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Closure Operators and Complete Embeddings of Residuated Lattices

Abstract. In this paper, a theorem on the existence of complete embedding of partially ordered monoids into complete residuated lattices is shown. From this, many interesting results on residuated lattices and substructural logics follows, including various types of completeness theorems of substructural logics.

Keywords: substructural logics, residuated lattices, complete embeddings, completeness.

1. Substructural Logics and Residuated Lattices

We will show in the present paper a theorem on the existence of complete embedding of partially ordered monoids into complete residuated lattices. Our theorem covers many of related results on embeddings. In particular, various completeness theorems for substructural logics follow from this, including e.g. *Kripke completeness* by Ono-Komori [6], *algebraic completeness of substructural predicate logics* in [5], *standard completeness* by Montagna-Ono [4], and *finite embeddability property* by Blok-van Alten [1]. [1]

Substructural logics are logics obtained from either classical logic or intuitionistic logic by deleting some or all of *structural rules*, when they are formalized in sequent calculi. The lack or the presence of structural rules effects sensitively the "meaning" of implication. Thus, within the framework of substructural logics, we can discuss nonclassical logics with various kinds of implications, in a uniform way. Substructural logics include Lambek calculus for categorial grammar, linear logic, which has only exchange rule, relevant logics, and the logic $\mathbf{FL_{ew}}$ without contraction rule. Here, the sequent system for $\mathbf{FL_{ew}}$ is obtained from Gentzen's sequent calculus **LJ** for intuitionistic logic by deleting contraction rule. For more information, see e.g. [7, 5].

In the following, we will treat mainly the logic $\mathbf{FL_{ew}}$ without contraction rule, and intuitionistic linear logic $\mathbf{FL_e}$. The latter is formalized as a sequent

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calculus obtained from $\mathbf{FL_{ew}}$ by deleting weakening rule. But as noticed in §§ 2 and 3, the same argument works also for other substructural logics, e.g. $\mathbf{FL}$ which has no structural rules, and $\mathbf{FL_{ec}}$ which is obtained from $\mathbf{FL_e}$ by adding contraction rule (see [5] for the precise definition).

In substructural logics, it is convenient and moreover natural to introduce a new logical connective $*$, called the *fusion* or the *multiplicative conjunction*. Rules for $*$ are given as:

$$\frac{\Gamma, \varphi, \psi, \Sigma \to \delta}{\Gamma, \varphi * \psi, \Sigma \to \delta}\ (* \to) \qquad \frac{\Gamma \to \varphi \quad \Sigma \to \psi}{\Gamma, \Sigma \to \varphi * \psi}\ (\to *)$$

Then, fusions represent commas in sequents. More precisely, the following holds:

> $\varphi_1, \ldots, \varphi_m \to \psi$ is provable if and only if $\varphi_1 * \ldots * \varphi_m \to \psi$ is provable.

Algebraic structures for these logics are residuated lattices. An algebra $\mathbf{P} = \langle P, \cap, \cup, \cdot, \to, 0, 1, \bot, \top \rangle$ is a *commutative residuated lattice* if it satisfies :

1. $\langle P, \cap, \cup, \bot, \top \rangle$ is a bounded lattice,
2. $\langle P, \cdot, 1 \rangle$ is a commutative monoid with the unit 1,
3. $x \cdot y \leq z$ iff $x \leq (y \to z)$ (the law of residuation).

In the above definition, 0 is an arbitrary element of P, which is used only for defining the interpretation of the negation. More precisely, the logical connective $\neg$ is interpreted as $\sim$ in a residuated lattice, where $\sim x$ denotes $x \to 0$. A commutative residuated lattice is *integral* if the unit 1 is equal to the greatest element $\top$ and 0 is equal to $\bot$, and it is *weakly idempotent* if $x \leq x \cdot x$ for each x. When a commutative residuated lattice is integral, both $\bot$ and $\top$ are usually omitted in its represention. A commutative residuated lattice is *complete* if it is complete as a lattice. We note that any complete, commutative residuated lattice satisfies the following distributivity of fusion over arbitrary join :

$$(\cup x_i) \cdot y = \cup (x_i \cdot y) \quad \text{for any } x_i, y.$$

A complete, commutative residuated lattice is sometimes called a *commutative unital quantale* (see [8]). A formula φ is said to be *valid* in a commutative residuated lattice $\mathbf{P}$ if $v(\varphi) \geq 1$ holds in $\mathbf{P}$ for any valuation

v on $\mathbf{P}$. Here, v is a *valuation* on $\mathbf{P}$ if v is any map from the set of propositional variables to P. As usual, v can be extended uniquely to a map from the set of formulas to P by interpreting logical connectives $\wedge, \vee, *, \supset$ and $\neg$ as $\cap, \cup, \cdot, \rightarrow$ and $\sim$, respectively. Clearly, we can replace the condition "$v(\varphi) \geq 1$" by "$v(\varphi) = 1$", when $\mathbf{P}$ is integral. By using Lindenbaum algebras, we can easily show the following.

PROPOSITION 1. *For any formula φ, φ is provable in $\mathbf{FL_e}$ iff it is valid in every commutative residuated lattice. Also, the similar relation holds between $\mathbf{FL_{ew}}$ and the class of commutative integral residuated lattices, and also between $\mathbf{FL_{ec}}$ and the class of commutative weakly idempotent residuated lattices.*

For a given commutative monoid $\mathbf{M} = \langle M, \cdot, 1\rangle$, we can construct a commutative residuated lattice, as shown below. First, define $*$ and $\Rightarrow$ for every $X, Y \subseteq M$ by

i. $X * Y = \{a \cdot b \in M : a \in X \text{ and } b \in Y\}$,
ii. $X \Rightarrow Y = \{a \in M : c \cdot a \in Y \text{ for each } c \in X\}$.

Then, we have the following (see e.g. [5]).

LEMMA 2. *For any given commutative monoid $\mathbf{M}$, the structure $\wp(\mathbf{M}) = \langle \wp(M), \cap, \cup, *, \Rightarrow, O, \{1\}, \emptyset, M\rangle$ forms a complete commutative residuated lattice, where O is an arbitrary subset of M.*

A map C on a commutative residuated lattice $\mathbf{P}$ is a *closure operator* if for all $x, y \in P$

1. $x \leq Cx$,
2. $x \leq y$ implies $Cx \leq Cy$,
3. $CCx \leq Cx$,
4. $Cx \cdot Cy \leq C(x \cdot y)$.

Note that the first three conditions correspond to the conditions for closure operators in the usual sense. Closure operators in our sense are called quantic nuclei in [8], when $\mathbf{P}$ is complete.

Suppose that C is a closure operator on a commutative residuated lattice $\mathbf{P}$. An element x of P is *C-closed* when $x = Cx$ holds. Let $C(P)$ be the set of all C-closed elements of P. It is easy to see that both $x \cap y$ and $x \rightarrow y$ are C-closed if both x and y are C-closed. But, this doesn't hold always for $\cup$ and $\cdot$. We define $\cup_C$ and $\cdot_C$ by

$$x \cup_C y = C(x \cup y) \text{ and } x \cdot_C y = C(x \cdot y).$$

Then, we can show the following.

Proposition 3. *Suppose that* $\mathbf{P}$ *is a commutative residuated lattice and* C *is a closure operator on* P*. Then, the algebra* $C(\mathbf{P}) = \langle C(P), \cap, \cup_C, \cdot_C, \rightarrow, d, C1, C\bot, \top\rangle$ *forms a residuated lattice where* d *is an arbitrary* C*-closed element. If* $\mathbf{P}$ *is integral then so is* $C(\mathbf{P})$ *if we take* $C0$ $(= C\bot)$ *for* d*, if* $\mathbf{P}$ *is weakly idempotent then so is* $C(\mathbf{P})$*, and also if* $\mathbf{P}$ *is complete then so is* $C(\mathbf{P})$*.*

By combining Proposition 3 with Lemma 2, we have the following.

Corollary 4. *If* $\mathbf{M}$ *is a commutative monoid and* C *is a closure operator on* $\wp(\mathbf{M})$*, then* $\mathbf{C_M} = \langle C(\wp(M)), \cap, \cup_C, *_C, \Rightarrow, D, C(\{1\}), C(\emptyset), M\rangle$ *is a complete commutative residuated lattice where* D *is any* C*-closed subset of* M*.*

We note that the converse of the above corollary holds. That is, any complete commutative residuated lattice is isomorphic to $\mathbf{C_M}$ for a commutative monoid $\mathbf{M}$ with a closure operator C on $\wp(\mathbf{M})$. See [5] for the proof.

2. Complete Embeddings

In the rest of the present paper, we assume moreover that $\mathbf{M}$ is a *partially ordered commutative monoid* (or simply, a commutative *po-monoid*). Here, we say that $\mathbf{M} = \langle M, \cdot, 1, \leq\rangle$ is a commutative po-monoid, when $\langle M, \cdot, 1\rangle$ is a commutative monoid with the unit element 1 and $\langle M, \leq, 1\rangle$ a partially ordered set in which the following monotonicity holds: for any $x, y, z \in M$, $x \leq y$ implies $x \cdot z \leq y \cdot z$. We say that a commutative po-monoid $\mathbf{M}$ is integral when the unit element 1 is equal to the greatest element, and it is weakly idempotent when $x \leq x \cdot x$ holds for every x.

Suppose that C is a closure operator on $\wp(\mathbf{M})$ for a commutative po-monoid $\mathbf{M}$, in which every C-closed subset X is *downward closed*, i.e. $x \in X$ and $y \leq x$ imply $y \in X$ for any $x, y \in M$. If $\mathbf{M}$ is integral then $C(\{1\}) = M$ holds, and hence the residuated lattice $\mathbf{C_M}$ with $D = C(\emptyset)$ becomes also integral. So, hereafter we take always $C(\emptyset)$ for D in $\mathbf{C_M}$ when $\mathbf{M}$ is integral. Similarly, if $\mathbf{M}$ is weakly idempotent then so is $\mathbf{C_M}$.

Now we have a way of constructing a complete residuated lattice from a given commutative po-monoid. Moreover, our construction respects the integrality and the weak idempotency. Let us take any commutative residuated lattice $\mathbf{P}$. Then, the po-monoid reduct $\mathbf{P}^\dagger$ of $\mathbf{P}$ determines another residuated lattice $\mathbf{C}_{\mathbf{P}^\dagger}$. Now, it is interesting to see how far the residuated lattice $\mathbf{C}_{\mathbf{P}^\dagger}$ reflects the structure of $\mathbf{P}$, in particular its lattice operations and residual. The following theorem, which is our main theorem, answers this.

For a given commutative po-monoid $\mathbf{M}$ and a closure operator C on $\wp(\mathbf{M})$, define a subset U of M by $U = \{b \in M : (b] \text{ is } C\text{-closed}\}$, where $(b] = \{x \in M : x \leq b\}$. We call U, the *base* for C in $\mathbf{M}$. When $\mathbf{M}$ is integral, the base U is non-empty, since (1] is C-closed as it is equal to M. We assume moreover the following closure conditions (1) and (2) for C ; for any C-closed subset X,

(1) X is downward closed,
(2) $\cup_i a_i \in X$ whenever $\{a_i : i \in I\} \subseteq X \cap U$ and $\cup_i a_i \in U$.

THEOREM 5. *Suppose that* $\mathbf{M}$ *is a commutative po-monoid and* C *is a closure operator on* $\wp(\mathbf{M})$ *for which closure conditions hold. Then the map* $h : U \to C(\wp(M))$ *defined by* $h(b) = (b]$ *is a complete embedding, i.e. an order isomorphism which preserves all existing products and residuals in* U*, and also all existing* (*infinite*) *joins and meets in* U*, where* U *is the base for* C *in* $\mathbf{M}$.

First, we give some comments on the theorem. It is not always the case that U is a submonoid of M. So, our theorem says that as long as $a \cdot b \in U$ for $a, b \in U$, $h(a) *_C h(b) = h(a \cdot b)$ holds. Also, if $a \to b$ is defined and belongs to U for $a, b \in U$ (and $\cup_i a_i$ or $\cap_i a_i$ is defined and belongs to U for $a_i \in U$ for each i) then it is preserved by the map h.

Now we will give a proof of our theorem. It is obvious that the map h is an order isomorphism, since the order relation of $\mathbf{C}_{\mathbf{M}}$ is the set inclusion. We need to show that h preserves existing products, residuals, joins and meets in U.

Suppose that $a \cdot b \in U$ for $a, b \in U$. It is clear that $(a \cdot b]$ is the least C-closed set which includes the set $(a]*(b]$. Therefore, $h(a)*_C h(b) = (a]*_C(b] = C((a] * (b]) = (a \cdot b] = h(a \cdot b)$.

Next suppose that $a \to b \in U$ for $a, b \in U$. Since $(a] * (a \to b] = \{x \cdot y : x \leq a \text{ and } y \leq a \to b\} \subseteq \{z : z \leq b\} = (b]$ and $(b]$ is C-closed,

$(a] *_C (a \to b] \subseteq (b]$. If $(a] *_C X \subseteq (b]$ for a C-closed set X, then, for any $c \in X$, $a \cdot c \leq b$ and therefore $c \leq a \to b$. Hence $X \subseteq (a \to b]$. This means that $(a \to b]$ is maximal among such C-closed sets X that $(a] *_C X \subseteq (b]$. That is, $(a] \Rightarrow (b] = (a \to b]$. Therefore, $h(a) \Rightarrow h(b) = (a \to b] = h(a \to b)$.

It is easily seen that h preserves all existing meets, since $\cap_i (a_i] = (\cap_i a_i]$ holds. It remains to show that h preserves all existing joins. Suppose that $\cup_i a_i$ exists and belongs to U where $\{a_i : i \in I\} \subseteq U$. Since $(a_j] \subseteq (\cup_i a_i]$ for each $j \in I$, $\cup_C (a_i] \subseteq (\cup_i a_i]$. For a given C-closed set X, suppose that $(a_j] \subseteq X$ for each $j \in I$. Then $a_j \in X$ for each $j \in I$. By our assumption (2), $\cup_i a_i$ must belong to X. Hence, $(\cup_i a_i] \subseteq X$. Therefore $\cup_C h(a_i) = \cup_C (a_i] = (\cup_i a_i] = h(\cup_i a_i)$.

Suppose that $\mathbf{P}^\dagger$ is the po-monoid reduct of a commutative residuated lattice $\mathbf{P}$. Then, the construction of a complete residuated lattice $\mathbf{C}_{\mathbf{P}^\dagger}$ and the complete embedding described in Theorem 5 say that the po-monoid $\mathbf{P}^\dagger$, but not the residuated lattice $\mathbf{P}$, plays an essential and special role in them. This will support the idea of substructural logics from algebraic point of view. In fact, in substructural logics, we attach special importance to structural rules, apart from rules for logical connectives, and structural rules determine the "po-monoid" part of a given logic.

As can be seen in our proof of Theorem 5, the closure condition (2) is used only in proving that h preserves all existing joins. In other words, the map h still preserves all existing products, residuals and meets (but not necessarily joins), even if we assume only that every C-closed subset is downward closed.

We note also that when $U = M$ holds, the second closure condition (2) becomes the following;

(2') if $\cup_i a_i$ exists for $\{a_i : i \in I\} \subseteq X$, then $\cup_i a_i \in X$,

Any subset X satisfying both (1) and (2') is called a complete ideal. The notion of complete ideals is used for getting *completion* of Heyting algebras (see e.g. [9]).

3. Consequences of Main Theorem

Our theorem in the above covers many results on the existence of a (complete) embedding of commutative po-monoids into complete residuated lattices, including Lemma 3.9 of [3] for example. We will show here that var-

ious completeness theorems follow from these embedding results, by taking a suitable closure operator to each case.

I. Kripke completeness of propositional logic $\mathbf{FL_{ew}}$

An algebraic structure $\mathbf{M} = \langle M, \cdot, 0, 1, \cup, \leq \rangle$ is a *semilattice-ordered, commutative integral monoid* (or, a commutative integral *so-monoid*) if it satisfies the following:

1. $\langle M, \cdot, 1, \leq \rangle$ is a commutative po-monoid with the greatest element 1 and smallest element 0,[2]
2. the join $x \cup y$ exists for every $x, y \in M$,
3. $x \cdot (y \cup z) = (x \cdot y) \cup (x \cdot z)$ holds.

Note that in [6], commutative integral so-monoids in our sense but with the reverse order are discussed and called so-monoids. Let $\mathbf{M}$ be a commutative integral so-monoid. A nonempty subset X of M is an *ideal*, if the following holds :

(1) X is downward closed,
(2) if both a and b are in X, then $a \cup b \in X$.

Now, for each $X \subseteq M$, define $C^1(X)$ to be the ideal generated by X, i.e. the smallest ideal containing X. Of course, $0 \in C^1(X)$ for any X, and in particular, $C^1(\emptyset) = \{0\}$. We can show that C^1 is a closure operator on $\wp(\mathbf{M})$. In fact, it is obvious that C^1 satisfies the first three conditions for closure operators. We show that $C^1X * C^1Y \subseteq C^1(X * Y)$. Take any $c \in C^1X$ and $d \in C^1Y$. Then there exist $\{a_i \in X : i = 1, \ldots, m\}$ and $\{b_j \in Y : j = 1, \ldots, n\}$ such that $c \leq \cup_i a_i$ and $d \leq \cup_j b_j$. Then $c \cdot d \leq (\cup_i a_i) \cdot (\cup_j b_j) = \cup_{i,j}(a_i \cdot b_j) \in C^1(X * Y)$, by using the distributivity of fusion over join, which is the third condition of integral so-monoids. Hence, $C^1X * C^1Y \subseteq C^1(X * Y)$ holds.

In [6], Kripke-type semantics for $\mathbf{FL_{ew}}$ is introduced by using commutative integral so-monoids. We will give here a brief explanation of the semantics based on commutative integral so-monoids. *Kripke frames* are just commutative integral so-monoids. A *valuation* on a Kripke frame $\mathbf{M}$ is any map v from the set of all propositional variables to $C^1(\wp(M))$. Then, v can be naturally extended to a map from the set of all formulas to $C^1(\wp(M))$, since $\mathbf{C^1}_{\mathbf{M}}$ is a commutative integral residuated lattice. In other words, a

[2]The existence of the smallest element 0 is not essential in the following argument, but this simplifies it.

valuation on a Kripke frame $\mathbf{M}$ is nothing but a valuation of a commutative integral residuated lattice $\mathbf{C^1}_{\mathbf{M}}$. Here, the value $v(\neg\varphi)$ for a valuation v is defined by $v(\neg\varphi) = v(\varphi) \Rightarrow C^1(\emptyset)$. In [6], valuations and Kripke frames in the above are called *strong valuations* and *total strong frames*, respectively, and the notation $a \models \varphi$ is used instead of writing $a \in v(\varphi)$, where $\models$ corresponds to a given valuation v. A formula φ is *valid* in a Kripke frame $\mathbf{M}$ ($\mathbf{M} \models \varphi$, in symbol) if $a \models \varphi$ for every $a \in M$.

The following lemma follows immediately from the definition.

Lemma 6. *Let* $\mathbf{M}$ *be an arbitrary Kripke frame. Then, for each formula* φ*,* $\mathbf{M} \models \varphi$ *if and only if* φ *is valid in a commutative integral residuated lattice* $\mathbf{C^1}_{\mathbf{M}}$.

Now let us apply our main theorem to the present case. Since the set $(b]$ is C^1-closed for every $b \in M$, the base U is equal to M. Since each C^1-closed set satisfies both of closure conditions (1) and (2) (for finite joins), we have the following, by Theorem 5.

Theorem 7. *For each commutative integral so-monoid* $\mathbf{M}$*, the map* $h : M \to C^1(\wp(M))$ *defined by* $h(b) = (b]$ *is an order isomorphism from* $\mathbf{M}$ *to* $\mathbf{C^1}_{\mathbf{M}}$ *preserving both products and joins, and moreover it preserves all exisiting residuals and meets. In particular, any commutative integral residuated lattice* $\mathbf{P}$ *can be embedded into a complete, commutative integral residuated lattice* $\mathbf{C^1}_{\mathbf{P}^\dagger}$ *by the residuated lattice monomorphism* h*, where* $\mathbf{P}^\dagger$ *is the so-monoid reduct of the residuated lattice* $\mathbf{P}$.

The second part of Theorem 7 is shown already in [6] as its Theorem 8.12. By using these results, we have the completeness of $\mathbf{FL_{ew}}$ with respect to Kripke frames.

Corollary 8. (Ono-Komori) *The propositional logic* $\mathbf{FL_{ew}}$ *is Kripke-complete.*

Proof. Suppose that a formula φ is not provable in $\mathbf{FL_{ew}}$. Then, φ is not valid in a commutative integral residuated lattice $\mathbf{P}$, e.g. in its Lindenbaum algebra. Therefore, it is not valid in $\mathbf{C^1}_{\mathbf{P}^\dagger}$, by Theorem 7. Hence, φ is not valid in a Kripke frame $\mathbf{P}^\dagger$, by Lemma 6. ■

Another immediate consequence of Theorem 7 is the conservativeness of $\mathbf{FL_{ew}}$ over its fragments, which can be shown e.g. by using cut elimination theorem for $\mathbf{FL_{ew}}$. (See e.g. Theorem 2.3 of [6]. Also cf. Theorem 8.12 of

[6].) Let F be any nonempty subset of logical connectives $\{\wedge, \vee, *, \supset, \neg\}$. A formula is a *$F$-formula* if every connective appearing in it belongs to F. Also, the *F-fragment* of the sequent system $\mathbf{FL_{ew}}$ is the sequent system obtained from the sequent sytem $\mathbf{FL_{ew}}$ by restricting its rules for connectives only to those for connectives in F. (As for the sequent system $\mathbf{FL_{ew}}$, see e.g. [6, 5]).

COROLLARY 9. *Let F be any subset of logical connectives which contains at least $*$. Then, for any F-formula φ, if φ is provable in $\mathbf{FL_{ew}}$ then it is provable in the F-fragment of $\mathbf{FL_{ew}}$.*

PROOF. Suppose otherwise. Then, φ is not valid in the Lindembaum algebra $\mathbf{P}$ of the F-fragment of $\mathbf{FL_{ew}}$. Suppose first that F contains $\cup$. Then, $\mathbf{P}$ forms a commutative integral so-monoid. Then, by Theorem 7 $\mathbf{P}$ is embedded into a complete, commutative integral residuated lattice $\mathbf{Q}$ by an embedding which preserves products, joins, and also existing residuals and meets. Thus, φ is not valid in $\mathbf{Q}$. Suppose next that F doesn't contain $\cup$. In this case, $\mathbf{P}$ forms a commutative integral po-monoid. As mentioned at the end of the previous section, if we define C^1X to be the smallest downward closed set including X, then we get a similar result to Theorem 7, which says the embeddability of a po-monoid $\mathbf{P}$ into a residuated lattice $\mathbf{Q}$ by an embedding which preserves products, and existing residuals and meets. But, this is enough to make φ invalid in $\mathbf{Q}$. Thus, φ is not provable in $\mathbf{FL_{ew}}$ in either case. ■

By obvious modifications of the above argument, we can obtain the similar results to Theorem 7 and its corollaries in the above, for non-integral (but weakly idempotent) case and for $\mathbf{FL_e}$ (and $\mathbf{FL_{ec}}$, respectively). To avoid unnecessary complications, we assume the commutativity of monoid structures, and accordingly the exchange rule in logics, throughout the present paper. But, we can extend our main theorem and most of its consequences in § 3 to non-commutative case.

II. Dedekind-MacNeille completion and completeness of substructural predicate logics

A similar but stronger result can be obtained by taking the closure operator C^2 determined by Dedekind-MacNeille completion.

Here, we will discuss the completeness of predicate logic $\mathbf{FL_{ew}}$, but by a slight modification of our proof, we can get the completeness of predicate logics $\mathbf{FL_e}$ and $\mathbf{FL_{ec}}$. Let $\mathbf{P}$ be any commutative integral residuated lattice

and V be a nonempty set. A map v is a *valuation* for first-order formulas on $\mathbf{P}$ with universe V if v is any map from the set of closed atomic first-order formulas with parameters in V to P. Then, v can be extended uniquely to a *partial* map from the set of first-order formulas with parameters in V to the set P, similarly to valuations for propositional formulas, and in addition, we claim the following for quantifiers :

i. $v(\forall x\theta(x))$ is defined and is equal to $\cap\{v(\theta(u)) : u \in V\}$, if for all u, $v(\theta(u))$ is defined, and $\cap\{v(\theta(u)) : u \in V\}$ exists in P,
ii. $v(\exists x\theta(x))$ is defined and is equal to $\cup\{v(\theta(u)) : u \in V\}$, if for all u, $v(\theta(u))$ is defined, and $\cup\{v(\theta(u)) : u \in V\}$ exists in P.

A valuation v is *safe* if $v(\varphi)$ is defined for every closed formula φ with parameters in V. A closed formula φ is *valid* in $\mathbf{P}$ if for any nonempty set V, $v(\varphi) = 1$ for every safe valuation on $\mathbf{P}$ with universe V. Clearly, every valuation becomes safe when $\mathbf{P}$ is complete.

For a given commutative integral residuated lattice $\mathbf{P}$, define a map C^2 on $\wp(\mathbf{P})$ by $C^2X = ((X^{\rightarrow})^{\leftarrow})$ for each $X \subseteq P$, where $Y^{\rightarrow}$ and $Z^{\leftarrow}$ denote the set of upper bounds of Y and of lower bounds of Z, respectively. It is easy to see that C^2 is a closure operator and that the base for C^2 in M is P. Moreover, every C^2-closed set satisfies both of closure conditions (1) and (2). Therefore, h is a residuated lattice monomorphism from $\mathbf{P}$ to $\mathbf{C^2}_{\mathbf{P}^\ddagger}$ which preserves all existing joins and meets, by Theorem 5. Here, $\mathbf{P}^\ddagger$ is the po-monoid reduct of $\mathbf{P}$. The complete residuated lattice $\mathbf{C^2}_{\mathbf{P}^\ddagger}$ is known as Dedekind-MacNeille completion of $\mathbf{P}$. Using this, we have the following result (see [5] for the details).

Corollary 10. (Ono) *The predicate logic* $\mathbf{FL_{ew}}$ *is complete with respect to the class of complete, commutative integral residuated lattices. Similarly, the predicate logics* $\mathbf{FL_e}$ *and* $\mathbf{FL_{ec}}$ *are complete with respect to the class of complete, commutative residuated lattices, and the class of complete, commutative weakly idempotent residuated lattices, respectively.*

Proof. If a first-order formula φ is not provable in the predicate $\mathbf{FL_{ew}}$, then φ is not valid in the Lindenbaum algebra $\mathbf{P}$ of $\mathbf{FL_{ew}}$. Since $\mathbf{P}$ is embedded into $\mathbf{C^2}_{\mathbf{P}^\ddagger}$ by a complete embedding h, φ is not valid also in $\mathbf{C^2}_{\mathbf{P}^\ddagger}$. (Note that h must preserve existing joins and meets in $\mathbf{P}$, since all quantifiers in φ should be calculated in $\mathbf{C^2}_{\mathbf{P}^\ddagger}$ just in the same way as in $\mathbf{P}$.) ■

III. Standard completeness of predicate logic $MTL\forall$ of left-continuous t-norms

Our theorem can be applied also to the proof of standard completeness of predicate logic $MTL\forall$. First, we will give a very brief survey of left-continuous t-norms and of predicate logic $MTL\forall$. As for the details, see e.g. [4].

When a commutative po-monoid $\mathbf{M}$ has a linear order, it is called a commutative *lo-monoid*. The product $\circ$ in a commutative integral lo-monoid $\mathbf{M} = \langle M, \circ, 1, \leq \rangle$ is *left-continuous*, if

$$\sup X \circ \sup Y = \sup(X * Y)$$

holds for all subsets X and Y of M whenever both $\sup X$ and $\sup Y$ exist. (On the other hand, it is said to be right-continuous when $\inf X \circ \inf Y = \inf(X * Y)$ holds). Here, $X * Y$ denotes the set $\{x \circ y : x \in X \text{ and } y \in Y\}$. When a commutative lo-monoid $\mathbf{M}$ with a left-continuous $\circ$ is moreover complete, i.e. both $\sup X$ and $\inf X$ exist for any subset X of M, $\mathbf{M}$ determines uniquely a complete, commutative integral residuated lattice. In fact, clearly it is a bounded lattice. Also, the residual $x \to y$ exists for all $x, y \in M$, since by using the left-continuity of $\circ$ we can show that $x \to y = \sup\{z : x \circ z \leq y\}$ holds. Thus, we can identify a complete, commutative lo-monoid with a left-continuous product, with the complete residuated lattice defined in this way.

Now let us consider the unit interval [0,1] of reals, which is partially ordered by natural order $\leq$. When the structure $\langle [0,1], \circ, 1, \leq \rangle$ forms a (complete) lo-monoid with a binary operation $\circ$ on [0,1], sometimes the product $\circ$ is called a *triangular norm* (or, simply a *t-norm*). As mentioned above, each complete, commutative integral lo-monoid $\langle [0,1], \circ, 1, \leq \rangle$ with a left-continuous t-norm $\circ$ forms a complete, commutative integral residuated lattice $\langle [0,1], \min, \max, \circ, 1, \to, 0, 1 \rangle$. We call any of such a residuated lattice as this, a *standard structure*. A logic L over $\mathbf{FL_{ew}}$ is *standard complete* if it is complete with respect to a class of standard structures.

An interesting problem is to find an axiom system L which is standard complete with respect to the class of all standard structures. This problem was solved for propositional case in [2]. That is, the logic, called MTL, which is obtained from propositional $\mathbf{FL_{ew}}$ by adding formulas of the form $(\varphi \supset \psi) \vee (\psi \supset \varphi)$ as axioms, is standard complete. Then, in [4] it is shown that the predicate logic $MTL\forall$ is standard complete, where $MTL\forall$ is obtained from predicate logic $\mathbf{FL_{ew}}$ by adding both formulas of the form $(\varphi \supset \psi) \vee (\psi \supset \varphi)$ and $\forall x(\theta \vee \sigma) \supset (\forall x\theta \vee \sigma)$ as axioms, where σ has no free occurrences of x in it.

Our proof of the standard completeness goes as follows. Suppose that a formula φ is not provable in $MTL\forall$. Then, it is shown that there exists a countable linearly ordered, commutative integral residuated lattice $\mathbf{P}$ in which φ is not valid. Next, it can be shown that there exists an order isomorphism g of the lo-monoid reduct of $\mathbf{P}$ into a countable dense lo-monoid $\mathbf{M}$ with a left-continuous product $\circ$ such that g preserves the smallest element 0, products, residuals and all existing joins and meets. This can be proved by using a technique introduced in [2]. Since the partially ordered set $\langle M, \leq\rangle$ is countable, dense and bounded, it is order isomorphic to $\mathbf{Q} \cap [0,1]$. So, we can assume that the domain of $\mathbf{M}$ is $\mathbf{Q} \cap [0,1]$.

Now, to complete the proof of the standard completeness, we need the following lemma. The proof described below is essentially the same as one given in [4], but here we use Theorem 5 explicitly.

Lemma 11. *Every countable dense, commutative integral lo-monoid* $\mathbf{M}$ *of the form* $\langle \mathbf{Q} \cap [0,1], \circ, \leq, 1\rangle$ *with a left-continuous product* $\circ$ *can be embedded into a standard structure by a complete embedding.*

Proof. For each subset X of $\mathbf{Q} \cap [0,1]$, define C^3X by $C^3X = \{q \in M : q \leq \sup X\}$. Here, $\sup X$ denotes the supremum of X in the set $\mathbf{R}$ of real numbers. We show that C^3 is a closure operator. To see this, we need to check that $C^3X * C^3Y \subseteq C^3(X * Y)$. Let α and β be real numbers defined by $\alpha = \sup X$ and $\beta = \sup Y$, respectively. It suffices to show that for all rational numbers r and s in M such that $r \leq \alpha$ and $s \leq \beta$, $r \circ s \leq \sup(X * Y)$. Suppose first that both α and β are irrational. Then $r < \alpha$ and $s < \beta$. Then there exist $r' \in X$ and $s' \in Y$ such that $r < r'$ and $s < s'$. Then, $r \circ s \leq r' \circ s' \in X * Y$. Thus, $r \circ s \leq \sup(X * Y)$. When both α and β are rational, it suffices to show that $\alpha \circ \beta \leq \sup(X * Y)$. Now, since $\alpha = \sup X$ and $\beta = \sup Y$, there exist subsets $\{r_m : m \in \mathbf{N}\}$ of X and $\{s_n : n \in \mathbf{N}\}$ of Y such that $\alpha = \sup\{r_m\}$ and $\beta = \sup\{s_n\}$. Then $\alpha \circ \beta = \sup\{r_m\} \circ \sup\{s_n\} = \sup\{r_m \circ s_n\}$ by the left-continuity of $\circ$. Since $r_m \in X$ and $s_n \in Y$, we have $\alpha \circ \beta \leq \sup(X * Y)$. For remaining cases, we can show also the above inclusion, by combining these arguments.

It is easy to see that a subset X of $\mathbf{Q} \cap [0,1]$ is C^3-closed if and only if it is a downward closed set such that $\sup X \in X$. Using this, we can show that the base U for C^3 is equal to $\mathbf{Q} \cap [0,1]$, and every C^3-closed set satisifies closure conditions (1) and (2). Thus, by Theorem 5, we have that $\mathbf{M}$ can be embedded into a complete, commutative integral residuated lattice $\mathbf{C^3_M}$ by a complete embedding. Now, define a map j from the set $C^3(\wp(\mathbf{Q} \cap [0,1]))$, which is the domain of $\mathbf{C^3_M}$, to [0,1] by $j(X) = \sup X$. Then, j induces

a surjective order-isomorphism from $\mathbf{C^3}_{\mathbf{M}}$ to a standard structure. This completes the proof. ■

COROLLARY 12. (Montagna-Ono) *The predicate logic $MTL\forall$ is standard complete.*

PROOF. As shown above, for each formula φ which is not provable in $MTL\forall$, there exists a countable linearly ordered, commutative integral residuated lattice $\mathbf{P}$ in which φ is not valid. The lo-monoid reduct of $\mathbf{P}$ is embedded into a countable dense, commutative integral lo-monoid $\mathbf{M}$ with a left-continuous product $\circ$ by a complete embedding, and then this $\mathbf{M}$ is embedded into a standard structure by a complete embedding. Clearly, φ is not valid in this standard structure. ■

IV. Finite Embeddability Property

As the last example of consequences of our main theorem, we will take the construction of residuated lattices discussed in Section 5 of [1], in particular Lemma 5.6.

We say that the class $\mathcal{K}$ of algebras has the *finite embeddability property* when for a given finite *partial* subalgebra $\mathbf{B}$ of an algebra $\mathbf{A}$ in $\mathcal{K}$, there exists a finite algebra $\mathbf{D}$ in $\mathcal{K}$ into which $\mathbf{B}$ can be embedded.

A typical example of finite embeddability property is the finite embeddability property of Heyting algebras. Let $\mathbf{B}$ be a finite partial subalgebra of a Heyting algebra $\mathbf{A}$. The Heyting subalgebra generated by the domain of $\mathbf{B}$ is not always finite, since the class of Heyting algebras is not locally finite. But, let us take the sublattice $\mathbf{D}$ generated by the domain of $\mathbf{B}$, instead. Since $\mathbf{D}$ is a finite distributive lattice, it is a finite Heyting algebra. Moreover, $\mathbf{B}$ can be embedded into $\mathbf{D}$.

A commutative integral po-monoid $\mathbf{M} = \langle M, \cdot, 1, \leq \rangle$ with a binary operation $\rightarrow$ on M is *residuated* if it satisfies the following law of residuation : $x \cdot y \leq z$ iff $x \leq (y \rightarrow z)$. Sometimes, residuated, commutative integral po-monoids, i.e. partially ordered commutative residuated integral monoids are called *pocrims*. In [1], the finite embeddability property of the class of pocrims and of the class of commutative integral residuated lattices is shown. We will give here a brief outline of the proof, and show how our main theorem works in it. These two results are proved essentially in the same way, here we consider pocrims.

Suppose that $\mathbf{B}$ is a partial subalgebra of a pocrim $\mathbf{A}$. Let $\mathbf{M} = \langle M, \cdot, 1, \leq \rangle$ be the po-submonoid generated by the domain B of $\mathbf{B}$. The set M is

not necessarily finite even if B is finite. For each $u \in M$ and $b \in B$, define $(u \rightsquigarrow b] = \{w \in M : uw \leq b\}$. Let $\overline{D} = \{(u \rightsquigarrow b] : u \in M \text{ and } b \in B\}$. For each subset X of M, define $C^4X = \bigcap\{Z \in \overline{D} : X \subseteq Z\}$. Then, C^4 is shown to be a closure operator on $\wp(\mathbf{M})$. Also, we can show that every C^4-closed subset is downward closed (and moreover satisfies the closure condition (2) for finite joins when $\mathbf{A}$ is a residuated lattice). Moreover, the domain B of $\mathbf{B}$ is a subset of the base U. Thus, by Theorem 5, $\mathbf{B}$ is embedded into a complete, commutative integral residuated lattice $\mathbf{C^4_M}$. To add to this, it is shown in [1] that when B is finite, $\mathbf{C^4_M}$ becomes finite. Thus, the following holds.

Corollary 13. (Blok-van Alten) *Both the class of pocrims and the class of commutative integral residuated lattices have the finite embeddability property. Therefore, propositional logic* $\mathbf{FL_{ew}}$ *is complete with respect to the class of finite, commutative integral residuated lattices.*

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