arbitrary $\psi \in \{\bigwedge A_a P' \wedge \bigwedge \neg A_a (P \backslash P') \wedge \bigwedge Q \wedge \bigwedge \neg (P' \backslash Q) \mid Q \subseteq P'\}$ such that $\varphi' \not\equiv \psi$. Then, $\models \varphi' \to p \not\Leftrightarrow \models \psi \to p$ for some $p \in P'$. Therefore, $\models \varphi' \wedge \psi \to \bot$.

- (**The case for** $n \neq 1$): Take an arbitrary positive integer $n \in \mathbb{N} \backslash \{0\}$ with $n \neq 1$ and two arbitrary formulas $\varphi_n, \varphi_n' \in \Sigma_n^a$ such that $\varphi_n \not\equiv \varphi_n'$. Then,

$$\varphi_n \equiv \psi_1 \wedge \bigwedge_{i \in \mathcal{G}} (\bigwedge_{\psi' \in \Sigma_i'} \Diamond_i \psi' \wedge \bigwedge_{\psi'' \in \Sigma_i''} \neg \Diamond_i \psi'')$$

  where $\psi_1 \in \Sigma_1^a$, $\Sigma_i' \subseteq \Sigma_{n-1}^a$ and $\Sigma_i'' = \Sigma_{n-1}^a \backslash \Sigma_i'$ for each $i \in \mathcal{G}$. Also,

$$\varphi_n' \equiv \psi_2 \wedge \bigwedge_{i \in \mathcal{G}} (\bigwedge_{\psi' \in \Pi_i'} \Diamond_i \psi' \wedge \bigwedge_{\psi'' \in \Pi_i''} \neg \Diamond_i \psi'')$$

  where $\psi_2 \in \Sigma_1^a$, $\Pi_i' \subseteq \Sigma_{n-1}^a$, and $\Pi_i'' = \Sigma_{n-1}^a \backslash \Pi_i'$ for each $i \in \mathcal{G}$.

If $\psi_1 \not\equiv \psi_2$, then $\models \varphi_n \wedge \varphi'_n \to \bot$ because $\models \psi_1 \wedge \psi_2 \to \bot$ by the proof of the case for $n = 1$.

Suppose $\psi_1 \equiv \psi_2$. By $\varphi_n \not\equiv \varphi'_n$, $\Sigma'_i \neq \Pi'_i$ for some $i \in \mathcal{G}$. Then, $\models \varphi_n \to \Diamond_i \psi' \not\Leftrightarrow \models \varphi'_n \to \Diamond_i \psi'$ for some $\psi' \in \Sigma^a_{n-1}$.

Therefore, $\models \varphi_n \wedge \varphi'_n \to \bot$. Since n is an arbitrary positive integer with $n \neq 1$, $\models \varphi \wedge \varphi' \leftrightarrow \bot$ for all $\varphi, \varphi' \in \Sigma^a_n$ such that $\varphi \not\equiv \varphi'$ for all $n \in \mathbb{N} \backslash \{0, 1\}$.

Thus, this lemma is valid. □

Lemma 7.3.9. Fix $a \in \mathcal{A}$. Let n be a positive integer. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. They are equivalent for all $w_n, v_n \in W$.

1. $(M, w_n) \models \varphi \Leftrightarrow (M, v_n) \models \varphi$ for all $\varphi \in \Sigma^a_n$.
2. For all w_1 such that there is a path from w_n to w_1 and the path length is n, there is a v_1 such that there is an equivalent path from v_n to v_1 and for all $p \in \mathscr{A}(a)$, $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$.

Proof. Fix $a \in \mathcal{A}$. Take an arbitrary positive integer n. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$.

- (**Base Case**): Suppose $n = 1$. Take an arbitrary $w_1 \in W$. Let $Q = \{p \mid (M, w_1) \models p \wedge A_a p\}$ and $\varphi \equiv \bigwedge A_a \mathscr{A}(a) \wedge \bigwedge \neg A_a(P \backslash \mathscr{A}(a)) \wedge \bigwedge Q \wedge \bigwedge \neg(\mathscr{A}(a) \backslash Q)$. Then, $(M, w_1) \models \varphi$ and $\varphi \in \Sigma^a_1$.

  Then, $(M, w_1) \models \varphi \Leftrightarrow (M, v_1) \models \varphi$ is equivalent to $(M, w_1) \models p \Leftrightarrow (M, v_1) \models p$ for all $p \in \mathscr{A}(a)$ because $M \models \bigwedge A_a \mathscr{A}(a) \wedge \bigwedge \neg A_a(P \backslash \mathscr{A}(a))$.

  - ($1 \Rightarrow 2$): Suppose $(M, w_1) \models \varphi' \Leftrightarrow (M, v_1) \models \varphi'$ for all $\varphi' \in \Sigma^a_1$. Then $(M, w_1) \models \varphi \Leftrightarrow (M, v_1) \models \varphi$. Therefore, $(M, w_1) \models p \Leftrightarrow (M, v_1) \models p$ for all $p \in \mathscr{A}(a)$.
  - ($2 \Rightarrow 1$): Suppose $(M, w_1) \models p \Leftrightarrow (M, v_1) \models p$ for all $p \in \mathscr{A}(a)$. Then, $(M, w_1) \models \varphi \Leftrightarrow (M, v_1) \models \varphi$. Since $(M, w') \models \varphi \Leftrightarrow (M, w') \not\models \psi$ for all $\psi \in \Sigma^a_1 \backslash \{\varphi\}$ for all $w' \in W$ by the lemma 7.3.8, $(M, w_1) \models \varphi' \Leftrightarrow (M, v_1) \models \varphi'$ for all $\varphi' \in \Sigma^a_1$.

  Then, the lemma holds if $n = 1$.

- (**Induction Hypothesis**): for all positive number k such that $k \leq n - 1$, they are equivalent for all w_k, v_k,

  1. $(M, w_k) \models \varphi \Leftrightarrow (M, v_k) \models \varphi$ for all $\varphi \in \Sigma^a_k$.

2. For all w_1 such that there is a path from w_k to w_1 and the path length is k, there is a v_1 such that there is an equivalent path from v_k to v_1 and for all $p \in \mathscr{A}(a)$, $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$.

- (**Induction Step**): Suppose $n = k+1$. Take arbitrary worlds $w_{k+1} \in W$. Let $\varphi \in \mathcal{L}_{ALGP}$ be a formula such that $(M, w_{k+1}) \models \varphi$ and $\varphi \in \Sigma^a_{k+1}$. For each $i \in \mathcal{G}$, let Σ'_i be the set of formulas such that $\Sigma'_i \subseteq \Sigma^a_k$ and $\psi' \in \Sigma'_i$ iff $(M, w_k) \models \psi'$ for some $w_k \in W$ with $w_{k+1} \sim_i w_k$. Also, let ψ be a formula such that $\psi \in \Sigma^a_1$ and $(M, w_{k+1}) \models \psi$. Then, $\varphi \equiv \psi \wedge \bigwedge_{i \in \mathcal{G}}(\bigwedge_{\psi' \in \Sigma'_i} \Diamond_i \psi' \wedge \bigwedge_{\psi'' \in \Sigma''_i} \neg \Diamond_i \psi'')$ where $\Sigma''_i = \Sigma^a_k \backslash \Sigma'_i$ for each $i \in \mathcal{G}$.

  - $(1 \Rightarrow 2)$: Take an arbitrary $v_{k+1} \in W$. Suppose $(M, w_{k+1}) \models \varphi' \Leftrightarrow (M, v_{k+1}) \models \varphi'$ for all $\varphi' \in \Sigma^a_{k+1}$. Then, $(M, w_{k+1}) \models \varphi \Leftrightarrow (M, v_{k+1}) \models \varphi$. Since $\varphi \equiv \psi \wedge \bigwedge_{i \in \mathcal{G}}(\bigwedge_{\psi' \in \Sigma'_i} \Diamond_i \psi' \wedge \bigwedge_{\psi'' \in \Sigma''_i} \neg \Diamond_i \psi'')$ where $\Sigma''_i = \Sigma^a_k \backslash \Sigma'_i$ for each $i \in \mathcal{G}$, for all $w_k \in W$, there is a $v_k \in W$ such that $w_{k+1} \sim_i w_k \Leftrightarrow v_{k+1} \sim_i v_k$ for all $i \in \mathcal{G}$ and $(M, w_k) \models \varphi_k \Leftrightarrow (M, v_k) \models \varphi_k$ for all $\varphi_k \in \Sigma^a_k$.

    By the induction hypothesis, for all w_1 such that there is a path from w_k to w_1 and the path length is k, there is a v_1 such that there is an equivalent path from v_k to v_1 and for all $p \in \mathscr{A}(a)$, $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$.

    Since $w_{k+1} \sim_i w_k \Leftrightarrow v_{k+1} \sim_i v_k$ for all $i \in \mathcal{G}$ and w_k is an arbitrary world, for all w_1 such that there is a path from w_{k+1} to w_1 and the path length is $k+1$, there is a v_1 such that there is an equivalent path from v_{k+1} to v_1 and for all $p \in \mathscr{A}(a)$, $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$.
  - $(2 \Rightarrow 1)$: Take an arbitrary $v_{k+1} \in W$. Suppose for all w_1 such that there is a path from w_{k+1} to w_1 and the path length is $k+1$, there is a v_1 such that there is an equivalent path from v_{k+1} to v_1 and for all $p \in \mathscr{A}(a)$, $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$. Take two arbitrary worlds $w_k, v_k \in W$ such that $w_{k+1} \sim_i w_k \Leftrightarrow v_{k+1} \sim_i v_k$ for all $i \in \mathcal{G}$. Then, for all w_1 such that there is a path from w_k to w_1 and the path length is k, there is a v_1 such that there is an equivalent path from v_k to v_1 and for all $p \in \mathscr{A}(a)$, $w_1 \in V(p) \Leftrightarrow v_1 \in V(p)$.

    By the induction hypothesis, $(M, w_k) \models \varphi_k \Leftrightarrow (M, v_k) \models \varphi_k$ for all $\varphi_k \in \Sigma^a_k$. Since $w_k, v_k \in W$ are two arbitrary worlds such that $w_{k+1} \sim_i w_k \Leftrightarrow v_{k+1} \sim_i v_k$ for all $i \in \mathcal{G}$, $(M, w_{k+1}) \models \varphi \Leftrightarrow (M, v_{k+1}) \models \varphi$. By the lemma 7.3.8, $(M, w_{k+1}) \models \varphi' \Leftrightarrow (M, v_{k+1}) \models \varphi'$ for all $\varphi' \in \Sigma^a_{k+1}$.

Therefore, this lemma holds. □

Theorem 7.3.10. Fix $a \in \mathcal{A}$. Let n be a positive integer. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Suppose that M is simple-path-length-bounded by n. They are equivalent:

1. $(M, w) \models \varphi \Leftrightarrow (M, v) \models \varphi$ for all $\varphi \in \Sigma_n^a$.
2. $w \approx_a v$.

Proof. It holds by the lemma 7.3.6 and 7.3.9. □

Theorem 7.3.11. Fix $a \in \mathcal{A}$. Let n be a positive integer. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Suppose that M is simple-path-length-bounded by n. Then,

$$(M, w) \models \langle \approx \rangle_a \varphi \;\Leftrightarrow\; (M, w) \models \bigvee_{\psi \in \Sigma_n^a} (\psi \wedge \Diamond_e(\psi \wedge \varphi)).$$

Proof. Fix $a \in \mathcal{A}$. Let n be a positive integer. Take an arbitrary epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Suppose that M is simple-path-length-bounded by n.

- ($\Rightarrow$): Suppose that $(M, w) \models \langle \approx \rangle_a \varphi$. Take a $\psi \in \Sigma_n^a$ such that $(M, w) \models \psi$.

  Since $(M, v) \models \psi$ for all $v \in w^a$ by the theorem 7.3.10, $(M, w) \models \psi \wedge \Diamond_e(\psi \wedge \varphi)$.

  Therefore, $(M, w) \models \bigvee_{\psi \in \Sigma_n^a} (\psi \wedge \Diamond_e(\psi \wedge \varphi))$.

- ($\Leftarrow$): Suppose that $(M, w) \models \bigvee_{\psi \in \Sigma_n^a} (\psi \wedge \Diamond_e(\psi \wedge \varphi))$. Then, $(M, w) \models \psi \wedge \Diamond_e(\psi \wedge \varphi)$ for some $\psi \in \Sigma_n^a$. By the theorem 7.3.10, $w \approx_a v$ for some $v \in W$ such that $(M, v) \models \psi \wedge \varphi$. By the semantics, $(M, w) \models \langle \approx \rangle_a \varphi$.

□

Chapter 8

Quotient Model

This chapter introduces an abstract model called "*the quotient model*". This chapter introduces two abstract models called "*the nested quotient model*" and "*the product quotient model*" for nested abstraction and mutually complementary concretization, respectively.

8.1 Quotient Model with Agent Expression

This section introduces quotient models with agent expressions. By adding agent expressions to quotient models, we can consider reductions from nested abstraction and mutually complementary concretization to one-time abstraction in the chapter 9.

Definition 8.1.1. (Equivalence class with respect to $\approx_a$)
Given an $a \in \mathcal{A}$ and an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. The equivalence class of $w \in W$ with respect to $\approx_a$ is denoted in the following:

$$w^a := \{v \in W \mid w \approx_a v\}.$$

Theorem 8.1.2.
For all $w \in W$ and all $a, b \in \mathcal{A}$,

$$w^{a+b} \;\subseteq\; w^a \cap w^b \;\subseteq\; w^a \;\subseteq\; w^a \cup w^b \;\subseteq\; w^{a \cdot b}.$$

Theorem 8.1.3.
For all $w \in W$ and all $a, b \in \mathcal{A}$ such that $\mathscr{A}(a) \subseteq \mathscr{A}(b)$,

$$w^{a+b} \;=\; w^a \cap w^b \;=\; w^b \;\subseteq\; w^a \;=\; w^a \cup w^b \;=\; w^{a \cdot b}.$$

Example 8.1.4.
There is an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ for some $w \in W$ and some $a, b \in \mathcal{A}$,

$$w^a \cup w^b = w^{a \cdot b} \not\Leftrightarrow \mathscr{A}(a) \subseteq \mathscr{A}(b) \text{ or } \mathscr{A}(b) \subseteq \mathscr{A}(a)$$

$$w^a \cap w^b = w^{a+b} \not\Leftrightarrow \mathscr{A}(a) \subseteq \mathscr{A}(b) \text{ or } \mathscr{A}(b) \subseteq \mathscr{A}(a)$$

Let $\mathcal{G} = \{i, j\}$ and $P = \{p, q\}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ consisting of:

1. $W = \{w\}$;
2. $\sim_i = \sim_j = \sim_e = \{(w, w)\}$.
3. $\mathscr{A}(i) = \{p\}$ and $\mathscr{A}(j) = \{q\}$.
4. $V(p) = \{w\}$ and $V(q) = \varnothing$.

Then, $w^i \cup w^j = w^{i \cdot j}$ and $w^i \cap w^j = w^{i+j}$ (See Figure 8.1).

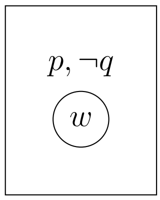

Figure 8.1: $w^i \cup w^j = w^{i \cdot j}$, $w^i \cap w^j = w^{i+j}$ and $\mathscr{A}(a) \cap \mathscr{A}(b) = \varnothing$

Definition 8.1.5. (**Quotient Frame**)
Given an agent expression a, a finite set of agents $\mathcal{G}$, and an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. A **quotient frame** based on M is a pair $\langle W/\approx_a, \sim/\approx_a \rangle$ consisting of:

1. $W/\approx_a := \{w^a \mid w \in W\}$;
2. $\sim/\approx_a : \mathcal{G} \longrightarrow 2^{(W/\approx_a) \times (W/\approx_a)}$, where for each $i \in \mathcal{G}$
   $(\sim/\approx_a)_i := \{(w^a, v^a) \mid (w, v) \in (\sim_i \circ \approx_a)^+, w, v \in W\}$.

$W/\approx_a$ and $(\sim/\approx_a)_i$ are denoted as W^a and $\sim_i^a$ respectively, too.

Definition 8.1.6. (**Quotient Model**)
Given a set of agent expressions $\mathcal{A}$, an agent expression $a \in \mathcal{A}$, an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and the quotient frame $\langle W^a, \sim^a\rangle$ of a based on M.
The quotient model of a based on M is a tuple $\langle W^a, \sim^a, \mathscr{A}/\approx_a, V/\approx_a\rangle$ where

- $\mathscr{A}/\approx_a : \mathcal{G} \longrightarrow 2^{\mathscr{A}(a)}$ with $\mathscr{A}/\approx_a (i) := \mathscr{A}(a) \cap \mathscr{A}(i)$;
- $V/\approx_a : \mathscr{A}(a) \longrightarrow 2^{W^a}$ with $V/\approx_a (p) := \{w^a \mid w \in V(p)\}$ for all $p \in P$;

$\mathscr{A}/\approx_a$ and $V/\approx_a$ are denoted as $\mathscr{A}^a$ and V^a respectively, too.

Definition 8.1.7. (**The Satisfaction Relation of $\mathcal{L}_{ALGP}$ in Quotient Model M^a**)
Fix $a \in \mathcal{A}$. Given epistemic state (M^a, w^a) with $M^a = \langle W^a, \sim^a, \mathscr{A}^a, V^a\rangle$, $w^a \in W^a$ and. For all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a)$,

$(M^a, w^a) \models^a_{ALGP} \top$;
$(M^a, w^a) \models^a_{ALGP} p \iff w^a \in V^a(p)$;
$(M^a, w^a) \models^a_{ALGP} \neg\varphi \iff (M^a, w^a) \not\models^a_{ALGP} \varphi$;
$(M^a, w^a) \models^a_{ALGP} \varphi \wedge \psi \iff (M^a, w^a) \models^a_{ALGP} \varphi$ and $(M^a, w^a) \models^a_{ALGP} \psi$;
$(M^a, w^a) \models^a_{ALGP} A_i\varphi \iff At(\varphi) \subseteq \mathscr{A}^a(i)$;
$(M^a, w^a) \models^a_{ALGP} \Box_i\varphi \iff$ for all $v^a \in W^a, w^a \sim^a_i v^a$ implies $(M^a, v^a) \models^a_{ALGP} \varphi$;

By the definition, the satisfaction relation for $\bot, \vee, \rightarrow, \Diamond_i, K_i, A_b$ is derived as follows: Given epistemic state (M^a, w^a) with $M^a = \langle W^a, \sim^a, \mathscr{A}^a, V^a\rangle$ and $w^a \in W^a$.

$(M^a, w^a) \not\models^a_{ALGP} \bot$;
$(M^a, w^a) \models^a_{ALGP} \varphi \vee \psi \iff (M^a, w^a) \models^a_{ALGP} \varphi$ or $(M^a, w^a) \models^a_{ALGP} \psi$;
$(M^a, w^a) \models^a_{ALGP} \varphi \rightarrow \psi \iff (M^a, w^a) \models^a_{ALGP} \varphi$ implies $(M^a, w^a) \models^a_{ALGP} \psi$;
$(M^a, w^a) \models^a_{ALGP} \Diamond_i\varphi \iff$ for some $v^a \in W^a, w^a \sim^a_i v^a$ and $(M^a, v^a) \models^a_{ALGP} \varphi$;
$(M^a, w^a) \models^a_{ALGP} K_i\varphi \iff (M^a, w^a) \models^a_{ALGP} \Box_i\varphi$ and $(M^a, w^a) \models^a_{ALGP} A_i\varphi$;
$(M^a, w^a) \models^a_{ALGP} A_b\varphi \iff At(\varphi) \subseteq \mathscr{A}^a(b)$;

Next, we show the lemma to prove that any accessibility relation on W^a is an equivalence relation.

Since $\sim_i$ and $\approx_a$ are equivalence relations, $\sim^a_i$ is an equivalence relation.

Theorem 8.1.8.
Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. For all $w \in W$,

$$\text{for all } \varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a), ((M, w) \models_{ALGP} \varphi \iff (M^a, w^a) \models^a_{ALGP} \varphi).$$

Proof. Given an arbitrary epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$, an $w \in W$,and an $a \in \mathcal{A}$. Let $M^a = \langle W^a, \sim^a, \mathscr{A}^a, V^a \rangle$ be the quotient model of a based on M.

- **Base Case** : Take an arbitrary atomic proposition $p \in \mathscr{A}(a)$. By the definition of $\approx_a$, if $x \approx_a y$, then $x \in V(p)$ iff $y \in V(p)$ for all $p \in \mathscr{A}(a)$. Then, for all $x, y \in w^a$, $x \in V(p)$ iff $y \in V(p)$ for all $p \in \mathscr{A}(a)$. By the definition of V^a and $w \in w^a$, $w^a \in V^a(p)$ iff $w \in V(p)$. Then, $(M, w) \models_{ALGP} p \Leftrightarrow (M^a, w^a) \models^a_{ALGP} p$.
- **Induction Hypothesis** : $(M, w) \models_{ALGP} \varphi \iff (M^a, w^a) \models^a_{ALGP} \varphi$.
- **Induction Step** : Take two arbitrary $\varphi, \psi \in \mathcal{L}_{ALGP}|\mathscr{A}(a)$ and $(M, w) \models_{ALGP} \varphi \iff (M^a, w^a) \models^a_{ALGP} \varphi$ and $(M, w) \models_{ALGP} \psi \iff (M^a, w^a) \models^a_{ALGP} \psi$.
  - **The case for** $\varphi \wedge \psi$:
    By the semantics, $(M, w) \models_{ALGP} \varphi \wedge \psi \iff (M, w) \models_{ALGP} \varphi$ and $(M, w) \models_{ALGP} \psi$. By the induction hypothesis, $(M, w) \models_{ALGP} \varphi$ and $(M, w) \models_{ALGP} \psi$ if and only if $(M^a, w^a) \models^a_{ALGP} \varphi$ and $(M^a, w^a) \models^a_{ALGP} \psi$. By the semantics, $(M^a, w^a) \models^a_{ALGP} \varphi$ and $(M^a, w^a) \models^a_{ALGP} \psi \iff (M^a, w^a) \models^a_{ALGP} \varphi \wedge \psi$. Then, $(M, w) \models_{ALGP} \varphi \wedge \psi \iff (M^a, w^a) \models^a_{ALGP} \varphi \wedge \psi$.
  - **The case for** $\neg\varphi$:
    By the semantics, $(M, w) \models_{ALGP} \neg\varphi \iff (M, w) \not\models_{ALGP} \varphi$. By the induction hypothesis, $(M, w) \not\models_{ALGP} \varphi$ if and only if $(M^a, w^a) \not\models^a_{ALGP} \varphi$. By the semantics, $(M^a, w^a) \not\models^a_{ALGP} \varphi \iff (M^a, w^a) \models^a_{ALGP} \neg\varphi$. Then, $(M, w) \models_{ALGP} \neg\varphi \iff (M^a, w^a) \models^a_{ALGP} \neg\varphi$.

- **The case for** $A_b\varphi$:
  By the semantics, $(M,w) \models_{ALGP} A_b\varphi$ iff $At(\varphi) \subseteq \mathscr{A}(b)$. Since $At(\varphi) \subseteq \mathscr{A}(a)$, $At(\varphi) \subseteq \mathscr{A}(b)$ iff $At(\varphi) \subseteq \mathscr{A}(a) \cap \mathscr{A}(b)$. By the definition of $\mathscr{A}^a$, $At(\varphi) \subseteq \mathscr{A}(a) \cap \mathscr{A}(b)$ iff $At(\varphi) \subseteq \mathscr{A}^a(b)$. By the semantics, $At(\varphi) \subseteq \mathscr{A}^a(b)$ iff $(M^a, w^a) \models^a_{ALGP} A_b\varphi$. Then, $(M,w) \models_{ALGP} A_b\varphi \iff (M^a, w^a) \models^a_{ALGP} A_b\varphi$.
- **The case for** $\Box_i\varphi$: Take an arbitrary $i \in \mathcal{G}$.
  * $(\Rightarrow)$:
    Suppose $(M,w) \models_{ALGP} \Box_i\varphi$. Since $\sim_i$ is an equivalence relation, for all $v \in W$ such that $w \sim_i v$, $(M,v) \models_{ALGP} \Box_i\varphi$. By $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a)$, for all $v' \in W$ such that $v \approx_a v'$, $(M,v') \models \Box_i\varphi$. Then, $(M,v') \models \Box_i\varphi$ for all $v' \in W$ such that $(w,v') \in (\sim_i \circ \approx_a)$. Similarly, $(M,w') \models \Box_i\varphi$ for all $w' \in W$ such that $(w,w') \in (\sim_i \circ \approx_a)^+$. By the definition of $\sim^a_i$, $(M^a, w^a) \models^a_{ALGP} \Box_i\varphi$.
  * $(\Leftarrow)$:
    By the contradiction. Suppose $(M,w) \not\models_{ALGP} \Box_i\varphi$. Then, there is a world $v \in W$ such that $(M,v) \not\models_{ALGP} \varphi$ and $w \sim_i v$. By $(M,v) \not\models_{ALGP} \varphi$ and $w \sim_i v$, $(M^a, v^a) \not\models^a_{ALGP} \varphi$ and $(w,v) \in (\sim_i \circ \approx_a)^+$. Then, there is a $v^a \in W^a$ such that $(M^a, v^a) \not\models^a_{ALGP} \varphi$ and $w^a \sim^a_i v^a$. By the semantics, $(M^a, w^a) \not\models_{ALGP} \Box_i\varphi$.

$\Box$

So, $(M,w) \models [\approx]_a\varphi \wedge A_a\varphi$ stands for 'φ holds' at w^a. We can introduce **an explicit abstraction operator** $[\approx]^E_a$ by expressing $[\approx]^E_a\varphi$ with $[\approx]_a\varphi \wedge A_a\varphi$.

Example 8.1.9. There are three children i_1, i_2, i_3. Let p_1, p_2 and p_3 be atomic propositions standing for "*agents* i_1, i_2, *and* i_3 *have mud on their faces*", respectively. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ consisting of (See Figure 8.2):

1. $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$,
2. $\sim_{i_1} = \{(w,v) \mid w,v \in \{w_1,w_3\}\} \cup \{(w,v) \mid w,v \in \{w_2,w_4\}\} \cup \{(w,v) \mid w,v \in \{w_5,w_6\}\}$,

$\sim_{i_2} = \{(w, v) \mid w, v \in \{w_2, w_5\}\} \cup \{(w, v) \mid w, v \in \{w_4, w_6\}\} \cup \{(w, w) \mid w \in W\}$,
$\sim_{i_3} = \{(w, v) \mid w, v \in \{w_1, w_5\}\} \cup \{(w, v) \mid w, v \in \{w_3, w_6\}\} \cup \{(w, w) \mid w \in W\}$,
$\sim_e = \{(w, v) \mid w, v \in W\}$,

3. $\mathscr{A}(i_1) = \{p_1\}$, $\mathscr{A}(i_2) = \{p_2\}$, $\mathscr{A}(i_3) = \{p_3\}, \mathscr{A}(e) = \{p_1, p_2, p_3\}$,
4. $V(p_1) = \{w_3, w_4, w_6\}, V(p_2) = \{w_1, w_3, w_5, w_6\}, V(p_3) = \{w_2, w_4, w_5, w_6\}$.

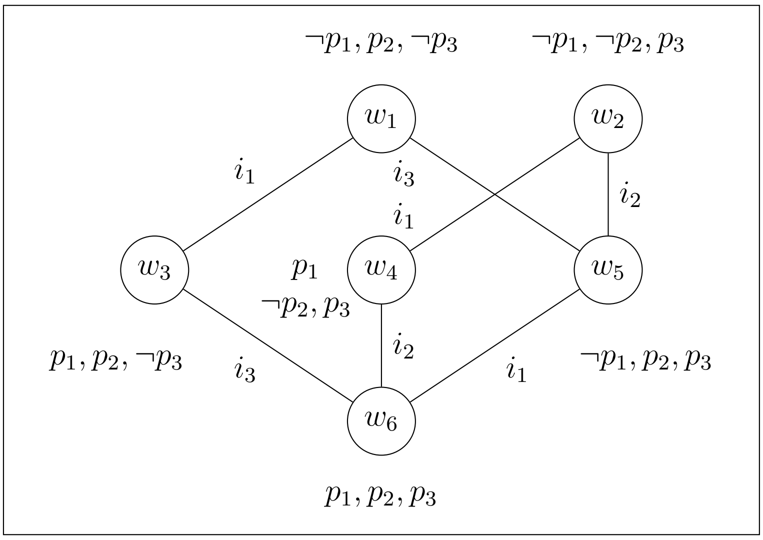

Figure 8.2: Kripke Model represented knowledge of i_1, i_2, and i_3

Then, M^{i_1}, M^{i_2}, and M^{i_3} are given as Figure 8.3, respectively.

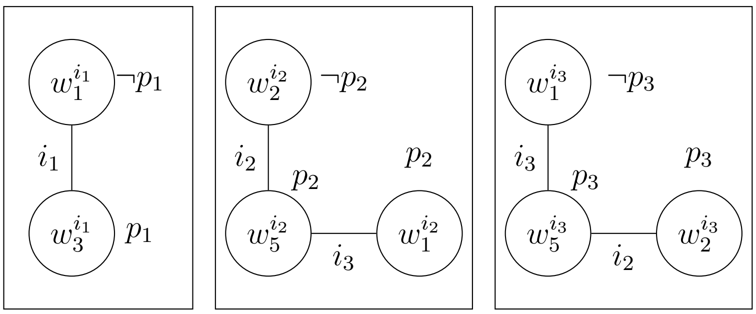

Figure 8.3: The Quotient Models M^{i_1}, M^{i_2}, and M^{i_3}

For example, $(M, w_5) \models_{ALGP} \neg\Box_{i_3}\neg\Box_{i_2} p_2$ and $(M^{i_2}, w_5^{i_2}) \models^{i_2}_{ALGP} \neg\Box_{i_3}\neg\Box_{i_2} p_2$.

Definition 8.1.10. (**b-equivalent of the quotient model** M^a)
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Let $M^a = \langle W^a, \sim^a, \mathscr{A}^a, V^a\rangle$ be the quotient model of a based on M. Then, agent expression b induces an equivalence relation $\approx_b^a$ on W^a: for all $w^a, v^a \in W^a$,

$$w^a \approx_b^a v^a \text{ iff } \text{ for all } \varphi \in \mathcal{L}_{ALGP}|\mathscr{A}^a(b), ((M^a, w^a) \models^a_{ALGP} \varphi \text{ iff } (M^a, v^a) \models^a_{ALGP} \varphi).$$

In other words, two equivalence classes are b-equivalent of the quotient model of a iff they satisfy the same formulas $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}^a(b)$.

Let $\mathcal{L}_{ALF}|Q$ mean the language $\mathcal{L}_{ALF}$ with the set of atomic propositions restricted to $Q \subseteq P$.

Definition 8.1.11. (**The Satisfaction Relation of $\mathcal{L}_{ALF}$ in Quotient Model** M^a)
Fix $a \in \mathcal{A}$. Given epistemic state (M^a, w^a) with $M^a = \langle W^a, \sim^a, \mathscr{A}^a, V^a\rangle$, $w^a \in W^a$. For all $\varphi \in \mathcal{L}_{ALF}|\mathscr{A}(a)$,

$$\begin{array}{lcl}
(M^a, w^a) \models^a \top \,; & & \\
(M^a, w^a) \models^a p & \iff & w^a \in V^a(p); \\
(M^a, w^a) \models^a \neg\varphi & \iff & (M^a, w^a) \not\models^a \varphi; \\
(M^a, w^a) \models^a \varphi \wedge \psi & \iff & (M^a, w^a) \models^a \varphi \text{ and } (M^a, w^a) \models^a \psi; \\
(M^a, w^a) \models^a A_i\varphi & \iff & At(\varphi) \subseteq \mathscr{A}^a(i); \\
(M^a, w^a) \models^a \Box_i\varphi & \iff & \text{for all } v^a \in W^a, w^a \sim_i^a v^a \text{ implies } (M^a, v^a) \models^a \varphi; \\
(M^a, w^a) \models^a [\approx]_b\varphi & \iff & \text{for all } v^a \in W^a, w^a \approx_b^a v^a \text{ implies } (M^a, v^a) \models^a \varphi.
\end{array}$$

By the definition, the satisfaction relation for $\bot, \vee, \to, \Diamond_i, K_i, A_b, \langle\approx\rangle_b$ is derived as follows: Given epistemic state (M^a, w^a) with $M^a = \langle W^a, \sim^a, \mathscr{A}^a, V^a\rangle$ and $w^a \in W^a$.

$$\begin{array}{lcl}
(M^a, w^a) \not\models^a \bot \,; & & \\
(M^a, w^a) \models^a \varphi \vee \psi & \iff & (M^a, w^a) \models^a \varphi \text{ or } (M^a, w^a) \models^a \psi; \\
(M^a, w^a) \models^a \varphi \to \psi & \iff & (M^a, w^a) \models^a \varphi \text{ implies } (M^a, w^a) \models^a \psi; \\
(M^a, w^a) \models^a \Diamond_i\varphi & \iff & \text{for some } v^a \in W^a, w^a \sim_i^a v^a \text{ and } (M^a, v^a) \models^a \varphi; \\
(M^a, w^a) \models^a K_i\varphi & \iff & (M^a, w^a) \models^a \Box_i\varphi \text{ and } (M^a, w^a) \models^a A_i\varphi; \\
(M^a, w^a) \models^a A_b\varphi & \iff & At(\varphi) \subseteq \mathscr{A}^a(b); \\
(M^a, w^a) \models^a \langle\approx\rangle_b\varphi & \iff & \text{for some } v^a \in W^a, w^a \approx_b^a v^a \text{ and } (M^a, v^a) \models^a \varphi.
\end{array}$$

We are motivated to share a mutually comprehensible model among different reasoning abilities. For this motivation, we introduce an abstract model called "*a nested quotient model*".

Definitions of an equivalence class, a nested quotient frame, a nested quotient model, and the satisfaction relation are the same as those of a quotient model, respectively. The denotations are listed as follows if we hope further abstraction by b's awareness of a quotient model $M^a = \langle W^a, \sim^a, \mathscr{A}^a, V^a \rangle$.:

- **Equivalence Class**: $(w^a)^b$ and $w^{a,b}$;
- **Nested Quotient Frame**: $(F^a)^b = \langle (W^a)^b, (\sim^a)^b \rangle$ and $F^{a,b} = \langle W^{a,b}, \sim^{a,b} \rangle$;
- **Nested Quotient Model**: $(M^a)^b = \langle (W^a)^b, (\sim^a)^b, (\mathscr{A}^a)^b, (V^a)^b \rangle$ and $M^{a,b} = \langle W^{a,b}, \sim^{a,b}, \mathscr{A}^{a,b}, V^{a,b} \rangle$;
- **Satisfaction Relation**: $\models^{a,b}_{ALGP}$

For example, $(w^a)^b$ is given in the same way as Definition 8.1.1.:

$$(w^a)^b := \{v^a \in W^a \mid w^a \approx^a_b v^a\}.$$

A nested quotient model is also given in the same way as Definition 8.1.5 and 8.1.6 if two $a, b \in \mathcal{A}$ and a quotient model $M^a = \langle W^a, \sim^a, \mathscr{A}^a, V^a \rangle$ of a are given.:

1. $(W^a)^b \; := \; \{w^{a,b} \mid w^a \in W^a\}$;
2. $(\sim^a)^b \; : \mathcal{G} \longrightarrow 2^{(W^a)^b \times (W^a)^b}$ where for each $i \in \mathcal{G}$, $(\sim^a)^b_i \; := \; \{(w^{a,b}, v^{a,b}) \mid (w^a, v^a) \in (\sim^a_i \circ \approx^a_b)^+, w^a, v^a \in W^a\}$.
3. $(\mathscr{A}^a)^b : \mathcal{G} \longrightarrow 2^{\mathscr{A}^a(b)}$ with $(\mathscr{A}^a)^b(i) := \mathscr{A}^a(i) \cap \mathscr{A}^a(b)$;
4. $(V^a)^b : \mathscr{A}^a(b) \longrightarrow 2^{(W^a)^b}$ with $(V^a)^b(p) := \{(w^a)^b \mid w^a \in V^a(p)\}$ for all $p \in \mathscr{A}^a(b)$.

The Satisfaction Relation of $\mathcal{L}_{ALGP}$ of $(M^a)^b$ is given for the truth of all formulas containing occurrences of $(\mathscr{A}^a)^b$ in the same way as Definition 8.1.11.

8.2 Product Quotient Model

The intuition that the product of models represents mutual complement is based on the idea of considering a Kripke model as a perspective view. For example, there is a cube and there are two agents who can only observe different projections of the cube. At this time, it is possible to grasp the perspective view of the cube by combining the information they have. When the Kripke model is considered as a perspective view of a cube, the quotient model by an agent's awareness can be considered as a projection of the cube that is visible to that agent. In this way, a mutual complement sometimes corresponds to the operation of the product of the quotient model.

Definition 8.2.1. (**Product Quotient Model**)
Given two agent expressions $a, b \in \mathcal{A}$, an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and two quotient model $M^a = \langle W^a, \sim^a, \mathscr{A}^a, V^a\rangle$ and $M^b = \langle W^b, \sim^b, \mathscr{A}^b, V^b\rangle$ of a and b based on M, respectively. The product quotient model of a and b based on M is a tuple $M^{a\otimes b} = \langle W^{a\otimes b}, \sim^{a\otimes b}, \mathscr{A}^{a\otimes b}, V^{a\otimes b}\rangle$ consisting of:

1. $W^{a\otimes b} \subseteq W^a \times W^b$ where $(w^a, v^b) \in W^{a\otimes b}$ iff

   $$\text{for all } \varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a \cdot b), (M^a, w^a) \models^a_{ALGP} \varphi \Leftrightarrow (M^b, v^b) \models^b_{ALGP} \varphi$$

2. $\sim^{a\otimes b} : \mathcal{G} \longrightarrow 2^{W^{a\otimes b} \times W^{a\otimes b}}$ with
   For all $w^a, s^a \in W^a$, $v^b, t^b \in W^b$ and $i \in \mathcal{G}$,
   $(w^a, v^b) \sim^{a\otimes b}_i (s^a, t^b)$ iff $w^a \sim^a_i s^a$ and $v^b \sim^b_i t^b$;

3. $\mathscr{A}^{a\otimes b} : \mathcal{G} \longrightarrow 2^{\mathscr{A}(a)\cup\mathscr{A}(b)}$ with
   $\mathscr{A}^{a\otimes b}(i) := \mathscr{A}^a(i) \cup \mathscr{A}^b(i)$ for all $i \in \mathcal{G}$;

4. $V^{a\otimes b} : \mathscr{A}(a) \cup \mathscr{A}(b) \longrightarrow 2^{W^{a\otimes b}}$ with
   $V^{a\otimes b}(p) := \{(w^a, v^b) \mid w^a \in V^a(p) \text{ or } v^b \in V^b(p)\}$ for all $p \in \mathscr{A}(a + b)$.

Definition 8.2.2. (**The Satisfaction Relation of $\mathcal{L}_{ALGP}$ in Product Quotient Model $M^{a\otimes b}$**)
Fix $a, b \in \mathcal{A}$. Given epistemic state $(M^{a\otimes b}, w^{a\otimes b})$ with $M^{a\otimes b} = \langle W^{a\otimes b}, \sim^{a\otimes b}$

$, \mathscr{A}^{a\otimes b}, V^{a\otimes b}\rangle$, $w^{a\otimes b} \in W^{a\otimes b}$. For all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a+b)$,

$$
\begin{aligned}
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \top \;; \\
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} p && \iff w^{a\otimes b} \in V^{a\otimes b}(p); \\
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \neg\varphi && \iff (M^{a\otimes b}, w^{a\otimes b})) \not\models^{a\otimes b}_{ALGP} \varphi; \\
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi \wedge \psi && \iff (M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi \\
& && \qquad \text{and } (M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \psi; \\
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} A_i\varphi && \iff At(\varphi) \subseteq \mathscr{A}^{a\otimes b}(i); \\
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \Box_i\varphi && \iff \text{for all } v^{a\otimes b} \in W^{a\otimes b}, w^{a\otimes b} \sim^{a\otimes b}_i v^{a\otimes b} \\
& && \qquad \text{implies } (M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi;
\end{aligned}
$$

By the definition, the satisfaction relation for $\bot, \vee, \rightarrow, \Diamond_i, K_i, A_c$ is derived as follows: Given epistemic state $(M^{a\otimes b}, w^{a\otimes b})$ with $M^{a\otimes b} = \langle W^{a\otimes b}, \sim^{a\otimes b}, \mathscr{A}^{a\otimes b}, V^{a\otimes b}\rangle$ and $w^{a\otimes b} \in W^{a\otimes b}$.

$$
\begin{aligned}
&(M^{a\otimes b}, w^{a\otimes b}) \not\models^{a\otimes b}_{ALGP} \bot \;; \\
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi \vee \psi && \iff (M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi \\
& && \qquad \text{or } (M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \psi; \\
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi \rightarrow \psi && \iff (M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi \\
& && \qquad \text{implies } (M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \psi; \\
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \Diamond_i\varphi && \iff \text{for some } v^{a\otimes b} \in W^{a\otimes b}, \\
& && \qquad w^{a\otimes b} \sim^{a\otimes b}_i v^{a\otimes b} \text{ and } (M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi; \\
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} K_i\varphi && \iff (M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \Box_i\varphi \\
& && \qquad \text{and } (M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} A_i\varphi; \\
&(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} A_c\varphi && \iff At(\varphi) \subseteq \mathscr{A}^{a\otimes b}(c);
\end{aligned}
$$

Next, we prove $\sim^{a\otimes b}_i$ is an equivalence relation for all $i \in \mathcal{G}$ and all $a, b \in \mathcal{A}$.

Theorem 8.2.3. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Then, $\sim^{a\otimes b}_i$ is an equivalence relation for all $i \in \mathcal{G}$.

Proof. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and an arbitrary agent $i \in \mathcal{G}$.

- (**Reflexive**) : Take an arbitrary world $(w^a, v^b) \in W^{a\otimes b}$. Since $w^a \sim^a_i w^a$ and $v^b \sim^b_i v^b$, $(w^a, v^b) \sim^{a\otimes b}_i (w^a, v^b)$. Since $(w^a, v^b) \in W^{a\otimes b}$ is an arbitrary world, $\sim^{a\otimes b}_i$ is reflexive.

- (**Transitive**) : Take three arbitrary worlds $(w_1^a, w_2^b), (v_1^a, v_2^b), (u_1^a, u_2^b) \in W^{a\otimes b}$. Suppose $(w_1^a, w_2^b) \sim_i^{a\otimes b} (v_1^a, v_2^b)$ and $(v_1^a, v_2^b) \sim_i^{a\otimes b} (u_1^a, u_2^b)$. Then, $w_1^a \sim_i^a v_1^a$ and $v_1^a \sim_i^a u_1^a$. Since $\sim_i^a$ is transitive, $w_1^a \sim^a u_1^a$. Similarly, $w_2^b \sim^a u_2^b$. By the definition of $\sim_i^{a\otimes b}$, $(w_1^a, w_2^a) \sim_i^{a\otimes b} (u_1^a, u_2^b)$. Since $(w_1^a, w_2^b), (v_1^a, v_2^b), (u_1^a, u_2^b) \in W^{a\otimes b}$ are arbitrary worlds, $\sim_i^{a\otimes b}$ is transitive.

- (**Symmetric**) : Take three arbitrary worlds $(w_1^a, w_2^b), (v_1^a, v_2^b) \in W^{a\otimes b}$. Suppose $(w_1^a, w_2^b) \sim_i^{a\otimes b} (v_1^a, v_2^b)$. By the definition of $\sim_i^{a\otimes b}$, $w_1^a \sim_i^a v_1^a$ and $w_2^b \sim_i^b v_2^b$. Since $\sim_i^a$ and $\sim_i^b$ are symmetric, $v_1^a \sim_i^a w_1^a$ and $v_2^b \sim_i^b w_2^b$. By the definition of $\sim_i^{a\otimes b}$, $(v_1^a, v_2^b) \sim_i^{a\otimes b} (w_1^a, w_2^b)$. Since $(w_1^a, w_2^b), (v_1^a, v_2^b) \in W^{a\otimes b}$ are two arbitrary worlds, $\sim_i^{a\otimes b}$ is symmetric.

Since $i \in \mathcal{G}$ is an arbitrary agent, $\sim_i^{a\otimes b}$ is an equivalence relation for all $i \in \mathcal{G}$. □

Theorem 8.2.4. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, three arbitrary worlds $w, w', v \in W$, and an agent $i \in \mathcal{G}$. Suppose M is image-finite. If $w \sim_i v$ and $w \approx_a w'$, then there is a world $v' \in W$ such that $w' \sim_i v'$ and $v \approx_a v'$.

Proof. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, an agent $i \in \mathcal{G}$ and two worlds $w, v \in W$ such that $w \sim_i v$. Suppose M is image-finite. By the Hennessy–Milner style contradiction. Suppose there is no $v' \in v^a$ with $w' \sim_i v'$ for some $w' \in w^a$. Take an arbitrary $w' \in w^a$ such that $w' \not\sim_i v'$ for all $v' \in v^a$. Let n be a positive integer such that $n = \#\{v'' \mid w' \sim_i v''\}$ and $v'_k \in \{v'' \mid w' \sim_i v''\}$ be a possible world for each positive integer $k \leq n$. By the assumption, $v'_k \notin v^a$. Then, there is a $\varphi_k \in \mathcal{L}_{ALGP}|\mathscr{A}(a)$ such that $(M, v) \models \varphi_k$ and $(M, v'_k) \not\models \varphi_k$ for every positive integer $k \leq n$. Since M is image-finite, $\Diamond_i(\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n)$ is a formula. Then, $(M, w) \models \Diamond_i(\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n)$. and $(M, w') \not\models \Diamond_i(\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n)$. Since $w \approx_a w'$, this is a contradiction. Therefore, If $w \sim_i v$, then for all $w' \in w^a$, there is a world $v' \in W$ such that $w' \sim_i v'$ and $v \approx_a v'$. □

Theorem 8.2.5. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, two arbitrary worlds $w, v \in W$, and an agent $i \in \mathcal{G}$. If $\mathscr{A}(b) \subseteq \mathscr{A}(a)$, then

$$w^a \sim_i^a v^a \text{ implies } w^b \sim_i^b v^b$$

Proof. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and an agent $i \in \mathcal{G}$. Take two arbitrary worlds $w, v \in W$. Suppose $\mathscr{A}(b) \subseteq \mathscr{A}(a)$ and $w^a \sim_i^a v^a$. By the definition of $\sim_i^a$, $(w, v) \in (\sim_i \circ \approx_a)^+$. By the lemma 5.3.1, $(w, v) \in (\sim_i \circ \approx_b)^+$. Then, $w^b \sim_i^b v^b$. □

Theorem 8.2.6. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, two arbitrary worlds $w, v \in W$, and an agent $i \in \mathcal{G}$. Suppose $\mathscr{A}(a) \subseteq \mathscr{A}(b)$ and M is image-finite. If $w_1^a \sim_i^a v_1^a$, then for all $w_2^b \subseteq w_1^a$, there is a $v_2^b \subseteq v_1^a$ such that $w_2^b \sim_i^b v_2^b$.

Proof. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and an agent $i \in \mathcal{G}$. Suppose $\mathscr{A}(a) \subseteq \mathscr{A}(b)$ and M is image-finite. Take three arbitrary worlds $w_1, w_2 \in W$ such that $w_2^b \subseteq w_1^a$.

Take arbitrary worlds $w_{1.1}, w_{1.2}, w_{1.3}, ... \in W$ such that $w_1 \approx_a w_{1.1}$, $w_{1.2k+1} \sim_i w_{1.2k+2}$, and $w_{1.2k+2} \approx_a w_{1.2k+3}$ for all $k \geq 0$.

- **Base Case**: Take an arbitrary $w_{2.1} \in w_2^b$. By the theorem 8.2.4 and $w_{1.1} \sim_i w_{1.2}$, there is $w_{2.2} \in w_{1.2}^a$ such that $w_{2.1} \sim_i w_{2.2}$. Then, there is $w_{2.2}^b \subseteq w_{1.2}^a$ such that $w_2^b \sim_i^b w_{2.2}^b$.
- **Induction Hypothesis**: There is $w_{2.2k}^b \subseteq w_{1.2k}^a$ such that $w_2^b \sim_i^b w_{2.2k}^b$ for all $k \in \mathbb{N}$ such that $1 \leq k \leq n$.
- **Induction Step**: Take an arbitrary $w_{2.2k+1} \in w_{2.2k}^b$. By the theorem 8.2.4 and $w_{1.2k+1} \sim_i w_{1.2k+2}$, there is $w_{2.2k+2} \in w_{1.2k+2}^a$ such that $w_{2.2k+1} \sim_i w_{2.2(k+1)}$. By $(w_{2.2k}, w_{2.2(k+1)}) \in (\sim_i \circ \approx_a)$, $w_2^b \sim_i^b w_{2.2(k+1)}^b$.

Since $w_{1.1}, w_{1.2}, w_{1.3}, ... \in W$ are arbitrary worlds such that $w_1 \approx_a w_{1.1}$, $w_{1.2k+1} \sim_i w_{1.2k+2}$, and $w_{1.2k+2} \approx_a w_{1.2k+3}$ for all $k \geq 0$, for all v_1^a such that $w_1 \sim_i^a v_1^a$, there is $v_2^b \subseteq v_1^a$ such that $w_2^b \sim_i^b v_2^b$. □

Lemma 8.2.7. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, three arbitrary worlds $w_1, w_2, v_1 \in W$, and an agent $i \in \mathcal{G}$. Suppose M is image-finite. If $(w_1^a, w_2^b) \in W^{a\otimes b}$ and $w_1^a \sim_i^a v_1^a$, then there is $v_2^b \subseteq v_1^{a\cdot b}$ such that $(w_1^a, w_2^b) \sim_i^{a\otimes b} (v_1^a, v_2^b)$.

Proof. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, an agent $i \in \mathcal{G}$ and three worlds $w_1, w_2, v_1 \in W$ such that $(w_1^a, w_2^b) \in W^{a\otimes b}$ and $w_1^a \sim_i^a v_1^a$. Suppose M is image-finite.

By the lemma 8.2.5, $w_1^{a\cdot b} \sim_i^{a\cdot b} v_1^{a\cdot b}$. Since M is image-finite and the lemma 8.2.6, there is $v_2^b \subseteq v_1^{a\cdot b}$ such that $w_2^b \sim_i^b v_2^b$. By the definition of $\sim_i^{a\otimes b}$, $(w_1^a, w_2^b) \sim_i^{a\otimes b} (v_1^a, v_2^b)$. Therefore, there is $v_2^b \subseteq v_1^{a\cdot b}$ such that $(w_1^a, w_2^b) \sim_i^{a\otimes b} (v_1^a, v_2^b)$. □

Theorem 8.2.8.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. If W is a image-finite set, then for all $w^a \in W^a$, all $v^b \in W^b$ such that $(w^a, v^b) \in W^{a\otimes b}$,

for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a), ((M^a, w^a) \models_{ALGP}^a \varphi \iff (M^{a\otimes b}, (w^a, v^b)) \models_{ALGP}^{a\otimes b} \varphi)$.

Proof.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Suppose M is image-finite.

- **Base Case**: Take an arbitrary atomic proposition $p \in \mathscr{A}(a)$. Suppose that there is a $w^a \in W^a$ such that $(w^a, v^b) \in V^{a\otimes b}(p)$ and $w^a \notin V^a(p)$. By the definition of $V^{a\otimes b}$, $v^b \in V^b(p)$. Then, $p \in \mathscr{A}(a) \cap \mathscr{A}(b)$. By the definition of $W^{a\otimes b}$ and $p \in \mathscr{A}(a) \cap \mathscr{A}(b)$, $(M^a, w^a) \models^a_{ALGP} p \Leftrightarrow (M^b, w^b) \models^b p$. Then, $w^a \in V^a(p)$. This is a contradiction. By the definition of $V^{a\otimes b}$, there is no $(w^a, v^b) \in W^{a\otimes b}(p)$ such that $(w^a, v^b) \notin V^{a\otimes b}(p)$ and $w^a \in V^a(p)$.
  Thus, $(w^a, v^b) \in V(p)$ if and only if $w^a \in V(p)$.

- **Induction Hypothesis**: $(M^a, w^a) \models^a_{ALGP} \varphi \iff (M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi$ for all $v^b \in W^b$.

- **Induction Step**: Take an arbitrary $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a)$ and $w^a \in W^a$ such that $(M^a, w^a) \models^a_{ALGP} \varphi \iff (M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi$ for all $v^b \in W^b$.

  - (**The case for** $\varphi \wedge \psi$) : By the semantics, $(M^a, w^a) \models^a_{ALGP} \varphi \wedge \psi \Leftrightarrow (M^a, w^a) \models^a_{ALGP} \varphi$ and $(M^a, w^a) \models^a_{ALGP} \psi$. By the induction hypothesis, $(M^a, w^a) \models^a_{ALGP} \varphi$ and $(M^a, w^a) \models^a_{ALGP} \psi$ if and only if $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi$ and $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \psi$. By the semantics, $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi \wedge \psi$.
  - (**The case for** $\neg\varphi$) : By the semantics, $(M^a, w^a) \models^a_{ALGP} \neg\varphi \Leftrightarrow (M^a, w^a) \not\models^a_{ALGP} \varphi$. By the induction hypothesis, $(M^a, w^a) \not\models^a_{ALGP} \varphi \Leftrightarrow (M^{a\otimes b}, (w^a, v^b)) \not\models^{a\otimes b}_{ALGP} \varphi$. By the semantics, $(M^{a\otimes b}, (w^a, v^b)) \not\models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \neg\varphi$. Then, $(M^a, w^a) \models^a_{ALGP} \neg\varphi \Leftrightarrow (M^{a\otimes a}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \neg\varphi$.
  - (**The case for** $A_c\varphi$) : By the semantics, $(M^a, w^a) \models^a_{ALGP} A_c\varphi$ if and only if $At(\varphi) \subseteq \mathscr{A}^a(c)$. Since $\mathscr{A}^a(c) = \mathscr{A}(a) \cap \mathscr{A}(c)$, $At(\varphi) \subseteq \mathscr{A}^a(c)$ if and only if $At(\varphi) \subseteq \mathscr{A}^{a\otimes b}(c)$ for all φ such that $At(\varphi) \subseteq \mathscr{A}(a)$.
    Then, $(M^a, w^a) \models^a_{ALGP} A_c\varphi$ if and only if $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} A_c\varphi$.
  - (**The case for** $\Box_i\varphi$) :
    - ($\Rightarrow$):
      Suppose that $(M^a, w^a) \models^a_{ALGP} \Box_i\varphi$.
      Suppose that there is a $(s^a, t^b) \in W^{a\otimes b}$ such that $(w^a, v^b) \sim^{a\otimes b}_i (s^a, t^b)$ and $(M^{a\otimes b}, (s^a, t^b)) \not\models^{a\otimes b}_{ALGP} \varphi$. By the definition of

$\sim^{a\otimes b}$, $w^a \sim_i^a s^a$. By the induction hypothesis, $(M^a, s^a) \not\models_{ALGP}^a \varphi$. By the semantics, $(M^a, w^a) \models_{ALGP}^a \neg\Box_i\varphi$. This is a contradiction. Then, there is no $(s^a, t^b) \in W^{a\otimes b}$ such that $(w^a, v^b) \sim_i^{a\otimes b} (s^a, t^b)$ and $(M^{a\otimes b}, (s^a, t^b)) \not\models_{ALGP}^{a\otimes b} \varphi$. Then, $(M^{a\otimes b}, (w^a, v^b)) \models_{ALGP}^{a\otimes b} \Box_i\varphi$.

* $(\Leftarrow)$:
  By the contraposition. Suppose $(M^a, w^a) \models^a \neg\Box_i\varphi$. Then, there is a $s^a \in W^a$ where $w^a \sim_i^a s^a$ and $(M^a, s^a) \not\models_{ALGP}^a \varphi$. By the lemma 8.2.7, there is a $t^a \in W^b$ such that $(w^a, v^b) \sim_i^{a\otimes b} (s^a, t^b)$. By the induction hypothesis, $(M^{a\otimes b}, (s^a, t^b)) \models_{ALGP}^{a\otimes b} \neg\varphi$. So, $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b} \neg\Box_i\varphi$.

□

Chapter 9

Reduction to Quotient Model with Agent Expression

9.1 Reduction of Nested Quotient Model by Common Awareness

We introduced the definition of a nested quotient model by transforming a quotient model. However, we want to represent nested abstractions in the semantics defined by the original epistemic awareness model. For this reason, this section shows a reduction from a nested quotient model to a quotient model.

Theorem 9.1.1. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. For all $w, v \in W$,

$$w \approx_{a \cdot b} v \iff w^a \approx_b^a v^a$$

Proof. Given two agent expressions $a, b \in \mathcal{A}$ and any epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Take two arbitrary worlds $w, v \in W$.

By the definition of $\approx_{a \cdot b}$, $w \approx_{a \cdot b} v$ if and only if $(M, w) \models_{ALGP} \varphi \Leftrightarrow (M, v) \models_{ALGP} \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a \cdot b)$. By the definition of $\mathscr{A}$ for agent expressions, $\mathscr{A}(a \cdot b) = \mathscr{A}(a) \cap \mathscr{A}(b)$. By the definition of $\mathscr{A}^a$, $\mathscr{A}(a) \cap \mathscr{A}(b) = \mathscr{A}^a(b)$. Then, $w \approx_{a \cdot b} v$ if and only if $(M, w) \models_{ALGP} \varphi \Leftrightarrow (M, v) \models_{ALGP} \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}^a(b)$.

Take an arbitrary $\varphi \in \mathcal{L}_{ALGP}$ such that $At(\varphi) \subseteq \mathscr{A}^a(b)$. By the theorem 8.1.8, $(M, w) \models_{ALGP} \varphi \Leftrightarrow (M^a, w^a) \models^a_{ALGP} \varphi$ and $(M, v) \models_{ALGP} \varphi \Leftrightarrow (M^a, v^a) \models^a_{ALGP} \varphi$. Then, $(M, w) \models_{ALGP} \varphi \Leftrightarrow (M, v) \models_{ALGP} \varphi$ if and only if $(M^a, w^a) \models^a_{ALGP} \varphi \Leftrightarrow (M^a, v^a) \models^a_{ALGP} \varphi$. By the definition of $\approx_b^a$, $(M, w) \models_{ALGP} \varphi \Leftrightarrow (M, v) \models_{ALGP} \varphi$ if and only if $w^a \approx_b^a v^a$. Since $w, v \in W$ are arbitrary worlds, $w \approx_{a \cdot b} v$ if and only if $w^a \approx_b^a v^a$ for all $w, v \in W$. □

Correspondence of Cardinality

Lemma 9.1.2.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Then,

$$\# W^{a,b} \ = \ \# W^{a \cdot b}.$$

Proof. By the lemma 9.1.1, $w \approx_{a \cdot b} v \Leftrightarrow w^a \approx_b^a v^a$ for all $w, v \in W$. Then, $s \in w^{a \cdot b} \Leftrightarrow s \in s^a \in w^{a,b}$ for all $s, w \in W$. Therefore, $\# W^{a,b} = \# W^{a \cdot b}$. $\square$

Correspondence of Relation

Lemma 9.1.3. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. For all $i \in \mathcal{G}$ and all $w, v \in W$,

$$w^{a,b} \sim_i^{a,b} v^{a,b} \ \Leftrightarrow \ w^{a \cdot b} \sim_i^{a \cdot b} v^{a \cdot b}.$$

Proof.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Take an arbitrary $i \in \mathcal{G}$ and two arbitrary worlds $w, v \in W$.

- $(\Rightarrow)$: Suppose that $w^{a,b} \sim_i^{a,b} v^{a,b}$. By the definition of $\sim_i^{a,b}$, $(w^a, v^a) \in (\sim_i^a \circ \approx_b^a)^+$. By the lemma 9.1.1 and definition of $\sim_i^a$, $(w, v) \in ((\sim_i \circ \approx_a)^+ \circ \approx_{a \cdot b})^+$. Then, $(w, v) \in (\sim_i \circ \approx_{a \cdot b})^+$. Therefore, $w^{a \cdot b} \sim_i^{a \cdot b} v^{a \cdot b}$.

- $(\Leftarrow)$: Suppose that $w^{a \cdot b} \sim_i^{a \cdot b} v^{a \cdot b}$. By the definition of $\sim_i^{a \cdot b}$, $(w, v) \in (\sim_i \circ \approx_{a \cdot b})^+$. Then, $(w, v) \in ((\sim_i \circ \approx_a)^+ \circ \approx_{a \cdot b})^+$. By the lemma 9.1.1 and definition of $\sim_i^a$, $(w^a, v^a) \in (\sim_i^a \circ \approx_b^a)^+$. By the definition of $\sim_i^{a,b}$, $w^{a,b} \sim_i^{a,b} v^{a,b}$.

Since $i \in \mathcal{G}$ and $w, v \in W$ are arbitrary, for all $i \in \mathcal{G}$ and all $w, v \in W$, $w^{a,b} \sim_i^{a,b} v^{a,b} \Leftrightarrow w^{a \cdot b} \sim_i^{a \cdot b} v^{a \cdot b}$. $\square$

Correspondence of Truth

Theorem 9.1.4.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. For all $w \in W$,

$$\text{for all } \varphi \in \mathcal{L}_{ALGP} | \mathscr{A}^a(b), ((M^a, w^a) \models_{ALGP}^a \varphi \iff (M^{a,b}, w^{a,b}) \models_{ALGP}^{a,b} \varphi).$$

Proof. The proof is similar to the proof of the theorem 8.1.8. $\square$

Theorem 9.1.5.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. For all $w \in W$,

$$\text{for all } \varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a \cdot b), ((M, w) \models_{ALGP} \varphi \iff ((M^a)^b, w^{a,b}) \models^{a,b}_{ALGP} \varphi).$$

Proof. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. By the theorem 8.1.8, $(M, w) \models_{ALGP} \varphi \Leftrightarrow (M^a, w^a) \models^a_{ALGP} \varphi$ for all $w \in W$ and all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a \cdot b)$.

By the theorem 9.1.4, $(M^a, w^a) \models^a_{ALGP} \varphi \Leftrightarrow (M^{a,b}, w^{a,b}) \models^{a,b} \varphi$ for all $w \in W$ and all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}^a(b)$.

Since $\mathscr{A}^a(b) = \mathscr{A}(a) \cap \mathscr{A}(b)$, $(M, w) \models_{ALGP} \varphi \Leftrightarrow (M^{a,b}, w^{a,b}) \models^{a,b} \varphi$ for all $w \in W$ and all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a \cdot b)$. □

Existence of Isomorphism

Now, $M \cong M'$ represents that there is an isomorphism between two models M and M'.

Theorem 9.1.6. (**Isomorphism between Quotient Model and Nested Quotient Model**)
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Then,

$$(M^a)^b \cong M^{a \cdot b}$$

Proof. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. It is shown that there is an isomorphism f from $(W^a)^b$ to $W^{a \cdot b}$ with

1. $w^{a,b} \sim_i v^{a,b} \Leftrightarrow f(w^{a,b}) \sim^{a \cdot b}_i f(v^{a,b})$,
2. $(M^{a,b}, w^{a,b}) \models^{a,b} \varphi \Leftrightarrow (M^{a \cdot b}, f(w^{a,b})) \models^{a \cdot b} \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a \cdot b)$.

Let $f : (W^a)^b \longrightarrow W^{a \cdot b}$ be a function with $f(w^{a,b}) = w^{a \cdot b}$ for each $w \in W$. By the lemma 9.1.5, $(M^{a,b}, w^{a,b}) \models^{a,b} \varphi \Leftrightarrow (M^{a \cdot b}, f(w^{a,b})) \models^{a \cdot b} \varphi$ for all $w \in W$ and all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a \cdot b)$. By the lemma 9.1.3, $w^{a,b} \sim_i v^{a,b} \Leftrightarrow f(w^{a,b}) \sim^{a \cdot b}_i f(v^{a,b})$ for all $i \in \mathcal{G}$ and all $w, v \in W$.
By the lemma 9.1.2, f is a bijection function. Since f is an isomorphism from $(W^a)^b$ to $W^{a \cdot b}$, $(M^a)^b \cong M^{a \cdot b}$. □

Example 9.1.7. There are three children i_1, i_2, i_3. Let p_1, p_2 and p_3 be atomic propositions standing for "*agents i_1, i_2, and i_3 have mud on their faces*", respectively. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ consisting of (See Figure 9.1):

1. $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$,

2. $\sim_{i_1} = \{(w,v) \mid w, v \in \{w_1, w_3\}\} \cup \{(w,v) \mid w, v \in \{w_2, w_4\}\} \cup \{(w,v) \mid w, v \in \{w_5, w_6\}\}$,
   $\sim_{i_2} = \{(w,v) \mid w, v \in \{w_2, w_5\}\} \cup \{(w,v) \mid w, v \in \{w_4, w_6\}\} \cup \{(w,w) \mid w \in W\}$,
   $\sim_{i_3} = \{(w,v) \mid w, v \in \{w_1, w_5\}\} \cup \{(w,v) \mid w, v \in \{w_3, w_6\}\} \cup \{(w,w) \mid w \in W\}$,
   $\sim_e = \{(w,v) \mid w, v \in W\}$,

3. $\mathscr{A}(i_1) = \{p_1, p_2\}$, $\mathscr{A}(i_2) = \{p_2, p_3\}$, $\mathscr{A}(i_3) = \mathscr{A}(e) = \{p_1, p_2, p_3\}$,

4. $V(p_1) = \{w_3, w_4, w_6\}$, $V(p_2) = \{w_1, w_3, w_5, w_6\}$, $V(p_3) = \{w_2, w_4, w_5, w_6\}$.

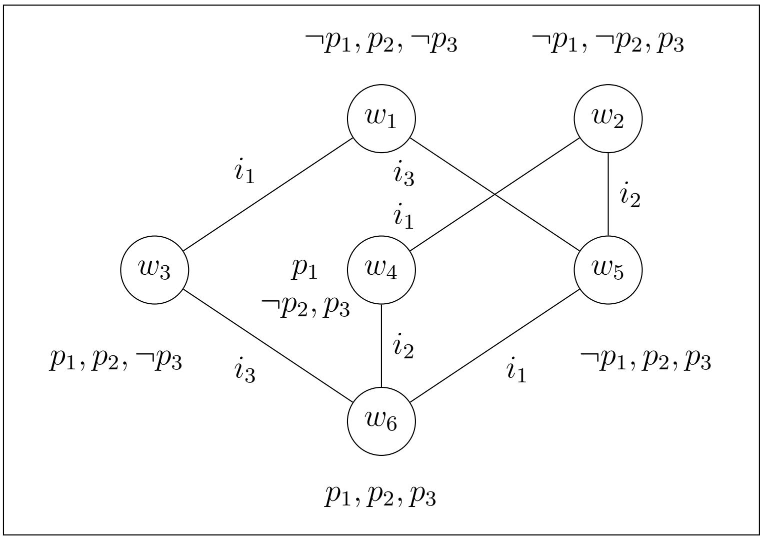

Figure 9.1: Kripke model represented knowledge of i_1, i_2, and i_3(Repeat)

Then, M^{i_1} is given as the Figure 9.2.

Since $\mathscr{A}^{i_1}(i_2) = \{p_2\}$, $(M^{i_1})^{i_2}$ is given as Figure 9.3.

Since $\mathscr{A}(i_1) = \{p_1, p_2\}$ and $\mathscr{A}(i_2) = \{p_2, p_3\}$, then $\mathscr{A}(i_1 \cdot i_2) = \{p_2\}$. Then $M^{i_1 \cdot i_2}$ is given as Figure 9.4.

Obviously, $(M^{i_1})^{i_2} \cong M^{i_1 \cdot i_2}$.

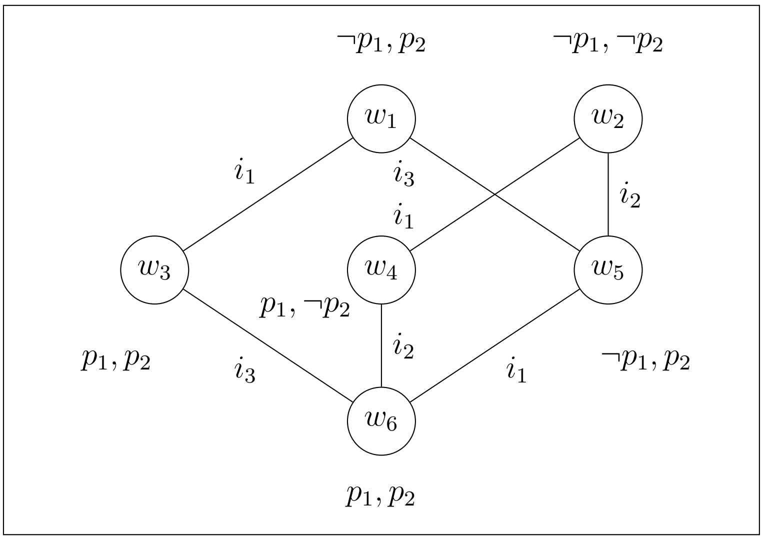

Figure 9.2: The Quotient Model M^{i_1}

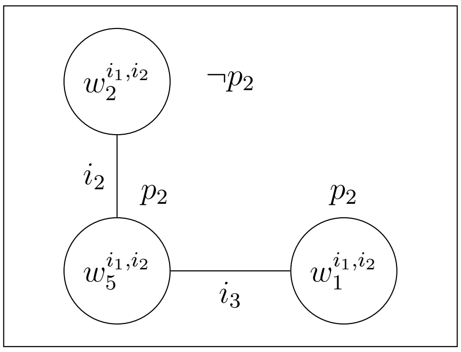

Figure 9.3: The Nested Quotient Model $(M^{i_1})^{i_2}$

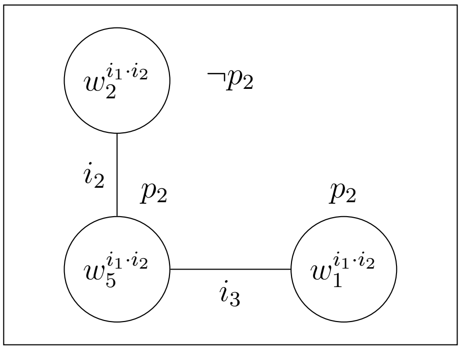

Figure 9.4: The Quotient Model $M^{i_1 \cdot i_2}$

9.2 Reduction of Product Quotient Model by Distributed Awareness

The section 8.2 introduced the definition of a product quotient model by transforming two quotient models. However, we want to represent mutually complementary concretizations in the semantics defined by the original epistemic awareness model. For this reason, this section shows a reduction from a product quotient model to a quotient model.

Correspondence of Cardinality

Lemma 9.2.1.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Then, for all $w, v \in W$, if $w^a \cap v^b \neq \varnothing$, then $(w^a, v^b) \in W^{a \otimes b}$.

Proof.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$ and two worlds $w, v \in W$. Suppose $w^a \cap v^b \neq \varnothing$. Take an arbitrary $s \in w^a \cap v^b$.

By the theorem 8.1.8, $(M^a, s^a) \models^a_{ALGP} \varphi \Leftrightarrow (M, s) \models \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a)$, and $(M^b, s^b) \models^b \psi \Leftrightarrow (M, s) \models \psi$ for all $\psi \in \mathcal{L}_{ALGP}|\mathscr{A}(b)$. Then, $(M^a, s^a) \models^a_{ALGP} \varphi \Leftrightarrow (M, s) \models \varphi$ and $(M^b, s^b) \models^b \varphi \Leftrightarrow (M, s) \models \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}|(\mathscr{A}(a) \cap \mathscr{A}(b))$. Thus, $(M^a, s^a) \models^a_{ALGP} \varphi \Leftrightarrow (M^b, s^b) \models^b \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a \cdot b)$. Therefore, $(s^a, s^b) \in W^{a \otimes b}$. Since $s \in w^a \cap v^b$, $s^a = w^a$ and $s^b = v^b$. Then, $(w^a, v^b) \in W^{a \otimes b}$. □

Theorem 9.2.2.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Let W' be a set of paired equivalence classes such that $W' = \{(w^a, v^b) \in W^{a \otimes b} \mid$

$w^a \cap v^b = \varnothing, w, v \in W\}$.
If $w^a \cap w^b = w^{a+b}$ for all $w \in W$, then

$$\#W^{a+b} \quad = \quad \#W^{a\otimes b} - \#W'.$$

Proof.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Suppose that $w^a \cap w^b = w^{a+b}$ for all $w \in W$.

- $\#W^{a+b} \leq \#W^{a\otimes b} - \#W'$:
  Take an arbitrary $s^{a+b} \in W^{a+b}$. By the theorem 8.1.2, $s^{a+b} \subseteq w^a, v^b$. Then, $s \in W^a \cap v^b$. By the lemma 9.2.1, $(w^a, v^b) \in W^{a\otimes b}$. Then, for all $s^{a+b} \in W^{a+b}$, there is just one $(w^a, v^b) \in W^{a\otimes b}$ such that $s^{a+b} \subseteq w^a, v^b$. Therefore, $\#W^{a+b} \leq \#W^{a\otimes b} - \#W'$.
- $\#W^{a+b} \geq \#W^{a\otimes b} - \#W'$:
  Take an arbitrary $(w^a, v^b) \in W^{a\otimes b}$ such that $w^a \cap v^b \neq \varnothing$. Take an arbitrary $s \in w^a \cap v^b$. By $w^a = s^a$ and $v^b = s^b$ and $s^a \cap s^b = s^{a+b}$, $s^{a+b} = w^a \cap v^b$. Since $s \in s^{a+b}$ is an arbitrary, for all $t \in w^a \cap v^b$, $s^{a+b} = t^{a+b}$. Then, for all $(w^a, v^b) \in W^{a\otimes b}$, there is just one $s^{a+b} \in W^{a+b}$ such that $s^{a+b} = w^a \cap v^b$.Therefore, $\#W^{a+b} \geq \#W^{a\otimes b} - \#W'$.

Thus, $\#W^{a+b} = \#W^{a\otimes b} - \#W'$. □

Correspondence of Relation

Lemma 9.2.3. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, an agent $i \in \mathcal{G}$, and two arbitrary worlds $s, t \in W$. If $s^{a+b} \sim_i^{a+b} t^{a+b}$, then $(w_1^a, v_1^b) \sim_i^{a\otimes b} (w_2^a, v_2^b)$ for all $w_1, w_2, v_1, v_2 \in W$ such that $s^{a+b} \subseteq w_1^a \cap v_1^b$ and $t^{a+b} \subseteq w_2^a \cap v_2^b$.

Proof. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, an agent $i \in \mathcal{G}$, and two arbitrary worlds $s, t \in W$. Suppose $s^{a+b} \sim_i^{a+b} t^{a+b}$.

By the lemma 8.2.5, $s^a \sim_i^a t^a$ and $s^b \sim_i^b t^b$. Take arbitrary worlds $w_1, w_2, v_1, v_2 \in W$ such that $s^{a+b} \subseteq w_1^a \cap v_1^b$ and $t^{a+b} \subseteq w_2^a \cap v_2^b$. Then, $w_1^a \sim_i^a w_2^a$ and $v_1^b \sim_i^b v_2^b$. By the definition of $\sim_i^{a\otimes b}$, $(w_1^a, v_1^b) \sim^{a\otimes b} (w_2^a, v_2^b)$. □

Lemma 9.2.4. Fix $a \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, an agent $i \in \mathcal{G}$, and arbitrary worlds $w_1, w_2, v_1, v_2 \in W$. Suppose M is image-finite. If $(w_1^a, v_1^b) \sim^{a\otimes b} (w_2^a, v_2^b)$, then for all $s^{a+b} \subseteq w_1^a \cap v_1^b$, there is $t^{a+b} \subseteq w_2^a \cap v_2^b$ such that $s^{a+b} \sim_i^{a+b} t^{a+b}$.

Proof. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$, an agent $i \in \mathcal{G}$, and arbitrary worlds $w_1, w_2, v_1, v_2 \in W$ such that $(w_1^a, v_1^b), (w_2^a, v_2^b) \in W^{a\otimes b}$ and $(w_1^a, v_1^b) \sim^{a\otimes b} (w_2^a, v_2^b)$. Then, $w_1^a \sim_i^a w_2^a$ and $v_1^b \sim_i^b v_2^b$. Suppose M is image-finite. Take an arbitrary $s^{a+b} \subseteq w_1^a \cap v_1^b$. By the lemma 8.2.6, there is $t^{a+b} \subseteq w_2^a \cap v_2^b$ such that $s^{a+b} \sim_i^{a+b} t^{a+b}$. Since $s^{a+b} \subseteq w_1^a \cap v_1^b$ is arbitrary world, for all $s^{a+b} \subseteq w_1^a \cap v_1^b$, there is $t^{a+b} \subseteq w_2^a \cap v_2^b$ such that $s^{a+b} \sim_i^{a+b} t^{a+b}$. □

Lemma 9.2.5. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and an agent $i \in \mathcal{G}$. Suppose M is image-finite. Suppose that

1. for all $w, v \in W$, if $(w^a, v^b) \in W^{a\otimes b}$, then $w^a \cap v^b \neq \varnothing$, and
2. for all $w \in W$, $w^a \cap w^b = w^{a+b}$.

Then, for all $w_1, w_2, v_1, v_2 \in W$,

$$(w_1^a, w_2^b) \ \sim_i^{a\otimes b} \ (v_1^a, v_2^b) \iff w_1^a \cap w_2^b \sim_i^{a+b} v_1^a \cap v_2^b.$$

Proof. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Suppose that

1. M is image-finite,
2. for all $w, v \in W$, if $(w^a, v^b) \in W^{a\otimes b}$, then $w^a \cap v^b \neq \varnothing$, and
3. for all $w \in W$, $w^a \cap w^b = w^{a+b}$. Take arbitrary worlds $w_1, w_2, v_1, v_2 \in W$.

- ($\Rightarrow$):
  Suppose that $(w_1^a, w_2^b) \sim_i^{a\otimes b} (v_1^a, v_2^b)$. Then, $w_1^a \sim_i^a v_1^a$ and $w_2^b \sim_i^b v_2^b$. By the lemma 9.2.4, for all $s^{a+b} \subseteq w_1^a \cap w_2^b$, there is $t^{a+b} \subseteq v_1^a \cap v_2^b$ such that $s^{a+b} \sim_i^{a+b} t^{a+b}$. By $s^{a+b} = s^a \cap s^b = w_1^a \cap w_2^b$ and $t^{a+b} = t^a \cap t^b = v_1^a \cap v_2^b$, $w_1^a \cap w_2^b \sim_i^{a+b} v_1^a \cap v_2^b$.
- ($\Leftarrow$):
  Suppose that $w_1^a \cap w_2^b \sim_i^{a+b} v_1^a \cap v_2^b$. Take arbitrary worlds $s, t \in W$ such that $s^{a+b} \subseteq w_1^a \cap w_2^b$ and $t^{a+b} \subseteq v_1^a \cap v_2^b$. By $s^{a+b} = s^a \cap s^b = w_1^a \cap w_2^b$ and $t^{a+b} = t^a \cap t^b = v_1^a \cap v_2^b$, $s^{a+b} \sim_i^{a+b} t^{a+b}$. By the lemma 8.2.5, $s^a \sim_i^a t^a$ and $s^b \sim_i^b t^b$. By the definition of $\sim_i^{a\otimes b}$, $(s^a, s^b) \sim_i^{a\otimes b} (t^a, t^b)$. By $s^{a+b} \subseteq w_1^a \cap w_2^b$ and $t^{a+b} \subseteq v_1^a \cap v_2^b$, $(w_1^a, w_2^b) \sim_i^{a\otimes b} (v_1^a, v_2^b)$.

Therefore, $(w_1^a, w_2^b) \sim_i^{a\otimes b} (v_1^a, v_2^b)$ if and only if $w_1^a \cap w_2^b \sim_i^{a+b} v_1^a \cap v_2^b$ for all $w_1, w_2, v_1, v_2 \in W$. □

Correspondence of Truth

Lemma 9.2.6.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and two worlds $w, v \in W$. If M is a image-finite set and $w^a \cap v^b \neq \varnothing$, then for all $s \in w^a \cap v^b$,

for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a) \cup \mathcal{L}_{ALGP}|\mathscr{A}(b)$,
$$((M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M, s) \models \varphi).$$

Proof.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$ and two worlds $w, v \in W$. Suppose M is image-finite.

By the theorem 8.2.8,

$(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M^a, w^a) \models^{a}_{ALGP} \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a)$ and $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M^b, v^b) \models^{b} \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(b)$.

Take an arbitrary $s \in w^a \cap v^b$. By the theorem 8.1.8, for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a)$, $(M^a, w^a) \models^{a}_{ALGP} \varphi \Leftrightarrow (M, s) \models \varphi$. By the theorem 8.1.8, for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(b)$, $(M^b, v^b) \models^{b} \varphi \Leftrightarrow (M, s) \models \varphi$. Therefore, for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a) \cup \mathcal{L}_{ALGP}|\mathscr{A}(b)$, $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M, s) \models \varphi$. □

Theorem 9.2.7.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Suppose that M is image-finite. Suppose that

1. for all $w, v \in W$, if $(w^a, v^b) \in W^{a\otimes b}$, then $w^a \cap v^b \neq \varnothing$, and
2. for all $w \in W$, $w^a \cap w^b = w^{a+b}$.

Then,

for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a{+}b), ((M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M^{a+b}, s^{a+b}) \models^{a+b}_{ALGP} \varphi)$

where $w^a \cap v^b = s^{a+b}$.

Proof.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Suppose M is image-finite. By the lemma 9.2.6, $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M, s) \models \varphi$ for all $s \in w^a \cap v^b$ and $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a) \cup \mathcal{L}_{ALGP}|\mathscr{A}(b)$. Since $s^{a+b} = w^a \cap v^b$, $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M^{a+b}, s^{a+b}) \models^{a+b}_{ALGP} \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a) \cup \mathcal{L}_{ALGP}|\mathscr{A}(b)$.

By the theorem 9.2.5, $(w^a, v^b) \sim^{a\otimes b}_i (s^a, t^b) \iff w^a \cap v^b \sim^{a+b}_i s^a \cap t^b$ for all $w, v, s, t \in W$.

Therefore, $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M^{a+b}, s^{a+b}) \models^{a+b}_{ALGP} \varphi$ for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a + b)$. □

Existence of Isomorphism

Theorem 9.2.8.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Suppose that M is image-finite. Suppose that

1. for all $w, v \in W$, if $(w^a, v^b) \in W^{a \otimes b}$, then $w^a \cap v^b \neq \varnothing$, and
2. for all $w \in W$, $w^a \cap w^b = w^{a+b}$.

Then,

$$M^{a\otimes b} \;\cong\; M^{a+b}.$$

Proof.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Suppose M is image-finite. It is shown that there is an isomorphism $f : W^{a\otimes b} \longrightarrow W^{a+b}$ with

- $f((w^a, v^b)) = s^{a+b}$ with $w^a \cap v^b = s^{a+b}$,
- $w^{a\otimes b} \;\sim_i^{a\otimes b}\; v^{a\otimes b}$ if and only if $f(w^{a\otimes b}) \;\sim_i^{a+b}\; f(v^{a\otimes b})$ and
- for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a+b)$,
  $(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi \;\Leftrightarrow\; (M^{a+b}, f(w^{a\otimes b})) \models^{a+b}_{ALGP} \varphi$.

Let f be a function from $W^{a\otimes b}$ to W^{a+b} such that $f((w^a, v^b)) = s^{a+b}$ for all $w, v, s \in W$ with $w^a \cap v^b = s^{a+b}$. By the Theorem 9.2.2, f is a bijective function.
By the theorem 9.2.5, $w^{a\otimes b} \;\sim_i^{a\otimes b}\; v^{a\otimes b}$ if and only if $f(w^{a\otimes b}) \;\sim_i^{a+b}\; f(v^{a\otimes b})$ for all $w, v \in W$.
By the theorem 9.2.7, for all $w \in W$ and all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a+b)$, $(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi \;\Leftrightarrow\; (M^{a+b}, f(w^{a\otimes b}) \models^{a+b}_{ALGP} \varphi$.
Therefore, f is an isomorphism from $W^{a\otimes b}$ to W^{a+b}. Thus, $M^{a\otimes b} \cong M^{a+b}$.
□

Theorem 9.2.9.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V \rangle$. Suppose that M is image-finite. If

$$M^{a\otimes b} \;\cong\; M^{a+b},$$

then

1. for all $w, v \in W$, if $(w^a, v^b) \in W^{a \otimes b}$, then $w^a \cap v^b \neq \varnothing$, and
2. for all $w \in W$, $w^a \cap w^b = w^{a+b}$.

Proof.
Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. Suppose that M is image-finite and there is no $s \notin w^a \cap v^b$ with $(M^{a\otimes b}, (w^a, v^b)) \models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M^{a+b}, s^{a+b}) \models^{a+b}_{ALGP} \varphi$ for all $w, v \in W$ and all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a+b)$. Then, $f((w^a, v^b)) \subseteq w^a \cap v^b$ or $f((w^a, v^b)) \notin W^{a+b}$.

1. Suppose that for some $w, v \in W$, $(w^a, v^b) \in W^{a\otimes b}$ and $w^a \cap v^b = \varnothing$. Since $f((w^a, v^b)) \in w^a \cap v^b$ or $f((w^a, v^b)) \notin W^{a+b}$, $f((w^a, v^b)) \notin W^{a+b}$. Then, f is not a morphism.

2. Suppose that $w^a \cap w^b \neq w^{a+b}$ for some $w \in W$. By the theorem 8.1.2, $w^{a+b} \subsetneq w^a \cap w^b$. Then, there are $v^{a+b} \subsetneq w^a \cap w^b$ such that $w^{a+b} \neq v^{a+b}$. Then, for some $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a+b)$, $(M^{a+b}, w^{a+b}) \models^{a+b}_{ALGP} \varphi \not\Leftrightarrow (M^{a+b}, v^{a+b}) \models^{a+b}_{ALGP} \varphi$. Since $f((w^a, v^b)) \subseteq w^a \cap w^b$ or $f((w^a, v^b)) \notin W^{a+b}$, f is not an isomorphism.

By the contraposition, if there is an isomorphism $f : W^{a\otimes b} \longrightarrow W^{a+b}$ with

- $w^{a\otimes b} \sim^{a\otimes b}_i v^{a\otimes b}$ if and only if $f(w^{a\otimes b}) \sim^{a+b}_i f(v^{a\otimes b})$ and
- for all $\varphi \in \mathcal{L}_{ALGP}|\mathscr{A}(a+b)$, $(M^{a\otimes b}, w^{a\otimes b}) \models^{a\otimes b}_{ALGP} \varphi \Leftrightarrow (M^{a+b}, f(w^{a\otimes b})) \models^{a+b}_{ALGP} \varphi$.

, then

1. for all $w, v \in W$, if $(w^a, v^b) \in W^{a\otimes b}$, then $w^a \cap v^b \neq \varnothing$, and
2. $w^a \cap w^b = w^{a+b}$ for all $w \in W$.

□

Example 9.2.10. Let $\mathcal{G} = \{e, i, j\}$ and $P = \{p, q, r\}$. Given an epistemic awareness model $M_1 = \langle W, \sim, \mathscr{A}, V\rangle$ consisting of (See the left model in Figure 9.5):

1. $W = \{w_1, w_2, w_3, w_4\}$;
2. $\sim_i = \{(w, v) \mid w, v \in \{w_1, w_2\}\} \cup \{(w, v) \mid w, v \in \{w_3, w_4\}\}$, $\sim_j = \{(w, w) \mid w \in W\}$, and $\sim_e = \{(w, v) \mid w, v \in W\}$.
3. $\mathscr{A}(i) = \{p, q\}$, $\mathscr{A}(j) = \{q, r\}$, and $\mathscr{A}(e) = P$.
4. $V(p) = \{w_1, w_3, w_4\}$, $V(q) = \{w_1, w_2\}$, and $V(r) = \{w_1, w_2, w_3\}$.

Further, given another epistemic awareness model $M_2 = \langle W', \sim', \mathscr{A}', V' \rangle$ consisting of (See the right model in Figure 9.5):

1. $W' = \{w_5, w_6, w_7\}$,
2. $\sim'_i = \{(w, w) \mid w \in W'\}, \sim'_j = \{(w, w) \mid w \in W'\}$, and $\sim'_e = \{(w, v) \mid w, v \in W'\}$.
3. $\mathscr{A}'(i) = \{p\}$, $\qquad \mathscr{A}'(j) = \{q\}$, and $\mathscr{A}'(e) = P$.
4. $V'(p) = \{w_5, w_7\}$, $\qquad V'(q) = \{w_5, w_6\}$.

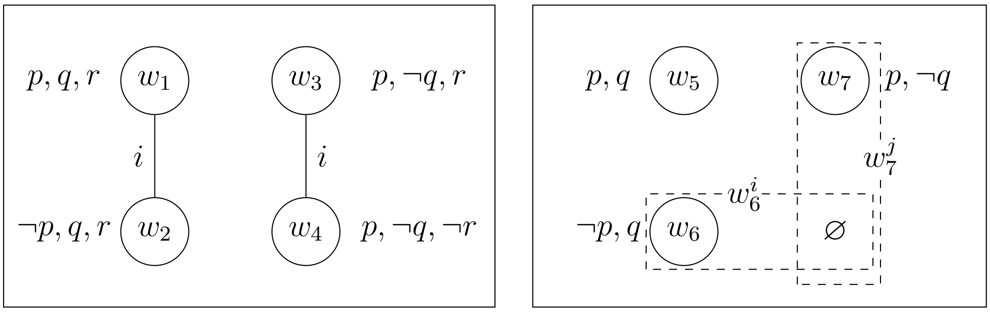

Figure 9.5: M_1 and M_2

Then, each quotient model M_1^i and M_1^j is given, respectively (See Figure 9.6).

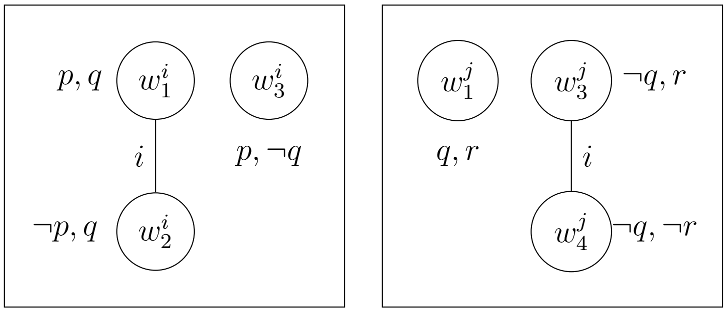

Figure 9.6: The Quotient Models M_1^i and M_2^j of The Left in Figure 9.5

Then, the product quotient model $M_1^{i \otimes j}$ is given as Figure 9.7.

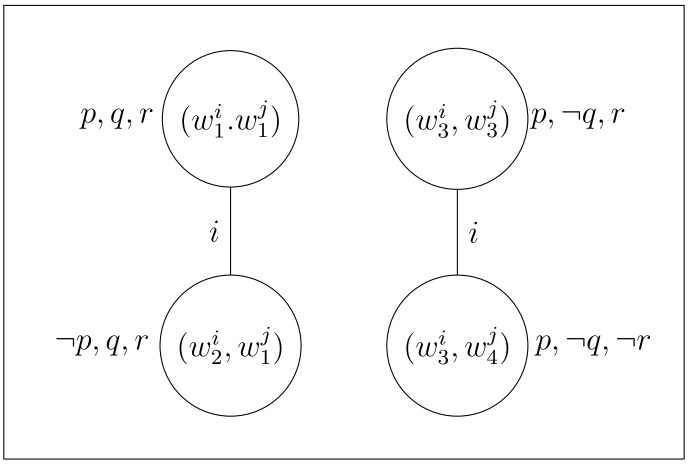

Figure 9.7: The Product Quotient Model $M_1^{i \otimes j}$ of Figure 9.5

Since $\mathscr{A}(i + j) = \{p, q, r\}$, the quotient model of $i + j$ is given as Figure 9.8.

Obviously, $M_1^{i \otimes j} \cong M_1^{i+j}$.

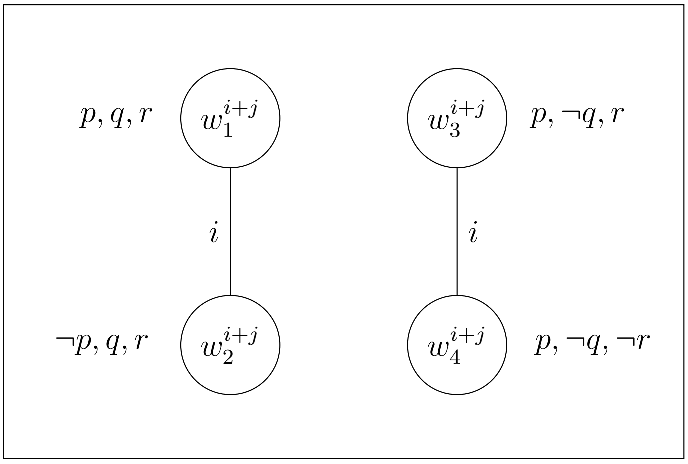

Figure 9.8: The Quotient Model M_1^{i+j} of The Left in Figure 9.5

$M_2^{i\otimes j}$ is shown in Figure 9.9:

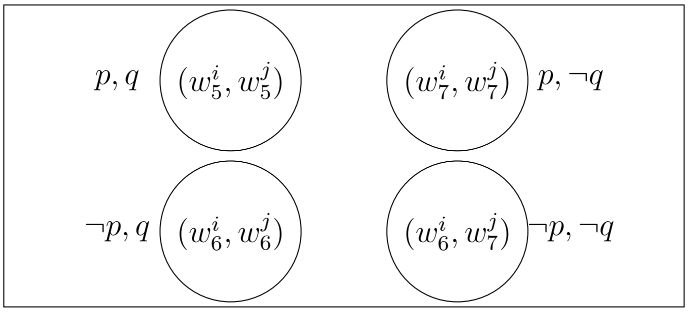

Figure 9.9: The Product Quotient Model $M_2^{i\otimes j}$ of The Right in Figure 9.5

Since $\mathscr{A}(i+j) = \{p, q\}$, $M^{i\otimes j} \not\cong M^{i+j}$.

9.3 Correspondence between Formulas and Reduction of Product Quotient Model

A product quotient model does not always correspond to a quotient model with distributed awareness. However, the conditions of reduction from a product quotient model to a quotient model with distributed awareness hold if and only if the following formulas are valid on a given model.:

1. $\langle\approx\rangle_a\varphi \wedge \langle\approx\rangle_b\varphi \rightarrow \langle\approx\rangle_{a+b}\varphi$, and

2. $\langle\approx\rangle_{a\cdot b}\varphi \to \langle\approx\rangle_a\langle\approx\rangle_b\varphi$.

$\langle\approx\rangle_a\varphi \wedge \langle\approx\rangle_b\varphi \to \langle\approx\rangle_{a+b}\varphi$ is valid if and only if $w^a \cap w^b \subseteq w^{a+b}$ for all $w \in W$. Also, $\langle\approx\rangle_{a\cdot b}\varphi \to \langle\approx\rangle_a\langle\approx\rangle_b\varphi$ is valid if and only if $v^a \cap u^b \neq \varnothing$ for all $v^a, u^b \subseteq w^{a\cdot b}$.

Since $\langle\approx\rangle_{a+b}\varphi \to \langle\approx\rangle_a\varphi \wedge \langle\approx\rangle_b\varphi$ and $\langle\approx\rangle_a\langle\approx\rangle_b\varphi \to \langle\approx\rangle_{a\cdot b}\varphi$ are valid on all epistemic awareness model, $\langle\approx\rangle_a\varphi \wedge \langle\approx\rangle_b\varphi \leftrightarrow \langle\approx\rangle_{a+b}\varphi$ and $\langle\approx\rangle_{a\cdot b}\varphi \leftrightarrow \langle\approx\rangle_a\langle\approx\rangle_b\varphi$ are valid on a given image-finite model if and only if the conditions of reduction from a product quotient model to a quotient model with distributed awareness hold.

The following theorem shows the correspondence between the condition and the above formula $\langle\approx\rangle_{a\cdot b}\varphi \to \langle\approx\rangle_a\langle\approx\rangle_b\varphi$.

Theorem 9.3.1. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$. They are equivalent to each other.

1. for all $w, v \in W$, if $(w^a, v^b) \in W^{a\otimes b}$, then $w^a \cap v^b \neq \varnothing$.
2. for all $w, v \in W$, if $w \approx_{a\cdot b} v$, then there is a $x \in W$ such that $w \approx_a x$ and $x \approx_b v$.
3. $\langle\approx\rangle_{a\cdot b}\varphi \to \langle\approx\rangle_a\langle\approx\rangle_b\varphi$ is valid for all $\varphi \in \mathcal{L}_{ALGP}$.

Proof. Fix $a, b \in \mathcal{A}$. Given an epistemic awareness model $M = \langle W, \sim, \mathscr{A}, V\rangle$.

- $(1 \Rightarrow 2)$: By the contraposition. Suppose that for some $w \in w^a$ and $v \in v^b$ such that $w \approx_{a\cdot b} v$, there is no $x \in W$ such that $w \approx_a x$ and $x \approx_b v$. Then, $w^a \cap v^b = \varnothing$. Since $w^a, v^b \in w^{a\cdot b}$, for all $\varphi \in \mathcal{L}_{ALGP}$ such that $At(\varphi) \subseteq \mathscr{A}(a) \cap \mathscr{A}(b)$, $(M, w) \models_{ALF} \varphi \Leftrightarrow (M, v) \models_{ALF} \varphi$. Then, $(w^a, v^b) \in W^{a\otimes b}$. Then, for some $w, v \in W$, $(w^a, v^b) \in W^{a\otimes b}$ and $w^a \cap v^b = \varnothing$.

- $(2 \Rightarrow 3)$: Suppose that for all $w, v \in W$, if $w \approx_{a\cdot b} v$, then there is a $x \in W$ such that $w \approx_a x$ and $x \approx_b v$. Take an arbitrary $w \in W$. Suppose that $(M, w) \models_{ALF} \langle\approx\rangle_{a\cdot b}\varphi$. Then, $(M, v) \models_{ALF} \varphi$ for some $v \in W$ such that $w \approx_{a\cdot b} v$. Since there is a $x \in W$ such that $w \approx_a x$ and $x \approx_b v$, $(M, w) \models_{ALF} \langle\approx\rangle_a\langle\approx\rangle_b\varphi$. Since w is an arbitrary world, $\langle\approx\rangle_{a\cdot b}\varphi \to \langle\approx\rangle_a\langle\approx\rangle_b\varphi$ is valid.

- $(3 \Rightarrow 1)$: Suppose that $\langle\approx\rangle_{a\cdot b}\varphi \to \langle\approx\rangle_a\langle\approx\rangle_b\varphi$ is valid for all $\varphi \in \mathcal{L}_{ALGP}$. Take two arbitrary possible worlds $w, v \in W$ such that $(w^a, v^b) \in W^{a\otimes b}$. Take an arbitrary $\psi \in \mathcal{L}_{ALGP}$ such that $(M, v) \models_{ALF} \psi$. By the definition of $W^{a\otimes b}$, $w \in v^{a\cdot b}$. Then, $(M, w) \models_{ALF} \langle\approx\rangle_{a\cdot b}\psi$. Since

$\langle\approx\rangle_{a\cdot b}\varphi \to \langle\approx\rangle_a\langle\approx\rangle_b\varphi$ is valid for all $\varphi \in \mathcal{L}_{ALGP}$, $(M,w) \models \langle\approx\rangle_a\langle\approx\rangle_b\psi$. By the semantics, there are two possible worlds $x, y \in w^{a\cdot b}$ such that $w \approx_a x$, $x \approx_b y$, and $(M,y) \models_{ALF} \psi$. Since $\psi \in \mathcal{L}_{ALGP}$ is an arbitrary formula such that $(M,v) \models_{ALF} \psi$, $v \approx_b y$. $\approx_b$ is an equivalence relation, $x \approx_b v$. Then, $x \in w^a \cap v^b$. Therefore, $w^a \cap v^b \neq \varnothing$.

□

Chapter 10

Conclusions and Further Directions

10.1 Conclusions

This thesis introduced **Awareness Logic with Global Propositional Awareness(ALGP)** and **Awareness Logic of Filtration(ALF)**. In summary. this study has contributed to the following.:

1. A sound and complete axiomatization of **ALGP** is given.
2. Common awareness and distributed awareness can be introduced to **ALGP** as a macro.
3. The non-compactness in **ALF** is shown.
4. A filtration of **ALF** corresponds to a restricted bisimulation(See Definition 6.2.1).
5. If all simple paths are bounded in a model, We can transform each formula in $\mathcal{L}_{ALF}$ to some equivalent formula in $\mathcal{L}_{ALGP}$.
6. A nested quotient model corresponds to a quotient model with common awareness.
7. For image-finite models, the condition of correspondence between a product quotient model and a quotient model with distributed awareness is shown.

Regarding the motivation of what kind of mutually comprehensible model for agents with different reasoning abilities, the answer to this thesis is to provide a quotient model based on the common awareness among the agents.

10.2 Future Directions

Currently, four main possible directions for the development of Awareness Logic of Abstraction are considered.

1. A sound and complete axiomatization of Awareness Logic of Abstraction.
2. Formalization of abstraction tailored to local awareness.
3. Development into dynamic epistemic logic.
4. Formalization of concretization as a dual of abstraction.

First, the details of item 1 are described. We suspect a complete axiomatization of ALF would not be shown because a canonical model covering all maximally consistent sets cannot be given without an axiom regarding image-finite. Currently, two directions to avoid this approach are considered.:

(a) Replacing filtration-based abstraction with bisimulation-based abstraction.

(b) Relax from propositional awareness to awareness closed under subformulas.

A fixed-point axiom would be needed for item (a). To give the fixed-point axiom, we expect it is better to formalize abstraction by **bisimulation** than filtration. There are two techniques for the proof of the completeness theorem of the modal logic without compactness. The first technique is the proof by constructing a finite canonical model tailored to a formula φ [10]. The second technique is to introduce inference rules deducing from infinite formulas to a formula [9]. We will mainly work on proof using those techniques.

For (b), global awareness closed under subformulas is considered a compatible restriction with filtration-based abstraction (Theorem 2.39 [8] p.79). If an agent's awareness set is finite, the results in [1] would be useful for **ALF** with global awareness closed under subformulas. Otherwise, a condition of compactness will be investigated.

We need to be very careful when introducing a new dynamic operator and providing a reduction axiom for it. For example, a complete axiomatization of Public Announcement Logic is shown by viewing updates as transitions among models in a "supermodel" [12]. A supermodel of Awareness Logic of Abstraction will be needed to consider, too.

Second, the details of item 2 are described. By restriction to global awareness, $\Diamond_i A_j p \wedge \Diamond_i \neg A_j p$ is always false for all agents $i, j \in \mathcal{G}$ and all atomic propositions $p \in P$ in the semantics, i.e., global awareness cannot deal with "*An agent is aware of* p *and does not know whether others are aware of* p *or not*". Global awareness is non-standard. We hope **ALF** is closer to that of traditional awareness logic. We consider that we cannot define the semantics of **ALF** with a local definition in the standard Kripke model. We would need to investigate semantics more generally in terms of the neighborhood model and coalgebras.

Third, the details of item 3 are described. When considering system behavior, the equality of behaviors that can be observed from outside the system is called **observational equivalence**. We consider that abstraction can give models regarding only behaviors that can be observed from outside the system. Further, we expect that awareness logic can be applied to formal verification or testing of black boxes. To bring **ALF** closer to program semantics, it is necessary to aim for development into dynamic epistemic logic.

Finally, the details of item 4 are described. It is motivated to understand more concrete information than what we are aware of. We often comprehend more concrete knowledge through mutual complement among different reasoning abilities. Currently, we could consider two approaches as formalization of concretization.: The first approach is research on a quotient model with distributed awareness. The second approach is research on a product quotient model. Section 9.2 showed that a quotient model with distributed awareness does not always correspond to a product quotient model. We consider a product of quotient models does not represent a mutual complement among agents with different reasoning abilities because an agent's knowledge and awareness do not interfere with the others in an operation represented by a product of models. We would investigate operations of awareness and models for non-interfering concretization and mutual complement.

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