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MATRIX ROUNDING UNDER THE L_p-DISCREPANCY MEASURE AND ITS APPLICATION TO DIGITAL HALFTONING*

TETSUO ASANO†, NAOKI KATOH‡, KOJI OBOKATA†, AND TAKESHI TOKUYAMA§

Abstract. We study the problem of rounding a real-valued matrix into an integer-valued matrix to minimize an L_p-discrepancy measure between them. To define the L_p-discrepancy measure, we introduce a family $\mathcal{F}$ of regions (rigid submatrices) of the matrix and consider a hypergraph defined by the family. The difficulty of the problem depends on the choice of the region family $\mathcal{F}$. We first investigate the rounding problem by using integer programming problems with convex piecewise-linear objective functions and give some nontrivial upper bounds for the L_p discrepancy. We propose "laminar family" for constructing a practical and well-solvable class of $\mathcal{F}$. Indeed, we show that the problem is solvable in polynomial time if $\mathcal{F}$ is the union of two laminar families. Finally, we show that the matrix rounding using L_1 discrepancy for the union of two laminar families is suitable for developing a high-quality digital-halftoning software.

Key words. approximation algorithm, digital halftoning, discrepancy, linear programming, matrix rounding, network flow, totally unimodular

AMS subject classifications. 68R05, 68W25, 68W40, 90C05, 90C27

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1. Introduction.

Rounding is an important operation in numerical computation and plays key roles in digitization of analog data. Rounding of a real number a is basically a simple problem: We round it to either $\lfloor a \rfloor$ or $\lceil a \rceil$, and we usually choose the one nearer to a. However, we often encounter a datum consisting of more than one real number instead of a singleton. If it has n numbers, we have 2^n choices for rounding since each number is rounded into either its floor or ceiling. If the original data set has some feature, we need to choose a rounding so that the rounded result inherits as much of the feature as possible. The feature is described by using some combinatorial structure; we indeed consider a hypergraph $\mathcal{H}$ on the set. A typical input set is a multidimensional array of real numbers, and we consider a hypergraph whose hyperedges are its subarrays with contiguous indices. In this paper, we focus on two-dimensional arrays; in other words, we consider rounding problems on matrices.

1.1. Rounding problem and discrepancy measure.

Given an $M \times N$ matrix $A = (a_{ij})_{1 \le i \le M, 1 \le j \le N}$ of real numbers, its rounding is a matrix $B = (b_{ij})_{1 \le i \le M, 1 \le j \le N}$ of integral values such that b_{ij} is either $\lfloor a_{ij} \rfloor$ or $\lceil a_{ij} \rceil$ for each (i, j). There are 2^{MN} possible roundings of a given A, and we would like to find an optimal rounding with respect to a given criterion. This is called the *matrix rounding problem.* Without loss of generality, we can assume that each entry of A is in the closed interval $[0, 1]$ and each entry is rounded to either 0 or 1.

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†School of Information Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Tatsunokuchi, 923-1292 Japan (t-asano@jaist.ac.jp, obokata@jaist.ac.jp).

‡Graduate School of Engineering, Kyoto University, Yoshida-Honmachi, Sakyoku, Kyoto, 606-8501 Japan (naoki@archi.kyoto-u.ac.jp).

§Graduate School of Information Sciences, Tohoku University, Aramaki, Aobaku, Sendai, 980-8579 Japan (tokuyama@dais.is.tohoku.ac.jp).

In order to give a criterion to determine the quality of roundings, we define a distance in the space of all $[0,1]$-valued $M \times N$ matrices. Let $n = MN$. Let $\mathcal{F}$ be a family of *regions* (i.e., subsets) of the $M \times N$ integer grid G_{MN}. Let $\mathcal{A} = \mathcal{A}(G_{MN})$ be the space of all $[0,1]$-valued matrices with the index set G_{MN}, and let $\mathcal{B} = \mathcal{B}(G_{MN})$ be its subset consisting of all $\{0,1\}$-valued matrices. Let R be a region in $\mathcal{F}$.[1] For an element $A \in \mathcal{A}$, let $A(R)$ be the sum of entries of A located in the region R, that is,

$$A(R) = \sum_{(i,j)\in R} a_{ij}.$$

We define a distance $Dist_p^{\mathcal{F}}(A, A')$ between two elements A and A' in $\mathcal{A}$ for a positive integer p by

$$Dist_p^{\mathcal{F}}(A, A') = \left[\sum_{R\in\mathcal{F}} |A(R) - A'(R)|^p\right]^{1/p}.$$

The distance is called the L_p distance with respect to $\mathcal{F}$. The L_∞ distance with respect to $\mathcal{F}$ is defined by

$$Dist_\infty^{\mathcal{F}}(A, A') = \lim_{p\to\infty} Dist_p^{\mathcal{F}}(A, A') = \max_{R\in\mathcal{F}} |A(R) - A'(R)|.$$

Using the notations above, we can formally define the matrix rounding problem.

L_p-optimal matrix rounding problem. $\mathcal{P}(G_{MN}, \mathcal{F}, p)$: Given a $[0,1]$-matrix $A \in \mathcal{A}$, a family $\mathcal{F}$ of subsets of G_{MN}, and a positive integer p, find a $\{0,1\}$-matrix $B \in \mathcal{B}$ that minimizes

$$Dist_p^{\mathcal{F}}(A, B) = \left[\sum_{R\in\mathcal{F}} |A(R) - B(R)|^p\right]^{1/p}.$$

Also, we are interested in the following combinatorial problem.

L_p-discrepancy bound. Given a $[0,1]$-matrix $A \in \mathcal{A}$, a family $\mathcal{F}$ of subsets of G_{MN}, and a positive integer p, investigate upper and lower bounds of

$$\mathcal{D}(G_{MN}, \mathcal{F}, p) = \sup_{A\in\mathcal{A}} \min_{B\in\mathcal{B}} Dist_p^{\mathcal{F}}(A, B).$$

The pair $(G_{MN}, \mathcal{F})$ defines a hypergraph on G_{MN}, and $\mathcal{D}(G_{MN}, \mathcal{F}, \infty)$ is called the *inhomogeneous discrepancy* of the hypergraph [6]. Abusing the notation, we call $\mathcal{D}(G_{MN}, \mathcal{F}, p)$ the (inhomogeneous) L_p discrepancy of the hypergraph and also often call $Dist_p^{\mathcal{F}}(A, B)$ the L_p-discrepancy measure of (the quality of) the output B with respect to $\mathcal{F}$.

1.2. Motivation and our application. The most popular example of the family $\mathcal{F}$ is the set of all rectangular subregions in G_{MN} (i.e., the set of all rigid submatrices), and the corresponding L_∞-discrepancy measure is utilized in many application areas such as Monte Carlo simulation and computational geometry. Unfortunately, if we consider the family of all rectangular subregions, the discrepancy bound (for the

[1] Strictly speaking, R can be any subset of G_{MN}. Although we implicitly assume that R forms some connected portion on the grid G_{MN}, the connectivity assumption is not used throughout the paper.

L_∞ measure) is known to be $\Omega(\log n)$ and $O(\log^3 n)$. See Beck and Sós's survey [6] for the theory. It seems hard to find an optimal solution to minimize the discrepancy. In fact, it is NP-hard [2].

Therefore, we seek a family of regions for which low discrepancy rounding is useful in an important application and also can be computed in polynomial time. For the application, L_∞ rounding is not always suitable, and L_p discrepancy (with $p = 1$ or 2) is preferable. For the purpose, we present a geometric structure of a family of regions reflecting the combinatorial discrepancy bound and computational difficulty of the matrix rounding problem.

In particular, we focus on the digital-halftoning application of the matrix rounding problem, where we should consider smaller families of rectangular subregions as $\mathcal{F}$. More precisely, the input matrix represents a digital (gray) image, where a_{ij} represents the brightness level of the (i, j)-pixel in the $M \times N$ pixel grid. Typically, M and N are between 256 and 4096, and a_{ij} is an integral multiple of 1/256: This means that we use 256 brightness levels. If we want to send an image using fax or print it out by a dot (or ink-jet) printer, brightness levels available are limited. Instead, we replace A by an integral matrix B so that each pixel uses only two brightness levels. Here, it is important that B looks similar to A; in other words, B should be a good approximation of A.

For each pixel (i, j), if the average brightness level of B in each of its neighborhoods (regions containing (i, j) in a suitable family of regions) is similar to that of A, we can expect that B is a good approximation of A. For this purpose, the set of all rectangles is not suitable (i.e., it is too large), and we may use a more compact family. Moreover, since human vision detects global features, the L_1 or L_2 measure should be better than the L_∞ measure to obtain a clear output image. This intuition is supported by our experimental results; for example, edges of objects are often blurred in the output based on the L_∞-discrepancy measure, while they are sharply displayed if we use the L_1-discrepancy measure.

1.3. Known results on L_∞ measure.

For the L_∞ measure, the following beautiful combinatorial result is classically known.

THEOREM 1.1 (Baranyai [5]). *Given a real-valued matrix $A = (a_{ij})$ and a family $\mathcal{F}$ of regions consisting of all rows, all columns, and the whole matrix, there exists an integer-valued matrix $B = (b_{ij})$ such that $|A(R) - B(R)| < 1$ holds for every $R \in \mathcal{F}$.*

Translating the theorem in our terminologies, the L_∞ discrepancy of the matrix rounding problem for the family of regions consisting of all rows, all columns, and the whole matrix is bounded by 1. Also, the combinatorial structure and algorithmic aspects of roundings of (one-dimensional) sequences with respect to the L_∞-discrepancy measure are investigated in recent studies [2, 19].

The *incidence matrix* $\mathcal{C}(G_{MN}, \mathcal{F}) = (\mathcal{C}_{ij})$ of the hypergraph $(G_{MN}, \mathcal{F})$ is defined by $\mathcal{C}_{ij} = 1$ if the jth element of G_{MN} belongs to the ith region R_i in $\mathcal{F}$ and 0 otherwise.[2] A hypergraph is called *unimodular* if its incidence matrix is totally unimodular, where a matrix $\mathcal{C}$ is *totally unimodular* if the determinant of each square submatrix of $\mathcal{C}$ is equal to 0, 1, or -1.

Both the Baranyai problem and the sequence rounding problems correspond to rounding problems with respect to unimodular hypergraphs. The L_∞-discrepancy problem can be formulated as an integer programming problem, and the unimodularity implies that its relaxation has an integral solution. A classical theorem of

[2]We implicitly assume a one-dimensional ordering of elements in G_{MN}.

Ghouila-Houri [10] implies that unimodularity is a necessary and sufficient condition for the existence of a rounding with a L_∞ discrepancy less than 1. Moreover, the following sharpened result is given by Doerr [7].

THEOREM 1.2. *If* $(G_{MN}, \mathcal{F})$ *is a unimodular hypergraph, there exists a rounding* $B = (b_{ij})$ *of* $A = (a_{ij})$ *satisfying*

$$|A(R) - B(R)| < \min\left\{1 - \frac{1}{n+1}, 1 - \frac{1}{m}\right\}$$

for every $R \in \mathcal{F}$*, where* $n = MN$*,* $m = |\mathcal{F}|$*.*

This bound is sharp. Moreover, L_∞-optimal rounding can be computed in polynomial time if $\mathcal{F}$ is unimodular.

1.4. Our results. We would like to consider the L_p-discrepancy measure instead of the L_∞-discrepancy measure. If the hypergraph is unimodular, an $|\mathcal{F}|^{1/p}$ upper bound for the L_p discrepancy can be derived from Theorem 1.2 trivially. We first improve the upper bound to $\frac{1}{2}|\mathcal{F}|^{1/p}$ for $p \leq 3$ and show that the bound is tight. We also consider the family $\mathcal{F}$, consisting of all 2×2 rigid submatrices, for which the matrix rounding problem is known to be NP-hard [2] (accordingly, the family is not unimodular).

Next, we consider the optimization problem. If the hypergraph is unimodular, the rounding minimizing the L_p discrepancy can be computed in polynomial time by translating it to a separable convex programming problem and applying known general algorithms [11, 12]. However, we want to define a class of region families for which we can compute the optimal solution more efficiently, as well as a class that is useful in applications (in particular, the digital-halftoning application). We consider the union of two laminar families (defined in section 3) and show that the matrix rounding problem can be formulated into a minimum cost flow problem, and hence solved in polynomial time. Finally, we implemented the algorithm using LEDA [14]. Some output pictures of the algorithm applying to the digital-halftoning problem are included.

2. Mathematical programming formulations.

2.1. Formulation as a piecewise-linear separable convex programming problem. We give a formulation of the L_p-discrepancy problem into an integer convex programming problem where the objective function is a separable convex function, i.e., a sum of univariate convex functions.

Introducing a new variable $y_i = B(R_i) = \sum_{(j,k)\in R_i} b_{jk}$ for each $R_i \in \mathcal{F}$, the problem $\mathcal{P}(G_{MN}, \mathcal{F}, p)$ is described in the following form:

$$\begin{aligned} \text{(P1)}: \text{minimize} \quad & \left[\sum_{R_i \in \mathcal{F}} |y_i - A(R_i)|^p\right]^{1/p} \\ \text{subject to} \quad & y_i = \sum_{(j,k)\in R_i} b_{jk}, \ i = 1, \ldots, m = |\mathcal{F}|, \\ \text{and} \quad & B \in \mathcal{B}(G_{MN}). \end{aligned}$$

When $p < \infty$, the objective function can be replaced with $\sum_{R_i \in \mathcal{F}} |y_i - c_i|^p$, where $c_i = A(R_i) = \sum_{(j,k)\in R_i} a_{jk}$ is a constant depending only on input values. Now $|y_i - c_i|^p$ is a convex function independent of other y_j's. The constraints $y_i = \sum_{(j,k)\in R_i} b_{jk}$,

$i = 1, \ldots, m$, are represented by $(-I, \mathcal{C}(G_{MN}, \mathcal{F}))Y = 0$ using the incidence matrix $\mathcal{C}(G_{MN}, \mathcal{F})$ defined in section 1.3, where $Y = (y_1, \ldots, y_m, b_{11}, \ldots, b_{MN})^T$ and I is an identity matrix.

Although the objective function is now a separable convex function, its nonlinearity makes it difficult to analyze the properties of the solution. Thus, we apply the idea of Hochbaum and Shanthikumar [11] to replace $|y_i - c_i|^p$ with a piecewise-linear convex continuous function $f_i(y_i)$ which is equal to $|y_i - c_i|^p$ for each integral value of y_i in $[0, |R_i|]$. This is because we need only integral solutions, and, if each b_{p_j} is integral, y_i must be a nonnegative integer less than or equal to $|R_i|$. Typically for $p = 1$, $f_i(y_i)$ is illustrated in Figure 1.

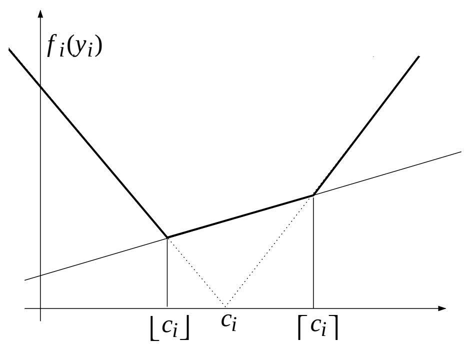

FIG. 1. *Conversion of the convex objective function* $|y_i - c_i|$ *into a piecewise-linear convex function* $f_i(y_i)$ *with integral breakpoints (shown in bold lines).*

Thus, we obtain the following problem (P2):

$$\begin{aligned} \text{(P2) : minimize} \quad & \sum_{R_i \in \mathcal{F}} f_i(y_i) \\ \text{subject to} \quad & y_i = \sum_{(j,k) \in R_i} b_{jk}, \; i = 1, \ldots, m = |\mathcal{F}|, \\ \text{and} \quad & B \in \mathcal{B}(G_{MN}). \end{aligned}$$

Thus, we can formulate the problem into an integer programming problem where the objective function is a separable piecewise-linear convex function.

2.2. Relaxation and totally unimodularity. Let (P3) be the continuous relaxation obtained from (P2) by replacing the integral condition of b_{ij} with the condition $0 \le b_{ij} \le 1$. Note that this is different from the continuous relaxation of (P1), since the objective function of (P2) is larger than that of (P1) at nonintegral values.

If the matrix is totally unimodular, (P3) has an integral solution by the theorem below. This is a key to derive discrepancy bounds and also algorithms.

THEOREM 2.1 (Hochbaum and Shanthikumar [11]). *A nonlinear separable convex optimization problem* $\min\{\sum_{i=1}^n f_i(x_i) \mid Ax \ge b\}$ *on linear constraints with a totally unimodular matrix* A *can be solved in polynomial time.*

This theorem is translated into our terminologies as follows.

COROLLARY 2.2. *The matrix rounding problem* $\mathcal{P}(G_{MN}, \mathcal{F}, p)$ *for* $p < \infty$ *is solved in polynomial time in* $n = MN$ *if its associated incidence matrix* $\mathcal{C}(G_{MN}, \mathcal{F})$ *is totally unimodular.*

3. Geometric families of regions defining unimodular hypergraphs. In this section we consider interesting classes of families whose associated incidence matrices are totally unimodular. We call such a family a *unimodular family*, since the associated hypergraph is unimodular. A family $\mathcal{F} = \{R_1, R_2, \ldots, R_m\}$ is a *partition family* (or a partition) of G_{MN} if $\bigcup_{i=1}^{m} R_i = G_{MN}$ and $R_i \cap R_j = \emptyset$ for any $R_i \neq R_j$ in $\mathcal{F}$. A *k-partition family* is a family of regions on a matrix which is the union of k different partitions of G_{MN}.

A family $\mathcal{F}$ of regions on a grid G_{MN} is a *laminar family* if one of the following holds for any pair R_i and R_j in $\mathcal{F}$: (1) $R_i \cap R_j = \emptyset$, (2) $R_i \subset R_j$, or (3) $R_j \subset R_i$. The family is also called a laminar decomposition of the grid G_{MN}. In general, a *k-laminar family* is a family of regions on a matrix which is the union of k different laminar families.

PROPOSITION 3.1. *A* 2*-laminar family is unimodular.*

Direct applications of Proposition 3.1 lead to various unimodular families of regions. The family of regions defined in Baranyai's theorem is a 2-laminar family. Also, take any 2-partition family consisting of 2×2 regions on a matrix. For example, take all 2×2 regions with their upper left corners located in even points (where the sums of their row and column indices are even). The set of all those regions defines 2-partition families $\mathcal{F}_{even}$ and $\mathcal{F}_{odd}$, where $\mathcal{F}_{even}$ (resp., $\mathcal{F}_{odd}$) consists of all 2×2 squares with their upper left corners lying at even (resp., odd) rows (see Figure 2). This kind of family plays an important role in section 5.2 and also in our experiment.

A 3-partition family is not unimodular in general. However, there are some families which are not 2-laminar but unimodular: For example, the set of all rectangular rigid submatrices of size 2 (i.e., domino tiles) is a 4-partition family, but it is unimodular.

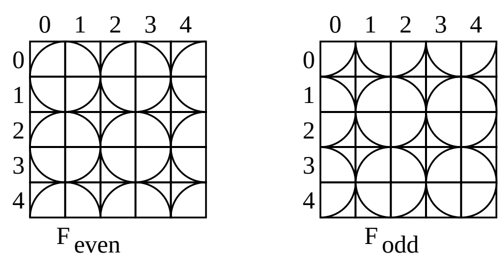

FIG. 2. 2*-partition family of* 2×2 *regions.*

4. Algorithms for computing the optimal rounding. The arguments so far guarantee the polynomial-time solvability of our problem. However, we needed a more practical algorithm for our experiments that runs fast for large-scale problem instances. In this section we will show how to solve the matrix rounding problem for a 2-laminar family based on the minimum-cost flow algorithm. To improve the readability, we mainly discuss the case for the L_1-discrepancy measure.

Our main result is the following.

THEOREM 4.1. *Given a* $[0, 1]$*-matrix* A *and a* 2*-laminar family* $\mathcal{F}$*, an optimal binary matrix* B *that minimizes the distance* $Dist_1^{\mathcal{F}}(A, B)$ *is computed in* $O(n^2 \log^2 n)$ *time, where* n *is the number of matrix elements.*

Proof. We can transform the problem into that of finding a minimum-cost circulation flow in the network defined as follows.

Let $\mathcal{F}$ be a 2-laminar family given as the union of two laminar families $\mathcal{F}_1 = \{R_0, R_1, \dots, R_m\}$ and $\mathcal{F}_2 = \{R'_0, R'_1, \dots, R'_{m'}\}$ over the grid G_{MN}, where R_0 and R'_0 are the entire region G_{MN}. The network to be constructed consists of three parts. The first part is an in-tree T_1 derived from $\mathcal{F}_1$ whose root is R_0; the second one is an out-tree T_2 from $\mathcal{F}_2$ whose root is R'_0; and the third part connects T_1 and T_2. The lattice structure implied by $\mathcal{F}_1$ naturally defines an in-tree T_1 such that the vertex set is the set of regions in $\mathcal{F}_1$, and there is a directed edge (R_i, R_j) if and only if $R_i \subseteq R_j$ and there is no other region R_k such that $R_j \subseteq R_k \subseteq R_i$. Then each region R_i, $i \geq 1$, has a unique outgoing edge, which is denoted by $e(R_i)$. We can similarly define T_2 for the laminar family $\mathcal{F}_2$, in which the edge direction is reversed in T_2; that is, each node R'_i, $i \geq 1$, has a unique incoming edge, which is denoted by $e(R'_i)$.

In addition, leaves of T_1 and T_2 are connected by edges corresponding to elements of G_{MN}. Because of the definition of $\mathcal{F}_1$ and $\mathcal{F}_2$, each element (k, l) of G_{MN} belongs to exactly one region in $\mathcal{F}_1$ which is a leaf in T_1 and to exactly one region in $\mathcal{F}_2$ which is a leaf in T_2. If (k, l) belongs to R_i and R'_j, then we have a directed edge $e(i, j)$ from R'_j to R_i. Finally, we draw an edge from R_0 to R'_0.

Now we define the capacity and cost coefficient of each edge. (The lower bound on the flow of each edge is defined to be 0.) The capacity of an edge $e(i, j)$ is determined simply as 1 because the value associated with an element of G_{MN} is to be rounded to 0 or 1.

Determining the cost coefficients of edges $e(R_i)$ and $e(R'_j)$ are not straightforward, although the cost coefficients of $e(i, j)$s and $e(R_0, R'_0)$ are defined to be 0s. This is because each term of the objective function depends on the difference between $B(R_i)$ and $A(R_i)$ or between $B(R'_j) - A(R'_j)$; that is, $|B(R_i) - A(R_i)|$ or $|B(R'_j) - A(R'_j)|$.

Recall the argument in section 2: To prove the polynomial-time solvability we have introduced a new variable $y_k = B(R_k) = \sum_{(i,j)\in R_k} b_{ij}$. $|B(R_k) - A(R_k)|$ is converted into $|y_k - c_k|$, where $c_k = A(R_k) = \sum_{(i,j)\in R_k} a_{ij}$ is a constant determined by input values. $|y_k - c_k|$ is further replaced by the piecewise-linear convex function $f_k(y_k)$ which coincides with $|y_k - c_k|$ at each integral value of y_k.

To reflect the new form of the objective function (see Figure 1), we replace each edge $e(R_k)$ by three parallel edges with different capacities and costs: $e_1(R_k)$ has capacity $\lfloor c_k \rfloor$ and cost $c^1 = -1$. $e_2(R_k)$ has capacity $\lceil c_k \rceil - \lfloor c_k \rfloor$ and cost $c^2 = \lfloor c_k \rfloor + \lceil c_k \rceil - 2c_k$. For the third edge $e_3(R_k)$, its capacity is ∞ and its cost is $c^3 = 1$. Since $c^1 \leq c^2 \leq c^3$, to minimize the overall cost for these three edges the flow at $e_2(R_k)$ is zero unless the first edge $e_1(R_k)$ is full; that is, the flow at $e_1(R_k)$ is $\lfloor c_k \rfloor$. Similarly, flow at $e_3(R_k)$ is positive only if the two edges $e_1(R_k)$ and $e_2(R_k)$ are both full.

The cost associated with an edge is determined by multiplying the above coefficient to the flow in the edge. When the total amount of flow in the three edges is given by y_k, the total cost is given by $f_k(y_k) - c_k$ in any case (see, e.g., Ahuja, Magnanti, and Orlin [1]). Since c_k is a constant, the constant term does not affect the optimality.

Once a network is constructed, we can find an optimal rounding in time $O(|E| \log U(|E| + |V| \log |V|))$ for a network with node set V and edge set E and the largest integral capacity U, using the scaling algorithm by Edmonds and Karp [8]. In our case, $|V|$, $|E|$, and U are all $O(n)$, and thus we have $O(n^2 \log^2 n)$, where n is the number of matrix elements. □

THEOREM 4.2. *Given a $[0, 1]$-matrix A and a 2-laminar family $\mathcal{F}$, an optimal binary matrix B that minimizes the distance $Dist_p^{\mathcal{F}}(A, B)$ $(p \geq 2)$ can be computed* *in* $O(n^2 \log^3 n)$ *time.*

Proof. In this case, f_i is a piecewise-linear convex function with $O(n)$ break points. We apply the convex-cost flow algorithm [18]. We omit details. □

5. Upper bounds for the L_p discrepancy.

5.1. L_p discrepancy for a unimodular hypergraph. In this subsection, we prove the following theorem for the L_p discrepancy of a unimodular family.

THEOREM 5.1. *If $\mathcal{F}$ is unimodular and $p \leq 3$, for any $A \in \mathcal{A}$ we have*

$$\min_{B \in \mathcal{B}} Dist_p^{\mathcal{F}}(A, B) \leq \frac{1}{2}|\mathcal{F}|^{1/p}.$$

Proof. There exists $\hat{B} \in \mathcal{B}$ such that $|\hat{B}(R_i) - A(R_i)| \leq 1$ holds for any R_i. The existence of such $\hat{B}$ is known by Theorem 1.2. However, for completeness, we shall give the proof. Consider the problem (P2) and its continuous relaxation (P3). It is then obvious that

$$b_{ij} = a_{ij} \text{ for every } i \text{ and } j \text{ and} \quad y_k = A(R_k) \text{ for every } R_k \in \mathcal{F} \tag{5.1}$$

is a feasible solution to (P3). Now we add lower and upper bound constraints for each variable y_k:

$$\lfloor A(R_k) \rfloor \leq y_k \leq \lceil A(R_k) \rceil.$$

Notice that the addition of these constraints to (P3) maintains total unimodularity of the constraints. Let (P4) denote the problem (P3) with these constraints. Since $f_k(y_k)$ is a linear function in the interval $[\lfloor A(R_k) \rfloor, \lceil A(R_k) \rceil]$, (P4) is a linear program.

Since (P3) has a feasible solution satisfying all constraints of (P4), (P4) also has a feasible solution. Since (P4) is a linear program over totally unimodular constraints, its optimal solution is an integral solution, and the corresponding objective value gives an upper bound on the optimal objective value of (P2). Thus, the objective value for (P4) of the above defined feasible solution gives an upper bound on the optimal objective value of (P2).

Let us now estimate the upper bound on $f_k(y_k)$ at $y_k = A(R_k)$. Let $a = A(R_k) - \lfloor A(R_k) \rfloor$. We then have $f_k(\lfloor A(R_k) \rfloor) = a^p$ and $f_k(\lceil A(R_k) \rceil) = (1-a)^p$. Therefore,

$$f_k(A(R_k)) = a^p(1-a) + (1-a)^p a \tag{5.2}$$

holds. We can see $f_k(A(R_k)) \leq (1/2)^p$ holds if $p \leq 3$. Thus, the optimal objective value of (P4) is at most $(1/2)^p|\mathcal{F}|$. Since the optimal objective value of (P4) is an upper bound on that for (P2), $(1/2)^p|\mathcal{F}|$ gives an upper bound on the optimal objective value for (P2). □

It is easy to give an instance to show that the bound is tight: Consider Baranyai's problem on a matrix having $\frac{1}{2}$ entries in its diagonal position (other entries are zeros).

For the case $p > 3$, we have the following.

THEOREM 5.2. *If $\mathcal{F}$ is unimodular and $p > 3$, for any $A \in \mathcal{A}$ we have*

$$\min_{B \in \mathcal{B}} Dist_p^{\mathcal{F}}(A, B) \leq (p^p/(p+1)^{p+1} + 2^p(p-1)/(p+1)^{p+1})^{1/p}|\mathcal{F}|^{1/p}.$$

Proof. This follows from the fact that $a^p(1-a)+(1-a)^p a < p^p/(p+1)^{p+1} + 2^p(p-1)/(p+1)^{p+1}$ for $0 \le a \le 1$. ∎

The term $(p^p/(p+1)^{p+1} + 2^p(p-1)/(p+1)^{p+1})^{1/p}$ is 0.550 and 0.587 if $p=4$ and $p=5$, respectively, and it is always less than $p/(p+1)$.

5.2. Discrepancy bounds for the family of 2 × 2 regions. The method in the previous subsection does not work for a nonunimodular case. A simple but interesting family defining a nonunimodular hypergraph is the family of all 2×2 regions of A. The known upper bound is merely $\frac{5}{3}|\mathcal{F}|^{1/p}$ from the corresponding L_∞ result [3]. We obtain the following result.

THEOREM 5.3. *For any $A \in \mathcal{A}(G_{MN})$ and a family $\mathcal{F}$ of 2×2 regions of the matrix, we have*

$$\min_{B\in\mathcal{B}} Dist_1^{\mathcal{F}}(A,B) \le \frac{3}{4}|\mathcal{F}|.$$

Proof. Let us consider the matrix rounding problem $P = \mathcal{P}(G_{MN}, \mathcal{F}, 1)$ for the family $\mathcal{F}$ of all 2×2 regions. We define another problem $\hat{P}$ defined over another family $\hat{\mathcal{F}}$ of regions consisting of two tiles:

$$\mathcal{T}_1 = \{(i,j),(i+1,j)\} \text{ for } (i,j)\in G_{MN},$$
$$\mathcal{T}_2 = \{(i,j),(i+1,j),(i,j+1),(i+1,j+1)\} \text{ for } (i,j)\in G_{MN} \text{ and } i+j=\text{even}.$$

Let B^* and $\hat{B}$ denote the optimal binary matrix for P and $\hat{P}$, respectively.

We now show

$$\sum_{R\in\mathcal{F}} |A(R)-B^*(R)| \le 3|\mathcal{F}|/4.$$

Let (R_1,R_2) be a partition of a 2×2 region R into two disjoint 2×1 regions. Then for any A and B we have

$$\begin{aligned}(5.3)\qquad &|A(R)-B(R)|\\ &\le |A(R_1)-B(R_1)| + |A(R_2)-B(R_2)|.\end{aligned}$$

Recall that $\mathcal{F}$ consists of all 2×2 regions and $\hat{\mathcal{F}}$ consists of constrained 2×2 regions and all 2×1 regions. Then $\mathcal{F}\backslash\hat{\mathcal{F}}$ consists of 2×2 regions that are not included in $\hat{\mathcal{F}}$, $\mathcal{F}\cap\hat{\mathcal{F}}$ consists of 2×2 regions that are included in $\hat{\mathcal{F}}$, and $\hat{\mathcal{F}}\backslash\mathcal{F}$ consists of all 2×1 regions. Now we have

$$\begin{aligned}(5.4)\qquad &\sum_{R\in\mathcal{F}} |A(R)-B^*(R)|\\ &= \sum_{R\in\mathcal{F}\backslash\hat{\mathcal{F}}} |A(R)-B^*(R)| + \sum_{R\in\mathcal{F}\cap\hat{\mathcal{F}}} |A(R)-B^*(R)|\\ &\le \sum_{R\in\mathcal{F}\backslash\hat{\mathcal{F}}} |A(R)-\hat{B}(R)| + \sum_{R\in\mathcal{F}\cap\hat{\mathcal{F}}} |A(R)-\hat{B}(R)|\\ &\le \sum_{R\in\mathcal{F}\cap\hat{\mathcal{F}}} |A(R)-\hat{B}(R)| + \sum_{R\in\hat{\mathcal{F}}\backslash\mathcal{F}} |A(R)-\hat{B}(R)|\\ &\quad \text{(from (5.3))}\\ &= \sum_{R\in\hat{\mathcal{F}}} |A(R)-\hat{B}(R)|.\end{aligned}$$

Since $\hat{\mathcal{F}}$ is a 2-laminar family, its associated incidence matrix is totally unimodular. As in the proof of Theorem 5.1, we consider the linear program $\hat{Q}'$ which is a continuous relaxation of $\hat{P}$ with the additional constraints

$$\lfloor A(R_k) \rfloor \le y_k \le \lceil A(R_k) \rceil$$

for all $R_k \in \hat{\mathcal{F}}$. From the total unimodularity of the incidence matrix, there exists an optimal solution to $\hat{Q}'$ such that it is integral. Let $\hat{B}'$ denote such solution. For a feasible solution to $\hat{Q}'$ defined by

$$\text{(5.5)} \qquad \begin{aligned} b_{ij} = a_{ij} &\text{ for every } i \text{ and } j \\ \text{and } y_k = A(R_k) &\text{ for every } R_k \in \hat{\mathcal{F}}, \end{aligned}$$

its objective value gives an upper bound on the optimal objective value of $\hat{Q}'$ which in turn gives an upper bound on the optimal objective value of $\hat{P}$. Therefore, we have

$$\text{(5.6)} \qquad \begin{aligned} \sum_{R_k \in \hat{\mathcal{F}}} |A(R_k) - \hat{B}(R_k)| &\le \sum_{R_k \in \hat{\mathcal{F}}} |A(R_k) - \hat{B}'(R_k)| \\ &\le \sum_{R_k \in \hat{\mathcal{F}}} f_k(A(R_k)). \end{aligned}$$

From Theorem 5.1, the rightmost term of (5.6) is bounded by $|\hat{\mathcal{F}}|/2$, which is almost equal to $3|\mathcal{F}|/4$ for a sufficiently large n. This completes the proof of the theorem. □

6. Application to digital halftoning. The quality of color printers has been drastically improved in recent years, mainly based on the development of the fine control mechanism. On the other hand, there seems to be no great invention on the software side of the printing technology. What is required is a technique to convert a continuous-tone image into a binary image consisting of black and white dots so that the binary image looks very similar to the input image. From a theoretical standpoint, the problem is how to approximate an input $[0, 1]$-array by a binary array. Since this is one of the central techniques in computer vision and computer graphics, a great number of algorithms have been proposed (see, e.g., [13, 9, 4, 15, 17]). However, there have been very few studies toward the goal of achieving an optimal binary image under some reasonable criterion; maybe it is because the problem itself is very practically oriented. A desired output image is the one which looks similar to the input image to the human visual system. The most popular distortion criterion that is used in practice is perhaps frequency weighted mean square error (FWMSE) [16], which is defined by

$$W(G, X) = \sum_{(i,j) \in G_{MN}} \left[\sum_{k=-K}^{K} \sum_{l=-K}^{K} v_{|k||l|} a_{i+k,j+l} - \sum_{k=-K}^{K} \sum_{l=-K}^{K} v_{|k||l|} b_{i+k,j+l} \right]^2 .$$

Here, $V = (v_{|k||l|})$, $-K \le k, l \le K$, is an impulse response that approximates the characteristics of the human visual system and K is some small constant, say 3. Our discrepancy measure which has been discussed in this paper is a hopeful replacement. Indeed, the L_2-discrepancy measure can be regarded as a simplified version of the FWMSE criterion.

We have implemented the algorithm using LEDA [14] functions for finding minimum-cost flow and applied it to several test images to compare its results with the error diffusion algorithm which is most commonly used in practice. The data we used for our experiments are *standard high precision picture data* created by the Institute of Image Electronics Engineers of Japan, which include four standard pictures called “Bride,” “Harbor,” “Wool,” and “Bottles.” They are color pictures of eight bits each in RGB. Their original picture size is 4096×3072. In our experiments we scaled them down to 1024×768 in order to shorten the running time of the program. Figure 3 shows experimental results for “Wool” to compare our algorithm with error diffusion. Our algorithm has been implemented using a 2-laminar family defined by the two tiles (b) and (c) depicted in Figure 4. We have used the L_1 measure. By our experience through experiments, it seems hard to have such a nice-looking output by the L_∞ measure.

FIG. 3. *Experimental results. Output images by the error diffusion algorithm (above) and the algorithm in this paper (below).*

7. Concluding remarks. We have considered the matrix rounding problem based on L_p-discrepancy measure. Although we have shown that the measure is useful in application to the digital-halftoning application, the current algorithm is too

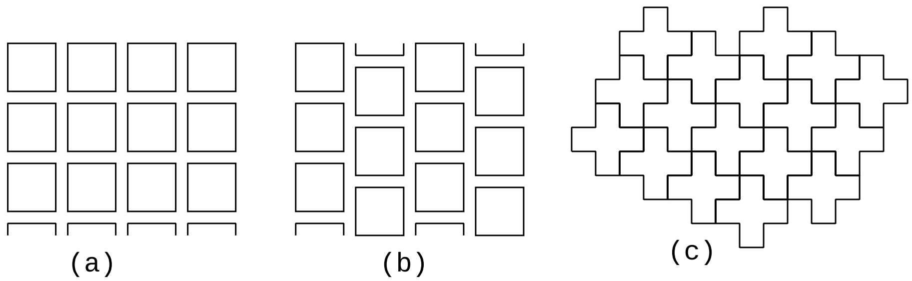

FIG. 4. *Three different partitions of the image plane* (a) *by* 2×2 *squares,* (b) *vertically shifted* 2×2 *squares, and* (3) *cross patterns consisting of five pixels.*

slow if we want to require speed together with the high-quality requirement. The problem comes from the quadratic time complexity. It is desired to design a faster algorithm (even an approximation algorithm). Moreover, it is an interesting question to investigate what kind of region families give the best criterion for the halftoning application. Once we know such a region family, it is valuable to design an algorithm (heuristic algorithm if the problem for solving the optimal solution is intractable) for the criterion.

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