Vol. 47 No. 3 IPSJ Journal Mar. 2006

Regular Paper

A Review of Recent Studies of Geographical Scale-Free Networks

YUKIO HAYASHI†

The scale-free (SF) structures that commonly appear in many complex networks are a hot topic in social, biological, and information sciences. These self-organized generation mechanisms are expected to be useful for efficient communication or robust connectivity in socio-technological infrastructures. This paper is the first review of geographical SF network models. We discuss the essential generation mechanisms for inducing the structures with power-law behavior, and consider the properties of planarity and link length. Distributed design of geographical SF networks without the crossing and long-range links that cause interference and dissipation problems is very important for many applications such as communications, power grids, and sensor systems.

1. Introduction

A breakthrough in network science[1] has been the discovery[2] that many real systems with social, technological, and biological origins have surprisingly common topological structures called *small-world* (SW)[3] and *scale-free* (SF)[4]. Such structures are characterized by the SW properties that the average path length over all nodes (vertices) is short, like that in random graphs, and that the clustering coefficient, defined as the average ratio of the number of links (edges) connecting a node to its nearest neighbor nodes to the number of possible links between all these nearest neighbors, is large, like that in regular graphs. A large *clustering coefficient* means a high frequency of the case that "the friend of a friend is also one's friend." The SF property is that, the degree distribution follows a power-law, $P(k) \sim k^{-\gamma}$, $2 < \gamma < 3$; the fat-tail distribution consists of many nodes with low degrees and a few hubs with very high degrees. Moreover, a proposal of universal mechanisms[4] for generating SF networks inspired elucidation of the topological properties of such networks. One of the advantages of SF networks is that they are optimal in minimizing both the effort of communication and the cost of maintaining the connections[5]. Intuitively, a SF network is positioned between a star or clique graph for minimizing the path length (the number of hops or legs) and a random tree for minimizing the number of links within the connectivity. Another important property is that SF networks are robust against random failures but vulnerable against the targeted attacks on hubs. This vulnerability, called "the Achilles' heel of the Internet,"[6] frightened us. Although the vulnerability is a double-edged sword for information delivery and spreading of viruses, we expect that the above properties will be useful for developing efficient and fault-tolerant networks with a defense mechanism based on the protection of hubs. Since the SF structure is at least selected with self-organized manners in social and biological environments, the evolutional mechanisms may provide insight into distributed network design or social management in communication or business.

On the other hand, in contrast to abstract graphs, many real networks are embedded in a metric space. It is therefore natural to investigate the possibility of embedding SF networks in space. The range of related applications is very wide, and includes Internet, power grids, airlines, mobile communication[7], and sensor networks[8]. However, most of the studies on SF networks do not take any acount of geographical space. In this paper, focusing on the SF structure found in many real systems, we consider rules for generation of geographical networks whose nodes are set on a Euclidean space and in which the undirected links between nodes are weighted by the Euclidean distance.

The organization of this paper is as follows. In Section 2, we introduce an example that shows the restriction of long-range links in real networks. Indeed, the decay of the connection probability for the distance between nodes follows an exponential or power-law. In Section 3, we review recent studies of geographical SF network models, which are categorized into three

† School of Knowledge Science, Japan Advanced Institute of Science and Technology

classes based on generation rules. We refer to the analytical forms of degree distributions that characterize the SF structure. In Section 4, we consider the relations among these models. In addition, we compare the properties of planarity and the distance of connections. Finally, in Section 5, we summarize the above properties and briefly discuss further issues.

2. Spatial Distribution in Real-world Networks

The restriction of long-range links has been observed in real networks, e.g., Internet at both router and autonomous system (AS) levels obtained by using the NETGEO tool to identify the geographical coordinates of 228,265 routers[9]. These data suggest that the distribution of link lengths (distances) is inversely proportional to the lengths, invalidating Waxman's exponential decay rule[10], which is widely used in traffic simulations. Other evidence has been reported for the real Internet data at the AS level (7,049 nodes and 13,831 links) compiled by the University of Oregon's Route Views project, the road network of US interstate highways (935 nodes and 1,337 links) extracted from GIS databases, and flight connections (187 nodes and 825 links) for a major airline[11]. It has been shown that all three networks have a clear bias towards shorter links to reduce the costs of construction and maintenance. However, some differences exist: the road network has only very short links on the order of 10 km to 100 km in the sharply decaying distribution, while the Internet and airline networks have much longer ones in the bimodal distribution, with distinct peaks around 2,000 km or less and 4,000 km. These differences may derive from physical constraints in the link cost or from the requirements of long distance direct connections.

As a similar example, we investigate the distributions of link lengths (distances of flights) in Japanese airline[12]. The network consists of 52 nodes (airports) and 961 links (flights) for Japan AirLines (JAL), 49 nodes and 909 links for All Nippon Airlines (ANA), and 84 nodes and 1,114 links for other flights including international ones. **Figure 1** shows the cumulative number of flights for the decreasing order of length measured in miles. We note an exponential decay in domestic flights (red and blue lines in Fig. 1), whereas it follows a power-law when international flights (green line in Fig. 1) are added. Note that the distribution of the

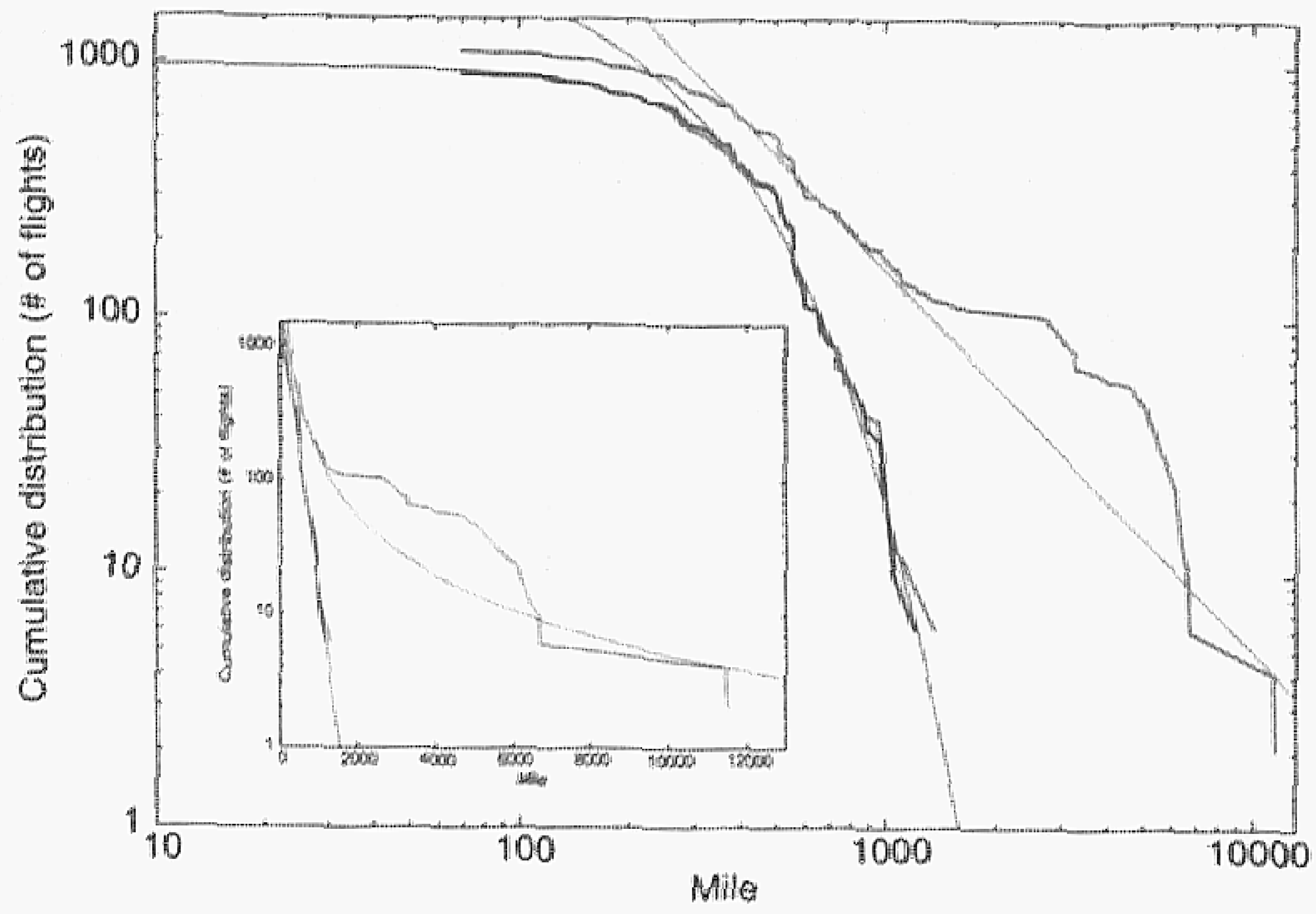


Fig. 1 Cumulative number of flights by Japanese airlines. The red, blue, and green lines correspond to domestic flights by JAL and ANA, and other flights including international flights (inset: semi-log scale). The magenta and cyan lines show the estimated exponential and power-law functions, respectively.

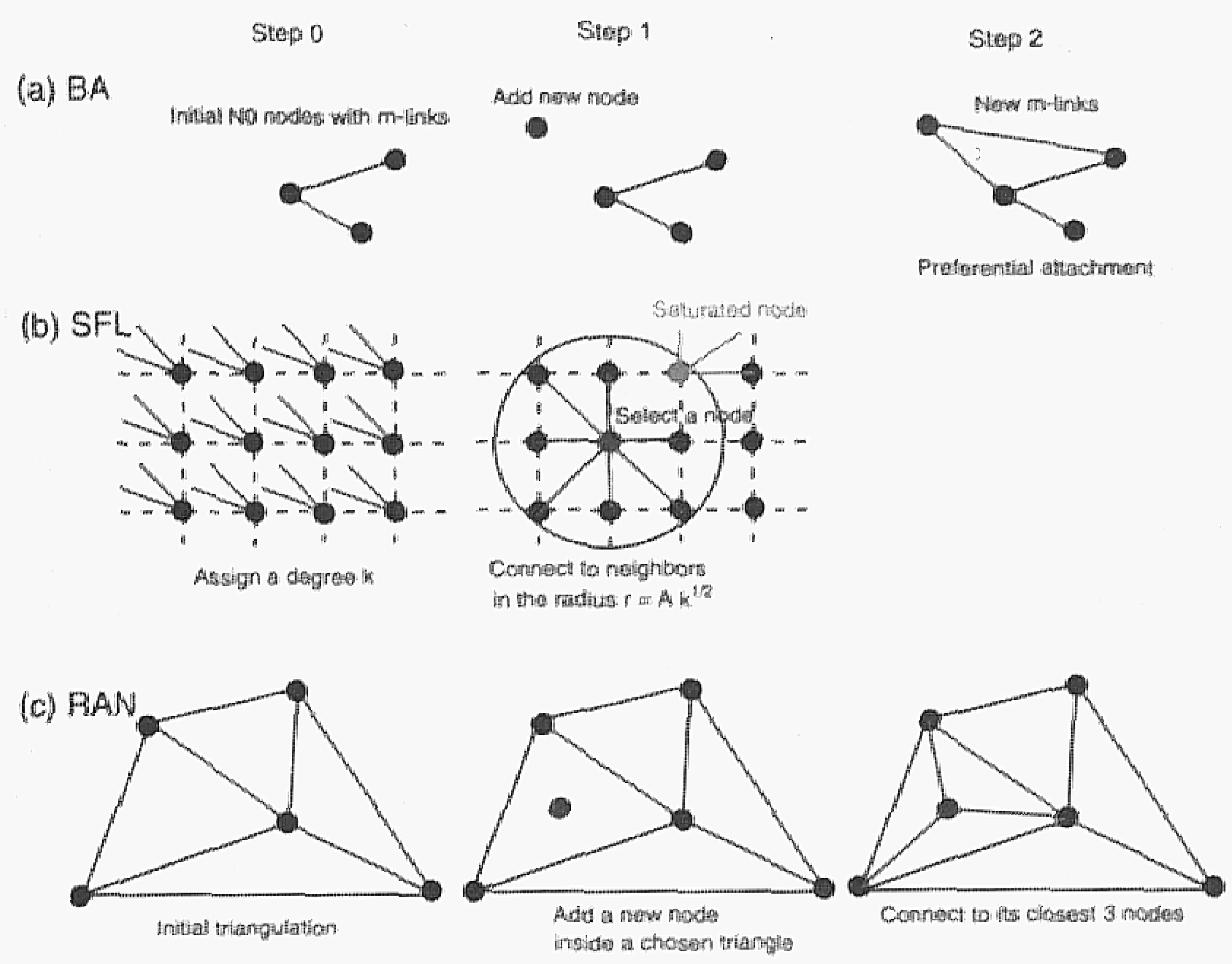


Fig. 2 Network generation in each model. The analytically obtained degree distributions for (a)-(c) follow $P(k) \sim k^{-3}$, $P(k) \sim k^{-\gamma}$ with a cutoff $k_c < K$, and $P(k) \sim k^{-\gamma_{RA}}$, $\gamma_{RA} \sim 3$, respectively.

link lengths is obtained from the differential of the cumulative one and that the decay form of the exponential or the power-law is invariant.

Thus, link lengths are restricted in real systems, although the distribution may have various forms, as the case of degree distribution[13].

3. Geographical SF Network Models

We review state-of-the-art geographical SF network models. Based on the generation rules of networks, they can be categorized into three classes, as shown in **Table 1**. The generation rules are explained by variations in the trade-off between minimizing the number of hops between nodes (benefits for transfers) and minimizing the link lengths.

Table 1 Summary of geographical SF network models. The symbols ○, △, and × denote goodness levels for P: the palanarity and S: the shortness of links.

Class of SF	Generation rule	P	S	Models
With disadvan. long-range links	Connect (i, t) with probability $\Pi_i(t) \sim k_i(t) l^{\alpha}$	×	○	Modulated BA [14),19)]
	Connect (i, j) iff $(w_i + w_j) h(r_{ij}) \geq \theta$	×	○	Geo. thresh. graph [18)]
Embedded in a lattice	With randomly assigned degree k_j restricted links in the radius $A k_j^{1/d}$	×	△	Warren [24)], Avraham [22),23)]
Created by space-filling	Triangulation (geo. attach. pref.)	○	△	Apollonian nets. [27)∼29)]
	Pref. attach. by selection of edges	×	△	Growing spatial SF nets [37)]

In this section, we refer to the generation rules and the power-law behavior only in their essential forms, because of the space limitation. The properties of planarity without crossing links and the link lengths will be discussed in the next section.

3.1 SF Networks with Disadvantaged Long-range Links

The modulated Barabási-Albert (BA) model [14)] and the geographical threshold graph [18)] belong to the first class: SF networks with disadvantaged long-range links between nodes whose positions are random on a space☆. They are natural extensions of the previous non-geographical SF network models, generated by adding competition between preferential linking based on the degree or weight and the restriction of link length (distance dependence).

3.1.1 Modulated BA Model in Euclidean Space

Before explaining the first class, we introduce the well-known BA model [4)] generated by the following rule: *growth* with a new node at each time step and *preferential attachment* of links to nodes with large degrees (see **Fig. 2** (a)).

BA-Step 0: A network grows from an initial N_0 nodes with $m < N_0$ links among them.

BA-Step 1: At every time step, a new node is introduced and is randomly connected to m previous nodes as follows.

BA-Step 2: Any of these m links of the new node introduced at time t is connected to a previous node i with an attachment probability $\Pi_i^{BA}(t)$ which is linearly proportional to the degree $k_i(t)$ of the i-th node at time t, $\Pi_i^{BA}(t) \sim k_i(t)$.

Preferential attachment creates a heterogeneous network with hubs. More precisely, the degree distribution $P(k) \sim k^{-3}$ is analytically obtained by using a mean-field approximation [4)] in the continuum approach [2)], in which the time dependence of the degree k_i of a given node i is calculated through the continuous real variables of degree and time.

Based on a competition between preferential attachment and the distance dependence of links, the modulated BA model on a space with physical distance has been considered [14)]. Note that the position of new nodes is random. The network is grown by introducing randomly positioned nodes on a Euclidean space (e.g., a two-dimensional square area), and the probability of connection is modulated according to $\Pi_i(t) \sim k_i(t) l^{\alpha}$, where l is the Euclidean distance between the t-th node introduced at time t and the older i-th node, and α is a parameter. The case of $\alpha = 0$ is the original BA model [4)]. In the limit of $\alpha \to -\infty$, only the smallest value of l corresponding to the nearest node will contribute with probability 1. Similarly, in the limit of $\alpha \to \infty$, only the furthest node will contribute. Indeed, it has been estimated that the distribution of link lengths follows a power-law $l^{-\delta}$ (long-range links are rare at $\delta > 0$), whose exponent is calculated as $\delta = \alpha + d - 1$ for all values of α [14)].

In the modulated BA model on a one-dimensional lattice (circumference), it has been numerically shown [19)] that for $-1 < \alpha < 0$ the degree distribution is close to a power-law, but for $\alpha < -1$ it is represented by a stretched exponential $P(k) = a \exp(-b k^{\gamma})$, where the parameters a, b, and γ depend on α and m, although the SW property [3)] is preserved at all values of α. For the transition from the stretched exponential to the SF behavior, the critical value is generalized to $\alpha_c = 1 - d$ in the embedded d-dimensional space [14)]. More systematic classification in a parameter space of the exponents of degree, distance, and fractal dimension has also been discussed [9)].

Other studies related to the form of connection probability $\Pi_i \sim k_i^{\beta} l^{\alpha}$ include a phase diagram of the clustering properties in the α-β plane [15)], a comparison of the topological properties for the special case where the connection probability is proportional to the distance ($\alpha = 1$, $\beta = 0$) and the inverse distance ($\alpha = -1$, $\beta = 0$) [16)], and a numerical inves-

☆ To simplify the discussion, we assume an uniformly random distribution of nodes on a space. However, the procedure can be generalized to any other distributions.

tigation of the scaling for the quantities (degree, degree-degree correlation, clustering coefficient) of the network generated by the connection probability proportional to the degree with the exponential decay of the distance [17].

3.1.2 Geographical Threshold Graphs

A geographical threshold graph [18] is a non-growing network model extended from the threshold SF network model [20),21]. It is embedded in the d-dimensional Euclidean space with disadvantaged long-range links. We briefly analyze the degree distribution.

Let us consider a set of nodes with size N. We assume that each node i is randomly distributed with uniform density ρ in a space whose coordinates are denoted by $x_1, x_2, \ldots, x_d$, and that it is assigned a weight $w_i \geq 0$ by a density function $f(w)$. According to the threshold mechanism [18], a pair of nodes (i, j) is connected iff

$$(w_i + w_j)h(r_{ij}) \geq \theta, \quad (1)$$

where θ is a constant threshold, and $h(r_{ij})$ is a decreasing function of the distance $r_{ij} > 0$ between the nodes.

If $f(w)$ is the Dirac delta function at $w* > 0$, then the condition of connection of Eq. (1) is equivalent to $r_{ij} \geq h^{-1}\left(\frac{\theta}{2w^*}\right) \stackrel{\text{def}}{=} r^*$, derived from the inverse function h^{-1}. This case is the unit disk graph, as a model of mobile and sensor networks, in which two nodes within the radius r^* are connected according to the energy consumption. However, the degree distribution $P(k)$ is homogeneous. We need more inhomogeneous weights.

Thus, if the exponential weight distribution

$$f(w) = \lambda e^{-\lambda w}, \quad (2)$$

and the power-law decay function $h(r_{ij}) = (r_{ij})^{-\beta}$, $\beta \geq 0$, are considered, then the degree is derived as a function of weight

$$k(w_i) = \int_0^\infty f(w_j)dw_j \times \int_{(w_i+w_j)/(r_{ij})^\beta \geq \theta} \rho dx_1 \ldots dx_d \sim e^{\lambda w_i}, \quad (3)$$

after slightly complicated calculations. The second integral in the r.h.s of Eq. (3) is the volume of a d-dimensional hypersphere. As in Refs. 18) and 21), by using the relation of cumulative distributions $\int_0^{k(w)} P(k)dk = \int_{-\infty}^{w} f(w')dw'$, we have

$$P(k) = f(w)\frac{dw}{dk}. \quad (4)$$

From Eqs. (3) and (4), we obtain the power-law degree distribution

$$P(k) \sim e^{-2\lambda w} \sim k^{-2}.$$

Note that this result is derived only if the value of β is sufficiently small; otherwise, the degree distribution has a stretched exponential decay or an exponential decay.

On the other hand, for the power-law weight distribution (called Parete distribution in this form)

$$f(w) = \frac{\alpha}{w^*}\left(\frac{w^*}{w}\right)^{\alpha+1}, \quad (5)$$

we similarly obtain

$$k(w) \sim w^{d/\beta}, \quad P(k) \sim k^{-(1+\alpha\beta/d)}.$$

The exponent $\gamma \stackrel{\text{def}}{=} 1 + \alpha\beta/d$ is a variable that depends on the parameters α and β.

Furthermore, we mention a gravity model with $h(r_{ij}) = 1/\log r_{ij}$. In this case, the condition of connection (1) is rewritten as $w_i + w_j \geq \theta \log r_{ij}$, and converted into

$$\frac{W_i W_j}{(R_{ij})^\beta} \geq \theta, \quad (6)$$

by the variable transformations $W_i \stackrel{\text{def}}{=} e^{w_i}$, $W_j \stackrel{\text{def}}{=} e^{w_j}$, and $(R_{ij})^\beta \stackrel{\text{def}}{=} (r_{ij})^\theta/\theta$. Equation (6) represents a physical, sociological, or chemical interaction with power-law distance dependence. From a combination of Eq. (6) and the weight distributions $f(w)$ in Eqs. (2) and (5), we can also derive the more complicated forms of $P(k)$. Thus, the choice of $f(w)$ matters for the SF properties, in contrast to an approximately constant exponent $\gamma \approx 2$ in non-geographical threshold graphs [21] without $h(r_{ij})$.

3.2 SF Networks Embedded in Lattices

The second class is based on the SF networks embedded in regular Euclidean lattices (SFL) accounting for graphical properties [22),23]. We distinguish this class from the first one, because the positions of nodes are not randomly distributed but well-ordered on a lattice with a scale (one-hop unit) that gives the minimum distance.

Let us consider a d-dimensional lattice of size R with periodic boundary conditions. The model is defined by the following configuration procedures (see Fig. 2 (b)) on an assumption of power-law degree distribution.

SFL-Step 0: To each node on the lattice, assign a random degree k taken from the distribution $P(k) = Ck^{-\lambda}$, $m \le k \le K$, $\lambda > 2$ (the normalization constant: $C \approx (\lambda-1)m^{\lambda-1}$ for large K).

SFL-Step 1: Select a node i at random, and connect it to its closest neighbors until its connectivity k_i is realized, or until all nodes up to a distance,

$$r(k_i) = Ak_i^{1/d} \tag{7}$$

have been explored: The connectivity quota k_j of the target node j is already saturated. Here $A > 0$ is a constant.

SFL-Step 2: Repeat the above process for all nodes.

As in Ref. 22), we derive the cutoff connectivity. Consider the number of links $n(r)$ entering a node from a surrounding neighborhood of radius r, when the lattice is infinite, $R \to \infty$. The probability of connections between the origin and nodes at distance r' is

$$P\left(k > \left(\frac{r'}{A}\right)^d\right) = \int_{(r'/A)^d}^{\infty} P(k')dk' \sim \begin{cases} 1 & r' < A \\ (r'/A)^{d(1-\lambda)} & r' > A. \end{cases}$$

Thus, from $n(r) = \int_0^r S_d r'^{d-1} dr' \int_{(r'/A)^d}^{\infty} P(k')dk'$, we obtain

$$n(r) = V_d r^d \left\{ \left(\frac{A}{r}\right)^d \int_0^{(r/A)^d} kP(k)dk + \int_{(r/A)^d}^{\infty} P(k)dk \right\},$$

where $V_d = S_d/d$ and S_d are the volume and the surface area of the d-dimensional unit sphere, respectively. The cutoff connectivity is then

$$k_c = \lim_{r \to \infty} n(r) = V_d A^d \langle k \rangle, \tag{8}$$

where $\langle k \rangle = \int kP(k)dk$ denotes the average connectivity.

If A is large enough such that $k_c > K$, the network can be embedded without cutoff. Otherwise, by substituting Eq. (8) into Eq. (7), the cutoff connectivity k_c implies a cutoff length

$$\xi = r(k_c) = (V_d \langle k \rangle)^{1/d} A^2. \tag{9}$$

The embedded network displays the original (power-law) distribution up to length scale ξ and repeats, statistically, at length scales larger than ξ.

Whenever the lattice is finite, $R < \infty$, the number of nodes is finite, $N \sim R^d$, which imposes a maximum connectivity

$$K \sim mN^{1/(\lambda-1)} \sim R^{d/(\lambda-1)}, \tag{10}$$

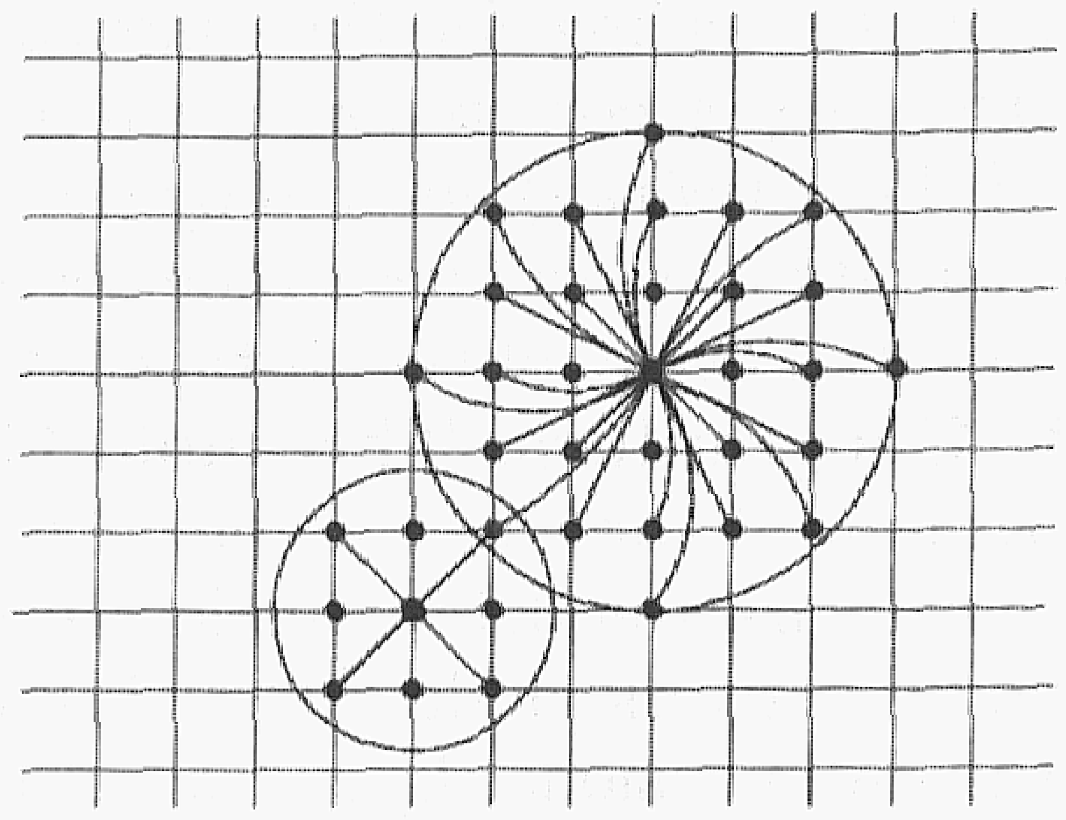


Fig. 3 Warren's SF network embedded in a two-dimensional lattice.

where the first approximation is obtained from $\int_K^{\infty} P(k)dk = \left[\frac{C}{1-\lambda}k^{1-\lambda}\right]_K^{\infty} = 1/N$. From Eqs. (7) and (10), a finite-size cutoff length is

$$r_{max} = r(K) \sim AR^{1/(\lambda-1)}. \tag{11}$$

These three length scales, R, ξ, r_{max}, determine the nature of networks. If the lattice is finite, then the maximum connectivity K is attained only if $r_{max} < \xi$. Otherwise ($r_{max} > \xi$), the cutoff k_c is imposed. As long as $\min(r_{max}, \xi) \ll R$, the lattice size R imposes no serious restrictions. Otherwise ($\min(r_{max}, \xi) \ge R$), finite-size effects bounded by R become important. In this regime, simulation results [22),23)] have also shown that for larger λ the network resembles the embedding lattice because of the rare long-range links, while the long-range links becomes noticeable as λ decreases.

Concurrently with the above work, Warren, et al. [24)] have proposed a similar embedding algorithm in a two-dimensional lattice. As shown in **Fig. 3**, the number of nodes in each circle is equal to the connectivity without cutoff. Thus, the main difference in their approach is that a node can be connected to as many of its closest neighbors as necessary, until its target connectivity is fulfilled.

In addition, Ref. 25) has discussed the shortest paths on d-dimensional lattices with the addition of an average of p long-range bonds (shortcuts) per site, whose length l is distributed according to $P_l \sim l^{-\mu}$.

3.3 Space-Filling Networks

The third class is related to space-filling packing in which a region is iteratively partitioned into subregions by adding new nodes and links.

3.3.1 Growing Small-World Networks

Let us consider a growing network with ge-

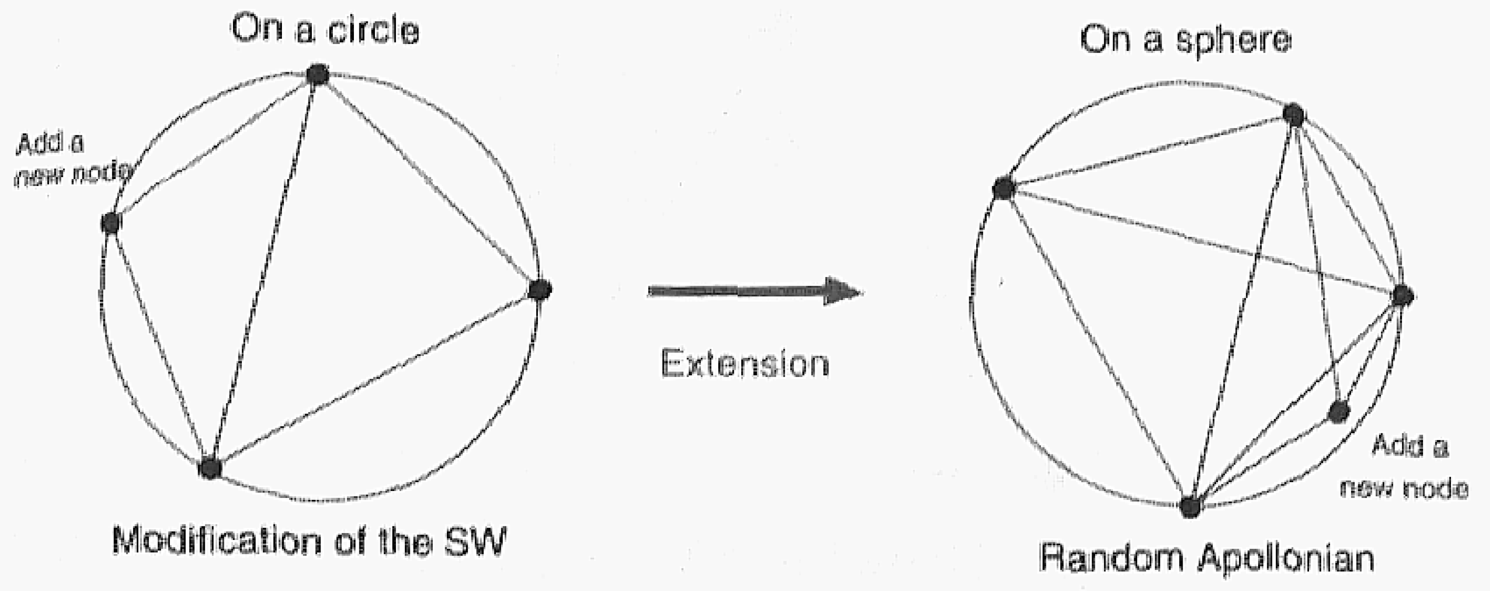


Fig. 4 Growing networks with geographical attachment preference.

ographical attachment preference[26] as a modification of the SW model[3]. In this network, from an initial configuration with $m+1$ completely connected nodes on the circumference of a circle, at each subsequent time step, a new node is added in an uniform-randomly chosen interval, and connects the new node to its m nearest neighbors w.r.t the distance along the circumference. **Figure 4** (left) illustrates the case of $m=2$. We denote $n(k,N)$ as the number of nodes with degree k when the size (or time) is N. At time N, a new node with degree m is added to the network, and if it connects to a preexisting node i, then the degree is updated by $k_i \to k_i+1$ with equal probability m/N to all nodes because of the uniform randomly chosen interval.

Thus, we have the following evolution equation:

$$n(k,N+1) = \left(1-\tfrac{m}{N}\right) n(k,N) + \tfrac{m}{N} n(k-1,N) + \delta_{k,m},$$

where $\delta_{k,m}$ is the Kronecker delta. Note that considering such an equation for the average number of nodes with k links at time N is called "the rate-equation approach," while considering the probability $p(k,t_i,t)$ that at time t a node i introduced at time t_i has a degree k is called "the master equation approach"[2].

When N is sufficiently large, $n(k,N)$ can be approximated as $NP(k)$. In the term of degree distribution, we obtain

$$P(k) = \frac{1}{m+1}\left(\frac{m}{m+1}\right)^{k-m}$$

for $k \gg m$ ($P(k)=0$ for $k<m$), although it is not a power-law.

3.3.2 Apollonian Networks

The growing small-world networks model[26] can be extended from polygonal divisions on a circle to polyhedral divisions on a sphere, as shown in Fig. 4. We should remark that the extended model becomes a planar graph when each node on the surface is projected onto a plane such as one from a Riemannian sphere. It is nothing but a random Apollonian network (RAN)[27),28], and also the dual version of Apollonian packing for space-filling disks into a sphere[29], whose hierarchical structure is related to the SF network formed by the minima and transition states on the energy landscape[30]. The power-law degree distribution has been analytically shown in RAN[27),28]. To derive the distribution $P(k)$, we consider the configuration procedure for a RAN as follows (see Fig. 2 (c)).

RAN-Step 0: Set an initial triangulation with N_0 nodes.

RAN-Step 1: At each time step, a triangle is randomly chosen, and a new node is added inside the triangle.

RAN-Step 2: The new node is connected to its three nodes of the chosen triangle.

RAN-Step 3: The processes in Steps 1 and 2 are repeated until the required size N is reached.

Since the probability of connection to a node increases with the number of its related triangles, it is proportional to its degree as the preferential attachment. Thus, we have the following rate-equation:

$$n(k+1,N+1) = \tfrac{k}{N_\triangle} n(k,N) + \left(1-\tfrac{k+1}{N_\triangle}\right) \times n(k+1,N), \qquad (12)$$

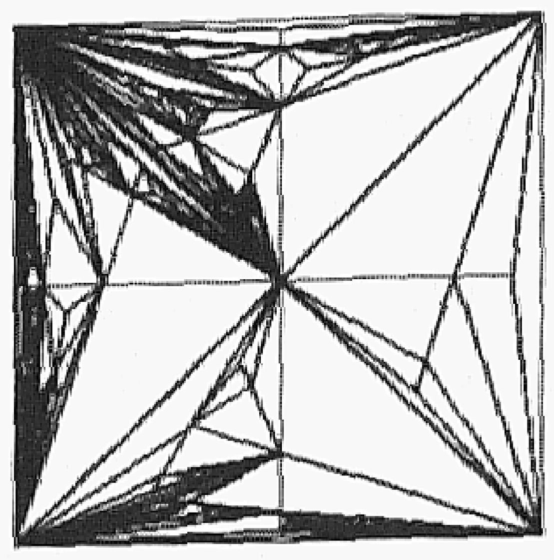
where the number of triangles $N_\triangle$ (at the grown size or time N) is defined as $N_\triangle = 2(N-4)+4$ for an initial tetrahedron, $N_\triangle = 2(N-3)+1$ for an initial triangle, and so forth.

In the term of $P(k) \approx n(k,N)/N$, Eq. (12) can be rewritten as

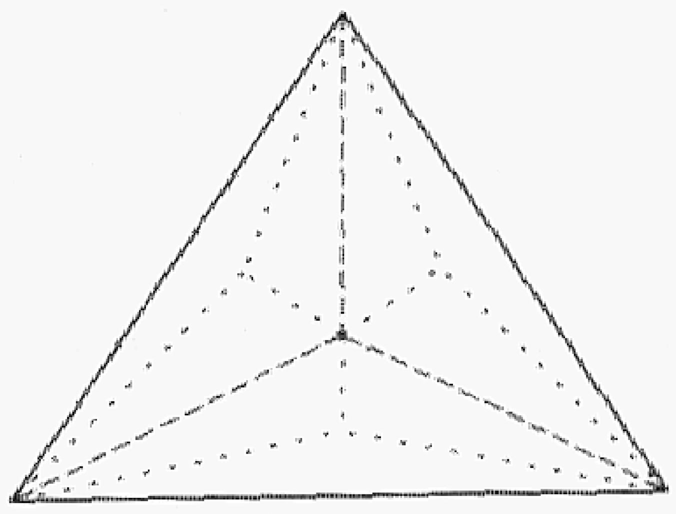
$$(N+1)P(k+1) = \tfrac{NkP(k)}{N_\triangle} + NP(k+1) - \tfrac{N(k+1)P(k+1)}{N_\triangle}.$$

By continuous approximation, we obtain the solution $P(k) \sim k^{-\gamma_{RA}}$ with $\gamma_{RA} = (N_\triangle + N)/N \approx 3$ for large N. **Figure 5** (a) shows an example of a RAN.

Moreover, in the deterministic version[29),31], analytical forms of the power-law degree distribution $P(k)$, clustering coefficient c_i, and degree-degree correlation $k_{nn}(k)$ can be derived[29], since the calculations are easier on the recursive structure without randomness for selection of subregions, as shown in Fig. 5 (b). Here, $k_{nn}(k)$ is defined by the the average de-



(a) RAN



(b) Deterministic AN

Fig. 5 Apollonian Networks: (a) Random AN generated from an initial triangulation of square and (b) deterministic AN generated from an initial triangle of back lines. The dashed and dotted lines show the added links at the first and second steps, respectively.

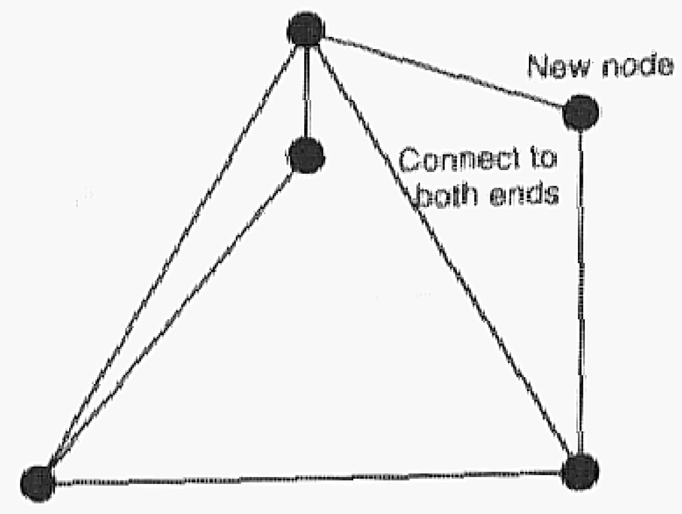
gree of the nearest neighbors of nodes with degree k. It has been observed in technological and biological networks and in social networks that two types of correlations exist, namely, dis-assortative and assortative mixings[32]. These types of networks tend to have connections between nodes with low-high degrees and with similar degrees, respectively. RANs exhibit dis-assortative mixing[29].

Similarly, the analytical forms in both high-dimensional random[33),34)] and deterministic[35)] Apollonian networks have been investigated by using slightly different techniques. They are more general space-filling models embedded in a high-dimensional Euclidean space, although the planarity is violated.

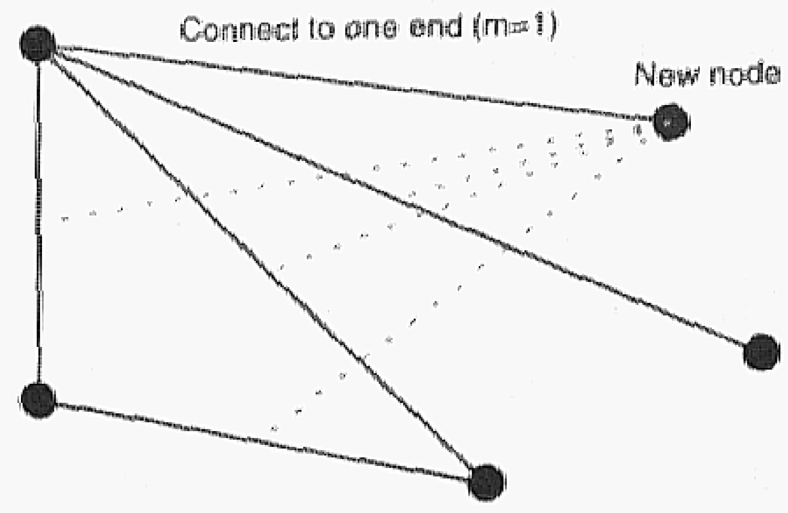
3.3.3 SF Networks Generated by Selecting Edges

Another modification of growing SW networks[26)] is based on random selection of edges[36),37)]. We classify them in the relation to their manner of partitioning an interval or region, as mentioned in Section 3.3.1, and a Voronoi diagram. The following two models give typical configurations (see **Fig. 6**).

The growing SW network generated by selecting edges[36)] is constructed as follows. Initially, the network has three nodes, each with degree two. As shown in Fig. 6 (a), at each time step, a new node is added, which is attached via two links to both ends of one randomly chosen link that has never been selected before. The



(a) Growing SW network



(b) Growing spatial SF network

Fig. 6 SW and SF networks generated by randomly selecting edges.

growth process is repeated until the required size N is reached. Since all nodes have two candidates links for selection at each time, an exponential degree distribution is analytically obtained. If the multi-selection is permitted for each link, it follows a power-law. The difference in the configuration procedures for a RAN is that, instead of triangulation with three links added, two links are added at each step. We assume that the position of added node is random (but the nearest to the chosen link) on a metric space, although only the topological properties are discussed in the original model[36)].

In a growing spatial SF network[37)] on a two-dimensional space, m links whose centers are nearest to an added new node (as indicated by the dashed lines in Fig. 6 (b)) are chosen at each time step. Both end nodes of the nearest link(s) have an equal probability of connection. If a Voronoi region[38),39)] for the centers of links is randomly chosen for the position of a new node in the region☆, the selection of a link is uniformly random, and therefore the probability of connection to each node is proportional to its degree. Thus, we can analyze the degree distribution. Note that any point in the Voronoi region is closer to the center (called the generator point) belong in it than to any other centers.

For the case of $m = 1$ as a tree, the number of nodes with degree k is evolved in the rate-

☆ Although the position of nodes is randomly selected on a two-dimensional space in the original paper[37)], it is modified to the random selection of a Voronoi region which is related to triangulation such as in a RAN. Note that it gives a heterogeneous spatial distribution of points.

equation

$$n(k, t+1) = n(k, t) + \frac{(k-1)}{2t} n(k-1, t) - \frac{k}{2t} n(k, t) + \delta_{k,1}, \quad (13)$$

where $n(k, t)$ denotes the number of nodes with degree k, and $2t$ is the total degree at time t.

In the term of degree distribution $P(k, t) \approx n(k, t)/t$ at time t, Eq. (13) is rewritten as

$$(t+1)P(k, t+1) - tP(k, t) = \frac{1}{2} [(k-1)P(k-1, t) - kP(k, t)] + \delta_{k,1}.$$

At the stationary value independent of time t, we have

$$P(k) = \frac{1}{2} [(k-1)P(k-1) - kP(k)] + \delta_{k,1}.$$

From the recursion formula and $P(1) = 2/3$, we obtain the solution

$$P(k) = \frac{k-1}{k+2} P(k-1) = \frac{4}{k(k+1)(k+2)} \sim k^{-3}.$$

4. Relations among the Models

We discuss the relations among the independently proposed models. Remember the summary of the geographical SF network models in Table 1.

The first class is based on a combination of preferential attachment or a threshold mechanism and a penalty for long-range links between nodes whose position is random, while the second one is based on embedding of a SF structure with a given power-law degree distribution in a lattice. Since the degree assigned to each node can be regarded as a fitness value[2], the SFL is considered as a special case of the fitness model[20] embedded in a lattice. In contrast, the penalty for age or distance dependence of each node can be regarded as an inverse of fitness value in general terms. If we neglect the differences among penalties, this explanation links the modulated BA[14),19)], SFL[22),23)], and aging models[40)] with a generalized fitness model. The crucial difference is the positioning of nodes: in one case they are randomly distributed on a space and in the other they are well-ordered on a lattice with the minimum unit distance between nodes. Moreover, the weight in the threshold graphs[18),21)] corresponds to a something of fitness value; however, the deterministic threshold and the attachment mechanisms should be distinguished in non-growing and growing networks. We also remark that, in the third class, the preferential attachment is implicitly performed, although the configuration procedures are more geometric, being based on triangulation[27)~29)] or selection of edges[37)]. In particular, the positions of nodes in the Apollonian networks are given by iterative subdivisions (as neither random nor fixed on a lattice), which may be related in practice to territories for communication or supply management.

Next, we qualitatively compare the properties of planarity without crossing links and link lengths. We emphasize that the planarity is an important and natural requirement to avoid interference of beams (or collision of particles) in wireless networks, airlines, layout of VLSI circuits, vas networks clinging to cutis, and other networks on the earth's surface[28)].

In modulated BA models and geographical threshold graphs, long-range links are restricted by the strong constraints with decay terms; however, crossing links may be generated. There exist longer links from hubs in the SFL, because such nodes have large numbers of links. Moreover, the positions of nodes are restricted on a lattice. The density of nodes is constant, and therefore they must connect to some nodes at long distances. More precisely, it depends on the exponent λ of the power-law degree distribution, as mentioned in Section 3.2. In addition, the planarity condition is not satisfied by the crossing between the lattice edges and the short-cuts. On the other hand, RANs have both good properties of planarity and averagely short links. However, in a narrow triangular region, long-range links are partially generated, as shown in Fig. 5. Similarly, SF networks generated by selection of edges may have long-range links, as shown in Fig. 6 (b): the chosen end point for connection is far from the newly added node at a random position, even though the selected edges have the nearest centers.

5. Conclusion

In this review of geographical SF network models, we have categorized them into three classes based on the generation rules: disadvantaged long-range links, embedding in a lattice, and space-filling. We have shown that these models have essential mechanisms for generating power-law degree distributions, whose analytical forms can be derived on an assumption that the restricted link lengths are consistent with real data. Furthermore, the basic topological properties of the planarity and link length have been discussed for each model. In particular, geographical threshold graphs and RANs are attractive because of the tunable exponent γ of $P(k) \sim k^{-\gamma}$ or the locality related to unit disk graphs, and the planarity of networks with heterogeneous positioning of nodes. However, they have drawbacks of crossing and long-range links, respectively. To avoid long-range links, an improvement using a combination of RANs and Delaunay triangulation based on diagonal flipping[38),39)] is under consideration[41)].

We have considered several configuration procedures for geographical SF networks and discussed their properties; however, these are still at the fundamental level. We must consider further issues, such as,

- Quantitative investigation of the topological properties, including the diameter of a network, the clustering coefficient, the degree-degree correlation, and the betweenness centrality (related to congestion of information flow).
- Analysis of the dynamics of traffic and the fault-tolerance, especially in disasters or emergent environments.
- Positioning of nodes with aggregations according to population density in evolutional and distributed manners.

We will progress from current stage, the observation of real networks, to the next stage, the development of future networks. The distributed design and management will have useful applications in many socio-technological infrastructures.

Acknowledgments This research is partially supported by Mitani foundation for research and development.

References

1) Buchanan, M.: Nexus: Small Worlds and the Groundbreaking Science of Networks, Norton & Company, Inc. (2002).
2) Albert, R. and Barabási, A.-L.: Statistical Mechanics of Complex Networks, *Rev. Mod. Phys.*, Vol.74, pp.47–97 (2002).
3) Watts, D.J. and Strogatz, S.H.: Collective Dynamics of Small-World Networks, *Nature*, Vol.393, p.440 (1998).
4) Barabási, A.-L., Albert, R. and Jeong, H.: Mean-Field Theory for Scale-Free Random Networks, *Physica A*, Vol.272, pp.173–187 (1999).
5) Cancho, R.F.i. and Solé, R.V.: Optimization in Complex Networks, *Statistical Mechanics of Complex Networks*, Pastor-Satorras, R., Rubi, M. and Diaz-Guilera, A. (Eds.), Chapter 6, pp.114–126 (2003).
6) Albert, R., Jeong, H. and Barabási, A.-L.: Error and Attack Tolerance of Complex Networks, *Nature*, Vol.406, pp.378–382 (2000).
7) Hong, X., Xu, K. and Gerla, M.: Scalable Routing Protocols for Mobile Ad Hoc Networks, *IEEE Network*, pp.11–21 (July/Aug. 2002).
8) Culler, D.E. and Mulder, H.: Smart Sensors to Network the World, *Scientific American*, pp.84–91 (June 2004).
9) Yook, S.-H., Jeong, H. and Barabási, A.-L.: Modeling the Internet's Large-Scale Topology, *PNAS*, Vol.99, No.21, pp.13382–13386 (2002).
10) Waxman, B.M.: Routing of Multipoint Connections, *IEEE Journal of Selected Areas in Communications*, Vol.6, pp.1617–1622 (1988).
11) Gastner, M.T. and Newman, M.E. J.: The Spatial Structure of Networks, *arXiv:cond-mat/0407680* (2004).
12) JAL Domestic Flight Schedule and Service Guide, ANA Group Flight Schedule (Apr.-May 2005).
13) Amaral, L.A., Scala, A., Barthélémy, M. and Stanley, H.E.: Classes of Behavior of Small-World Networks, *PNAS*, Vol.97, p.11149 (2000).
14) Manna, S.S. and Parongama, S.: Modulated Scale-Free Networks in Euclidean Space, *Phys. Rev. E*, Vol.66, p.066114 (2002).
15) Sen, P. and Manna, S.S.: Clustering Properties of a Generalized Critical Euclidean Network, *Phys. Rev. E*, Vol.68, p.026104 (2003).
16) Jost, J. and Joy, M.P.: Evolving Networks with Distance Preferences, *Phys. Rev. E*, Vol.66, p.036126 (2002).
17) Barthélemy, M.: Crossover from Scale-Free to Spatial Networks, *Europhys. Lett.*, Vol.63, No.6, pp.915–921 (2003).
18) Masuda, N., Miwa, H. and Konno, N.: Geographical Threshold Graphs with Small-World and Scale-Free Properties, *Phys. Rev. E*, Vol.71, p.036108 (2005).
19) Xulvi-Brunet, R. and Sokolov, I.M.: Evolving Networks with Disadvantaged Long-Range Connections, *Phys. Rev. E*, Vol.66, p.026118 (2002).
20) Caldareli, G, Capocci, A., Rios, P. DeLos and Muñoz, M.A.: Scale-free Networks from Varying Vertex Intrinsic Fitness, *Physical Review Letters*, Vol.89, p.258702 (2002).
21) Masuda, N., Miwa, H. and Konno, N.: Analysis of Scale-Free Networks Based on a Threshold Graph with Intrinsic Vertex Weights, *Phys.*

Rev. E, Vol.70, p.036124 (2004).
22) ben-Avraham, D., Rozenfeld, A.F., Cohen, R. and Havlin, S.: Geographical Embedding of Scale-Free Networks, *Physica A*, Vol.330, pp.107–116 (2003).
23) Rozenfeld, A.F., Cohen, R., ben-Avraham, D. and Havlin, S.: Scale-Free Networks on Lattices, *Phys. Rev. Lett.*, Vol.89, p.218701 (2002).
24) Warren, C.P., Sander, L.M. and Sokolov, I.M.: Geography in a Scale-Free Network Model, *Phys. Rev. E*, Vol.66, p.056105 (2002).
25) Moukarzel, C.F. and Argollo de Menzes, M.: Shortest Paths on Systems with Power-Law Distributed Long-Range Connections, *Phys. Rev. E*, Vol.65, p.056709 (2002).
26) Ozik, J., Hunt, B.R. and Ott, E.: Growing Networks with Geographical Attachment Preference: Emergence of Small Worlds, *Phys. Rev. E*, Vol.69, p.026108 (2004).
27) Zhou, T., Yan, G., Zhou, P.-L., Fu, Z.-Q. and Wang, B.-H.: Random Apollonian Networks, *arXiv:cond-mat/0409414* (2004).
28) Zhou, T., Yan, G. and Wang, B.-H.: Maximal Planar Networks with Large Clustering Coefficient and Power-Law Degree Distribution, *Phys. Rev. E*, Vol.71, p.046141 (2005).
29) Doye, J.P.K. and Massen, C.P.: Self-Similar Disk Packings as Model Spatial Scale-Free Networks, *Phys. Rev. E*, Vol.71, p.016128 (2005).
30) Doye, J.P.K.: Network Topology of a Potential Energy Landscape: A Static Scele-Free Network, *Phys. Rev. Lett.*, Vol.88, p.238701 (2002).
31) Andrade, Jr., J.S., Herrmann, H.J., Andrade, R.F.S. and da Silva, L.R.: Apollonian Networks: Simultaneously Scale-Free, Small World, Euclidean, Space Filling, and with Matching Graphs, *Phys. Rev. Lett.*, Vol.94, p.018702 (2005).
32) Newman, M.E.J.: Mixing Patterns in Networks, *Phys. Rev. E*, Vol.67, p.026126 (2003).
33) Zhang, Z., Rong, L. and Comellas, F.: High Dimensional Random Apollonian Networks, *arXiv:cond-mat/0502591* (2005).
34) Gu. Z.-M., Zhou, T., Wang, B.-H., Yan, G., Zhu, C.-P. and Fu, Z.-Q.: Simplex Triangulation Induced Scale-Free Networks, *arXiv:cond-mat/0505175* (2005).
35) Zhang, Z., Comellas, F., Fertin, G. and Rong, L.: High Dimensional Apollonian Networks, *arXiv:cond-mat/0503316* (2005).
36) Zhang, Z. and Rong, L.: Growing Small-World Networks Generated by Attaching to Edges, *arXiv:cond-mat/0503637* (2005).
37) Mukherjee, G. and Manna, S.S.: Growing Spatial Scale-Free Graphs by Selecting Local Edges, *arXiv:cond-mat/0503697* (2005).
38) Imai, K: Structures of Triangulations of Points, *IEICE Trans. Inf. Syst.*, Vol.83-D, No.3, pp.428–437 (2000).
39) Okabe, A., Boots, B., Sugihara, K. and Chiu, S.N.: Spatial Tessellations, 2nd ed., John Wiley (2000).
40) Dorogovtsev, S.N. and Mendes, J.F.F.: Evolution of networks with aging of sites, *Phys. Rev. E*, Vol.62, p.1842 (2000).
41) Hayashi, Y. and Matsukubo, J.: Scale-Free Networks on a Geographical Planar Space for Efficient Ad Hoc Communication, *Proc. Int. Symp. on NOLTA*, pp.118–121 (2005).

(Received May 24, 2005)
(Accepted January 6, 2006)
(Online version of this article can be found in the IPSJ Digital Courier, Vol.2, pp.155–164.)

Yukio Hayashi received the B.E. and M.E. degrees in electrical and electronic engineering from Toyohashi University of Technology in 1985 and 1987, and Dr.E degree in applied mathematics and physics from Kyoto University in 1995. In 1987, he joined the Fuji Xerox Corporation. From 1991 to 1994, he was with the ATR Auditory and Visual Perception Research Labs. and ATR Human Information Processing Research Labs. Since 1997 he has been an Associate Professor in School of Knowledge Science, Japan Advanced Institute of Science and Technology. His current research interests include network science, especially, evolutional networks, distributed systems, ad hoc networks, P2P, social networking, community analysis, webology, and so on. He is a member of JSIAM and IEICE.