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On Vector Network Equilibrium Problems

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ABSTRACT

In this paper we define a concept of weak equilibrium for vector network equilibrium problems. We obtain sufficient conditions of weak equilibrium points and establish relation with vector network equilibrium problems and vector variational inequalities.

Keyword: Network Equilibrium Problem, Vector Variational Inequality, Weak Equilibrium.

1 INTRODUCTION

The earliest network equilibrium model was proposed by Wardrop [1] for a transportation network. Since then, many other equilibrium models have also been proposed in the economics literature (see Nagurney [2]). Until only recently, all these equilibrium models are based on single cost or utility function. Recently, equilibrium models based on multicriteria consideration or vector-valued cost functions have been proposed. In Chen and Yen [3], a multicriteria traffic equilibrium model was proposed and the relationship between this model and the vector variational inequality problem was considered under a singleton assumption. Other papers that consider multicriteria equilibrium models can be found in Brenninger-Göthe et al [4], Chen, Goh and Yang [5], Dial [6], Goh and Yang [10], Leurent [8], and Yang and Goh [9]. In particular, the multicriteria network equilibrium model was formulated as a vector variational inequality problem in Goh and Yang [10] via a vector optimization approach, but without the singleton assumption.

In this paper, we consider weak vector network equilibrium, vector network equilibrium and dynamic vector equilibrium problems. We establish their relations with vector variational inequalities and vector optimization problems.

2 WEAK VECTOR EQUILIBRIUM PROBLEM

Consider a transportation network $G = (\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ denotes the set of nodes and $\mathcal{A}$ denotes the set of arcs. Let $\mathcal{I}$ be the set of origin-destination (O-D) pair and P_i, $i \in \mathcal{I}$ be the set of paths joining O-D pair i. For a given path $k \in P_i$, let h_k denote the traffic flow on this path and $h = (h_1, h_2, \cdots, h_M) \in \mathbb{R}^M$, where $M = \sum_{i \in \mathcal{I}} |P_i|$. The path flow vector h induces a flow v_a on each arc $a \in \mathcal{A}$ given by

$$v_a = \sum_{i \in \mathcal{I}} \sum_{k \in P_i} \delta_{ak} h_k,$$

where $\Delta = [\delta_{ak}] \in \mathbb{R}^{|\mathcal{A}| \times M}$ is the arc path incidence matrix with $\delta_{ak} = 1$ if the arc belongs to path k and 0 otherwise. Let $v = [v_a : a \in \mathcal{A}] \in \mathbb{R}^{|\mathcal{A}|}$ be the vector of arc flow. Succinctly

$$v = \Delta h. \tag{1}$$

We will assume that the demand of traffic flow is fixed for each O-D pair, i.e., $\sum_{k \in P_i} h_k = d_i$, where d_i is a given demand of each O-D pair i. A flow $h \geq 0$ satisfying the demand is called a feasible flow. Let $\mathcal{H} = \{h : h \geq 0, \sum_{k \in P_i} h_k = d_i, \forall i \in \mathcal{I}\}$ be the set of feasible flows. $\mathcal{H}$ is clearly a closed and convex set. Let $t_a : \mathbb{R}^{|\mathcal{A}|} \to R^\ell$ be a vector-valued cost function for the arc a and it is in general a function of all the arc flows, and let metric $t(v) = [t_a(v) : a \in \mathcal{A}] \in \mathbb{R}^{\ell \times |\mathcal{A}|}$. The vector-valued cost function along the path k, we denote τ_k, $\tau_k : \mathbb{R}^M \to \mathbb{R}^\ell$ is assumed to be the sum of all the arc cost along this path, thus

$$\tau_k(h) = \sum_{a \in \mathcal{A}} \delta_{ak} t_a(v).$$

Let $T(h) = [\tau_k(h) : k \in P_i, i \in \mathcal{I}] \in \mathbb{R}^{\ell \times \mathcal{M}}$. Succinctly

$$T(h) = t(v)\Delta. \tag{2}$$

In this section, we consider an equilibrium problem defined on transportation network with vector-valued cost functions. In this model, the cost space is ℓ-dimensional Euclidean space $\mathbb{R}^\ell$, with the ordering cone C, a pointed, closed and convex cone with nonempty interior $intC$.

Definition 1 *Given a flow h, we say that a path $p \in P_i$ for an O-D pair i is a weakly minimal one if there does not exist another path $p' \in P_i$ such that $\tau_{p'}(h) - \tau_p(h) \leq_{intC} 0$.*

Let $\Gamma_i(h) = \{\tau_p(h) : p \in P_i\}$ denote the (discrete) set of vector costs for all paths for O-D pair i, and

$$\mathcal{I}_i(h) = \{k \in P_i \mid \tau_k(h) - \tau_p(h) \not\geq_{intC} 0, \forall p \in P_i\} \subseteq P_i$$

denote the set of all weakly minimal paths for O-D pair i.

We define the weakly minimal frontier for O-D pair i to be the set of weakly minimal points in the cost-space of O-D pair i:

$$\text{Min}_{intC}(\Gamma_i(h)) = \{\xi \in \mathbb{R}^\ell \mid \xi = \tau_p(h) \text{ where } p \in \mathcal{I}_i(h)\}.$$

Note that $\text{Min}_{intC}(\Gamma_i(h))$ is a discrete set because it is a subset of the discrete set $\mathcal{I}_i(h)$.

The following weak vector equilibrium principle is a generalization of the well-known Wardrop's equilibrium principle (see Wardrop [1]):

Definition 2 *A flow $h \in H$ is said to be in weak vector equilibrium if*

$$\forall i \in \mathcal{I}, \forall k, l \in P_i, \quad \tau_k(h) \geq_{intC} \tau_l(h) \Longrightarrow h_k = 0. \tag{3}$$

A flow h in weak vector equilibrium is often referred to as a weak vector equilibrium flow.

In terms of the weakly minimal frontier for O-D pair i, the weak vector equilibrium principle can be stated in an equivalent form:

Definition 3 *[Equivalent weak vector equilibrium principle] The path flow vector h is in weak vector equilibrium if*

$$\begin{aligned} &\forall i \in \mathcal{I}, \forall p \in P_i, \ h_p = 0 \\ &\textit{whenever } \ \tau_p(h) \notin Min_{intC}(\Gamma_i(h)). \end{aligned} \tag{4}$$

These definitions are natural generalizations of the Wardrop equilibrium principle for a scalar valued cost, in which case, a strict inequality $>$ is used in (3). The motivation for both the scalar and the vector cost cases is provided by the fact that an user will not choose to travel on a path if it is cheaper (both in the scalar and the vector sense) to travel on another path that links the same origin and destination.

We shall investigate weak vector equilibrium flows by virtue of linear scalarization function and nonlinear scalarization function, respectively.

Linear Scalarization Approach

Let us first introduce the concept of a parametric equilibrium flow.

Definition 4 *(Weak parametric equilibrium principle) Let a parameter $\lambda \in C^*$ be given. A path flow vector h is in weak λ-equilibrium if $\forall i \in \mathcal{I}$, $\forall p \in P_i$, $h_p = 0$ whenever $\exists\, e_i \in Min_{intC}(\Gamma_i(h))$, such that $\lambda^\top \tau_p(h) > \lambda^\top e_i$.*

Note that a parametric equilibrium flow is based on a scalar cost, as in the case of Wardrop's equilibria. In the case of scalarization for vector optimization, it is known that certain convexity assumption is necessary before the scalar optimal solution is necessarily a weakly minimal solution for the vector problem. In the present context, however, the set of concern $\Gamma_i(h)$ is discrete and hence convexity has no meaning. To get around this, we make the following assumption.

Assumption 1

$$Min_{intC}(\Gamma_i(h)) \subseteq Min_{intC}(co(\Gamma_i(h))),$$

where $co(\Gamma_i(h))$ is the convex hull of the discrete set $\Gamma_i(h)$.

The following result establishes relationships between a weak vector equilibrium flow and a parametric equilibrium flow.

We need the following scalarization result.

Lemma 1 *Let $A \subset \mathbb{R}^\ell$ be a nonempty and convex set and $a^* \in Min_{intC}A$. Then, there exists $\lambda \in C^* \backslash \{0\}$ such that*

$$\lambda^\top a^* = \min_{a \in A} \lambda^\top a.$$

Theorem 1 *(i) If h is in weak vector equilibrium and Assumption 1 holds, then there exists $\lambda \in C^* \setminus \{0\}$ such that the path flow h is in weak λ-equilibrium;*

(ii) If h is in weak λ-equilibrium for some $\lambda \in C^* \backslash \{0\}$, then h is in weak vector equilibrium.

For $\lambda \in C^*$, we define the minimum scalarized cost for O-D pair i as:

$$u_i(\lambda) = \min_{p \in P_i} \lambda^\top \tau_p(h). \tag{5}$$

Lemma 2 *If $\lambda \in C^* \backslash \{0\}$, then $u_i(\lambda) = \lambda^\top e_i$ for some $e_i \in Min_{intC}(\Gamma_i(h))$.*

Theorem 2 *(i) Let $\lambda \in C^*$. Then h is in weak λ-equilibrium if the following condition holds:*

$$\forall i \in \mathcal{I}, \forall p \in P_i, \ h_p = 0 \textit{ whenever } \lambda^\top \tau_p(h) > u_i(\lambda); \tag{6}$$

(ii) If $\lambda \in C^*\backslash\{0\}$ and h is in weak λ-equilibrium, then condition (6) holds.

Next, necessary and sufficient optimality conditions of weak vector traffic equilibrium in terms of vector variational inequalities are given.

Theorem 3 *Let Assumption 1 hold , the cost function t_a be integrable and the cost matrix $t(v)$ be C-monotone. If h is in weak vector equilibrium, then h is a solution of the following (WVVI) of finding $h \in \mathcal{H}$:*

$$T(h)(g-h) \not\leq_{intC} 0, \ \forall g \in \mathcal{H}.$$

We may now establish a sufficient condition for a flow h to be in weak vector equilibrium.

Theorem 4 *$h \in \mathcal{H}$ is in weak vector equilibrium if h solves the (WVVI) of finding $h \in \mathcal{H}$:*

$$T(h)(\bar{h}-h) \not\leq_{intC} 0, \ \forall \bar{h} \in \mathcal{H}. \qquad (7)$$

Proof. Let h satisfy (7). Choose $\bar{h}$ to be such that

$$\bar{h}_j = \begin{cases} h_j, & \text{if } j \neq k \text{ or } j, \\ 0, & \text{if } j = k, \\ h_k + h_j, & \text{if } j = j. \end{cases} \qquad (8)$$

Clearly, $\bar{h} \in \mathcal{H}$ since $\forall i \in \mathcal{I}$, $\sum_{j \in P_i} h_j = \sum_{j \in P_i} \bar{h}_j = d_i$. Now

$$\begin{aligned} T(h)(\bar{h}-h) &= \sum_{i \in I} \sum_{j \in P_i} (\bar{h}_j - h_j)\tau_j(h) \\ &= (\bar{h}_k - h_k)\tau_k(h) + (\bar{h}_j - h_j)\tau_j(h) \\ &= h_k(\tau_j(h) - \tau_k(h)) \not\leq_{intC} 0. \qquad (9) \end{aligned}$$

If

$$\tau_k(h) - \tau_j(h) \geq_{intC} 0, \qquad (10)$$

then (9) and (10) together imply that $h_k = 0$ since C is a pointed cone. ∎

3 NONLINEAR SCALARIZATION APPROACH

In this subsection, we assume that $C = \mathbb{R}^\ell_+$. Choose any $a \in \mathbb{R}^\ell$ and $e \in int\mathbb{R}^\ell_+$. By using the nonlinear scalarization function ξ_{ea}, define a function $\xi^k_{ea} : \mathbb{R}^M \to \mathbb{R}$ by:

$$\xi^k_{ea}(h) = \xi_{ea}(\tau_k(h)), \quad k \in P_i, i \in \mathcal{I}.$$

The vector-valued function $\bar{\xi}_{ea} : \mathcal{H} \to \mathbb{R}^M$ and the scalar-valued function $u^i_{ea} : \mathcal{H} \to \mathbb{R}$, $i \in \mathcal{I}$ are defined, respectively, by

$$\bar{\xi}_{ea}(h) = [\xi^k_{ea}(h) : k \in P_i, \ i \in \mathcal{I}] \qquad (11)$$

and

$$u^i_{ea}(h) = \min_{k \in P_i} \xi_{ea}(\tau_k(h)), \quad i \in \mathcal{I}. \qquad (12)$$

Definition 5 *The path flow $h \in \mathcal{H}$ is said to be in ξ_{ea}-equilibrium if there exist $e \in int\mathbb{R}^\ell_+$ and $a \in \mathbb{R}^\ell$ such that*

$$\forall i \in \mathcal{I}, \forall k, l \in P_i, \quad \xi_{ea}(\tau_k(h)) > \xi_{ea}(\tau_l(h)) \Longrightarrow h_k = 0. \qquad (13)$$

Consider the following vector optimization problem (VO):

$$(VO) \qquad \operatorname*{Min}_{x \in X}{}_C \, f(x),$$

where $f : \mathbb{R}^M \to \mathbb{R}^\ell$, $X \subset \mathbb{R}^M$ is a possibly finite set. Note that neither f nor X is required to be convex.

We have the following non-convex scalarization theorem.

Theorem 5 (Non-convex Scalarization Theorem) *Let $A \subset \mathbb{R}^\ell$ be a $\mathbb{R}^\ell_+$ order lower bounded subset. Then $y^* \in Min_{int\mathbb{R}^\ell_+} A$ if and only if, for some $a \in \mathbb{R}^\ell$ and $e \in intR^\ell_+$,*

$$\xi_{ea}(y^*) = \min \xi_{ea}(A).$$

We may now use Theorem 5 to establish an equivalent condition for a weak vector equilibrium in terms of a scalar variational inequality.

Theorem 6 *The path flow $h \in \mathcal{H}$ is in weak vector equilibrium if and only if h is in ξ_{ea}-equilibrium for some $e \in int\mathbb{R}^\ell_+$ and $a \in \mathbb{R}^\ell$.*

Remark 1 *It is important to note that the set K_i in the above proof is a discrete set, in which convexity has no meaning. The converse proof would not have worked if we had used the linear scalarization instead, since this would have required the set K_i to be infinite and cone convex.*

The problem of finding a ξ_{ea}-equilibrium for given $e \in \mathbb{R}^\ell_+$ and $a \in \mathbb{R}^\ell$ is still not directly solvable. We now reduce the ξ_{ea}- equilibrium to a scalar variational inequality and consequently well-known techniques for solving variational inequalities can be applied accordingly.

Theorem 7 *The path flow $h \in \mathcal{H}$ is in ξ_{ea}-equilibrium if and only if there exist $e \in int\mathbb{R}^{\ell}_{+}$ and $a \in \mathbb{R}^{\ell}$ such that h solves the following (scalar) variational inequality:*

$$\bar{\xi}_{ea}(h)^{\top}(\bar{h} - h) \geq 0, \quad \forall \bar{h} \in \mathcal{H}, \tag{14}$$

where $\bar{\xi}_{ea}(h) = [\xi^k_{ea}(h) : k \in P_i, i \in \mathcal{I}]$ and $\xi^k_{ea}(h) = \xi_{ea}(\tau_k(h))$.

Proof. $(\Longleftarrow)$

Assume that h solves the variational inequality (14). Choose the special $\bar{h}$ defined by (8), then

$$\begin{aligned} \bar{\xi}_{ea}(h)^{\top}(\bar{h} - h) &= \sum_{i\in\mathcal{I}} \sum_{j\in P_i} (\bar{h}_j - h_j)\xi^j_{ea}(h) \\ &= (\bar{h}_k - h_k)\xi^k_{ea}(h) + (\bar{h}_l - h_l)\xi^l_{ea}(h) \\ &= h_k(\xi^l_{ea}(h) - \xi^k_{ea}(h)) \\ &= h_k(\xi_{ea}(\tau_l(h)) - \xi_{ea}(\tau_k(h))) \\ &\geq 0. \end{aligned} \tag{15}$$

Thus if $\xi_{ea}(\tau_k(h)) - \xi_{ea}(\tau_l(h)) > 0$, (15) and $h_k \geq 0$ implies that $h_k = 0$, i.e., h is in weak vector equilibrium.

$(\Longrightarrow)$

Conversely, we assume that $h \in \mathcal{H}$ is in ξ_{ea}-equilibrium and define,

$$\begin{aligned} P^1_i &:= \{k \in P_i : \xi_{ea} \circ \tau_k(h) = u^i_{ea}(h)\}, \\ P^2_i &:= \{k \in P_i : \xi_{ea} \circ \tau_k(h) > u^i_{ea}(h)\}. \end{aligned} \tag{16}$$

Then for any $\bar{h} \in \mathcal{H}$, we have

$$\begin{aligned} & \bar{\xi}_{ea}(h)^{\top}(\bar{h} - h) \\ = & \sum_{i\in\mathcal{I}} \sum_{k\in P_i} \xi^k_{ea} \circ \tau_k(h)(\bar{h}_k - h_k) \\ = & \sum_{i\in\mathcal{I}} \left\{ \sum_{k\in P^1_i} u^i_{ea}(h)(\bar{h}_k - h_k) + \sum_{k\in P^2_i} u^i_{ea}(h)\bar{h}_k \right\} \\ = & \sum_{i\in\mathcal{I}} u^i_{ea}(h) \sum_{k\in P_i} (\bar{h}_k - h_k) \\ = & \sum_{i\in\mathcal{I}} u^i_{ea}(h)(d_i - d_i) \\ = & \ 0, \end{aligned}$$

i.e., h solves the variational inequality (14). ■

Corollary 1 *Let $D \subset \mathbb{R}^{\ell}$ be a base of $\mathbb{R}^{\ell}_{+}$. Then the path flow $h \in \mathcal{H}$ is in weak vector equilibrium if and only if there exists a $d \in D \cap int\mathbb{R}^{\ell}_{+}$ such that h solves*

$$\bar{\xi}_{d0}(h)^{\top}(\bar{h} - h) \geq 0, \quad \forall \bar{h} \in \mathcal{H}. \tag{17}$$

Proof. Since $\xi_{e0}(y)$ is positively homogeneous for $\alpha > 0$ we have $\xi_{e0}(\alpha y) = \alpha\xi_{e0}(y)$. Since D is a base, for $e \in int\mathbb{R}^{\ell}_{+}$, there exist $\alpha_1 > 0$ and $d \in D$ such that $e = \alpha_1 d$, and we have $\xi_{e0}(y) = \dfrac{1}{\alpha_1}\xi_{d0}(y)$. Thus, by Theorem 6 and Theorem 7, the result of this Corollary holds. ■

4 VECTOR EQUILIBRIUM PROBLEM

In this section, we consider an equilibrium problem defined on transportation networks with vector-valued cost functions. In this model, the cost space is again ℓ-dimensional Euclidean space $\mathbb{R}^{\ell}$, with the ordering cone C, a pointed, closed and convex cone with nonempty interior int$C$.

Definition 6 *Given a flow h, we say that a path $p \in P_i$ for an O-D pair i is a minimal one if there does not exist another path $p' \in P_i$ such that $\tau_{p'}(h) - \tau_p(h) \leq_{C\setminus\{0\}} 0$.*

Let $\Gamma_i(h) = \{\tau_p(h) : p \in P_i\}$ denote the (discrete) set of vector costs for all paths for O-D pair i, and

$$\mathcal{I}'_i(h) = \{k \in P_i \mid \tau_k(h) - \tau_p(h) \not\geq_{C\setminus\{0\}} 0, \ \forall p \in P_i\} \subseteq P_i$$

denote the set of all minimal paths for O-D pair i.

We define the minimal frontier for O-D pair i to be the set of minimal points in the cost-space of O-D pair i:

$$\mathrm{Min}_C(\Gamma_i(h)) = \{\xi \in \mathbb{R}^{\ell} \mid \xi = \tau_p(h) \text{ where } p \in \mathcal{I}'_i(h)\}.$$

Note that $\mathrm{Min}_C(\Gamma_i(h))$ is a discrete set because it is a subset of $\mathcal{I}'_i(h)$ and $\mathcal{I}'_i(h)$ is a discrete set.

The following vector equilibrium principle is a generalization of the well-known Wardrop's equilibrium principle (see Wardrop [1]):

Definition 7 *A flow $h \in \mathcal{H}$ is said to be in vector equilibrium if*

$$\forall i \in \mathcal{I}, \forall k, l \in P_i, \quad \tau_k(h) \geq_{C\setminus\{0\}} \tau_l(h) \Longrightarrow h_k = 0.$$

A flow h in vector equilibrium is often referred to as a vector equilibrium flow.

In terms of the minimal frontier for O-D pair i, the vector equilibrium principle can be stated in an equivalent form:

Definition 8 *(Equivalent vector equilibrium principle) The path flow vector h is in vector equilibrium if:*

$$\forall i \in \mathcal{I},\ \forall p \in P_i,\ h_p = 0 \ \ \textit{whenever} \ \ \tau_p(h) \notin Min_C(\Gamma_i(h)). \tag{18}$$

Definition 9 *(Parametric equilibrium principle) Let a parameter $\lambda \in C^*$ be given. A path flow vector h is in λ-equilibrium if*

$$\forall i \in \mathcal{I},\ \forall p \in P_i,\ h_p = 0 \ \ \textit{whenever} \ \ \exists\ e_i \in Min_C(\Gamma_i(h)),$$
$$\textit{such that } \lambda^\top \tau_p(h) > \lambda^\top e_i.$$

Assumption 2 $Min_C(\Gamma_i(h)) \subseteq Min_C(co(\Gamma_i(h)))$.

We need the following scalarization result.

Lemma 3 *Let $A \subset I\!R^\ell$ be a nonempty and convex set and $a^* \in Min_C A$. Then, there exists $\lambda \in intC^*$ such that*

$$\lambda^\top a^* = \min_{a \in A} \lambda^\top a.$$

The following result establishes relationships between a vector equilibrium flow and a parametric equilibrium flow.

Theorem 8 *(i) If h is in vector equilibrium and Assumption 2 holds, then there exists $\lambda \in C^* \setminus \{0\}$ such that the path flow h is in λ-equilibrium;*

(ii) If h is in λ-equilibrium for some $\lambda \in$ intC, then h is in vector equilibrium.*

Proof. (i) Similar to the proof of Theorem 1 (i), but using Lemma 3 instead.

(ii) Let $\lambda \in \text{int}\, C^*$ and let h be in λ-equilibrium. Suppose that h is not in vector equilibrium, then by Definition 7, there exists $i \in \mathcal{I}, p \in P_i$ such that,

$$h_p > 0 \ \text{ and } \ \tau_p(h) \notin \text{Min}_C(\Gamma_i(h)).$$

Thus

$$h_p > 0 \ \text{ and } \lambda^\top \tau_p(h) > \lambda^\top e_i, \ \text{ for some } \ e_i \in \text{Min}_C(\Gamma_i(h)).$$

Hence h is not in λ-equilibrium, a contradiction. ■

Lemma 4 *Let $u_i(\lambda)$ be defined. If $\lambda \in$ int C^*, then $u_i(\lambda) = \lambda^\top e_i$ for some $e_i \in Min_C(\Gamma_i(h))$.*

Proof. From (5), let $p \in P_i$ be such that $u_i(\lambda) = \lambda^\top \tau_p(h)$. Choose $e_i := \tau_p(h)$. Suppose now that $e_i \notin \text{Min}_C(\Gamma_i(h))$, then there exists $\bar{p} \in P_i$, such that $\tau_p(h) \geq_{C\setminus\{0\}} \tau_{\bar{p}}(h)$. Since $\lambda \in intC^*$, $\lambda^\top \tau_p(h) > \lambda^\top \tau_{\bar{p}}(h)$, a contradiction. Therefore $e_i \in \text{Min}_C(\Gamma_i(h))$. ■

Theorem 9 *(i) Let $\lambda \in C^*$. Then h is in λ-equilibrium if the following condition holds:*

$$\forall i \in \mathcal{I}, \forall p \in P_i,\ h_p = 0 \ \textit{whenever}\ \lambda^\top \tau_p(h) > u_i(\lambda); \tag{19}$$

(ii) If $\lambda \in$ int C^* and h is in λ-equilibrium, then condition (19) holds.

Proof. (i) If there exists $e_i \in \text{Min}_C(\Gamma_i(h))$ such that $\lambda^\top \tau_p(h) > \lambda^\top e_i$, say $e_i = \tau_q(h)$ for some $q \in P_i$, then $\lambda^\top \tau_p(h) > \lambda^\top \tau_q(h),\ q \in P_i$. Thus, clearly,

$$\lambda^\top \tau_p(h) > u_i(\lambda) = \min_{p \in P_i} \lambda^\top \tau_p(h),$$

by (19), $h_p = 0$, so h is in λ-equilibrium.

(ii) Let h be a λ-equilibrium flow. By Lemma 4, there exists $e_i \in \min_C(\Gamma_i(h))$ such that $u_i(\lambda) = \lambda^\top e_i$. Suppose that $\lambda^\top \tau_p(h) > u_i(\lambda)$. Then

$$\lambda^\top \tau_p(h) > \lambda^\top e_i.$$

By Definition 9, $h_p = 0$ and hence (19) holds. ■

We may now establish a sufficient condition for a flow h to be in vector equilibrium.

Theorem 10 *$h \in \mathcal{H}$ is in vector equilibrium if h solves the (VVI) of finding $h \in \mathcal{H}$ such that*

$$T(h)(\bar{h} - h) \not\leq_{C\setminus\{0\}} 0,\ \forall \bar{h} \in \mathcal{H}. \tag{20}$$

Proof. Let h satisfy (20). Choose $\bar{h}$ to be such that

$$\bar{h}_j = \begin{cases} h_j, & \text{if } j \neq k \text{ or } j, \\ 0, & \text{if } j = k, \\ h_k + h_j, & \text{if } j = j. \end{cases}$$

Clearly, $\bar{h} \in \mathcal{H}$ since $\forall i \in \mathcal{I},\ \sum_{j \in P_i} \bar{h}_j = \sum_{j \in P_i} h_j = d_i$. Now

$$\begin{aligned} T(h)(\bar{h} - h) &= \sum_{i \in \mathcal{I}} \sum_{j \in P_i} (\bar{h}_j - h_j)\tau_j(h) \\ &= (\bar{h}_k - h_k)\tau_k(h) + (\bar{h}_j - h_j)\tau_j(h) \\ &= h_k(\tau_j(h) - \tau_k(h)) \not\leq_{C\setminus\{0\}} 0. \end{aligned} \tag{21}$$

If

$$\tau_k(h) - \tau_j(h) \geq_{C\setminus\{0\}} 0, \tag{22}$$

then (21) and (22) together imply that $h_k = 0$ since C is a pointed cone. Thus, h is in vector equilibrium. ■

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