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Network Structures in a Society Composed of Individuals with Utilities Depending on Their Reputation

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ABSTRACT

We observe structures of social networks in the real world which are not always simple such as the empty, the complete, or the star network. In this paper, by introducing effect of individuals' reputation in the society, we show that a decentralized connected star network which consists of multiple disjoint components connecting some star networks can be also stable.

Keywords: Social networks, reputation, stability, decentralization.

1. INTRODUCTION

In this paper, we deal with forming a social network. A network is defined as a set of the members (players) belonging to a society where the network is formed and the links connecting players. In published papers on networks, using the framework of game theory, a number of studies have been made on the payoff allocation rules among players taking the network structure into account and the stability and efficiency of the network [3, 4].

From the viewpoints of stability and efficiency of networks, Jackson and Wolinsky [3] give a network model leading to some of the empty, the complete, or the star network. However, structures of social networks in the real world such as networks of groups with interest in environmental issues are not always one of them, and it is observed that such a network consists of multiple components in different regions and nations.

In most of models for analyzing the stability and efficiency of networks, a utility of a player of the network is composed of the benefit from the network and the cost to form links. For analyzing communities with individuals interested in social issues, it is natural that a member in the community ought to take into account the benefit from the network which deals with the social issues as well as the private gain or loss obtained by participating the network, and he or she may care about the reputation in the community. In this paper, by introducing effect of individuals' reputation in the community suggested by Akerlof [1], we define a utility function including a term of the reputation and show that a general network can be stable such as a decentralized connected star network which consists of multiple disjoint components connecting some star networks.

2. PRELIMINARIES

Let $N = \{1, \dots, n\}$ be the set of all players or members in a community. A network is characterized by nodes representing players and arcs representing links connecting any two players.

Definition 1 A subset containing two elements $\{i, j\}$ of the set N is called a *link* and it is denoted by ij. A *network* g is represented by a set of links. Especially, the set of all the possible links is referred to as the *complete network* and it is denoted by g^N, and the empty set $g = \emptyset$ of links is called the *empty network*.

The complete network and the empty network are shown in Figure 1.

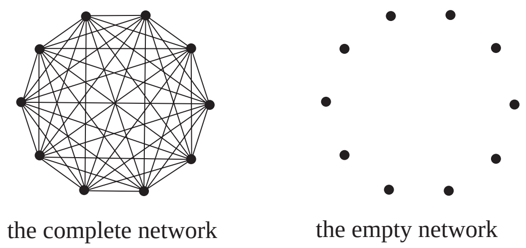


Figure 1: The complete network and the empty network

For any network $g \subseteq g^N$, if $ij \in g$, a link is formed between players i and j, and if $ij \notin g$, any link is not formed between them.

Definition 2 A function v, called a *value function* of a network, is a real-valued function which associates any network $g \subseteq g^N$ with its real number $v(g) \in \mathbb{R}$, and we always let $v(\emptyset) = 0$.

A value $v(g)$ of a network g is interpreted as a benefit obtained from the network g.

Definition 3 A network g is said to be *efficient* if $v(g) \geq v(g')$ for all $g' \subseteq g^N$.

Definition 4 For a given set of players $\{i_k, \dots, i_l\} \subseteq N$, if a pair of the players i_k and i_l is connected through a set of links $\{i_k i_{k+1}, \dots, i_{l-1} i_l\}$, then such a connection is said to be a *path* connecting i_k and i_l, which is denoted by $i_k \overset{g}{\leftrightarrow} i_l$.

From Definition 4, if $ij \in g$, we have $i \overset{g}{\leftrightarrow} j$, and the link ij is a trivial path directly connecting players i and j.

Definition 5 For a given network g, let $N(g) = \{i \mid$ there exists j such that $ij \in g\}$ denote a set of players possessing a link in the network g. Then, a subnetwork $g' \subseteq g$ is called a *component*, if the following two conditions are satisfied:

1. Any pair of players $i, j \in N(g'), i \neq j$ have a path.
2. If, for players $i \in N(g')$ and $j \in N(g)$, $ij \in g$, then $ij \in g'$.

Especially, a component is said to be *minimal* if the component is divided into two components by deleting any one of the links in the component.

An example of a minimal component is shown in Figure 2.

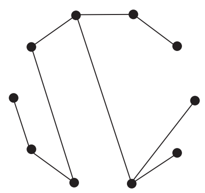


Figure 2: A minimal component

Definition 6 A function π_i, called a *utility function* of a player i, is a real-valued function which associates any network $g \subseteq g^N$ with its real number $\pi_i(g) \in \mathbb{R}$.

For a given network g, let $\Delta_{i,g}^{+ij}$ denote an increment of the utility of player i by forming a new link $ij \notin g$, and let $\Delta_{i,g}^{-ij}$ denote an increment of the utility of player i by deleting an existing link $ij \in g$. These values are given as follows:

$$\Delta_{i,g}^{+ij} = \pi_i(g + ij) - \pi_i(g), \; ij \notin g, \quad (1)$$

$$\Delta_{i,g}^{-ij} = \pi_i(g - ij) - \pi_i(g), \; ij \in g. \quad (2)$$

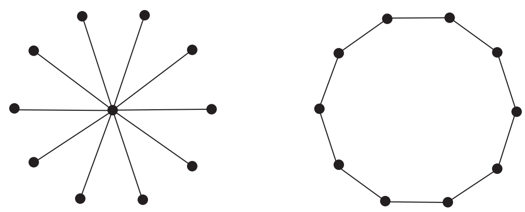
It is assumed that, for a given network g, a new link $ij \notin g$ is formed if $\Delta_{i,g}^{+ij} \geq 0$ and $\Delta_{j,g}^{+ij} \geq 0$, and an existing link $ij \in g$ is deleted if $\Delta_{i,g}^{-ij} > 0$ or $\Delta_{j,g}^{-ij} > 0$. Namely, we assume mutual link formation where a new link is formed if each of the utilities of both players connected does not decrease by the link formation, and unilateral link deletion where an existing link is deleted without consideration of the utility status of the connected player if the utility of one player increases by the link deletion.

Definition 7 Player i is said to be *stable* if $\Delta_{i,g}^{+ij} < 0$ and $\Delta_{i,g}^{-ij} \leq 0$ for any player $j \in N$.

Definition 8 A network g is said to be *stable* if a new link is not formed and an existing link is not deleted in the network g.

Although a network is stable if each of the players in the network is stable, all the player in a network are not always stable even if the network is stable as shown in a following section.

Jackson and Wolinsky [3] show that a stable network is not always efficient under the condition of the mutual link formation and the unilateral link deletion, and that a star network shown in Figure 3 with links between a certain central player and the other players can be stable as well as the complete and the empty networks in their network model. Assuming another conditions on the link formation, Hummon [2] shows that a ring network as shown in Figure 3 can be also stable.



a star network　　　a ring network

Figure 3: A star network and a ring network

As mentioned above, although each of the complete network, the empty network, the star network, and the ring network can be stable, in the real world we observe more general shapes of networks such as a decentralized connected star network which consists of multiple disjoint components connecting some star networks shown in Figure 4. In the following section we intend to consider the possibility of formation of more general networks.

3. NETWORK STRUCTURE

Jackson and Wolinsky [3] define the utility of a player as a function of the benefit from a network and the cost of network formation, and in the model the benefit of a

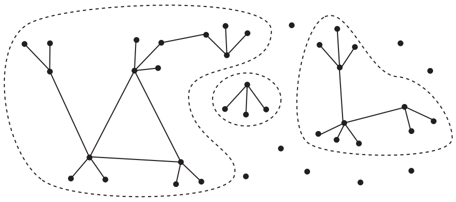


Figure 4: A decentralized connected star network

network depends on distances between two players connected through a path. In this paper, however, because we focus on networks providing commons and in such a network each of players equally benefits from the commons, we newly define the utility function of a player to suit to this situation.

We assume that the value $v(g)$ of a network g increases by ε_{ij} if a new link $ij \notin g$ is formed, and it additionally increases by δ_{ij} if there does not exist a path between the players i and j before they are tied with the link. The increment ε_{ij} means an increase of the value of the network by simply adding one link ij, and the increment δ_{ij} means an increase of the value of the network by additionally connecting two distinct components with the exception of the effect of adding one link.

Let a modified network g to which a new link $ij \notin g$ is added be denoted by $g + ij \equiv g \cup \{ij\}$, and let a modified network g from which an existing link $ij \in g$ is deleted be denoted by $g - ij \equiv g \setminus \{ij\}$. After forming a new link ij, the value of a modified network $g + ij$ is represented as

$$v(g+ij) = \begin{cases} v(g) + \varepsilon_{ij} & \text{if } i \overset{g}{\leftrightarrow} j \\ v(g) + \varepsilon_{ij} + \delta_{ij} & \text{if } i \overset{g}{\not\leftrightarrow} j. \end{cases} \quad (3)$$

Moreover, we formulate the utility function of a player by taking into account not only the benefit from a network and the cost of link formation but also the social norm in a society, which means the individual reputation in the network, and then we examine the effect of the individual reputation with respect to stability of the network structure.

3.1. The Utility Function without the Social Norm

First we consider a model with a utility function of a player depending only on the benefit from a network and the cost of network formation.

Let c_{ij} denote the cost for forming the link ij, where the cost for maintaining it is also included. Let $\beta_i \in [0,1]$ denote the marginal benefit of player i from the value of a network g. Then, we define a utility $\pi_i(g)$ of player i from the network g as follows:

$$\pi_i(g) = \beta_i v(g) - \sum_{j \in P_i^g} c_{ij}, \quad (4)$$

where P_i^g is a set of players who are directly connected by a link with player i in the network g, i.e., $P_i^g = \{j \mid ij \in g\}$.

We assume the symmetric cost of forming a link and focus on the simple and symmetric benefit networks, and therefore let $\beta_i = 1$, $\varepsilon_{ij} = \varepsilon > 0$, $\delta_{ij} = \delta > 0$, $c_{ij} = c > 0$ for all $i, j \in N$. Let the number of elements of the set P_i^g be denoted by $|P_i^g| = e_i^g$, and then the utility function (4) is simply rewritten as

$$\pi_i(g) = v(g) - e_i^g c. \quad (5)$$

Theorem 1 For a given network g with the utility function (5), the following three statements hold.

(i) If $c \leq \varepsilon$, the complete network is uniquely stable.

(ii) If $\varepsilon < c \leq \delta + \varepsilon$, a network which is a single minimal component containing all the players is stable.

(iii) If $\delta + \varepsilon < c$, the empty network is uniquely stable.

Proof From (1) and (3) , we have

$$\Delta_{i,g}^{+ij} = \begin{cases} \delta + \varepsilon - c & \text{if } i \overset{g}{\not\leftrightarrow} j \\ \varepsilon - c & \text{if } i \overset{g}{\leftrightarrow} j, \end{cases} \quad (6)$$

and similarly from (2) and (3) , we have

$$\Delta_{i,g}^{-ij} = \begin{cases} -(\delta + \varepsilon) + c & \text{if } i \overset{g-ij}{\not\longleftrightarrow} j \\ -\varepsilon + c & \text{if } i \overset{g-ij}{\longleftrightarrow} j. \end{cases} \quad (7)$$

If $c \leq \varepsilon$, from (6), for all $ij \notin g$, we have $\Delta_{i,g}^{+ij} \geq 0$, $g \subseteq g^N$, $i \in N$. Similarly, from (7), for all $ij \in g$, we have $\Delta_{i,g}^{-ij} \leq 0$, $g \subseteq g^N$, $i \in N$. Namely, all possible links are formed, and any existing link is not deleted. Therefore, the complete network is formed and then statement (i) holds.

If $\varepsilon < c \leq \delta + \varepsilon$, from (6), for all $ij \notin g$, we have $\Delta_{i,g}^{+ij} \geq 0$ for all i such that $i \overset{g}{\not\leftrightarrow} j$, and we have $\Delta_{i,g}^{+ij} < 0$ for all i such that $i \overset{g}{\leftrightarrow} j$. Similarly, from (7), for all $ij \in g$, we have $\Delta_{i,g}^{-ij} \leq 0$ for all i such that $i \overset{g-ij}{\not\longleftrightarrow} j$, and we have $\Delta_{i,g}^{-ij} > 0$ for all i such that $i \overset{g-ij}{\longleftrightarrow} j$. Namely, when there does not exist any path between players i and j, the link ij is formed, and when there exists a nontrivial path between players i and j, the link ij is deleted. Thus, a single minimal component containing all the players is formed and then statement (ii) holds.

If $\delta+\varepsilon<C$, from (6), for all $ij\notin g$, we have $\Delta^{+ij}_{i,g}<0$, $g\subseteq g^N$, $i\in N$. Similarly, from (7), for all $ij\in g$, we have $\Delta^{-ij}_{i,g}>0$, $g\subseteq g^N$, $i\in N$. Namely, no link is formed, and each of the existing links are deleted. Therefore, the empty network is formed and then statement (iii) holds. ■

Although Jackson and Wolinsky [3] show that the complete, the empty, or the star network is stable, in our model we have shown that the complete or the empty network is also stable but a single minimal component containing all the players can be stable instead of the star network. The cost of link formation in our model is the same as that of Jackson and Wolinsky, but the benefit from a network is not the same. Because, in the Jackson and Wolinsky model, a player receives a more benefit from neighboring players than players in the distance, we can intuitively understand that the star network in which a distance between any pair of players is one or two link length is likely to be formed. In contrast, because we suppose that each player equally receives a benefit from the commons provided by the network in our model, a distance between players need not be smaller for the sake of stability. Simply, the number of components and the number of links become smaller and then it follows that a single minimal component containing all the players is formed.

3.2. The Utility Function with the Social Norm

In this subsection we consider a model with a utility function of a player depending not only on the benefit from a network and the cost of network formation but also on the individual reputation in the network.

Let $R_i:\{g\mid g\subset g^N\}\to\mathbb{R}$ denote a real-valued function of the reputation of player i in the network g, and let $a_i\in[0,1]$ denote the personal tastes representing a rate how player i cares about his or her reputation in the network. Then, the utility function of player i including the effect of the reputation $a_iR_i(g)$ in the network is represented by

$$\pi_i(g)=a_iR_i(g)+\beta_i v(g)-\sum_{j\in P_i^g}c. \qquad (8)$$

As we assumed in the previous subsection, we consider a simple model with the symmetric cost and benefit networks, and therefore let $\beta_i=1$, $\varepsilon_{ij}=\varepsilon>0$, $\delta_{ij}=\delta>0$, $c_{ij}=c>0$ for all $i,j\in N$. Then the utility function (8) is simply rewritten as

$$\pi_i(g)=a_iR_i(g)+v(g)-e_i^g c. \qquad (9)$$

Akerlof [1] assumes that the reputation of an individual depends on his or her obedience of the code of behavior and also on the portion of the population who believe in that code. Nyborg and Rege [5] deal with the formation of social norms for considerate smoking behavior, and express the reduced social approval of being inconsiderate as the product of the public's belief about adverse health effects for passive smoking and the average consideration level in society.

We try to formulate the reputation of an individual suitable for the context of network formation. In our social network model, the link formation corresponds to the code of behavior, and the proportion of the population who make links corresponds to the portion of the population who believe in the code. Moreover, because it is natural that the reputation of a player relates to the number of links between the player and the other players as well as whether there is any link, we define the reputation as the ratio of the number of links of the player to the average number of links in the network. Namely, the average of links in the network g is $\overline{e^g}=\sum_{k\in N}e_k^g/n$, and then the reputation $R_i(g)$ of player i is represented by

$$R_i(g)=e_i^g/\overline{e^g}. \qquad (10)$$

To consider stable networks, we give conditions for forming and deleting a link in the following.

3.2.1 Link Formation

For a given network g, from the value function (3), the utility function (9), and the reputation (10), an increment of the utility of player i by forming a new link $ij\notin g$ is rewritten as

$$\Delta^{+ij}_{i,g}=\begin{cases}\dfrac{n\overline{e^g}-2e_i^g}{\overline{e^g}(n\overline{e^g}+2)}a_i+\delta+\varepsilon-c & \text{if } i\overset{g}{\not\leftrightarrow}j,\ e_i^g\geq 0\\ \dfrac{n\overline{e^g}-2e_i^g}{\overline{e^g}(n\overline{e^g}+2)}a_i+\varepsilon-c & \text{if } i\overset{g}{\leftrightarrow}j,\ e_i^g\geq 1.\end{cases} \qquad (11)$$

In (11), if there exists a path connecting i and j, we have $e_i^g\geq 1$ because there exists at least one link. Then, a condition $\Delta^{+ij}_{i,g}\geq 0$ of player i for forming a new link $ij\notin g$ is given as the following two inequalities

$$0\leq e_i^g\leq -\frac{\overline{e^g}(n\overline{e^g}+2)(c-(\delta+\varepsilon))}{2}\frac{1}{a_i}+\frac{n\overline{e^g}}{2}\quad \text{if } i\overset{g}{\not\leftrightarrow}j, \qquad (12)$$

$$1\leq e_i^g\leq -\frac{\overline{e^g}(n\overline{e^g}+2)(c-\varepsilon)}{2}\frac{1}{a_i}+\frac{n\overline{e^g}}{2}\quad \text{if } i\overset{g}{\leftrightarrow}j. \qquad (13)$$

Because the conditions (12) and (13) of player i depend only on the personal tastes a_i and the number of links e_i^g, a set of points (a_i,e_i^g) satisfying the conditions (12) and

(13) is given in the a-e^g plain. Let a and e^g be generic representations of a_i and e_i^g, and we define the right hand sides of (12) and (13) as follows:

$$f_0^g(a) \equiv -\frac{\bar{e}^g(n\bar{e}^g+2)(c-(\delta+\varepsilon))}{2}\frac{1}{a}+\frac{n\bar{e}^g}{2}, \quad (14)$$

$$f_1^g(a) \equiv -\frac{\bar{e}^g(n\bar{e}^g+2)(c-\varepsilon)}{2}\frac{1}{a}+\frac{n\bar{e}^g}{2}. \quad (15)$$

Using $f_0^g(a)$ and $f_1^g(a)$, we define the areas satisfying the conditions (12) and (13) as

$$F_0^g = \{(a,e^g) \mid e^g \leq f_0^g(a)\}, \quad (16)$$

$$F_1^g = \{(a,e^g) \mid e^g \leq f_1^g(a)\}. \quad (17)$$

From $\delta > 0$, we have $f_0^g(a) - f_1^g(a) = \delta\bar{e}^g(n\bar{e}^g + 2)/(2a) > 0$, and therefore $f_0^g(a) > f_1^g(a)$ holds. From this fact, we have $F_1^g \subset F_0^g$. Then, from (12), (13) and the fact that $F_1^g \subset F_0^g$, for any player i, the followings hold.

$$(a_i,e_i^g) \in F_0^g \iff \Delta_{i,g}^{+ij} \geq 0,\ j \in \left\{ j \in N \mid ij \notin g,\ i \overset{g}{\not\leftrightarrow} j \right\}, \quad (18)$$

$$(a_i,e_i^g) \in F_1^g \iff \Delta_{i,g}^{+ij} \geq 0,\ j \in \{ j \in N \mid ij \notin g \}. \quad (19)$$

The statement (18) means that a point (a_i,e_i^g) of player i in a network g is in the area F_0^g if and only if player i is ready to form a link with each of players who are not connected through a nontrivial path, and the statement (19) means that a point (a_i,e_i^g) is in the area F_1^g if and only if player i is ready to form a link with anyone of players who are still not connected directly by a link.

The functions f_0^g and f_1^g are hyperbolic functions of a, and the second term is the common constant $n\bar{e}^g/2$. Signs of the coefficients of the first terms depend on the relation of ε, δ, and c. Namely, if $c \leq \varepsilon$, the coefficients of the first terms of f_0^g and f_1^g are nonnegative. If $\varepsilon < c \leq \delta+\varepsilon$, the coefficient of f_0^g is nonnegative, and that of f_1^g is negative. If $\delta+\varepsilon < c$, both of them are negative. Moreover, from $f_0^g(a) > f_1^g(a)$, for a given network g, the areas of link formation can be expressed as in Figure 5.

In Figure 5, the values of $\hat{f}_{00}^g$ and $\hat{f}_{11}^g$ are given as

$$\hat{f}_{00}^g = \frac{(c-\delta-\varepsilon)(n\bar{e}^g+2)}{n}, \quad (20)$$

$$\hat{f}_{11}^g = \frac{\bar{e}^g(c-\varepsilon)(n\bar{e}^g+2)}{n\bar{e}^g-2}. \quad (21)$$

From $f_1^g(a) > f_2^g(a)$, we have $\hat{f}_{00}^g < \hat{f}_{11}^g$. When $\delta+\varepsilon < c$, we have $\hat{f}_{00}^g > 0$, and when $\varepsilon < c$, because, for the point $(\hat{f}_{11}^g, 1)$, $n\bar{e}^g = \sum_{k\in N} e_k^g \geq 2$, we have $\hat{f}_{11}^g \geq 0$.

3.2.2 Link Deletion

For a given network g, from the value function (3), the utility function (9), and the reputation (10), an increment of the utility of player i by deleting an existing link $ij \in g$ is rewritten as

$$\Delta_{i,g}^{-ij} = \begin{cases} \dfrac{n\bar{e}^g - 2e_i^g}{\bar{e}^g(n\bar{e}^g-2)}a_i - (\delta+\varepsilon) + c & \text{if } i \overset{g-ij}{\not\leftrightarrow} j,\ e_i^g \geq 1 \\ \dfrac{n\bar{e}^g - 2e_i^g}{\bar{e}^g(n\bar{e}^g-2)}a_i - \varepsilon + c & \text{if } i \overset{g-ij}{\longleftrightarrow} j,\ e^g \geq 2. \end{cases} \quad (22)$$

In (22), we have $e_i^g \geq 1$ because there exists at least one link to delete, and if there exists a path connecting i and j, we have $e_i^g \geq 2$. Then, a condition $\Delta_{i,g}^{-ij} \geq 0$ of player i for deleting an existing link $ij \in g$ is given as the following two inequalities

$$1 \leq e_i^g \leq -\frac{\bar{e}^g(n\bar{e}^g-2)(c-\varepsilon)}{2}\frac{1}{a_i}+\frac{n\bar{e}^g}{2} \ \text{ if } i \overset{g-ij}{\not\leftrightarrow} j, \quad (23)$$

$$2 \leq e_i^g \leq -\frac{\bar{e}^g(n\bar{e}^g-2)(c-(\delta+\varepsilon))}{2}\frac{1}{a_i}+\frac{n\bar{e}^g}{2} \ \text{ if } i \overset{g-ij}{\longleftrightarrow} j. \quad (24)$$

We give a set of points (a_i,e_i^g) satisfying the conditions (23) and (24) in the a-e^g plain in a way similar to that of the link formation. Let the right hand sides of (23) and (24) be defined as follows:

$$d_0^g(a) \equiv -\frac{\bar{e}^g(n\bar{e}^g-2)(c-\varepsilon)}{2}\frac{1}{a}+\frac{n\bar{e}^g}{2}, \quad (25)$$

$$d_1^g(a) \equiv -\frac{\bar{e}^g(n\bar{e}^g-2)(c-(\delta+\varepsilon))}{2}\frac{1}{a}+\frac{n\bar{e}^g}{2}. \quad (26)$$

Using $d_0^g(a)$ and $d_1^g(a)$, we define the areas satisfying the conditions (23) and (24) as

$$D_0^g = \{(a,e^g) \mid e^g > d_0^g(a)\}, \quad (27)$$

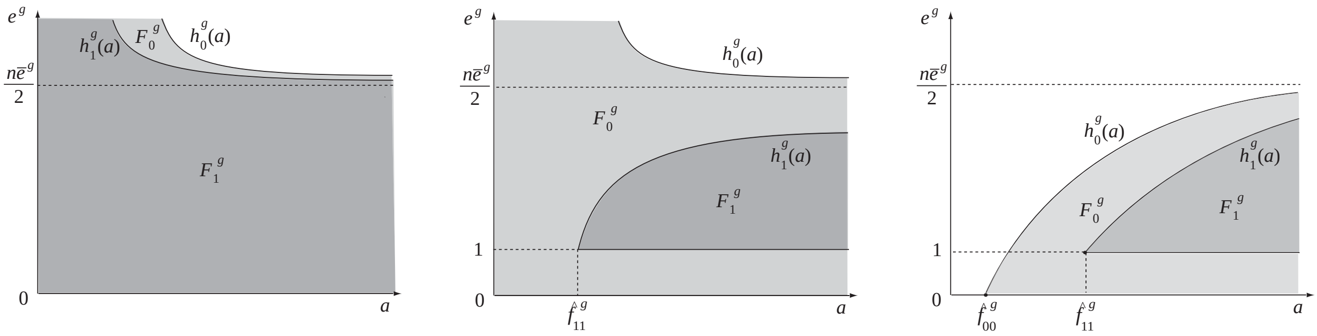
$$D_1^g = \{(a,e^g) \mid e^g > d_1^g(a)\}. \quad (28)$$

If $e_i^g > d_0^g(a)$ or $e_i^g > d_1^g(a)$, then $e_i^g \geq 1$ or $e_i^g \geq 2$, respectively, and therefore we have $n\bar{e} = \sum_{k\in N} e_k^g \geq 2$. From this fact and $\delta > 0$, we have $d_0^g(a) - d_1^g(a) = \delta\bar{e}^g(n\bar{e}^g - 2)/(2a) > 0$, and therefore $d_0^g(a) > d_1^g(a)$ holds. Thus, we have $D_0^g \subset D_1^g$ in the area of $e_i^g \geq 2$. From (23), (24) and the fact that $D_0^g \subset D_1^g$, for any player i, the followings hold.

$$(a_i,e_i^g) \in D_0^g \iff \Delta_{i,g}^{-ij} > 0,\ j \in \{ j \in N \mid ij \in g \}, \quad (29)$$

$$(a_i,e_i^g) \in D_1^g \iff \Delta_{i,g}^{-ij} > 0,\ j \in \left\{ j \in N \mid ij \in g,\ i \overset{g-ij}{\longleftrightarrow} j \right\}. \quad (30)$$

The statement (29) means that a point (a_i,e_i^g) of player i in a network g is in the area D_0^g if and only if player



(a) $C \leq \varepsilon$ (b) $\varepsilon < C \leq \delta+\varepsilon$ (c) $\delta+\varepsilon < C$

Figure 5: Areas of link formation

i is ready to delete a link with anyone of players who are connected directly by the link, and the statement (30) means that a point (a_i, e_i^g) is in the area D_1^g if and only if player i is ready to delete a link with each of players who are connected not only directly by the link but also by another nontrivial path.

Similarly to the functions f_0^g and f_1^g, if $c \leq \varepsilon$, the coefficients of the first terms of d_0^g and d_1^g are nonpositive. If $\varepsilon < c \leq \delta+\varepsilon$, the coefficient of d_0^g is positive, and that of d_1^g is nonpositive. If $\delta+\varepsilon < c$, both of them are positive. Moreover, from $d_0^g(a) > d_1^g(a)$, for a given network g, the areas of link deletion can be expressed as in Figure 6.

In Figure 6, the values of $\hat{d}_{01}^g$ and $\hat{d}_{12}^g$ are given as

$$\hat{d}_{01}^g = \overline{e}^g(c-(\delta+\varepsilon)), \quad (31)$$

$$\hat{d}_{12}^g = \frac{\overline{e}^g(c-\varepsilon)(n\overline{e}^g-2)}{n\overline{e}^g-4}. \quad (32)$$

From $d_0^g(a) > d_1^g(a)$, we have $\hat{d}_{01}^g < \hat{d}_{12}^g$, and when $\delta+\varepsilon < c$, we have $\hat{d}_{01}^g > 0$.

3.2.3 Network Formation

We consider network structures for the three cases: (a) $c \leq \varepsilon$, (b) $\varepsilon < c \leq \delta+\varepsilon$, and (c) $\delta+\varepsilon < c$. Before examining each of the three cases, we give the following lemma.

Lemma 1 For a given network g, the number of links in g is not larger than $n\overline{e}^g/2$.

Proof First, consider the empty network $g = \emptyset$. For any player $i \in N$, because $e_i^{\emptyset} = 0 = n\overline{e}^{\emptyset}$, we have $e_i^g \leq n\overline{e}^g/2$.

For a given network g, we assume that $e_i^g \leq n\overline{e}^g/2$ for any $i \in N$.

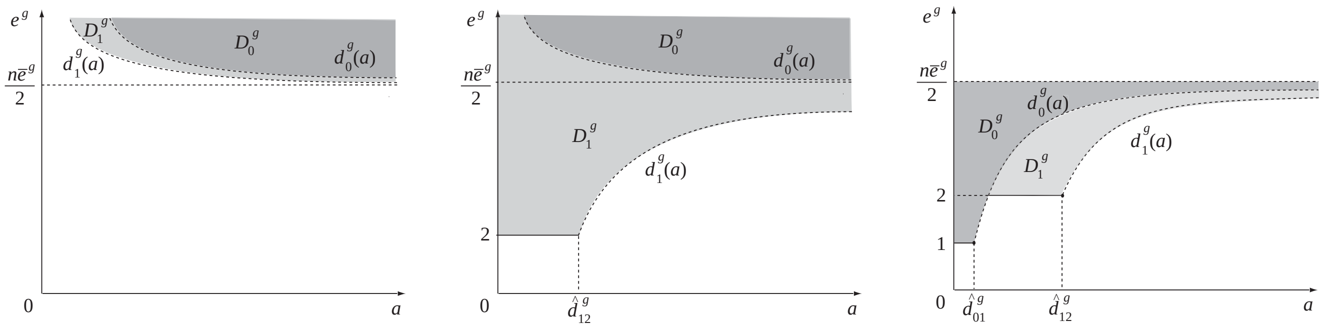
Consider a new network g' such that the number of links increases by one from the given network g. For a player who increases one link, we have $e_i^{g+ij} = e_i^g + 1 \leq n\overline{e}^g/2 + 1 = (n\overline{e}^g+2)/2 = n\overline{e}^{g+ij}/2$. For a player with the same links, we have $e_k^{g+ij} = e_k^g \leq n\overline{e}^g/2 = (n\overline{e}^{g+ij} - 2)/2 < n\overline{e}^{g+ij}/2$. Therefore, for any player i, we have $e_i^{g'} \leq n\overline{e}^{g'}/2$, and the lemma is proved. ■

For networks satisfying (a) $c \leq \varepsilon$, the following theorem is obtained.

Theorem 2 For a society with the utility function with a term of reputation (9) and the symmetric network benefit (3), if the condition (a) $c \leq \varepsilon$ is satisfied, the complete network is uniquely stable.

Proof From Lemma 1, we consider only the area of $0 \leq e^g \leq n\overline{e}^g/2$ in the a-e^g plain. As seen in (a) of Figure 5, the area of $0 \leq e^g \leq n\overline{e}^g/2$ is contained in F_1^g, and therefore each of all the players tries to make a link with each of players who are not connected directly by a link. As seen in (a) of Figure 6, there does not exist the intersection between the area of $0 \leq e^g \leq n\overline{e}^g/2$ and D_1^g, and therefore none of all the players delete any link. Thus, the complete network is uniquely stable. ■

For networks such that (b) $\varepsilon < c \leq \delta+\varepsilon$ is satisfied, no stable network could be formed in the following case. For a network g, assume that all the players except for the players i and j are stable. Moreover, for the players i and j, assume that $(a_i, e_i^g) \in F_1^g$ and $(a_j, e_j^g) \in F_1^g$, and the players i and j do not have a link, but they have a path. Then, the link ij is formed and the network g is changed to the network $g+ij$. If this results in $(a_i, e_i^{g+ij}) \in F_0^{g+ij} \cap D_1^{g+ij}$ for the player i or $(a_j, e_j^{g+ij}) \in F_0^{g+ij} \cap D_1^{g+ij}$ for the player j, the player i or j deletes the link ij and then the network $g+ij$ returns to the original network g. In this manner, if such two networks appear by turns, no stable networks appear. The areas of link formation and deletion on the a-e^g plain in the case of $\varepsilon < c \leq \delta+\varepsilon$ and players i and j are depicted in Figure 7.



(a) $c \leq \varepsilon$ (b) $\varepsilon < c \leq \delta+\varepsilon$ (c) $\delta+\varepsilon < c$

Figure 6: Areas of link deletion

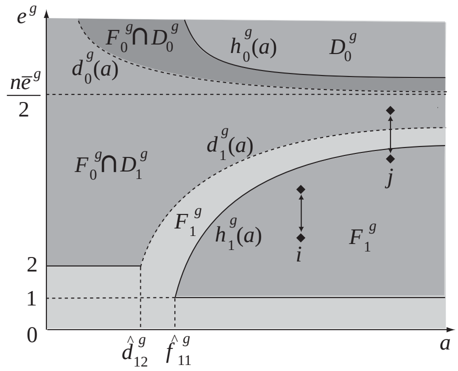


Figure 7: Areas of link formation and deletion in the case of $\varepsilon < c \leq \delta+\varepsilon$

In the case where (b) $\varepsilon < c \leq \delta+\varepsilon$, although there would be no stable networks, a network shown in the following theorem is formed.

Theorem 3 For a society with the utility function with a term of reputation (9) and the symmetric network benefit (3), if the condition (b) $\varepsilon < c \leq \delta+\varepsilon$ is satisfied, a single component network including all the players is formed.

Proof From Lemma 1, we consider only the area of $0 \leq e^g \leq n\overline{e}^g/2$ in the plain of a-e^g. From $\varepsilon < c \leq \delta+\varepsilon$, we have $h_1^g < d_1^g$, and then the area of $0 \leq e^g \leq n\overline{e}^g/2$ is divided into three areas F_1^g, F_0^g, and $F_0^g \cap D_1^g$ as seen in Figure 7.

From the statement (19), players corresponding to the area F_1^g make a new link with other players, and from the statement (18), players corresponding to the area F_0^g make a new link with each of players who are not connected through any path. For players corresponding to the area $F_0^g \cap D_1^g$, the following holds:

$$(a_i, e_i^g) \in F_0^g \cap D_1^g \Longleftrightarrow \Delta_{i,g}^{+ij} \geq 0,\ j \in \{j \in N \mid ij \notin g,\ i \overset{g}{\not\leftrightarrow} j\},\ \text{and}\ \Delta_{i,g}^{-ij} > 0,\ j \in \{j \in N \mid ij \in g,\ i \overset{g-ij}{\longleftrightarrow} j\}. \quad (33)$$

Namely, players corresponding to the area $F_0^g \cap D_1^g$ make a new link with each of players who are not connected through any path, and delete an existing link with each of players who are also connected through another nontrivial path.

Therefore, each player in some component tries to form a link with a player in the other component, who is not connected by any path, and then because any pair of players are connected by a path, a single component network including all the players is formed. ■

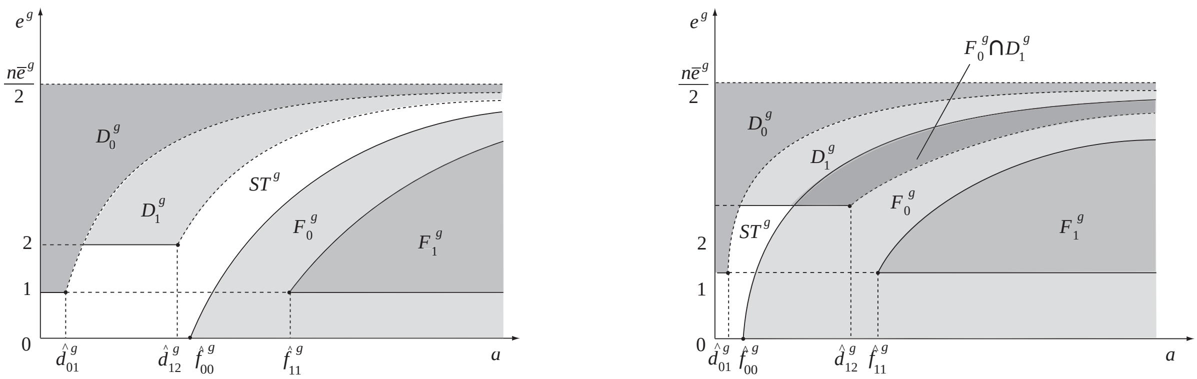
For networks such that (c) $\delta+\varepsilon < c$ is satisfied, similarly no stable network could be formed. Namely, because there would exist players who form and delete links in turn, there would be no stable networks.

Conversely, we consider stable networks when the condition (c) $\delta+\varepsilon < c$ is satisfied. From $\delta+\varepsilon < c$, we have $h_1^g < d_1^g$ and $h_0^g < d_0^g$. Moreover, if $n\overline{e}^g/2 < (2c-(\delta+2\varepsilon))/\delta$, we have $h_0^g < d_1^g$ and if $n\overline{e}^g/2 \geq (2c-(\delta+2\varepsilon))/\delta$, we have $h_0^g \geq d_1^g$. Thus, the area of $0 \leq e^g \leq n\overline{e}^g/2$ can be depicted in two cases shown in Figure 8. The area ST^g in Figure 8 is defined by

$$ST^g = \{(a, e^g) \mid (a, e^g) \notin F_0^g \cup D_0^g \cup D_1^g\}. \quad (34)$$

For players corresponding to the area ST^g, the following holds:

$$(a_i, e_i^g) \in ST^g \Longleftrightarrow \Delta_{i,g}^{+ij} < 0,\ j \in \{j \in N \mid ij \notin g\},\ \text{and}\ \Delta_{i,g}^{-ij} \leq 0,\ j \in \{j \in N \mid ij \in g\}. \quad (35)$$



(a) $\frac{n\bar{e}^g}{2} < \frac{2c-(\delta+2\varepsilon)}{\delta}$ (b) $\frac{n\bar{e}^g}{2} \geq \frac{2c-(\delta+2\varepsilon)}{\delta}$

Figure 8: Areas of link formation and deletion in the case of $\delta + \varepsilon < c$

Namely, players corresponding to the area ST^g do not make a new link and do not intend to delete an existing link, and then they are stable players.

Although players in F_0^g or D_1^g intend to form or delete links, they do not so if there does not exist an appropriate partner. Therefore when all the players are in F_0^g, D_1^g, or ST^g, a stable network can be formed. In such a case, although the empty network is uniquely stable in a model with a utility function not depending on the reputation in the network, there is some chance that more general networks are stable in a model with a utility function with a term of the reputation.

To give a stable network in a shape of a general structure, we conduct the following computational experiment. Let the number of players be $n = 25$, and let the parameters of the value and the cost of a network be set at $\delta = 0.01$, $\varepsilon = 0.03$, and $c = 0.5$. The personal tastes a_i of players are uniformly distributed in the interval $[0, 1]$. The initial state is the empty network, and players make their decisions in the increasing order of their indices. Then, we obtain a stable network shown in Figure 9.

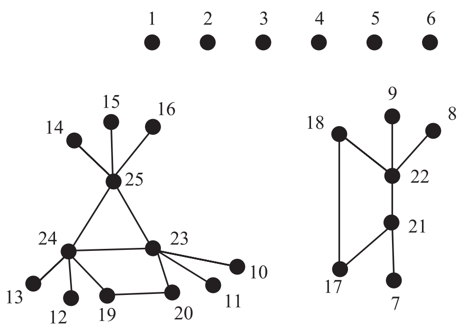


Figure 9: A stable network in a shape of a general structure

The structure of a stable network shown in Figure 9 can be expressed as a decentralized connected star network which consists of multiple disjoint components connecting some star networks, and we can observe such structure of networks in the real world.

4. CONCLUSIONS

In this paper, we focus on networks for providing public goods, and develop a model for analyzing network formation. First, we examine a model with a utility function with the benefit from the network and the cost to form links, and it is found that three types of simple networks can be stable. Next, introducing effect of the reputation in the network, we define a utility function including a term of the reputation and show that a general network can be stable such as a decentralized connected star network which consists of multiple disjoint components connecting some star networks.

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