Gathering Asynchronous Mobile Robots with Inaccurate Compasses

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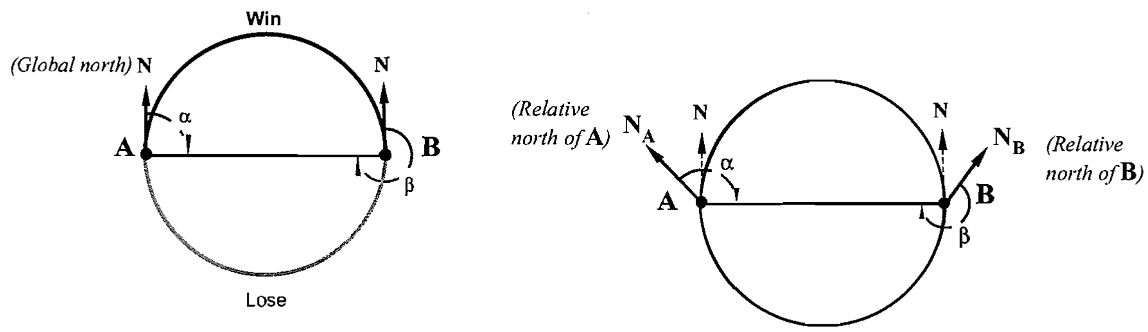
Abstract

This paper considers a system of asynchronous autonomous mobile robots that can move freely in a two-dimensional plane with no agreement on a common coordinate system. Starting from any initial configuration, the robots are required to eventually gather at a single point, not fixed in advance (gathering problem).

Prior work has shown that gathering oblivious (i.e., stateless) robots cannot be achieved deterministically without additional assumptions. In particular, if robots can detect multiplicity (i.e., count robots that share the same location) gathering is possible for three or more *robots. Similarly, gathering of any number of robots is possible if the robots share a common direction, as given by compasses, with* no errors.

Our work is motivated by the pragmatic standpoint that (1) compasses are error-prone devices in reality, and (2) multiplicity detection, while being easy to achieve, allow gathering for situations with more than two robots. Consequently, this paper focusses on gathering two asynchronous robots equipped with inaccurate *compasses. In particular, we provide a self-stabilizing algorithm to gather, in a finite number of steps, two oblivious robots equipped with compasses that can differ by as much as* $\pi/4$.

Keywords: Mobile cooperative computing, distributed algorithms, autonomous robots, gathering, inaccurate compasses, oblivious computations, self-stabilization.



(a) Robot A and B share an accurate compass, i.e. a global north. Thus, we can compare them in a consistent manner.

(b) Robot A and B are equipped with inaccurate compasses, i.e. North of A is different of north of B ($N_A \neq N_B$).

Figure 1. Difficulty of gathering two robots with inaccurate compasses.

1 Introduction

Background. The problem of reaching agreement among autonomous robots has attracted considerable attention within the last few years. One problem of particular interest is the gathering problem, where robots are required to meet at a single, not predetermined location, with no agreement on a common coordinate system. This problem has been studied extensively in the literature, under different models and various assumptions [3, 4, 9, 15]. In fact, several factors render this problem difficult to solve. In particular, in all these studies, the problem has been solved only by making some additional assumptions regarding robots' capabilities.

In this paper, we focus on solving the gathering problem in asynchronous models. In their asynchronous model CORDA [11], Prencipe [12] has shown that there exists no deterministic algorithm to solve the gathering problem in finite time with oblivious robots. Cieliebak et al. [4] have introduced multiplicity and have shown that gathering is possible for three or more robots when they are able to detect multiple robots at a single point.

Afterward, Flocchini et al. [9] have solved the gathering for any number of robots when they share a common direction as provided by a compass. However, their result hold apply when compasses are perfectly consistent (i.e., with no errors). However, in practice, sensors are error-prone and sensitive to magnetic interferences. Consequently, in this paper, we concentrate on the gathering of two asynchronous mobile robots when their compasses are subject to errors.

This work is motivated by the fact that: (1) in practice, compasses are sensors that are sensitive to errors. For example, for low-cost sensors, accuracy of typical sensors may vary from 1 to more than 10 degrees, depending on sensor quality (cost) and environment conditions. (2) with multiplicity detection, the gathering is solvable only for more than two robots. Therefore, our aim is to fill this gap, and to provide effective answers to the following two questions. First, is it possible to gather two asynchronous mobile robots when their compasses are inaccurate by some unknown angle? Second, what is the bound of that angle?

In particular, we address the problem when robots are oblivious (or memoryless), meaning that, they can not remember their previous states, their previous actions or the previous positions of the other robots. While this is somewhat over-restrictive assumption, developing algorithms in this model is interesting because any algorithm that works correctly for oblivious robots is intrinsically self-stabilizing, [1] i.e., it withstands transient failures. We thus, provide an algorithm that gathers, in a finite number of steps, two asynchronous oblivious mobile robots equipped with compasses that can differ by as much as $\pi/4$.

[1] Self-stabilization is the property of a system which, starting in an arbitrary state, always converges toward a desired behavior [7, 13].

In the asynchronous model CORDA, where robots are equipped with inaccurate compasses, it is difficult to gather two robots or compare them in a consistent manner. This is mainly due to the issue of breaking the symmetry between these robots. Let us illustrate this point using a simple example. Assume that there exists a naive algorithm for comparing two asynchronous robots A and B in a consistent manner when their compasses are inaccurate. First, consider that A and B are equipped with accurate compasses and place them at the two endpoints of a horizontal diameter of a unit circle. Then, a naive algorithm can be based on the comparison of the angles that A and B form respectively with some global North N (i.e., they share the same north) and the segment $\overline{AB}$ in clockwise direction. For instance, the robot with an angle less than or equal to $\pi/2$ wins, otherwise loses. Then, a robot, say A wins. Then, we rotate the diameter to exchange the positions of A and B. Now B wins. We thus, color the perimeter of the circle by Win and Lose, where at any point which is colored Win or Lose, A wins or loses (refer to Figure 1(a)). Then, there is a point p that is a boundary between a Win and a Lose segment. By introducing error to their compasses, at p, we can derive a contradiction. That is, we can not decide which robot wins, and which one loses (see Figure 1(b)). [2].

Contribution. The main contribution of this paper is to study the solvability of the gathering of two asynchronous oblivious mobile robots in the face of compass inaccuracies. In particular, we show that with inaccurate compasses, we can gather two asynchronous oblivious mobile robots in a finite number of steps. This result holds for an angle inaccuracy of at most $\pi/4$ of their compasses.

Related work. In their SYm model [15], referred to a semi-synchronous model, Suzuki and Yamashita proposed an algorithm to solve the gathering problem deterministically in the case where robots have unlimited visibility. For a system with two robots, they have proven that it is impossible to achieve the gathering of two *oblivious* mobile robots that have no common orientation under their semi-synchronous model, in a finite time. The difficulty of the problem is inherent in breaking the symmetry between the two robots.

Using the same model, Ando et al. [2] proposed an algorithm to address the gathering problem in systems wherein robots have limited visibility. Their algorithm *converges* toward a solution to the problem, but it does not solve it deterministically.

Cielibak et al. [4] proposed in the asynchronous model CORDA [11] a deterministic algorithm that gathers the robots at a point, in systems where they have unlimited visibility. Among other things, in order to solve this problem, robots must have the ability to detect a multiplicity of robots at a single point.

Later on, Flocchini et al. [9] proposed an algorithm for solving the gathering problem in finite time, in the oblivious and limited visibility settings. However, the proposed algorithm requires robots to share a compass that provides perfectly accurate information on direction.

The gathering problem also has been studied in the presence of faulty robots by Agmon and Peleg [1] in synchronous and asynchronous settings. In particular, they proposed an algorithm that tolerates one crash-faulty robot in a system of three or more robots. They also showed that in an asynchronous environment, it is impossible to perform a successful gathering in a 3-robot system with one Byzantine[3] failure. Later on, Défago et al [6] strengthen the impossibility of gathering in systems with Byzantine robots, by showing that it still holds in stronger models. They also show the existence of randomized solutions for systems with Byzantine-prone robots.

In some of our recent work [14], we introduced the notion of unreliable compasses for robots, and we studied the solvability of the gathering problem in the face of compass instabilities. In particular, we proposed a gathering algorithm that solves the problem in the semi-synchronous model SYm, with compasses that are eventually stabilizing.

[2] The argument is similar to the bi-valent argument in the impossibility result of the consensus problem [8]

[3] A robot is said to be Byzantine if it executes arbitrary steps that are not in accordance with its local algorithm [16].

Recently, Cohen and Peleg [5] addressed the issue of analyzing the effect of errors in solving the gathering and convergence problem. In particular, they studied imperfections in robot measurements, calculations and movements. They showed that gathering cannot be guaranteed in environments with errors, and illustrated how certain existing geometric algorithms, including ones designed for fault-tolerance, fail to guarantee even convergence in the presence of small errors. One of their main positive results is an algorithm for convergence under bounded measurement, movement and calculation errors. However, their algorithm is based on error measurement in general, and does not consider errors of the compasses.

Structure. The remainder of this paper is organized as follows. In Section 2, we describe the system model and the basic terminology. Section 3 describes the algorithm to gather two asynchronous oblivious mobile robots under compasses inaccuracies, and Section 4 proves its correctness. Finally, Section 5 concludes the paper.

2 System model and definitions

2.1 System model

In this paper, we consider the CORDA model [10, 11] of Prencipe, which is defined as follows. The system consists of a set of autonomous mobile robots $\mathcal{R} = \{r_1, \cdots, r_n\}$ that are modelled as units having computational capabilities, and are able to move freely in the two-dimensional plane. In addition, robots are equipped with sensorial capabilities to observe the positions of other robots, and form a local view of the world. The robots are modelled and viewed as points in the Euclidean plane.[4] The local view of each robot includes a unit of length, an origin and the directions and orientations of the two x and y coordinate axes as given by a compass.

The robots are completely *autonomous*. Moreover, they are *anonymous*, in the sense that they are a priori indistinguishable by their appearances, and they do not have any kind of identifiers that can be used during their computation. Furthermore, there is no direct means of communication among them.

We further assume that the robots are *oblivious*, meaning that they don't remember any previous observations nor computations performed in the previous steps.

The cycle of a robot consists of four states: Wait-Look-Compute-Move.

- *Wait.* In this state, robot is idle. A robot cannot stay permanently idle (see Assumption 2) below. At the beginning all the robots are in Wait state.
- *Look.* Here, a robot *observes* the world by activating its sensors, which will return a snapshot of the positions of all other robots with respect to its local coordinate system. Since each robot is viewed as a point, the positions in the plane are just the set of robots' coordinate.
- *Compute.* In this state, a robot *performs* a local computation according to its deterministic, oblivious algorithm. The algorithm is the same for all robots, and the result of the compute state is a destination point.
- *Move.* The robot *moves* toward its computed destination. If the destination is its current location, then the robot is said to perform a *null movement*; otherwise, it is said to execute a *real movement.* The robot moves toward the computed destination, but the distance it moves is unmeasured; neither infinite, nor infinitesimally small (see Assumption 1). Hence, the robot can only go towards its goal, but the move can end anywhere before the destination.

[4] We assume that there are no obstacles that can obstruct vision. Moreover, robots do not obstruct the view of other robots and can "see through" other robots.

The (global) time that passes between two successive states of the same robot is finite, but unpredictable. In addition, no time assumption within a state is made. This implies that the time that passes after the robot starts observing the positions of all others and before it starts moving is arbitrary, but finite. That is, the actual movement of a robot may be based on a situation that was observed arbitrarily far in the past, and therefore it may be totally different from the current situation.

In the model, there are two limiting assumptions related to the cycle of a robot.

Assumption 1 *It is assumed that the distance travelled by a robot r in a move is not infinite. Furthermore, it is not infinitesimally small: there exists a constant $\delta_r > 0$, such that, if the target point is closer than δ_r, r will reach it; otherwise, r will move towards it by at least δ_r.*

Assumption 2 *The time required by a robot r to complete a cycle (wait-look-compute-move) is not infinite. Furthermore, it is not infinitesimally small; there exists a constant $\epsilon_r > 0$, such that the cycle will require at least ϵ_r time.*

2.2 Definitions

Definition 1 (Absolute north) *An absolute north $\overrightarrow{N}$ is a vector that indicates a fixed north direction. The absolute north is collocated with an absolute y positive axis.*

It is important to stress that the absolute north is not known to the robots, and is used only for the sake of explanation.

Definition 2 (Compass) *A compass is a function of robots and time. The function outputs a relative north direction $\overrightarrow{N_r}(t)$ for some robot r at time t.*

Definition 3 (γ^*-Inaccurate compasses) *Compasses are* γ^*-Inaccurate *iff., for every robot r, the absolute difference between the compass of r and $\overrightarrow{N}$ is at most γ^*. That is, $\forall r \in \mathcal{R}, \forall t, |\measuredangle \overrightarrow{N}\overrightarrow{N_r}(t)| \leq \gamma^*$.*

In other words, a pair of γ^*-Inaccurate compasses can differ by as much as $2\gamma^*$. The special case when $\gamma^* = 0$ represents perfect compasses.

2.3 Notations

Given some robot r, $r(t)$ is the position of r at a time t. Let A and B be two points, with $\overline{AB}$, we will indicate the segment starting at A and terminating at B, and $\|\overline{AB}\|$ is the length of such a segment. Given three distinct points A, B, and C, we denote by $\triangle(A, B, C)$, the triangle having them as corners, and by $\widehat{BAC}$, the angle formed by A, B and C, and centered at A. Finally, given a region $X(t)$ at time t, we denote by $|X(t)|$, the number of robots in that region at time t. The parameter t is omitted whenever clear from the context.

3 Gathering with Inaccurate compasses

The basic intuition behind the algorithm is to forbid symmetric configurations of two robots. More precisely, with a perfect compass, it is easy to break the symmetry between two robots by making one move and the other stay still. However, by introducing errors in their compasses, it is difficult to break the symmetry between the two, as they can be in a situation where both of them can move or both stay still. To do so, a robot needs to partition the plane into three different zones, so as two similar zones for two different robots should not overlap, and then it decides its movement. The partition of the plane, and the movements are designed so as to avoid deadlock situations and infinite executions by the robots due to compasses's inaccuracies.

Before we describe the algorithm in more details, we first explain how robots divide the plane.

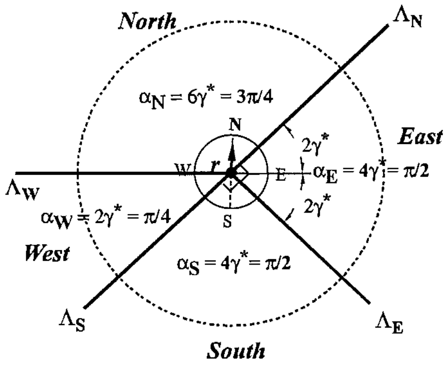


Figure 2. The four sectors *North/South/East/West* **for robot** r.

3.1 Partitions

Let us first assume that there exits a constant $\gamma^* \geq 0$, that represents the maximum angle inaccuracy between the relative north $\overrightarrow{N_r}$ of some robot r, and the absolute north $\overrightarrow{N}$.

In order to avoid symmetric configurations between robots r and r' due to their compasses inaccuracies, a robot needs to partition the plane into four sectors or zones that do not overlap; we call them *North*, *South*, *East* and *West* sectors. Let α_N, α_S, α_E and α_W be the angular measurement of these sectors, respectively. Then, the following conditions must be satisfied:

$$\alpha_N \leq \pi - 2\gamma^* \tag{1}$$
$$\alpha_S \leq \pi - 2\gamma^* \tag{2}$$
$$\alpha_E \leq \pi - 2\gamma^* \tag{3}$$
$$\alpha_W \leq \pi - 2\gamma^* \tag{4}$$

The above conditions are essential in order to avoid that both robots see each other on the same sector, for instance the *North/North* or *South/South*, or *East/East* or *West/West* due to compasses inconsistencies. From Equation 2 and Equation 3, we derive the condition on the *East* sector:

$$\alpha_E \geq 4\gamma^* \tag{5}$$

We further set the following conditions on the sectors. These conditions will help to avoid the occurrence of infinite executions, i.e., having robots looping on the same configuration.

$$\alpha_E + \alpha_S \leq \pi \tag{6}$$
$$\alpha_N + \alpha_W \leq \pi \tag{7}$$

By summation of Equation 2 and Equation 6, we get:

$$\begin{aligned}
\alpha_N + \alpha_E + \alpha_S &\leq 2\pi - 2\gamma^* \text{ then,} \\
\alpha_N + \alpha_E + \alpha_S + \alpha_W &\leq 2\pi - 2\gamma^* + \alpha_W \\
2\pi &\leq 2\pi - 2\gamma^* + \alpha_W \\
2\gamma^* &\leq \alpha_W
\end{aligned}$$

Table 1. Different configurations and movements of robot r and r' ($\gamma^* = \pi/8$).

Robot r/Robot r'	*North* (*no movement*)	*South* (*direct move*)	*East* (*side move up*)	*West* (*side move down*)
North (*no movement*)	no	○	○	○
South (*direct move*)	○	no	○	no
East (*side move up*)	○	○	no	○
West (*side move down*)	○	no	○	no

We take the minimum value of α_W. That is: $\alpha_W = 2\gamma^*$.

From Equation 6, and Equation 5, we obtain: $\alpha_S \leq 4\gamma^*$. We choose, $\alpha_E = \alpha_S = 4\gamma^* = \pi/2$. This means that $\gamma^* = \pi/8$. Then, $\alpha_W = 2\gamma^* = \pi/4$. Also, from Equation 2, and the fact that the sum of the four sectors is equal to 2π, we get, $\alpha_N = \pi - 2\gamma^* = 3\pi/4$.

We have derived the condition that $\gamma^* \leq \pi/8$. Thus, in the remainder of the paper, we consider the largest inaccuracy, and take $\gamma^* = \pi/8$.

Now, we will describe how robots divide the plane, and the features of each sector. Depending on the orientation of its East, $\overrightarrow{E_r}$, robot r divides the plane into four sectors namely, the *North*, *South*, *East* and *West* sectors (refer to Figure 2).

By $\Lambda_N(r)$, $\Lambda_S(r)$, $\Lambda_E(r)$ and $\Lambda_W(r)$, we denote respectively, the rays delimiting the *North*, *South*, *East* and *West* sectors for robot r, as depicted in Figure 2. Each sector is described as follows:

- $East(r)$ sector: it is centered at r, has the East direction $\overrightarrow{E_r}$ as its bisector, and its angular sector α_E is equal to $4\gamma^*$ which is $\pi/2$. Note that $East(r)$ is delimited by $\Lambda_N(r)$ and $\Lambda_E(r)$. However, it is important to point out that only $\Lambda_E(r)$ is a part of $East(r)$.

- $South(r)$ sector: it is adjacent to $East(r)$ in clockwise direction, and its angular sector α_S is equal to α_E, which is equal to $4\gamma^*$ (i.e., $\pi/2$). Note that $South(r)$ is delimited by $\Lambda_E(r)$ and $\Lambda_S(r)$. However, it is important to mention that only $\Lambda_S(r)$ is included in $South(r)$.

- $West(r)$ sector: it is adjacent to $South(r)$ in clockwise direction and its angular sector α_W is equal to $2\gamma^*$ that is $\pi/4$. Note that $West(r)$ is delimited by $\Lambda_W(r)$ and $\Lambda_N(r)$. However, it is important to stress that only $\Lambda_W(r)$ is part of $West(r)$ sector.

- $North(r)$ sector: it is the remaining sector, and its angular sector α_N is equal to $6\gamma^*$, that is $3\pi/4$. Note that $North(r)$ is delimited by $\Lambda_N(r)$ and $\Lambda_W(r)$. However, it is important to stress that only $\Lambda_N(r)$ is included in $North(r)$ sector.

In the following, we will describe the possible configurations of the two robots, given the above partitions.

3.2 Valid configurations

Under our algorithm, a valid configuration is a configuration that does not cause deadlock situations or infinite executions of the algorithm by the robots. We consider two robots r and r' that are equipped with compasses that can diverge by as much as $2\gamma^*$ that is $\pi/4$. Let r and r' divide the plane as described in Section 3.1. Then, r and r' can only be in one of the following valid configurations or their symmetric ones:

1. **Configuration** *North/South*: $r' \in South(r)$ (i.e., robot r sees r' on its *South* sector) and $r \in North(r')$, or vice versa.
2. **Configuration** *North/East*: $r' \in East(r)$ and $r \in North(r')$, or vice versa.
3. **Configuration** *North/West*: $r' \in West(r)$ and $r \in North(r')$, or vice versa.
4. **Configuration** *East/West*: $r' \in West(r)$ and $r \in East(r')$, or vice versa.
5. **Configuration** *East/South*: $r' \in South(r)$ and $r \in East(r')$, or vice versa.

Based on the partitions described in Section 3.1, Table 1 summarizes the possible and incompatible configurations when robots's compasses are inaccurate by $\gamma^* = \pi/8$.

By design, the partitions prevent the occurrence of some undesirable configurations, such as *North/North*, that could lead to deadlock situations (see Section 3.3). [5]

Algorithm 1 Gathering two robots with $\pi/8$-Inaccurate compasses

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Algorithm:

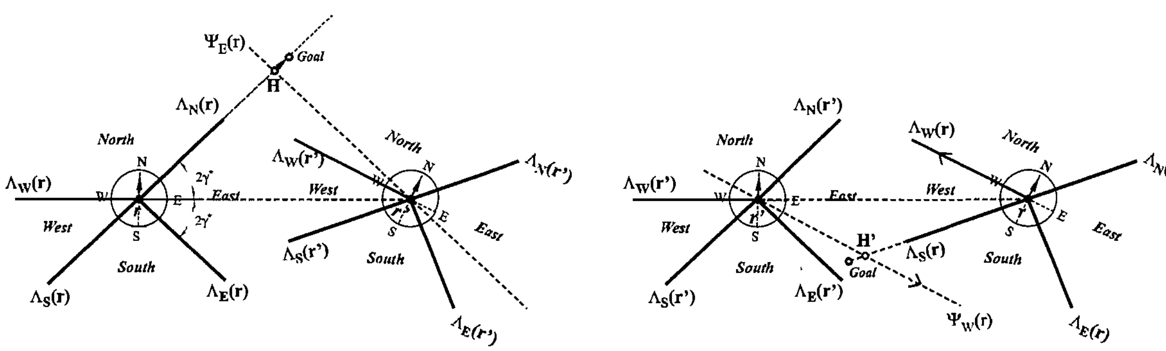
1: $\overrightarrow{N_r}$ = $compass_r(t)$.query();
2: Robot r divides the plane into three sectors: *North*, *South* and *East*. (see Section 3.1);
3: $S :=$ the set of robots visible to r at time t;
4: **if** ($|S| = 1$) **then** {Gathering terminated}
5: Do_nothing();
6: **else**
7: $r' := \{r \in S\}$
8: **if** ($|South(r)| > 0$) **then** {*Other robot is to the South: Direct move*}
9: Move(r');
10: **else if** ($|East(r)| > 0$) **then** {*Other robot is to the East: Side move up*}
11: $\Psi_E(r) :=$ the parallel axis to $\Lambda_E(r)$ passing through r';
12: $H := \Lambda_N(r) \cap \Psi_E(r)$ (see Figure 3(a));
13: *Goal* := first point on $\Lambda_N(r)$ above H;
14: Move (*Goal*);
15: **else if** ($|West(r)| > 0$) **then** {*Other robot is to the West: Side move down*}
16: $\Psi_W(r) :=$ the parallel axis to $\Lambda_W(r)$ passing through r';
17: $H' := \Lambda_S(r) \cap \Psi_W(r)$ (see Figure 3(b));
18: *Goal* := first point on $\Lambda_S(r)$ below H';
19: Move (*Goal*);
20: **else** {*Other robot is to the North: No movement.*}
21: Do_nothing();
22: **end if**
23: **end if**

3.3 Movements

The algorithm is given in Algorithm 1, and Table 1 summarizes the different movements of robot r and r' (the table is symmetrical). Let us consider the movement of robot r, first robot r creates the four sectors, and then it decides its movement based on which sector it locates robot r'.

- *No movement* (Algorithm1:line 21): If $r' \in North(r)$, then r does not move. That is if r sees r' on its *North* sector, it remains stationary.

[5] It is important to mention that when γ^* is equal to zero, that is when the compasses of r and r' are consistent or correct, the configurations *East/South* and *North/West* are impossible.



(a) Side move up on $\Lambda_N(r)$: $r' \in East(r)$, then r performs a *side move up* to *Goal*. *Goal* is the first point on $\Lambda_N(r)$ that is above $\Psi_E(r)$.

(b) Side move down on $\Lambda_S(r)$: $r' \in West(r)$, then r performs a *side move down* to *Goal*. *Goal* is the first point on $\Lambda_S(r)$ that is below $\Psi_W(r)$.

Figure 3. Principle of the algorithm.

- *Direct move* (Algorithm1:line 8): If $r' \in South(r)$, then r moves directly on a a linear movement to r'.
- *Side move up* (Algorithm1:line 10): If $r' \in East(r)$, then r performs a *side move up*. The need for such a move is explained as follows:

  Given the valid configurations described in Section 3.2, if $r' \in East(r)$, then $r \in South(r')$ or $r \in North(r')$, or $r \in West(r')$. Since, robot r can not figure out in which configuration they are (also r'); i.e., $East/South$ or $North/East$ configuration for instance, then, if we let robot r make a *direct move* toward r', then, if both robots are in the configuration $East/South$, then, they will swap their positions endlessly. Also, if we make robot r stay still, then, if both robots are in the configuration $North/East$, none of the robots will ever move, and they will always remain in a deadlock situation. Therefore, the aim of this *side move up* is to bring both robots eventually into the configuration $North/South$, where one robot can move, and the other remains stationary, which can lead to the gathering by our algorithm.

  A *side move up* is computed by robot r as follows: let H be the intersection of the two axes $\Lambda_N(r)$ and $\Psi_E(r)$, with $\Psi_E(r)$ is the parallel to $\Lambda_E(r)$ passing through robot r' (refer to Figure 3(a)). Then, the destination point *Goal* of robot r is the first point above H that belongs to $\Lambda_N(r)$.
- *Side move down* (Algorithm1:line 15): If $r' \in West(r)$, then r performs a *side move down*.

  The aim of this move is similar to the *side move up*, and it is computed by robot r as follows: let H' be the intersection of the two axes $\Lambda_S(r)$ and $\Psi_W(r)$, with $\Psi_W(r)$ is the parallel to $\Lambda_W(r)$ passing through robot r' (refer to Figure 3(b)). Then, the destination point *Goal* of robot r is the first point below H' that belongs to $\Lambda_S(r)$.

4 Correctness

In this section, we will prove that our algorithm solves the problem of gathering two robots in a finite time assuming $\pi/8$-Inaccurate compasses. We first state some lemmas to illustrate that some incompatible configurations are ruled out by the algorithm. Second, we show how any valid configuration under the algorithm is transformed into the gathering in a finite time. Figure 4 summarizes the different possible configurations and their transformation to the gathering.

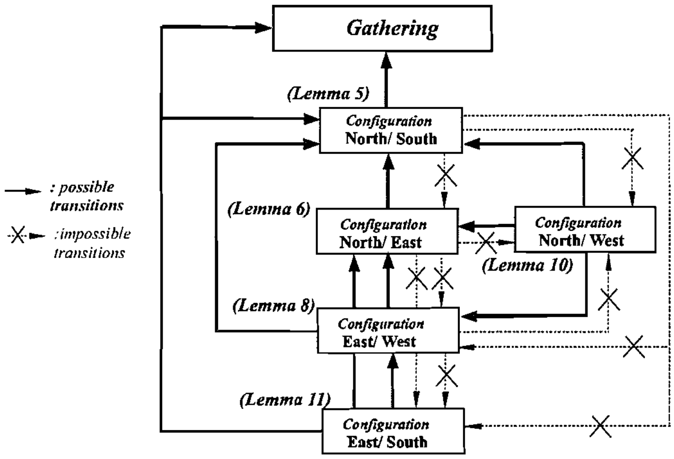


Figure 4. Different configurations allowed by Algorithm 1 and their transformation to the gathering.

Trivially, under the partitions described in Section 3.1 and by considering $\gamma^* = \pi/8$, we derive the following two lemmas:

Lemma 1 *Under the partitions, and assuming $\pi/8$-Inaccurate compasses, the system can not be in the configuration $North/North$ or $East/East$ or $South/South$ or $West/West$ at any time t.*

Lemma 2 *Under the partitions, and assuming $\pi/8$-Inaccurate compasses, the system can not be in the configuration $West/South$ at any time t.*

From the above lemmas, we derive the following theorem:

Theorem 1 *By the algorithm, the possible configurations are $North/South$, $North/East$, $North/West$, $East/West$ and $East/South$, and their symmetric ones (i,e. $South/North$, $East/North$, $West/North$, $West/East$ and $South/East$).*

Lemma 3 *Given a robot r, and its target point H, with $r \neq H$, r reaches its target in a finite number of steps.*

PROOF. The proof derives from Assumption 1. In one cycle, r travels at least $\delta_r > 0$. Besides, by Assumption 2, the cycle of a robot is finite. Thus, the number of steps required for robot r to reach its destination H is at most $\|\overline{rH}\|/\delta_r$, which is finite. $\Box_{\text{Lemma 3}}$

Lemma 4 *Given two robots r and r' at some time t_0, where r and r' are in the configuration $North/East$ or $East/West$ or $East/South$, with $r' \in East(r)$ and either $r \in North(r')$ or $r \in West(r')$ or $r \in South(r')$. Then, the destination Goal computed by robot r (resulting from its side move up) belongs to $North(r')$.*

PROOF.

We will make the proof for the $North/East$ configuration only. For the $East/West$ and $East/South$ configurations, they can be proved in a similar way.

Assume that $r' \in East(r)$ and $r \in North(r')$ at time t_0. Let $H = \Psi_E(r) \cap \Lambda_N(r)$ (refer to Figure 5).

By Algorithm 1, $Goal$ is the first point on $\Lambda_N(r)$ above H. We will prove that $Goal \in North(r')$.

First, observe that if $\gamma^* = 0$, then $\Lambda_N(r)$ is parallel to $\Lambda_N(r')$. Then, since $r \in North(r')$ by hypothesis. In addition, $r \in \Lambda_N(r)$ and $Goal \in \Lambda_N(r)$. Consequently, $Goal \in North(r')$.

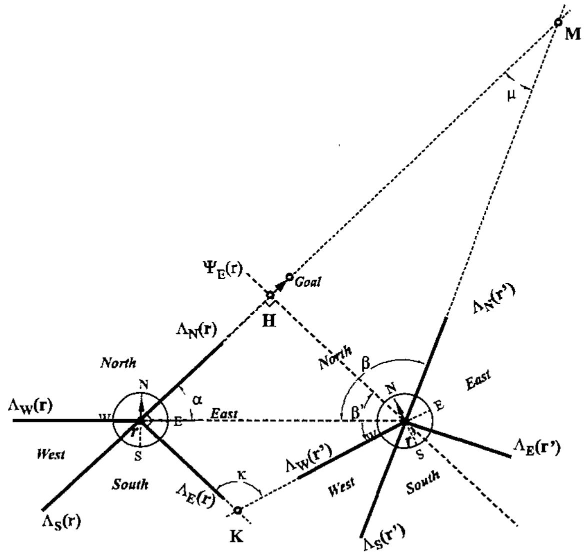


Figure 5. Transformation of $North/East$ configuration.

Assume now that $\gamma^* \neq 0$, and $\Lambda_N(r) \cap \Lambda_N(r') = M$ (see Figure 5). To show that $Goal \in North(r')$, we will show that always $Goal \in \triangle(r, r', M)$. In other words, we need to show that always $H \in \triangle(r, r', M)$, and the distance $\|\overline{HM}\| \neq 0$.

Consider the triangle $\triangle(r, r', M)$. Let α, β, and μ denote the angles at r, r' and M that are within the triangle $\triangle(r, r', M)$, respectively.

First, if all the three angles α, β, and μ are acute. Then, obviously, the foot H of the perpendicular starting form r' is inside $\triangle(r, r', M)$, and $\|\overline{HM}\| \neq 0$. Second, if the angle β at r' is obtuse, then again the foot H of the perpendicular starting form r' is inside $\triangle(r, r', M)$, and $\|\overline{HM}\| \neq 0$. Now consider the angle α at r. By hypothesis α_E is at most $\pi/2$. This means that α can not be an obtuse angle, and α is at most $\pi/2$. In this later case, we have the foot H of the perpendicular starting form r' is equal to r (in this case $\Lambda_E(r)$ passes by r'), and the triangle $\triangle(r', r, M)$ is angular at r. Consequently, $\|\overline{rM}\| = \|\overline{HM}\| \neq 0$, and $Goal \in \triangle(r, r', M)$.

Now, we will prove that the angle μ at M can not be an obtuse angle (because if μ is an obtuse angle, H is outside $\triangle(r, r', M)$).

Let $K = \Lambda_E(r) \cap \Lambda_W(r')$ and κ be the angle at K. We also denote by β', the angle at r' formed by $\Psi_E(r)$ and $\Lambda_W(r')$. Consider, the quadrilateral formed by r, H, r' and K. Then, we have: (1) $\kappa + \beta' = \pi$, since the respective angle at r and H is equal to $\pi/2$. Consider now the quadrilateral formed by r, K, r' and M. Then, we have: (2) $\kappa + \mu = 3\pi/4$, since the angle at r (α_E) is equal to $\pi/2$, and the angle at r' ($\alpha_N = 3\pi/4$) by hypothesis.

By subtraction of (1) from (2), we get: (3) $\beta' - \mu = \pi/4$. By assumption, $\beta' < 3\pi/4$ because $\Psi_E(r)$ can not be equal to $\Lambda_N(r')$ ($\Lambda_N(r')$ can not be perpendicular to $\Lambda_N(r)$ by the partitions described in Section 3.1). Consequently, the angle μ at M is less than $\pi/2$. Thus, μ can not be an obtuse angle. As a result, in all cases, the foot H of the perpendicular starting from r' is inside the triangle $\triangle(r, r', M)$, and $\|\overline{HM}\| \neq 0$. This proves that always $Goal \in \triangle(r, r', M)$, and thus $Goal \in North(r')$. This completes the proof. $\square_{\text{Lemma 4}}$

In the following, for each configuration, we will show to what configuration it leads in a finite time. That is, we show the possible transitions that each configuration can take in order to reach the gathering configuration. For the impossible transitions, they can be derived implicitly, thus, we don't prove them explicitly.

4.1 Transition of *North/South* configuration to gathering

Lemma 5 *Let r and r' be two robots that are in the configuration North/South at some time t_0. Then, there is a time $\bar{t} > t_0$ when r and r' gather at the same point. Moreover, r and r' can not slide to any other configuration except the gathering.*

PROOF.
The proof of this lemma is trivial. Let $r' \in South(r)$ and $r \in North(r')$ at time t_0. Assume that r' performs a look operation at time $t' \geq t_0$. Then, r' will stay still, since $r \in North(r')$. Let r also perform a look operation at time $t \geq t_0$. Then, by the algorithm, r will perform a *direct move* toward r'. Let $t'' > t$, be the time when r completes its move toward r'. By Lemma 3, r reaches r' in a finite time. Also, between t and t'', r' is unable to move. Thus, at t'', $r = r'$. Consequently, from the configuration *North/South*, r and r' can only shift to the gathering configuration, which is done in a finite time. This completes the proof. $\Box_{\text{Lemma 5}}$

4.2 Transition of *North/East* configuration to gathering

Lemma 6 *Let r and r' be two robots that are in the configuration North/East with $r' \in East(r)$, and $r \in North(r')$ at some time t_0. Then, there is a finite time $\bar{t}$ in which this configuration is transformed into North/South configuration with $r' \in South(r)$. Moreover, r and r' can not slide to any other configuration except the North/South configuration.*

PROOF.
The proof is a direct consequence from Lemma 4. Assume that $r' \in East(r)$ and $r \in North(r')$ at some time t_0 (refer to Figure 5). Let r perform a look operation at time $t \geq t_0$, and let *Goal* be its destination.

Let also $\bar{t}$ be the time when r reaches its destination *Goal*. By hypothesis, $r \in North(r')$ at time t_0. By Lemma 4, $Goal \in North(r')$. This means that between t_0 and $\bar{t}$, r' is unable to move because $\forall p \in \overline{rH}$, $p \in North(r')$. At $\bar{t}$, r reaches its destination *Goal*, and $\Lambda_E(r)$ is above r'. Thus, at $\bar{t}$, $r' \in South(r)$. Consequently, at $\bar{t}$, r and r' become in the configuration *North/South* in a finite time, and the proof holds. $\Box_{\text{Lemma 6}}$

From Lemma 5 and Lemma 6, we conclude that:

Theorem 2 *Any North/East configuration of two robots equipped with $\pi/8$-Inaccurate compasses is transformed after a finite time to the gathering.*

4.3 Transition of *East/West* configuration to gathering

Lemma 7 *Given two robots r and r' at some time t_0, where r and r' are in the configuration East/West, with $r \in West(r')$ and $r' \in East(r)$. Then, the destination $Goal'$ computed by robot r' (resulting from its side move down) belongs to $East(r)$ or $South(r)$.*

PROOF. The idea of this proof is as follows. There are two cases to consider. First, $\Lambda_W(r')$ is parallel to $\Lambda_E(r)$. Then, $\Lambda_E(r) = \Psi_W(r')$. Since, the destination $Goal'$ of r' is the first point on $\Lambda_S(r')$ below $\Psi_W(r')$. Then, $Goal'$ is below $\Lambda_E(r)$. Consequently, in this case, $Goal' \in South(r)$. Second, $\Lambda_W(r')$ is not parallel to $\Lambda_E(r)$. Then, $\Psi_W(r')$ is above $\Lambda_E(r)$. Thus, $Goal'$ is between $\Psi_W(r')$ and $\Lambda_E(r)$. Consequently, $Goal' \in East(r)$ in this case.

Assume that $r \in West(r')$ and $r' \in East(r)$ at some time t_0. Let $H' = \Psi_W(r') \cap \Lambda_S(r')$ (refer to Figure 6). We denote by β, the angle at H' that is inside the triangle $\triangle(r, r', H')$.

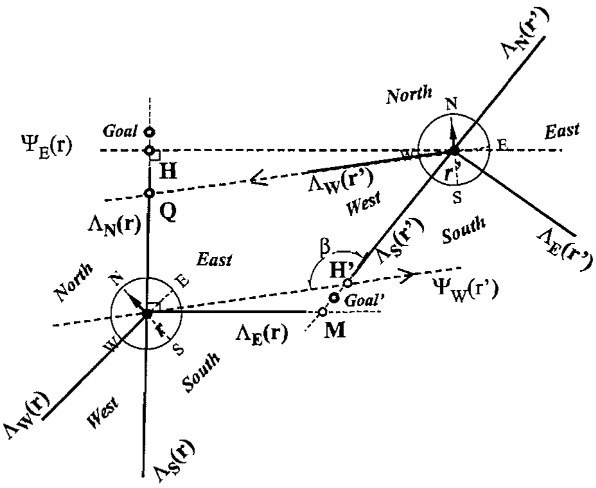


Figure 6. Transformation of $East/West$ **configuration.**

By Algorithm 1, *Goal'* is the first point on $\Lambda_S(r')$ below H'. We will prove that $Goal' \in East(r)$ or $Goal' \in South(r)$.

Let $M = \Lambda_E(r) \cap \Lambda_S(r')$. Let also $Q = \Lambda_W(r') \cap \Lambda_N(r)$. We have, $\widehat{Qr'M} = \alpha_W = \pi/4$ by hypothesis. Then, $\beta = 3\pi/4$ because $\Lambda_W(r')$ is parallel to $\Psi_W(r')$. We thus, derive that $\widehat{rMr'} + \widehat{MrH'} = 3\pi/4$. Thus, two cases follow:

If $\widehat{rMr'} = 0$. Then, $\Lambda_E(r) = \Psi_W(r')$. Since, the destination *Goal'* of r' is the first point on $\Lambda_S(r')$ below $\Psi_W(r')$. Then, *Goal'* is below $\Lambda_E(r)$. Consequently, in this case, $Goal' \in South(r)$.

If $\widehat{rMr'} > 0$, this means that $\Psi_W(r')$ is above $\Lambda_E(r)$. Then, *Goal'* is between $\Psi_W(r')$ and $\Lambda_E(r)$. Consequently, $Goal' \in East(r)$. This completes the proof.

$\Box_{\text{Lemma 7}}$

Lemma 8 *Let r and r' be two robots that are in the configuration $East/West$ with $r' \in East(r)$, and $r \in West(r')$ at some time t_0. Then, there is a finite time $\bar{t}$ in which this configuration is transformed into $North/East$ or $North/South$ configuration. Moreover, r and r' can not enter any other configuration except the $North/East$ or $North/South$ configuration.*

PROOF.

Assume that $r' \in East(r)$ and $r \in West(r')$ at time t_0.

By the algorithm, r will make a *side move up*, and r' will make a *side move down*. We distinguish the following cases depending on the movement of each robot, and where it ends its move:

1. r **moves/** r' **does not move**:

   Let r perform a look operation at time $t \geq t_0$, and let *Goal* be its destination. Let also $\bar{t}$ be the time when r finishes its move. Assume that between t_0 and $\bar{t}$, r' does not perform any look operation. Let $Q = \Lambda_W(r') \cap \Lambda_N(r)$ and $H = \Psi_E(r) \cap \Lambda_N(r)$ (refer to Figure 6). Thus, three cases follow depending where r ends its move:

   - *r stops at $q \in \overline{rQ}$, and $q \neq r$.*

At time $\bar{t}$, $r' \in East(r)$ still, since r does not reach $Goal$. r' remains stationary by hypothesis. Thus, r and r' remain in the configuration $East/West$. By assumption, robot r can be activated infinitely often, and by Lemma 3, it can reach its target in a finite time, thus robot r can change the current configuration $East/West$ into a different configuration in a finite time.

- *r stops at $q \in \overline{QH}$, and $q \neq Q$.*

  At time $\bar{t}$, $r' \in East(r)$ still, since r did not reach $Goal$. However, at time $\bar{t}$, r becomes above $\Lambda_W(r')$. Thus, at $\bar{t}$, $r \in North(r')$. Consequently, r and r' become in the configuration $North/East$ with $r \in North(r')$.

- *r stops at Goal (Goal is the first point above $\Psi_E(r)$).*

  At $\bar{t}$, we have $r' \in South(r)$ (since at $\bar{t}$, r' becomes below $\Lambda_E(r)$). In addition, by Lemma 4, $Goal \in North(r')$. Then, at $\bar{t}$, $r \in North(r')$. Consequently, r and r' become in the configuration $North/South$ in a finite time.

2. **r' moves/ r does not move**:

   Let r' perform a look operation at time $t' \geq t_0$, and let $Goal'$ be its destination. Let also $\bar{t'}$ be the time when r' finishes its move. Assume that between t_0 and $\bar{t'}$, r does not perform any look operation. Let $H' = \Psi_W(r') \cap \Lambda_S(r')$ (refer to Figure 6).

   Assume first that r' stops at $Goal'$, then at time $\bar{t'}$, r is above $\Lambda_W(r')$, thus $r \in North(r')$. In addition, at $\bar{t'}$, by Lemma 7, $r' \in East(r)$ or $r' \in South(r)$. Consequently, r and r' leave the configuration $East/West$ in a finite number of steps, and become in the configuration $East/North$ or $North/South$.

   Assume now that r' stops before $Goal'$, then at time $\bar{t'}$, r and r' remain in the configuration $East/West$. By assumption, robot r' can be activated infinitely often, and by Lemma 3, it can reach its target in a finite time, thus robot r' can reach its target $Goal'$ in a finite time. Consequently, r and r' can leave the configuration $East/West$ in a finite time.

3. **both r and r' move**:

   Let r perform a look operation at time $t \geq t_0$, and let $Goal$ be its destination. Let also r' perform a look operation at time $t' \geq t_0$, and let $Goal'$ be its destination. We denote by $\bar{t}$ and $\bar{t'}$, the time when r and r' end their moves, respectively.

   In this case, we assume that both r and r' reach their respective destinations $Goal$ and $Goal'$. All other cases, where r or r' end their moves before destination are covered and proved in the previous cases.

   At $\bar{t}$, r reaches its destination $Goal$. Hence, $\forall p$ that is below $\Lambda_E(r(\bar{t}))$, and to the right of $\Lambda_E(r\bar{t})$, $p \in South(r)$. At $\bar{t}$, $r' \in \overline{r'Goal'}$. By Lemma 7, $Goal' \in East(r(t_0))$ or $Goal' \in South(r(t_0))$. Since, at $\bar{t}$, $\Lambda_E(r)$ is above $Goal'$. Thus, at $\bar{t}$, $r' \in South(r)$. Now, at $\bar{t'}$, r' reaches $Goal'$. At $\bar{t'}$, r is above $\Lambda_W(r')$. Consequently, at $\bar{t'}$, $r \in North(r')$. Since, r and r' reach their respective target in a finite time, we hence conclude that r and r' become in the configuration $North/South$ in a finite time.

$\Box_{\text{Lemma 8}}$

From Lemma 8, Lemma 5 and Theorem 2, we conclude:

Theorem 3 *Any East/West configuration of two robots equipped with $\pi/8$-Inaccurate compasses is transformed after a finite time to the gathering.*

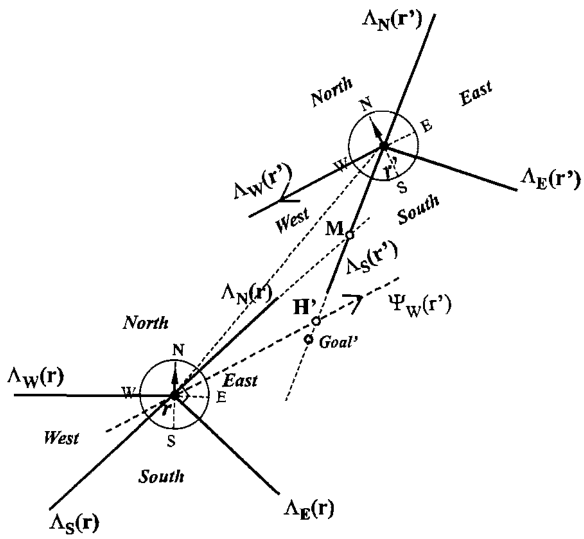


Figure 7. Transformation of $North/West$ configuration.

4.4 Transition of $North/West$ configuration to gathering

Lemma 9 *Given two robots r and r' at some time t_0, where r and r' are in the configuration $North/West$, with $r \in West(r')$ and $r' \in North(r)$. Then, the destination $Goal'$ computed by robot r' (resulting from its side move down) always belong to $East(r)$.*

PROOF. The idea of this proof is as follows. There are two cases to consider. First, if $\Lambda_W(r')$ is parallel to $\Lambda_N(r)$. Then, $\Lambda_N(r) = \Psi_W(r')$. Since, the destination $Goal'$ of r' is the first point on $\Lambda_S(r')$ below $\Psi_W(r')$. Then, $Goal'$ is below $\Lambda_N(r)$. Consequently, $Goal' \in East(r)$. Second, $\Lambda_W(r')$ is not parallel to $\Lambda_N(r)$, then we show that $\Psi_W(r')$ is below $\Lambda_N(r)$ and above $\Lambda_E(r)$. Consequently, $Goal' \in East(r)$.

Assume that $r \in West(r')$ and $r' \in North(r)$ at some time t_0. Let $H' = \Psi_W(r') \cap \Lambda_S(r')$ (refer to Figure 7). We denote by β, the angle at H' that is inside the triangle $\triangle(r, r', H')$. By Algorithm 1, $Goal'$ is the first point on $\Lambda_S(r')$ below H'. We will prove that $Goal' \in East(r)$ always.

First, if $\Lambda_W(r')$ is parallel to $\Lambda_N(r)$. Then, $\Lambda_N(r) = \Psi_W(r')$. Since, the destination $Goal'$ of r' is the first point on $\Lambda_S(r')$ below $\Psi_W(r')$. Then, $Goal'$ is below $\Lambda_N(r)$. Consequently, $Goal' \in East(r)$.

Now consider the case when $\Lambda_W(r')$ is not parallel to $\Lambda_N(r)$. Let $M = \Lambda_N(r) \cap \Lambda_S(r')$.

We have, $\alpha_W = \pi/4$ by hypothesis. Then, $\beta = 3\pi/4$ because $\Lambda_W(r')$ is parallel to $\Psi_W(r')$.

Consider the triangle $\triangle(r, M, r')$. We have, $\widehat{rMr'} > \widehat{rH'r'}$. Thus, H' is below M. Then, $H' \in East(r)$, since $0 < \widehat{MrH'} < \pi/2$. Then, $Goal'$ is between $\Psi_W(r')$ and $\Lambda_E(r)$. Consequently, $Goal' \in East(r)$. This completes the proof. $\square_{\text{Lemma 9}}$

Lemma 10 *Let r and r' be two robots that are in the configuration $North/West$ with $r \in West(r')$, and $r' \in North(r)$ at some time t_0. Then, there is a finite time $\bar{t}$ in which this configuration is transformed into $North/East$ or $East/West$ or $North/South$ configuration. Moreover, r and r' can not enter any other configuration except the $North/East$ or $East/West$ or $North/South$ configuration.*

PROOF.

Let r' perform a look operation at time $t' \geq t_0$, and let $Goal'$ be its destination. We denote by $\bar{t}$ the time when r' finishes its move. Let also $H' = \Psi_W(r') \cap \Lambda_S(r')$ (refer to Figure 7)

First, assume that between t_0 and $\bar{t}$, r does not perform any look operation, this means that r remains stationary. Thus, when r' reaches $Goal'$, r is above $\Lambda_W(r')$. Consequently $r \in North(r')$. In addition, by Lemma 9, at $\bar{t}$, $r' \in East(r)$. Also, by Lemma 3, r' reaches $Goal'$ in a finite time, then r and r' become in the configuration $East/North$ in a finite time.

Now assume that between t_0 and $\bar{t}$, r performs some look operation. Let $M = \Lambda_N(r) \cap \Lambda_S(r')$. Then, two cases follow:

- *r sees r' at position $q \in \overline{r'M}$.*

  In this case, r' still belongs to $North(r)$. Thus, r stays still. By hypothesis, r' can be activated infinitely often, and thus, it can pass M in a finite time. When r' passes the point M, $r' \in East(r)$. Consequently, r and r' can leave the configuration $North/West$ in a finite time.

- *r sees r' at position $q \in \overline{M\,Goal'}$, with $q \neq \{M, Goal'\}$.*

  In this case, $r' \in East(r)$ because r' is below $\Lambda_N(r)$, and $r \in West(r')$, since r' did not reach its target $Goal'$. Consequently, r and r' become in the configuration $East/West$, which is done in a finite time.

  Let also $Goal$ be the target of r, and t'' be the time when it reaches its target, then if r reaches its destination before r' (i.e., $t'' < \bar{t}$), then at t'', $r' \in South(r)$, and $r \in North(r')$. Consequently, r and r' become in the configuration $North/South$ in a finite time.

$\Box_{\text{Lemma 10}}$

From Lemma 10, Theorem 2 and Theorem 3, we conclude:

Theorem 4 *Any North/West configuration of two robots equipped with $\pi/8$-Inaccurate compasses is transformed after a finite time to the gathering.*

4.5 Transition of $East/South$ configuration to gathering

Lemma 11 *Let r and r' be two robots that are in the configuration East/South at some time t_0 with $r' \in East(r)$ and $r \in South(r')$ Then, there is a finite time in which this configuration is transformed into North/South or North/East or East/West or the gathering configuration.*

PROOF.

Let $r' \in East(r)$ and $r \in South(r')$ at time t_0 (refer to Figure 8). Let also $Q = \Lambda_N(r) \cap \Lambda_S(r')$, $H = \Lambda_N(r) \cap \Psi_E(r)$ and $G = \Lambda_N(r) \cap \Lambda_W(r')$.

By the algorithm, r' will make a *direct move* toward r, and r will make a *side move up*. Then, we distinguish several cases, depending where each robot sees the other one, and where it ends its move.

1. **r does not move/ r' moves.**

   Assume that r' performs a look operation at time $t' \geq t_0$. Then, r' will make a *direct move* toward r. By Lemma 3, r' reaches r in a finite time, let $\bar{t'}$ be such time. Assume that r does not perform any look operation between t_0 and $\bar{t'}$, then r' will gather with r at time $\bar{t'}$, and the lemma holds for this case.

2. **r moves/ r' does not move.**

   Assume that r performs a look operation at time $t \geq t_0$. Then, r will make a *side move up*. Let $Goal$ be its destination and $\bar{t}$ be the time when r finish its move.

   Assume that between t and $\bar{t}$, r' does not make any look operation, thus r' remains stationary. Three cases follow depending where robot r stops:

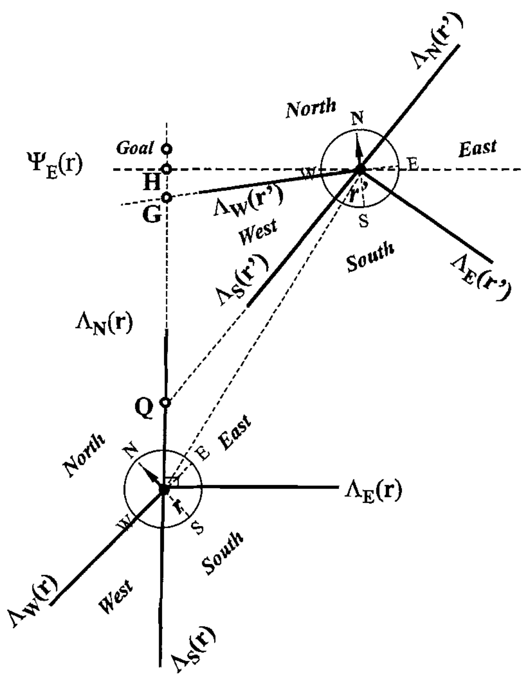


Figure 8. Transformation of the $East/South$ **configuration.**

- *Robot r stops at Goal.*
  By Lemma 3, r reaches *Goal* in finite time. By the algorithm, at time $\bar{t}$, r' is below $\Lambda_E(r)$, thus, $r' \in South(r)$ at $\bar{t}$. Moreover, By Lemma 4, at time $\bar{t}$, $r \in North(r')$. Thus, r and r' become in the configuration $North/South$ with $r' \in South(r)$ at $\bar{t}$. This completes the proof for this case.
- *Robot r stops at* $q \in \overline{GH}$, *and* $q \neq G$.
  This case is similar to the previous one, except that robot r will end its move before reaching its destination *Goal*. Assume that at time $\bar{t}$, r finishes its move at any point that is between H and G. At time $\bar{t}$, r' is still above or on $\Lambda_E(r)$, thus, $r' \in East(r)$. Moreover, at time $\bar{t}$, $r \in North(r')$ since $r(\bar{t})$ is above $\Lambda_W(r')$. Thus, r and r' enter the configuration $East/North$ with $r \in North(r')$ in a finite time.
- *Robot r stops at* $q \in \overline{GQ} \wedge q \neq Q$.
  In this case, at time $\bar{t}$, $r' \in East(r)$ still. In addition, $r \in West(r')$, since r is above $\Lambda_S(r')$. By Lemma 3, r' ends its move in a finite time. Consequently, r and r' become in the configuration $East/West$ in a finite time.
- *Robot r stops at* $p \in \overline{rQ} \wedge p \neq \{r, Q\}$.
  In this case, assume that at time $\bar{t}$, r finishes its move at any point that is between r and Q. At time $\bar{t}$, $r' \in East(r)$ still, since r' does not reach its destination *Goal*. In addition, $r \in South(r')$ because r stops at p, which is below $\Lambda_S(r')$, and r' stays still by hypothesis. As a result, both robots remain in the same configuration $East/South$ at time $\bar{t}$. By assumption, robot r can be activated infinitely often, and by Lemma 3, it can reach its target in a finite time, thus robots r and r' can leave the current configuration $East/South$ in a finite time.

3. **both r and r' move: cases of "seen while moving".**

Let r perform a look operation at time $t \geq t_0$, and start to move toward its destination $Goal$. Let also r' perform a look operation at time $t' \geq t$, and see r while it is moving. Then, three cases follow:

- *r' sees r at position $q \in \overline{rQ}$:*
  In this case, r' will perform a direct move to q. Let $\bar{t}$ and $\bar{t'}$ be the times when r and r' respectively, finish their moves. Assume also, robot r will reach its destination $Goal$ (robot r' may stop before q). Thus, $\forall p \in \triangle(r(t_0), r'(t_0), Q)$, $p \in South(r)$. Also, at $\bar{t'}$, $r \in North(r')$ because r is above $\Lambda_W(r')$. Consequently, r and r' enter the configuration $North/South$ with $r' \in South(r)$ in a finite time.
- *r' sees r at position $q \in \overline{QG}$ and $q \neq Q$:*
  At time t', where r' performs its look operation, we have $r \in West(r')$ because r is between $\Lambda_S(r')$ and $\Lambda_W(r')$. Also, at time t', r still did not reach its target, thus, $r' \in East(r)$ still. As a result, r and r' enter the configuration $East/West$ in a finite time, which is in turn can be transformed by Theorem 3 to the gathering in a finite time.
- *r' sees r at position $q \in \overline{GH}$ and $q \neq G$:*
  At time t', where r' performs its look operation, we have $r \in North(r')$ because r is above $\Lambda_W(r')$. Also, at time t', r still did not reach its target, thus, $r' \in East(r)$ still. As a result, r and r' enter the configuration $North/East$ in a finite time, which is in turn can be transformed by Theorem 2 to the gathering in a finite time.
- *r' sees r at $Goal$:*
  At time t', where r' performs its look operation, r already reaches $Goal$. Thus, by Lemma 4, $r \in North(r')$. Also, at time t', $r' \in South(r)$, since r' becomes below $\Lambda_E(r)$ at t'. As a result, r and r' enter the configuration $North/South$ in a finite time, which is in turn can be transformed by Lemma 5 to the gathering in a finite time.

In all cases, r and r' can only shift to the $North/South$ or $North/East$ or $East/West$ or the gathering configuration in a finite time. This completes the proof. $\square_{\text{Lemma 11}}$

From Lemma 5, Lemma 11, Theorem 2 and Theorem 3, we conclude that:

Theorem 5 *Any $East/South$ configuration of two robots equipped with $\pi/8$-Inaccurate compasses is transformed in a finite time to the gathering.*

Theorem 6 *In a system, with 2 anonymous, oblivious mobile robots relying on inaccurate compasses, the gathering problem is solvable in a finite time for $\pi/8$-Inaccurate compasses.*

PROOF.
Theorem 1 states the different valid configurations by the algorithm. Besides, from Lemma 5, Theorem 2, Theorem 3, Theorem 4 and Theorem 5, any valid configuration is transformed into the gathering in a finite time (see Figure 4) , thus the theorem holds. $\square_{\text{Theorem 6}}$

5 Conclusion

In this paper, we have studied the solvability of the gathering problem of asynchronous mobile robots relying on oblivious computations. While previous work [3] has shown that gathering can be achieved for three or more robots if robots are able to detect multiplicity (i,e. multiple robots at a single point), the case of two robots is left as an open question. Later on, Flocchini et al. [9] proposed a gathering algorithm for any number of robots when they share a common direction as provided by compasses with no errors. In this paper, we concentrate on the gathering of two robots when compasses are subject to errors. In particular, we have presented an algorithm that gathers two asynchronous mobile robots in a finite time when their compasses differ by an angle of at most $\pi/4$. The benefit of our algorithm is that we solve the problem with inexact compasses. Moreover, our algorithm is self-stabilizing and tolerates any number of transient errors. We can also argue that even with a weaker compass that fluctuate for some arbitrary periods, and then it stabilizes eventually to an error of $\pi/8$ with respect to a global north, our algorithm is still valid and solves the problem in a finite time.

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