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Enumeration of Polyabolos Constructible with Tangram and Sei Shonagon Chie no Ita

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1. Introduction and Research Background

Silhouette puzzles represent a classic category of geometric tiling problems where a specific set of polygons, referred to as “pieces,” must be arranged to form a target silhouette without overlapping. Two of the most representative examples in this domain are “Tangram,” which originated in China (first documented in 1813), and “Sei Shonagon Chie no Ita,” which originated in Japan (documented in 1742). Despite their distinct historical origins, these two puzzles share remarkable geometric similarities. Both consist of a square dissected into exactly seven pieces. Furthermore, they are mathematically equivalent in terms of fundamental properties: the total area of their pieces is identical, and the sum of the constituent edge lengths is also exactly the same (consisting of 20 edges of length 1 and 10 edges of length $\sqrt{2}$).

However, a discrepancy in their combinatorial expressiveness—the variety of shapes they can create—has long been intuitively felt but insufficiently quantified. Previous mathematical comparisons have been confined to the limited scope of convex polygons. Wang and Hsiung (1942) proved that Tangram can form 13 types of convex polygons, while Fox-Epstein et al. (2016) demonstrated that Sei Shonagon Chie no Ita can form 16 types. While these studies provided foundational insights, focusing solely on convex polygons ignores the vast majority of possible configurations. The objective of this study is to transcend this limitation by extending the comparison to “polyabolos”—general geometric shapes formed by connecting isosceles right triangles. By employing exhaustive computational enumeration, this research aims to quantitatively elucidate the total geometric expressiveness of both puzzles, covering non-convex and multiply connected (holey) shapes.

2. Mathematical Modeling and Proposed Methodology

In this study, the fundamental unit of geometry is defined as the “polyabolo.” A polyabolo is a geometric figure formed by connecting n congruent isosceles right triangles edge-to-edge. Within this framework, the pieces of Tangram, Sei Shonagon Chie no Ita, and their variants are modeled as subsets of unit triangles. Since the total area of the original square for both puzzles is equal to 16 unit triangles, the target shapes for enumeration are polyabolos of order $n = 16$. The enumeration considers shapes identical if they coincide after rotation or reflection.

To achieve efficient exhaustive enumeration for a problem of this magnitude, we proposed a novel computational framework centering on a "4-bit mask encoding" method on a 2D integer grid.

- **4-Bit Mask Encoding:** We manage the placement state of isosceles right triangles at each edge (left, top, right, bottom) of a grid cell using a 4-bit integer. For example, a triangle in the top-left is encoded as 1100_2 (12), while one in the bottom-right is 0011_2 (3). This representation allows geometric operations to be replaced by high-speed bitwise operations. Collision detection becomes a bitwise check, and piece integration becomes a bitwise OR or XOR operation (e.g., $1100_2 \oplus 0011_2 = 1111_2$, forming a complete square). To accommodate the maximum spatial extension of $n = 16$ polyabolos, the grid size L was set to satisfy $L \geq 9$.
- **Search Algorithm:** Rather than a brute-force placement approach, we adopted a "connectivity-based sequential construction" method. This functions as a search tree that starts with an arbitrary initial piece and sequentially adds remaining pieces to the existing shape according to edge-adjacency rules.

3. Optimization Strategies

The search space for $n = 16$ polyabolos is massive. To make the computation feasible on a single machine, we implemented four key optimization strategies:

1. **Fixed Initial Piece:** By fixing the first piece of the puzzle to the origin, we eliminated redundancy caused by translation and reduced the permutation space from 7! to 6!.
2. **Canonicalization and Pruning:** We applied a canonicalization procedure to every intermediate partial shape. Shapes were normalized to a relative coordinate system (using an offset $\delta = 2$ to ensure non-negative indices). We then generated a unique "minimum signature" for each shape by comparing the lexicographical order of its coordinate sequences across all 8 possible orientations (rotation and reflection). This allowed us to identify and prune isomorphic duplicates early in the search tree.
3. **Dynamic Bounding Box:** We maintained a dynamic bounding box for the current partial shape. When attempting to add a new piece, the

search was restricted to a local range adjacent to the existing boundaries (specifically $[0, x_{\max} + 1] \times [0, y_{\max} + 1]$), significantly reducing invalid placement attempts.

4. **Direct Insertion Strategy:** A major bottleneck was memory consumption during the final step of recursion. Instead of storing the millions of generated candidate shapes in an intermediate list, we implemented a "direct insertion strategy" for the addition of the 7th piece. Generated signatures were checked directly against the final result set; if a signature did not exist, it was added, otherwise it was discarded. This prevented memory overflow and overhead.

4. Verification and Experimental Results

The correctness of the implemented algorithm was rigorously verified through a benchmark experiment: the exhaustive enumeration of unconstrained polyabolos formed by 16 unit triangles ($n = 16$). The algorithm reported a total of **28,616,815** unique polyabolos. This figure perfectly matches sequence **A006074** in the On-Line Encyclopedia of Integer Sequences (OEIS). Since previous entries in the OEIS were primarily theoretical counts without accessible geometric datasets, this result serves as strong evidence for the accuracy of our proposed framework.

Using this verified framework, we performed the comparative enumeration for the two target puzzles:

- **Tangram:** The enumeration yielded a total of **5,583,516** distinct constructible polyabolos.
- **Sei Shonagon Chie no Ita:** The enumeration yielded a total of **10,889,227** distinct constructible polyabolos.

These results quantitatively prove that Sei Shonagon Chie no Ita possesses approximately **1.95 times** the generative capacity of Tangram. While the pieces look similar, the Japanese puzzle exhibits a drastically higher level of geometric expressiveness when constraints are lifted to allow general non-convex shapes. All generated datasets and a custom web-based viewer have been made publicly available.

5. Analysis of Geometric Factors

To understand the geometric origins of this discrepancy, we extended our analysis to seven additional puzzle variants (Special Puzzles A–G), including those proposed by Fox-Epstein et al. and novel designs. By correlating the piece characteristics with the enumeration results, we identified three primary geometric factors governing expressiveness:

1. **Constraint of Large-Area Pieces:** There is a negative correlation between the presence of large monolithic pieces and the number of configurations. Tangram contains two large triangles (area 4). Puzzles that replace these with smaller components (area ≤ 3) showed significantly higher generation counts (e.g., Sp. G reached approximately 17.96 million shapes). Large pieces lack the flexibility to fill fine gaps or form complex local boundaries.

2. **Piece Shape Diversity:** Higher diversity in the shapes of constituent pieces reduces symmetry-induced redundancies. For instance, comparing Variant Sp.F (four identical trapezoids) and Sp.G (one trapezoid replaced by an isosceles trapezoid of equal area), the latter produced nearly double the configurations. Diverse pieces reduce the likelihood that swapping two pieces results in an identical geometric state.

3. **Effectiveness of Asymmetry:** Asymmetric pieces contribute more to expressiveness than symmetric ones. Comparing puzzles with identical area distributions, those containing asymmetric pieces (like parallelograms) outperformed those with symmetric pieces (like squares). Asymmetry allows a piece to take on distinct states via reflection (flipping), effectively increasing the degrees of freedom in combination. Sei Shonagon Chie no Ita benefits significantly from this factor compared to the more symmetric composition of Tangram.

6. Conclusion

This study provides the first exhaustive quantitative comparison of the geometric expressiveness of Tangram and Sei Shonagon Chie no Ita beyond the limited scope of convex polygons. By generating tens of millions of shapes, we established that Sei Shonagon Chie no Ita is nearly twice as expressive as Tangram and isolated the geometric principles—piece granularity, diversity, and asymmetry—that drive this combinatorial richness.