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Description	

On Geometric Structure of Global Roundings for Graphs and Range Spaces

Tetsuo Asano* Naoki Katoh† Hisao Tamaki‡ Takeshi Tokuyama§

Abstract

Given a hypergraph $\mathcal{H} = (V, \mathcal{F})$ and a $[0,1]$-valued vector $\mathbf{a} \in [0,1]^V$, its global rounding is a binary (i.e.,$\{0,1\}$-valued) vector $\alpha \in \{0,1\}^V$ such that $|\sum_{v \in F}(\mathbf{a}(v) - \alpha(v))| < 1$ holds for each $F \in \mathcal{F}$. We study geometric (or combinatorial) structure of the set of global roundings of $\mathbf{a}$ using the notion of compatible set with respect to the discrepancy distance. We conjecture that the set of global roundings forms a simplex if the hypergraph satisfies "shortest-path" axioms, and prove it for some special cases including some geometric range spaces and the shortest path hypergraph of a series-parallel graph.

1 Introduction

Rounding problem is a central problem in computer science and computer engineering. Given a real number a, its rounding is either its floor $\lfloor a \rfloor$ or ceiling $\lceil a \rceil$. Then, we want to consider how to round a set of n real numbers each of which is assigned to an element of a set $V = \{v_1, v_2, \ldots, v_n\}$ with a given structure. We can assume that each number is in the range $[0,1]$, so that the input set can be considered as $\mathbf{a} \in [0,1]^V$ and the output rounding is $\alpha \in \{0,1\}^V$. [1]

We assume that the structure on V is represented by a hypergraph $\mathcal{H} = (V, \mathcal{F})$ where $\mathcal{F} \subset 2^V$ is the set of hyperedges. For simplicity, we assume without loss of generality that $\mathcal{F}$ contains all the singletons. We say α is a *global rounding* of $\mathbf{a}$ iff $w_F(\alpha) = \sum_{v \in F} \alpha(v)$ is a rounding (i.e., either floor or ceiling) of $w_F(\mathbf{a}) = \sum_{v \in F} \mathbf{a}(v)$ for each $F \in \mathcal{F}$. Let $\Gamma_{\mathcal{H}}(\mathbf{a})$ be the set of all global roundings of $\mathbf{a}$.

We can rephrase the global rounding condition as $D_{\mathcal{H}}(\mathbf{a}, \alpha) < 1$, where $D_{\mathcal{H}}$ is the *discrepancy distance* between $\mathbf{a}$ and $\mathbf{b}$ in $[0,1]^V$ defined by

$$D_{\mathcal{H}}(\mathbf{a}, \mathbf{b}) = \max_{F \in \mathcal{F}} |w_F(\mathbf{a}) - w_F(\mathbf{b})|.$$

Thus, $\Gamma_{\mathcal{H}}(\mathbf{a})$ is the set of integral points in the open unit ball about $\mathbf{a}$ by considering $D_{\mathcal{H}}$ as the distance. $\Gamma_{\mathcal{H}}(\mathbf{a}) \neq \emptyset$ for every $\mathbf{a}$ iff $\mathcal{H}$ is unimodular [5]. However, except for the above unimodular condition for the nonemptiness and some results on its cardinality, little is known on the structure of $\Gamma_{\mathcal{H}}(\mathbf{a})$. We remark that $\sup_{\mathbf{a} \in [0,1]^V} \min_{\alpha \in \{0,1\}^V} D_{\mathcal{H}}(\mathbf{a}, \alpha)$ is the *linear discrepancy* of $\mathcal{H}$, and considered as a key concept in hypergraph theory and combinatorial geometry [5, 7, 11].

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[1] Throughout this paper, we use a Greek (resp. bold) character for representing a binary (resp. real-valued) function on V.

In this paper, we study the geometric property of $\Gamma_{\mathcal{H}}(\mathbf{a})$. We say that a hypergraph $\mathcal{H}$ has the *simplex property* if $\Gamma_{\mathcal{H}}(\mathbf{a})$ is (the vertex set of) a simplex (possibly a degenerate one or empty) regarding it as a set of n-dimensional points for any $\mathbf{a} \in [0,1]^V$. Our main aim is to investigate classes of hypergraphs that have the simplex property. The global rounding condition is directly written in an integer programming formula, and thus from the viewpoint of mathematical programming, we have interesting classes of integer programming problems for which the solution space is a simplex while the corresponding LP polytope is not always a simplex.

The simplex property is motivated by recent results on the maximum number $\mu(\mathcal{H}) = \max_{\mathbf{a}\in[0,1]^V} |\Gamma_{\mathcal{H}}(\mathbf{a})|$ of global roundings. $\mu(\mathcal{H})$ can never become less than $n+1$ for any hypergraph since n unit vectors and the zero vector always form $\Gamma_{\mathcal{H}}(\mathbf{a})$ for a suitable $\mathbf{a}$. In general, $\mu(H)$ may become exponential in n. However, Sadakane *et al.*[13] discovered that $\mu(\mathcal{I}_n) = n+1$ where $\mathcal{I}_n$ is a hypergraph on $V = \{1, 2, .., n\}$ with edge set $\{[i, j]; 1 \leq i \leq j \leq n\}$ consisting of all subintervals of V. A corresponding global rounding is called *sequence rounding*, which is a convenient tool in digitization of a sequence analogue data.

Given this discovery, it is natural to ask for which class of hypergraphs the property $\mu(\mathcal{H}) = n+1$ holds. Moreover, there should be combinatorial (or geometric) reasoning why $\mu(\mathcal{H}) = n+1$ holds for those hypergraphs. Naturally, the simplex property implies that $\mu(\mathcal{H}) = n+1$ since a d-dimensional simplex has $d+1$ vertices, and indeed $\mathcal{I}_n$ has the simplex property.

Shortest-path hypergraphs and range spaces

$\mathcal{I}_n$ has $n(n+1)/2$ hyperedges, and the authors do not know any hypergraph with less than $n(n+1)/2$ hyperedges (including n singletons) that has the simplex property. Thus, it is reasonable to consider some natural classes of hypergraphs with $n(n+1)/2$ hyperedges.

Consider a connected graph $G = (V, E)$ in which each edge e has a positive length $\ell(e)$. We fix a total ordering on V, and write $V = \{v_1, v_2, \ldots, v_n\}$. This ordering is inherited to any subset of V. For each pair (v_i, v_j) of vertices in V such that $i < j$, let $p(v_i, v_j)$ be the shortest path between them. If there are more than one shortest paths between them, we consider the lexicographic ordering among the paths induced from the ordering on V, and select the one with the first one in this ordering. Let $P(v_i, v_j)$ be the set of vertices on $p(v_i, v_j)$ including the terminal nodes v_i and v_j. We also define $P(v, v) = \{v\}$ for each $v \in V$. Let $\mathcal{F}(G) = \{P(v_i, v_j) : 1 \leq i \leq j \leq n\}$, and call $\mathcal{H}(G) = (V, \mathcal{F}(G))$ the *shortest-path hypergraph* associated with G.

It is conjectured that $\mu(\mathcal{H}(G)) = n+1$ if G is a connected graph with n vertices [1]. Note that $\mathcal{H}(G) = \mathcal{I}_n$ if G is a path. The conjecture has been proved for for trees, cycles, and outerplanar graphs [1, 14]. However, those proofs are complicated and case dependent. We try to establish a more structured theory considering the following deeper conjecture.

Conjecture 1.1 *For any connected graph G, $\mathcal{H}(G)$ satisfies the simplex property.*

This conjecture was proposed in [1] by the authors where the simplex property was called "affine independence property" since vertices of a simplex are affine independent as a set of vectors. So far, the conjecture has been proved only for trees, unweighted complete graphs, and unweighted (square) meshes. We prove that the simplex property is invariant under some graph-theoretic connection operations, and as a consequence, we show that the conjecture holds for series-parallel graphs.

In addition to significantly extending the verified classes of hypergraphs for both of the weaker and stronger conjectures, our theory also simplifies the proofs of known results. For example, that the weaker conjecture holds for cycles is one of the main results of [1] and its proof therein is quite involved. In our framework, it is almost trivial that the stronger conjecture holds for cycles (see Section 3).

From a computational-geometric viewpoint, $\mathcal{I}_n$ can be considered as the 1-dimensional range space corresponding to intervals, and thus we try to extend the theory to geometric range spaces. We generalize the argument for $\mathcal{H}(G)$ to *axiomatic shortest-path* hypergraphs (defined later), and prove the simplex property for some geometric range spaces such as the space of isothetic right-angle triangles.

Algorithmic implication

The theory is not only combinatorially interesting but is applied to algorithm design on the rounding problems. The algorithmic question of how to obtain a low-discrepancy rounding of given $\mathbf{a}$ is important in several applications. For example, consider the problem of digital halftoning in image processing, where the gray-scale value of each pixel has to be rounded into a binary value. This problem is formulated as that of obtaining a low-discrepancy rounding, in which the hypergraph corresponds to a family of certain local sets of pixels, and several methods have been proposed[2, 3, 12]. Unfortunately, for a general hypergraph, it is NP-complete to decide whether a given input $\mathbf{a}$ has a global rounding or not, and hence it is NP-hard to compute a rounding with the minimum discrepancy. Thus, a practical approach is to consider a special hypergraph for which we can compute a low-discrepancy rounding efficiently.

It is folklore that the unimodularity condition means that the vertices of the ball (w.r.t. $D_{\mathcal{H}}$) are integral, and an LP solution automatically gives an IP solution. Thus, a global rounding always exists and can be computed in polynomial time if $\mathcal{H}$ is unimodular, and therefore in the literature [2, 3, 8] unimodular hypergraphs are mainly considered.

Here, we consider another case where an integer programming problem can be solved in polynomial time: If the number of integral points in the solution space is small (i.e. of polynomial size), and there is an enumeration algorithm that is polynomial in the output size (together with the input size), we can solve the problem in polynomial time. We show that enumeration of all global roundings can be done in polynomial time for several (non-unimodular) hypergraphs with the simplex property by applying this framework.

2 Combinatorial and linear algebraic tools

2.1 Compatible set representing global roundings

The set of binary functions on V can be regarded as the n-dimensional hypercube $C_n = \{0,1\}^n$, where $n = |V|$. Consider an integer-valued distance f on C_n. We call a subset A of C_n a *compatible set* with respect to f if $f(x,y) = 1$ for any pair $x \neq y$ of A. In other words, A is a compatible set if and only if it is a unit diameter set. Property of a compatible set is highly dependent on f: If f is the L_∞ distance, the hypercube itself is a compatible set, while the cardinality of a compatible set for the Hamming distance is at most two. By definition[2] , $D_{\mathcal{H}}$ gives an integer-valued distance on the hypercube C_n.

Definition 1 *A set of binary functions on V is called $\mathcal{H}$-compatible if it is a compatible set with respect to $D_{\mathcal{H}}$. In other words, $|w_F(\alpha) - w_F(\beta)| \leq 1$ holds for every hyperedge F of $\mathcal{H}$ for any elements α and β of the set.*

$\Gamma_{\mathcal{H}}(\mathbf{a})$ is always an $\mathcal{H}$-compatible set, since the $D_{\mathcal{H}}$ distance between two global roundings must be integral and less than 2. Conversely, any maximal $\mathcal{H}$-compatible set is $\Gamma_{\mathcal{H}}(\mathbf{g})$, where $\mathbf{g}$

[2]Here, we use the assumption that $\mathcal{H}$ contains all singleton hyperedges.

is the center of gravity of the compatible set. Thus, it suffices to show the simplex property for compatible sets instead of sets of global roundings.

2.2 General results on simplex property

It is obvious that the simplex property is monotone, that is,

Lemma 2.1 *If $\mathcal{H} = (V, \mathcal{F})$ has the simplex property and $\mathcal{F} \subset \mathcal{F}'$ then $\mathcal{H}' = (V, \mathcal{F}')$ does, too.*

Recall that a set $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ of vectors is affine dependent if and only if there are real numbers $c_1, c_2, \ldots, c_m$ satisfying (1) at least one of them is non-zero, (2) $\sum_{1 \le i \le m} c_i = 0$, and (3) $\sum_{1 \le i \le m} c_i \mathbf{a}_i = \mathbf{0}$.

A set A of binary assignments on V is called *minimal affine dependent* if it is an affine dependent set as a set of vectors in the n-dimensional real vector space ($n = |V|$) and every proper subset of it is affine independent.

For a binary assignment α on V and a subset X of V, $\alpha|_X$ denotes the restriction of α on X. Given a set A of binary assignments on V, its restriction to X is $A|_X = \{\alpha|_X : \alpha \in A\}$. Note that the set is not a multi-set, and we only keep a single copy even if $\alpha|_X = \beta|_X$ for different α and β in A.

For binary assignments α on X and β on Y $\alpha \oplus \beta$ is a binary assignment on $V = X \cup Y$ obtained by concatenating α and β: That is, $\alpha \oplus \beta(v) = \alpha(v)$ if $v \in X$, and $\alpha \oplus \beta(v) = \beta(v)$ if $v \in Y$. By definition, $\alpha \oplus \beta$ is only defined if $\alpha(v) = \beta(v)$ for each $v \in X \cap Y$.

The following is our key lemma:

Lemma 2.2 *Let A be a minimal affine dependent set on V, and let $V = X \cup Y$. If $A|_X$ and $A|_Y$ are affine independent, then $A|_{X \cap Y}$ has only one assignment.*

Proof We construct a bipartite graph $W = (U_X, U_Y, E)$, where U_X and U_Y correspond to the assignments in $A|_X$ and $A|_Y$, respectively. The vertices $u \in U_X$ and $v \in U_Y$ are connected by an edge in E if and only if the concatenation of corresponding assignments is an element of A. Thus, there is a one-to-one correspondence between E and A, and let α^e be the element of A associated with an edge $e \in E$. Thus, $\alpha^e = \beta^u \oplus \gamma^v$ if $e = (u, v)$, where β^u and γ^v are the assignments corresponding to $u \in U_X$ and $v \in U_Y$, respectively.

If $e = (u, v)$ is an edge, β^u and γ^v must coincide on $X \cap Y$. This coincidence condition is transitive, and hence on each connected component of W, all the assignment take the same value on $X \cap Y$.

Since A is affine dependent, there exists a constant $c(e)$ for each e such that $\sum_{e \in E} c(e) = 0$ and $\sum_{e \in E} c(e)\alpha^e = \mathbf{0}$, and at least one $c(e)$ is nonzero. For a vertex u, let $E(u)$ be the set of edges adjacent to u. Let $C(u) = \sum_{e \in E(u)} c(e)$. Then, we can easily see that $\sum_{u \in U_X} C(u) = \sum_{e \in E} c(e) = 0$. Also, $\sum_{u \in U_X} C(u)\beta^u = \mathbf{0}$, since the left-hand side is the restriction of $\sum_{e \in E} c(e)\alpha^e$ to X. Because of the affine independence of $A|_X$, it follows that $C(u) = 0$ for each $u \in U_X$. Similarly, $C(v) = 0$ for each $v \in U_Y$.

We now claim that W cannot be a forest. Indeed, if W is a forest, we can find a vertex v of degree 1. Thus, for the edge e incident to v, $c(e) = C(v) = 0$. We can continue this removal operation of leaf vertices to find that $c(e) = 0$ for every edge e; a contradiction.

Thus, W has a cycle as its subgraph. Let $e_1, e_2, \ldots e_m$ be the list of edges of the cycle numbered in the order on the cycle (starting from an arbitrary edge). Since W is bipartite, m is even. Then, if we consider $\sum_{i=1}^{m} (-1)^i \alpha^{e_i}$, its restrictions to X and Y are zero vectors. Indeed, for each vertex $u \in U_X$ (resp. $v \in U_Y$)in the cycle, $\beta(u)$ (resp. γ^v) appears twice in the sum so that one is multiplied by 1 and the other is multiplied by -1. Thus, $\sum_{i=1}^{m} (-1)^i \alpha^{e_i} = \mathbf{0}$ and it is clear that $\sum_{i=1}^{m}(-1)^i = 0$. Therefore, the set $\{\alpha^{e_i} | i = 1, 2, \ldots, m\}$ is affine dependent. Since A is a minimal affine dependent set, we conclude that W itself must be a cycle, and thus it is connected. Thus, all assignments of A are equal to each other on $X \cap Y$. □

Given a subset S of V, we can consider the induced hypergraph $\mathcal{H}|_S = (S, \mathcal{F} \cap 2^S)$. Naturally, if a set A of binary assignments on V is compatible for $\mathcal{H}$, $A|_S$ is compatible for $\mathcal{H}|_S$.

By definition, a subset of a compatible set is also a compatible set. Thus, the concept of minimal affine dependent compatible set (possibly an empty set) is well defined. We have the following corollary of Lemma 2.2:

Corollary 2.3 *Consider a hypergraph $\mathcal{H} = (V, \mathcal{F})$ and a minimal affine dependent compatible set A. Suppose that $V = X \cup Y$ and each of $\mathcal{H}|_X$ and $\mathcal{H}|_Y$ has the simplex property. Then, for any pair α and α' in A, $\alpha(v) = \alpha'(v)$ for each $v \in X \cap Y$.*

Definition 2 *A vertex v of a hypergraph $\mathcal{H}$ is called a double-covered vertex if there exist suitable subsets X and Y such that $V = X \cup Y$, $v \in X \cap Y$, and both of $\mathcal{H}|_X$ and $\mathcal{H}|_Y$ have the simplex property. We say $S \subset V$ is double-covered if every element of S is double-covered.*

Definition 3 *For a subset S of vertices of a hypergraph $\mathcal{H} = (V, \mathcal{F})$, a set A of assignments on V is called S-contracted if $\alpha(v) = 0$ for each pair $v \in S$ and $\alpha \in A$.*

Theorem 2.4 *Let $\mathcal{H} = (V, \mathcal{F})$ be a hypergraph, and let $S \subset V$ be a double-covered set. Then, if every S-contracted compatible set is affine independent, $\mathcal{H}$ has the simplex property.*

Proof Assume on the contrary that $\mathcal{H}$ does not have the simplex property. Thus, we have an affine dependent compatible set, and hence have a minimal affine dependent compatible set A. From Corollary 2.3, we can assume that all assignments of A take the same value on each element of S. Thus, if we replace the value to 0 at every $v \in S$, the revised set $\tilde{A}$ is also compatible and minimal affine dependent, since we subtract the same vector from each member of A to obtain $\tilde{A}$. However, $\tilde{A}$ is S-contracted, and hence contradicts the hypothesis. □

Corollary 2.5 *If V itself is double-covered, $\mathcal{H} = (V, \mathcal{F})$ has the simplex property.*

Proof The (unique) V-contracted set is $\{\mathbf{0}\}$, thus is affine independent. □

2.3 Axiomatic shortest path hypergraph

Definition 4 *A hypergraph $\mathcal{H} = (V, \mathcal{F})$ is called an ASP (axiomatic shortest path) hypergraph if $\mathcal{F} = \{f(u, v) | u, v \in V \times V\}$ satisfies the following conditions: (1): $f(u, u) = \{u\}$.*
(2): $f(u, v) = f(u', v')$ if and only if $(u, v) = (u', v')$ as unordered pairs (one-to-one property).
(3): For any $s, t \in f(u, v)$, $f(s, t) \subset f(u, v)$ (monotonicity).

It is clear that the shortest-path hypergraph $\mathcal{H}(G)$ becomes an ASP hypergraph for any connected graph G with any edge-length function.

Definition 5 *Given an ASP hypergraph $\mathcal{H} = (V, \mathcal{F})$, a subset S of V is called a shortest-path-closed subset (SPC subset) if $f(u, v) \subseteq S$ for any pair u and v in S.*

The following lemma is immediate from definitions.

Lemma 2.6 *Given an ASP hypergraph $\mathcal{H} = (V, \mathcal{F})$ and an SPC subset S of V, $\mathcal{H}|_S$ is also an ASP hypergraph.*

3 Shortest path hypergraphs with the simplex property

The following lemma has been given in [1]:

Lemma 3.1 *Let $G = (V, E)$ be a connected graph, and let $V = X \cup Y$ be a partition of V. Let α_1 and α_2 be different assignments on X and let β_1 and β_2 be different assignments on Y. Then, the set $\mathcal{F} = \{\ \alpha_1 \oplus \beta_1,\ \alpha_1 \oplus \beta_2,\ \alpha_2 \oplus \beta_1,\ \alpha_2 \oplus \beta_2\ \}$ cannot be $\mathcal{H}(G)$-compatible.*

Definition 6 *A subgraph $G' = (V', E')$ of $G = (V, E)$ is called an SPC subgraph if any shortest path in G' is a shortest path in G.*

The following two lemmas are immediate from definitions:

Lemma 3.2 *If $G' = (V', E')$ is an SPC subgraph of $G = (V, E)$, V' is an SPC subset of V with respect to $\mathcal{H}(G)$, and $\mathcal{H}(G)|_{V'} = \mathcal{H}(G')$.*

Lemma 3.3 *Consider $\mathcal{H} = \mathcal{H}(G)$ for $G = (V, E)$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be SPC subgraphs such that $V_1 \cup V_2 = V$. Then, if both $\mathcal{H}(G_1)$ and $\mathcal{H}(G_2)$ have the simplex property, each vertex in $V_1 \cap V_2$ is double-covered.*

Proposition 3.4 *If G is a cycle, $\mathcal{H}(G)$ has the simplex property.*

Proof We give a cyclic ordering $v_1, v_2, \ldots, v_n$ of the vertices. For the vertex v_1, let $V_1 = \{v_1, v_2, v_3, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, v_{k+2}, \ldots, v_n, v_1\}$ where k is the largest index for which the shortest path from v_1 to v_k goes through v_2. Let G_1 and G_2 are induced subgraphs associated with V_1 and V_2, respectively. Since G_1 and G_2 are paths, it is known [1] that $\mathcal{H}(G_1)$ and $\mathcal{H}(G_2)$ have the simplex property. It is clear the G_1 and G_2 are SPC subgraphs, and $V_1 \cap V_2 = \{v_1\}$, and $V_1 \cup V_2 = V$. Thus, from Lemma 3.3, v_1 is double covered. This argument holds for any cyclic ordering, and thus every vertex of V is double-covered. Thus, from Corollary 2.5, $\mathcal{H}(G)$ has the simplex property. □

A graph $G = (V, E)$ is a series connection of two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ if there exists a vertex (joint vertex) v such that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \{v\}$, and $E_1 \cup E_2 = E$.

Lemma 3.5 *If G is a series connection of G_1 and G_2 where G_2 is a path and $\mathcal{H}(G_1)$ has the simplex property, $\mathcal{H}(G)$ has the simplex property.*

Proof We can prove the lemma in a routine way by using Lemma 3.1, although we omit the proof in this version. □

The following theorem has been given in [1]. Our framework allows a clearer proof below:

Theorem 3.6 *Let G be a series connection of two connected graphs G_1 and G_2. Then, if both $\mathcal{H}(G_1)$ and $\mathcal{H}(G_2)$ have the simplex property, $\mathcal{H}(G)$ does.*

Proof Consider the joint vertex v and any vertex $w \in V$. Without loss of generality, we can assume that $w \in G_1$. Consider the shortest path $p(w, v)$ between w and v, and let $P(w, v)$ be the set of vertices on $p(w, v)$. Consider the set $V_2 \cup P(w, v)$. Then, we can see that any shortest path between a pair of vertices in $V_2 \cup P(w, v)$ cannot go through a vertex in $V_1 \setminus P(w, v)$.

Thus, $\tilde{G_2} = G_2 \cup P(w, v)$ is an SPC subgraph, and also G_1 is an SPC subgraph. From Lemma 3.5, $\mathcal{H}(\tilde{G_2})$ has the simplex property. Thus, from Lemma 3.3, w is double-covered. Since w is arbitrarily chosen, V itself is a double-covered set. Thus, from Corollary 2.5, $\mathcal{H}(G)$ has the simplex property. □

Definition 7 *A connected graph $G = (V, E)$ has a 3-parallel decomposition if there exist two vertices u and v such that G is decomposed into nonempty connected graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G_3 = (V_3, E_3)$ such that (1) $V = V_1 \cup V_2 \cup V_3$, (2) $V_1 \cap V_2 = V_2 \cap V_3 = V_1 \cap V_3 = \{u, v\}$, and (3) E is the disjoint union of E_1, E_2, and E_3. (see Fig. 1).*

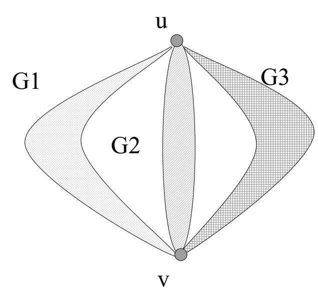

Figure 1: 3-parallel decomposition of G.

Consider a family Ψ of connected graphs, and assume that it is *closed under the subgraph operation*; that is, any connected subgraph of $G \in \Psi$ is also in Ψ. A graph $G \in \Psi$ is a *minimal counterexample* for the simplex property in Ψ if $\mathcal{H}(G)$ does not satisfy the simplex property but $\mathcal{H}(G')$ has the simplex property for every connected subgraph G' of G.

Theorem 3.7 *A minimal counterexample G for the simplex property in Ψ is 2-connected, and does not have a 3-parallel decomposition.*

Proof 2-connectivity follows from Theorem 3.6. Thus, we assume that G has a 3-parallel decomposition at u and v, and derive a contradiction. We define the following three subgraphs of G: $G_{(1,2)}$ is the union of G_1 and G_2, $G_{(1,3)}$ is the union of G_1 and G_3, and $G_{(2,3)}$ is the union of G_2 and G_3. These graphs are connected and hence satisfy the simplex property because of the minimality of G.

By symmetry, we can assume that the shortest path between u and v is in G_1. Then, both $G_{(1,2)}$ and $G_{(1,3)}$ are SPC subgraphs. Thus, $\mathcal{H}(G)|_{V(G_{(1,2)})} = \mathcal{H}(G_{(1,2)})$ and $\mathcal{H}(G)|_{V(G_{(1,3)})} = \mathcal{H}(G_{(1,3)})$, where $V(G_{(i,j)})$ is the vertex set of $G_{(i,j)}$. Thus, it is clear that each vertex of G_1 is double-covered.

A vertex x in $V(G_{(2,3)})$ is called *biased* if either the shortest path in G from x to u goes through v or the shortest path from x to v goes through u. We claim that a biased vertex is double-covered. Without loss of generality, we assume that x is a vertex of G_2 and the shortest path $\mathbf{p}$ from x to v goes through u. Then, any vertex of G_2 on $\mathbf{p}$ is also biased, and $G_{(1,3)} \cup \mathbf{p}$ is an SPC subgraph. Thus, v is in the intersection of two SPC subgraphs $G_{(1,3)} \cup \mathbf{p}$ and $G_{(1,2)}$, and hence double-covered. Thus, $S = V(G_1) \cup B$ is double-covered, where B is the set of all biased vertices.

Now, we are ready to apply Theorem 2.4. Consider an arbitrary S-contracted compatible set A of $\mathcal{H}(G)$. We claim that A is also $\mathcal{H}(G_{(2,3)})$ compatible. If this claim is true, A must be affine independent (since $\mathcal{H}(G_{(2,3)})$ has the simplex property), and we can conclude that $\mathcal{H}(G)$ has the simplex property from Theorem 2.4, so that we have contradiction.

We give a proof for the claim. Let α and β be any two members of A. Consider any shortest path $\mathbf{p}$ of $G_{(2,3)}$. Let x and y be endpoints of $\mathbf{p}$, and let P be the vertex set of the path. It suffices to show the compatibility $|\alpha(P) - \beta(P)| \leq 1$.

If all the vertices on $\mathbf{p}$ are in S, $\alpha(P) = \beta(P) = 0$, and the compatibility condition is trivial. Thus, we assume there exist vertices in $V \setminus S$ on $\mathbf{p}$. Let x_0 be the nearest vertex in $(V \setminus S) \cap P$ to x. The subpath $\mathbf{p}_0$ of $\mathbf{p}$ between x_0 and y is the shortest path in $G_{(2,3)}$ between them. Let P_0 be the set of vertices on it.

Consider the shortest path $\mathbf{q}$ with respect to G between x_0 and y. If it contains both u and v on it, either the shortest path between x_0 and u contains v or that between x_0 and v contains u. Thus, x_0 must be biased, and hence in S, contradicting our hypothesis. Therefore, without loss of generality, we can assume $\mathbf{q}$ does not contain v. This means that $\mathbf{q}$ contains no vertex of $G_1 \setminus \{u\}$, since otherwise $\mathbf{q}$ must go through u twice. Thus, $\mathbf{q}$ is in $G_{(2,3)}$, and hence $\mathbf{q} = \mathbf{p}_0$ since the shortest path between given two vertices is unique in our definition of the shortest path hypergraph.

Thus, from the compatibility on a shortest path of G, $|\alpha(P_0) - \beta(P_0)| \leq 1$. Since assignment on each vertex of S is 0 for each of α and β, we have the compatibility $|\alpha(P) - \beta(P)| \leq 1$ on $\mathbf{p}$. Thus, A is $\mathcal{H}(G_{(2,3)})$ compatible, and we have the claim. □

Thus, the simplex property holds for a graph that is constructed by applying a series of 3-parallel connections and series connections from pieces (such as paths, cycles, unit edge-length complete graphs, and unit edge-length meshes) for which the simplex property is known to hold. We give a typical example in the following.

A graph is *series-parallel* if it does not have a subdivision of the complete graph K_4 as its subgraph. Here, a subdivision of a graph is obtained by replacing edges of the original graph with chains. A connected graph is *outerplanar* if and only if it has a planar drawing in which every vertex lies on the outerface boundary (the boundary is a cycle if the graph is 2-connected). An edge that is not on the outerface boundary is called a *chord*. It is known that a series parallel graph is a planar graph, and an outerplanar graph is series-parallel.

Theorem 3.8 *If G is a connected series-parallel graph, $\mathcal{H}(G)$ has the simplex property.*

Proof Clearly, the family of connected series-parallel graphs is closed under the subgraph operation, and we consider its minimal counterexample G. By Theorem 3.7, G is 2-connected. If G is not outerplanar, G has a vertex v in the interior of the outerface cycle C. Since G is 2-connected, v is connected to at least two vertices of C without using edges on C. If v is connected to three vertices of C, the union of these paths and C contains a subdivision of K_4, and we have contradiction. Thus, v is connected to exactly two vertices u_1 and u_2 of C, and we have 3-parallel decomposition at u_1 and u_2. Thus, G must be outerplanar. If G has a chord, G has 3-parallel decomposition at the end vertices of the chord. Thus, G does not have a chord. However, a 2-connected outerplanar graph without a chord must be a cycle, and we have already shown the simplex property for cycles. Thus, we have the theorem. □

As a corollary, we have $\mu(G) = n + 1$ for a series-parallel graph, extending the result for an outerplanar graph given in [14].

Moreover, it can be observed that any (non-cycle) 2-connected series parallel graph has a 3-parallel decomposition in which two of the components are paths from a classification of substructures of series parallel graphs given by Juvan *et al.* [10] (also see [15]). Using this observation and the argument given in [14] for outerplanar graphs, we have the following (we omit details in this version):

Theorem 3.9 *We can enumerate all the global roundings of an input* **a** *for the shortest path hypergraph of a series-parallel graph with n vertices in $O(n^3)$ time.*

4 Geometric problems

We consider some geometric hypergraphs that are ASP hypergraphs. Consider a set V of n points on a plane. For each pair $u = (x_u, y_u)$ and $v = (x_v, y_v)$ of points, uv is the line segment connecting them. Let $B(u, v)$ be the region below the segment uv; that is, $B(u, v) = \{(x, y)|x \in [x_u, x_v], y - y_u \leq \frac{y_v - y_u}{x_v - x_u}(x - x_u)\}$ if $x_u \neq x_v$. If $x_u = x_v$, we define $B(u, v) = uv$. Let $R(u, v)$ be the closed isothetic rectangle which has u and v in its diagonal position, and let $T(u, v) = B(u, v) \cap R(u, v)$ be the lower right-angle isothetic triangle which has uv as its longest boundary edge. We define $T(u, u) = R(u, u) = B(u, u) = uu = \{u\}$.

We consider hypergraphs $\mathcal{S} = (V, \{V \cap uv : u, v \in V\})$, $\mathcal{B} = (V, \{V \cap B(u, v) : u, v \in V\})$, $\mathcal{R} = (V, \{V \cap R(u, v) : u, v \in V\})$, and $\mathcal{T} = (V, \{V \cap T(u, v) : u, v \in V\})$. See Fig. 2 to get intuition.

They are typical examples of range spaces. $\mathcal{B}$ becomes the hypergraph $\mathcal{I}_n$ consisting of all intervals if the point set is convex (i.e., it is on the lower chain of the convex hull of itself) and arranged with respect to the x-coordinate values. Thus, the global rounding for $\mathcal{B}$ is a natural extension of the $\mathcal{I}_n$-global rounding (sequence rounding). We also remark that $\mathcal{S}$ corresponds to the stabbed sets by segments, and equals $\mathcal{K}_n = (V, V \times V)$ if the point set is in general position.

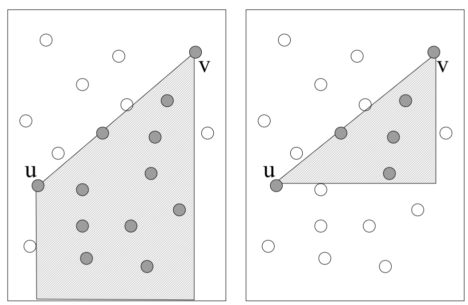

Figure 2: $B(u, v)$ (left) and $T(u, v)$ (right).

Lemma 4.1 *$\mathcal{S}$, $\mathcal{B}$, and $\mathcal{T}$ are ASP hypergraphs for any point set V. $\mathcal{R}$ is an ASP hypergraph if there are no four points of V forming corners of an isothetic rectangle.*

4.1 Simplex property of range spaces

Theorem 4.2 *Each of $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{S}$ have the simplex property. If there are no four points of V forming corners of an isothetic rectangle, $\mathcal{R}$ has the simplex property.*

Proof We prove the theorem for $\mathcal{T}$, and only briefly explain for other cases. We prove the simplex property by induction on the number of horizontal lines and that of vertical lines on which V lies. If V lies on a horizontal line ℓ, the problem is reduced to the sequence rounding problem, since $T(u, v)$ is the set of points on the interval uv on ℓ. Let X_j (resp. Y_j) be the vertex set whose x-coordinate (resp. y-coordinate) value is the j-th smallest among V. Suppose that the statement holds if the point set lies on less than M horizontal lines or less than N vertical lines. We consider the case where V lies on M horizontal lines and also lies on N vertical lines.

Let $X_{\geq j} = \cup_{i \geq j} X_i$ and $X_{\leq j} = \cup_{i \leq j} X_i$. It is easy to see that they are SPC subsets of V. In particular, we consider $X_{\geq 2}$ and $X_{\leq N-1}$. Let $\mathcal{T}^+ = \mathcal{T}|_{X_{\geq 2}}$ and $\mathcal{T}^- = \mathcal{T}|_{X_{\leq N-1}}$. From

Lemma 2.6, they are ASP hypergraphs, and by induction hypothesis, have the simplex property. Thus, $X_{\geq 2} \cap X_{\leq N-1} = V \setminus (X_1 \cup X_N)$ is double-covered.

Similarly, we can see that $V \setminus (Y_1 \cup Y_M)$ is double-covered. Since union of two double-covered sets is also double-covered, $S = [V \setminus (X_1 \cup X_N)] \cup [V \setminus (Y_1 \cup Y_M)]$ is double-covered. Thus, we can apply Theorem 2.4, and consider the restriction of $\mathcal{T}$ to $V \setminus S$.

Any point in $V \setminus S$ must be at a corner of the minimum enclosing isothetic rectangle of V, thus $V \setminus S$ has at most four points, for which we can directly show the simplex property of the restriction of $\mathcal{T}$: For example, if it has four points, the hypergraph is equivalent to the shortest path hypergraph for a cycle (on four vertices) considered in the previous section. Thus, $\mathcal{T}$ has the simplex property.

For the case of $\mathcal{S}$, the only difference is that its restriction on $V \setminus S$ is the complete graph (regarded as a special hypergraph) on at most four vertices, and the affine independence for its compatible set is easy to see. The case for $\mathcal{R}$ is analogous, where $V \setminus S$ contains at most two points. For $\mathcal{B}$, we only divide V by using vertical lines, and reduce the problem to the sequence rounding problem. We omit details for it. □

We remark that $\mathcal{R}$ is smaller than the range space corresponding to all isothetic rectangles. However, since $\mathcal{R}$ has the simplex property, the range space of all isothetic rectangles also has the simplex property because of Lemma 2.1. Similarly, since $\mathcal{T}$ has the simplex property, the range space of all isothetic right-angle triangles has the simplex property.

If we consider the digital halftoning application, it is important to consider the case where V is the set of points of an $M \times N$ grid and the hyperedge is a set of rectangles. Let $v_{i,j}$ be the point at the (i, j) position. Given two points $v = v_{s,t}$ and $w = v_{k,\ell}$ such that $s \leq k$, let $R(v, w)$ be the set of points in the rectangle which has v and w as corners.

Unfortunately, if we consider the range space $\mathcal{R}$ on the set of grid points, $\mathcal{R}$ is not an ASP hypergraph, since $R(v_{i,j}, v_{k,\ell}) = R(v_{i,\ell}, v_{k,j})$ and the one-to-one property does not hold. Indeed, this hypergraph does not have the simplex property (we have a counterexample).

However, if we give a slight modification, we can apply our theory. The chipped rectangle $\tilde{R}(v, w)$ is obtained by removing the upper corner point that is neither v nor w if v and w are neither on the same row nor on the same column. We define $\tilde{R}(v, w) = R(v, w)$ if v and w are either on the same row or on the same column. We define $\mathcal{CR} = (V, \{\tilde{R}(v, w) | v, w \in V\})$.

Theorem 4.3 *$\mathcal{CR}$ has the simplex property.*

4.2 Algorithms for computing roundings

We can design a polynomial-time algorithm for enumerating all the global roundings of an input real assignment **a** for each of $\mathcal{B}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{CR}$. We briefly explain the algorithm for $\mathcal{T}$. Basically, we can apply a building-up (or divide-and-conquer) strategy, in which we first compute the restrictions on $X_{\geq \lceil n/2 \rceil}$ and $X_{\leq \lceil n/2 \rceil - 1}$ recursively, and check the rounding condition for $\mathcal{T}$ on each possible concatenated rounding. It takes $O(n^2)$ time for testing each concatenated rounding by using an efficient range-searching method, and hence the total time complexity becomes $O(n^4)$. We can improve the complexity to $O(n^3)$ for $\mathcal{CR}$ (we omit details).

The algorithm implies that we can decide whether a given input has a global rounding or not in polynomial time for these region families. This is highly contrasted to the fact that it is NP hard to decide the existence of a global rounding for the family of all 2×2 square regions in a grid [3, 4]. If we consider $\mathcal{CR}$, the linear discrepancy is known to be $O(\log^3 n)$ and $\Omega(\log n)$ [6] , and hence it is expected that a given input may have no global rounding. Thus, we may consider a heuristic algorithm for computing a nice (not necessarily global) rounding by using the building-up strategy in which we select K best roundings (with respect to the discrepancy) from those obtained by concatenating pairs of assignments constructed in the previous stage to proceed to the next stage. Our theorem implies that if we set $K \geq n+1$, we never miss a global rounding if it exists.

5 Concluding remarks

If we can replace 3-parallel decomposition with 2-parallel decomposition in Theorem 3.7, we can prove the conjecture, since any 2-connected graph is decomposed into e and $G \setminus \{e\}$ at the endpoints of any edge e. For a special input where each entry of $\mathbf{a}$ is $0.5 + \epsilon$, it has been shown that there are at most $m+1$ global roundings for unit edge-length connected graphs (except trees) with m edges [9]. However, it is not known whether $\mu(\mathcal{H}(G))$ is polynomially bounded in general. Another interesting question is whether there is a hypergraph with the simplex property with less than $n(n+1)/2$ hyperedges (including singletons).

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