Distributing Distinct Integers Uniformly over a Square Matrix based on Discrepancy Measure to Construct Dither Matrix

Shinya HASHIMA (210069)

School of Information Science,
Japan Advanced Institute of Science and Technology

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The problem studied in this thesis is about distributing n^2 distinct integers between 0 and n^2-1 as uniformly as possible over an $n \times n$ square matrix.

To measure the uniformity, we consider a discrepancy-based measure. There are various definitions for discrepancy measure. In this thesis, we define the discrepancy measure as follows. Let a matrix P be an $n \times n$ square matrix containing integers $0, \cdots, n^2-1$ and a submatrix R be all 2×2 regions which consists of contiguous elements of P. Let $P(R)$ denote the sum of the elements of R. We define the discrepancy $\mathcal{D}_{2,n}(P)$ of the matrix P as

$$\mathcal{D}_{2,n}(P) \stackrel{\text{def.}}{=} \max_R P(R) - \min_R P(R).$$

We also define that the lower the discrepancy, the better is the uniformity. Therefore, in this thesis, we try to reduce $\mathcal{D}_{2,n}(P)$ as much as possible.

In this problem, it is known that if n is an even integer, then we can construct 0-discrepancy matrices. However, if n is an odd integer, there is no matrix of the same property. Here, we try to sequence elements of odd sized matrices to reduce the value of discrepancy as much as possible.

However, it is too hard to find low discrepancy matrices by programming. For example, there will be 25! sequence patterns for a 5×5 matrix. It is impossible to verify all the sequences by computer. In fact, although we verified a million sequences for a 5×5 matrix, it was impossible to improve the previous discrepancy.

In this thesis, we propose two methods. One is an alternating diagonal sequencing method where we present a scheme for discrepancy $2n + 4$. The second method is based on an n-ary number system method, and can achieve a discrepancy of $2n + 2$. Furthermore, using the second method, we improve the discrepancy to $2n$.

This study is related to digital halftoning. Digital halftoning is a technique to convert a continuous-tone image into a binary image consisting of black and white dots. This technique converts each pixel of an input image into a 0 or 1 pixel by comparing with a threshold. There are a number of methods for digital halftoning. One such method is ordered dither algorithm.

Ordered dither algorithm uses different thresholds for pixels instead of using a fixed threshold over an entire image. A submatrix called dither matrix is tiled periodically to cover the image. Each pixel in the image is compared with the corresponding threshold from the dither matrix to decide the value of the binary pixel at that location. Dither matrix D_k is recursively defined as follows:

$$D_0 = (1). \quad D_k = \begin{pmatrix} 4D_{k-1} - 3U_{k-1} & 4D_{k-1} - U_{k-1} \\ 4D_{k-1} & 4D_{k-1} - 2U_{k-1} \end{pmatrix}. \quad (k = 1, 2, \cdots)$$

where U_k is a $2^k \times 2^k$ square matrix consisting of all 1s.

The dither matrix constructed in the above fashion is not always optimal for ordered dither algorithm. This is caused by the difference between density of pixels in the converted binary image. Such a difference between the densities is captured by the discrepancy measure.

Low discrepancy matrices can be used to construct dither matrices to improve the outcome of the ordered dither algorithm.