An algebraic approach to substructural logics

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January 9, 2008

Abstract

Substructural logics are logic obtained from classical logic **LK** or intuitionistic logic **LJ** by deleting some of structural rules. This study is started from study of **FL** by Lambek. They include relevant logics, linear logic and BCK-logics. By introducing sequent calculi which have the cut elimination theorem we can show various syntactical results. But syntactical methods work well only for particular logics, so we cannot use them for general discussions. So we need to find useful semantical methods. Since Kripke-type semantics is quite powerful in the study of modal logics, it does not work well for substructural logics. In recent years, algebraic methods have been developed as a powerful tool for investigating substructural logics. In this thesis we study two topics by using algebraic methods. One is an algebraic characterization of a logical property. The another is about maximal consistent logics. We show the detail these topics in following.

Disjunction property

For modal logics and intermediate logics Maksimova and Wroński show algebraic characterization of some logical properties. Some of the basic substructural logics are shown to have the disjunction property (DP) by using cut elimination of sequent calculi for these logics. On the other hand, this syntactic method works only for a limited number of substructural logics. Here, we show that Maksimova's criterion on the DP of superintuitionistic logics can be naturally extended to one on the DP of substructural logics over **FL**. By using this, we show the DP for some of the substructural logics for which syntactic methods do not work well. From algebraic characterization we show that substructural logic $\mathbf{FL}[\mathrm{E}_{\mathrm{n}}^{\mathrm{m}}]$ and $\mathbf{FL}[\mathrm{DN}]$ which does not have cut-elimination theorem have the disjunction property,

Minimal subvarieties

It is known that classical logic **CL** is the single maximal consistent logic over intuitionistic logic **Int**, which is moreover the single one even over the substructural logic $\mathbf{FL_{ew}}$. On the other hand, if we consider maximal consistent logic over a weaker logic the number of them can be uncountably many. Since the subvariety lattice of a given variety $\mathcal{V}$ of residuated lattices is dually isomorphic to the lattice of logics over the corresponding substructural logic $\mathbf{L}(\mathcal{V})$, the number of maximal consistent logics is equal to the number of minimal subvarieties (atoms) of the subvariety lattice of $\mathcal{V}$.

Tsinakis and Wille have shown that there exist uncountably many atoms in the subvariety lattice of the variety of involutive residuated lattices. We will show that while there exist uncountable many atoms in the subvariety lattice of the variety $\mathcal{V}_{\Updownarrow}$ of bounded representable involutive residuated lattices with mingle axiom $x^2 \leq x$, only two atoms exists in the subvariety lattice of the variety $\mathcal{V}_{\rangle}$ of bounded representable involutive residuated lattices with the idempotency $x = x^2$.

Key Words: **substructural logics, residuated lattices, disjunction property, well-connectedness, minimal subvariety, maximal consistent logic, involutive, representable**