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PAPER

Generalized Hi-Q is NP-Complete

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SUMMARY This paper deals with a popular puzzle known as Hi-Q. The puzzle is generalized : the board is extended to the size $n \times n$, an initial position of the puzzle is given, and a place is given on which only one token is finally placed. The complexity of the generalized Hi-Q is proved NP-complete.

1. Introduction

In general, combinatorial puzzles and games are hard to analyze, since we have to cope with enormous number of positions of the board. It is one of the main themes in artificial intelligence to solve these problems by heuristic methods. It is important at the same time to show the difficulties of these puzzles and games.

Complexity of problems concerning various generalized puzzles and games have been studied[(1)-(3)]. Most of these generalizations were, however, artificial rather than natural, i. e., they were games on propositional formulas, sets, or graphs, or sometimes played three dimensionally. Complexity results on generalized popular two-person games such as Chess, Checkers, Go, Shogi, Gomoku have been presented[(4)-(8)].

These generalization is natural in the sense that the board is extended to the size $n \times n$ with many pieces. Few results, however, have been presented concerning popular puzzles, which are naturally generalized.

We consider a puzzle commonly known as Hi-Q in this paper. The board of the puzzle consists of 33 points arranged as in Fig. 1 with all but one point placed with tokens. This special point, the center of the board, is named the goal. A move consists of jumping one token over an adjacent one onto an empty point. The jumping must be made either horizontally or vertically.When a token is jumped it is removed from the board. The objective of the puzzle is to make the board to a position with just one token on the goal. Thus a sequence of jumps which leads the board consisting of no tokens but on the goal will be an answer for the puzzle.

The puzzle is generalized in such a way that an initial position is given on the extended board of size $n \times n$, and a goal is also given on which only one token will finally be placed. We show that the problem to determine whether there is an answer for a given generalized Hi-Q is NP-complete. The NP-hardness can be obtained by reducing from a variation of the hamiltonian cycle problem.

2. Complexity Result

We extend the size of the board of Hi-Q to $n \times n$, and assume further that both a position and a goal of the puzzle are given. Now call the puzzle generalized Hi-Q. We will show,

[Theorem] The question : Given generalized Hi-Q, is there an answer for the generalized puzzle ? is NP-complete.

(Proof) It is easy to see that the decision problem belongs to NP. It then remains to prove the completeness. We provide a polynomial time reducibility from the following known NP-complete problem : To determine whether a given planar digraph contains a hamiltonian cycle, where each node of the digraph is either indegree one and outdegree two, or indegree two and outdegree one[(9)].

Let $\tilde{G}$ be the class of planar digraphs where each node is either indegree one and outdegree two, called type ∇ (Fig. 2(a)), or indegree two and outdegree one, called type Δ (Fig. 3(a)).

Hamiltonian Cycle Problem of $\tilde{G}$ is

Given : an element G of $\tilde{G}$.
Question : Does G contain a hamiltonian cycle ?

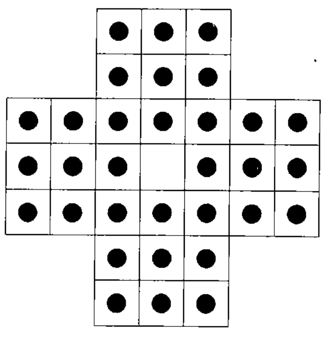

Fig. 1 An initial board of Hi-Q.

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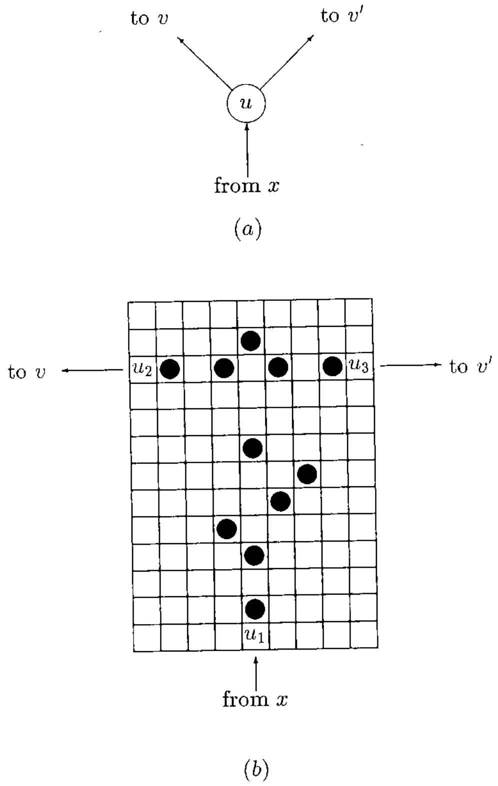

Fig. 2 A type ∇ node u of indegree one and outdegree two.

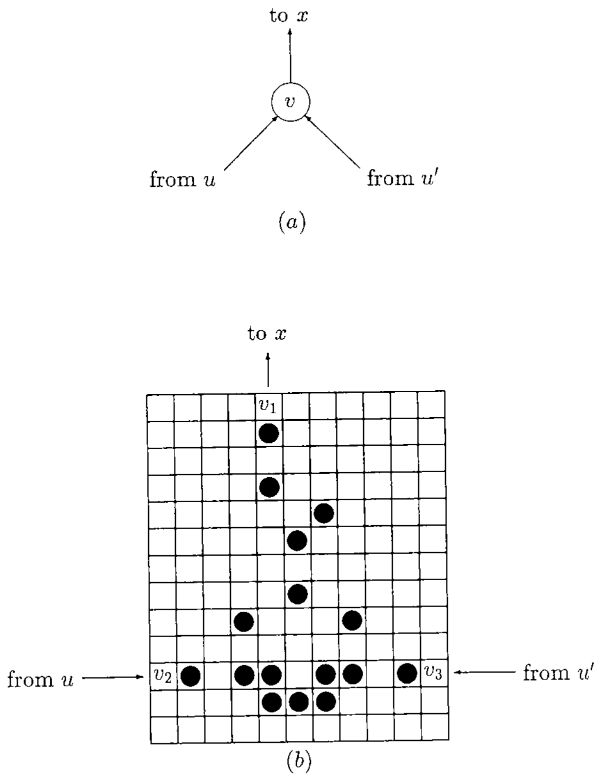

Fig. 3 A type Δ node v of indegree two and outdegree one.

Let G constitute a given instance of the above problem. We construct from G a corresponding instance of generalized Hi-Q in polynomiall time such that there is a hamiltonian cycle in G if and only if there is an answer in the constructed Hi-Q.

The board of the generalized puzzle will be constructed as follows:

(1) For each node u of type ∇, construct a pattern of token placement shown in Fig. 2(b) and call it the pattern u. The points u_2 and u_3 are said to be the exits and u_1 be the entry of the pattern u. For each node v of type Δ, construct similarly a pattern v as in Fig. 3(b). The points v_2 and v_3 are the entries and the point v_1 is the exit of the pattern v.

(2) For each edge (u, v), connect two patterns corresponding to u and v between the exit of u and the entry of v, using extension modules shown in Fig. 4. Call c_1 the entry, and c_2 the exit in Fig. 4. Two extension modules can be connected by identifying the exit of one extension module with the entry of another module. An extension module and a pattern of Fig. 2(b) or 3(b) can be similarly connected.

Note that since G is planar, we can put patterns on

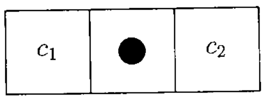

Fig. 4 An extension module.

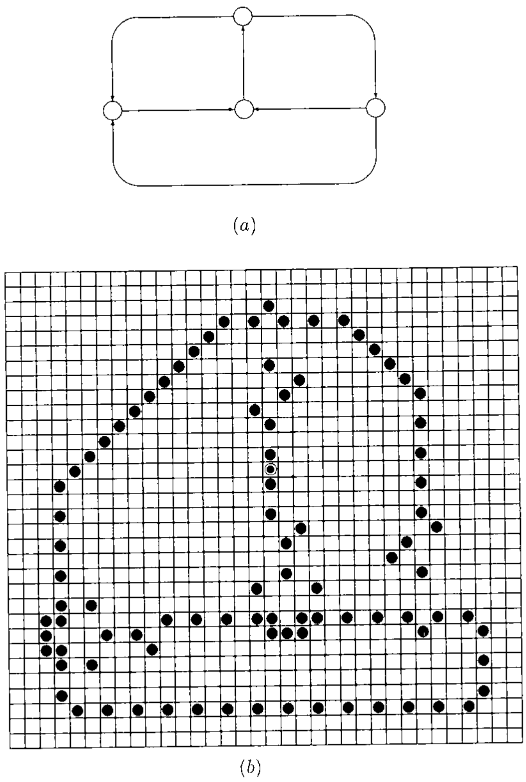

Fig. 5 An example of the construction.

the board so that the connection between them by the procedure (2) above may not cross.

Select one node of type ∇, say z, and put a token on the entry z_1 of the pattern z. The point z_1 is the goal of the constructed generalized Hi-Q.

As an example, for a digraph shown in Fig. 5(a), the constructed puzzle is shown in Fig. 5(b) where a token surrounded by a circle denotes the goal.

Now we will show that G has a hamiltonian cycle if and only if there is an answer in the constructed puzzle.

Assume that $G=(V, E)$ has a hamiltonian cycle. For each edge $e=(\tilde{u}, \tilde{v})$ not on the hamiltonian cycle, we first remove the tokens related to e from the board: we note that $\tilde{u}$ should be of type ∇ and $\tilde{v}$ be of type Δ. Hence, the edge $e=(\tilde{u}, \tilde{v})$ should be either (u, v) or (u, v') in Fig. 2(a), and be either (u, v) or (u', v) in Fig. 3(a). Now suppose that $e=(u, v)$ is not on the hamiltonian cycle, and that u, v are nodes as shown in Fig. 2(a) and Fig. 3(a), respectively. First, we make six jumps from Fig. 3(b) to place a token on the entry v_2 from u as in Fig. 6(a). (If e is of the form (u', v), a token is placed on the entry v_3 from u' as in Fig. 6(b).) Next, the token on the entry jumps to remove the tokens on the extension modules from u to v, and it moves to the exit v_2 in the placement of Fig. 2(b); with the token on the exit, we make three moves and the placement can be the one in Fig. 7(a). (If e is of the form (u, v'), a token is placed on the exit u_3 to v', and then become the one in Fig. 7(b).) After executing this procedure for every edge not on the hamiltonian cycle, we note that all the placements of the forms Figs. 2(b) and 3(b) would be the ones shown in Fig. 7(a) or 7(b), and Fig. 6(a) or 6(b), respectively, and that the tokens which remain on extension modules are of the edges belonging to the hamiltonian cycle.

Then the token on the goal begins to jump to remove the tokens on the board along the direction of the edges in the hamiltonian cycle, and finally one token can remain on the goal.

Assume that there is an answer in the constructed puzzle. We show that there is a hamiltonian cycle in G. In order to prove this, we need some lemmas:

[Lemma 1] Consider a placement of tokens constructed for nodes u and v shown in Fig. 2(b) and Fig. 3(b), respectively. We say that u_i and v_i, $1 \leqq i \leqq 3$ is a gate. Let $S \subseteqq \{u_1, u_2, u_3\}$, and $T \subseteqq \{v_1, v_2, v_3\}$. With an additional token on each gate of $S(T)$, assume that every token on the figure is removed by a series of jumps, and that a token remains on each gate $\{u_1, u_2, u_3\}-S(\{v_1, v_2, v_3\}-T$, respectively) thereafter. Then $S=\{u_1, u_2\}$ or $S=$

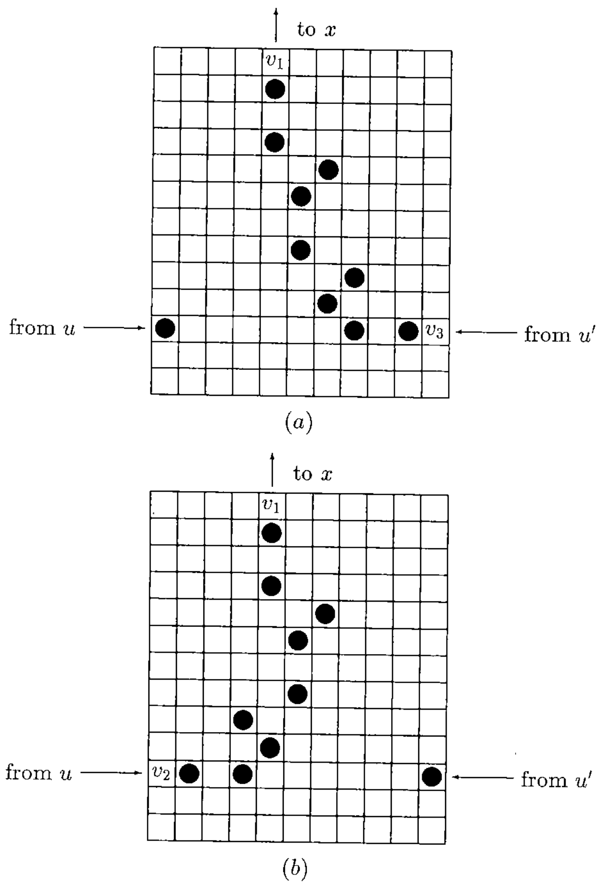

Fig. 6 Placement after six jumps from Fig. 3(b).

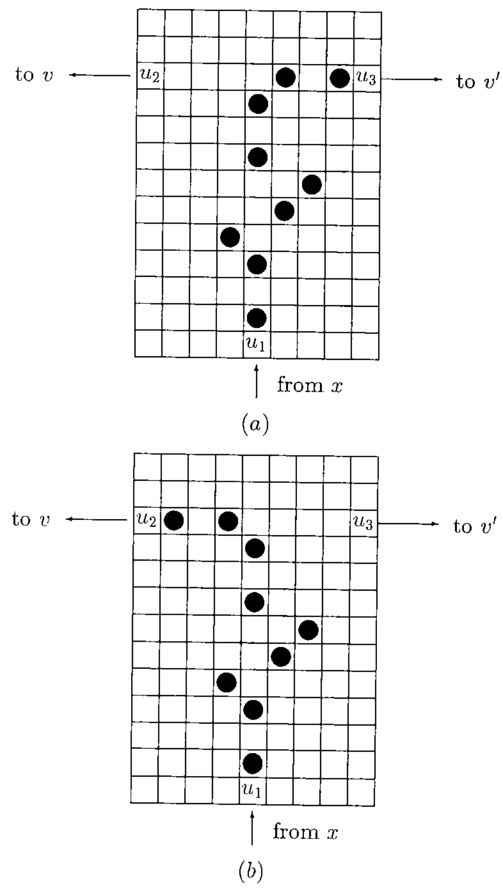

Fig. 7 Placement after three jumps from Fig. 2(b) with additional token on u_2 or u_3.

$\{u_1, u_3\}$, and $T=\{v_2\}$ or $T=\{v_3\}$.
(Proof outline) We first show that $T=\{v_2\}$ or $T=\{v_3\}$. It is obvious that $T \neq \phi$ and $v_1 \notin T$ from the construction of the placement of Fig. 3(b). For $T=\{v_2, v_3\}$, we have examined every possible sequence of jumps by using a computer from the placement of Fig. 3(b) with two additional tokens on v_2 and v_3, and found that no matter how tokens jump, at least one token remains on non-gate point in the figure. Thus, $T \neq \{v_2, v_3\}$.

We now show the proof on S. By the construction of the placement of Fig. 2(b), we obtain that $u_1 \in S$, $S \neq \{u_1, u_2, u_3\}$, and $S \neq \{u_1\}$. Then the lemma follows.
(Remark) We note that for $T=\{v_2\}$ ($T=\{v_3\}$) in the placement of Fig. 3(b), there are two possible independent moves which make all tokens of the placement removed: (1) a move for a token to jump to v_3 (v_2), and (2) having an additional token on v_2 (v_3, respectively), a move for a token to come up on v_1.

We say that a series of jumps is an edge-directed move if it removes tokens of placement of Fig. 3(b) or Fig. 2(b) in the direction from an entry to an exit, or if it removes tokens of extension modules in the same direction as the edge in G, where the extension modules are constructed from.
[Lemma 2] There is exactly one edge-directed move in the constructed puzzle during the sequence of jumps to leave only one token on the goal.
(Proof) Assume that an edge-directed move passes a placement of the form Fig. 2(b) or Fig. 3(b). Then Lemma 1 insists that after the passage there is still one edge-directed move. At the beginning of the puzzle, there is an edge-directed move starting from the goal. Therefore the lemma follows.

We now proceed with the proof of the theorem. The existence of the sequence of jumps to remove all tokens but one on the goal implies that the edge-directed move visits every placement of Figs. 2(b) and 3(b) one after another. Also the edge-directed move starts from the goal and ends at the goal. We have selected the node z. of G of the form Fig. 2(a) to choose z_1 as the goal. Now we can make a hamiltonian cycle in G starting and ending at z by traversing corresponding nodes in order of placements that the edge-directed move visits.

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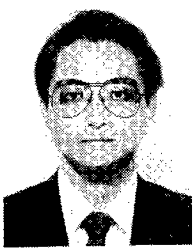

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