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Description	

Three-player partizan games

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Abstract

Conway's theory of partizan games is both a theory of games and a theory of numbers. We present here an extension such a theory to classify three-player partizan games. We apply this extension to solve a restricted version of three-player hackenbush.

1 Introduction

Games represent a conflict of interests between two or more parties and, as a consequence, they are a good framework to study complex problem-solving strategies. Typically, a real-world economical, social or political conflict involves more than two parties and a winning strategy is often the result of coalitions. For this reason, it is important to determine the winning strategy of a player in the worst scenario, i.e., assuming the all his/her opponents are allied against him/her.

It is therefore, a challenging and fascinating problem to extend the field of combinatorial game theory so to allow more than two players. Past effort to classify impartial three-player combinatorial games (the theories of Li [4] and Straffin [7]) have made various restrictive assumptions about the rationality of one's opponents and the formation and behavior of coalitions. Loeb [5] introduces the notion of a stable winning coalition in a multi-player game as a new system of classification of games. Differently, J. Propp [6] adopts in his work an agnostic attitude toward such issues, and seeks only to understand in what circumstances one player has a winning strategy against the combined forces of the other two.

In this paper we present a theoretical framework to classify three-player partizan games and we adopt the same attitude. Such theory represents a possible

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extension of Conway's theory of partizan games [2,3], and it is therefore both a theory of games and a theory of numbers. We apply our theoretical model to classify three-player hackenbush instances, that is to say the three-player version of hackenbush, a classical combinatorial game defined in [1].

1.1 Outline

In section 2 we introduce a number-based notation. In accordance with Conway's proposal, a number is defined as a triple of sets of numbers previously defined. In a typical two-player zero-sum game, the advantage of a player is the disadvantage for his opponent. In general, in a three-player game the advantage of the first player can be a disadvantage for the second player and an advantage for the third player or vice versa, or a disadvantage for both of the opponents. For this reason we introduce three different relations ($\geq_L$,$\geq_C$, $\geq_R$) that represent the subjective point of view of every player which is independent from the point of view of the other players.
In section 3 we introduce games. A game is defined like a triple of sets of games previously defined where every set represents the different moves of every single player. The main difference between number and games is that numbers are totally ordered with respect to every relation defined in section 2, whereas games are not. We also introduce a sum operation and discuss some of its properties.
In section 4 we show that it is possible to classify numbers in 11 sub-classes representing a partition of our set of numbers.
In section 5 we show what happens when we add two numbers and in which cases it is possible to determine the sub-class of the sum.
In section 6 we provide the relations between numbers and games. In other words, we try to understand when it is possible to determine the outcome of a game represented by a number that belongs to a specific sub-class. Knowing the outcome of a game means that we are able to determine the player that has a winning strategy once we fixed the player that starts the game. Moreover, we prove that there exists just one zero-game, i.e., a game that does not affect the outcome of another game when we add them together.
In section 7 we prove that every instance of three-player hackenbush is a number and present a theorem that is very useful in practice to classify such instances.
Section 8 summarizes the results obtained so far.

2 Basic Definitions

2.1 Construction

If L, C, R are any three sets of number, and

- no element of L is $\geq_L$ any element of $C \cup R$, and
- no element of C is $\geq_C$ any element of $L \cup R$, and
- no element of R is $\geq_R$ any element of $L \cup C$,

then $\{L|C|R\}$ is a number. All numbers are constructed in this way.

2.2 Convention

If $x = \{L|C|R\}$ we denote by x^L, x^C and x^R respectively the typical member of L, C, and R. x can therefore be written as $\{x^L|x^C|x^R\}$.
The notation $x = \{a, b, c, \ldots|d, e, f, \ldots|g, h, i, \ldots\}$, considering that $x = \{L|C|R\}$, tells us that $a, b, c, \ldots$ are the typical elements of L, $d, e, f, \ldots$ are the typical elements of C, and $g, h, i, \ldots$ the typical elements of R.

2.3 Definitions

Definition of $x \geq_L y, x \leq_L y, x \geq_C y, x \leq_C y, x \geq_R y, x \leq_R y$.
We say that $x \geq_L y$ iff ($y \geq_L$ no x^C and $y \geq_L$ no x^R and no $y^L \geq_L x$), and $x \leq_L y$ iff $y \geq_L x$.
We write $x \not\leq_L y$ to mean that $x \leq_L y$ does not hold.
We say that $x \geq_C y$ iff ($y \geq_C$ no x^L and $y \geq_C$ no x^R and no $y^C \geq_C x$), and $x \leq_C y$ iff $y \geq_C x$.
We write $x \not\leq_C y$ to mean that $x \leq_C y$ does not hold.
We say that $x \geq_R y$ iff ($y \geq_R$ no x^L and $y \geq_R$ no x^C and no $y^R \geq_R x$), and $x \leq_R y$ iff $y \geq_R x$.
We write $x \not\leq_R y$ to mean that $x \leq_R y$ does not hold.

Definition of $x =_L y, x >_L y, x <_L y, x =_C y, x >_C y, x <_C y, x =_R y, x >_R y, x <_R y$.
$x =_L y$ iff ($x \geq_L y$ and $y \geq_L x$).
$x >_L y$ iff ($x \geq_L y$ and $y \not\geq_L x$).
$x <_L y$ iff $y >_L x$.
$x =_C y$ iff ($x \geq_C y$ and $y \geq_C x$).
$x >_C y$ iff ($x \geq_C y$ and $y \not\geq_C x$).

$x <_C y$ iff $y >_C x$.
$x =_R y$ iff ($x \geq_R y$ and $y \geq_R x$).
$x >_R y$ iff ($x \geq_R y$ and $y \not\geq_R x$).
$x <_R y$ iff $y >_R x$.

Definition of $x = y$.
$x = y$ iff $(x =_L y, x =_C y, x =_R y)$.

Definition of $x + y$.
$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^R + y, x + y^R\}$.

All the given definitions are inductive, so that to decide, for instance, whether $x \leq_L y$ we check the pairs $(x^C, y), (x^R, y)$, and (x, y^L) and so on.

2.4 Examples of numbers, and some of their properties

According to the construction procedure, every number has the form $\{L|C|R\}$, where L, C, and R are three sets of earlier constructed numbers. At day zero, we have only the empty set $\emptyset$. So the earliest constructed number could only be $\{L|C|R\}$ with $L = C = R = \emptyset$, or in the simplified notation $\{\ |\ |\ \}$. We denote it by 0.
Is 0 a number ? The answer is yes, since we cannot have any inequality of the form

$$0^L \geq_L 0^C, 0^L \geq_L 0^R, 0^C \geq_C 0^L, 0^C \geq_C 0^R, 0^R \geq_R 0^L, 0^R \geq_R 0^C$$

because $0^L, 0^C$ and 0^R are all the empty set. For the same reason we can observe that $0 \geq_L 0$ so we have $0 =_L 0$. Moreover $0 =_C 0$ and $0 =_R 0$ so we have $0 = 0$.
We can now use the sets $\{\}$, i.e. the empty set, and $\{0\}$ for L, C and R to obtain

$$\{\ |\ |\ \}, \{0|\ |\ \}, \{\ |0|\ \}, \{\ |\ |0\}, \{0|0|\ \}, \{0|\ |0\}, \{\ |0|0\}, \{0|0|0\}.$$

We have only three new numbers, which we call $1_L = \{0|\ |\ \}, 1_C = \{\ |0|\ \}$, and $1_R = \{\ |\ |0\}$. It can easily be checked that $\{0|0|\ \}, \{0|\ |0\}, \{\ |0|0\}$, and $\{0|0|0\}$ are not numbers.
In table 1 we have summarized the relations between the numbers so far created. At this point, we have 4 numbers and, using them appropriately, we can create 18 numbers which are shown in table 2.

	0	1_L	1_C	1_R
0	$=$	$<_L$,$>_C$,$>_R$	$>_L$,$<_C$,$>_R$	$>_L$,$>_C$,$<_R$
1_L	$>_L$,$<_C$,$<_R$	$=$	$>_L$,$<_C$,$=_R$	$>_L$,$=_C$,$<_R$
1_C	$<_L$,$>_C$,$<_R$	$<_L$,$>_C$,$=_R$	$=$	$=_L$,$>_C$,$<_R$
1_R	$<_L$,$<_C$,$>_R$	$<_L$,$=_C$,$>_R$	$=_L$,$<_C$,$>_R$	$=$

Table 1

$\{1_L\| \ \| \ \}$	$\{ \ \|1_C\| \ \}$	$\{ \ \| \ \|1_R\}$
$\{1_C\| \ \| \ \}$	$\{ \ \|1_L\| \ \}$	$\{ \ \| \ \|1_L\}$
$\{1_R\| \ \| \ \}$	$\{ \ \|1_R\| \ \}$	$\{ \ \| \ \|1_C\}$
$\{ \ \|0\|1_C\}$	$\{0\| \ \|1_L\}$	$\{0\|1_L\| \ \}$
$\{ \ \|1_R\|0\}$	$\{1_R\| \ \|0\}$	$\{1_C\|0\| \ \}$
$\{ \ \|1_R\|1_C\}$	$\{1_R\| \ \|1_L\}$	$\{1_C\|1_L\| \ \}$

Table 2

3 Relations and operations

The construction for numbers generalizes immediately to the following construction for what we call *games.*
Construction. If L, C and R are any three sets of games, then there is a game $\{L|C|R\}$. All games are constructed in this way. Order-relations and arithmetic operations on games are defined analogously to numbers. The most important distinction between numbers and general games is that numbers are totally ordered but games are not, e.g. there exist games x and y for which we have neither $x \geq_L y$ nor $y \geq_L x$. To show that a game $x = \{x^L|x^C|x^R\}$ is a number, we must show that the games x^L, x^C, x^R are numbers, and that there is no inequality of the form

$$x^L \geq_L x^C, x^L \geq_L x^R, x^C \geq_C x^L, x^C \geq_C x^R, x^R \geq_R x^L \text{ e } x^R \geq_R x^C.$$

3.1 Identity

We shall call games x and y *identical* ($x \equiv y$) if their left, center, and right sets are identical, that is, if x^L is identical to y^L, x^C is identical to y^C, and x^R is identical to y^R. Thus, $x \equiv y \Rightarrow x = y$.
Finally, we note that almost all our proofs are inductive, so that, for instance, in proving something about the pair (x, y) we can suppose that it is

already known about all pairs $(x^L, y), (x^C, y), (x^R, y), (x, y^L), (x, y^C), (x, y^R)$. The games x^L, x^C and x^R will be called the left, center, and right *options* of x.

3.2 Properties of order and equality

Recall that $x \geq_L y$ if we have no inequality of the form

$$x^C \leq_L y, x^R \leq_L y, x \leq_L y^L.$$

Theorem 1 *For all games x we have*

(1) $x^L \ngeq_L x$, $x \ngeq_L x^C$, and $x \ngeq_L x^R$.
(2) $x^C \ngeq_C x$, $x \ngeq_C x^L$, and $x \ngeq_C x^R$.
(3) $x^R \ngeq_R x$, $x \ngeq_R x^L$, and $x \ngeq_R x^C$.
(4) $x \geq_L x$, $x \geq_C x$, and $x \geq_R x$.
(5) $x = x$.

PROOF.

(1) If $x^L \geq_L x$ was true we could not have $x^L \geq_L x^L$ which is true using inductive hypothesis. The same reasoning holds for the other two cases.
(2) Analogous to (1).
(3) Analogous to (1).
(4) It follows from (1), (2) and (3).
(5) It follows from (4).

□

Theorem 2 *For any three games x,y and z we have*

(1) If $x \geq_L y$ and $y \geq_L z$, then $x \geq_L z$.
(2) If $x \geq_C y$ and $y \geq_C z$, then $x \geq_C z$.
(3) If $x \geq_R y$ and $y \geq_R z$, then $x \geq_R z$.

PROOF.

(1) To show that $x \geq_L z$ we have to show that there is no inequality of the form $z^L \geq_L x$, $z \geq_L x^C$ and $z \geq_L x^R$. If the first inequality is true and since $x \geq_L y$ is true, it would follow, using inductive hypothesis, that $z^L \geq_L y$ which is impossible because $y \geq_L z$. If the second inequality is true since $y \geq_L z$ and using inductive hypothesis, it would follow that

$y \geq_L x^C$ which is impossible because $x \geq_L y$. Analogously, for the third inequality.

(2) Analogous to (1).

(3) Analogous to (1).

□

Summing up, we can claim that $\geq_L$,$\geq_C$, and $\geq_R$ are partial order relations on games.

Theorem 3 *For any number x*

(1) $x^L \leq_L x, x \leq_L x^C, x \leq_L x^R$.
(2) $x^C \leq_C x, x \leq_C x^L, x \leq_C x^R$.
(3) $x^R \leq_R x, x \leq_R x^L, x \leq_R x^C$

and, for any two numbers x and y

(4) $x \leq_L y$ or $x \geq_L y$.
(5) $x \leq_C y$ or $x \geq_C y$.
(6) $x \leq_R y$ or $x \geq_R y$.

PROOF.

(1) Let us consider the first inequality. We know that $x^L \ngeq_L x$ so we only have to show that $x^L \leq_L x$. If the latter inequality was false then one of the inequalities $x^{LL} \geq_L x$, $x^C \leq_L x^L$, $x^R \leq_L x^L$ would have be true. If $x^{LL} \geq_L x$ was true, since $x^L \geq_L x^{LL}$ is true for inductive hypothesis, we would have for transitivity that $x^L \geq_L x$ that we know is false. The other two inequalities are both false by the definition of number. The same reasoning can be applied for the second and the third inequalities of (1).

(2) Analogous to (1).

(3) Analogous to (1).

(4) The inequality $x \ngeq_L y$ implies $x^R \leq_L y$ or $x^C \leq_L y$ or $x \leq_L y^L$. Hence we have $x <_L x^R \leq_L y$ or $x <_L x^C \leq_L y$ or $x \leq_L y^L <_L y$.

(5) Analogous to (4).

(6) Analogous to (4).

□

Thus we can claim that numbers are totally ordered with respect to $\leq_L$, $\leq_C$, and $\leq_R$.

3.3 Properties of addition

Theorem 4 *For all x, y, z we have*

(1) $x + 0 \equiv x$.
(2) $x + y \equiv y + x$.
(3) $(x + y) + z \equiv x + (y + z)$.

PROOF.

(1) $x + 0 \equiv \{x^L + 0 | x^C + 0 | x^R + 0\} \equiv \{x^L | x^C | x^R\} \equiv x$

(2) $x + y \equiv$
$\{x^L + y, x + y^L | x^C + y, x + y^C | x^R + y, x + y^R\} \equiv$
$\{y + x^L, y^L + x | y + x^C, y^C + x | y + x^R, y^R + x\} \equiv$
$y + x$

(3) $(x + y) + z \equiv$
$\{(x + y)^L + z, (x + y) + z^L | \ldots | \ldots\} \equiv$
$\{(x^L + y) + z, (x + y^L) + z, (x + y) + z^L | \ldots | \ldots\} \equiv$
$\{x^L + (y + z), x + (y^L + z), x + (y + z^L) | \ldots | \ldots\} \equiv$
$\{x^L + (y + z), x + (y + z)^L | \ldots | \ldots\} \equiv$
$x + (y + z)$

In each case the middle identity follows from the inductive hypothesis.

□

3.4 Properties of addition and order

Theorem 5 *If x and y are numbers then*

(1) $y \geq_L z$ *iff* $x + y \geq_L x + z$.
(2) $y \geq_C z$ *iff* $x + y \geq_C x + z$.
(3) $y \geq_R z$ *iff* $x + y \geq_R x + z$.

PROOF.

(1) If $x + y \geq_L x + z$ then the following inequalities are false

$$x + y^R \leq_L x + z, x + y^C \leq_L x + z, x + y \leq x + z^L$$

and so, by induction, the following inequality are also false

$$y^R \leq_L z, y^C \leq_L z, y \leq_L z^L$$

Therefore, we have $y \geq_L z$.
Let us suppose now that $x + y \not\geq_L x + z$. It follows, that at least one of the following inequalities must be true

$$x^R + y \leq_L x + z, x + y^R \leq_L x + z, x^C + y \leq_L x + z,$$
$$x + y^C \leq_L x + z, x + y \leq_L x^L + z, x + y \leq_L x + z^L$$

If we suppose by contradiction that $y \geq_L z$, by transitivity we have that at least one of the following inequalities must be true

$$x^R + y \leq_L x + y, x + y^R \leq_L x + y, x^C + y \leq_L x + y,$$
$$x + y^C \leq_L x + y, x + z \leq_L x^L + z, x + z \leq_L x + z^L$$

all of which give us a contradiction by the cancellation law on the partial order relation.

(2) Analogous to (1).
(3) Analogous to (1).

□

As a corollary of the above theorem we have

Corollary 6 *If x, y, and z are numbers then $y = z$ iff $x + y = x + z$.*

Theorem 7 *If x and y are numbers then $x + y$ is a number.*

PROOF. By induction we have that

$$x^L + y, x + y^L \leq_L x + y \leq_L x^C + y, x + y^C, x^R + y, x + y^R$$
$$x^C + y, x + y^C \leq_C x + y \leq_C x^L + y, x + y^L, x^R + y, x + y^R$$
$$x^R + y, x + y^R \leq_R x + y \leq_R x^L + y, x + y^L, x^C + y, x + y^C$$

and, since by induction $x^L + y$, etc., are numbers, we have that $x + y$ is a number.

□

3.5 The simplicity theorem

Theorem 8 *Let $x = \{x^L | x^C | x^R\}$ and suppose that for a number z the following properties hold:*

- $x^L \not\geq_L z \not\geq_L x^C, x^R$ *for all x^L, x^C, and x^R, but that no option of z satisfies the same condition.*
- $x^C \not\geq_C z \not\geq_C x^L, x^R$ *for all x^L, x^C, and x^R, but that no option of z satisfies the same condition.*

- *$x^R \not\geq_R z \not\geq_R x^L, x^C$ for all x^L, x^C, and x^R, but that no option of z satisfies the same condition.*

Then $x = z$.

PROOF. We have $x \geq_L z$ unless one of the following is true

$$x^R \leq_L z, x^C \leq_L z, x \leq_L z^L$$

The first and the second inequalities are false by hypothesis, moreover if the third was true

$$x^L \not\geq_L x \leq_L z^L < z \not\geq_L x^R, x^C$$

it would follow

$$x^L \not\geq_L z^L \not\geq_L x^R, x^C$$

which contradicts the hypotheses on z. Analogously, we can show that $z \geq_L x$ obtaining in turn that $x =_L z$. Similar reasoning let us conclude that $x =_C z$ and $x =_R z$. So it follows that $x = z$.

□

The following theorem holds.

Theorem 9 *Let $x = \{a, b, \ldots | c, d, \ldots | e, f, \ldots\}$ and $y = \{b, \ldots | d, \ldots | f, \ldots\}$ be numbers.*

(1) If $a \leq_L b$, $d \leq_L c$, and $f \leq_L e$ then $x =_L y$.
(2) If $b \leq_C a$, $c \leq_C d$, and $f \leq_C e$ then $x =_C y$.
(3) If $b \leq_R a$, $d \leq_R c$, and $e \leq_R f$ then $x =_R y$.

PROOF.

(1) We know that
$a, b, \ldots <_L x <_L c, d, \ldots, e, f, \ldots$ and $b, \ldots <_L y <_L d, \ldots, f, \ldots$
By hypothesis $a \leq_L b$, $d \leq_L c$, and $f \leq_L e$ so it follows by transitivity $a <_L y$, $y <_L c$, and $y <_L e$.
By definition we have $x \leq_L y$ and $y \leq_L x$.
(2) Analogous to (1).
(3) Analogous to (1).

□

G	Left starts	Right starts
> 0	Left wins	Left wins
< 0	Right wins	Right wins
$= 0$	Right wins	Left wins
$\parallel 0$	Left wins	Right wins

Table 3

1	$<_L, <_C, <_R$	10	$=_L, <_C, <_R$	19	$>_L, <_C, <_R$
2	$<_L, <_C, =_R$	11	$=_L, <_C, =_R$	20	$>_L, <_C, =_R$
3	$<_L, <_C, >_R$	12	$=_L, <_C, >_R$	21	$>_L, <_C, >_R$
4	$<_L, =_C, <_R$	13	$=_L, =_C, <_R$	22	$>_L, =_C, <_R$
5	$<_L, =_C, =_R$	14	$=_L, =_C, =_R$	23	$>_L, =_C, =_R$
6	$<_L, =_C, >_R$	15	$=_L, =_C, >_R$	24	$>_L, =_C, >_R$
7	$<_L, >_C, <_R$	16	$=_L, >_C, <_R$	25	$>_L, >_C, <_R$
8	$<_L, >_C, =_R$	17	$=_L, >_C, =_R$	26	$>_L, >_C, =_R$
9	$<_L, >_C, >_R$	18	$=_L, >_C, >_R$	27	$>_L, >_C, >_R$

Table 4

4 Outcome classes

We recall that in a two-player combinatorial game theory we can classify all games into four *outcome classes*, which specify who has the winning strategy when Left starts and who has the winning strategy when Right starts, as shown in table 3. If we consider three-player games the situation is more complicated because we can have at most 27 *outcome classes*.

We will classify only cold games, i.e. games whose value is a number. We know that numbers are totally ordered so if we compare (for Left) a generic number x with 0 we have one of the three following cases: $x <_L 0$, $x =_L 0$, or $x >_L 0$. Analogously, we have three different cases for Center and Right. Moreover, we recall that $>_L$ represents the subjective point of view of the Left player which is independent from the Central and Right players. Hence, when we compare a number x with 0 we have 27 possible outcomes that are represented in table 4.

Theorem 10 *Ninth, twenty-first and twenty-fifth classes are empty.*

PROOF. Suppose that there exists a number x such that $x <_L 0$, $x >_C 0$, and $x >_R 0$. We have

- $x <_L 0 \Rightarrow x^L <_L 0$
- $x >_C 0 \Rightarrow x^L >_C 0$
- $x >_R 0 \Rightarrow x^L >_R 0$

Which contradicts the inductive hypothesis. In the same way for $x >_L 0$, $x <_C 0$, $x >_R 0$ and $x >_L 0$, $x >_C 0$, $x <_R 0$.

□

Theorem 11 *Eighteenth, twenty-fourth and twenty-sixth classes are empty.*

PROOF. Suppose that there exists a number x such that $x =_L 0$, $x >_C 0$, and $x >_R 0$. We have

- $x =_L 0 \Rightarrow x^L <_L 0$
- $x >_C 0 \Rightarrow x^L >_C 0$
- $x >_R 0 \Rightarrow x^L >_R 0$

which is a contradiction by theorem 10. Analogously for $x >_L 0$, $x =_C 0$, $x >_R 0$ and $x >_L 0$, $x >_C 0$, and $x =_R 0$.

□

Theorem 12 *The twenty-seventh class is empty.*

PROOF. Suppose that there exists a number x such that $x >_L 0$, $x >_C 0$, and $x >_R 0$. We have

- $x >_L 0 \Rightarrow x^R >_L 0$
- $x >_C 0 \Rightarrow x^R >_C 0$

By theorem 10 we cannot have $x^R <_R 0$ and by theorem 11 we cannot have $x^R =_R 0$ so we have $x^R >_R 0$ which contradicts the inductive hypothesis.

□

Theorem 13 *Fifteenth, seventeenth and twenty-third classes are empty.*

PROOF. Suppose that there exists a number x such that $x =_L 0$, $x =_C 0$, and $x >_R 0$. We have

- $x =_L 0 \Rightarrow x^L <_L 0$
- $x =_C 0 \Rightarrow x^L >_C 0$
- $x >_R 0 \Rightarrow x^L >_R 0$

which is a contradiction by theorem 10. Analogously for $x =_L 0$, $x >_C 0$, $x =_R 0$ and $x >_L 0$, $x =_C 0$, and $x =_R 0$.

$\square$

Theorem 14 *Sixth, eighth, twelfth, sixteenth, twentieth and twenty-second classes are empty.*

PROOF. Suppose that there exists a number x such that $x <_L 0$, $x =_C 0$, and $x >_R 0$. We have

- $x <_L 0 \Rightarrow x^L <_L 0$
- $x =_C 0 \Rightarrow x^L >_C 0$
- $x >_R 0 \Rightarrow x^L >_R 0$

and we obtain a contradiction by theorem 10. Analogously for the other classes.

$\square$

Theorem 15 *First, second, third, fourth, fifth, seventh, tenth, eleventh, thirteenth, fourteenth, nineteenth classes are not empty.*

PROOF. It sufficient to note that

- $\{\{\ |1_R|1_C\}|\{1_R|\ |1_L\}|\{1_C|1_L|\ \}\}$ belongs to the first class
- $\{\ |\ |\{1_C|1_L|\ \}\}$ belongs to the second class
- $\{\ |\ |1_R\}$ belongs to the third class
- $\{\ |\{1_R|\ |1_L\}|\ \}$ belongs to the fourth class
- $\{\ |1_R|1_C\}$ belongs to the fifth class
- $\{\ |1_C|\ \}$ belongs to the seventh class
- $\{\{\ |1_R|1_C\}|\ |\ \}$ belongs to the tenth class
- $\{1_R|\ |1_L\}$ belongs to the eleventh class
- $\{1_C|1_L|\ \}$ belongs to the thirteenth class
- $\{\ |\ |\ \}$ belongs to the fourteenth class
- $\{1_L|\ |\ \}$ belongs to the nineteenth class

$\square$

Theorem 16 0 *is the only number that belongs to the fourteenth class.*

Class	Short notation
$=_L, =_C, =_R$	$=$
$>_L, <_C, <_R$	$>_L$
$<_L, >_C, <_R$	$>_C$
$<_L, <_C, >_R$	$>_R$
$=_L, =_C, <_R$	$=_{LC}$
$=_L, <_C, =_R$	$=_{LR}$
$<_L, =_C, =_R$	$=_{CR}$
$=_L, <_C, <_R$	$<_{CR}$
$<_L, =_C, <_R$	$<_{LR}$
$<_L, <_C, =_R$	$<_{LC}$
$<_L, <_C, <_R$	$<$

Table 5

PROOF. Suppose that x belongs to fourteenth class. We have

- $x =_L 0 \Rightarrow x^L <_L 0, x^C >_L 0, x^R >_L 0$
- $x =_C 0 \Rightarrow x^C <_C 0, x^L >_C 0, x^R >_C 0$
- $x =_R 0 \Rightarrow x^R <_R 0, x^L >_R 0, x^C >_R 0$

It follows that $x^L = x^C = x^R = \emptyset$ hence $x = \{ \mid \mid \} = 0$.

$\square$

Summarizing we have 11 outcome classes that are shown in table 5.

5 Sum of cold games

In this section we give first some results that will help us sum two cold games. Subsequently, we will give the complete table for all possible cases.

Theorem 17 *If x, y are numbers then*

(1) $x \geq_L 0, y \geq_L 0 \Rightarrow x + y \geq_L 0$
(2) $x \geq_L 0, y >_L 0 \Rightarrow x + y >_L 0$

PROOF.

(1) By hypothesis

$$x \geq_L 0 \Rightarrow x^C, x^R >_L 0$$

and

$$y \geq_L 0 \Rightarrow y^C, y^R >_L 0$$

We recall that

$$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^C + y, x + y^C\}$$

By inductive hypothesis the following inequalities are true

$$x + y^C >_L 0, x^C + y >_L 0, x + y^R >_L 0, x^R + y >_L 0$$

so we have

$$x + y \geq_L 0$$

(2) By hypothesis

$$x \geq_L 0 \Rightarrow x^C, x^R >_L 0$$

and

$$y >_L 0 \Rightarrow y^C, y^R >_L 0$$

and there exists at least

$$y^L \geq_L 0$$

We also recall that

$$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^C + y, x + y^C\}$$

To show that $x + y >_L 0$ it is sufficient to note that $x + y^L \geq_L 0$.

□

The following two theorems can be proven in an analogous way.

Theorem 18 *If x, y are numbers then*

(1) $x \geq_C 0, y \geq_C 0 \Rightarrow x + y \geq_C 0$
(2) $x \geq_C 0, y >_C 0 \Rightarrow x + y >_C 0$

Theorem 19 *If x, y are numbers then*

(1) $x \geq_R 0, y \geq_R 0 \Rightarrow x + y \geq_R 0$
(2) $x \geq_R 0, y >_R 0 \Rightarrow x + y >_R 0$

We also have

Theorem 20 *If x, y are numbers then*

(1) $x \leq_L 0, y \leq_L 0 \Rightarrow x + y \leq_L 0$

(2) $x \leq_L 0, y <_L 0 \Rightarrow x + y <_L 0$

PROOF.

(1) By hypothesis

$$x \leq_L 0 \Rightarrow x^L <_L 0$$

and

$$y \leq_L 0 \Rightarrow y^L <_L 0$$

We recall that

$$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^C + y, x + y^C\}$$

By inductive hypothesis the following inequalities are true

$$x^L + y <_L 0, x + y^L <_L 0$$

so we have

$$x + y \leq_L 0$$

(2) By hypothesis

$$x \leq_L 0 \Rightarrow x^L <_L 0$$

and

$$y <_L 0 \Rightarrow y^L <_L 0$$

and there exists at least

$$y^C \cup y^R \leq_L 0$$

We recall that

$$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^C + y, x + y^C\}$$

To show that $x + y <_L 0$ it is sufficient to note that $x + y^C \vee x + y^R \leq_L 0$.

□

The following two theorems can be proven in an analogous way.

Theorem 21 *If x, y are numbers then*

(1) $x \leq_C 0, y \leq_C 0 \Rightarrow x + y \leq_C 0$
(2) $x \leq_C 0, y <_C 0 \Rightarrow x + y <_C 0$

Theorem 22 *If x, y are numbers then*

(1) $x \leq_R 0, y \leq_R 0 \Rightarrow x + y \leq_R 0$
(2) $x \leq_R 0, y <_R 0 \Rightarrow x + y <_R 0$

Theorem 23 *If x, y are numbers then*

(1) $x =_{LC} 0, y =_{LC} 0 \Rightarrow x + y =_{LC} 0$
(2) $x =_{LR} 0, y =_{LR} 0 \Rightarrow x + y =_{LR} 0$
(3) $x =_{CR} 0, y =_{CR} 0 \Rightarrow x + y =_{CR} 0$

PROOF.

(1) By hypothesis

$$x \geq_L 0, y \geq_L 0 \Rightarrow x + y \geq_L 0$$

and

$$x \geq_C 0, y \geq_C 0 \Rightarrow x + y \geq_C 0$$

If we consider the possible legal cases there is only one possibility

$$x + y =_{LC} 0$$

(2) Analogous to (1)
(3) Analogous to (1)

□

Theorem 24 *If x, y are numbers then*

(1) $x =_{LC} 0, y =_{LR} 0 \Rightarrow x + y <_{CR} 0$
(2) $x =_{LC} 0, y =_{CR} 0 \Rightarrow x + y <_{LR} 0$
(3) $x =_{LR} 0, y =_{CR} 0 \Rightarrow x + y <_{LC} 0$

PROOF.

(1) By hypothesis

$$x \leq_L 0, y \leq_L 0 \Rightarrow x + y \leq_L 0$$

$$x \geq_L 0, y \geq_L 0 \Rightarrow x + y \geq_L 0$$

Moreover

$$x \leq_C 0, y <_C 0 \Rightarrow x + y <_C 0$$

and

$$x <_R 0, y \leq_R 0 \Rightarrow x + y <_R 0$$

It follows $x + y <_{CR} 0$
(2) Analogous to (1)
(3) Analogous to (1)

□

	$=$	$>_L$	$>_C$	$>_R$	$=_{LC}$	$=_{LR}$	$=_{CR}$	$<_{CR}$	$<_{LR}$	$<_{LC}$	$<$
$=$	$=$	$>_L$	$>_C$	$>_R$	$=_{LC}$	$=_{LR}$	$=_{CR}$	$<_{CR}$	$<_{LR}$	$<_{LC}$	$<$
$>_L$	$>_L$	$>_L$	?	?	$>_L$	$>_L$	?	$>_L$	?	?	?
$>_C$	$>_C$	?	$>_C$	?	$>_C$	?	$>_C$	?	$>_C$	?	?
$>_R$	$>_R$	?	?	$>_R$	?	$>_R$	$>_R$	?	?	$>_R$	?
$=_{LC}$	$=_{LC}$	$>_L$	$>_C$	?	$=_{LC}$	$<_{CR}$	$<_{LR}$	$<_{CR}$	$<_{LR}$	$<$	$<$
$=_{LR}$	$=_{LR}$	$>_L$	?	$>_R$	$<_{CR}$	$=_{LR}$	$<_{LC}$	$<_{CR}$	$<$	$<_{LC}$	$<$
$=_{CR}$	$=_{CR}$	?	$>_C$	$>_R$	$<_{LR}$	$<_{LC}$	$=_{CR}$	$<$	$<_{LR}$	$<_{LC}$	$<$
$<_{CR}$	$<_{CR}$	$>_L$	?	?	$<_{CR}$	$<_{CR}$	$<$	$<_{CR}$	$<$	$<$	$<$
$<_{LR}$	$<_{LR}$	?	$>_C$	?	$<_{LR}$	$<$	$<_{LR}$	$<$	$<_{LR}$	$<$	$<$
$<_{LC}$	$<_{LC}$	?	?	$>_R$	$<$	$<_{LC}$	$<_{LC}$	$<$	$<$	$<_{LC}$	$<$
$<$	$<$	?	?	?	$<$	$<$	$<$	$<$	$<$	$<$	$<$

Table 6

In table 6 we show all the possible cases when we sum two numbers. The entries '?' are unrestricted and indicate that we can have more than one outcome, e.g., if $x = \{1_L| \mid \} = 2_L$ and $y = 1_C$ then $x + y >_L 0$ but if $x = 1_L$ and $y = 1_C$ then $x + y =_{LC} 0$.

6 Winning strategies

In this section we give some results that help us in better understanding the relations between a number and the possible winning strategies in the game that this number represents. Players take turns making legal moves in a cyclic fashion $(\ldots, L, C, R, L, C, R, \ldots)$ until one of the players is unable to move. Then that player leaves the game and the other two continue in alternation until one of them cannot move. Then that player leaves the game, and the remaining player is the winner.

Theorem 25 *Let x be a number. If $x = 0$ then there exists a winning strategy for the player who moves last.*

PROOF. We recall that the $x = \{ \mid \mid \}$ is the only number which is equal to 0.

□

Theorem 26 *Let* $x = \{x^L|x^C|x^R\}$ *be a number. If* $x >_L 0$ *then there exists a winning strategy for Left.*

PROOF. By hypothesis $x >_L 0$, so it follows that $x^C >_L 0$ and $x^R >_L 0$. Therefore by inductive hypothesis if Center or Right play first, Left has a winning strategy.
Moreover, there exists at least an option x^L for Left such that $x^L \geq_L 0$. If $x^L >_L 0$ Left has a winning strategy by inductive hypothesis, so if Left starts first will certainly choose this option.
If $x^L =_L 0$ we have $x^L = \{x^{LL}|x^{LC}|x^{LR}\} =_L 0$ and it is Center or Right turn to play because Left just played. But $x^{LC} >_L 0$ and $x^{LR} >_L 0$ and so, by inductive hypothesis, there exists a winning strategy for Left.
We conclude by remarking that, obviously, even if x^C and/or x^R are empty, Left has still a winning strategy.

□

The following two theorems can be proven analogously.

Theorem 27 *Let* $x = \{x^L|x^C|x^R\}$ *be a number. If* $x >_C 0$ *then there exists a winning strategy for Center.*

Theorem 28 *Let* $x = \{x^L|x^C|x^R\}$ *be a number. If* $x >_R 0$ *then there exists a winning strategy for Right.*

The following theorem also holds.

Theorem 29 *Let* $x = \{x^L|x^C|x^R\}$ *be a number. If* $x =_{LC} 0$ *then there exists a winning strategy for player (Left or Center) who moves last.*

PROOF. By hypothesis we have $x^C >_L 0$, $x^R >_L 0$, $x^L >_C 0$, and $x^R >_C 0$. It follows that $x^R = \emptyset$ and we can observe that

- If Left starts, Center has a winning strategy because $x^L >_C 0$.
- If Center starts, Left has a winning strategy because $x^C >_L 0$.
- If Right starts, Left plays and Center wins.

If x^L or x^C are empty we have the same result.

□

The following two theorems can be proven in the same way.

x^L	Left starts	Center starts	Right starts
$>_C$	Center wins	Left wins	Left wins
$>_R$	Right wins	Left wins	Left wins
$=_{CR}$	Right wins	Left wins	Left wins
$<_{LC}$	Right wins	Left wins	Left wins
$<_{LR}$	?	Left wins	Left wins
$<$	?	Left wins	Left wins

Table 7

x^C	Left starts	Center starts	Right starts
$>_L$	Center wins	Left wins	Center wins
$>_R$	Center wins	Right wins	Center wins
$=_{LR}$	Center wins	Left wins	Center wins
$<_{LC}$	Center wins	?	Center wins
$<_{CR}$	Center wins	Left wins	Center wins
$<$	Center wins	?	Center wins

Table 8

Theorem 30 *Let $x = \{x^L|x^C|x^R\}$ be a number. If $x =_{LR} 0$ then there exists a winning strategy for the player (Left or Right) who moves last.*

Theorem 31 *Let $x = \{x^L|x^C|x^R\}$ be a number. If $x =_{CR} 0$ then there exists a winning strategy for the player (Center or Right) who moves last.*

The following theorems hold.

Theorem 32 *Let $x = \{x^L|x^C|x^R\} <_{CR} 0$ be a number. If either Center or Right starts the game then Left has a winning strategy.*

PROOF. By hypothesis $x =_L 0$ hence $x^C >_L 0$ and $x^R >_L 0$. When Left starts the outcome depends on x^L. Table 7 shows the outcomes.

□

Theorem 33 *Let $x = \{x^L|x^C|x^R\} <_{LR} 0$ be a number. If either Left or Right starts the game then Center has a winning strategy.*

x^R	Left starts	Center starts	Right starts
$>_L$	Right wins	Right wins	Left wins
$>_C$	Right wins	Right wins	Center wins
$=_{LC}$	Right wins	Right wins	Center wins
$<_{LR}$	Right wins	Right wins	Center wins
$<_{CR}$	Right wins	Right wins	?
$<$	Right wins	Right wins	?

Table 9

x^L	x^C	x^R
$>_C$	$>_L$	$>_L$
$>_R$	$>_R$	$>_C$
$=_{CR}$	$=_{LR}$	$=_{LC}$
$<_{LR}$	$<_{LC}$	$<_{LR}$
$<_{LC}$	$<_{CR}$	$<_{CR}$
$<$	$<$	$<$

Table 10

PROOF. From the hypothesis $x =_C 0$ hence $x^L >_C 0$ and $x^R >_C 0$. When Center starts the outcome depends on x^C. Table 8 shows the outcomes.

□

Theorem 34 *Let $x = \{x^L|x^C|x^R\} <_{LC} 0$ be a number. If either Left or Center starts the game then Right has a winning strategy.*

PROOF. From the hypothesis $x =_R 0$ hence $x^L >_R 0$ and $x^C >_R 0$. When Right starts the outcome depends on x^R. Table 9 shows the outcomes.

□

We investigate now what happens when $x < 0$.
In this case the outcome depends on x^L, x^C, and x^R. Table 10 shows the possible different classes for every option. We can summarize all the results we have obtained so far in table 11.

	Left starts	Center starts	Right starts
$=$	Right wins	Left wins	Center wins
$>_L$	Left wins	Left wins	Left wins
$>_C$	Center wins	Center wins	Center wins
$>_R$	Right wins	Right wins	Right wins
$=_{LC}$	Center wins	Left wins	Center wins
$=_{LR}$	Right wins	Left wins	Left wins
$=_{CR}$	Right wins	Right wins	Center wins
$<_{CR}$	?	Left wins	Left wins
$<_{LR}$	Center wins	?	Center wins
$<_{LC}$	Right wins	Right wins	?
$<$	?	?	?

Table 11

Since for the first 7 classes only one outcome is possible, the following theorem holds.

Theorem 35 *Let x and y be numbers. If $x = y$ and they both belong to one of the first 7 classes then x and y have the same outcome.*

6.1 About zero-games

Definition. A game which does not affect the outcome of another game when added to it, is called *zero-game*. Formally, y is a zero-game iff for every game x, the games x and $x + y$ have the same outcome.
Since $(\forall x)x + 0 \equiv x$ we have that 0 is a zero-game. Let now x be a zero-game. By definition, the outcome of $x + 0$ must be equal to the outcome of 0. On the other hand, $x + 0 \equiv x$ therefore the outcome of every zero-game must be equal to the outcome of 0. Thus, except for 0, all zero-games, if any, belong to the class $<$. The following holds.

Theorem 36 *The class $<$ does not contain any zero-game.*

PROOF. Let $z = \{z^L|z^C|z^R\} < 0$ be a zero-game and let m be the maximum number of moves that Right can do in z^L in the best case. Let us now consider the number $x = \{\{\ |1_R|1_C\}|(m+2)_L 1_R|2_L 1_C\} <_{CR} 0$. We observe that Right has a winning strategy when Left starts. Given $x + z$ and supposing that Left plays in z and Center in x, we obtain $x^C + z^L$ where Left has at least one move more than Right. Therefore adding z affected the outcome of the game, and so it is not a zero game.

□

As a corollary of the above theorem we have

Corollary 37 *The only zero-game is* 0.

7 Three-player hackenbush

Hackenbush is a classical combinatorial game.

Definition. Three-player hackenbush is the natural extension of hackenbush where we introduce a third player Center.
Notation. In a three-player hackenbush instance

- L represents a left edge
- C represents a center edge
- R represents a right edge

Theorem 38 *If G is a connected graph representing an instance of three-player hackenbush then G is a number.*

PROOF. Let $G = \{G^L|G^C|G^R\}$ be a connected graph representing an instance of three-player hackenbush. By inductive hypothesis, G^L, G^C, and G^R are numbers. Let $G - G^L$ and $G - G^C$ be respectively the set of edges deleted after a left move and a center move. By definition G^{LC} is the set of arcs, subset of G, connected to the ground by at least one path which does not contain either L or C. The same definition applies to G^{CL}, thus $G^{LC} \equiv G^{CL}$.
It follows that

$$G^L <_L G^{LC} \equiv G^{CL} <_L G^C \Rightarrow G^L <_L G^C$$

.
In the same way, we prove that $G^L <_L G^R$, $G^C <_C G^L$, $G^C <_C G^R$, $G^R <_R G^L$, and $G^R <_R G^C$.

□

Since the sum of two numbers is a number, the following corollary is true.

Corollary 39 *Let $G = G_1 \cup G_2 \cup \ldots \cup G_n$ be a general instance of three-player hackenbush where G_i is a connected graph for all $1 \le i \le n$. Then, G is a number.*

Definition. Let G be a general instance of three-player hackenbush.
We define

- $M^{RC}(G)$ as the instance obtained from G by changing all the right edges into center edges;
- $M^{CR}(G)$ as the instance obtained from G by changing all the center edges into right edges.

Analogously we can define M^{RL}, M^{LR}, M^{CL} and M^{LC}.
So the function M changes a general three player instance of hackenbush in a two player version.
Properties. Let G be an instance of three-player hackenbush, it can easily be seen that

- $M^{RC}(G^L) \equiv M^{RC}(G)^L$
- $M^{RC}(G)^C \equiv M^{RC}(G^C) \cup M^{RC}(G^R)$
- $M^{RL}(G^C) \equiv M^{RL}(G)^C$
- $M^{RL}(G)^L \equiv M^{RL}(G^L) \cup M^{RL}(G^R)$
- $M^{CL}(G^R) \equiv M^{CL}(G)^R$
- $M^{CL}(G)^L \equiv M^{CL}(G^L) \cup M^{CL}(G^C)$

We have the following theorem.

Theorem 40 *Let G and H be two instances of three-player hackenbush.*

(1) $G \le_L H \Leftrightarrow M^{RC}(G) \le_L M^{RC}(H)$
(2) $G \le_C H \Leftrightarrow M^{RL}(G) \le_L M^{RL}(H)$
(3) $G \le_R H \Leftrightarrow M^{CL}(G) \le_L M^{CL}(H)$

PROOF.

(1) To prove the first implication we start by recalling that by definition of $G \le_L H$ we have

$$\text{no } G^L \ge_L H \Rightarrow \text{no } M^{RC}(G^L) \ge_L M^{RC}(H) \Rightarrow \text{no } M^{RC}(G)^L \ge_L M^{RC}(H)$$

$$\left.\begin{array}{l}\text{no } H^C \le_L G \Rightarrow \text{no } M^{RC}(H^C) \le_L M^{RC}(G)\\ \text{no } H^R \le_L G \Rightarrow \text{no } M^{RC}(H^R) \le_L M^{RC}(G)\end{array}\right\} \Rightarrow \text{no } M^{RC}(H)^C \le_L M^{RC}(G)$$

$$\text{no } M^{RC}(H)^R \le_L M^{RC}(G)$$

Conversely, let us assume that $M^{RC}(G) \le_L M^{RC}(H)$

$$\text{no } M^{RC}(G)^L \geq_L M^{RC}(H) \Rightarrow \text{no } M^{RC}(G^L) \geq_L M^{RC}(H) \Rightarrow \text{no } G^L \geq_L H$$

$$\text{no } M^{RC}(H)^C \leq_L M^{RC}(G) \Rightarrow \begin{cases} \text{no } M^{RC}(H^C) \leq_L M^{RC}(G) \Rightarrow \text{no } H^C \leq_L G \\ \text{no } M^{RC}(H^R) \leq_L M^{RC}(G) \Rightarrow \text{no } H^R \leq_L G \end{cases}$$

(2) Analogous to (1).
(3) Analogous to (1).

□

Note 1 *In the next two theorems we use $\leq$ and $\geq$ to represent the relations defined in Conway's theory.*

Theorem 41 *Let G and H be two instances of three-player hackenbush.*

(1) $M^{RC}(G) \leq_L M^{RC}(H) \Leftrightarrow M^{RC}(G) \leq M^{RC}(H)$
(2) $M^{RL}(G) \leq_C M^{RL}(H) \Leftrightarrow M^{RL}(G) \leq M^{RL}(H)$
(3) $M^{CL}(G) \leq_R M^{CL}(H) \Leftrightarrow M^{CL}(G) \leq M^{CL}(H)$

PROOF.

(1) To prove the first implication we assume first that $M^{RC}(G) \leq_L M^{RC}(H)$

$$\text{no } M^{RC}(G)^L \geq_L M^{RC}(H) \Rightarrow \text{no } M^{RC}(G)^L \geq M^{RC}(H)$$
$$\text{no } M^{RC}(H)^C \leq_L M^{RC}(G) \Rightarrow \text{no } M^{RC}(H)^C \leq M^{RC}(G)$$

Conversely, let us assume that $M^{RC}(G) \leq M^{RC}(H)$

$$\text{no } M^{RC}(G)^L \geq M^{RC}(H) \Rightarrow \text{no } M^{RC}(G)^L \geq_L M^{RC}(H)$$
$$\text{no } M^{RC}(H)^C \leq M^{RC}(G) \Rightarrow \text{no } M^{RC}(H)^C \leq_L M^{RC}(G)$$
$$\text{no } M^{RC}(H)^R \leq_L M^{RC}(G)$$

(2) Analogous to (1).
(3) Analogous to (1).

□

From the last two theorems it follows

Theorem 42 *Let G and H be two instances of three-player hackenbush.*

(1) $G \leq_L H \Leftrightarrow M^{RC}(G) \leq M^{RC}(H)$
(2) $G \leq_C H \Leftrightarrow M^{RL}(G) \leq M^{RL}(H)$
(3) $G \leq_R H \Leftrightarrow M^{CL}(G) \leq M^{CL}(H)$

Such a result is very useful in practice since given an instance of three-player hackenbush G, if we can calculate $M^{RC}(G)$, $M^{RL}(G)$, and $M^{CL}(G)$ then it is possible to calculate immediately which class G belongs to. We recall that if a number belongs to one of the first 7 classes, then we know the outcome of the game represented by this number.
The following theorems will allow us to extend such a result to the first 10 classes.

Theorem 43 *Let $G = G_1 \cup G_2 \cup \ldots \cup G_n$ be a general instance of three-player hackenbush where G_i is a connected graph for all $1 \leq i \leq n$. If $G <_{CR} 0$ and Left starts the game then Left has a winning strategy.*

PROOF. If $G <_{CR} 0$ then $M^{RC}(G) =_L 0$.
Let us suppose that Left makes the move with the smallest absolute value in $M^{RC}(G)$. We prove that $M^{RC}(G^{LC}) \geq_L 0$. Two cases are possible.
In the first case, the left edge belongs to $M^{RC}(G^C)$. Under this condition, we know that $M^{RC}(G^{LC}) \equiv M^{RC}(G^{CL})$ where $M^{RC}(G^{CL}) \geq_L 0$ because $M^{RC}(G)$ is a zero game in two-player version.
In the second case we observe that $M^{RC}(G^{LC}) \equiv M^{RC}(G^C) >_L 0$.
If $M^{RC}(G^{LC}) >_L 0$ then Left has a winning strategy in G^{LC}. If $M^{RC}(G^{LC}) =_L 0$ then Left has a winning strategy in G^{LC} because it is Right's turn to play.
We need also to underline that the following facts are true:

- after Left's move, Center can still make a move.
  - Let us suppose that the removal of the left edge L by Left causes as well the removal of the last edge C for Center. We recall that $M^{RC}(G)$ is a zero game, so before Left's move, if Center removes such an edge C, it would exist in G another left move L' (different from L because $M^{RC}(G^{CL}) \equiv M^{RC}(G^L) <_L 0$) such that $M^{RC}(G^{CL'}) \geq_L 0$.
- after Left's move, Right can still make a move.
  - Analogous to the previous case. In particular, we observe that the move that does not delete the last right edge, cannot delete the last center edge because otherwise it would be $G >_L 0$.
- after Center's move, Right can still make a move.
  - Since $M^{RC}(G^{LC}) =_L 0$, the absolute value of the left move is equal to the absolute value of the center move. Now, if there was a right edge R (which is a center edge in $M^{RC}(G)$) deleted by the center move, its absolute value would be less than the absolute value of the center move, which is impossible because it would be $M^{RC}(G^{RL}) <_L 0$ while $M^{RC}(G)$ is a zero game.

□

The following two theorems can be proven in the same way.

	Left starts	Center starts	Right starts
$=$	Right wins	Left wins	Center wins
$>_L$	Left wins	Left wins	Left wins
$>_C$	Center wins	Center wins	Center wins
$>_R$	Right wins	Right wins	Right wins
$=_{LC}$	Center wins	Left wins	Center wins
$=_{LR}$	Right wins	Left wins	Left wins
$=_{CR}$	Right wins	Right wins	Center wins
$<_{CR}$	Left wins	Left wins	Left wins
$<_{LR}$	Center wins	Center wins	Center wins
$<_{LC}$	Right wins	Right wins	Right wins
$<$	?	?	?

Table 12

Theorem 44 *Let $G = G_1 \cup G_2 \cup \ldots \cup G_n$ be a general instance of three-player hackenbush where G_i is a connected graph for all $1 \leq i \leq n$. If $G <_{LR} 0$ and Center starts the game then Center has a winning strategy.*

Theorem 45 *Let $G = G_1 \cup G_2 \cup \ldots \cup G_n$ be a general instance of three-player hackenbush where G_i is a connected graph for all $1 \leq i \leq n$. If $G <_{LC} 0$ and Right starts the game then Right has a winning strategy.*

8 Conclusions and future works

Table 12 summarizes the results obtained so far about a general instance of three-player hackenbush G. If we can solve $M^{RC}(G)$, $M^{RL}(G)$, and $M^{CL}(G)$, we are able to tell which class G belongs to, otherwise we can say nothing about G. If G belongs to one of the first 10 classes, we know immediately the outcome of the game.

8.1 Open questions

- Which instances it is possible to solve in $<$?
- Does it exist a NP-complete instance which belongs to $<$?

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References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways For Your Mathematical Plays*,Academic Press, London, 1982.

[2] J. H. Conway, *On Numbers And Games*, Academic Press, London, 1976.

[3] D. Knuth, *Surreal Numbers*, Addison-Wesley, Reading, Mass., 1974.

[4] S. Y. R. Li, N-person Nim and N-person Moore's games, *Internat. J. Game Theory*, **7** (1978), 31-36.

[5] D. E. Loeb, Stable Winning Coalitions, *Games of No Chance*, Proc. MSRI Workshop on Combinatorial Games, July, 1994, Berkeley, CA (R. J. Nowakowski, ed.), MSRI Publ. Vol. 29, Cambridge University Press, Cambridge, pp. 451-471.

[6] J. Propp, Three-player impartial games, *J. Theor. Comp. Sci.*, **233** (2000), 263-278.

[7] P. D. Straffin, Jr., Three-person winner-take-all games with Mc-Carthy's revenge rule, *College J. Math.*, **16** (1985), 386-394.