An algebraic approach to substructural logics

by

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Abstract

Substructural logics are logic obtained from classical logic $\mathbf{LK}$ or intuitionistic logic $\mathbf{LJ}$ by deleting some of structural rules. This study is started from study of $\mathbf{FL}$ by Lambek. They include relevant logics, linear logic and BCK-logics. By introducing sequent calculi which have the cut elimination theorem we can show various syntactical results. But syntactical methods work well only for particular logics, so we cannot use them for general discussions. So we need to find useful semantical methods. Since Kripke-type semantics is quite powerful in the study of modal logics, it does not work well for substructural logics. In recent years, algebraic methods have been developed as a powerful tool for investigating substructural logics. In this thesis we study two topics by using algebraic methods. One is an algebraic characterization of a logical property. The another is about maximal consistent logics. We show the detail these topics in following.

Disjunction property For modal logics and intermediate logics Maksimova and Wroński show algebraic characterization of some logical properties [13, 22, 12]. Some of the basic substructural logics are shown to have the disjunction property (DP) by using cut elimination of sequent calculi for these logics [16, 15]. On the other hand, this syntactic method works only for a limited number of substructural logics. Here, we show that Maksimova's criterion [13] on the DP of superintuitionistic logics can be naturally extended to one on the DP of substructural logics over $\mathbf{FL}$. By using this, we show the DP for some of the substructural logics for which syntactic methods do not work well. From algebraic characterization we show that substructural logic $\mathbf{FL}[\mathrm{E}_\mathrm{n}^\mathrm{m}]$ and $\mathbf{FL}[\mathrm{DN}]$ which does not have cut-elimination theorem have the disjunction property,

Minimal subvarieties It is known that classical logic $\mathbf{CL}$ is the single maximal consistent logic over intuitionistic logic $\mathbf{Int}$, which is moreover the single one even over the substructural logic $\mathbf{FL_{ew}}$. On the other hand, if we consider maximal consistent logic over a weaker logic the number of them can be uncountably many. Since the subvariety lattice of a given variety $\mathcal{V}$ of residuated lattices is dually isomorphic to the lattice of logics over the corresponding substructural logic $\mathbf{L}(\mathcal{V})$, the number of maximal consistent logics is equal to the number of minimal subvarieties (atoms) of the subvariety lattice of $\mathcal{V}$.

Tsinakis and Wille have shown that there exist uncountably many atoms in the subvariety lattice of the variety of involutive residuated lattices. We will show that while there exist uncountable many atoms in the subvariety lattice of the variety $\mathcal{V}_m$ of bounded representable involutive residuated lattices with mingle axiom $x^2 \leq x$, only two atoms exists in the subvariety lattice of the variety $\mathcal{V}_i$ of bounded representable involutive residuated lattices with the idempotency $x = x^2$.

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Contents

Chapter 1

Introduction

1.1 Introduction

1.1.1 Background and purpose

The proof theoretic methods and algebraic methods are two important basic ways in study of logic. The former focuses on finite syntactically structures and the latter take methods like set theory and so on. So relation of these had not discussed. But for example in these years there are studies in which the cut elimination theorem is used in proving the finite model property and the cut elimination is proved by algebraic methods.

The study of logics by algebraic methods studied actively from 1950 to 1960. After that Grippe semantics become mainstream of semantical study. The conventional algebraic study turn off study of logics and develop as universal algebra. In 1990s , it is used for study of substructural logic and modal logic.

Roughly substructural logics are logics obtained from intuitionistic logic $\mathbf{LJ}$ and classical logic $\mathbf{LK}$ by deleting structural rules. The study starts from the study of categorical grammar by Lambek and it get active in 1990.

An algebraic study for substructural logics has been developed remarkably in these years. Also, collaborations of logicians with algebraists who interested in ordered algebraic structures are on-going. A syntactical proof of the cut elimination is not necessarily easy to understand for algebraist. Recently we get purely algebraic proof of cut elimination theorem.

In this thesis we principally take up residuated lattices which does not necessarily assume integrality, commutativity and contractivity. This corresponds to substructural logic $\mathbf{FL}$.

1.1.2 Topics of this thesis

In this thesis, we study two topics.

By applying Maksimova's result on algebraic characterization of the disjunction property to substructural logics, we show the disjunction property of many substructural logics, for which proof-theoretic methods are intractable.

Next, we show that the number of minimal subvarieties of involutive representable residuated lattices is uncountable even if we assume the mingle axiom $x^2 \leq x$. This strengthens a related result on representable residuated lattices by Jipsen and Tsinakis, and also one on involutive residuated lattices by Tsinakis and Wille. Moreover we show that the number becomes only two if we assume the idempotent axiom $x = x^2$ instead of the mingle axiom. The result makes an interesting contrast with a result by Galatos, which says that the number of uncountable when the involutiveness is omitted.

Our thesis is organized as follows. The first four chapters are denoted to explanations of the background; and the last two chapters consist of our main results.

In Chapter 2 we introduce a substructural logic **FL** and its extensions. First we introduce sequent calculus **LJ**. Roughly speaking sequent calculus **FL** is obtained from **LJ** by deleting all structure rules. To introduce sequent calculus **FL** we will show that what will happen when deleting structure rules and why we must introduce new logical connectives fusion, two implications and propositional constants. Then we define logics over **FL** as sets of formulas and they form a complete lattice.

In Chapter 3 we give a brief survey of some results on algebras from a view point of universal algebra.

In Chapter 4 we introduce residuated lattice which is an algebraic structure of substructural logics. We will show completeness theorem and discuss about lattice of logics is dually isomorphic to lattice of varieties of residuated lattices.

In Chapter 5 we give an algebraic proof of the disjunction property of **FL**, $\mathbf{FL}[\mathrm{E}_\mathrm{n}^\mathrm{m}]$ and $\mathbf{FL}[\mathrm{DN}]$. Some basic substructural logics are shown to have the disjunction property by using cut elimination theorem. Thus, substructural logic which does not hold cut elimination theorem we cannot prove disjunction property. The algebraic characterization of disjunction property for logics over intermediate logic is shown by L. L. Maksimova [13]. We extends this result to logics over substructural logic **FL**. We construct a suitable well-connected residuated lattice. It satisfies the condition of the algebraic characterization of disjunction property. These results in Chapter 5 will be appeared soon in Notre Dame Journal of Formal Logic as “An algebraic approach to the disjunction property of substructural logics” [20]. Moreover, some parts of these results are already announced in Chapter 5 of [6].

In Chapter 6 we discuss the number of minimal subvarieties of involutive representable residuated lattices. First we give a sketch of related results. We discuss the number of minimal subvarieties of two classes of representable residuated lattices. One is that there are uncountably many minimal subvarieties of bounded representable 3-potent residuated lattices, shown by P. Jipsen and C. Tsinakis [11]. The another is that there are uncountably many minimal subvarieties of representable residuated lattice with idempotent axiom, shown by N. Galatos [5]. Next, we explain a result by C. Tsinakis and A. Wille [21] that there exists uncountably many minimal subvarieties of involutive residuated lattice. We show that there are uncountably many minimal subvarieties of bounded involutive representable residuated lattice with mingle axiom. On the other hand, we show that there exists only two minimal subvarieties of bounded involutive representable residuated lattice with idempotent axiom. These results are presented at the conference “Algebraic and Topological Methods in Non-Classical Logics III”, in Oxford.

In Chapter 7 we coclude these results with future works.

Chapter 2

Sequent calculus of the substructural logic FL

In this chapter, we introduce a substructural logic **FL** and its extensions. We introduce sequent calculus **FL** is obtained from **LJ** by deleting all structure rules. To introduce sequent calculus **FL** we will show that what will happen when deleting structure rules. Then we define logics over **FL** as sets of formulas and they form a complete lattice.

2.1 Sequent calculus LJ for intuitionistic logic

First, we introduce sequent calculus **LJ** for intuitionistic logic **Int**, see [16, 17]. We use $\wedge$(conjunction), $\vee$(disjunction), $\rightarrow$(implication) and $\neg$(negation) as logical connectives. By using these connectives we define *formulas* inductively as follows.

Definition 2.1.1 (formula) Formulas are defined inductively as follows;

i. all propositional variables and propositional constants($\top, \bot$) are formulas,

ii. if α, β are formulas then $\alpha \wedge \beta, \alpha \vee \beta, \alpha \rightarrow \beta$ and $\neg\alpha$ are formulas.

A *sequent* is an expression of the following form.

$$\alpha_1, \alpha_2, \ldots, \alpha_m \Rightarrow \beta$$

where $\alpha_1, \ldots, \alpha_m, \beta$ are formulas and $m \geq 0$. β can be empty. Hereafter we use capital Greek letters $\Gamma, \Delta, \ldots$ for finite sequences of formulas, separated by commas. Next we define the sequent calculus **LJ**. The sequent calculus **LJ** contains initial sequents and inference rules.

Initial sequents The *initial sequents* of **LJ** are following.

1. $\alpha \Rightarrow \alpha$
2. $\Gamma \Rightarrow \top$
3. $\bot, \Gamma \Rightarrow \gamma$

Inference rules

Weakening rules:

$$\frac{\Gamma \Rightarrow \beta}{\alpha, \Gamma \Rightarrow \beta}\ (\textit{left-weakening}) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (\textit{right-weakening})$$

Contraction rule:

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \beta}{\alpha, \Gamma \Rightarrow \beta}\ (\textit{contraction})$$

Exchange rule:

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma}\ (\textit{exchange})$$

Cut rule:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta}\ (\textit{cut})$$

Logical rule:

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma}\ (\textit{left-}\wedge\textit{1}) \quad \frac{\Gamma, \beta, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma}\ (\textit{left-}\wedge\textit{2})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\textit{right-}\wedge)$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha \vee \beta, \Gamma \Rightarrow \gamma}\ (\textit{left-}\vee)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\textit{left-}\vee\textit{1}) \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\textit{left-}\vee\textit{2})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha \rightarrow \beta, \Gamma, \Delta \Rightarrow \gamma}\ (\textit{left-}\rightarrow) \quad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}\ (\textit{right-}\rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\neg\alpha, \Gamma \Rightarrow}\ (\textit{left-}\neg) \quad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\alpha}\ (\textit{right-}\neg)$$

A sequent $\Gamma \Rightarrow \varphi$ is *provable* in **LJ** if it can be obtained from initial sequents by applying rules of inference repeatedly. A formula φ in **LJ** is *provable* if a sequent $\Rightarrow \varphi$ is provable. A figure which shows how a given sequent $\Gamma \Rightarrow \varphi$ is obtained is called a *proof* of $\Gamma \Rightarrow \varphi$. We say that a formula φ is *provably equivalent* to another formula ψ in a given sequent calculus, when both sequents $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$ are provable in it. For more information on **LJ**, see [16]

2.2 Substructural logics over FL

In this section, we define substructural logics over **FL**. The calculus **FL** is obtained from **LJ** by deleting all structure rules. Roughly speaking extension of **FL** is called substructural logic over **FL**. Before giving the definition, we explain briefly some ideas behind it.

2.2.1 Weakening rules and propositional constants

Here we explain relations between propositional constants ($\top, \bot$) and weakening rule. In **LJ** $\Gamma \Rightarrow \top$ is an initial sequent. We can show the following.

A formula φ is provable in **LJ** iff φ is provably equivalent to $\top$ in **LJ**.

($\Rightarrow$)

$$\dfrac{\varphi \Rightarrow \top}{\Rightarrow \varphi \to \top} \qquad \dfrac{\dfrac{\Rightarrow \varphi}{\top \Rightarrow \varphi}}{\Rightarrow \top \to \varphi}$$

($\Leftarrow$)

$$\dfrac{\Rightarrow \top \to \varphi \qquad \dfrac{\Rightarrow \top \quad \varphi \Rightarrow \varphi}{\top \to \varphi \Rightarrow \varphi}}{\Rightarrow \varphi}$$

Moreover we can show that $\neg\varphi$ is provably equivalent to $\varphi \to \bot$ in **LJ**.

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}}{\neg\varphi, \varphi \Rightarrow \bot}}{\neg\varphi \Rightarrow \varphi \to \bot} \qquad \dfrac{\dfrac{\varphi \Rightarrow \varphi \quad \bot \Rightarrow}{\varphi \to \bot, \varphi \Rightarrow}}{\varphi \to \bot \Rightarrow \neg\varphi}$$

As shown in these proofs, weakening rules are essentially used to show these equivalences. Since $\mathbf{FL_e}$ does not have weakening rules we cannot show these equivalences. To keep the similar equivalences even for logics without weakening rules we introduce new propositional constants $1, 0$ and initial sequents and inference rules for them.

1. $\Rightarrow 1$
2. $0 \Rightarrow$

$$\dfrac{\Gamma, \Delta \Rightarrow \gamma}{\Gamma, 1, \Delta \Rightarrow \gamma}\ (1w) \qquad \dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0w)$$

Roughly speaking the constant 1 is the weakest provable formula and 0 is the strongest contradictory formula. In fact, by the help of these initial sequent and rule for 0, we can show that $\neg\varphi$ is equivalent to $\varphi \to 0$ as follows.

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}}{\neg\varphi, \varphi \Rightarrow 0}}{\neg\varphi \Rightarrow \varphi \to 0} \qquad \dfrac{\dfrac{\varphi \Rightarrow \varphi \quad 0 \Rightarrow}{\varphi \to 0, \varphi \Rightarrow}}{\varphi \to 0 \Rightarrow \neg\varphi}$$

2.2.2 Structural rules and commas

In **LJ** we can show the following.

$\varphi_1, \varphi_2, \ldots, \varphi_m \Rightarrow \psi$ is provable iff $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_m \Rightarrow \psi$ is provable.

To show the only-if part we need contraction rule, and to show the if-part we need weakening rule. On the other hand in substructural logics either without contraction rule or without left-weakening rule, commas on the left-hand side of sequents don't behave as conjunctions. To represent commas in sequents, it is convenient to introduce a new logical connective $\cdot$ called *fusion*. We add following rules for $\cdot$.

$$\dfrac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \gamma}\ (\textit{left-fusion}) \qquad \dfrac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\textit{right-fusion})$$

2.2.3 Exchange rules and implications

Suppose that we have exchange rule. We can show that if $\alpha, \Gamma \Rightarrow \beta$ is provable then $\Gamma \Rightarrow \alpha \rightarrow \beta$ is provable. But lack of exchange rule we cannot show it. Thus, in sequent calculi without exchange rule, we introduce two implication connectives $\backslash$ and $/$. Moreover it is natural to introduce two negation connectives $\sim$ and $-$.

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta, \Delta \Rightarrow \gamma}{\Sigma, \Gamma, \alpha\backslash\beta, \Delta \Rightarrow \gamma}\ (\textit{left-}\backslash) \quad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\textit{right-}\backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta, \Delta \Rightarrow \gamma}{\Sigma, \beta/\alpha, \Gamma, \Delta \Rightarrow \gamma}\ (\textit{left-}/) \quad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\textit{right-}/)$$

2.2.4 Sequent calculus substructural logic FL

The sequent calculus **FL** has the following initial sequents and rules;

Initial sequents:

1. $\alpha \Rightarrow \alpha$
2. $\Rightarrow 1$
3. $0 \Rightarrow$

Rules:

Cut rule:

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \alpha, \Delta \Rightarrow \beta}{\Sigma, \Gamma, \Delta \Rightarrow \beta}\ (\textit{cut})$$

Logical rule:

$$\frac{\Gamma, \Delta \Rightarrow \gamma}{\Gamma, 1, \Delta \Rightarrow \gamma}\ (1w) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0w)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \gamma}\ (\textit{left-}\wedge\textit{1}) \quad \frac{\Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \gamma}\ (\textit{left-}\wedge\textit{2})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\textit{right-}\wedge)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma \quad \Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \gamma}\ (\textit{left-}\vee)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\textit{left-}\vee\textit{1}) \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\textit{left-}\vee\textit{2})$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \gamma}\ (\textit{left-fusion}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\textit{right-fusion})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta, \Delta \Rightarrow \gamma}{\Sigma, \Gamma, \alpha\backslash\beta, \Delta \Rightarrow \gamma}\ (\textit{left-}\backslash) \quad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\textit{right-}\backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta, \Delta \Rightarrow \gamma}{\Sigma, \beta/\alpha, \Gamma, \Delta \Rightarrow \gamma}\ (\textit{left-}/) \quad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\textit{right-}/)$$

Note that we define negation rules $\sim$ and $-$ by $\sim \varphi \Leftrightarrow \varphi\backslash 0$ and $-\varphi \Leftrightarrow 0/\varphi$.

We denote the sequent calculus $\mathbf{FL_e}$ is obtained from **FL** by adding the exchange rule and $\mathbf{FL_{ew}}$ is obtained from $\mathbf{FL_e}$ by adding weakening rule. Note that if we add exchange rule then $\alpha\backslash\beta$ and β/α are provably equivalent in **FL**. Thus, we use $\rightarrow$.

2.3 Substructural logics over FL

Formally, *substructural logics over* **FL** (or simply, *logics over* **FL**) are defined to be axiomatic extensions of **FL**, i.e. sequent calculi obtained from **FL** by adding some additional initial sequents $\{\Rightarrow \varphi_i | i \in I\}$. These formulas φ_i should be regarded as schemes, called *axiom schemes*, and therefore every substitution instance φ of them can be taken as an initial sequent. We denote logic which obtained from **FL** by adding axiom schemes $\{\varphi_1, \ldots, \varphi_n\}$ is denoted by $\mathbf{FL}[\varphi_1, \ldots, \varphi_n]$.

Similarly, we can introduce the notion of substructural logics over $\mathbf{FL_e}$ etc. The calculus $\mathbf{FL_e}$ can be also regarded as a logic over **FL**, since the exchange rule can be expressed by a formula $(\varphi \cdot \psi) \rightarrow (\psi \cdot \cdot \cdot \varphi)$. Also, the contraction rule, the left- and right-weakening rules are expressed as $\varphi \rightarrow (\varphi \cdot \varphi)$, $(\varphi \cdot \psi) \rightarrow \varphi$ and $(\varphi \cdot \psi) \rightarrow \psi$, and $0 \rightarrow \varphi$, respectively.

But, sometimes it is more convenient to define substructural logics as sets of **L** formulas. In fact, for each substructural logics over **FL** (by the above definition), let L be the set of all formulas provable in **L**. Then, we can show that the set L satisfies the following, where FL in (1) denotes the set of all formulas provable in **FL**.

1. $\mathrm{FL} \subseteq L$.

2. L is closed under substitution, i.e. if a formula $\varphi(p)$ which includes a propositional variable p belongs to L then $\varphi(\psi)$ belongs also to L for any formula ψ. Here $\varphi(\psi)$ expresses the formula obtained from $\varphi(p)$ by replacing every occurrence of p in $\varphi(p)$ by the formula ψ.

3. If $\varphi, \varphi\backslash\psi \in L$ then $\psi \in L$.

4. If $\varphi \in L$ then $\varphi \wedge 1 \in L$.

5. If $\varphi \in L$ and ψ is an arbitrary formula of the form $\psi\backslash(\varphi \cdot \psi), (\psi \cdot \varphi)/\psi \in L$.

From now on, we will take these conditions to give an alternative definition of substructural logics. That is,

Definition 2.3.1 A set L of formulas is a substructural logic over **FL**, if it satisfies all of conditions from (1) to (5).

Hereafter, we identify the calculus **L** with the corresponding set L of formulas, and in most cases, logics are represented by using bold face letters. By replacing FL in the condition (1) by another substructural logic L_0, we can introduce the notion of logic over $\mathbf{L_0}$. Clearly, every logic over $\mathbf{FL_e}$ is a logic over $\mathbf{FL}$, and so on.

The set $\mathcal{L}$ of all logics is ordered by the set inclusion. The maximum logic is Φ. The logic Φ is called the *inconsistent logic*. Other logics are said to be consistent. Then it is clear that $L_1 \cap L_2 \in \mathcal{L}$ for any $L_1, L_2 \in \mathcal{L}$. In general, for any $\{\mathrm{L_i} \in \mathcal{L} | i \in I\}$ the intersection $\bigcap_{i \in I} \mathrm{L_i}$ is in $\mathcal{L}$. But union $L_1 \cup L_2$ is not necessarily. So we define $L_1 \vee L_2$ as the minimum logic including $L_1 \cup L_2$. Thus $\langle \mathcal{L}, \cap, \vee, \mathbf{FL}, \Phi \rangle$ forms a bounded lattice whose greatest element is Φ and the least element $\mathbf{FL}$.

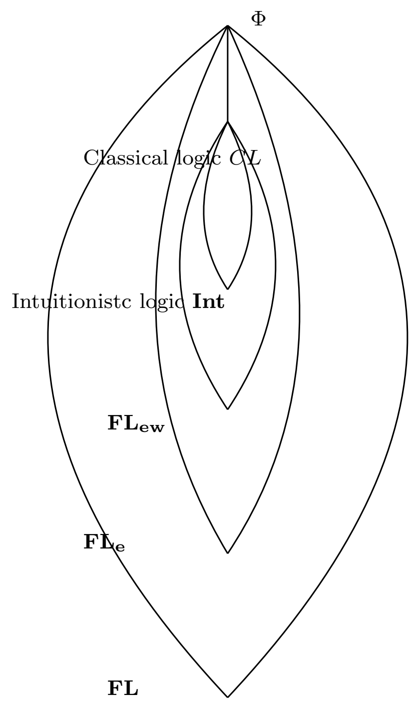

Figure 2.1.

Chapter 3

Universal algebra

As the title of our thesis shows, a special feature of our approach to substructural logics is to use heavily concepts and tools of algebra. In this chapter we give a brief survey of some basic results on algebras from a view point of universal algebra [3], which will be necessary in later chapters.

3.1 Lattices and Boolean algebras

Definition 3.1.1 (partial order) A structure $\mathbf{A} = \langle \mathrm{A}, \leq \rangle$ is a *partially ordered set* (p.o.set) if the binary relation $\leq$ satisfies the following. For all $x, y, z \in \mathrm{A}$.

(P1) $x \leq x$.

(P2) If $x \leq y$ and $y \leq x$ then $x = y$.

(P3) If $x \leq y$ and $y \leq z$ then $x \leq z$.

Moreover if a p.o.set $\mathbf{A} = \langle \mathrm{A}, \leq \rangle$ satisfies

(P4) $x \leq y$ or $y \leq x$ for every $x, y \in \mathrm{A}$

then $\mathbf{A}$ is a *totally ordered set*.

Definition 3.1.2 (lattice) A structure $L = \langle \mathrm{L}, \wedge, \vee \rangle$ is a *lattice* if it satisfies the following. For all $x, y, z \in \mathrm{L}$.

(L1) $x \wedge x = x$, $x \vee x = x$.

(L2) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \vee (y \vee z) = (x \vee y) \vee z$.

(L3) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$.

(L4) $x \wedge (x \vee y) = x$, $x \vee (x \wedge y) = x$.

Let $\mathbf{L} = \langle \mathrm{L}, \wedge, \vee \rangle$ be a lattice. Define a binary relation $\leq$ by

$$x \leq y \Leftrightarrow x \wedge y = x.$$

Then, we can show that $\leq$ is a partial order. We note that $x \wedge y = x$ is equivalent to the condition $x \vee y = y$. Thus each lattice induces always a partial order on it.

Definition 3.1.3 Let X be a set. The *Boolean algebra of subsets of* X, $\mathbf{Su}(X)$, has as its universe $\mathrm{Su}(X)$ which is the power set of X and as operations $\cup, \cap, ', \emptyset, X$.

3.2 Concepts from universal algebra

Subalgebras, homomorphisms and so on which play an important role in study of algebra can be introduced into algebras. In this section we show some basic properties. A language (or type) of algebras is a set $\mathcal{F}$ of n-ary operation symbol f. Algebra $\mathbf{A}$ of type $\mathcal{F}$ is an pair $\langle \mathrm{A}, \mathcal{F} \rangle$ where A is a nonempty set and where there is an n-ary operation $f^{\mathbf{A}}$ on A. Here after we say that algebra $\mathbf{A}$ means $\mathbf{A}$ of type $\mathcal{F}$. For further information, see [3].

3.2.1 Homomorphism and isomorphism

Definition 3.2.1 Let $\mathbf{A}$ and $\mathbf{B}$ be algebras. A mapping $\alpha : \mathbf{A} \to \mathbf{B}$ is a *homomorphism* if α satisfies the following conditions.

for any $a_1, a_2 \in \mathrm{A}$, $\alpha(f^{\mathbf{A}}(a_1, \ldots, a_n)) = f^{\mathbf{B}}(\alpha(a_1), \ldots, \alpha(a_2))$.

Furthermore,

1. if α is an one-to-one mapping then α is called a *monomorphism* or a *embedding*.
2. if α is an onto mapping then α is called a *epimorphism* or an *onto homomorphism*.
3. if α is an one-to-one and onto mapping then α is called an *isomorphism*. If there exist an isomorphism α from $\mathbf{A}$ to $\mathbf{B}$ then $\mathbf{A}$ is said to be *isomorphic* to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$.

Definition 3.2.2 Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then the *kernel of* α, written $ker(\alpha)$, and the *image of* α, written $Im(\alpha)$, are defined by

$$ker(\alpha) = \{\langle a, b \rangle \in \mathrm{A}^2 : \alpha(a) = \alpha(b)\}, \quad Im(\alpha) = \{\alpha(a) \in \mathrm{B} : a \in \mathrm{A}\}.$$

If α is a surjective then $Im(\alpha)$ is equal to $\mathbf{B}$ and we say that $\mathbf{B}$ is the homomorphic image of $\mathbf{A}$. Sometime $Im(\alpha)$ is expressed also by $\alpha(\mathbf{A})$.

3.2.2 Subalgebra and quotient algebra

Definition 3.2.3 Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras. Then $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $\mathrm{B} \subseteq \mathrm{A}$ and every operation $f_{\mathbf{B}}$ of $\mathbf{B}$ is the restriction of the corresponding operation of $\mathbf{A}$. We write simply $\mathbf{B} \leq \mathbf{A}$ when $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. A subuniverse of $\mathbf{A}$ is a subset B of A which is closed under the operations of $\mathbf{A}$, i.e. if $f_{\mathbf{A}}$ is a operation of $\mathbf{A}$ and $a_1, \ldots, a_n \in \mathrm{B}$ we would require $f^{\mathbf{A}}(a_1, \ldots, a_n) \in \mathrm{B}$.

Definition 3.2.4 Given an algebra $\mathbf{A}$ define, for every $X \subseteq \mathrm{A}$,

$$Sg(X) = \bigcap\{\mathrm{B} : X \subseteq \mathrm{B} \text{ and } \mathrm{B} \text{ is a subuniverse of } \mathbf{A}\}$$

We read $Sg(X)$ as the subuniverse generated by X.

For information on Sg, see [3].

Definition 3.2.5 Let $\mathbf{A}$ be an algebra and let θ is an equivalence relation on $\mathbf{A}$. Then θ is a *congruence* on $\mathbf{A}$ if θ satisfies the following *compatibility property*:

CP: For each operation $f^{\mathbf{A}}$ and elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathrm{A}$, if $a_i \theta b_i$ holds for any $i \in \{1, \ldots, n\}$ then

$$f^{\mathbf{A}}(a_1, \ldots, a_n) \theta f^{\mathbf{A}}(b_1, \ldots, b_n)$$

holds.

We can consider that congruences on $\mathbf{A}$ are a subset of $\mathbf{A} \times \mathbf{A}$ and thus they are ordered by the set inclusion. Hence we define maximum congruence ∇, called the full congruence and minimum congruence Δ as follows.

$$\nabla = \{\langle a, b \rangle; a, b \in \mathrm{A}\}$$
$$\Delta = \{\langle a, a \rangle; a \in \mathrm{A}\}$$

The set of all congruences on $\mathbf{A}$ is denoted by $\mathrm{Con}\,\mathbf{A}$. Then we can easily show that $\mathrm{Con}\,\mathbf{A}$ is a bounded lattice which has the maximum element ∇ and the minimum element Δ. So the congruence lattice on $\mathbf{A}$ denoted by $\mathbf{Con}\,\mathbf{A}$. The following is the definition of $\wedge$ and $\vee$, where $\theta_1 \circ \theta_2$ denotes the set $\{\langle a, b \rangle \mid \exists c \in \mathbf{A}$ such that $a\theta_1 c \theta_2 b\}$.

$$\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2$$
$$\theta_1 \vee \theta_2 = \theta_1 \cup (\theta_1 \circ \theta_2) \cup (\theta_1 \circ \theta_2 \circ \theta_1) \cup (\theta_1 \circ \theta_2 \circ \theta_1 \circ \theta_2) \cup \ldots.$$

Definition 3.2.6 Let $\mathbf{A}$ be a algebra and $a_1, \ldots, a_n \in \mathrm{A}$. Then $\Theta(a_1, \ldots, a_n)$ is the minimum congruence such that $a_1, \ldots, a_n$ are contained in a same equivalence class.

Definition 3.2.7 An algebra $\mathbf{A}$ is *congruence-distributive* if $\mathbf{Con}\,\mathbf{A}$ is a distributive lattice. Moreover a class $\mathcal{K}$ of algebras is congruence-distributive if every algebra in $\mathcal{K}$ is congruence-distributive.

Proposition 3.2.1 *Let* $\alpha : \mathbf{A} \to \mathbf{B}$ *be a homomorphism. Then* $ker(\alpha)$ *is actually a congruence on* $\mathbf{A}$.

Proof If $\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \in ker(\alpha)$, then

$$\begin{aligned} \alpha(f^{\mathbf{A}}(a_1, \ldots, a_n)) &= f^{\mathbf{B}}(\alpha(a_1), \ldots, \alpha(a_2)) \\ &= f^{\mathbf{B}}(\alpha(b_1), \ldots, \alpha(b_2)) \\ &= \alpha(f^{\mathbf{A}}(b_1, \ldots, b_2)) \end{aligned}$$

hence

$\langle f^{\mathbf{A}}(a_1, \ldots, a_2), f^{\mathbf{B}}(b_1, \ldots, b_2)\rangle \in ker(\alpha)$.

Clearly $ker(\alpha)$ is an equivalence relation, so it follows that $ker(\alpha)$ is actually a congruence on $\mathbf{A}$.

□

Let θ is a congruence on a algebra $\mathbf{A}$. Then θ is an equivalence relation. So we define an equivalence class (a/θ) include $a \in \mathrm{A}$ as follows.

$$a/\theta = \{x \in \mathrm{A}; x\theta a\}.$$

In addition we define quotient set A/θ as follows.

$$\mathrm{A}/\theta = \{a/\theta; a \in \mathrm{A}\}.$$

Definition 3.2.8 Let θ be a congruence on $\mathbf{A}$. Then the *quotient algebra of* $\mathbf{A}$ *by* θ, written $\mathbf{A}/\theta$, is the algebra whose universe is A/θ and whose operations satisfy

$$f^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = (f^{\mathbf{A}}(a_1, \ldots, a_n))/\theta$$

where $a_1, \ldots, a_n \in \mathrm{A}$.

Definition 3.2.9 Let $\mathbf{A}$ be an algebra and let $\theta \in \mathrm{Con}\ \mathbf{A}$. The *natural map* $\nu_\theta : \mathrm{A} \to \mathrm{A}/\theta$ is defined by $\nu_\theta(a) = a/\theta$ for any $a \in \mathrm{A}$. (When there is no ambiguity we write simply ν instead of ν_θ.)

Proposition 3.2.2 *A natural map from* $\mathbf{A}$ *to* $\mathbf{A}/\theta$ *is an onto homomorphism.*

Proof It is clear that the natural map is onto. For any $a_1, \ldots, a_n \in \mathrm{A}$

$$\begin{aligned} v_\theta(f^{\mathbf{A}}(a_1, \ldots, a_n)) &= (f^{\mathbf{A}}(a_1, \ldots, a_n))/\theta \\ &= f^{\mathbf{A}/\theta}(a_n/\theta, \ldots, a_n/\theta) \\ &= f^{\mathbf{A}/\theta}(v_\theta(a), \ldots, v_\theta(a_n)) \end{aligned}$$

Thus v_θ is a homomorphism.

□

Proposition 3.2.3 (Homomorphism theorem) *Let* $\alpha : \mathbf{A} \to \mathbf{B}$ *be an onto homomorphism. Then there is an isomorphism* β *from* $\mathbf{A}/ker(\alpha)$ *to* $\mathbf{B}$ *defined by* $\alpha = \beta \circ \nu$*, where* ν *is the natural homomorphism from* $\mathbf{A}$ *to* $\mathbf{A}/ker(\alpha)$.

Proof First note that if $\alpha = \beta \circ \nu$ then we must have $\beta(a/\theta) = \alpha(a)$. The second of these equalities does indeed define a function β, and β satisfies $\alpha = \beta \circ \nu$. It is not difficult to verify that β is a bijection. To show that β is actually an isomorphism, suppose $a_1, \ldots, a_n \in \mathrm{A}$. Then

$$\begin{aligned} \beta(f^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta)) &= \beta((f^{\mathbf{A}}(a_1, \ldots, a_n))/\theta) \\ &= \alpha(f^{\mathbf{A}}(a_1, \ldots, a_2)) \\ &= f^{\mathbf{B}}(\alpha(a_1), \ldots, \alpha(a_n)) \\ &= f^{\mathbf{B}}(\beta(a_1/\theta), \ldots, \beta(a_2/\theta)). \end{aligned}$$

□

Let $\mathbf{A}$ be a algebra and $\phi, \theta \in \mathrm{Con}\ \mathbf{A}$ and $\theta \subseteq \phi$. Then we define ϕ/θ as follows.

$$\phi/\theta = \{\langle a/\theta, b/\theta\rangle \in (\mathbf{A}/\theta)^2 : \langle a, b\rangle \in \phi\}.$$

The next proposition holds.

Proposition 3.2.4 *Let $\phi, \theta \in \mathrm{Con}\ \mathbf{A}$ and $\theta \subseteq \phi$. Then ϕ/θ is a congruence on $\mathbf{A}/\theta$.*

Proof Let $\langle a_1/\theta, b_1/\theta\rangle, \ldots, \langle a_n, b_n\rangle \in \phi/\theta$. Then $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \phi$ from definition of ϕ/θ. So

$$\langle f^{\mathbf{A})}(a_1, \ldots, a_n), f^{\mathbf{B}}(b_1, \ldots, b_n)\rangle \in \phi.$$

Hence

$$\langle f^{\mathbf{A})}(a_1, \ldots, a_n)/\theta, f^{\mathbf{B}}(b_1, \ldots, b_n)/\theta\rangle \in \phi/\theta.$$

Form this

$$\langle f^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_2/\theta), f^{\mathbf{B}/\theta}(b_1/\theta, \ldots, b_2/\theta)\rangle \in \phi.$$

Thus ϕ/θ is a congruence on $\mathbf{A}/\theta$.

□

Let $\mathbf{A}$ be an algebra and $\theta \in Con(\mathbf{A})$. Then we define a sublattice $[\theta, \nabla]$ of $\mathrm{Con}\ \mathbf{A}$ as follows.

$$[\theta, \nabla] = \{\phi \in \mathrm{Con}\ \mathbf{A} : \theta \subseteq \phi \subseteq \nabla\}$$

Proposition 3.2.5 (Correspondence theorem) *Let $\mathbf{A}$ be a algebra and $\theta \in \mathrm{Con}\ \mathbf{A}$. Then a mapping α on $[\theta, \nabla]$ defined by*

$$\alpha(\phi) = \phi/\theta$$

is a isomorphism from $[\theta, \nabla]$ to $\mathrm{Con}\ \mathbf{A}/\theta$

Proof First we show α is one-to-one. Let $\phi, \psi \in [\theta, \nabla]$ $(\phi \neq \psi)$. Suppose that $\phi \not\subseteq \psi$. Then there are $a, b \in \mathrm{A}$ such that $\langle a, b\rangle \in \phi - \psi$. Hence

$$\langle a/\theta, b/\theta\rangle \in (\phi/\theta) - (\psi/\theta)$$

So

$$\alpha(\phi) \neq \alpha(\psi)$$

Thus α is one-to-one. Next we show α is onto. Let $\psi \in \mathrm{Con}\mathbf{A}/\theta$, and $\phi = ker(\nu_\psi \circ \nu_\theta)$, where ν_ψ is a natural homomorphism from $\mathbf{Con\ A}/\theta$ to $(\mathbf{Con\ A}/\theta)/\psi$. Hence for any $a, b \in \mathrm{A}$

$$\begin{array}{ll} & \langle a/\theta, b/\theta\rangle \in \phi/\theta \\ \Leftrightarrow & \langle a, b\rangle \in \phi \\ \Leftrightarrow & \langle a/\theta, b/\theta\rangle \in \psi. \end{array}$$

So

$$\psi = \phi/\theta = \alpha(\phi).$$

Thus α is onto. Finally we show α is an isomorphism.

$$\begin{aligned} (\phi \cap_{\mathbf{A}} \psi)/\theta &= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \langle a, b\rangle \in \phi \cap_{\mathbf{A}} \psi\} \\ &= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \langle a, b\rangle \in \phi \text{ and } \langle \mathrm{a}, \mathrm{b}\rangle \in \psi\} \\ &= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \langle a, b\rangle \in \phi\} \text{ and } \{\langle \mathrm{a}/\theta, \mathrm{b}/\theta\rangle \in (\mathrm{A}/\theta)^2 : \langle \mathrm{a}, \mathrm{b}\rangle \in \psi\} \\ &= \phi/\theta \text{ and } \psi/\theta \\ &= \phi/\theta \cap_{\mathrm{Con}\ \mathbf{A}/\theta} \psi/\theta \end{aligned}$$

$$\begin{aligned} (\phi \vee_{\mathbf{A}} \psi)/\theta &= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \langle a, b\rangle \in \phi \vee_{\mathbf{A}} \psi\} \\ &= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \exists c_0 = a, c_1, \ldots, c_k = b \in \mathrm{A} \\ &\qquad s.t.\ \langle c_i, c_{i+1}\rangle \in \phi \text{ or } \langle \mathrm{c_i}, \mathrm{c_{i+1}}\rangle \in \psi\ (0 \leq \mathrm{i} \leq \mathrm{k} - 1)\} \\ &= \{\langle a/\theta, b/\theta\rangle \in (\mathrm{A}/\theta)^2 : \exists c_0/\theta = a/\theta, c_1/\theta, \ldots, c_k/\theta = b/\theta \in \mathrm{A}/\theta \\ &\qquad s.t.\ \langle c_i/\theta, c_{i+1}/\theta\rangle \in \phi/\theta \text{ or } \langle \mathrm{c_i}/\theta, \mathrm{c_{i+1}}/\theta\rangle \in \psi/\theta\ (0 \leq \mathrm{i} \leq \mathrm{k} - 1)\} \\ &= \phi/\theta \vee_{\mathrm{Con}\ \mathbf{A}/\theta} \psi/\theta \end{aligned}$$

Thus $\alpha(f^{\mathbf{A}}(\phi, \psi)/\theta = f\mathrm{Con}\mathbf{A}/\theta(\phi/\theta, \psi/\theta)$ holds.

□

3.2.3 Direct product and subdirect product

Definition 3.2.10 Let $(\mathbf{A}_i)_{1\leq i\leq n}$ is an indexed family of algebras. Define the *direct product* $\prod_{1\leq i\leq n} \mathbf{A}_i$ to be the algebra whose universe is the set $\prod_{1\leq i\leq n} \mathrm{A}_i$ and such that $a_i^1, a_i^m \in \mathrm{A_i}$, $1 \leq i \leq n$,

$$f^{\prod_{1\leq i\leq n}}(\langle a_1^1, \ldots, a_n^1\rangle, \ldots, \langle a_1^m, \ldots, a_n^m\rangle) = \langle f^{\mathbf{A}_1}(a_1^1, \ldots, a_1^m), \ldots, f^{\mathbf{A}_n}(a_n^1, \ldots, a_n^m)\rangle.$$

After here $x(j)$ means jth element of x.

Proposition 3.2.6 *Let $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}$ be algebras. Then the following isomorphic relations hold.*

1. $\mathbf{A_1} \times \mathbf{A_2} \cong \mathbf{A_2} \times \mathbf{A_1}$

2. $\mathbf{A_1} \times (\mathbf{A_2} \times \mathbf{A_3}) \cong \mathbf{A_1} \times \mathbf{A_2} \times \mathbf{A_3}$

Proof Let homomorphisms of 1 and 2 be $\alpha(\langle a_1, a_2 \rangle) = \langle a_2, a_1 \rangle$ and $\alpha(\langle a_1, \langle a_2, a_3 \rangle \rangle) = \langle a_1, a_2, a_3 \rangle$, respectively. Clearly that α_1, α_2 are isomorphisms.

□

The mapping $\pi_i : \prod_{1 \leq j \leq n} \mathbf{A}_j \longrightarrow \mathbf{A}_i$ $(1 \leq i \leq n)$ defined by

$$\pi_i(\langle a_1, a_2, \dots, a_n \rangle) = a_i$$

is called the *projection map on the* ith *coordinate* of $\prod_{1 \leq i \leq n} \mathrm{A}_i$. We can easily show that each projection map is an onto homomorphism.

Definition 3.2.11 An algebra $\mathbf{A}$ is a *subdirect product* of an indexed family $(\mathbf{A}_i)_{i \in I}$ of algebras if

$$\mathbf{A} \leq \prod_{i \in I} \mathbf{A}_i$$

and

$$\pi_i(\mathbf{A}) = \mathbf{A}_i \text{ for each } i \in I.$$

A subdirect product of $(\mathbf{A}_i)_{i \in I}$ is an algebra which is a subalgebra of $\prod_{i \in I} \mathbf{A}_i$ and satisfies the condition 2. Moreover because $\mathbf{A}$ satisfies the condition 2, it is not necessarily that $\mathbf{A}$ is isomorphic to $\prod_{i \in I} \mathbf{A}_i$. For example, if $\mathrm{A}_1 = \{a, b\}$, $\mathrm{A}_2 = \{c, d, e\}$, $\mathrm{A} = \{(a, c), (a, d), (b, c), (b, e)\}$, then it satisfies 2. But $\mathbf{A}$ is not isomorphic to $\prod_{i \in I} \mathbf{A}_i$ from $\prod_{i \in \{1,2\}} \mathrm{A}_i = \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$. An intuitive meaning of subdirect products is that they are sufficiently large subalgebra among direct products.

Definition 3.2.12 The mapping $\alpha : \mathbf{A} \to \prod_{i \in I} \mathbf{A}_i$ is a *subdirect embedding* if α is a embedding and $\alpha(\mathbf{A})$ is a subdirect product of $\prod_{i \in I} \mathbf{A}_i$.

Proposition 3.2.7 *Let* $\theta \in \mathrm{Con}\,\mathbf{A}$ *(*$i \in I$*) and* $\bigcap_{i \in I} \theta_i = \Delta$. *Then a homomorphism* $\nu : \mathbf{A} \to \prod_{i \in I} \mathbf{A}/\theta_i$ *defined by*

$$\nu(\alpha)(i) = a/\theta_i$$

is a subdirect embedding.

Proof If we define the ν by $\nu_i = \pi_i \circ \nu$ for any $i \in I$ then the ν_i is a natural homomorphism from $\mathbf{A}$ to $\mathbf{A}/\theta_i$. First we show that $\nu(\mathbf{A})$ is a subalgebra of $\prod_{i \in I} \mathbf{A}/\theta_i$.
For all $\nu(a_1), \dots, \nu(a_n) \in \nu(\mathbf{A})$ $(a_1, \dots, a_n \in \mathrm{A})$

$$f^{\prod_{i \in I} \mathbf{A}/\theta_i}(\nu(a_1), \dots, \nu(a_n)) = \nu(f^{\mathbf{A}}(a_1, \dots, a_n)) \in \nu(\mathbf{A}).$$

Furthermore

$$\{\top_{\prod_{i \in I} \mathbf{A}/\theta_i}, \bot_{\prod_{i \in I} \mathbf{A}/\theta_i}, 1_{\prod_{i \in I} \mathbf{A}/\theta_i}, 0_{\prod_{i \in I} \mathbf{A}/\theta_i}\} = \{\nu(\top_{\mathbf{A}}), \nu(\bot_{\mathbf{A}}), \nu(1_{\mathbf{A}}, \nu(0_{\mathbf{A}})\} \subseteq \nu(\mathbf{A}).$$

Hence $\nu(\mathbf{A})$ is a subalgebra of $\prod_{i\in I}\mathbf{A}/\theta_i$.
Moreover for all $i \in I$ $\nu(\mathbf{A})$ is a subdirect product of $\prod_{i\in I}\mathbf{A}/\theta_i$ from $\nu_i(\mathbf{A}) = \mathbf{A}/\theta_i$.
Next we show that ν is an embedding. For all $a, b \in \mathrm{A}$ ($a \neq b$)

$$\langle a, b\rangle \notin \bigcap_{i\in I}\theta_i$$

from $\bigcap_{i\in I}\theta_i = \Delta$. Hence there exist some $j \in I$ such that

$$\langle a, b\rangle \notin \theta_j.$$

From this $\nu_j(a) \neq \nu_j(b)$. So $\nu(a) \neq \nu(b)$. Thus ν is an embedding.

□

Definition 3.2.13 (subdirectly irreducible) An algebra $\mathbf{A}$ is *subdirectly irreducible* if for every subdirect embedding

$$\alpha : \mathbf{A} \to \prod_{i\in I}\mathbf{A}_i$$

there is an $i \in I$ such that

$$\pi_i \circ \alpha : \mathbf{A} \to \mathbf{A}_i$$

is an isomorphism.

Next lemma is most useful for understanding subdirect irreducible algebra.

Lemma 3.2.8 *An algebra $\mathbf{A}$ is subdirectly irreducible if and only if $\mathbf{A}$ is trivial or there is a minimum congruence in $Con\mathbf{A} - \{\Delta\}$. In the latter case the minimum element is $\cap(Con\mathbf{A} - \{\Delta\})$.*

Proof First we show only-if part. If $\mathbf{A}$ is not trivial and $Con\mathbf{A} - \{\Delta\}$ has no minimum element then $\bigcap(Con\mathbf{A} - \{\Delta\}) = \Delta$. Let $I = Con\mathbf{A} - \{\Delta\}$. Then the natural map $\alpha : \mathbf{A} \to \prod_{\theta\in I}\mathbf{A}/\theta$ is a subdirect embedding by Lemma 3.2.7, and as the natural map $\mathbf{A} \to \mathbf{A}/\theta$ is not injective for $\theta \in I$, it follows that $\mathbf{A}$ is not subdirect irreducible.

Next we show if part. If $\mathbf{A}$ is trivial and $\alpha : \mathbf{A} \to \prod_{i\in I}\mathbf{A}_i$ is a subdirect embedding then each $\mathbf{A}_i$ is trivial; hence each $\pi_i \circ \alpha$ is an isomorphism. So suppose $\mathbf{A}$ is not trivial, and let $\theta = \bigcap(Con\mathbf{A} - \{\Delta\}) \neq \Delta$. Choose $\langle a, b\rangle \in \theta$, $a \neq b$. If $\alpha : \mathbf{A} \to \prod_{i\in I}\mathbf{A}_i$ is a subdirect embedding then for some i, $(\alpha a)(i) \neq (\alpha b)(i)$; hence $(\pi_i \circ \alpha)(a) \neq (\pi_i \circ \alpha)(b)$. Thus $\langle a, b\rangle \notin ker(\pi_i \circ \alpha)$ so $\theta \nsubseteq ker(\pi_i \circ \alpha)$. But this implies $ker(\pi_i \circ \alpha) = \Delta$, so $\pi_i \circ \alpha : \mathbf{A} \to \mathbf{A}_i$ is an isomorphism. Consequently $\mathbf{A}$ is subdirect irreducible.

If $Con\mathbf{A} - \{\Delta\}$ has a minimum element θ then for $a \neq b$ and $\langle a, b\rangle \in \theta$ we have $\Theta(a, b) \subseteq \theta$, hence $\theta = \Theta(a, b)$.

□

Lemma 3.2.9 (Birkhoff) *Every algebra $\mathbf{A}$ is isomorphic to a subdirect product of subdirectly irreducible algebra.*

Proof The trivial algebra are subdirectly irreducible. Let $\mathbf{A}$ be a non-trivial $\mathbf{A}$. For $a, b \in A$ with $a \neq b$ we can find a congruence $\theta_{a,b}$ on $\mathbf{A}$ which is maximal with respect to the property $\langle a, b\rangle \notin \theta_{a,b}$ by using Zorn's lemma. Then clearly $\Theta(a, b) \vee \theta_{a,b}$ is the smallest congruence in $[\theta_{a,b}, \nabla] - \{\theta_{a,b}\}$, so by lemma 3.2.5 and 3.2.8 we see that $\mathbf{A}/\theta_{a,b}$ is subdirectly irreducible. As $\bigcap\{\theta_{a,b}\ a \neq b\} = \Delta$ we can apply proposition 3.2.7 to show that $\mathbf{A}$ is subdirectly embeddable in the product of the indexed family of subdirectly irreducible algebra $(\mathbf{A}/\theta_{a,b})_{a\neq b}$.

□

3.3 Varieties

In the previous section we show some properties of algebra. In this section we show properties of classes of algebras.

Definition 3.3.1 We define mappings from class $\mathcal{K}$ of algebras to class $\mathsf{I}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$, $\mathsf{H}(\mathcal{K})$, $\mathsf{P}(\mathcal{K})$ and $\mathsf{P_s}(\mathcal{K})$ as follows.

- $\mathbf{A} \in \mathsf{I}(\mathcal{K}) \Leftrightarrow \mathbf{A}$ is isomorphic to some member of $\mathcal{K}$.
- $\mathbf{A} \in \mathsf{S}(\mathcal{K}) \Leftrightarrow \mathbf{A}$ is a subalgebra of some member of $\mathcal{K}$.
- $\mathbf{A} \in \mathsf{H}(\mathcal{K}) \Leftrightarrow \mathbf{A}$ is a homomorphic image of some member of $\mathcal{K}$.
- $\mathbf{A} \in \mathsf{P}(\mathcal{K}) \Leftrightarrow \mathbf{A}$ is a direct product of a nonempty family of algebras in $\mathcal{K}$.
- $\mathbf{A} \in \mathsf{P_s}(\mathcal{K}) \Leftrightarrow \mathbf{A}$ is a subdirect product of a nonempty family of algebras in $\mathcal{K}$.

Let O_1 and O_2 are two operators on classes of algebras.We write O_1O_2 for the composition and $\leq$ denotes the usual partial order, i.e. $O_1 \leq O_2$ if $O_1(\mathcal{K}) \subseteq O_2(\mathcal{K})$ for every class $\mathcal{K}$ of algebras.

Definition 3.3.2 Let $\mathcal{K}$ be a class of algebras and O be a operator on class of algebras. Then O is a *idempotent* if $O^2 = O$, and $\mathcal{K}$ is *closed* under O if $O(\mathcal{K}) \subseteq \mathcal{K}$.

Proposition 3.3.1 *Following inequalities hold.*

$$\mathsf{SH} \leq \mathsf{HS}$$
$$\mathsf{PS} \leq \mathsf{SP}$$
$$\mathsf{PH} \leq \mathsf{HP}$$

Also the operators H, S, *and* IP *are idempotent.*

Proof Suppose $\mathbf{A} = \mathsf{SH}(\mathcal{K})$. Then for some $\mathbf{B} \in \mathcal{K}$ and onto homomorphism $\alpha : \mathbf{B} \to \mathbf{C}$, we have $\mathbf{A} \leq \mathbf{C}$. Thus $\alpha^{-1}(\mathbf{A}) \leq \mathbf{B}$, and as $\alpha(\alpha^{-1}(\mathbf{A})) = \mathbf{A}$, we have $\mathbf{A} \in \mathsf{HS}(\mathcal{K})$.
If $\mathbf{A} \in \mathsf{PS}(\mathcal{K})$ then $\mathbf{A} = \prod_{i\in I} \mathbf{A}_i$ for suitable $\mathbf{A}_i \leq \mathbf{B}_i \in \mathcal{K}$, $i \in I$. As $\prod_{i\in I} \mathbf{A}_i \leq \prod_{i\in I} \mathbf{B}_i$, we have $\mathbf{A} \in \mathsf{SP}(\mathcal{K})$.
Next if $\mathbf{A} \in \mathsf{PH}(\mathcal{K})$, then there are algebras $\mathbf{B}_i \in \mathcal{K}$ and epimorphisms $\alpha_i : \mathbf{B}_i \to \mathbf{A}_i$ such that $\mathbf{A} = \prod_{i\in I} \mathbf{A}_i$. It is easy to check that the mapping $\alpha : \prod_{i\in I} \mathbf{B}_i \to \prod_{i\in I} \mathbf{A}_i$ defined by $\alpha(b)(i) = \alpha_i(b(i))$ is an epimorphism; hence $\mathbf{A} \in \mathsf{HP}(\mathcal{K})$.
We show $\mathsf{H} = \mathsf{H}^2$. $\mathsf{H} \subseteq \mathsf{H}^2$ is clear. If $\mathbf{A} \in \mathsf{H}^2(\mathcal{K})$ then there exist onto homomorphisms $\alpha : \mathbf{B} \to \mathbf{C}$, $\beta : \mathbf{C} \to \mathbf{A}$ and $\mathbf{B} \in \mathcal{K}$. So $\beta \circ \alpha$ is an onto homomorphism. Thus $\mathbf{A} \in \mathsf{H}(\mathcal{K})$.
We can show that S, and IP are idempotent in the same way.

$\square$

A class of algebras, called a variety, defined by the following.

Definition 3.3.3 (variety) Let $\mathcal{K}$ is a nonempty class of algebras. $\mathcal{K}$ is a *variety* if it is closed under class operators $\mathsf{S}, \mathsf{H}, \mathsf{P}$.

If $\mathcal{K}$ is a class of algebras let $\mathsf{V}(\mathcal{K})$ denote the smallest variety containing $\mathcal{K}$. We say that $\mathsf{V}(\mathcal{K})$ is the *variety generated by* $\mathcal{K}$. If $\mathcal{K}$ consists of a single member $\mathbf{A}$ then we write simply $\mathsf{V}(\mathbf{A})$.

Proposition 3.3.2 (Tarski) $\mathsf{V} = \mathsf{HSP}$.

Proof Since $\mathsf{HV} = \mathsf{SV} = \mathsf{IPV} = \mathsf{V}$ and $\mathsf{I} \leq \mathsf{V}$, it follows that $\mathsf{HSP} \leq \mathsf{HSPV} = \mathsf{V}$. From above lemma we see that $\mathsf{H}(\mathsf{HSP}) = \mathsf{HSP}$, $\mathsf{S}(\mathsf{HSP}) \leq \mathsf{HSSP} = \mathsf{HSP}$, and

$$\begin{aligned} \mathsf{P}(\mathsf{HSP}) &\leq \mathsf{HPSP} \\ &\leq \mathsf{HSPP} \\ &\leq \mathsf{HSIPIP} \\ &= \mathsf{HSIP} \\ &\leq \mathsf{HSHP} \\ &\leq \mathsf{HHSP} \\ &= \mathsf{HSP}. \end{aligned}$$

Hence for any $\mathcal{K}$, $\mathsf{HSP}(\mathcal{K})$ is closed under H, S, and P. As $\mathsf{V}(\mathcal{K})$ is the smallest class containing $\mathcal{K}$ and closed under H, S, and P, we must have $\mathsf{V} = \mathsf{HSP}$.

$\square$

The following lemma is another version of Birkhoff's Theorem 3.2.9.

Lemma 3.3.3 *If $\mathcal{K}$ is a variety, then every member of $\mathcal{K}$ is isomorphic to a subdirect product of subdirectly irreducible member of $\mathcal{K}$.*

Corollary 3.3.4 *A variety is determined by its subdirectly irreducible members.*

3.4 Jónsson's Lemma

Definition 3.4.1 Let $\mathbf{B} = \langle \mathrm{B}, \wedge, \vee, ', 0, 1 \rangle$ be a Boolean algebra. A *filter* F of $\mathbf{B}$ is defined by

1. $1 \in \mathrm{F}$.
2. $a, b \in \mathrm{F} \Rightarrow a \wedge b \in \mathrm{F}$.
3. $a \in \mathrm{F}$ and $a \leq b \Rightarrow b \in \mathrm{F}$.

A filter F is called *proper* if $\mathrm{F} \neq \mathrm{A}$. A filter F is called *prime* if for any $a, b \in \mathrm{F}$, $a \vee b \in \mathrm{F}$ implies $a \in \mathrm{F}$ or $b \in \mathrm{F}$.

The collection of all filters of $\mathbf{A}$ forms a complete lattice denoted by $\mathbf{Fil}(\mathbf{A})$. For, if $\{\mathrm{F_I}|i \in I\}$ is a family of filters then $\bigcap_{i\in I}\mathrm{F_i}$ is also a filter.

Definition 3.4.2 A filter F of a Boolean algebra $\mathbf{B}$ is an *ultrafilter* if F is maximal with respect to the property that $0 \notin \mathrm{F}$. Let X be a set and $\mathcal{P}(X)$ be a powerset of X. A subset U of $\mathcal{P}(X)$ is called an *ultrafilter over* X, if it is a filter of $\mathbf{Su}(X)$ which is maximal with respect to the property that $\emptyset \notin \mathrm{U}$.

Proposition 3.4.1 *Let* F *be a filter of a Boolean algebra* $\mathbf{B}$. *Then the following are equivalent:*

1. F *is an ultrafilter of* $\mathbf{B}$*,*
2. *for any* $a \in \mathrm{B}$*, exactly one of* a *and* a' *belongs to* F*,*
3. $a \vee b \in \mathrm{F} \Leftrightarrow a \in \mathrm{F}$ *and* $b \in \mathrm{F}$ *for any* $a, b \in \mathrm{B}$.

Proof $1 \Rightarrow 2$. If F is an ultrafilter then $\mathbf{B}/\theta_F \cong \mathbf{2}$ since $\mathbf{B}/\theta_F$ is simple, where $\mathbf{2}$ is the tow element Boolean algebra. Let $\nu : \mathbf{B} \to \mathbf{B}/\theta_F$ be the natural homomorphism. For $a \in \mathrm{B}$, $\nu(a') = \nu(a)'$ so

$$\nu(a) = 1/\theta_F \text{ or } \nu(a') = 1/\theta_F,$$

as $\mathbf{B}/\theta_F \cong \mathbf{2}$; hence

$$ainF \text{ or } a' \in F.$$

If there exists $ain\mathrm{B}$ such that $ainF$ and $a' \in F$ then $0 = a \wedge a' \in F$, so this is a contradiction.

$2 \Rightarrow 3$. Suppose F is a filter with $a \vee b \in F$. By 2, $(a \vee b)' = (a' \wedge b') \notin F$, so $a' \notin F$ or $b' \notin F$ Thus, either $a \in F$ or $b \in F$.

$3 \Rightarrow 1$. Suppose that F' is a filter of $\mathbf{B}$ such that $F \subset F'$. If $a \in F - F'$ then $a' \in F$, since $1 = a \vee a' \in F$ and $a \notin F$, by 3. Hence, $a' \in F' \subset F$, so $0 = a \wedge a' \in F'$. Thus $F' = \mathrm{B}$.

□

Definition 3.4.3 Let $\mathcal{A} = \{\mathbf{A}_i|i \in I\}$ be a an indexed set of algebras and U be a ultrafilter over I. Then we define the *ultraproduct* $\prod_{i\in I}\mathbf{A}_i/\mathrm{U}$ to be $\prod_{i\in I}\mathbf{A}_i/\theta_\mathrm{U}$ where θ_U is the binary relation on $\prod_{i\in I}\mathbf{A}_i$ by

$$\langle a, b\rangle \Leftrightarrow \{i \in I|a_i = b_i\} \in \mathrm{U}.$$

The elements of $\prod_{i\ inI}\mathrm{A}_i/\mathrm{U}$ are denoted by a/U, where $a \in \prod_{i\in I}\mathrm{A}_i$.

Lemma 3.4.2 *If* $\{\mathbf{A}_i|i \in I\}$ *is a finite set of finite algebras, say* $\{\mathbf{B}_1, \ldots, \mathbf{B}_k\}$*, (I can be infinite), and* U *is an ultrafilter over* I*, then* $\prod_{i\ inI}\mathbf{A}_i/\mathrm{U}$ *is isomorphic to one of the algebra* $\mathbf{B}_1, \ldots, \mathbf{B}_k$*, namely to that* $\mathbf{B}_j$ *such that* $\{i \in I|\mathbf{A}_i = \mathbf{B}_j\} \in U\}$.

Proof Let $S_i = \{i \in I|\mathbf{A}_i = \mathbf{B}_j\}$. Then $I = S_1 \cup \cdots \cup S_m$, so by Proposition 3.4.1, there is some j $(1 \leq j \leq m)$ such that $S_j \in U$. Let $\mathrm{B}_j = \{b_1, \ldots, b_k\}$, where the b's are all distinct, and choose $a_1, \ldots, a_k \in \prod_{i\in I}\mathrm{A}_i$ such that $a_1(i) = b_1, \ldots, a_k(i) = b_k$ if $i \in S_j$. Then, for every element $a \in \prod_{i\in I}\mathrm{A}_i$,

$$\{i \in I | a(i) = a_1(i)\} \cup \cdots \cup \{i \in I | a(i) = a_k(i)\} \supseteq S_j.$$

Since $S_j \in U$, $\{i \in I | a(i) = a_1(i)\} \cup \cdots \cup \{i \in I | a(i) = a_k(i)\} \in U$, this follows $\{i \in I | a(i) = a_1(i)\} \in U$ or $\cdots$ or $\{i \in I | a(i) = a_k(i)\} \in U$; hence

$$a/\theta_U = a_1/\theta_U \text{ or } \ldots \text{ or } a/\theta_U = a_k/\theta_U.$$

Also it is evident that $a_1/\theta_U, \ldots, a_k/\theta_U$ are all distinct. Thus $\prod_{i \in I} \mathrm{A}_i/\theta_U$ to B_j defined by

$$\alpha(a_t/\theta_U) = b_t,\ 1 \leq t \leq k.$$

Then it is easy to see that α is an isomorphism.

□

Lemma 3.4.3 (Jónsson) *Let W be a family of subsets of $I(\neq \emptyset)$ such that*

1. *$I \in W$,*
2. *if $J \in W$ and $J \subseteq K \subseteq I$ then $K \in W$ and*
3. *if $J_1 \cup J_2 \in W$ then $J_1 \in W$ or $J_2 \in W$.*

Then there is an ultrafilter U over I with $U \subseteq W$.

Proof If $\emptyset \in W$ then $W = \mathrm{Su}(I)$, so any ultrafilter is in W. If $\emptyset \notin W$, then $\mathrm{Su}(I) - W$ is a proper ideal. Hence it is extended to a maximal ideal and by taking the complementary ultrafilter we can obtain an ultrafilter.

□

Definition 3.4.4 The class of ultraproducts of members of $\mathcal{K}$ is denoted by $\mathsf{P_U}(\mathcal{K})$.

Proposition 3.4.4 (Jónsson) *Let $\mathsf{V}(\mathcal{K})$ be a congruence-distributive variety. If $\mathbf{A}$ is a subdirectly irreducible algebra in $\mathsf{V}(\mathcal{K})$, then $\mathbf{A} \in \mathsf{HSP_U}(\mathcal{K})$.*

Proof Suppose that $\mathbf{A}$ is a nontrivial subdirectly irreducible algebra in $\mathsf{V}(\mathcal{K})$. Then for some $\mathbf{A}_i \in \mathcal{K}$, $i \in I$, and for some $\mathbf{B} \leq \prod_{i \in I} \mathbf{A}_i$ there is a surjective homomorphism $\alpha : \mathbf{B} \to \mathbf{A}$. Let $\theta = ker(\alpha)$. For $J \subseteq I$ let

$$\theta_J = \{\langle a, b\rangle \in (\prod_{i \in I} \mathrm{A}_i)^2 | J \subseteq \{i \in I | a(i) = b(i)\}\}.$$

It is easy to see that for any $J(\subseteq I)$, θ_J is a congruence on $\prod_{i \in I} \mathbf{A}_i$. Let $\theta_J \uparrow_{\mathrm{B}} = \theta_J \cap \mathrm{B}^2$ be the restriction of θ_J to $\mathbf{B}$, and $W = \{J \subseteq I | \theta_J \uparrow_{\mathrm{B}} \subseteq \theta\}$. Clearly

$$I \in W, \emptyset \notin W$$

and

$$J \in W \text{ and } J \subseteq K \subseteq I \text{ implies } \theta_J \uparrow_{\mathrm{B}} \subseteq \theta,$$

as $\theta_{\mathcal{K}} \upharpoonright_{\mathrm{B}} \subseteq \theta_J \upharpoonright_{\mathrm{B}}$. Now suppose $J_1 \cup J_2 \in W$, i.e., $\theta_{J_1 \cup J_2} \upharpoonright_{\mathrm{B}} \subseteq \theta$. As $\theta_{J_1 \cup J_2} = \theta_{J_1} \cap \theta_{J_2}$, it follows that

$$\theta_{J_1 \cup J_2} \upharpoonright_{\mathrm{B}} = \theta_{J_1} \upharpoonright_{\mathrm{B}} \cap \theta_{J_2} \upharpoonright_{\mathrm{B}}.$$

Since $\theta = \theta \vee (\theta_{J_1} \upharpoonright_{\mathrm{B}} \cap \theta_{J_2} \upharpoonright_{\mathrm{B}})$ it follows that

$$\theta = (\theta \vee \theta_{J_1} \upharpoonright_{\mathrm{B}}) \cap (\theta \vee \theta_{J_2} \upharpoonright_{\mathrm{B}})$$

by distributivity. Since $\mathbf{B}/\theta$ is isomorphic to $\mathbf{A}$ $\theta = \theta \vee \theta_{J_i} \upharpoonright_{\mathrm{B}}$ for $i = 1$ or 2. Thus $\theta_{J_i} \upharpoonright_{\mathrm{B}} \subseteq \theta$ for $i = 1$ or 2, so either J_1 or j_2 is in W. By Lemma 3.4.3, there is an ultrafilter U contained in W. Form the definition of W we have

$$\theta_U \upharpoonright_{\mathrm{B}} \subseteq \theta$$

as $\theta_U = \bigcup\{\theta_J | J \in U\}$. Let ν be the natural homomorphism from $\prod_{i \in I} \mathbf{A}_i$ to $\prod_{i \in I} \mathbf{A}_i/U$. Then let $\beta : \mathbf{B} \to \nu(\mathbf{B})$ be the restriction of ν to $\mathbf{B}$. As $ker(\beta) = \theta_U \upharpoonright_{\mathrm{B}} \subseteq \theta$ we have

$$\mathbf{A} \cong \mathbf{B}/\theta \cong (\mathbf{B}/ker(\beta))/(\theta/ker(\beta)).$$

Now $\mathbf{B}/ker(\beta) \cong \nu(\mathbf{B}) \leq \prod_{i \in I} \mathbf{A}_i/U$ so $\mathbf{B}/ker(\beta) \in \mathsf{ISP_U}(\mathcal{K})$, hence

$$\mathbf{A} \in \mathsf{HSP_U}(\mathcal{K}).$$

□

3.5 Free algebras and universal mapping property

Definition 3.5.1 Let X be a set of (distinct) objects called *variables*. The set $T(X)$ of *terms over* X is the smallest set such that

1. $X \cup \{0, 1\} \subseteq T(X)$.
2. If $p_1, \ldots, p_n \in T(X)$ then the "string" $f(p_1, \ldots, p_n) \in T(X)$.

For $p \in T(X)$ we often write p as $p(x_1, \ldots, x_n)$ to indicate that the variables occurring in p are among $x_1, \ldots, x_n$.

Definition 3.5.2 Given a term $p(x_1, \ldots, x_n)$ over some set X and given an algebra $\mathbf{A}$ we define a mapping $p^{\mathbf{A}} : \mathrm{A}^n \to \mathbf{A}$ as follows:
(1) if p is a variable x_i, then

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = a_i$$

for $a_1, \ldots, a_n \in \mathrm{A}$, i.e., $p^{\mathbf{A}}$ is the *i*th projection map;
(2) if p is of the form $f(p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n))$ then

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{A}}(p_1^{\mathbf{A}}(x_1, \ldots, x_n), \ldots, p_m^{\mathbf{A}}(x_1, \ldots, x_n)).$$

In particular if p is f then $p^{\mathbf{A}}$ is $f^{\mathbf{A}}$. The expression $p^{\mathbf{A}}$ is called the *term function* on $\mathbf{A}$ corresponding to the term p. (Often we will drop the superscript $\mathbf{A}$).

The next proposition gives some useful properties of term functions.

Proposition 3.5.1 *For any algebra* $\mathbf{A}$ *and* $\mathbf{B}$ *we have the following.*
(a) Let p *be an n-ary term, let* $\theta \in \operatorname{Con} \mathbf{A}$*, and suppose* $\langle a_i, b_i \rangle \in \theta$ *for* $1 \leq i \leq n$*. Then*

$$p^{\mathbf{A}}(a_1, \ldots, a_n) \theta p^{\mathbf{A}}(b_1, \ldots, b_n).$$

(b) If p *is an n-ary term and* $\alpha : \mathbf{A} \to \mathbf{B}$ *is a homomorphism, then*

$$\alpha(p^{\mathbf{A}}(a_1, \ldots, a_n)) = p^{\mathbf{B}}(\alpha(a_1), \ldots, \alpha(a_n))$$

for $a_1, \ldots, a_n \in \mathrm{A}$.
(c) Let S *be a subset of* A*. Then*

$$\operatorname{Sg}(S) = \{p^{\mathbf{A}}(a_1, \ldots, a_n) | p \text{ is an n-ary term, } n \leq \omega \text{, and } a_1, \ldots, a_n \in S\}.$$

One can, in a natural way, transform the set $T(X)$ into an algebra.

Definition 3.5.3 Given X, if $T(X) \neq \emptyset$ then the *term algebra* over X, written $\mathbf{T}(X)$, has as its universe the set $T(X)$, and operations satisfy

$$f^{\mathbf{T}(X)}(p_1, \ldots, p_m) = f(p_1, \ldots, p_2)$$

for $p_1, \ldots, p_m \in T(X)$.

Definition 3.5.4 (universal mapping property) Let $\mathcal{K}$ be a class of algebras and let $\mathbf{U}(X)$ be an algebra which is generated by X. If for every $\mathbf{A} \in \mathcal{K}$ and for every map

$$\alpha : X \to \mathrm{A}$$

there is a homomorphism

$$\beta : \mathbf{U}(X) \to \mathbf{A}$$

which extends α (i.e., $\beta(x) = \alpha(x)$ for $x \in X$), then we say $\mathbf{U}(X)$ has the *universal mapping property for* $\mathcal{K}$ *over* X, X is called a set of *free generators* of $\mathbf{U}(X)$, and $\mathbf{U}(X)$ is said to be *freely generated* by X.

Lemma 3.5.2 *Suppose* $\mathbf{U}(X)$ *has the universal mapping property for* $\mathcal{K}$ *over* X*. Then if we are given* $\mathbf{A} \in \mathcal{K}$ *and* $\alpha : X \to \mathrm{A}$*, there is a unique extension* β *of* α *such that* β *is a homomorphism from* $\mathbf{U}(X)$ *to* $\mathbf{A}$.

Thus given any class $\mathcal{K}$ of algebras the term algebra provide algebra which have the universal mapping property for $\mathcal{K}$. To study properties of classes of algebras we often try to find special kinds of algebra in these classes which yield the desired information. In order to find algebra with the universal mapping property for $\mathcal{K}$ which give more insight into $\mathcal{K}$ we will introduce $\mathcal{K}$-free algebra. Unfortunately not every class $\mathcal{K}$ contains algebras with the universal mapping property for $\mathcal{K}$. Nonetheless we will be able to show that any class closed under I, S, and P contains its $\mathcal{K}$-free algebra. There is reasonable difficulty in providing transparent descriptions of $\mathcal{K}$-free algebra for most $\mathcal{K}$. However, most of the applications of $\mathcal{K}$-free algebra come directly from the universal mapping property, the fact that they exist in varieties, and their relation to identities holding in $\mathcal{K}$.

Definition 3.5.5 Let $\mathcal{K}$ be a family of algebras. Given a set X of variables define the congruence $\theta_{\mathcal{K}}(X)$ on $\mathbf{T}(X)$ by

$$\theta_{\mathcal{K}}(X) = \cap\Phi_{\mathcal{K}}(X)$$

where

$$\Phi_{\mathcal{K}}(X) = \{\phi \in \text{Con } \mathbf{T}(X) | \mathbf{T}(X)/\phi \in IS(\mathcal{K})\};$$

and then define $\mathbf{F}(\overline{X})$, *the $\mathcal{K}$-free algebra over $\overline{X}$*, by

$$\mathbf{F}_{\mathcal{K}}(\overline{X}) = \mathbf{T}(X)/\theta_{\mathcal{K}}(X),$$

where

$$\overline{X} = X/\theta_{\mathcal{K}}(X).$$

Proposition 3.5.3 (Birkhoff) *Suppose $\mathbf{T}(X)$ exists. Then $\mathbf{F}_{\mathcal{K}}(\overline{X})$ has the universal mapping property for $\mathcal{K}$ over $\overline{X}$.*

Corollary 3.5.4 *IF $\mathcal{K}$ is a class of algebras and $\mathbf{A} \in \mathcal{K}$, then for sufficiently large$X$, $\mathbf{A} \in H(\mathbf{F}_{\mathcal{K}}(\overline{X}))$.*

The next proposition says that there exists a free algebra in varieties.

Proposition 3.5.5 (Birkhoff) *Suppose $\mathbf{T}(X)$ exists. Then for $\mathcal{K} \neq \emptyset$, $\mathbf{F}_{\mathcal{K}}(\overline{X}) \in \mathsf{ISP}(\mathcal{K})$. Thus if $\mathcal{K}$ is closed under I, S, and P, in particular if $\mathcal{K}$ is a variety, then $\mathbf{F}_{\mathcal{K}}(\bar{X}) \in \mathcal{K}$.*

Proof As

$$\theta_{\mathcal{K}}(X) = \cap\Phi_{\mathcal{K}}(X)$$

it follows that

$$\mathbf{F}_{\mathcal{K}}(\overline{X}) = \mathbf{T}(X)/\theta_{\mathcal{K}}(X) \in \mathsf{IP_s}(\{\mathbf{T}(X)/\theta | \theta \in \Phi_{\mathcal{K}}(X)\}),$$

so

$$\mathbf{F}_{\mathcal{K}}(\overline{X}) \in \mathsf{IP_S IS}(\mathcal{K}),$$

and thus by Proposition 3.3.1 and the fact that $\mathsf{P_S} \leq \mathsf{SP}$,

$$\mathbf{F}_{\mathcal{K}}(\overline{X}) \in \mathsf{ISP}(\mathcal{K}).$$

□

3.6 Birkhoff's theorem

In this section we show the famous theorem of Birkhoff. The Birkhoff theorem says that a class of algebras defined by equations is a variety.

Definition 3.6.1 An *identity* (or *equation*) over X is an expression of the form

$$p \approx q$$

where $p, q \in T(X)$. Let $Id(X)$ be the set of identities over$X$. An algebra $\mathbf{A}$ satisfies an identity

$$p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$$

(or the identity *is true in* $\mathbf{A}$, or *holds in* $\mathbf{A}$), abbreviated by

$$\mathbf{A} \models p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n),$$

or more briefly

$$\mathbf{A} \models p \approx q,$$

if for every choice of $a_1, \ldots, a_n \in \mathrm{A}$ we have

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = q^{\mathbf{A}}(a_1, \ldots, a_n).$$

A class $\mathcal{K}$ of algebras satisfies $p \approx q$, written

$$\mathcal{K} \models p \approx q,$$

if each member of $\mathcal{K}$ satisfies $p \approx q$. If Σ is a set of identities, we say $\mathcal{K}$ satisfies Σ, written

$$\mathcal{K} \models \Sigma,$$

if $\mathcal{K} \models p \approx q$ for each $p \approx q \in \Sigma$. Given $\mathcal{K}$ and X let

$$Id_{\mathcal{K}}(X) = \{p \approx q \in Id(X) | \mathcal{K} \models p \approx q\}.$$

We use the symbol $\not\models$ for "does not satisfy."

We can reformulate the above definition of satisfaction using the notion of homomorphism.

Lemma 3.6.1 *If* $\mathcal{K}$ *is a class of algebras and* $p \approx q$ *is an identity over* X*, then*

$$\mathcal{K} \models p \approx q$$

iff for every$\mathbf{A} \in \mathcal{K}$ *and for every homomorphism* $\alpha : \mathbf{T}(X) \to \mathbf{A}$ *we have*

$$\alpha(p) = \alpha(q)$$

Proof ($\Rightarrow$) Let $p = p(x_1, \ldots, x_n)$, $q = q(x_1, \ldots, x_n)$. Suppose $\mathcal{K} \models p \approx q$, $\mathbf{A} \in \mathcal{K}$, and $\alpha : \mathbf{T}(X) \to \mathbf{A}$ is a homomorphism. Then

$$\begin{aligned} & p^{\mathbf{A}}(\alpha(x_1), \ldots, \alpha(x_n)) = p^{\mathbf{A}}(\alpha(x_1), \ldots, \alpha(x_n)) \\ \Rightarrow \quad & \alpha(p^{\mathbf{T}(X)}(x_1, \ldots, x_n)) = \alpha(q^{\mathbf{T}(X)}(x_1, \ldots, x_n)) \\ \Rightarrow \quad & \alpha(p) = \alpha(q). \end{aligned}$$

($\Leftarrow$) For the converse choose $\mathbf{A} \in \mathcal{K}$ and $a_1, \ldots, a_n \in \mathrm{A}$. By the universal mapping property of $\mathbf{T}(X) \to \mathbf{A}$ such that

$\alpha(x_i) = a_i, \quad 1 \le i \le n.$

But then

$$\begin{aligned} p^{\mathbf{A}}(a_1, \ldots, a_n) &= p^{\mathbf{A}}(\alpha(x_1), \ldots, \alpha(x_n)) \\ &= \alpha(p) \\ &= \alpha(q) \\ &= q^{\mathbf{A}}(\alpha(x_1), \ldots, \alpha(x_n)) \\ &= q^{\mathbf{A}}(a_1, \ldots, a_n), \end{aligned}$$

so $\mathcal{K} \models p \approx q$.

□

Next we see that the basic class operators preserve identities.

Proposition 3.6.2 *For any class $\mathcal{K}$, all of the classes $\mathcal{K}$, $\mathsf{I}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$, $\mathsf{H}(\mathcal{K})$, $\mathsf{P}(\mathcal{K})$ and $V(\mathcal{K})$ satisfy the same identities over any set of variables X.*

Proof Clearly $\mathcal{K}$ and $\mathsf{I}(\mathcal{K})$ satisfy the same identities. As

$\mathsf{I} \le \mathsf{IS}$, $\mathsf{I} \le \mathsf{H}$, and $\mathsf{I} \le \mathsf{IP}$,

we must have

$Id_{\mathcal{K}}(X) \supseteq Id_{\mathsf{S}(\mathcal{K})}(X)$, $Id_{\mathsf{H}(\mathcal{K})}(X)$, and $Id_{\mathsf{P}(\mathcal{K})}(X)$.

For the remainder of the proof suppose

$\mathcal{K} \models p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n).$

Then if $\mathbf{B} \le \mathbf{A} \in \mathcal{K}$ and $b_1, \ldots, b_n \in \mathrm{B}$, then as $b_1, \ldots, b_n \in \mathbf{A}$ we have

$p^{\mathbf{A}}(b_1, \ldots, b_n) = q^{\mathbf{A}}(b_1, \ldots, b_n);$

hence

$p^{\mathbf{B}}(b_1, \ldots, b_n) = q^{\mathbf{B}}(b_1, \ldots, b_n),$

so

$\mathbf{B} \models p \approx q.$

Thus

$Id_{\mathcal{K}}(X) = Id_{\mathsf{S}(\mathcal{K})}(X).$

Next suppose $\alpha : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism with $\mathbf{A} \in \mathcal{K}$. If $b_1, \ldots, b_n \in \mathrm{B}$, choose $a_1, \ldots, a_n \in \mathrm{A}$ such that

$\alpha(a_1) = b_1, \ldots, \alpha(a_n) = b_n.$

Then

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = q^{\mathbf{A}}(a_1, \ldots, a_n)$$

implies

$$\alpha(p^{\mathbf{A}}(a_1, \ldots, a_n)) = \alpha(q^{\mathbf{A}}(a_1, \ldots, a_n));$$

hence

$$p^{\mathbf{B}}(b_1, \ldots, b_n) = q^{\mathbf{B}}(b_1, \ldots, b_n)$$

Thus

$$\mathbf{B} \models p \approx q,$$

so

$$Id_{\mathcal{K}}(X) = Id_{\mathsf{H}(\mathcal{K})}(X).$$

Lastly, suppose $\mathbf{A}_i \in \mathcal{K}$ for $i \in I$. Then for $a_1, \ldots, a_n \in \mathrm{A} = \prod_{i \in I} \mathrm{A}_i$ we have

$$p^{\mathbf{A}_i}(a_1(i), \ldots, a_n(i)) = q^{\mathbf{A}_i}(a_1(i), \ldots, a_n(i));$$

hence

$$p^{\mathbf{A}}(a_1, \ldots, a_n)(i) = q^{\mathbf{A}}(a_1, \ldots, a_n)(i)$$

for $i \in I$, so

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = q^{\mathbf{A}}(a_1, \ldots, a_n).$$

Thus

$$Id_{\mathcal{K}}(X) = Id_{\mathsf{P}(\mathcal{K})}(X).$$

As $V = \mathsf{HSP}$ by 3.3.2, the proof is complete.

□

Now we will formulate the crucial connection between $\mathcal{K}$-free algebra and identities.

Lemma 3.6.3 *Given a class* $\mathcal{K}$ *of algebras and terms* $p, q \in T(X)$ *we have*

$$\begin{aligned} & \mathcal{K} \models p \approx q \\ \Leftrightarrow \quad & \mathbf{F}_{\mathcal{K}}(\bar{X}) \models p \approx q \\ \Leftrightarrow \quad & \bar{p} = \bar{q} \text{ in } \mathbf{F}_{\mathcal{K}}(\bar{X}) \\ \Leftrightarrow \quad & \langle p, q \rangle \in \theta_{\mathcal{K}}(X). \end{aligned}$$

Proof Let $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(\overline{X})$, $p = p(x_1, \ldots, x_n)$, $q = q(x_1, \ldots, x_n)$, and let

$$\nu : \mathbf{T}(X) \to \mathbf{F}$$

be the natural homomorphism. Certainly $\mathcal{K} \models p \approx q$ implies $\mathbf{F} \models p \approx q$ as $\mathbf{F} \in \mathsf{ISP}(\mathcal{K})$. Suppose next that $\mathbf{F} \models p \approx q$. Then

$$p^{\mathbf{F}}(\overline{x}_1, \ldots, \overline{x}_n) = q^{\mathbf{F}}(\overline{x}_1, \ldots, \overline{x}_n),$$

hence $\overline{p} = \overline{q}$. Now suppose $\overline{p} = \overline{q}$ in $\mathbf{F}$. Then

$$\nu(p) = \overline{p} = \overline{q} = \nu(q),$$

so

$$\langle p, q \rangle \in ker(\nu) = \theta_{\mathcal{K}}(X).$$

Finally suppose $\langle p, q \rangle \in \theta_{\mathcal{K}}(X)$. Given $\mathbf{A} \in \mathcal{K}$ and $a_1, \ldots, a_n \in \mathrm{A}$ choose $\alpha : \mathbf{T}(X) \to \mathbf{A}$ such that $\alpha(x_i) = a_i$, $1 \leq i \leq n$. As $ker(\alpha) \in \Phi_{\mathcal{K}}(X)$ we have

$$ker(\alpha) \supseteq ker(\nu) = \theta_{\mathcal{K}}(X),$$

so it follows that is a homomorphism $\beta : \mathbf{F} \to \mathbf{A}$ such that $\alpha = \beta \circ \nu$. Then

$$\alpha(p) = \beta \circ \nu(p) = \beta \circ \nu(q) = \alpha(q).$$

Consequently

$$\mathcal{K} \models p \approx q$$

by reformulation of definition of satisfaction.

□

Let $\mathcal{E}$ be a set of identities, and define $Mod(\mathcal{E})$ to be the class of all algebras satisfying $\mathcal{E}$. A class $\mathcal{K}$ of algebras is an *identity class* (or *equational class*) if there is a set $\mathcal{E}$ of identities such that $\mathcal{K} = Mod(\mathcal{E})$. In this case, we say that $\mathcal{K}$ is *axiomatized* by $\mathcal{E}$.

Proposition 3.6.4 (Birkhoff) *$\mathcal{K}$ is an equational class if and only if $\mathcal{K}$ is a variety.*

Proof Suppose that $\mathcal{K} = Mod(\mathcal{E})$. Then, by Proposition 3.6.2, $V(\mathcal{K}) \models \mathcal{E}$. Hence

$$V(\mathcal{K}) \subseteq Mod(\mathcal{E}) = \mathcal{K} \subseteq V(\mathcal{K}),$$

so $\mathcal{K}$ is a variety.

Let $\mathcal{K}$ be a variety and X an infinite set of variables. Then we can show $\mathcal{K} = Mod(Id_{\mathcal{K}}(X))$. For detail, see [3].

□

Chapter 4

Relations between logics and algebras

In this chapter, first we introduce residuated lattices which are algebras for substructural logics. We will show completeness theorem and discuss about lattice of logics is dually isomorphic to lattice of varieties of residuated lattices.

4.1 Monoids

Definition 4.1.1 A structure $\mathbf{A} = \langle \mathrm{A}, \cdot, 1 \rangle$ is a *monoid* if it satisfies the following. For all $x, y, z \in \mathrm{A}$.

(M1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(M2) there exists some $e \in \mathrm{A}$ such that $e \cdot a = a \cdot e = a$.

It is easily seen that such an element e exists uniquely. Therefore, this element e is called the *identity element*. In the following, we consider only commutative monoids, i.e. monoids satisfying $x \cdot y = y \cdot x$ for all x, y.

4.2 Residuated lattices

In this section we introduce algebras corresponding to substructural logics over **FL**. They are called *residuated lattices* (RLs). We show basic properties of residuated lattice from the viewpoint of universal algebra.

Definition 4.2.1 An algebra $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ is a RL if $\mathbf{A}$ satisfies the following three conditions.

(R1) $\langle \mathrm{A}, \wedge, \vee, 1 \rangle$ is a lattice,

(R2) $\langle \mathrm{A}, \cdot, 1 \rangle$ is a monoid with the unit 1,

(R3) for $x, y, z \in \mathrm{A}$, $x \cdot y \leq z \Leftrightarrow y \leq x \backslash z \Leftrightarrow x \leq z / y$.

When $\langle \mathrm{A}, \wedge, \vee, 1 \rangle$ is a bounded lattice with the greatest element 1, $\mathbf{A}$ is called a *integral residuated lattice* (IRL). When RL $\mathbf{A}$ is satisfies the condition $x \leq x^2$, $\mathbf{A}$ is called a *contractive residuated lattice* (KRL). When monoid operation is commutative, $\mathbf{A}$ is called a *commutative* *residuated lattice* (CRL). It is easy to see that a commutative integral residuated lattice is a *Heyting algebra* if and only if the semigroup operation $\cdot$ is equal to $\wedge$. Moreover $\mathbf{B}$ is a *Boolean algebra* if and only if $\mathbf{B}$ is Heyting algebra and satisfies $x \vee x' = 1$ where $x' = x \rightarrow \bot$. The set A of $\mathbf{A}$ is called the *universe* of $\mathbf{A}$.

An *involutive residuated lattice* (InRL) is an algebra with a fundamental unary operation $'$, whose $\{'\}$-free reduct is a RL, and which satisfies

1. for any $x'' = x$.
2. for any $x \backslash y' = x'/y$.

We call the operation $'$ an involution. In InRL let us define $0 = 1'$. We call 0 the involution constant.

An RL is called *representable,* if it can be represented as subdirect product of totally ordered algebras.

The condition (R3) of this definition is called *residuation.* This condition means that $\backslash$ and $/$ behaves similarly to an inverse operation of $\cdot$.

Definition 4.2.2 An algebra $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ is called a FL-algebra, if

1. $\langle \mathrm{A}, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is a residuated lattice,
2. 0 is an arbitrary element of A.

We define negation operations in FL-algebra by $\sim x = x \backslash 0$ and $-x = 0/x$.

Suppose that FL-algebra $\mathbf{A}$ satisfies $\sim x = -x$ and $\sim -x = x$ (and also $- \sim x = x$). Then we can show

$$\begin{aligned} x \backslash -y \leq x \backslash -y &\Leftrightarrow x \backslash (0/y) \leq x \backslash (0/y) \\ &\Leftrightarrow x \cdot (x \backslash (0/y)) \leq 0/y = y \backslash 0 \\ &\Leftrightarrow x \cdot (x \backslash (0/y)) \cdot y \leq 0 \\ &\Leftrightarrow (x \backslash (0/y)) \cdot y \leq x \backslash 0 \\ &\Leftrightarrow x \backslash (0/y) \leq (x \backslash 0)/y \\ &\Leftrightarrow x \backslash -y \leq \sim x/y. \end{aligned}$$

From this result FL-algebras which satisfy $\sim x = -x$ and $\sim -x = x$ (and also $- \sim x = x$) can be considered as involutive residuated lattices.

Next lemma shows basic properties of residuated lattice and FL-algebras, see [7, 8] for the details.

Lemma 4.2.1 *All residuated lattices and all* FL*-algebras satisfy the following identities:*

1. $x(y \vee z) = (xy) \vee (xz)$ *and* $(y \vee z)x = (yx) \vee (zx)$
2. $x \backslash (y \wedge z) = (x \backslash y) \wedge (x \backslash z)$ *and* $(y \wedge z)/x = (y/x) \wedge (z/x)$
3. $x/(y \vee z) = (x/y) \wedge (x/z)$ *and* $(y \vee z) \backslash x = (y \backslash x) \wedge (z \backslash x)$

4. $(x/y)y \leq x$ *and* $y(y\backslash x) \leq x$

5. $x(y/z) \leq (xy)/z$ *and* $(z\backslash y)x = z\backslash(yx)$

6. $(x/y)/z = x/(zy)$ *and* $z\backslash(y\backslash x) = (yz)\backslash x$

7. $x\backslash(y/z) = (x\backslash y)/z$

8. $x/1 = x = 1\backslash x$

9. $1 \leq x/x$ *and* $1 \leq x\backslash x$

10. $x \leq y/(x\backslash y)$ *and* $x \leq (y/x)\backslash y$

11. $y/((y/x)\backslash y) = y/x$ *and* $(y/(x\backslash y))\backslash y = x\backslash y$

12. $x/(x\backslash x) = x$ *and* $(x/x)\backslash x = x$

13. $(z/y)(y/x) \leq z/x$ *and* $(x\backslash y)(y\backslash z) \leq x\backslash z$

Lemma 4.2.2 *Every involutive residuated lattice satisfies*

$$x \cdot y \approx (y\backslash x')' \approx (y'/x)'.$$

The following lemma says that the class of residuated lattices and FL-algebras are equational classes.

Lemma 4.2.3 *An algebra is a residuated lattice or an* FL*-algebra if and only if it satisfies the lattice equations, the monoid equations and the following equations.*

1. $x(x\backslash z \wedge y) \leq z$

2. $(y \wedge z/x)x \leq z$

3. $y \leq x\backslash(xy \vee z)$

4. $y \leq (z \vee yx)/x$

Definition 4.2.3 For any algebra $\mathbf{A}$, a nonempty subset F of A is called a *deductive filter* of $\mathbf{A}$ if it satisfies the following conditions.

1. if $1 \leq x$ then $x \in \mathrm{F}$.

2. if $x, x\backslash y \in \mathrm{F}$ then $y \in \mathrm{F}$.

3. if $x, y \in \mathrm{F}$ then $a \wedge b \in \mathrm{F}$.

4. if $x \in \mathrm{F}$ then $z\backslash xz, zx/z \in \mathrm{F}$.

Proposition 4.2.4 *Let* $\mathbf{A}$ *be an algebra and* F *be a deductive filter of* $\mathbf{A}$. *Then for any* $a, b \in \mathrm{A}$,

1. *if* $a \in \mathrm{F}$ *and* $a \leq b$ *then* $b \in \mathrm{F}$ *and*

2. *if* $a, b \in \mathrm{F}$ *then* $a \cdot b \in \mathrm{F}$.

3. $a\backslash b \in \mathrm{F}$ *if and only if* $b/a \in \mathrm{F}$.

Proof 1.
Let $a \leq b$. We have $1 \leq a\backslash b$. Thus $a\backslash b \in \mathrm{F}$ from definition of 1. By definition of 2, we show $b \in \mathrm{F}$.
2.
Let $a, b \in \mathrm{F}$. From definition of 4, we have $b\backslash ab \in \mathrm{F}$. By definition of 2, we show $ab \in \mathrm{F}$.
3.
Let $a\backslash b \in \mathrm{F}$. We have $a(a\backslash b)/a \in \mathrm{F}$ from definition of 4. We can show that $a(a\backslash b)/a \leq b/a$. Thus $b/a \in \mathrm{F}$. Converse is show by same way.

□

4.3 Relations between logics over FL and FL-algebras.

In this section we show some relations between logics over **FL** and residuated lattices. We will write $\mathbf{\Lambda}(\mathcal{V})$ for the lattice of subvarieties of a variety $\mathcal{V}$ and $\mathbf{\Lambda}(\mathbf{L})$ for the lattice of a logic $\mathbf{L}$.

Definition 4.3.1 A formula φ is *valid* in a FL-algebra $\mathbf{A}$ if $v(\varphi) \geq 1$ for every valuation v.

Note that a formula φ is valid in $\mathbf{A}$ if and only if the identity $\varphi \wedge 1 \approx 1$ is valid in $\mathbf{A}$, i.e., $\mathbf{A} \models \varphi \wedge 1 \approx 1$. A sequent $\Gamma \Rightarrow \alpha$ is valid in $\mathbf{A}$ if and only if a formula $\Gamma^*\backslash\alpha$ is valid in $\mathbf{A}$ where Γ^* is $\gamma_1 \cdot \ldots \cdot \gamma_n$ for $\gamma_1, \ldots, \gamma_n \in \Gamma$.

Definition 4.3.2 Let $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, \rightarrow, 0, 1 \rangle$ is a FL-algebra. A *valuation* v is a mapping from set of all propositional variable to A. Furthermore this v is extended to a mapping from set of all formulas to A as follows.

1. $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$
2. $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$
3. $v(\varphi\backslash\psi) = v(\varphi)\backslash v(\psi)$
4. $v(\psi/\varphi) = v(\psi)/v(\varphi)$
5. $v(\varphi \cdot \psi) = v(\varphi) \cdot v(\psi)$
6. $v(\sim \varphi) = v(\varphi)\backslash 0$
7. $v(-\varphi) = 0/v(\varphi)$

For a given FL-algebra $\mathbf{A}$, and let $L(\mathbf{A})$ be the set of all formulas such that $v(\varphi)$ for any valuation v on $\mathbf{A}$.

Proposition 4.3.1 *For each* FL*-algebra* $\mathbf{A}$, $L(\mathbf{A})$ *is a substructural logics over* **FL**.

Proof It is easy to see that $\mathbf{FL} \subseteq L(\mathbf{A})$

Let $\varphi(p)$ be a formula containing a propositional variable p and $\varphi(p) \in L(\mathbf{A})$. By our assumption, $v(\varphi(p)) \geq 1$ for any valuation v. Now, consider any substitution instance $\varphi(\alpha)$ of $\varphi(p)$ and any valuation w on $\mathbf{A}$. Let $w(\alpha) = a$. Then, define a valuation v' by $v'(p) = a$ and $v'(q) = w(q)$ if q is different from p. Then, $q \leq v'(\varphi(p)) = w(\varphi(\alpha))$. Therefore, $\varphi(\alpha)$ is valid in $\mathbf{A}$. Thus $\varphi(\alpha) \in L(\mathbf{A})$.

Let $\varphi, \varphi\backslash\psi \in L(\mathbf{A})$. Then for any valuation v $v(\varphi) \geq 1$, $v(\varphi\backslash\psi) \geq 1$. By the definition of valuations we can show $v(\varphi) \leq v(\psi)$. So from $v(\varphi) \geq 1$ and $v(\varphi) \leq v(\psi)$ $v(\psi) \geq 1$. Hence $v(\psi) \geq 1$ for any valuation v. Thus $\psi \in L(\mathbf{A})$. Thus $L(\mathbf{A})$ is closed under modus ponens.

Let $\varphi \in L(\mathbf{A})$. Then $v(\varphi) \geq 1$ for any valuation v. By the definition of valuations $v(\varphi \wedge 1) = v(\varphi) \wedge v(1)$. So $v(\varphi) \wedge v(1) \geq 1$ from $v(\varphi) \geq 1$. Hence $v(\varphi \wedge 1) \geq 1$ for any valuation. Thus $\varphi \wedge 1 \in L(\mathbf{A})$.

Let $\varphi \in L(\mathbf{A})$ and ψ is an arbitrary formula. Then $v(\varphi) \geq 1$ for any valuation v. By the definition of valuations

$$
\begin{aligned}
1 \leq v(\varphi) &\Rightarrow v(\psi) \leq v(\varphi) \cdot v(\psi) \\
&\Leftrightarrow 1 \leq v(\psi)(v(\varphi) \cdot v(\psi)) = v(\psi\backslash(\varphi \cdot \psi)).
\end{aligned}
$$

Thus $\psi\backslash(\varphi \cdot \psi) \in L(\mathbf{A})$.

Therefore $L(\mathbf{A})$ is a logic.

□

The logic $L(\mathbf{A})$ is called the logic characterized by $\mathbf{A}$.

Proposition 4.3.2 *For any logic* $\mathbf{L}$ *over* $\mathbf{FL}$ *there exists some* FL*-algebra* $\mathbf{A}$ *such that*

$$\mathbf{L} = L(\mathbf{A}).$$

(Out line of proof) We show this by constructing the Lindenbaum algebra of $\mathbf{L}$. First we define a binary relation $\equiv$ between formula φ and ψ as follows.

$$\varphi \equiv \psi \Leftrightarrow \varphi\backslash\psi \in \mathbf{L} \text{ and } \psi\backslash\varphi \in \mathbf{L}$$

$\varphi \equiv \psi$ means that φ and ψ are logically equivalent. It is clear that $\equiv$ is an equivalence relation. We can show moreover that $\equiv$ is a congruence relation, i.e. if $\varphi \equiv \psi$, $\varphi' \equiv \psi'$ then $\varphi \oplus \varphi' \equiv \psi \oplus \psi'$ for any logical connectives $\oplus$.

Next by using this congruence relation $\equiv$, construct the quotient set $\varphi/\equiv$ where φ is a set of all formulas. We write $[\varphi]$ equivalence class including φ. We can show that $\mathbf{A} = \langle \varphi/\equiv, \wedge, \vee, \cdot, \backslash, /, [\top], [\bot] \rangle$ is a FL-algebra where $\wedge, \vee, \cdot, \backslash, /$ are defined as follows.

$$
\begin{aligned}
[\varphi] \cup [\psi] &= [\varphi \wedge \psi] \\
[\varphi] \cap [\psi] &= [\varphi \vee \psi] \\
[\varphi] \cdot [\psi] &= [\varphi \cdot \psi] \\
[\varphi]\backslash[\psi] &= [\varphi\backslash\psi] \\
[\psi]/[\varphi] &= [\psi/\varphi]
\end{aligned}
$$

Finally we show that $\mathbf{L}$ and $L(\mathbf{A})$ correspond to each other, i.e.

$\varphi \in \mathbf{L} \Leftrightarrow$ for any valuation on $\mathbf{A}$, $v(\mathbf{A}) \geq [1]$

This algebra $\mathbf{A}$ is called the *Lindenbaum algebra* of a logic $\mathbf{L}$.

Proposition 4.3.3 (completeness theorem) *For any formula φ, φ is provable in $\mathbf{FL}$ if and only if for any FL-algebra $\mathbf{A}$ and for any valuation v on $\mathbf{A}$, $v(\varphi) \geq 1$.*

(Outline of proof) We show only-if part.
We define a valuation v for a sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ as follows.

$$v(\alpha_1, \ldots, \alpha_n \Rightarrow \beta) = v(\alpha_1 \cdot \ldots \cdot \alpha_n) \backslash v(\beta)$$

However if left-hand side of a sequent is empty then $v(\Rightarrow \beta) = 1 \backslash v(\beta)$ and if right-hand side of a sequent is empty then $v(\alpha_1, \ldots, \alpha_n \Rightarrow) = v(\alpha_1 \cdot \ldots \cdot \alpha_n) \backslash 0$.
We show this by induction on the construction of a proof of a formula φ. That is, for a given valuation v, every sequent S in a proof $v(S) \geq 1$. First we show base case. Initial sequents are satisfies following condition. For example,

1. $v(\alpha \Rightarrow \alpha) \geq 1$,
2. $v(\Gamma \Rightarrow \top) \geq 1$,
3. $v(\Gamma, \bot, \Delta \Rightarrow \gamma) \geq 1$,
4. $v(\Rightarrow 1) \geq 1$,
5. $v(0 \Rightarrow) \geq 1$.

Second we show induction case. Let for each inference rule upper sequents S_1 and S_2 are satisfies $v(S_1) \geq 1$ and $v(S_2) \geq 1$. Then lower sequent S is satisfies $v(S) \geq 1$.

Next we show if-part.
We show the contraposition of if-part. Suppose that for a given a formula φ such that φ is not provable in $\mathbf{FL}$. Then by using Lindenbaum algebra of $\mathbf{FL}$ there exist some FL-algebra $\mathbf{A}$ and some valuation v such that $v(\varphi) \not\geq 1$.

From a Proposition 4.3.3 we transcribe a Proposition 4.3.1 as follows.

Proposition 4.3.4 *A logic $L(\mathbf{A})$ characterized by a FL-algebra $\mathbf{A}$ is a logic over $\mathbf{FL}$.*

Proof It is clear from a Proposition 4.3.3 that $\mathbf{FL} \subseteq L(\mathbf{A})$ for all FL-algebra $\mathbf{A}$.

□

4.4 Algebraic operations and inclusion relation among logics

In previous section,we show that the $L(\mathbf{A})$ is a logic $L(\mathbf{A})$ over **FL** for each FL-algebra $\mathbf{A}$, and conversely every logic $\mathbf{L}$ over **FL** can be represented as $L(\mathbf{A})$ for some FL-algebra $\mathbf{A}$. Hereinafter we show the relations between three basic algebraic operations and logic.

Proposition 4.4.1 (subalgebras) *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *are* FL*-algebras and* $\mathbf{A} \leq \mathbf{B}$*. Then* $L(\mathbf{B}) \subseteq L(\mathbf{A})$ *hold.*

Proof Suppose that$\mathbf{A} \leq \mathbf{B}$. Then any valuation on $\mathbf{A}$ can be considered to be the restriction of a valuation on $\mathbf{B}$ of A. So if φ is a element of $L(\mathbf{B})$, i.e. $v(\varphi) \geq 1$ for any valuation v, then $u(\varphi) \geq 1$ for any valuation u on $\mathbf{A}$. Thus $L(\mathbf{B}) \subseteq L(\mathbf{A})$.

□

Proposition 4.4.2 (quotient algebras) *Let* $\mathbf{A}$ *is* FL*-algebra and* θ *is a congruence on* $\mathbf{A}$*. Then* $L(\mathbf{A}) \subseteq L(\mathbf{A}/\theta)$

Proof Let $\varphi(p_1, \dots p_n)$ be a formula, where $p_1, \dots p_n$ are all propositional variables appearing in φ. And for some formula $\varphi(p_1, \dots p_n)$ we express a replacing logical connectives $\wedge$, $\vee$, $\cdot$, $\backslash$,$/$ with $\wedge$, $\vee$, $\cdot$, $\backslash$, $/$ and a replacing propositional variables p_i with x_i by $f_\varphi(x_1, \dots, x_n)$. Then this is a element of FL-algebra.
Suppose that $\varphi(p_1, \dots p_n) \in L(\mathbf{A})$. In other words for any valuation v on $\mathbf{A}$

$$v(\varphi) = f_\varphi^{\mathbf{A}}(v(p_1), \dots, v(p_n)) \geq 1_{\mathbf{A}}.$$

Let θ is a congruence on $\mathbf{A}$. Then we can get

$$\begin{aligned} f_\varphi^{\mathbf{A}}(x_1, \dots, x_n)/\theta &= f_\varphi^{\mathbf{A}/\theta}(x_1/\theta, \dots, x_n/\theta) \\ &\geq 1_{\mathbf{A}}/\theta \\ &= 1_{\mathbf{A}/\theta}. \end{aligned}$$

This holds for any $x_1/\theta, \dots, x_n/\theta \in \mathbf{A}/\theta$. So $\varphi \in L(\mathbf{A}/\theta)$. Thus $L(\mathbf{A}) \subseteq L(\mathbf{A}/\theta)$.

□

Proposition 4.4.3 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *are* FL*-algebras and* $\alpha : \mathbf{A} \longrightarrow \mathbf{B}$ *is a homomorphism. Then the following holds.*

1. *If* α *is surjective then* $L(\mathbf{A}) \subseteq L(\mathbf{B})$.
2. *If* α *is injective then* $L(\mathbf{B}) \subseteq L(\mathbf{A})$.
3. *If* α *is bijective then* $L(\mathbf{A}) = L(\mathbf{B})$.

Proof They follow from previous two propositions. In fact if α is surjective then $\mathbf{A}/ker(\alpha) \simeq \mathbf{B}$ and if α is injective then $\mathbf{A} \simeq Im(\alpha) \leq \mathbf{B}$. Moreover 3 is clear from 1 and 2.

□

Proposition 4.4.4 $L(\prod_{i\in I}\mathbf{A}_i)=\bigcap_{i\in I}L(\mathbf{A}_i)$

Proof We show this only for the case of $I=\{1,2\}$.
($\subseteq$)
$\alpha_1:\mathbf{A}_1\times\mathbf{A}_2\longrightarrow\mathbf{A}_1$, $\alpha_2:\mathbf{A}_1\times\mathbf{A}_2\longrightarrow\mathbf{A}_2$ are onto homomorphism. So from the previous Proposition 4.4.3 $L(\mathbf{A}_1\times\mathbf{A}_2)\subseteq L(\mathbf{A}_1)$, $L(\mathbf{A}_1\times\mathbf{A}_2)\subseteq L(\mathbf{A}_2)$. Thus $L(\mathbf{A}_1\times\mathbf{A}_2)\subseteq L(\mathbf{A}_1)\cap L(\mathbf{A}_2)$.
($\supseteq$)
Let $\varphi\in L(\mathbf{A}_1)$ and $\varphi\in L(\mathbf{A}_2)$. In other words Let $v_1(\varphi)\geq 1_{\mathbf{A}_1}$ and $v_2(\varphi)\geq 1_{\mathbf{A}_2}$ for any valuation v_1 and v_2 on $\mathbf{A}_1$ and $\mathbf{A}_2$ respectively. Since any valuation v on $\mathbf{A}_1\times\mathbf{A}_2$ can be expressed as $v(\varphi)=\langle v_1(\varphi),v_2(\varphi)\rangle$ for valuations v_1 and v_2 on $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. So from an assumption

$$\begin{aligned}v(\varphi)&=\langle v_1(\varphi),v_2(\varphi)\rangle\\&\geq\langle 1_{\mathbf{A}_1},1_{\mathbf{A}_2}\rangle\\&=1_{\mathbf{A}_1\times\mathbf{A}_2}.\end{aligned}$$

Hence $\varphi\in L(\mathbf{A}_1\times\mathbf{A}_2)$.

$\square$

Above propositions intuitively mean that if an algebra become bigger (smallen) then a logic become smaller (bigger, respectively).

4.5 Logics over FL and varieties of FL-algebras

In previous two sections we discuss relations between logics and algebras. In this section we discuss relation between logics and classes of FL-algebras.

4.5.1 From logics to varieties

Definition 4.5.1 Let $\mathbf{L}$ be a logic over $\mathbf{FL}$. We define a class $\mathcal{V}_\mathbf{L}$ of FL-algebras by

$$\mathcal{V}_\mathbf{L}=\{\mathbf{Q}:\mathbf{L}\subseteq L(\mathbf{Q})\}.$$

Proposition 4.5.1 *For every logic over* $\mathbf{FL}$ *a class* $\mathcal{V}_\mathbf{L}$ *of* FL*-algebras is a variety.*

Proof It is enough to show that $\mathcal{V}_\mathbf{L}$ is closed under homomorphic images, subalgebras, direct products.
(homomorphic images)
Let $\mathbf{A}\in\mathcal{V}_\mathbf{L}$.Then $\mathbf{L}\subseteq L(\mathbf{A})$. If $\alpha(\mathbf{A})$ is a homomorphic image of $\mathbf{A}$ then by Proposition 4.4.3 we can get $L(\mathbf{A})\subseteq L(\alpha(\mathbf{A}))$. So $\mathbf{L}\subseteq L(\alpha(\mathbf{A}))$. Thus we can get $\alpha(\mathbf{A})\in\mathcal{V}_\mathbf{L}$.
(subalgebras)
Let $\mathbf{A}\in\mathcal{V}_\mathbf{L}$ and $\mathbf{B}\leq\mathbf{A}$. Then by Proposition 4.4.1

$$\mathbf{L}\subseteq L(\mathbf{A})\subseteq L(\mathbf{B})$$

Thus $\mathbf{B} \in \mathcal{V}_{\mathbf{L}}$.
(direct products)
Let $\mathbf{A} \in \mathcal{V}_{\mathbf{L}}$ for each $i \in I$. Then by Proposition 4.4.4 we can get $L(\prod_{i\in I} \mathbf{A}_i) = \bigcap_{i\in I} L(\mathbf{A}_i)$. Moreover from our assumption 2 $\mathbf{L} \subseteq L(\mathbf{A}_i)$ for any $i \in I$. So we can get

$$\mathbf{L} \subseteq \bigcap_{i\in I} L(\mathbf{A}_i) = L(\prod_{i\in I} \mathbf{A}_i).$$

Thus $\prod_{i\in I} \mathbf{A}_i \in \mathcal{V}_{\mathbf{L}}$.

□

4.5.2 From varieties to logics

Definition 4.5.2 Let $\mathcal{K}$ be a class of FL-algebras. $\mathcal{K}$ is an *identity class* if there exists some sets Σ of identities, i.e.

$$\mathcal{K} = \{\mathbf{A} : \mathbf{A} \models s \approx t \text{ for any } s \approx t \in \Sigma\}.$$

We note that all identities can be expressed of a form $1 \leq r$ for a term r. Because for any identity $s \approx t$

$$\begin{aligned} s \approx t \quad &\Leftrightarrow \quad s \leq t \; and \; t \leq s \\ &\Leftrightarrow \quad 1 \leq s\backslash t \; and \; 1 \leq t\backslash s \\ &\Leftrightarrow \quad 1 \leq (s\backslash t) \wedge (t\backslash s). \end{aligned}$$

Let $\mathcal{V}$ is a variety of FL-algebras. We define the set of formulas $\mathbf{L}_{\mathcal{V}}$ as follows.

$$\mathbf{L}_{\mathcal{V}} = \{\varphi | \mathcal{V} \models \varphi \wedge 1 \approx 1\}.$$

Then a following proposition holds.

Proposition 4.5.2 *For any variety $\mathcal{V}$ of FL-algebras $\mathbf{L}_{\mathcal{V}}$ is a logic over $\mathbf{FL}$.*

Proof Let $\varphi \in \mathrm{FL}$. Then from completeness theorem for any FL-algebra and for any valuation v, $v(\varphi) \geq 1_{\mathbf{A}}$. For any FL-algebra $\mathbf{A} \in \mathcal{V}$

$$v(\varphi) \geq 1_{\mathbf{A}}.$$

$\mathbf{L}_{\mathcal{K}}$ is a logic which is a set of any formula ψ satisfying $\mathcal{K} \models v(\psi)$. So

$$\varphi \in \mathbf{L}_{\mathcal{K}}$$

Thus $\mathbf{FL} \subseteq \mathbf{L}_{\mathcal{V}}$.

Let $\varphi(p_1, \ldots p_n) \in \mathbf{L}_{\mathcal{V}}$. Suppose that substitution instance $\varphi(\psi_1, \ldots, \psi_n) \notin \mathbf{L}_{\mathcal{V}}$. Then there exists some $\mathbf{A} \in \mathcal{V}$ and valuation v on $\mathbf{A}$ such that $v(\varphi(\psi_1, \ldots, \psi_n)) \not\geq 1_{\mathbf{A}}$. Let v' be a valuation on $\mathbf{A}$ defined by $v'(p_i) = v(\psi_i)$ for $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned} v'(\varphi(p_1, \ldots, p_n)) &= \varphi(v'(p_1), \ldots, v'(p_n)) \\ &= \varphi(v(\psi_1), \ldots, v(\psi_n)) \\ &= v(\varphi(\psi_1, \ldots, \psi_n)) \\ &\not\geq 1_{\mathbf{A}} \end{aligned}$$

This is a contradiction. Thus $\mathbf{L}_{\mathcal{V}}$ is closed under substitution.

Let $\varphi, \varphi\backslash\psi \in \mathbf{L}_{\mathcal{V}}$. Then $v(\varphi) \geq 1_{\mathbf{A}}, v(\varphi\backslash\psi) \geq 1_{\mathbf{A}}$ for any $\mathbf{A} \in \mathcal{V}$ and valuation v. Hence

$$\begin{aligned} 1_{\mathbf{A}} \leq v(\varphi\backslash\psi) &\Leftrightarrow 1_{\mathbf{A}} \leq v(\varphi)\backslash v(\psi) \\ &\Leftrightarrow v(\varphi) \leq v(\psi) \\ &\Leftrightarrow 1_{\mathbf{A}} \leq v(\psi). \end{aligned}$$

Thus $\psi \in \mathbf{L}_{\mathcal{V}}$.

Let $\varphi \in \mathbf{L}_{\mathcal{K}}$. Then $v(\varphi) \geq 1_{\mathbf{A}}$ for any $\mathbf{A} \in \mathcal{V}$ and valuation v. Hence $v(\varphi \wedge 1) = v(\varphi) \wedge 1_{\mathbf{A}} = 1_{\mathbf{A}}$. Thus $\varphi \wedge 1 \in \mathbf{L}_{\mathcal{K}}$.

Let $\varphi \in \mathbf{L}_{\mathcal{K}}$ and ψ be an arbitrary formula. Then $v(\varphi) \geq 1_{\mathbf{A}}$ for any $\mathbf{A} \in \mathcal{V}$ and valuation v. Hence

$$\begin{aligned} 1_{\mathbf{A}} \leq v(\varphi) &\Rightarrow v(\psi) \leq v(\varphi) \cdot v(\psi) \\ &\Leftrightarrow 1_{\mathbf{A}} \leq v(\psi)\backslash(v(\varphi) \cdot v(\psi)) \\ &\Leftrightarrow 1_{\mathbf{A}} \leq v(\psi\backslash(\varphi \cdot \psi)). \end{aligned}$$

Thus $\psi\backslash(\varphi \cdot \psi) \in \mathbf{L}_{\mathcal{K}}$. Similarly we can show $(\psi \cdot \varphi)/\psi \in \mathbf{L}_{\mathcal{K}}$

□

Proposition 4.5.3 *The maps* $\mathbf{\Lambda}(\mathrm{RL}) \rightarrow \mathbf{\Lambda}(\mathbf{FL})$ *and* $\mathbf{\Lambda}(\mathbf{FL}) \rightarrow \mathbf{\Lambda}(\mathrm{RL})$ *are mutually inverse, dual lattice isomorphism.*

Proof Let $\mathbf{L}_1 \subseteq \mathbf{L}_2$. Then $\varphi \in \mathbf{L}_2$ for every $\varphi \in \mathbf{L}_1$. Thus, $V(\mathbf{L}_2) \models \varphi \wedge 1 \approx 1$, so, $V(\mathbf{L}_2) \subseteq V(\mathbf{L}_1)$. Let $\mathcal{V}_1 \subseteq \mathcal{V}_2$. If $\mathcal{V}_2 \models \varphi \wedge 1 \approx 1$ then $\mathcal{V}_1 \models \varphi \wedge 1 \approx 1$ for any $\varphi \in V(\mathbf{L}_2)$. Thus $L(\mathcal{V}_2) \subseteq L(\mathcal{V}_1)$.

If $\varphi \in \mathbf{L}$ then $V(\mathbf{L}) \models \varphi \wedge 1 \approx 1$. Hence $\varphi \in L(V(\mathbf{L}))$. If $\varphi \notin \mathbf{L}$ then there is some FL-algebra $\mathbf{A}$ such that $\mathbf{A} \not\models \varphi \wedge 1 \approx 1$ and $\mathbf{A} \in V(\mathbf{L})$ by Proposition 4.3.4. Hence $V(\mathbf{L}) \not\models \varphi \wedge 1 \approx 1$, so $\varphi \notin L(V(\mathbf{L}))$.

Let $\mathbf{A} \in \mathcal{V}$. Then for any $\varphi \wedge 1 \approx 1$, if $\mathcal{V} \models \varphi \wedge 1 \approx 1$ then $\mathbf{A} \models \varphi \wedge 1 \approx 1$. Thus $\mathbf{A} \in Mod(\{\varphi \wedge 1 \approx 1 | \mathcal{V} \models \varphi \wedge 1 \approx 1\})$, so $\mathbf{A} \in V(L(\mathcal{V}))$.

Conversely, let $\mathbf{A} \in V(L(\mathcal{V}))$. Then $\mathcal{V} \models \varphi \wedge 1 \approx 1 \Rightarrow \mathbf{A} \models \varphi \wedge 1 \approx 1$ for any $\varphi \wedge 1 \approx 1$. Note that for any identity $t \approx s$ and FL-algebra $\mathbf{B}$,

$$\mathbf{B} \models t \approx s \Leftrightarrow \mathbf{B} \models ((t\backslash s) \wedge (s\backslash t)) \wedge 1 \approx 1.$$

Thus

$$\begin{aligned} \mathcal{V} \models t \approx s &\Leftrightarrow \mathcal{V} \models ((t\backslash s) \wedge (s\backslash t)) \wedge 1 \approx 1 \\ &\Rightarrow \mathbf{A} \models ((t\backslash s) \wedge (s\backslash t)) \wedge 1 \approx 1 \\ &\Leftrightarrow \mathbf{A} \models t \approx s \\ &\Rightarrow \mathbf{A} \in Mod(\{t \approx s | \mathcal{V} \models t \approx s\}). \end{aligned}$$

By Proposition **??**, $\mathcal{V} = Mod(\{t \approx s | \mathcal{V} \models t \approx s\})$, therefore $\mathbf{A} \in \mathcal{V}$.

□

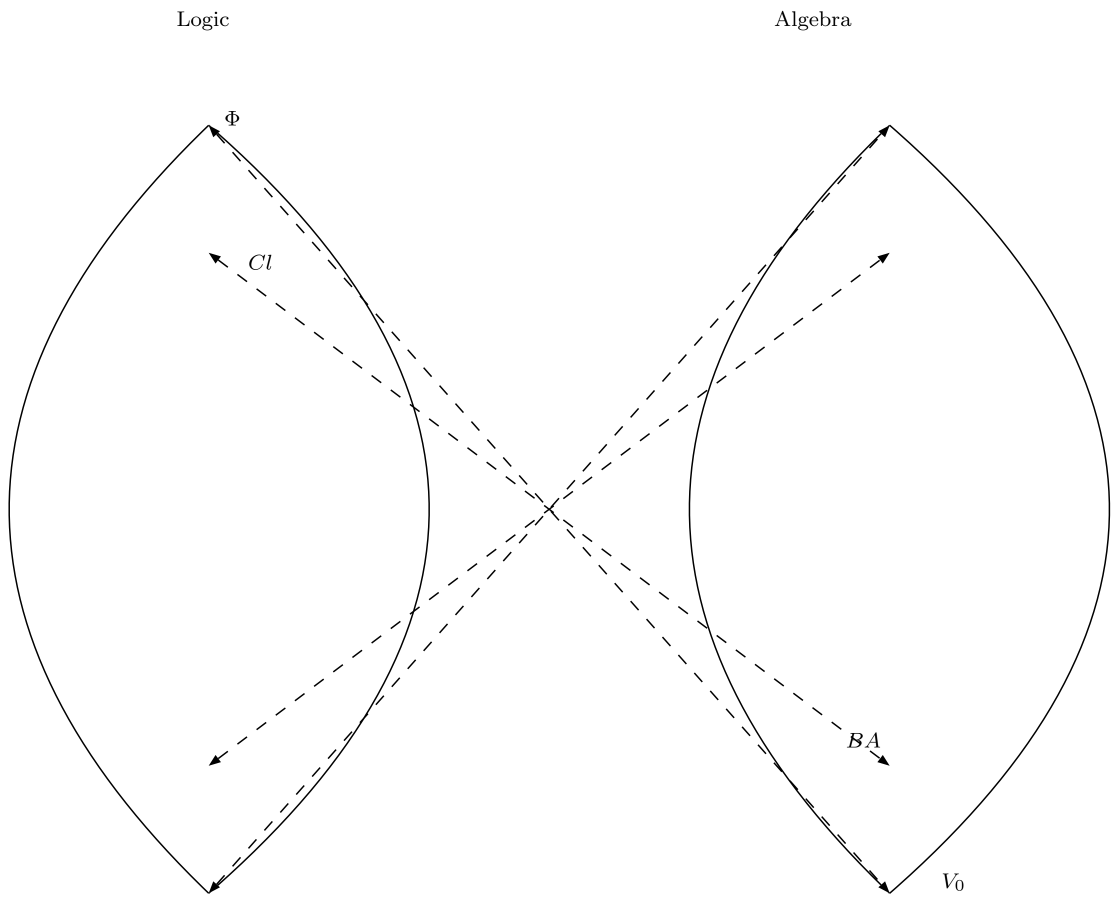

Figure 4.1.

Chapter 5

Algebraic characterization of disjunction property

In this chapter, we will discuss about the disjunction property (DP). Some of the basic substructural logics are shown to have the DP by using cut elimination of sequent calculi for these logics [16, 10, 1, 2]. On the other hand, this syntactic method works only for a limited number of substructural logics. Here, we show first that Maksimova's algebraic criterion on the DP of superintuitionistic logics can be naturally extended to one on the DP of substructural logics over $\mathbf{FL}$. By using this criterion, we show the DP for some of the substructural logics for which syntactic methods do not work well.

5.1 Disjunction property as a consequence of cut elimination

Definition 5.1.1 (disjunction property) A logic L has the *disjunction property*, when for any formula φ and ψ if $\varphi \vee \psi$ is provable in L then at least one of the formulas ϕ and ψ is provable in it.

Classical logics does not have the disjunction property, as $p \vee \neg p$ is provable but neither of p or $\neg p$ are provable.

Theorem 5.1.1 *Intuitionistic logic has the disjunction property.*

Proof We give here a syntactic proof, by using the sequent calculus $\mathbf{LJ}$ for intuitionistic logic and the cut elimination. Suppose that the sequent $\to \varphi \vee \psi$ is provable in $\mathbf{LJ}$. Then there exists cut-free proof P of $\to \varphi \vee \psi$. Let I be the last inference of P. I will be either $\to w$ or $\to \vee$. If I is $\to w$ then the upper sequent is $\to$. It is impossible since $\mathbf{LJ}$ is consistent. Hence I must be $\to \vee$. Then the upper sequent is $\to \varphi$ or $\to \psi$. Thus φ or ψ is provable.

□

We can show the following theorem in the same way as above proof, using the fact that cut elimination holds in each of them.

Theorem 5.1.2 *Each of* $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$ *and* $\mathbf{FL_{ec}}$ *has the disjunction property.*

Then, why the above proof does not work for $\mathbf{LK}$? Let us consider of any cut-free proof of $\to \varphi \vee \neg\varphi$, and let I be the last inference rule. Then I will be $(\to w)$, $(\to \vee)$ or $(\to c)$. When I is $(\to c)$, then

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \to \varphi}{\to \varphi, \neg\varphi}\,(\to \neg)}{\to \varphi, \varphi \vee \neg\varphi}\,(\to \vee)}{\to \varphi \vee \neg\varphi, \varphi \vee \neg\varphi}\,(\to \vee)}{\to \varphi \vee \neg\varphi}\,(\to c)$$

.

Thus, the similar argument as in the proof of the previous theorems does not work. On the other hand, the above proof of DP as shown in Theorem 5.1.1 works for a logics over $\mathbf{CFL}$ with a cut-free system, as long as it does not have $(\to c)$.

Theorem 5.1.3 *Both* $\mathbf{CFL_e}$ *and* $\mathbf{CFL_{ew}}$ *have the DP.*

5.2 Algebraic characterization of disjunction property for logics over $\mathbf{FL}$

In the previous section we show the DP as a consequence of cut elimination theorem. But having a cut-free sequent calculus is rather exceptional. In [13], Maksimova gave an algebraic characterization of the disjunction property for *superintuitionistic logics*, i.e. logics over intuitionistic logic $\mathbf{Int}$. In this section we extend the Maksimova's result to logics over $\mathbf{FL}$.

Definition 5.2.1 (well-connectedness) A RL $\mathbf{A}$ is *well-connected* if for any $x, y \in \mathbf{A}$ $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$.

In every Heyting algebra the unit element 1 is always the greatest element. Thus a Heyting algebra $\mathbf{A}$ is well-connected iff for all $x, y \in A$, $x \vee y = 1$ implies either $x = 1$ or $y = 1$. In [13], L. Maksimova showed the following.

Theorem 5.2.1 (Maksimova) *Suppose that a logic L over* $\mathbf{Int}$ *is complete with respect to a class $\mathcal{K}$ of Heyting algebras. Then, the following are equivalent;*

1. *L has the disjunction property,*
2. *For all Heyting algebras $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exist a well-connected Heyting algebra $\mathbf{C}$ such that L is valid in $\mathbf{C}$, and there is a surjective homomorphism from $\mathbf{C}$ onto the direct product $\mathbf{A} \times \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$.*

In the same way as this, we can show the following Theorem 5.2.2.

Theorem 5.2.2 *Suppose that a logic* $\mathbf{L}$ *over* $\mathbf{FL}$ *is complete with respect to a class $\mathcal{K}$ of* $\mathbf{FL}$-*algebras. Then, the following are equivalent:*

1. $\mathbf{L}$ *has the disjunction property,*

2. *For all* $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ *there exist a well-connected* $\mathbf{FL}$*-algebra* $\mathbf{C} \in V(\mathbf{L})$*, and a surjective homomorphism from* $\mathbf{C}$ *onto the direct product* $\mathbf{A} \times \mathbf{B}$ *of* $\mathbf{A}$ *and* $\mathbf{B}$.

Proof We will show the Theorem, following Maksimova's idea, but in the context of logics over $\mathbf{FL}$. We show first that 2 implies 1. Suppose that 2 holds and that neither ϕ nor ψ is valid in L for some formulas ϕ and ψ. We show that $\varphi \vee \psi$ is not valid.

Because of the completeness of L with respect to $\mathcal{K}$ we have $v_{\mathbf{A}}(\phi) \not\geq_{\mathbf{A}} 1_{\mathbf{A}}$ and $v_{\mathbf{B}}(\psi) \not\geq_{\mathbf{B}} 1_{\mathbf{B}}$ for some valuations $v_{\mathbf{A}}$ in $\mathbf{A}$ and $v_{\mathbf{B}}$ in $\mathbf{B}$ where $\mathbf{A}, \mathbf{B} \in \mathcal{K}$. From 2 there exist a well-connected CRL $\mathbf{C}$ such that L is valid in $\mathbf{C}$, and a surjective homomorphism α from $\mathbf{C}$ onto $\mathbf{A} \times \mathbf{B}$. We define a valuation v in $\mathbf{C}$ as follows. For any propositional variable p, define $v(p) = a$, where a is an arbitrary element in $\alpha^{-1}(\langle v_{\mathbf{A}}(p), v_{\mathbf{B}}(p)\rangle)$. Thus for any variable p,

$$\alpha(v(p)) = \langle v_{\mathbf{A}}(p), v_{\mathbf{B}}(p)\rangle.$$

Then we can show inductively that for any formula δ, $\alpha(v(\delta)) = \langle v_{\mathbf{A}}(\delta), v_{\mathbf{B}}(\delta)\rangle$. In particular,

$$\alpha(v(\phi)) = \langle v_{\mathbf{A}}(\phi), v_{\mathbf{B}}(\phi)\rangle < \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}\rangle \text{ and } \alpha(v(\psi)) = \langle v_{\mathbf{A}}(\psi), v_{\mathbf{B}}(\psi)\rangle < \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}\rangle.$$

Hence, we have $v(\phi) < 1_{\mathbf{C}}$ and $v(\psi) < 1_{\mathbf{C}}$ as $\alpha(1_{\mathbf{C}}) = \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}\rangle$. Thus $v(\phi \vee \psi) < 1_{\mathbf{C}}$ by the well-connectedness of $\mathbf{C}$. Thus $\phi \vee \psi$ is not valid in $\mathbf{C}$.

Next we show that 1 implies 2. First, note that if L has the disjunction property, then all free algebras of the corresponding variety $V(L) = \{\mathbf{C} \mid L \text{ is valid in } \mathbf{C}\}$ are well-connected. For give $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, their direct product $\mathbf{A} \times \mathbf{B}$ belongs to $V(L)$. By the universal mapping property of free algebras, if we take an enough big free algebra in $V(L)$ there exists a surjective homomorphism from $\mathbf{C}$ to $\mathbf{A} \times \mathbf{B}$. This completes the proof.

□

5.3 Disjunction property of various substructural logics

As an application of Theorem 5.2.2, we show the disjunction property of some logics over $\mathbf{FL}$. First we show the disjunction property of $\mathbf{FL}[\mathrm{E}_\mathrm{n}^\mathrm{m}]$ and $\mathbf{FL_e}[\mathrm{E}_\mathrm{n}^\mathrm{m}]$ where $\mathrm{E}_\mathrm{n}^\mathrm{m}$: $(p^m \backslash p^n)$ $(m \geq 0, n \geq 0)$. Note that $\mathrm{E}_\mathrm{n}^\mathrm{m}$ corresponds to the contractiveness when $m = 1$ and $n = 2$.

Theorem 5.3.1 (Disjunction property for $\mathbf{FL}[\mathrm{E}_\mathrm{n}^\mathrm{m}]$.) *Both* $\mathbf{FL}[\mathrm{E}_\mathrm{n}^\mathrm{m}]$ *and* $\mathbf{FL_e}[\mathrm{E}_\mathrm{n}^\mathrm{m}]$ *have the disjunction property for every* m, n.

The sequent calclus $\mathbf{FL}[\mathrm{E}_\mathrm{n}^\mathrm{m}]$ does not hold the cut-elimination theorem. To prove this theorem we construct a suitable RL $\mathbf{C}$, which satisfies the conditions stated in Theorem 5.2.2 for give RLs $\mathbf{A}$ and $\mathbf{B}$. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are given as follows.

- $\mathbf{A} = \langle \mathrm{A}, \wedge_{\mathbf{A}}, \vee_{\mathbf{A}}, \cdot_{\mathbf{A}}, /_{\mathbf{A}}, \backslash_{\mathbf{A}}, 0_{\mathbf{A}}, 1_{\mathbf{A}}\rangle$,
- $\mathbf{B} = \langle \mathrm{B}, \wedge_{\mathbf{B}}, \vee_{\mathbf{B}}, \cdot_{\mathbf{B}}, /_{\mathbf{B}}, \backslash_{\mathbf{B}}, 0_{\mathbf{B}}, 1_{\mathbf{B}}\rangle$.

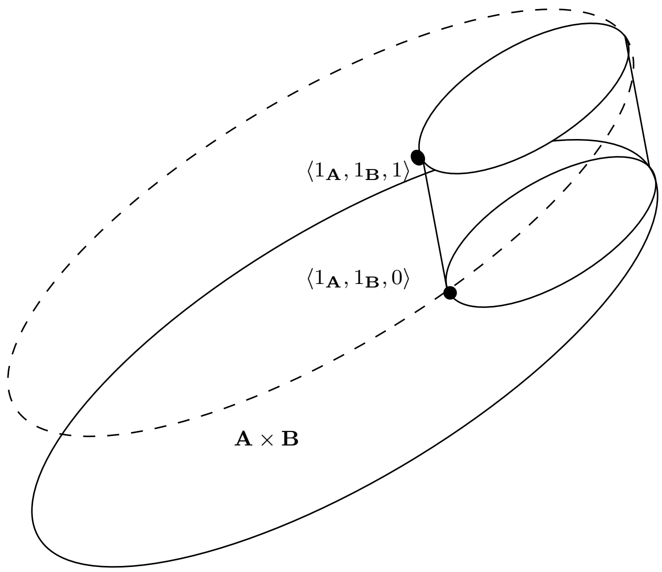

Figure 5.1.

Define a RL $\mathbf{C} = \langle \mathrm{C}, \wedge, \vee, \cdot, /, \backslash, \mathbf{0}, \mathbf{1}\rangle$ as follows;

Let $\mathbf{2}$ be the two element Boolean algebra with the universe $\{0, 1\}$. Take the direct product $\mathbf{A}\times\mathbf{B}\times\mathbf{2}$. Consider a subset $\mathrm{C} = \{(a, b, 0)|a \in \mathrm{A},\ b \in \mathrm{B}\} \cup \{(a, b, 1)|a \in \mathrm{A},\ b \in \mathrm{B},\ a \geq_{\mathbf{A}} 1_{\mathbf{A}},\ b \geq_{\mathbf{B}} 1_{\mathbf{B}}\}$ of $\mathrm{A}\times\mathrm{B}\times 2$. Define $\mathbf{1} = \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$ and $\mathbf{0} = \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle$. Note that the operations $\cdot$, $\backslash$, $/$ on $\mathbf{2}$ are defined by $x \cdot x = x$, $1 \cdot 0 = 0 \cdot 1 = 0$, $x\backslash x = 1$, $1\backslash 0 = 0$ and $0\backslash 1 = 1$ ($/$ is defined in the same way as $\backslash$).

Obviously, $\mathbf{A}\times\mathbf{B}\times\mathbf{2}$ is a lattice. Since C is closed under lattice operations, the algebra $\langle \mathrm{C}, \wedge, \vee\rangle$ can be regarded as a sublattice of $\mathbf{A}\times\mathbf{B}\times\mathbf{2}$.

Define the multiplication $\cdot$ on C as follows:

$$\langle a, b, i\rangle \cdot \langle a', b', j\rangle = \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', i \cdot j\rangle.$$

Define the left residual $\backslash$ on C as follows:

$$\langle a, b, i\rangle\backslash\langle a', b', j\rangle = \begin{cases} \langle a\backslash_{\mathbf{A}} a', b\backslash_{\mathbf{B}} b', i\backslash j\rangle & \text{if } \langle a\backslash_{\mathbf{A}} a', b\backslash_{\mathbf{B}} b', i\backslash j\rangle \in \mathrm{C} \\ \langle a\backslash_{\mathbf{A}} a', b\backslash_{\mathbf{B}} b', 0\rangle & \text{if } \langle a\backslash_{\mathbf{A}} a', b\backslash_{\mathbf{B}} b', i\backslash j\rangle \notin \mathrm{C} \end{cases}$$

Similarly for the right residual $/$.

We show that $\mathbf{C} = \langle \mathrm{C}, \wedge, \vee, \cdot, \backslash, /, 0, 1\rangle$ is a RL.

Lemma 5.3.2 *The algebra $\mathbf{C}$ satisfies the law of residuation.*

Proof First we prove that $\langle a_1, b_1, i\rangle\cdot\langle a_2, b_2, j\rangle \leq \langle a_3, b_3, k\rangle$ implies $\langle a_2, b_2, j\rangle \leq \langle a_1, b_1, i\rangle\backslash\langle a_3, b_3, k\rangle$.

Suppose that $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_3, b_3, k\rangle$ holds, we can easily show $a_1 \cdot_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_3$, $b_1 \cdot_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_3$ and $i \cdot j \leq k$. Hence we can get $a_2 \leq_{\mathbf{A}} a_1 \backslash_{\mathbf{A}} a_3$, $b_2 \leq_{\mathbf{B}} b_1 \backslash_{\mathbf{B}} b_3$, $j \leq i \backslash k$. By the definition of $\backslash$ of $\mathbf{C}$, we need to consider the following two cases.

- Suppose that $\langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle = \langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, i \backslash k\rangle$. Then

$$\langle a_2, b_2, j\rangle \leq \langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, i \backslash k\rangle = \langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle.$$

- Suppose that $\langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle = \langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, 0\rangle$. Then we can prove $a_2 \not\geq_{\mathbf{A}} 1_{\mathbf{A}}$, since $a_2 \leq_{\mathbf{A}} a_1 \backslash_{\mathbf{A}} a_3$ and $a_1 \backslash_{\mathbf{A}} a_3 \not\geq_{\mathbf{A}} 1_{\mathbf{A}}$. Similarly we can prove $b_2 \not\geq_{\mathbf{B}} 1_{\mathbf{B}}$. So j cannot be 1. Thus

$$\langle a_2, b_2, j\rangle \leq \langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, 0\rangle = \langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle.$$

Next we prove that $\langle a_2, b_2, j\rangle \leq \langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle$ implies $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_3, b_3, k\rangle$.

- Suppose that $\langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle = \langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, i \backslash k\rangle$. From $\langle a_2, b_2 j\rangle \leq \langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, i \backslash k\rangle$, we can prove $a_1 \cdot_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_3$, $b_1 \cdot_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_3$, $i \cdot j \leq k$ easily. Thus

$$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_3, b_3, k\rangle.$$

- Suppose that $\langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle = \langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, 0\rangle$. Then similarly we can prove $a_1 \cdot_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_3$, $b_1 \cdot_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_3$. Since $i \leq 0$ we can show $i \cdot j = 0 \leq k$. Thus

$$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \leq \langle a_3, b_3, k\rangle.$$

□

Now we show that $\mathbf{C}$ satisfies the condition $\mathrm{E}_{\mathrm{n}}^{\mathrm{m}}$, assuming that both $\mathbf{A}$ and $\mathbf{B}$ satisfy $\mathrm{E}_{\mathrm{n}}^{\mathrm{m}}$.

Lemma 5.3.3 *If both $\mathbf{A}$ and $\mathbf{B}$ satisfy the condition $\mathrm{E}_{\mathrm{n}}^{\mathrm{m}}$ then so does $\mathbf{C}$.*

Proof From our assumption $1_{\mathbf{A}} \leq a^m \backslash a^n$, $1_{\mathbf{B}} \leq b^m \backslash b^n$ for all $a \in \mathrm{A}$ and $b \in \mathrm{B}$. For all $\langle a, b, i\rangle \in \mathrm{C}$,

$$\begin{aligned} & \langle a, b, i\rangle^m \backslash \langle a, b, i\rangle^n \\ = \ & \langle a^m, b^m, i^m\rangle \backslash \langle a^n, b^n, i^n\rangle \\ = \ & \langle a^m \backslash_{\mathbf{A}} a^n, b^m \backslash_{\mathbf{B}} b^n, i^m \backslash i^n\rangle \\ \geq \ & \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle. \end{aligned}$$

Hence $\langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle \leq \langle a, b, i\rangle^m \backslash \langle a, b, i\rangle^n$. Thus $\mathbf{C}$ satisfies $\mathrm{E}_{\mathrm{n}}^{\mathrm{m}}$.

□

Thus $\mathbf{C} = \langle \mathrm{C}, \wedge, \vee, \cdot, \backslash, /, 0, 1\rangle$ is a RL.

Lemma 5.3.4 *The residuated lattice $\mathbf{C}$ is well-connected.*

Proof Suppose that $\langle a, b, i\rangle, \langle a', b', j\rangle \in \mathrm{C}$. If $\langle a, b, i\rangle \not\geq \mathbf{1}$, $\langle a', b', j\rangle \not\geq \mathbf{1}$ then $i = j = 0$. Then

$$\begin{array}{rl} & \langle a, b, i\rangle \vee \langle a', b', j\rangle \\ = & \langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b', i \vee j\rangle \\ = & \langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b', 0\rangle \\ \not\geq & \mathbf{1}. \end{array}$$

Thus $\mathbf{C}$ is well-connected.

□

It is clear that if both $\mathbf{A}$ and $\mathbf{B}$ are commutative then so is $\mathbf{C}$.

Lemma 5.3.5 *A mapping α from $\mathbf{C}$ to $\mathbf{A} \times \mathbf{B}$ defined by*

$$\alpha(\langle a, b, i\rangle) = \langle a, b\rangle$$

is a surjective homomorphism.

Proof The mapping α is clearly surjective. Thus, it remains to show that α is a homomorphism.

$$\begin{array}{rcl} \alpha(\langle a, b, i\rangle \vee \langle a', b', j\rangle) & = & \alpha(\langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b', i \vee j\rangle) \\ & = & \langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b'\rangle \\ & = & \langle a, b\rangle \vee \langle a', b'\rangle \\ & = & \alpha(\langle a, b, i\rangle) \vee \alpha(\langle a', b', j\rangle) \end{array}$$

We can show $\alpha(\langle a, b, i\rangle \wedge \langle a', b', j\rangle) = \alpha(\langle a, b, i\rangle) \wedge \alpha(\langle a', b', j\rangle)$ and $\alpha(\langle a, b, i\rangle \cdot \langle a', b', j\rangle) = \alpha(\langle a, b, i\rangle) \cdot \alpha(\langle a', b', j\rangle)$ in the same way as above.

$\alpha(\rangle a, b, i\langle\backslash\langle a', b', j\rangle)$ is either $\alpha(\langle a\backslash_{\mathbf{A}} a', b\backslash_{\mathbf{B}} b', i\backslash j\rangle)$ or $\alpha(\langle a\backslash_{\mathbf{A}} a', b\backslash_{\mathbf{B}} b', 0\rangle)$. Therefore

$$\begin{array}{rcl} \alpha(\langle a, b, i\rangle\backslash\langle a', b', j\rangle) & = & \langle a\backslash_{\mathbf{A}} a', b\backslash_{\mathbf{B}} b'\rangle \\ & = & \langle a, b\rangle\backslash\langle a', b'\rangle \\ & = & \alpha(\langle a, b, i\rangle)\backslash\alpha(\langle a', b', j\rangle). \end{array}$$

Similarly we can show $\alpha(\langle a', b', j\rangle/\langle a, b, i\rangle) = \alpha(\langle a', b', j\rangle)/\alpha(\langle a, b, i\rangle)$.
Thus α is surjective homomorphism.

□

Next we consider the following axioms:

- $\mathrm{E_k}$: $p^{k+1}\backslash p^k$ (*weak k-potency*, i.e.,$\mathrm{E_k^{k+1}}$)
- Dis: $((p \vee q) \wedge (p \vee r))\backslash(p \vee (q \wedge r))$ (*distributivity*)

Corollary 5.3.6 **FL**, $\mathbf{FL_e}$, $\mathbf{FL}[\mathrm{E_k}]$ *and* $\mathbf{FL_e}[\mathrm{E_k}]$ *have the disjunction property for every* k

Remark that since E_2^1 is the axiom of contraction, the DP of $\mathbf{FL_{ec}}$ follows also.

We note that in this proof if we assume moreover that both $\mathbf{A}$ and $\mathbf{B}$ satisfy Dis, i.e. their lattice reducts are distributive, then $\mathbf{A} \times \mathbf{B} \times \mathbf{2}$ is distributive. The lattice reduct of $\mathbf{C}$ is a sublattice of lattice reduct of $\mathbf{A} \times \mathbf{B} \times \mathbf{2}$. Since the distributivity is represented as identity, $\mathbf{C}$ satisfy Dis. Hence we have also the following theorem.

Corollary 5.3.7 *Both* $\mathbf{FL}[\mathrm{E}_\mathrm{n}^\mathrm{m}, \mathrm{Dis}]$ *and* $\mathbf{FL_e}[\mathrm{E}_\mathrm{n}^\mathrm{m}, \mathrm{Dis}]$ *have the disjunction property. In particular* $\mathbf{FL}[\mathrm{Dis}]$ *have the disjunction property.*

As the existence of the zero element $\mathbf{0}$ of $\mathbf{C}$ does not play any particular role in the proof of Theorem 5.3.1, we can derive that each positive fragment of these logics has also the disjunction property. It is well-known that the positive relevant logic $\mathbf{R}^+$ is equal to the positive fragment of $\mathbf{FL_e}[\mathrm{E}_2^1, \mathrm{Dis}]$. Hence we can get an alternative proof of the disjunction property of $\mathbf{R}^+$, which was firstly proved by R. K. Meyer in [14].

5.4 Disjunction property of involutive substructural logics

In this section we discuss about involutive substructural logics. Here, we say that a substructural logic is a *involutive*, when the following law of double negation

DN: $(\sim -p\backslash p) \wedge (- \sim p\backslash p)$

holds in it.

In general, the RL $\mathbf{C}$ in the proof of Theorem 5.3.1 does not satisfy DN, even if both $\mathbf{A}$ and $\mathbf{B}$ satisfy DN. For example if $1_\mathbf{A} \leq_\mathbf{A} a$, $1_\mathbf{B} \leq_\mathbf{B} b$ then

$$\begin{aligned} \sim -\langle a,b,0\rangle\backslash\langle a,b,0\rangle &= \langle \sim_\mathbf{A} -_\mathbf{A} a, \sim_\mathbf{B} -_\mathbf{B} b, 1\rangle\backslash\langle a,b,0\rangle \\ &= \langle \sim_\mathbf{A} -_\mathbf{A} a\backslash_\mathbf{A} a, \sim_\mathbf{B} -_\mathbf{B} b\backslash_\mathbf{B} b, 0\rangle \\ &\not\geq \langle 1_\mathbf{A}, 1_\mathbf{B}, 1\rangle. \end{aligned}$$

So we need to introduce a different RL $\mathbf{C}$ in proving the disjunction property of $\mathbf{FL}[\mathrm{DN}]$.

Note that $\mathbf{FL_e}[\mathrm{DN}]$ is nothing but the multiplicative additive linear logic MALL. As mentioned in section 5.1 and [16], $\mathbf{FL_e}[\mathrm{DN}]$ has the DP since it is formulated by a cut-free sequent system without the right contraction rule. Here, we give an algebraic proof of it.

Theorem 5.4.1 *Both* $\mathbf{FL}[\mathrm{DN}]$ *and* $\mathbf{FL_e}[\mathrm{DN}]$ *have the disjunction property.*

We prove this theorem in the same way as Theorem 5.3.1. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are given. Define a RL $\mathbf{D} = \langle \mathrm{D}, \wedge, \vee, \cdot, \backslash, /, \mathbf{0}, \mathbf{1}\rangle$ as follows;

Let $\mathbf{C_3}$ be the three element MV-algebra with the universe $\{0, \frac{1}{2}, 1\}$. Take the direct product $\mathbf{A} \times \mathbf{B} \times \mathbf{C_3}$. Consider a subset $\mathrm{D} = \{(a, b, \frac{1}{2})|a \in \mathrm{A},\ b \in \mathrm{B}\} \cup \{(a, b, 1)|a \in \mathrm{A},\ b \in \mathrm{B},\ a \geq_\mathbf{A} 1_\mathbf{A},\ b \geq_\mathbf{B} 1_\mathbf{B}\} \cup \{(a, b, 0)|a \in \mathrm{A},\ b \in \mathrm{B},\ a \leq_\mathbf{A} 0_\mathbf{A},\ b \leq_\mathbf{B} 0_\mathbf{B}\}$ of $\mathrm{A} \times \mathrm{B} \times \mathrm{C}_3$. Note that since the $\mathbf{C_3}$ is a residuated lattice, $\cdot$, $\backslash$ and $/$ on $\mathbf{C_3}$ are defined by $x \cot 0 = 0 \cdot x = x$, $x \cdot 1 = 1 \cdot x = x$, $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$, $x \leq y$ implies $x\backslash y = 1$, $1\backslash\frac{1}{2} = \frac{1}{2}$ and $1\backslash 0 = \frac{1}{2}\backslash 0 = 0$.

It is easy to see that the set D is closed under the lattice operations of $\mathbf{A} \times \mathbf{B} \times \mathbf{C_3}$. Thus, $\mathbf{D}$ can be regarded as a sublattice of the lattice reduct of $\mathbf{A} \times \mathbf{B} \times \mathbf{C_3}$.

Define multiplication on D as follows:

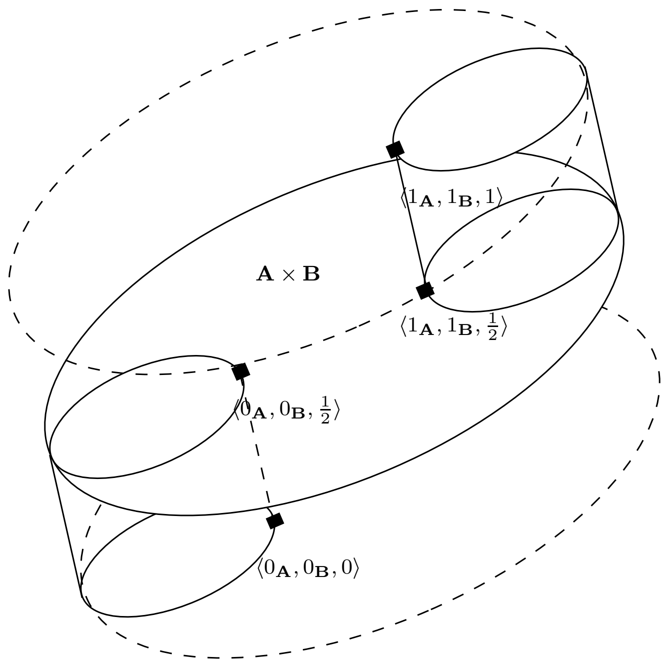

Figure 5.2.

- If $a \cdot_{\mathbf{A}} a' \leq 0_{\mathbf{A}}$, $b \cdot_{\mathbf{B}} b' \leq_{\mathbf{B}} 0_{\mathbf{B}}$ and $i, j \in \{0, \frac{1}{2}\}$ then

$$\langle a, b, i\rangle \cdot \langle a', b', j\rangle = \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', 0\rangle.$$

- Otherwise

$$\langle a, b, i\rangle \cdot \langle a', b', j\rangle = \begin{cases} \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', i \cdot j\rangle & \text{if } \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', i \cdot j\rangle \in \mathrm{D} \\ \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', \frac{1}{2}\rangle & \text{if } \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', i \cdot j\rangle \notin \mathrm{D} \end{cases}$$

Next define residuals on D as follows:

- If $i = \frac{1}{2}$ and $\langle a', b', j\rangle \leq \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle$ then

$$\langle a, b, i\rangle \backslash \langle a', b', j\rangle = \langle a \backslash_{\mathbf{A}} a', b \backslash_{\mathbf{B}} b', \tfrac{1}{2}\rangle.$$

- Otherwise,

$$\langle a, b, i\rangle \backslash \langle a', b', j\rangle = \begin{cases} \langle a \backslash_{\mathbf{A}} a', b \backslash_{\mathbf{B}} b', i \backslash j\rangle & \text{if } \langle a \backslash_{\mathbf{A}} a', b \backslash_{\mathbf{B}} b', i \backslash j\rangle \in \mathrm{D} \\ \langle a \backslash_{\mathbf{A}} a', b \backslash_{\mathbf{B}} b', \frac{1}{2}\rangle & \text{if } \langle a \backslash_{\mathbf{A}} a', b \backslash_{\mathbf{B}} b', i \backslash j\rangle \notin \mathrm{D} \end{cases}$$

Similarly for the left residuals.

Lemma 5.4.2 *The tuple* $\langle \mathrm{D}, \cdot, \mathbf{1}\rangle$ *is a monoid.*

Lemma 5.4.3 *The algebra* $\mathbf{D}$ *satisfies the law of residuation.*

Lemma 5.4.2 and 5.4.3 can be shown by long, tedious calculations. So, we put them in the Appendix of our thesis.

Definition 5.4.1 For any $\langle a, b, i\rangle \in \mathrm{D}$ we define unary operations $\sim, -$ by following;

$$\sim \langle a, b, 1\rangle = \langle \sim_{\mathbf{A}} a, \sim_{\mathbf{B}} b, 0\rangle$$
$$\sim \langle a, b, 0\rangle = \langle \sim_{\mathbf{A}} a, \sim_{\mathbf{B}} b, 1\rangle$$
$$\sim \langle a, b, \tfrac{1}{2}\rangle = \langle \sim_{\mathbf{A}} a, \sim_{\mathbf{B}} b, \tfrac{1}{2}\rangle.$$

Note that if $a \geq_{\mathbf{A}} 1_{\mathbf{A}}$ then $\sim_{\mathbf{A}} a \leq_{\mathbf{A}} 0_{\mathbf{A}}$ and if $a \leq_{\mathbf{A}} 0_{\mathbf{A}}$ then $\sim_{\mathbf{A}} a \geq_{\mathbf{A}} 1_{\mathbf{A}}$. Similarly we define $-$.

Lemma 5.4.4 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *satisfy the condition* DN *then so does* $\mathbf{D}$.

Proof For all $\langle a, b, i\rangle \in \mathrm{D}$,

$$\begin{aligned} \sim -\langle a, b, i\rangle \backslash \langle a, b, i\rangle &= \langle \sim_{\mathbf{A}} -_{\mathbf{A}} a, \sim_{\mathbf{B}} -_{\mathbf{B}} b, i\rangle \backslash \langle a, b, i\rangle \\ &= \langle \sim_{\mathbf{A}} -_{\mathbf{A}} a \backslash_{\mathbf{A}} a, \sim_{\mathbf{B}} -_{\mathbf{B}} b \backslash b, i \backslash i\rangle \\ &\geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle. \end{aligned}$$

Similarly we can easily show $- \sim \langle a, b, i\rangle \backslash \langle a, b, i\rangle \geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$. So $(\sim -\langle a, b, i\rangle \backslash \langle a, b, i\rangle) \wedge (- \sim \langle a, b, i\rangle \backslash \langle a, b, i\rangle) \geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$. Thus the algebra $\mathbf{D}$ satisfies DN.

□

Lemma 5.4.5 *The residuated lattice* $\mathbf{D}$ *is well-connected*

Proof Suppose that $\langle a, b, i\rangle, \langle a', b', j\rangle \in \mathrm{D}$. If $\langle a, b, i\rangle \not\geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$ and $\langle a', b', j\rangle \not\geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$ then $i = j = \frac{1}{2}$.

$$\begin{aligned} & \langle a, b, i\rangle \vee \langle a', b', j\rangle \\ = & \langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b', i \vee j\rangle \\ = & \langle a \vee_{\mathbf{A}} a', b \vee_{\mathbf{B}} b', \frac{1}{2}\rangle \\ \not\geq & \mathbf{1}. \end{aligned}$$

Thus $\mathbf{D}$ is well-connected.

□

Lemma 5.4.6 *A mapping* α *from* $\mathbf{D}$ *to* $\mathbf{A} \times \mathbf{B}$ *defined by*

$$\alpha(\langle a, b, i\rangle) = \langle a, b\rangle.$$

is a surjective homomorphism.

Proof We can easily show that

$$\begin{aligned}\alpha(\langle a,b,i\rangle \oplus \langle a',b',j\rangle) &= \alpha(\langle a \oplus_{\mathbf{A}} a', b \oplus_{\mathbf{B}} b', k\rangle) \\ &= \langle a \oplus_{\mathbf{A}} a', b \oplus_{\mathbf{B}} b'\rangle \\ &= \langle a,b,i\rangle \oplus_{\mathbf{A}\times\mathbf{B}} \langle a',b',j\rangle \\ &= \alpha(\langle a,b,i\rangle) \oplus_{\mathbf{A}\times\mathbf{B}} \alpha(\langle a',b',j\rangle)\end{aligned}$$

(for $\oplus \in \{\wedge, \vee, \cdot, \backslash, /\}$). Hence α is a homomorphism. The mapping α is clearly surjective.

□

Now, by Theorem 5.2.2, $\mathbf{FL}[\mathrm{DN}]$ has the DP. It is clear that if both $\mathbf{A}$ and $\mathbf{B}$ are commutative then so is $\mathbf{D}$. Thus, the DP of $\mathbf{FL_e}[\mathrm{DN}]$ follows also. Hence we have Theorem 5.4.1.

In the proof of Theorem 5.4.1, suppose moreover that both $\mathbf{A}$ and $\mathbf{B}$ satisfy the formula Dis. It means that both $\mathbf{A}$ and $\mathbf{B}$ are distributive. Since $\mathbf{C_3}$ is distributive, the product $\mathbf{A}\times\mathbf{B}\times\mathbf{C_3}$ is also distributive. The lattice $\langle \mathrm{D}, \wedge, \vee\rangle$ is a sublattice of $\mathbf{A}\times\mathbf{B}\times\mathbf{C_3}$. Thus, $\mathbf{D}$ is distributive. Hence the following corollary holds.

Corollary 5.4.7 *Both* $\mathbf{FL}[\mathrm{Dis}, \mathrm{DN}]$ *and* $\mathbf{FL_e}[\mathrm{Dis}, \mathrm{DN}]$ *have the disjunction property.*

Note that $\mathbf{FL_e}[\mathrm{Dis}, \mathrm{DN}]$ is equivalent to the contraction-less relevant logic $\mathbf{RW}$, whose disjunction property is shown in [18] by using *metavaluations*.

We can show the following by extending Theorems 5.3.1 and 5.4.1 when we have weakening rules, i.e. when we assume $x \le 1$ and $0 \le x$ for any x in algebras.

Corollary 5.4.8 $\mathbf{FL_{ew}}[\mathrm{E_n^m}]$, $\mathbf{FL_{ew}}[\mathrm{DN}]$, $\mathbf{FL_{ew}}[\mathrm{E_n^m}, \mathrm{Dis}]$ *and* $\mathbf{FL_{ew}}[\mathrm{DN}, \mathrm{Dis}]$ *have the disjunction property.*

Since $\mathbf{FL_{ew}}[\mathrm{E_2^1}]$ is equal to intuitionistic logic $\mathbf{Int}$, the above corollary also covers the DP of $\mathbf{Int}$.

On the other hand, these proofs cannot always be combined together. That is, the argument does not work well for $\mathbf{FL_x}[\mathrm{E_n^m}, \mathrm{DN}]$, where $\mathbf{x}$ is either empty or $\mathbf{e}$ or $\mathbf{ew}$. In fact, the DP does not hold for cases like $\mathbf{FL_{ew}}[\mathrm{E_2^1}, \mathrm{DN}]$, since the latter is equal to classical logic. Note that in the proof of Theorem 5.4.1, $\mathbf{D}$ is not always a Boolean algebra even if both $\mathbf{A}$ and $\mathbf{B}$ are Boolean algebras.

Our proof works well for $\mathbf{FL_{ew}}$, $\mathbf{FL_{ew}}[\mathrm{E_k}]$ and $\mathbf{FL_{ew}}[\mathrm{DN}]$ as we have mentioned in the above [19]. In these cases, $1_{\mathbf{A}}$ and $1_{\mathbf{B}}$ are the greatest, $0_{\mathbf{A}}$ and $0_{\mathbf{B}}$ are the least elements of $\mathbf{A}$ and $\mathbf{B}$, respectively. In such a case residuated lattices shown in Figure 5.3 become the following.

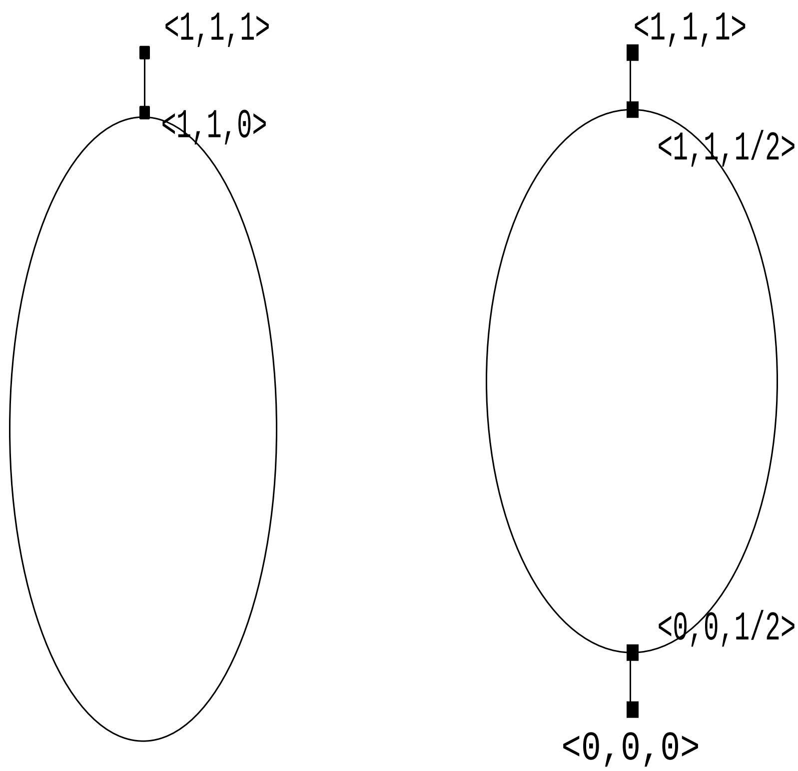

Figure 5.3.

Chapter 6

Minimal subvarieties of $\mathcal{RL}$

In this chapter we discuss about minimal subvarieties of the subvariety lattice of residuated lattice. As we mentioned before (Theorem 4.5.3), the lattice of substructural logics is dually isomorphic to the lattice of $\mathbf{FL}$-algebras. So the number of minimal subvarieties corresponds to the number of maximal consistent logics. The maximal consistent logic over intuitionistic logic $\mathbf{Int}$ (even over $\mathbf{FL_{ew}}$) is only classical logic $\mathbf{Cl}$. The goal of this chapter is to show that there exist uncountably many minimal subvarieties of bounded involutive representable residuated lattices with mingle axiom, but there are only two minimal subvarieties of bounded involutive representable residuated lattice with idempotent axiom.

6.1 General facts about minimal subvarieteis

A non-trivial variety $\mathcal{V}$ is called *minimal* iff the trivial variety is only one proper subvariety of $\mathcal{V}$.

A non-trivial algebra $\mathbf{A}$ is a *strictly simple*, if it has neither non-trivial proper subalgebra nor non-trivial congruences. Note that for infinite algebras, the notion of proper subalgebras is defined in such a way that a subalgebra $\mathbf{B}$ of $\mathbf{A}$ is *proper* if $\mathbf{B}$ is not isomorphic to $\mathbf{A}$. The fact that an algebra has no non-trivial proper subalgebra is enough to establish strict simplicity for RL but not in general for FL. For, congruences on residuated lattices correspond to convex normal subalgebras and thus the lack of non-trivial proper subalgebras is enough to establish strict simplicity.

The element $\bot \in \mathbf{A}$ is a *nearly term-definable lower bound*, if $\bot$ is the bottom element of $\mathbf{A}$ and there is an n-ary term-operation $t(\bar{x})$ on A such that for any $\bar{x} \neq (\underbrace{1, \dots, 1}_{n\text{-times}})$, $t(\bar{x}) = \bot$ holds.

We write the variety of InRL with mingle axiom: $x^2 \leq x$ by $\mathcal{InRL}\text{-}mingle$. The following result was proved in [5].

Lemma 6.1.1 *Let $\mathbf{A}$ be a strictly simple algebra with the bottom element $\bot$ nearly term definable by an n-ary term t. Then, $V(\mathbf{A})$ is a minimal subvariety.*

Proof Let $\mathcal{V}$ be a variety generated by $\mathbf{A}$, i.e. $\mathcal{V} = V(\mathbf{A})$. By Jónsson's Lemma (see 3.4.4), for congruence distributive varieties, the subdirect irreducible algebras of $\mathcal{V}$ are contained in $\mathsf{HSP_U}(\mathbf{A})$. Therefore if $\mathbf{D}$ is subdirect irreducible algebra of $\mathcal{V}$ then there exist an ultrapower

$\mathbf{B} = \mathbf{A}^I/U$ and a non-trivial subalgebra $\mathbf{C}$ of $\mathbf{B}$ such that $\mathbf{D} = f(\mathbf{C})$ for some homomorphism f. Since $\mathbf{A}$ is strictly simple, $\mathbf{A}$ is generated by $\bot$. Note that $\mathbf{A}$ satisfies

$$(\forall x_1, \dots x_n)((x_1 \neq 1 \text{ or } \dots \text{ or } x_n \neq 1) \text{ implies } (t(x_1, \dots, x_n) = \bot)). \ (*)$$

Therefore $\mathbf{B}$ satisfies $(*)$, with $\bot$ being the element $(\bot : i \in I)/U$ by properties of ultraproducts. Since $(*)$ is a universal formula, for any subalgebra $\mathbf{C}'$ of $\mathbf{B}$ satisfies $(*)$. Hence any non-trivial subalgebra $\mathbf{E}$ of $\mathbf{B}$ contains $a \neq 1$. Then $\mathbf{E}$ satisfies $t(a, \dots, a) = \bot$. Thus $\mathbf{E}$ contains $\bot$. Since $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ generated by $\bot$, every non-trivial subalgebra of $\mathbf{B}$ contains $\mathbf{A}$ as a subalgebra. In particular we take $\mathbf{C}$ as such subalgebra then $\mathbf{C}$ contains $\mathbf{A}$ as a subalgebra.

Suppose that $f(u) = f(v)$ for distinct elements $u, v \in \mathrm{A} \subseteq \mathrm{C}$. For any $x, y \in \mathrm{A}$, if we define $x \sim y$ by $f(x) = f(y)$ then $\sim$ is a congruence. Since $\mathbf{A}$ is simple, congruences on $\mathbf{A}$ is only identity relation Δ and full relation ∇. $\sim$ is not the identity relation since $u \neq v$. Thus for any $x, y \in \mathrm{A}$, $x \sim y$, i.e. $f(x) = f(y)$. In particular $f(\bot) = f(1) = 1$. But $\bot$ is the bottom element of $\mathbf{C}$. Let $\bot \sim 1$. Then $\bot = \bot \cdot x \sim 1 \cdot x = x$. Since $x, y \in \mathrm{C}$ $x \sim 1 \sim y$, $\mathrm{Cg}^{\mathbf{C}}(u, v) = \mathrm{Cg}^{\mathbf{C}}(\bot, 1)$ is the full congruence. So $f(\mathrm{C}) = \mathrm{D}$ is a singleton. It contradicts that $\mathbf{D}$ is a subdirect irreducible algebra. Therefore f is injective on $\mathbf{A}$ and $f(\mathbf{A})$ is a subalgebra of $\mathbf{D}$. Thus $\mathbf{A}$ is isomorphic to a subalgebra of every subdirect irreducible member of $\mathcal{V}$. Hence $\mathcal{V}$ is a minimal.

□

6.2 Representable minimal subvarieties

Our results shown in Section 6.4 are the number of minimal subvarieties of some classes of involutive representable residuated lattices. In the present section, we discuss two important results on minimal subvarieties of some classes of representable residuated lattices.

6.2.1 Bounded representable 3-potent minimal subvarieties

Here we show that there exists uncountable many minimal subvarieties of bounded representable 3-potent residuated lattices, where 3-potent axiom is $x^3 = x^4$, shown by P. Jipsen and C. Tsinakis, in [11]. To prove this we construct uncountably many strictly simple residuated lattices and show that they generate distinct varieties.

Let S be any subset of ω. Define the algebra $\mathbf{J}_S = \langle \mathrm{J}_S, \vee, \wedge, \cdot, \backslash, /, 1, \bot, \top \rangle$. The universe of $\mathbf{J}_S$ is the set

$$\{\bot, a, b, 1, \top\} \cup \{c_i | i \in \omega\} \cup \{d_i | i \in \omega\}.$$

The order is defined by

$$\bot \leq a \leq b \leq c_0 \leq c_1 \leq \cdots \leq \cdots \leq d_1 \leq d_0 \leq e \leq \top.$$

The monoid operation is defined by (1) $ex = x = xe$, (2) $\bot x = \bot = x\bot$, (3) $\top x = x = x\top$ and $ax = \bot = xa$ and $bx = \bot = xb$ for any $x \notin \{1, \top\}$. Moreover, for any $i, j \in \omega$,

$$c_i c_j = \bot$$
$$d_i d_j = b$$

$$c_i d_j = \begin{cases} \bot & \text{if } i < j \\ a & \text{if } i = j \text{ or } i = j+1 \text{ and } j \in S \\ b & \text{otherwise,} \end{cases}$$

$$d_i c_j = \begin{cases} \bot & \text{if } i \geq j \\ b & \text{otherwise.} \end{cases}$$

The following table is monoid operation table for $\mathbf{J}_S$. The element s_i in the table are equal to a (if $i \in S$) or b (if $i \notin S$).

$\cdot$	$\top$	1	d_0	d_1	d_2	$\cdots$	$\cdots$	c_2	c_1	c_0	b	a	$\bot$
$\top$	$\top$	$\top$	d_0	d_1	d_2	$\cdots$	$\cdots$	c_2	c_1	c_0	b	a	$\bot$
1	$\top$	1	d_0	d_1	d_2	$\cdots$	$\cdots$	c_2	c_1	c_0	b	a	$\bot$
d_0	d_0	d_0	b	b	b	$\cdots$	$\cdots$	b	b	$\bot$	$\bot$	$\bot$	$\bot$
d_1	d_1	d_1	b	b	b	$\cdots$	$\cdots$	b	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
d_2	d_2	d_2	b	b	b	$\cdots$	$\cdots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_2	c_2	c_2	b	s_1	a	$\cdots$	$\cdots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
c_1	c_1	c_1	s_0	a	$\bot$	$\cdots$	$\cdots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
c_0	c_0	c_0	a	$\bot$	$\bot$	$\cdots$	$\cdots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	b	b	$\bot$	$\bot$	$\bot$	$\cdots$	$\cdots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	a	$\bot$	$\bot$	$\bot$	$\cdots$	$\cdots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\cdots$	$\cdots$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

It is easy to show that $xyz = \bot$ whenever $1, \top \notin \{x, y, z\}$ and $\mathbf{J}_S$ satisfies residuation law. Thus $\mathbf{J}_S$ is bounded 3-potent representable residuated lattice.

Now $\top = \bot\backslash\bot$, $d_0 = \top\backslash 1$, $c_i = d_i\backslash\bot$ and $d_{i+1} = c_i\backslash\bot$, so the algebra $\mathbf{J}_S$ generated by $\bot$ and $\bot$ is nearly term-definable by $t(x) = (x\backslash 1 \wedge x)^3$. Hence $\mathbf{J}_S$ is strictly simple residuated lattice with nearly term definable lower bound. Moreover for any distinct set $S_1, S_2 \in \omega$, $\mathbf{J}_{S_1}$ and $\mathbf{J}_{S_2}$ generates distinct varieties. Then the following theorem holds.

Theorem 6.2.1 *There are uncountably many minimal subvarieties of bounded 3-potent representable residuated lattices.*

6.2.2 Representable idempotent minimal subvarieties

Next we explain the result that there exists uncountably many representable idempotent minimal subvarieties, shown by N. Galatos, in [5, 4].

Define idempotent representable RL $\mathbf{N}_S = \langle \mathrm{N}, \wedge, \vee, \cdot_{\mathrm{S}}, \backslash, /, 1\rangle$. Let us define a set N.

$$\mathrm{N} = \{a_i | i \in \mathbb{Z}\} \cup \{b_i | i \in \mathbb{Z}\} \cup \{1\}.$$

We define order on N as follows;

$$b_i \leq b_j \leq 1 \leq a_k \leq a_l \Leftrightarrow \text{for all } i, j, k, l \in \mathbb{Z},\ i \leq j \text{ and } k \geq l.$$

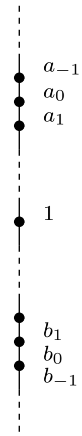

Figure 6.1.

Obviously, this is a total order. Let $S \subseteq \mathbb{Z}$. We define multiplication on $\mathbf{N}$ depending of S, by

$$a_i \cdot_{\mathrm{S}} a_j = a_{min\{i,j\}}$$

$$b_i \cdot_{\mathrm{S}} b_j = b_{min\{i,j\}}$$

$$b_j \cdot_{\mathrm{S}} a_i = \begin{cases} b_j & \text{if } j < i \text{ or } i = j \in S \\ a_i & \text{if } i < j \text{ or } i = j \notin S \end{cases}$$

$$a_i \cdot_{\mathrm{S}} b_j = \begin{cases} a_i & \text{if } i < j \text{ or } i = j \in S \\ b_j & \text{if } j < i \text{ or } i = j \notin S \end{cases}$$

Finally, we define two division operations, by

$$x \backslash y = \bigvee \{z | x \cdot_{\mathrm{S}} z \leq y\}$$
$$y / x = \bigvee \{z | z \cdot_{\mathrm{S}} x \leq y\}$$

It is easy to see that multiplication is associative and residuated by the division operations. So $\mathbf{N}_S$ is a bounded RL (a_1 is the top element and b_0 is the bottom element). Moreover it satisfies idempotent axiom as $x \cdot_{\mathrm{S}} x = x$.

We define following terms

$l(x) = x \backslash 1$, $r(x) = 1/x$,
$t(x) = 1/x \vee x \backslash 1$,
$m(x) = ll(x) \wedge lr(x) \wedge rl(x) \wedge rr(x)$,
$p(x) = ll(x) \vee lr(x) \vee rl(x) \vee rr(x)$

Moreover, consider binary relations defined by,

$x \xrightarrow{r} y \Leftrightarrow r(x) = y$,
$x \xrightarrow[l]{} y \Leftrightarrow l(x) = y$,
$x \rightarrow y \Leftrightarrow r(x) = y$ or $l(x) = y$.

A *word* over $\{0,1\}$ is a function $w : \mathrm{A} \to \{0,1\}$, where A is a subinterval of $\mathbb{Z}$. A is called *support*, $supp(w)$, of w. If $|\mathrm{A}| < \mathbb{N}$ ($|\mathrm{A}| = \mathbb{N}^+$, $|\mathrm{A}| = \mathbb{Z}$) then we call w is *finite* (*infinite*, *bi-infinite*, respectively). Let w is a word and v is a finite word. If there exists an integer k such that $v(i) = w(i+k)$ for any $i \in supp(v)$ then we say that v is a *subword* of w. Note that w_S of $S \subseteq \mathbb{Z}$ is a bi-infinite word where $w_S(i) = 1 \Leftrightarrow i \in S$. Define preorder $\leq$ by for any word w_1, w_2 every finite subword of w_1 is a subword of w_2. Define $w_1 \cong w_2$ by $w_1 \leq w_2 \leq w_1$. We say that a bi-infinite word w is minimal, if $w' \leq w \Leftarrow w \cong w'$ for every bi-infinite word w' .

In the following $x \prec y$ means that $x < y$ and $x \leq z \leq y \Rightarrow z = x$ or $z = y$.

Lemma 6.2.2 *For any $S \subseteq \mathbb{Z}$ the following properties hold for $\mathbf{N}_S$.*

1. *For any $i \in \mathbb{Z}$, $m(b_i) = b_{i-1}$, $p(b_i) = b_{i+1}$, $m(a_i) = a_{i+1}$, $p(a_i) = a_{i-1}$, $t(a_i) = b_i$ and $t(b_i) = a_i$.*
2. *For any x, y, $x \leq y$ or $y \leq x$.*
3. *For every x, $\{xt(x), t(x)x\} = \{x, t(x)\}$.*
4. *If $x < 1 < y$, then $m(x) \prec x \prec p(x) < 1 < m(y) \prec y \prec p(y)$ and $t(y) < 1 < t(x)$.*
5. *For every x, $m(t(x)) = t(p(x))$, $p(t(x)) = t(m(x))$, $m(p(x)) = p(m(x)) = x$ and $t(t(x)) = x$.*
6. *If x is negative, then*

$$xy = yx = \begin{cases} x & \text{for } x \leq y < t(x) \\ y & \text{for } y \leq x \text{ or } t(x) < y \end{cases}$$

   If x is positive, then

$$xy = yx = \begin{cases} x & \text{for } t(x) < y \leq x \\ y & \text{for } y < t(x) \text{ or } x \leq y \end{cases}$$

7. *For any x, y, $x \wedge y, x \vee y, xy \in \{x, y\}$.*
8. *For any x, y, $x/y, y\backslash x \in \{x, m(x), p(x), t(x), m(t(x)), t(y), m(t(y)), p(t(y))\}$.*
9. *For every finite word v there exists a universal first order formula φ_v, such that v is not a subword of w_S iff φ_v is satisfied in $\mathbf{N}_S$.*

Proof It is easy to see that

$$b_{i-1} \underset{l}{\leftarrow} a_i \overset{\overset{r}{\rightarrow}}{\underset{l}{\leftarrow}} b_i \overset{r}{\rightarrow} a_{i+1} \ (i \in S)$$
$$b_{i-1} \overset{r}{\leftarrow} a_i \overset{\overset{r}{\leftarrow}}{\underset{l}{\rightarrow}} b_i \underset{l}{\rightarrow} a_{i+1} \ (i \notin S)$$

Thus we can show that $t(b_i) = a_i$ and $t(a_i) = b_i$. Moreover,

$$\{r(r(b_i)), r(l(b_i)), l(r(b_i)), l(l(b_i))\} = \{b_{i-1}, b_i, b_{i+1}\},$$

so $m(b_i) = b_{i-1}$ and $p(b_i) = b_{i+1}$. Similarly, we can show $m(a_i) = a_{i+1}$ and $p(a_i) = a_{i-1}$. Thus 1 holds. 2 is shown by the definition. 3-7 are shown by 1. 8 is routine to check. Finally we show 9. The first order formula associated to a finite word v is φ_v defined as

$$\forall x_1, \dots, x_n, y_1, \dots, y_n(x_1 \prec \cdots \prec x_n < 1 < y_n \prec y_1 \text{ and}$$
$$t(x_1) = y_1 \text{ and } \dots \text{ and } t(x_n) = y_n)$$
$$\Rightarrow \text{ not } (x_1 y_1 = s_1) \text{ and } \dots \text{ and } (x_n y_n = s_n),$$

where n is the length of v, if $v(i) = 1$ then $s_i = x_i$ and if $v(i) = 0$ then $s_i = y_i$. Since we can replace $x_i \prec x_{i+1}$ by $x_i = m(x_{i+1})$ and $y_{i+1} \prec y_i$ by $y_i = p(y_{i+1})$, φ_v is equivalent to a universal quantified first-order formula in the language of residuated lattices.

□

For any $a, b \in \mathrm{N}_S \backslash \{1\}$, (a, b) is transitive closure of the relation $\rightarrow$. So the following corollary holds.

Corollary 6.2.3 *The residuated lattice $\mathbf{N}_S$ is strictly simple.*

Lemma 6.2.4 *Every non-trivial one-generated subalgebra of an ultrapower of $\mathbf{N}_S$ is isomorphic to $\mathbf{N}_{S'}$, for some set of integers S'.*

Proof Every first order formula which is true in $\mathbf{N}_S$ is also true in an ultrapower of it. Since properties 2-8 of lemma 6.2.2 can be expressed as first order formula, they hold in ultrapower of $\mathbf{N}_S$.

By 2, any ultrapower $\mathbf{B}$ of $\mathbf{N}_S$ is totally ordered, so same holds for every subalgebra of $\mathbf{B}$. Let $\mathbf{A}$ be a non-trivial one-generate subalgebra of $\mathbf{B}$ and a be a generator for $\mathbf{A}$. The element a can be taken to be negative. If a is positive then $t(a)$ is negative and it generates $\mathbf{A}$, because $t(t(a)) = a$.

By 7 and 8, A is the set of evaluations of the terms composed by the term m, p, t and 1. By 5, these compositions reduce to one of the forms $m^n(x)$, $p^n(x)$, $p^n(t(x))$ and $m^n(t(x))$ for n a natural number.

For any natural number n, set $b_{-n} = m^n(a)$, $b_n = p^n(a)$, $a_{-n} = p^n(t(a))$ and $a_n = m^n(t(a))$. A consists of exactly these elements together with 1. Define a subset S' of $\mathbb{Z}$ by $S' = \{m | b_m a_m = b_m\}$, and the map $f : \mathrm{A} \rightarrow \mathrm{N}_{S'}$ by $f(b_i) = b'_i$, $f(a_i) = a'_i$ and $f(1) = 1'$ for $b'_i, a'_i, 1 \in \mathrm{N}_{S'}$.

By 4, f is an order isomorphism and consequently, a lattice isomorphism. By,3 and 6, f is monoid homomorphism. Every lattice isomorphisms preserve existing join, so f preserve division operation. Thus $\mathbf{A}$ is isomorphic to $\mathbf{N}_{S'}$.

□

Lemma 6.2.5 *Let $\mathbf{A}$ be a one-generated residuated lattice and S a subset of $\mathbb{Z}$. Then $\mathbf{A} \in \mathsf{HSP_U}(\mathbf{N}_S) \Leftrightarrow \mathbf{A} \cong \mathbf{N}_{S'}$, for some S' such that $w_{S'} \leq w_S$.*

Proof First we show if part. Let $\mathbf{B} = (\mathbf{N}_S)^{\mathbb{N}}/U$ where U is a non principal ultrafilter over $\mathbb{N}$ and $\mathrm{N}_S = \{b_i | i \in \mathbb{Z}\} \cup \{a_i | i \in \mathbb{Z}\} \cup \{1\}$. We will show that $\mathbf{N}_{S'} \in \mathsf{ISP_U}(\mathbf{N}_S)$.

For any natural number n v_n is a finite approximation of the bi-infinite word $w_{S'}$ defined by $v_n(i) = w_{S'}(i)$ for any $i \in [-n, n]_{\mathbb{Z}}$. Since $w_{S'} \leq w_S$, the words v_n are subwords of w_S. So for any $n \in \mathbb{N}$ there exists $K_n \in \mathbb{N}$ such that $v_n(i) = w_S(K_n + i)$ for any $i \in supp(v_n) = [-n, n]_{\mathbb{Z}}$.

Let $\bar{b} = (b_{K_n})_{n\in\mathbb{N}}$ where $b_{K_n} \in \mathrm{N}_S$. By lemma 6.2.5 the subalgebra of $\mathbf{B}$ generated by $\tilde{b} = [\bar{b}]$ is isomorphic to $\mathbf{N}_{\tilde{S}}$ with $\mathrm{N}_{\tilde{S}} = \{\tilde{b}_i | i \in \mathbb{Z}\} \cup \{\tilde{a}_i | i \in \mathbb{Z}\} \cup \{\tilde{1}\}$ for some subset $\tilde{S}$ of $\mathbb{Z}$. $[\bar{b}]$ is equivalence class of $\bar{b}$ under U. We identify the subalgebra generated by $\tilde{b}$ with $\mathbf{N}_{\tilde{S}}$ and we choose $\tilde{S}$ such that $\tilde{b}_0 = \tilde{b}$. We will show that $\tilde{S} = S'$.

There exists $\bar{b}_{mn}, \bar{a}_{mn} \in \mathbf{N}_S$ such that $\tilde{b}_m = [(b_{K_n+m})_{n\in\mathbb{N}}]$ and $\tilde{a}_m = [(a_{K_n+m})_{n\in\mathbb{N}}]$. By using the definition of $\tilde{b}$, lemma 6.2.2, induction and following facts

$$\begin{aligned}
&\tilde{a}_m = t(\tilde{b}_m) = t([(\bar{b}_{mn})_{n\in\mathbb{N}}]) = [(t(\tilde{b}_{mn}))_{n\in\mathbb{N}}]\\
&\tilde{b}_{m+1} = p(\tilde{b}_m) = p([(\bar{b}_{mn})_{n\in\mathbb{N}}]) = [(p(\tilde{b}_{mn}))_{n\in\mathbb{N}}]\\
&\tilde{b}_{m-1} = m(\tilde{b}_m) = m([(\bar{b}_{mn})_{n\in\mathbb{N}}]) = [(m(\tilde{b}_{mn}))_{n\in\mathbb{N}}]
\end{aligned}$$

it is easy to prove that $\tilde{b}_m = [(b_{k_n+m})_{n\in\mathbb{N}}]$ and $\tilde{a}_m = [(a_{k_n+m})_{n\in\mathbb{N}}]$.

Now, for $|m| < n$, i.e., $m \in supp(v_n)$, we have

$$\begin{aligned}
K_n + m \in S &\iff w_s(K_n + m) = 1\\
&\iff v_n(m) = 1\\
&\iff w_{S'}(m) = 1\\
&\iff m \in S'
\end{aligned}$$

When $K_n+m \in S$, $b_{K_n+m}a_{K_n+m} = b_{K_n+m}$ exactly. We get that if $|m| < n$ then $b_{K_n+m}a_{K_n+m} = b_{K_n+m}$ is equivalent to $m \in S'$.

In other words, $\{n||m| < n\} \subseteq \{n|b_{K_n+m}a_{K_n+m} = b_{K_n+m} \Leftrightarrow m \in S'\}$. Since $\{n||m| < n\} \in U$, $\{n|b_{K_n+m}a_{K_n+m} = b_{K_n+m} \Leftrightarrow m \in S'\} \in U$. It is not hard to check that $\{n|b_{K_n+m}a_{K_n+m} = b_{K_n+m}\} \in U$ is equivalent to $m \in S'$. $\tilde{b}_m\tilde{a}_m = \tilde{b}_m$ is equivalent to $m \in S'$, hence $m \in \tilde{S} \Leftrightarrow m \in S'$. Thus $\tilde{S} = S'$.

Next we show only if part. We will prove the implication for $\mathbf{A} \in \mathsf{SP_U}(\mathbf{N}_S)$. Since under homomorphism every one-generated subalgebra will either map isomorphically or to the identity element because of the strictly simple nature of the algebras $\mathbf{N}_{S'}$, it is sufficient.

Let $\mathbf{A}$ be a subalgebra of an ultrapower of $\mathbf{N}_S$. By lemma 6.2.5, $\mathbf{A}$ is isomorphic to $\mathbf{n}_{S'}$, for some subset S' of $\mathbb{Z}$. To show $w_{S'} \le w_S$ it is suffices to show that for any finite word v if v is not a subword of w_S then $\mathbf{N}_S$ satisfies φ_v of lemma 6.2.2. Hence every ultrapower of $\mathbf{N}_S$ satisfies φ_v. Since φ_v is universal formula and $\mathbf{N}_{S'}$ is subalgebra of ultrapower of $\mathbf{N}_S$, $\mathbf{N}_{S'}$ satisfies φ_v. Thus by lemma 6.2.2 v is not a subword of $w_{S'}$.

□

Corollary 6.2.6 *Let $S, S' \subseteq \mathbb{Z}$.*

1. *$\mathcal{V}(\mathbf{N}_{S'}) \subseteq \mathcal{V}(\mathbf{N}_S) \Leftrightarrow w_{S'} \le w_S$,*
2. *If w_S is minimal with respect to $\le$, then $\mathcal{V} = \mathcal{V}(\mathbf{N}_S)$ is a minimal subvariety of $\mathcal{RL}$.*

Proof 1.
First we show if part. By lemma 6.2.5 we can show $\mathbf{N}_{S'} \in \mathsf{HSP_U}(\mathbf{N}_S) \subseteq \mathcal{V}(\mathbf{N}_S)$. Thus $\mathcal{V}(\mathbf{N}_{S'}) \subseteq \mathcal{V}(\mathbf{N}_S)$. Next we show only if part. $\mathbf{N}_{S'} \in \mathcal{V}(\mathbf{N}_S)$ since $\mathbf{N}_{S'}$ is subdirect irreducible by lemma 6.2.2. So by Jónsson's lemma $\mathbf{N}_{S'} \in \mathsf{HSP_U}(\mathbf{N}_S)$. Thus $w_{S'} \le w_S$ by lemma 6.2.5.

2.
Let $\mathbf{L}$ be a subdirect irreducible algebra from $\mathcal{V}$. then $\mathbf{L} \in \mathsf{HSP_U}(\mathbf{N}_S)$ by Jónsson's lemma. For any subalgebra $\mathbf{A}$ of $\mathbf{L}$ $\mathbf{A} \in \mathsf{SHSP_U}(\mathbf{N}_S) \subseteq \mathsf{HSP_U}(\mathbf{N}_S)$ so $\mathbf{A}$ is isomorphic to some $\mathbf{N}_{S'}$,

where $w_{S'} \leq w_S$. We have $w_{S'} \cong w_S$ since w_S is minimal. Hence $\mathcal{V}(\mathbf{N}_S) = \mathcal{V}(\mathbf{N}_{S'})$ by 1. Thus $\mathcal{V} = \mathcal{V}(\mathbf{N}_{S'}) = \mathcal{V}(\mathbf{A}) \subseteq \mathcal{V}(\mathbf{L}) \subseteq \mathcal{V}$. Since $\mathcal{V} = \mathcal{V}(\mathbf{L})$ for every subdirect irreducible $\mathbf{L}$ in $\mathcal{V}$, $\mathcal{V}$ is a minimal subvariety.

□

Lemma 6.2.7 *There are uncountably many minimal subvarieties in* $\mathcal{RRL} + (x = x^2)$

6.3 Involutive minimal subvarieties

In the previous section, to show the number of minimal subvarieties we construct strictly simple residuated lattices. In this section we introduce a way of constructing a bounded involutive residuated lattice from a given residuated lattice and we show that residuated lattices thus obtained generate minimal subvarieties shown by C. Tsinakis and A. Wille.

6.3.1 From modules to dualizing RL

Definition 6.3.1 Let $\mathbf{L}$ be a residuated lattice and $\mathbf{M} = \langle \mathrm{M}, \wedge, \vee \bot \rangle$ be a lower bounded lattice. For any $x, y \in \mathrm{M}$ and $a, b \in \mathrm{L}$, a *right module* action of $\mathbf{L}$ into $\mathbf{M}$ is a map $* : \mathrm{M} \times \mathrm{L} \longrightarrow \mathrm{M}$ satisfying the following conditions.

(1) $x * e = x$,

(2) $x * (a * b) = (x * a) * b$,

(3) $x * a \leq y \Leftrightarrow x \leq y/_{*}a$.

We call $\mathbf{M}$ a *right* $\mathbf{L}$*-module*. A *left* $\mathbf{L}$*-module* is defined analogously with the module action on the left. A $\mathbf{L}$*-bimodule* is a left and right $\mathbf{L}$-module which satisfies the following condition, for any $x \in \mathrm{M}$ and $a, b \in \mathrm{L}$.

(4) $(a * x) * b = a * (x * b)$.

Lemma 6.3.1 *Let* $\mathbf{M}$ *be a right* $\mathbf{L}$*-module. Then, for any* $a \in \mathrm{L}$ *and* $x, y \in \mathrm{M}$*, the following conditions hold.*

1. $\bot * a = a * \bot$.
2. $(x \vee y) * a = x * a \vee y * a$.

Proof First we show 1. Since $\bot$ is the least element of $\mathbf{M}$ and $\bot * a, \bot/_{*}a \in \mathrm{M}$, $\bot \leq \bot * a$ and $\bot \leq \bot/_{*}a$. Thus we can show $\bot * a = a * \bot$.
Next we show 2.

$$\begin{aligned} (x \vee y) * a \leq (x \vee y) * a &\iff (x \vee y) \leq ((x \vee y) * a)/_{*}a \\ &\implies x \leq ((x \vee y) * a)/_{*}a \\ &\iff x * a \leq (x \vee y) * a. \end{aligned}$$

We can get $y * a \leq (x \vee y) * a$ by same way. Thus we can show $x * a \vee y * a \leq (x \vee y) * a$. From residuation we can show $x \leq (x * a \vee y * a)/_{*}a$ and $y \leq (x * a \vee y * a)/_{*}a$. Hence we can show $(x \vee y) * a \leq x * a \vee y * a$.

$\square$

The corresponding condition hold for a left L-module.

Lemma 6.3.2 *For any* $\mathbf{L}$*-bimodule* $\mathbf{M}$ *gives rise to a residuated lattice* $\mathbf{L}\diamond\mathbf{M} = \langle \mathrm{L}\times\mathrm{M}, \wedge, \vee, \cdot, \backslash, /, (e, \bot)\rangle$ *defined by follows.*

$$\begin{aligned}
(a,x)\wedge(b,y) &= (a\wedge b, x\wedge y)\\
(a,x)\vee(b,y) &= (a\vee b, x\vee y)\\
(a,x)(b,y) &= (ab, a*y\wedge x*b)\\
(a,x)\backslash(b,y) &= (a\backslash b\wedge x\backslash_* y, a\backslash_* y)\\
(a,x)/(b,y) &= (a/b\wedge x/_* y, x/_* b)
\end{aligned}$$

Proof $\langle \mathrm{L}\diamond\mathrm{M}, \vee, \wedge\rangle$ is clearly a lattice. For any $a\in\mathrm{L}$ and $x\in\mathrm{M}$,

$$\begin{aligned}
&(a,x)(e,\bot) = (ae, a*\bot\vee x*e) = (a,\bot\vee x) = (a,x),\\
&(e,\bot)(a,x) = (ea, e*x\vee\bot*a) = (a, x\vee\bot) = (a,e).
\end{aligned}$$

Hence $(e,\bot)$ is a identity element. For any $a,b,c\in\mathrm{L}$ and $x,y,z\in\mathrm{M}$,

$$\begin{aligned}
((a,x)(b,y))(c,z) &= ((ab)c, ab*z\vee(a*y\vee x*b)*c)\\
&= ((ab)c, a*(b*z)\vee(a*y)*c\vee(x*b)*c)\\
&= (a(bc), a*(b*z)\vee a*(y*c)\vee x*bc)\\
&= (a(bc), a*(b*z\vee y*c)\vee x*bc)\\
&= (a,x)((b,y)(c,z))
\end{aligned}$$

Thus $\langle \mathrm{L}\diamond\mathrm{M}, \cdot, (e,\bot)$ is a monoid.
It remains to prove that the residuation law. For any $a,b,c\in\mathrm{L}$ and $x,y,z\in\mathrm{M}$,

$$\begin{aligned}
(a,x)(b,y)\le(c,z) &\Leftrightarrow ab\le c \text{ and } a*y\vee x*b\le z\\
&\Leftrightarrow ab\le c \text{ and } a\le z/_* y \text{ and } x\le z/_* b\\
&\Leftrightarrow a\le c/b\wedge z/_* y \text{ and } x\le z/_* b\\
&\Leftrightarrow (a,x)\le(c,z)/(b,y)
\end{aligned}$$

$(a,x)(b,y)\le(c,z)\Leftrightarrow(b,y)\le(a,x)\backslash(c,z)$ is obtained by a similar way.
Thus $\mathbf{L}\diamond\mathbf{M}$ is a residuated lattice.

$\square$

By using this lemma, we can easily show the next corollary.

Corollary 6.3.3 *Let* $\mathbf{L}$ *be a upper bounded residuated lattice. Then* $\hat{\mathbf{L}} = \langle \mathrm{L}\times\mathrm{L}, \wedge, \vee, \cdot, \backslash, /, (e, \top)\rangle$ *is a residuated lattice defined by follows.*

$$\begin{aligned}
(a,x)\wedge(b,y) &= (a\wedge b, x\vee y)\\
(a,x)\vee(b,y) &= (a\vee b, x\wedge y)\\
(a,x)(b,y) &= (ab, y/a\wedge b\backslash x)\\
(a,x)\backslash(b,y) &= (a\backslash b\wedge x/y, ya)\\
(a,x)/(b,y) &= (a/b\wedge x\backslash y, bx)
\end{aligned}$$

Corollary 6.3.4 *Maintaining the notation established in Corollary 6.3.3, we have the following:*

1. *The element* $D = (\top, e)$ *is a involutive constant of* $\hat{\mathbf{L}}$*. More specifically, for all* $a, x \in \mathrm{L}$,

   $$(a, x)\backslash(\top, e) = (x, a) = (\top, e)/(a, x).$$

2. $\tilde{\mathbf{L}} = \langle \mathrm{L} \times \mathrm{L}, \wedge, \vee, \cdot, \backslash, /, E, D\rangle$ *is a involutive residuated lattice, where* $E = (e, \top)$, $D = (\top, e)$ *and the order operations are defined as in Corollary 6.3.3.*

3. *Let* $\hat{\mathbf{L}}^{\star} = \langle \hat{\mathrm{L}}^{\star}, \wedge, \vee, \cdot, \backslash^{\star}, /^{\star}, E\rangle$*, where*

   $$\hat{\mathrm{L}}^{\star} = \mathrm{L} \times \{\top\}$$
   $$\mathrm{B}/^{\star}\mathrm{A} = \mathrm{B}/\mathrm{A} \wedge (\top, \top),$$
   $$A\backslash^{\star}B = A\backslash B \wedge (\top, \top).$$

   Then the map $\epsilon : \mathbf{L} \to \hat{\mathbf{L}}^{\star}$*,defined by* $\epsilon(a) = (a, \top)$ *for all* $a \in \mathrm{L}$*, is a residuated lattice isomorphism. Furthermore, it restricts to a residuated lattice isomorphism from* $\mathbf{L}^{-}$ *to* $\hat{\mathbf{L}}^{\star}$.

Proof 1 is shown by following.

$$\begin{aligned} (a, x)\backslash(\top, e) &= (a\backslash\top \wedge x/e, ea) \\ &= (x, a) \\ &= (\top/a \wedge e\backslash x, ae) \\ &= (\top, e)/(a, x) \end{aligned}$$

Thus 2 is follows from 1 and Corollary 6.3.3. To prove 3, note that, for any $a, b \in \mathrm{L}$,

$$\epsilon(a)\epsilon(b) = (a, \top)(b, \top) = (ab, \top/a \wedge b\backslash\top) = (ab, \top),$$

and

$$\epsilon(a)/^{\star}\epsilon(b) = (a, \top)/^{\star}(b, \top) = (a/b \wedge \top\backslash\top, b\top) \wedge (\top, \top) = (a/b, \top).$$

Furthermore,

$$\begin{aligned} \epsilon(a)\backslash^{\star}\epsilon(b) &= (a, \top)\backslash^{\star}(b, \top) = & (a\backslash b, \top), \\ \epsilon(a) \wedge \epsilon(b) &= (a, \top) \wedge (b, \top) = & (a \wedge b, \top), \\ \epsilon(a) \vee \epsilon(b) &= (a, \top) \vee (b, \top) = & (a \vee b, \top), \\ \epsilon(e) &= (e, \top) = & E. \end{aligned}$$

Since ϵ is clearly a bijection, hence $\hat{\mathbf{L}}^{\star}$ is a residuated lattice and ϵ is a residuated lattice isomorphism. Lastly, it is also clear that ϵ restricts to a residuated lattice isomorphism from $\mathbf{L}^{-}$ to $\hat{\mathbf{L}}^{\star}$.

□

6.3.2 Minimal subvarieties of involutive residuated lattices

By using results in 6.3.1 we show that there exists uncountably many involutive minimal subvarieties. The result is shown by C. Tsinakis and A. Wille, in [21].

Let S be a subset of ω, $\mathbf{J}_S$ be a strictly simple residuated lattice constructed in 6.2.1 and $\tilde{\mathbf{J}}_S$ be defined in Corollary 6.3.4.

Let $\mathbf{L}_S$ be the subalgebra of $\tilde{\mathbf{J}}_S$ generated by E and D. Since $D \cdot D = (\top, e/\top \wedge \top\backslash e) = (\top, d_0) \in \mathrm{L}_S$, $\mathbf{L}_S$ has elements other than E and D.

Lemma 6.3.5 $\{(x,\top)|x \in \mathrm{J}_S\backslash\{\top\}\} \subseteq \mathrm{L}_S$ *and* $\{(\top, x)|x \in \mathrm{J}_S\backslash\{\top\}\} \subseteq \mathrm{L}_S$. *Furthermore,* $\{(x,\top)|x \in \mathrm{J}_S\backslash\{\top\}\}$ *is closed under monoid operation.*

Proof First we show $\{(x,\top)|x \in \mathrm{J}_S\backslash\{\top\}\} \subseteq \mathrm{L}_S$ and $\{(\top, x)|x \in \mathrm{J}_S\backslash\{\top\}\} \subseteq \mathrm{L}_S$.

We know that $E, D, (\top, d_0) \in \mathrm{L}_S$.

$D/(\top, d_0) = (D_0, \top) \in \mathrm{L}_S$,
$(d_0, \top)^3 = (\bot, \top) \in \mathrm{L}_S$,
$D/(\bot, \top) = (\top, \bot) \in \mathrm{L}_S$,
$(\bot, \top)/(d_0, \top) = (c_0, d_0) \in \mathrm{L}_S$,
$(c_0, d_0) \wedge (E) = (c_0, \top) \in \mathrm{L}_S$, and
$D/(c_0, \top) = (\top, c_0) \in \mathrm{L}_S$.

For any $i \in \omega$, we can show

$(\bot, \top)/(d_i, \top) \wedge E = (c_i, d_i) \wedge E = (c_i, \top)$, and
$(\bot, \top)/(c_i, \top) \wedge E = (d_{i+1}, c_i) \wedge E = (d_{i+1}, \top)$.

Since $(d_0, \top), (c_0, \top) \in \mathrm{L}_S$, for any $i \in \omega$ we can show $(d_i, \top), (c_i, \top) \in \mathrm{L}_S$ inductively. Furthermore,

$D/(c_i, \top) = (\top, c_i) \in \mathrm{L}_S$ and
$D/(d_i, \top) = (\top, d_i) \in \mathrm{L}_S$.

Moreover,

$(d_0, \top)^2 = (b, \top) \in \mathrm{L}_S$,
$D/(b, \top) = (\top, b) \in \mathrm{L}_S$,
$(c_0, \top)(d_0, \top) = (a, \top) \in \mathrm{L}_S$ and
$D/(a, \top) = (\top, a) \in \mathrm{L}_S$.

Second we show $\{(x,\top)|x \in \mathrm{J}_S\backslash\{\top\}\}$ is closed under monoid operation.
Let $x, y \in \mathrm{J}_S\backslash\{\top\}$. Then $xy \in \mathrm{J}_S\backslash\{\top\}$ and $(x, \top)(y, \top) = (xy, \top/x \wedge y\backslash\top) = (xy, \top)$.

□

Note that $(\bot, \top)$ is the least element and $(\top, \bot)$ is the greatest element of $\mathbf{J}_S$.

Theorem 6.3.6 *There are uncountably many minimal subvarieties of involutive residuated lattices.*

Proof To prove this theorem we show following three conditions.

1. For any $S \subseteq \omega$, $\mathbf{L}_S$ is strictly simple.

2. $(\bot, \top) \in \mathrm{L}_S$ is a nearly term definable.

3. For any pair of distinct sets $S_1, S_2 \subseteq \omega$, $\mathbf{L}_{S_1}$ and $\mathbf{L}_{S_2}$ generate distinct varieties.

First we show (2). Let $(x, y) \in \mathrm{L}_S$. If $(x, y) \not\geq E$ then $(x, y) \wedge E < E$. If $(x, y) \geq E$ then $E/(x, y) \wedge E < E$. Thus $(((x, y) \wedge E) \wedge (E/(x, y) \wedge E))^3 = (\bot, \top)$.

Second we prove (1). Since $\mathbf{L}_S$ is generated by $\{E, D\}$ it has no proper subalgebras.

Finally we show (3). Let S_1 and S_2 be tow distinct subsets of ω. From lemma 6.3.5 we can find constant terms $q_{(c_{i+1},\top)}$, $q_{(d_i,\top)}$, $q_{(a,\top)}$ and $q_{(b,\top)}$ such that

$$f(q_{(c_{i+1},\top)}) = (c_{i+1}, \top)$$
$$f(q_{(d_i,\top)}) = (d_i, \top)$$
$$f(q_{(a,\top)}) = (a, \top)$$
$$f(q_{(b,\top)}) = (b, \top)$$

for any assignment f of $\mathbf{L}_S$. Without loss of generality, we can assume that there exists $i \in \omega$ such that $i \in S_1$ and $i \notin S_2$. By the definition $(c_{i+1}, \top) \cdot_1 (d_i, \top) = (a, \top)$ but $(c_{i+1}, \top) \cdot_1 (d_i, \top) = (b, \top)$. Then

- $\mathbf{L}_S \models q_{(c_{i+1},\top)} \cdot q_{(d_i,\top)} \approx q_{(a,\top)}$ and $\mathbf{L}_S \not\models q_{(c_{i+1},\top)} \cdot q_{(d_i,\top)} \approx q_{(b,\top)}$
- $\mathbf{L}_S \not\models q_{(c_{i+1},\top)} \cdot q_{(d_i,\top)} \approx q_{(a,\top)}$ and $\mathbf{L}_S \models q_{(c_{i+1},\top)} \cdot q_{(d_i,\top)} \approx q_{(b,\top)}$

Hence $\mathbf{L}_{S_1}$ and $\mathbf{L}_{S_2}$ generate distinct varieties.

□

Proposition 6.3.7 *Each subvariety generated by $\mathbf{L}_S$ satisfies the identity $x^4 \approx x^5$.*

Proof For any $x \in \mathrm{J}_S$ with $x < e$, $x^3 = \bot$. Moreover we have $(x, y)^2 = (x^2, y/x \wedge x\backslash y)$ and $(x, y)^4 = (x^4, (y/x \wedge x\backslash y)/x^2 \wedge x^2\backslash(y/x \wedge x\backslash y))$ for any $(x, y) \in \mathrm{L}_S$. If $x \neq e$ then $\top \cdot x = x \cdot \top = x$. It follows that

$$x \neq e \text{ and } x \leq y \Longrightarrow y/x \wedge x\backslash y = \top \tag{6.1}$$

and therefore,

$$x < e \text{ and } x^2 \leq y/x \wedge x\backslash y \Longrightarrow (x, y)^4 = (x, y)^5, \tag{6.2}$$

where $(x, y)^4 = (\bot, \top)$.

(case $x = \top$)
If $y \neq e$ then $(\top, y)^2 = (\top, y)$. If $y = e$ then $(\top, e)^4 = (\top, d_0)^2 = (\top, d_0)$ and $(\top, e)^5 = (\top, d_0)(\top, e) = (\top, e)$.

(case $x = e$)
$(e, y)^2 = (e^2, y/e \wedge e\backslash y) = (e, y)$.

(case $x < e$)
We show $x^2 \leq y/x \wedge x\backslash y$ then by (6.2) $(x, y)^4 = (x, y)^5$ holds. If $x \leq y$ then from (6.1) $x^2 \leq y/x \wedge x\backslash y$.
If $x = d_i$ and $x > y$ then $x^3 = d_i^3 = bd_i = \bot$ and $b \leq y/d_i \wedge d_i\backslash y$.
If $x < d_i$ then $x^2 = \bot$. Thus $x^2 \leq y/x \wedge x\backslash y$.

□

6.4 Main theorems

In the previous sections the number of minimal subvarieties of the following varieties is investigated, and is shown to be uncountably many in all of these cases.

- $\mathcal{RRL}_\bot + Mod(x^3 = x^4)$
- $\mathcal{RRL} + Mod(x^2 = x)$
- $\mathcal{I}n\mathcal{RL}$

It is natural to ask what will happen if these two conditions, i.e., representability and involutiveness, are combined. Main results of this chapter shown in this section answer this question.

First, we show in Theorem 6.4.1 that the number of minimal subvarieties of bounded involutive representable residuated lattices is still uncountably many, even if the mingle axiom $x^2 \leq x$ is assumed. Interestingly enough, if we replace the mingle axiom by the idempotent axiom $x = x^2$, the number becomes only two.

6.4.1 Adding involution (preserving mingle axiom)

To show the first result we construct a strictly simple bounded involutive representable residuated lattice with mingle axiom from a given upper-bounded residuated lattice with mingle axiom. This construction is given by N. Galatos and J. G. Raftery in [9].

Let $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, /, \backslash, 1\rangle$ be an upper-bounded RL. Let $\mathrm{A}^- = \{a^- | a \in \mathrm{A}\}$ be a disjoint copy of A and $\mathrm{A}^* = \mathrm{A} \cup \mathrm{A}^-$.
We extend the lattice order $\leq$ on A to A^* by stipulating that for any $a, b \in \mathrm{A}$,

1. $a^- < b$ and
2. $a^- \leq b^- \leftrightarrow b \leq a$.

Thus, $\langle \mathrm{A}^*, \leq\rangle$ is order-isomorphic to the ordinal sum of the dual poset of $\langle \mathrm{A}, \leq\rangle$ and $\langle \mathrm{A}, \leq\rangle$ itself. Let $\top$ be the greatest element of A. We define $\bot = \top^-$ and $0 = 1^-$. Then $\bot$ is the least element of A^*. For any $x \in \mathrm{A}$, we define $(x^-)' = x$ and $x' = x^-$. Then $'$ becomes a unary operation on A^* and satisfies an equation $x'' \approx x$. From now on we identify $^-$ and $'$.

Next we extend the monoid operation on A to A^* as follows: if $a, b \in \mathrm{A}$ then

1. $a \cdot b' = (b/a)'$, $b' \cdot a = (a\backslash b)'$ and
2. $a' \cdot b' = \bot$.

Finally, we extend the division operation as follows: for all $a, b \in \mathrm{A}$

1. $a\backslash b' = a'/b = (b \cdot a)'$,
2. $b'\backslash a = a/b' = \top$,
3. $a'\backslash b' = a/b$,
4. $b'/a' = b\backslash a$.

Then we can show that associative law and residuation law holds. $x'' = x$ and $x\backslash y' = x'/y$ hold from definition of $'$. If $\mathbf{A} \in \mathcal{I}n\mathcal{RL} + Mod(x^2 \leq x)$ then for any $x \in \mathrm{A}$, $x \cdot x \leq x$ and $x' \cdot x' = \bot \leq x'$. Thus $\mathbf{A}^* \in \mathcal{I}n\mathcal{RL} + Mod(x^2 \leq x)$.

6.4.2 Bounded representable involutive residuated lattice with mingle axiom

In the Sections 6.2, we already show that there exists uncountably many minimal subvarieties in the subvariety lattice of both bounded representable 3-potent residuated lattices and involutive residuated lattices. We show now that there exists uncountably many minimal subvarieties in the subvariety lattice of bounded representable involutive residuated lattices with mingle axiom, too.

Define an idempotent representable RL $\mathbf{D}_S = \langle \mathrm{D}, \wedge, \vee, \cdot_\mathrm{S}, \backslash, /, 1\rangle$ as follows. Let us define a set D.

$$\mathrm{D} = \{a_i | i \in \mathbb{N}^+\} \cup \{b_i | i \in \mathbb{N}\} \cup \{1\}.$$

We define an order $\leq$ on D as follows;

$$b_0 \leq b_i \leq b_j \leq 1 \leq a_k \leq a_l \Leftrightarrow \text{for all } i, j, k, l \in \mathbb{N},\ i \leq j \text{ and } k \geq l.$$

Obviously, this is a total order. Let $S \subseteq \omega$. We define a multiplication $\cdot_S$ on $\mathbf{D}$ depending of S,

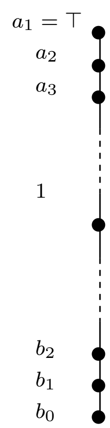

Figure 6.2.

by

$$a_i \cdot_\mathrm{S} a_j = a_{min\{i,j\}}$$

$$b_i \cdot_\mathrm{S} b_j = b_{min\{i,j\}}$$

$$b_j \cdot_\mathrm{S} a_i = \begin{cases} b_j & \text{if } j < i \text{ or } i = j \in S \\ a_i & \text{if } i < j \text{ or } i = j \notin S \end{cases}$$

$$a_i \cdot_\mathrm{S} b_j = \begin{cases} a_i & \text{if } i < j \text{ or } i = j \in S \\ b_j & \text{if } j < i \text{ or } i = j \notin S \end{cases}$$

Finally, we define two division operations, by

$$x \backslash_S y = \bigvee \{z | x \cdot_\mathrm{S} z \leq y\}$$
$$y /_S x = \bigvee \{z | z \cdot_\mathrm{S} x \leq y\}$$

(For the simplicity's sake, we omit the subscript S of $\backslash_S$ and $/_S$ below.) It is easy to see that the multiplication is associative and is residuated by the division operations. So $\mathbf{D}_S$ is a bounded RL (a_1 is the top element and b_0 is the bottom element). Moreover it satisfies the idempotent axiom as $x \cdot_{\mathrm{S}} x = x$.

We construct a bounded InRL $\mathbf{D}^*_{\mathbf{S}}$ from algebra $\mathbf{D}_S$ by the Galatos-Raftery construction mentioned in 6.4.1. Note that $\mathbf{D}^*_{\mathbf{S}}$ is representable, bu for any $x \in D^-_S$ $x \cdot x = \bot$. Thus, it does not satisfy the idempotent axiom any more, but still it satisfies the mingle axiom $x^2 \leq x$.

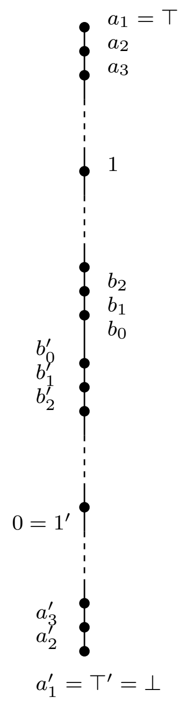

Figure 6.3.

Theorem 6.4.1 *There are uncountably many minimal subvarieties of bounded involutive representable residuated lattices with mingle axiom.*

Proof It is enough to prove the following:

1. For any $S \subseteq \omega$, $\mathbf{D}^*_{\mathbf{S}}$ is a strictly simple algebra.
2. The element $\bot \in \mathbf{D}^*_{\mathbf{S}}$ is nearly term definable lower bound.
3. If S_1 and S_2 are distinct subset of ω then $\mathbf{D}^*_{\mathbf{S_1}}$ and $\mathbf{D}^*_{\mathbf{S_2}}$ generate distinct varieties.

To prove that $\mathbf{D}^*_{\mathbf{S}}$ is strictly simple, it suffices to show that $\mathbf{D}^*_{\mathbf{S}}$ is generated by 1. Obviously, $0 = 1'$ and $0\backslash 1 = \top$. We have

if $i \in S_w$ then $1/a_i = b_i$ and $1/b_i = a_{i+1}$, if $i \notin S_w$ then $a_i \backslash 1 = b_i$ and $b_i \backslash 1 = a_{i+1}$

and $1/a_1 \wedge a_1 \backslash 1 = b_0$. We can generate all elements of $\mathbf{D}_S$ inductively. Finally, we can get $a_i{}'$ and $b_i{}'$ by

$a_i \backslash 0 = a_i{}'$ and $b_i \backslash 0 = b_i{}'$.

Hence $\mathbf{D}_S^*$ is strictly simple.

Now we define a term $q_\perp(x)$ as follows;

$q_\perp(x) = (x \wedge x')^2$.

Suppose that $x \neq 1$. If $x \in \mathrm{D}_S$ then $x > x' \in \mathrm{D}'_S$. If $x \in \mathrm{D}'_S$ then $x < x' \in \mathrm{D}_S$. Hence $(x \wedge x')^2 = \perp$. Thus $\perp$ is nearly term-definable lower bound.

Now we show that for any pair of distinct sets $S_1, S_2 \in \omega$, $V(D_{S_1})$ and $V(D_{S_2})$ generate distinct varieties. We define terms t_a, t_b and t as follows.

$t_a(x) = 1/x \wedge x \backslash 1$
$t_b(x) = 1/x \vee x \backslash 1$
$t(x) = t_a(t_b(x))$

Let S_1 and S_2 be distinct sets. Without loss of generality, we can assume that there exists $i \in \mathbb{N}^+$ such that $i \in S_1$, $i \notin S_2$. Then $b_i \cdot_1 a_i = b_i$ but $b_i \cdot_2 a_i = a_i$. Now we define terms

$q_{b_i} = t_b(t^{i-1}(1' \backslash 1))$
$q_{a_i} = t^{i-1}(1' \backslash 1)$.

Thus the following holds.

The equation $q_{b_i} \cdot q_{a_i} \approx q_{b_i}$ holds in $\mathbf{D}^*_{\mathbf{S_1}}$, but not in $\mathbf{D}^*_{\mathbf{S_2}}$ since $b_i \cdot_2 a_i = a_i \neq b_i$.

So $V(\mathbf{D}^*_{\mathbf{S_1}})$ satisfies the equation $q_{b_i} \cdot q_{a_i} \approx q_{b_i}$, but $V(\mathbf{D}^*_{\mathbf{S_2}})$ does not satisfy it. Hence $V(\mathbf{D}^*_{\mathbf{S_1}}) \neq V(\mathbf{D}^*_{\mathbf{S_2}})$.

□

6.4.3 Bounded representable idempotent involutive residuated lattice

On the contrary, we show that the number of minimal subvarieties of bounded representable idempotent involutive residuated lattices is only two.

First we define three bounded representable InRLs with idempotent axiom $x = x^2$ as follows.
$\mathbf{2} = \langle 2, \wedge_{\mathbf{2}}, \vee_{\mathbf{2}}, \cdot_{\mathbf{2}}, /_{\mathbf{2}}, \backslash_{\mathbf{2}}, 1, 1' \rangle$,
$\mathbf{3} = \langle 3, \wedge_{\mathbf{3}}, \vee_{\mathbf{3}}, \cdot_{\mathbf{3}}, /_{\mathbf{3}}, \backslash_{\mathbf{3}}, 1, 1 \rangle$ and
$\mathbf{4} = \langle 4, \wedge_{\mathbf{4}}, \vee_{\mathbf{4}}, \cdot_{\mathbf{4}}, /_{\mathbf{4}}, \backslash_{\mathbf{4}}, 1, 1' \rangle$,
where 2, 3 and 4 denote underlying sets defined by $2 = \{1', 1\}$, $3 = \{\perp, 1, \top\}$ and $4 = \{\perp, 1', 1, \top\}$, respectively. We define orders on 2, 3 and 4 by

- $\perp = 1' \leq 1 = \top$

- $\bot \leq 1' = 1 \leq \top$
- $\bot \leq 1' \leq 1 \leq \top$

We define also monoid operation on $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$ by the following tables.

$\cdot_2$	1	1'
1	1	1'
1'	1'	1'

$\cdot_3$	$\top$	1	$\bot$
$\top$	$\top$	$\top$	$\bot$
1	$\top$	1	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\cdot_4$	$\top$	1	1'	$\bot$
$\top$	$\top$	$\top$	$\top$	$\bot$
1	$\top$	1	1'	$\bot$
1'	$\top$	1'	1'	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Involution is defined as follows.

x	x'
1	$1'$
$1'$	1

x	x'
$\top$	$\bot$
1	1
$\bot$	$\top$

x	x'
$\top$	$\bot$
1	$1'$
$1'$	1
$\bot$	$\top$

Note that involution is defined by $1'' = 1$, $\top' = \bot$ and $\bot' = \top$ in all of these algebras. We can show easily that the residuation law holds in all of $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$. Thus they are bounded involutive representable residuated lattices with idempotent axiom.

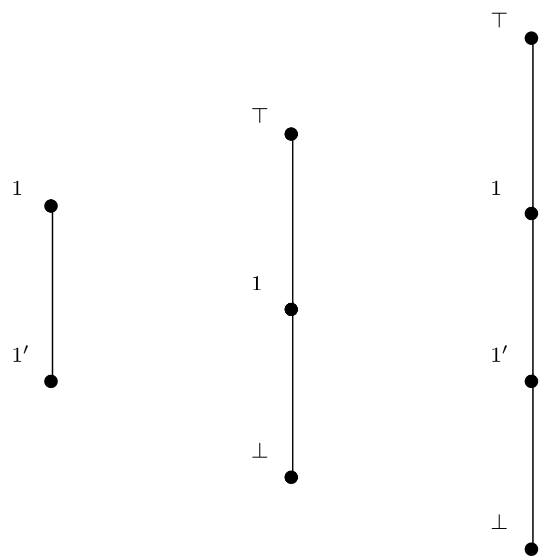

Figure 6.4.

By using theses algebras, we can show the following theorem.

Theorem 6.4.2 *There exists only two minimal subvarieties of bounded involutive representable residuated lattice with idempotent axiom.*

Proof First we show that any subdirect irreducible $\mathbf{A} \in \mathcal{I}n\mathcal{RRL}_\perp + Mod(x = x^2)$ has a subalgebra which is isomorphic to one of $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$. Since $\mathbf{A}$ satisfies the idempotent axiom we can show $1 = 1'' = (1' \cdot 1')' = (1' \backslash 1'')'' = 1' \backslash 1$. Thus $1' \leq 1$. Also, it is

$$\begin{aligned} \perp = 1' &\iff \perp' = 1'' \\ &\iff \top = 1. \end{aligned}$$

Suppose that $\mathbf{A}$ satisfies $1' = 1$. Clearly $\{\perp, 1, \top\} \subseteq A$ and it is closed under involution as mentioned before. $\top \cdot \perp = \perp$. Moreover, $\top \backslash 1 = (\top \backslash 1')'' = (\top 1)' = \top' = \perp$ holds from lemma 4.2.2. Hence $\{\perp, 1, \top\}$ is a subalgebra of $\mathbf{A}$ which is isomorphic to $\mathbf{3}$.

Suppose that $\mathbf{A}$ satisfies $1' < 1$ and $\top = 1$. Then 1 is the greatest and $1'$ is the least element of $\mathbf{A}$. Clearly $\{1', 1\}$ is closed under monoid operation, residuation and involution. Hence $\{1', 1\}$ is a subalgebra of $\mathbf{A}$ which is isomorphic to $\mathbf{2}$.

Finally suppose that $\mathbf{A}$ satisfies $1' < 1$ and $\top \neq 1$. We have $\perp \neq 1'$. Clearly $\{\perp, 1', 1, \top\} \subseteq A$ and it is closed under involution. Let $1' \backslash \perp = x$. If $x \geq 1'$ then $1' = 1'^2 \leq 1' \cdot x = \perp$. This is a contradiction. So $x < 1'$. Then $x = x^2 \leq x \cdot 1' = \perp$. Thus $1' \backslash \perp = \perp$. By lemma 4.2.2 we have

$$\top \cdot 1' = (1' \backslash \top')' = (1' \backslash \perp)' = \perp' = \top.$$

Hence $\{\perp, 1', 1, \top\}$ is closed under monoid operation. Moreover,

$$\top \backslash 1 = (\top \backslash 1'')'' = (1' \cdot \top)' = \top' = \perp \text{ and}$$
$$1' \backslash 1 = (1' \backslash 1'')'' = (1' \cdot 1')' = 1'' = 1$$

hold by lemma 4.2.2. We can show that it is closed under residuation. Hence $\{\perp, 1', 1, \top\}$ is subalgebra of $\mathbf{A}$ which is isomorphic to $\mathbf{4}$.

On the other hand, we show that the algebra $\mathbf{3}$ is a homomorphic image of $\mathbf{4}$. In fact, the map f defined by $f(\top) = \top$, $f(1) = f(1') = 1$ and $f(\perp) = \perp$ gives such a homomorphism. So $\mathbf{3}$ is a element of the subvariety generated by $\mathbf{4}$. Moreover it is clear that

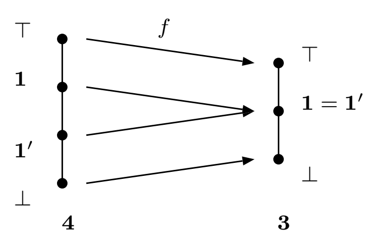

Figure 6.5. homomorphism form $\mathbf{4}$ onto $\mathbf{3}$

$$\top \cdot_{\mathbf{2}} 1' = 1 \cdot_{\mathbf{2}} 1' = 1'$$
$$\top \cdot_{\mathbf{3}} 1' = \top \cdot_{\mathbf{3}} 1 = \top.$$

Hence $V(\mathbf{2})$ and $V(\mathbf{3})$ are distinct varieties. It is easy to see that $\mathbf{2}$ and $\mathbf{3}$ has no proper subalgebras. Therefore, only $\mathcal{V}(\mathbf{2})$ and $\mathcal{V}(\mathbf{3})$ are minimal subvarieties of $\mathcal{I}n\mathcal{RRL}_\perp + (x = x^2)$.

□

The following table shows conclusion of this chapter.

variety	minimal subvarieties
$\mathcal{P}_\ni\mathcal{RRL}_\perp$	uncountably many (Jipsen-Tsinakis)
$\mathcal{RRL} + (x = x^2)$	uncountably many (Galatos)
$\mathcal{I}n\mathcal{RL}$	uncountably many (Tsinakis-Wille)
$\mathcal{I}n\mathcal{RRL}_\perp + (x^2 \leq x)$	uncountably many
$\mathcal{I}n\mathcal{RRL}_\perp + (x = x^2)$	only 2 ($\mathcal{V}(\mathbf{2})$ and $\mathcal{V}(\mathbf{3})$)

6.5 Logical consequences

In this section we show that what our theorems mean from a logical point of view. First we introduce the logic $\mathbf{InFL}'$ which corresponds to variety of involutive residuated lattices. The logic $\mathbf{InFL}'$ is introduced as a sequent calculus obtained from $\mathbf{FL}$ by deleting an initial sequent and an inference rule for the logical constant 0. Moreover we add following an initial sequent and inference rules.

$$\neg\neg\alpha \Rightarrow \alpha$$

$$\frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\alpha}\ (\Rightarrow \neg) \quad \frac{\Gamma \Rightarrow \alpha}{\neg\alpha, \Gamma \Rightarrow}\ (\neg \Rightarrow) \quad \frac{\Sigma, \Gamma \Rightarrow}{\Gamma, \Sigma \Rightarrow}\ (cycling)$$

We show the following lemma.

Lemma 6.5.1 *(1)* $L(\mathcal{I}n\mathcal{RL}) = \mathbf{InFL}'$. *(2)* $V(\mathbf{InFL}') = \mathcal{I}n\mathcal{RL}$.

Proof First we show that

(a) $\mathbf{InFL}' \subseteq L(\mathcal{I}n\mathcal{RL})$.

It is enough to show that $\Gamma \Rightarrow \beta$ is provable in $\mathbf{InFL}'$ implies that $\Gamma \Rightarrow \beta$ is valid in $\mathcal{I}n\mathcal{RL}$, i.e. $\mathbf{A} \models \Gamma^* \leq \beta$ in every $\mathbf{A} \in \mathcal{I}n\mathcal{RL}$. To prove this we use induction on the length of a proof of $\Gamma \Rightarrow \beta$. It is trivial that initial sequents and inference rules of $\mathbf{FL}$. We discuss only about above an initial sequent and inference rules.

(Initial sequent $\neg\neg\alpha \Rightarrow \alpha$)

From definition of involution we can show $x'' = x$. Thus, $\neg\neg\alpha \Rightarrow \alpha$ is valid.

(Inference rule $\Rightarrow \neg$) Let $x, a \in \mathrm{A}$. By the hypothesis of induction, we can assume that $\alpha, \Gamma \Rightarrow$ is valid. We have $\mathbf{A} \models a \cdot x \leq 1'$. Then

$$a \cdot x \leq 1' \Rightarrow x \leq a\backslash 1' = a'/1 = a'.$$

Thus $\mathbf{A} \models x \leq a'$. Hence $\Gamma \Rightarrow \neg\alpha$ is valid.

(Inference rule $\neg \Rightarrow$) Let $x, a \in \mathrm{A}$. By the hypothesis of induction, we can assume that $\Gamma \Rightarrow \alpha$ is valid. We have $\mathbf{A} \models x \leq a$. Then

$$\begin{aligned} x \leq a &\Rightarrow x \leq a'' = a''/1 = a'\backslash 1' \\ &\Rightarrow a' \cdot x \leq 1'. \end{aligned}$$

Thus $\mathbf{A} \models a' \cdot x \leq 1'$. Hence $\neg\alpha, \Gamma \Rightarrow$ is valid.

(Inference rule $cycling$) Let $x, y \in \mathrm{A}$. By the hypothesis of induction, we can assume that $\Sigma, \Gamma \Rightarrow$ is valid. We have $\mathbf{A} \models x \cdot y \leq 1'$. Then

$$\begin{aligned} x \cdot y \leq 1' &\Rightarrow y \leq x\backslash 1' = x'/1 = x' = 1\backslash x' = 1'/x \\ &\Rightarrow y \cdot x \leq 1'. \end{aligned}$$

Thus $\mathbf{A} \models y \cdot x \leq 1'$. Hence $\Gamma, \Sigma \Rightarrow$ is valid.

Next we show that

(b) $\mathcal{I}n\mathcal{RL} \supseteq V(\mathbf{InFL}')$.

It is enough to show that $s \Rightarrow t$ and $t \Rightarrow s$ is provable in $\mathbf{InFL}'$ for every equations $s \approx t$ of $\mathcal{I}n\mathcal{RL}$. It is clear that $s \Rightarrow t$ and $t \Rightarrow s$ is provable for any equation $s \approx t$ of $\mathcal{RL}$. The following proofs

$$\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \quad \dfrac{\beta \Rightarrow \beta}{\neg\beta, \beta \Rightarrow}\,(\neg \Rightarrow)}{\alpha, \alpha\backslash\neg\beta, \beta \Rightarrow}\,(\backslash \Rightarrow)}{\alpha\backslash\neg\beta, \beta \Rightarrow \neg\alpha}\,(\Rightarrow \neg)}{\alpha\backslash\neg\beta \Rightarrow \neg\alpha/\beta}\,(\Rightarrow /) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta \quad \dfrac{\alpha \Rightarrow \alpha}{\neg\alpha, \alpha \Rightarrow}\,(\neg \Rightarrow)}{\neg\alpha/\beta, \beta, \alpha \Rightarrow}\,(/ \Rightarrow)}{\beta, \alpha, \neg\alpha/\beta \Rightarrow}\,(cycling)}{\alpha, \neg\alpha/\beta \Rightarrow \neg\beta}\,(\Rightarrow \neg)}{\neg\alpha/\beta \Rightarrow \alpha\backslash\neg\beta}\,(\Rightarrow \backslash)$$

show that $\alpha\backslash\neg\beta \Rightarrow \neg\alpha/\beta$ and $\neg\alpha/\beta \Rightarrow \alpha\backslash\neg\beta$ are provable in $\mathbf{InFL}'$.

From (b) we can show $L(\mathcal{I}n\mathcal{RL}) \subseteq L(V(\mathbf{InFL}')) = \mathbf{InFL}'$. Since (a) and above holds we have $\mathbf{InFL}' = L(\mathcal{I}n\mathcal{RL})$. Moreover from (a) we can show $V(\mathbf{InFL}') \supseteq V(L(\mathcal{I}n\mathcal{RL})) = \mathcal{I}n\mathcal{RL}$, as $\mathcal{I}n\mathcal{RL}$ is a variety. Since (b) and above holds we have $V(\mathbf{InFL}') = \mathcal{I}n\mathcal{RL}$.

□

Note that the logic $\mathbf{InFL}' + exchange$ is corresponding to the logic $\mathbf{InFL_e}$ since $\neg 1$ is defined by $1 \rightarrow 0 (= 0)$ in $\mathbf{FL_e}$. Next we give an axiomatization of the logic determined by $\mathcal{RRL}$ and $\mathcal{RL}_\perp$ respectively. The variety $\mathcal{RRL}$ is axiomatized by

$\lambda_z((x \vee y)/x) \vee \rho_w((x \vee y)/y) \equiv 1.$

Thus the sequent calculus of the logic determined by the variety $\mathcal{RRL}$ has

(R) $\Rightarrow \lambda_\alpha((\varphi \vee \psi)/\varphi) \vee \rho_\beta((\varphi \vee \psi)/\psi)$

as initial sequents where λ_z (ρ_w) is a left conjugate (and right conjugate, respectively).

The sequent calculus of the logic determined by the variety $\mathcal{RL}_\perp$ has the following initial sequents.

(T) $\Gamma \Rightarrow \top$

(B) $\Gamma, \bot, \Delta \Rightarrow \gamma$

Thus from a logical point of view our theorems mean the following.

- The number of maximal consistent logics over $\mathbf{InFL}'+$ (R) $+$ (T) $+$ (B) $+(\alpha \cdot \alpha \Rightarrow \alpha)$ is uncountably many.
- There exists only two maximal consistent logics over $\mathbf{InFL}'+$ (R) $+$ (T) $+$ (B) $+(\alpha \cdot \alpha \Rightarrow \alpha) + (\alpha \Rightarrow \alpha \cdot \alpha)$.

Chapter 7

Conclusions and future works

In this thesis we have two topics about logics over $\mathbf{FL}$. In Chapter 5, we show the algebraic characterization of the disjunction properties. Moreover we show that many substructural logics have the disjunction property by applying the algebraic characterization.

- An algebraic characterization of the disjunction property is given.
- The disjunction property of many substructural logics is shown by applying it.

In Chapter 6, we show the number of the maximal consistent logics by using dual isomorphism between lattice of logics and subvariety lattice of residuated lattice.

- The existence of uncountably many minimal subvarieties in $\mathcal{I}n\mathcal{RRL}_\perp \cap Mod(x^2 \leq x)$ is shown.
- On the other hand only two exist in $\mathcal{I}n\mathcal{RRL}_\perp \cap Mod(x = x^2)$.

Algebraic characterizations of some logical properties, for example the Halldén completeness and deductive principle of variable separations. Algebraic characterization of Harrop-style disjunction property is not given. Thus we have following future works about this topic.

- Give an algebraic characterization of Harrop-style disjunction property.
- How to extend algebraic characterization of disjunction property to modal substructural logics?

Results of this thesis we discuss about mingle axiom and idempotent axiom but we do not discuss about contraction axiom. Commutative representable residuated latice has at least countably many minimal subvarieties (see **??**). In this thesis we discuss about only non-commutative case. Thus we have follwoing future works about minimal subvarieties.

- How many minimal subvarieties are there in $\mathcal{I}n\mathcal{RRL}_\perp \cap Mod(x \leq x^2)$ (contraction)?
- How many minimal subvarieties are there in $\mathcal{I}n\mathcal{CRRL}_\perp$?
- Is there a natural condition that the number of minimal subvarieties is countably many? Find such a condition, if any.
- Axiomatize the logic determined by the variety $V(\mathbf{3})$.

Publications

Refereed International Journals

1. Daisuke Souma, An algebraic approach to the disjunction property of substructural logics, Notre Dame Journal of Formal Logic, 48(4), pp.489-495, 2007.

Refereed Conference talks at international conferences

2. Daisuke Souma, Minimal subvarieties of involutive residuated lattices, International Conference"Algebraic and Topological Methods in Non-Classical Logics III", Oxford, August 2007.

Other Conferences Papers

3. Daisuke Souma, Algebraic approach to disjunction property of substructural logics, Proceedings of the 38th MLG meeting at Gamagori (2004), pp.26-28.

4. Daisuke Souma, Minimal subvarieties of involutive residuated lattices with mingle axiom", Proceedings of the 40th MLG meeting at Yufuin (2006), pp.20-23.

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Appendix A

A.1 The proof of Lemma 5.4.2

Lemma 5.4.2 The tuple $\langle \mathrm{D}, \cdot, \mathbf{1} \rangle$ is a monoid.

Proof For every $\langle a, b, i \rangle$ in D, $\langle a, b, i \rangle \cdot \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1 \rangle = \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1 \rangle \cdot \langle a, b, i \rangle = \langle a, b, i \rangle$. Thus $\langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1 \rangle$ is the identity element.

Next we prove that the associative law holds.

1. $\langle a_1.b_1.i \rangle \cdot \langle a_2, b_2, j \rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, i \cdot j \rangle$
   (a) $i, j \in \{0, \frac{1}{2}\}$ or ($i = 1$ and $j = 0$)
   (b) $i = 0$ and $j = 1$
   (c) $i = \frac{1}{2}$ and $j = 1$
   (d) $i = 1$ and $j = \frac{1}{2}$
   (e) $i = j = 1$
2. $\langle a_1.b_1.i \rangle \cdot \langle a_2, b_2, j \rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2} \rangle$
   (a) $k = 1$
   (b) $k = \frac{1}{2}$
   (c) $k = 0$
3. $\langle a_1.b_1.i \rangle \cdot \langle a_2, b_2, j \rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0 \rangle$
   (a) $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0 \rangle \cdot \langle a_3, b_3, k \rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 0 \rangle$
   (b) $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0 \rangle \cdot \langle a_3, b_3, k \rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2} \rangle$

(case 1a) Let

$$\langle a_2, b_2, j \rangle \cdot \langle a_3, b_3, k \rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m \rangle$$

where $m \in \{0, \frac{1}{2}\}$.

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0 \rangle \cdot \langle a_3, b_3, k \rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 0 \rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, i \rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m \rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0 \rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \cdot \langle a_3, b_3, k\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq 0_{\mathbf{A}}$ or $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

(case 1b) Let $k = 1$.

$$\langle a_2, b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, 1\rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \cdot \langle a_3, b_3, 1\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 0\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, 0\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, 1\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \cdot \langle a_3, b_3, 1\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq 0_{\mathbf{A}}$ or $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, 0\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, 1\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

Let $k = \frac{1}{2}$.

$$\langle a_2, b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \cdot \langle a_3, b_3, \frac{1}{2}\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 0\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, 0\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \cdot \langle a_3, b_3, \frac{1}{2}\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq 0_{\mathbf{A}}$ or $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, 0\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

Let $k = 0$.

$$\langle a_2, b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$$

where $m \in \{0, \frac{1}{2}\}$.

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \cdot \langle a_3, b_3, 0\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 0\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, 0\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \cdot \langle a_3, b_3, 0\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq 0_{\mathbf{A}}$ or $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, 0\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

(case 1c) Let $k = 1$. Then $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle \cdot \langle a_3, b_3, 1\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. Since $\langle a_2, b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, 1\rangle$. We can show

$$\langle a_1, b_1, \tfrac{1}{2}\rangle \cdot (\langle a_2, b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle) = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

Let $k = \frac{1}{2}$.

$$\langle a_2, b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle \cdot \langle a_3, b_3, \frac{1}{2}\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 0\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, \tfrac{1}{2}\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle \cdot \langle a_3, b_3, \frac{1}{2}\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq 0_{\mathbf{A}}$ or $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, \tfrac{1}{2}\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

Let $k = 0$.

$$\langle a_2, b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$$

where $m \in \{0, \frac{1}{2}\}$.

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle \cdot \langle a_3, b_3, 0\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 0\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, \tfrac{1}{2}\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle \cdot \langle a_3, b_3, 0\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq 0_{\mathbf{A}}$ or $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, \tfrac{1}{2}\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

(case 1d) Let $k = 1$. Then $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle \cdot \langle a_3, b_3, 1\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. Since $\langle a_2, b_2, \frac{1}{2}\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. We can show

$$\langle a_1, b_1, 1\rangle \cdot (\langle a_2, b_2, \tfrac{1}{2}\rangle \cdot \langle a_3, b_3, 1\rangle) = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

Let $k \in \{0, \frac{1}{2}\}$.

$$\langle a_2, b_2, \tfrac{1}{2}\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$$

where $m \in \{0, \frac{1}{2}\}$.

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle \cdot \langle a_3, b_3, k\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 0\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, 1\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle \cdot \langle a_3, b_3, k\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq 0_{\mathbf{A}}$ or $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, 1\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

(case 1e) Let $k \in \{\frac{1}{2}, 1\}$. Then $(\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle) \cdot \langle a_3, b_3, k\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, k\rangle$. We can show

$$\langle a_2, b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, k\rangle$$

Thus $\langle a_1, b_1, i\rangle \cdot (\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, k\rangle) = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), k\rangle$

Let $k = 0$. Then

$$\langle a_2, b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$$

where $m \in \{0, \frac{1}{2}\}$.

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, 0\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, 1\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

Suppose that $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1\rangle \cdot \langle a_3, b_3, k\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle$. Then $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq 0_{\mathbf{A}}$ or $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq 0_{\mathbf{B}}$. Thus we can show

$$\langle a_1, b_1, 1\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

(case 2a) We can show

$$(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ or } (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$$

from $a_1 \cdot_{\mathbf{A}} a_2 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $b_1 \cdot_{\mathbf{B}} b_2 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$. So

$$\begin{aligned}(\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle) \cdot \langle a_3, b_3, k\rangle &= \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle \cdot \langle a_3, b_3, k\rangle \\ &= \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, \frac{1}{2}\rangle.\end{aligned}$$

We can easily show $\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, l\rangle$ such that $l = j$ or $l = \frac{1}{2}$.
If $l = j$ then from $i \cdot j = 0$, $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$ we can show

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, j\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

If $l = \frac{1}{2}$ then from $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$ we can get

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

(case 2b)

$$\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \tfrac{1}{2}\rangle \cdot \langle a_3, b_3, \tfrac{1}{2}\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, l\rangle$$

such that $l = \frac{1}{2}$ or $l = 0$.

If $l = \frac{1}{2}$ then

$$(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ or } (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

Let $\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, k\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$ such that $m \in \{0, \frac{1}{2}\}$.
If $i = 0$ then from $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ or $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$,

$$\langle a_1, b_1, 0\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

If $i \neq 0$, i.e. $j = 0$. Then in the same way as the previous case

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), \tfrac{1}{2}\rangle.$$

If $l = 0$ then

$$(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ and } (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

Suppose that $\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, 0\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$ such that $m \in \{0, \frac{1}{2}\}$. So

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_A a_3, b_2 \cdot_B b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle$$

by $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq_{\mathbf{B}} 0_{\mathbf{B}}$.

(case 2c)

$$\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \tfrac{1}{2}\rangle \cdot \langle a_3, b_3, 0\rangle = \langle (a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3, (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3, l\rangle$$

such that $l = \frac{1}{2}$ or $l = 0$.
If $l = 0$ then

$$\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, 0\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle \text{ such that } m \in \{0, \tfrac{1}{2}\}.$$

So

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

by $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \leq_{\mathbf{B}} 0_{\mathbf{B}}$

If $l = \frac{1}{2}$ then

$$(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ and } (b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

Suppose that $\langle a_2, b_2, j\rangle \cdot \langle a_3, b_3, 0\rangle = \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle$ such that $m \in \{0, \frac{1}{2}\}$. So

$$\langle a_1, b_1, i\rangle \cdot \langle a_2 \cdot_{\mathbf{A}} a_3, b_2 \cdot_{\mathbf{B}} b_3, m\rangle = \langle a_1 \cdot_{\mathbf{A}} (a_2 \cdot_{\mathbf{A}} a_3), b_1 \cdot_{\mathbf{B}} (b_2 \cdot_{\mathbf{B}} b_3), 0\rangle.$$

by $(a_1 \cdot_{\mathbf{A}} a_2) \cdot_{\mathbf{A}} a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $(b_1 \cdot_{\mathbf{B}} b_2) \cdot_{\mathbf{B}} b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$

□

A.2 The proof of Lemma 5.4.3

Lemma 5.4.3 The algebra $\mathbf{D}$ satisfies the law of residuation.

Proof First we show the only-if part. It is enough to prove that the following cases.

1. $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, i \cdot j\rangle$
   (a) $\langle a_1, b_1, j\rangle \backslash \langle a_3, b_3, k\rangle = \langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, i \backslash k\rangle$
   (b) $\langle a_1, b_1, j\rangle \backslash \langle a_3, b_3, k\rangle = \langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, \frac{1}{2}\rangle$
2. $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle$
   (a) $j = 0$
   (b) $j = 1$
   (c) $j = \frac{1}{2}$

3. $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle$

   (a) $k \geq \frac{1}{2}$

   (b) $k = 0$

Here we show each case.
(case 1a) Then clearly

$$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_3, b_3, k\rangle \text{ if and only if } \langle a_2, b_2, j\rangle \leq \langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle.$$

(case 1b) Then

$$a_2 \leq_{\mathbf{A}} a_1 \backslash_{\mathbf{A}} a_3,\ b_2 \leq_{\mathbf{B}} b_1 \backslash_{\mathbf{B}} b_3,\ j \leq i \backslash k.$$

If $i\backslash k = 0$ then

$$j \leq i\backslash k \leq \tfrac{1}{2}.$$

If $i\backslash k = 1$ then

$$a_1 \backslash_{\mathbf{A}} a_3 \not\geq_{\mathbf{A}} 1_{\mathbf{A}},$$
$$b_1 \backslash_{\mathbf{B}} b_3 \not\geq_{\mathbf{B}} 1_{\mathbf{B}}.$$

So

$$a_2 \not\geq_{\mathbf{A}} 1_{\mathbf{A}} \text{ and } b_2 \not\geq_{\mathbf{B}} 1_{\mathbf{B}}.$$

Thus $j \neq 1$. Hence

$$\langle a_2, b_2, j\rangle \leq \langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle.$$

(case 2a) Clearly

$$\langle a_2, b_2, j\rangle \leq \langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle$$

Note that if $j \neq 0$ then $i = 0$.

(case 2b) We can get

$$i\backslash k = 1,$$
$$1_{\mathbf{A}} \leq_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_1 \backslash_{\mathbf{A}} a_3,$$
$$1_{\mathbf{B}} \leq_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_1 \backslash_{\mathbf{B}} b_3.$$

So

$$\langle a_2, b_2, j\rangle \leq \langle a_1, b_1, 0\rangle \backslash \langle a_3, b_3, k\rangle.$$

(case 2c) By $i\backslash k = 1$,

$$\langle a_2, b_2, j\rangle \leq \langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle \leq \langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle.$$

(case 3a) We can get

$$\langle a_1 \backslash_{\mathbf{A}} a_3, b_1 \backslash_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle \leq \langle a_1, b_1, i\rangle \backslash \langle a_3, b_3, k\rangle$$

from $i\backslash k = 1$. So

$$\langle a_2, b_2, j\rangle \leq \langle a_1\backslash_{\mathbf{A}} a_3, b_1\backslash_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle \leq \langle a_1, b_1, i\rangle\backslash\langle a_3, b_3, k\rangle.$$

(case 3b) If $i = 0$ then

$$\langle a_1\backslash_{\mathbf{A}} a_3, b_1\backslash_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle \leq \langle a_1, b_1, i\rangle\backslash\langle a_3, b_3, k\rangle$$

by $i\backslash k = 1$.

If $i = \frac{1}{2}$ then $\langle a_1, b_1, i\rangle\backslash\langle a_3, b_3, k\rangle = \langle a_1\backslash_{\mathbf{A}} a_3, b_1\backslash_{\mathbf{B}} b_3, \frac{1}{2}\rangle$ from definition. So

$$\langle a_2, b_2, j\rangle \leq \langle a_1\backslash_{\mathbf{A}} a_3, b_1\backslash_{\mathbf{B}} b_3, \tfrac{1}{2}\rangle \leq \langle a_1, b_1, i\rangle\backslash\langle a_3, b_3, k\rangle.$$

Next we prove if-part. It is enough to prove that the following cases.

1. $\langle a_1, b_1, i\rangle\backslash\langle a_3, b_3, k\rangle = \langle a_1\backslash_{\mathbf{A}} a_3, b_1\backslash_{\mathbf{B}} b_3, i\backslash k\rangle$
   (a) $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, i \cdot j\rangle$ or $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle$
   (b) $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle$
2. $\langle a_1, b_1, i\rangle\backslash\langle a_3, b_3, k\rangle = \langle a_1\backslash_{\mathbf{A}} a_3, b_1\backslash_{\mathbf{B}} b_3, \frac{1}{2}\rangle$
   (a) $i\backslash k = 1$
      i. $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \frac{1}{2}\rangle$
      ii. $\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle$ or $\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 1\rangle$
   (b) $i\backslash k = 0$

Here we show the each cases.
(case 1a) We can show $a_1 \cdot_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_3$, $b_1 \cdot_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_3$ and $0 \leq i \cdot j \leq k$.
Thus

$$\langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \leq \langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_3, b_3, k\rangle.$$

(case 1b) We can show

$$a_2 \leq_{\mathbf{A}} a_1\backslash_{\mathbf{A}} a_3,\ b_2 \leq_{\mathbf{B}} b_1\backslash_{\mathbf{B}} b_3,\ a_1 \cdot_{\mathbf{A}} a_2 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}\ b_1 \cdot_{\mathbf{B}} b_2 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

So clearly $a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}$. Hence $k \neq 0$. Thus

$$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, \tfrac{1}{2}\rangle \leq \langle a_3, b_3, k\rangle.$$

(case 2(a)i) We can show

$$a_1 \cdot_{\mathbf{A}} a_2 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ or } b_1 \cdot_{\mathbf{B}} b_2 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

So

$$a_3 \not\leq_{\mathbf{A}} 0_{\mathbf{A}} \text{ or } b_3 \not\leq_{\mathbf{B}} 0_{\mathbf{B}}.$$

Hence $k \neq 0$.

(case 2(a)ii) Clearly

$$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle \leq \langle a_3, b_3, k\rangle.$$

(case 2b) We can show

$$a_1 \cdot_{\mathbf{A}} a_2 \leq_{\mathbf{A}} a_3 \leq_{\mathbf{A}} 0_{\mathbf{A}},$$
$$b_1 \cdot_{\mathbf{B}} b_2 \leq_{\mathbf{B}} b_3 \leq_{\mathbf{B}} 0_{\mathbf{B}}$$

from $a_2 \leq_{\mathbf{A}} a_1 \backslash_{\mathbf{A}} a_3$ and $b_2 \leq_{\mathbf{B}} b_1 \backslash_{\mathbf{B}} b_3$ respectively. Hence

$$\langle a_1, b_1, i\rangle \cdot \langle a_2, b_2, j\rangle = \langle a_1 \cdot_{\mathbf{A}} a_2, b_1 \cdot_{\mathbf{B}} b_2, 0\rangle \leq \langle a_3, b_3, k\rangle.$$

□