

Title	A parametric representation of linguistic hedges in Zadeh's fuzzy logic
Author(s)	Huynh, V. N.; Ho, T. B.; Nakamori, Y.
Citation	International Journal of Approximate Reasoning, 30(3): 203-223
Issue Date	2002-09
Type	Journal Article
Text version	author
URL	http://hdl.handle.net/10119/4997
Rights	NOTICE: This is the author's version of a work accepted for publication by Elsevier. V. N. Huynh, T. B. Ho and Y. Nakamori, International Journal of Approximate Reasoning, 30(3), 2002, 203-223, http://dx.doi.org/10.1016/S0888-613X(02)00075-0
Description	

A Parametric Representation of Linguistic Hedges in Zadeh's Fuzzy Logic

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Abstract

This paper proposes a model for the parametric representation of linguistic hedges in Zadeh's fuzzy logic. In this model each linguistic truth value, which is generated from a primary term of the linguistic truth variable, is identified by a real number r depending on the primary term. It is shown that the model yields a method of efficiently computing linguistic truth expressions accompanied with a rich algebraic structure of the linguistic truth domain, namely De Morgan algebra. Also, a fuzzy logic based on the parametric representation of linguistic truth values is introduced.

Key words: Linguistic hedges, linguistic variable, distributive lattice, De Morgan algebra, fuzzy logic, approximate reasoning

1 Introduction

In 1970s, L. Zadeh introduced and developed the theory of approximate reasoning based on the notions of linguistic variable and fuzzy logic [19–23]. Informally, by a linguistic variable we mean a variable whose values are words in a natural or artificial language. For example, *Age* is a linguistic variable whose values are linguistic such as *young*, *old*, *very young*, *very old*, *quite young*, *more or less young*, *not very young* and *not very old*, etc. As is well-known, the values of a linguistic variable are generated from primary terms (e.g., *young* and *old* in the case of linguistic variable *Age*) by various linguistic hedges (e.g., *very*, *more or less*, etc.) and connectives (e.g., *and*, *or*, *not*).

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In Zadeh’s view of fuzzy logic, the truth-values are linguistic, e.g., of the form “*true*”, “*very true*”, “*more or less true*”, “*false*”, “*possible false*”, etc., which are expressible as values of the linguistic variable *Truth*, and the rules of inference are approximate rather than exact. In this sense, approximate reasoning (also called fuzzy reasoning) is, for the most part, qualitative rather than quantitative in nature, and almost all of it falls outside of the domain of applicability of classical logic (see Zadeh [2,22,23]). The primary aim of the theory of approximate reasoning is to mimic human linguistic reasoning particularly in describing the behaviour of human-centered systems.

Throughout this paper, by a fuzzy logic we mean a fuzzy logic in the sense of Zadeh, that is, its truth-values are linguistic values of the linguistic truth variable, which are represented by fuzzy sets in the interval $[0, 1]$.

According to Zadeh’s rule for truth qualification [23], a proposition such as “Lucia is very young” is considered as being *semantically equivalent* with the proposition “Lucia is young is very true”. This semantic equivalence relation plays an important role in approximate reasoning. In fuzzy set based approaches to fuzzy reasoning [7,22,23,2], the primary linguistic truth-values such as *true* and *false* are correspondingly assigned to fuzzy sets defined over the interval $[0, 1]$, which are designed to interpret the meaning of these primary terms. The composite linguistic truth-values are then computed by using the following procedure:

- Linguistic hedges¹, for example *very* and *more or less*, are defined as unary operators on fuzzy sets, for example CON, DIL, respectively;
- The logic connectives such as *and*, *or*, *not* and *if . . . then* are defined generally as operators such as *t-norm*, *t-conorm*, *negation*, and *implication* respectively.

As is well known, one of inherent problems in a model of fuzzy reasoning is that of linguistic approximation, i.e., how to name by a linguistic term a resulted fuzzy set of the deduction process. This depends on the shape of the resulted fuzzy set in relation with the primary fuzzy sets and the operators.

Based on two characteristics of linguistic variables introduced by Zadeh (namely, the context-independent meaning of linguistic hedges and connectives, and the universality of linguistic domains), and the meaning of linguistic hedges in natural language, Nguyen and Wechler [15,16] proposed an algebraic approach to the structure of linguistic domains (term-sets) of linguistic variables. It is shown in [12–14] that the obtained structure is rich enough for the investigation of some kinds of fuzzy logic. Furthermore, the approach also provides a possibility for introducing methods of linguistic reasoning that allow us to handle linguistic terms directly, and hence, to avoid the problem of linguistic

¹ also called linguistic modifiers [6].

Fig. 1. Membership functions of truth-values by Lascio's model

approximation [10,11].

It is of interest that in [6], Lascio et al. have proposed a model for representation of linguistic terms satisfying the hypotheses imposed on linguistic hedges introduced by Nguyen and Wechler [15]. In their model, each linguistic term of a linguistic variable is characterised by three parameters and can be identified by only a positive real number. It is shown that the set of linguistic terms of the linguistic truth variable in Lascio's model exhibited interesting semantic properties justified by intuitive meaning of linguistic hedges, which were axiomatically formulated in the terms of hedge algebras [15]. However, going back to the membership function representation, Lascio's model does not give a good interpretation at the intuitive level on logical basis behind the shape of membership functions of linguistic truth values (see Fig. 1).

It is important to note that in the conventional approach to fuzzy reasoning, fuzzy logic, which a method of fuzzy reasoning bases on, can be viewed as a fuzzy extension of a underlying multi-valued logic (i.e., *base logic*), in which the truth-values are fuzzy sets of the truth-value set of the base logic (see, e.g., [2,22,23]). Although membership functions of primary terms such as *true* or *false* are defined subjectively, it will be natural to hope that a fuzzy logic should meet the base logic at the limited cases. For example, for membership function of the **unitary** truth-value *u-true* [23], that is $\mu_{u-true}(v) = v$ for $v \in [0, 1]$, and the linguistic hedge *very* defined by CON operation, we have $very^n true$ tends to *Absolutely true* as n tends to infinity, where *Absolutely true* is identified with 1 as a nonfuzzy truth-value, see Fig. 2. Unfortunately, this is not the case for Lascio's model, again see Fig. 1.

In this paper, we introduce a new representation model for linguistic terms of the linguistic truth variable in fuzzy logic. In this model, each linguistic truth value generated from a primary term of the linguistic truth variable is identified by a real number r depending on the primary term. It will be shown that the model not only satisfies the interesting semantic properties justified by intuitively meaning of linguistic hedges as Lascio's model, but also meets

Fig. 2. Membership functions of unitary truth-values [4,23]

in the special cases the well-known models in the literature.

The paper is organised as follows. In Section 2, we briefly present some preliminary notions on linguistic variables, the fuzzy set based interpretation of linguistic hedges, as well as the related work in the literature. A new representation model for linguistic terms of the linguistic truth variable will be introduced in Section 3. The model allows to represent two ordered sets of linguistic terms generated from two primary terms *true* and *false*; each linguistic truth value is associated with a real number depending on the primary term from which it is generated. Section 4 introduces a fuzzy logic based on this model in comparison with the models already known in literature. Finally, some concluding remarks will be given in Section 5.

2 Preliminaries

2.1 Linguistic variables

In this subsection, we briefly recall the notion of linguistic variables and the fuzzy set theoretic interpretation of linguistic hedges introduced by Zadeh in 1970s. More details can be referred to [5,19–21,23].

Formally, a *linguistic variable* is characterised by a quintuple $(\mathcal{X}, T(\mathcal{X}), U, R, M)$, where:

\mathcal{X} is the name of the variable such as age variable *Age*, truth variable *Truth*, etc.;

$T(\mathcal{X})$ denotes the **term-set** of \mathcal{X} , that is, the set of linguistic values of the linguistic variable \mathcal{X} ;

U is a universe of discourse of the base variable;

Fig. 3. Hierarchical structure of the linguistic variable *Age*

R is a syntactic rule for generating linguistic terms of $T(\mathcal{X})$;

M is a semantic rule assigning to each linguistic term a fuzzy set on U .

As an illustration, we consider an example of a linguistic variable *Age*, i.e. $\mathcal{X} = \text{Age}$, taken from [20]. The term-set $T(\mathcal{X})$ is represented as follows

$T(\text{Age}) = \{ \text{young, very young, not young, very very young, not very young, } \dots, \text{old, very old, not old, } \dots, \text{not very young and not very old, } \dots, \text{extremely young, } \dots, \text{more or less young, } \dots \}$.

The universe of discourse for *Age* may be taken to be the interval $U = [0, 100]$, with the base variable u ranging over U . Then, a linguistic value of *Age*, for example, *young* is viewed as a name of a fuzzy set of U which is designed to define the meaning of *young*. That is, the meaning of the linguistic value *young* is characterized by its compatibility function $c : U \rightarrow [0, 1]$, with $c(u)$ representing the compatibility of a numerical age u with the label *young*. For example, the compatibilities of the numerical ages 20, 25, 30, and 35 with *young* may be 1, 0.9, 0.8, and 0.6, respectively. As such, the meaning of a linguistic value can be regarded as the membership function of a *fuzzy restriction* on the values of the base variable u . Fig. 3 sketches the above mentioned relationships [24,25].

Typically, the values of a linguistic variable such as *Age* are built up of one or more primary terms such as *young* and *old*, with one being an antonym of the other, together with a set of linguistic hedges, such as *very*, *more or less*, *quite*, *extremely*, etc., and connectives which allow a composite linguistic value to be generated from primary terms.

Assume that the meaning of a linguistic value X is defined by the membership

function $\mu_X(u)$ of U , then linguistic hedges *very*, *more or less*, *slightly* are constructed by mathematical representations as follows [18].

Concentration: *very* $X = \text{CON}(X)$, where $\mu_{\text{CON}(X)}(u) = (\mu_X(u))^2$;

Dilation: *more or less* $X = \text{DIL}(X)$, where $\mu_{\text{DIL}(X)}(u) = (\mu_X(u))^{0.5}$;

Intensification: denote by $\text{INT}(X)$, and

$$\mu_{\text{INT}(X)}(u) = \begin{cases} 2(\mu_X(u))^2 & \text{if } 0 \leq \mu_X(u) \leq 0.5, \\ 1 - 2(1 - \mu_X(u))^2 & \text{if } 0.5 \leq \mu_X(u) \leq 1. \end{cases}$$

And the hedge *slightly* may be defined by one of the following expressions

$$\textit{slightly } X = \text{NORM}(X \text{ and not } \textit{very} X),$$

$$\textit{slightly } X = \text{INT}(\text{NORM}(\textit{plus } X \text{ and not } \textit{very} X)),$$

$$\textit{slightly } X = \text{INT}(\text{NORM}(\textit{plus } X \text{ and not } \textit{plus } \textit{very} X)),$$

where NORM is the operation of normalization and *plus* is an artificial hedge defined by

$$\mu_{\text{NORM}(X)}(u) = (\sup_U \mu_X(u))^{-1} \mu_X(u), \text{ and } \mu_{\textit{plus } X}(u) = (\mu_X(u))^{1.25}.$$

A more detailed discussion of linguistic hedges from a fuzzy set theoretic point of view can be found in [5,18].

A linguistic variable is called to be a Boolean linguistic variable provided that its values are Boolean expressions in variables of the form X_p, hX_p, X or hX , where h is a linguistic hedge or a string of linguistic hedges, X_p is a primary term and hX is the name of a fuzzy set resulting from acting with h on X . For example, in the case of the linguistic variable *Age* whose term-set is defined previously, the term *not very young and not very old* is of this form with $h = \textit{very}$, $X_p = \textit{young}$ and $X_{p'} = \textit{old}$. Similarly, it is also the case for the term *very very young*, here $h = \textit{veryvery}$ and $X_p = \textit{young}$. It was shown in [20] that we can construct a context-free grammar for generating the term-set of a Boolean linguistic variable.

2.2 Mathematical representation of linguistic hedges in fuzzy logic

In the conventional approach to fuzzy logic, each primary linguistic truth value such as *true* or *false* is semantically assigned by a fuzzy set in the interval $[0, 1]$. A well known form of membership function of *true* is defined by

$\mu_{true}(u) = u$ for $u \in [0, 1]$, and the membership function of *false* is defined by $\mu_{false}(u) = \mu_{true}(1 - u)$ for $u \in [0, 1]$. Linguistic hedges are then defined as operators on these primary fuzzy sets to form fuzzy sets for composite linguistic truth values. For example, linguistic hedges such as *very* and *more or less* (or, *fairly* [1]) are mainly modeled as CON and DIL operators, respectively, [1,8,22,25]. However, the definition of a linguistic hedge as order of the power of a truth value *true* or *false* as in [1,8,22] suffers from an intuitive criterion when applied infinitely to linguistic hedges [6,15]. For example, it is intuitively agreed that *true* is more true than $(very)^n approximately true$, for any natural number n . Then it should be intuitively suitable if $(very)^n approximately true$ tends to *true* as n tends to infinity. However, when we interpret *very* as the CON operator, we get both $(very)^n approximately true$ and $(very)^n true$ tend to *Abs.true* as n tends to infinity. This causes a discrepancy between the intuitive utilization made in natural language of linguistic truth values and the mathematical representation obtained using CON and DIL operators.

To cancel the above mentioned discrepancy, Lascio et al. have proposed in [6] a model for representation of linguistic hedges, within which each linguistic value of the truth linguistic variable is characterized by three parameters and can be identified by a positive real number n . It was shown that the set of linguistic terms of the linguistic truth variable in this model exhibited interesting semantic properties justified by intuitively meaning of linguistic hedges, which were axiomatically formulated by Nguyen and Wechler in [15] in the terms of hedge algebras.

To represent the meaning of linguistic values of the linguistic truth variable, Lascio et al. introduced the following characteristic function, for $n \in R^+$,

$$\mu_n(u) = \begin{cases} \min(1, nu) & \text{for } 0 \leq u \leq 0.5, \\ \min(1, -n(u - 1)) & \text{for } 0.5 \leq u \leq 1. \end{cases}$$

Note that they utilized only this function for a generic linguistic term of the linguistic truth variable irrespective of a linguistic value generating from *true* or *false*. This is essentially different from conventional approaches to fuzzy logic in the literature. For $n \rightarrow \infty$, and $n = 0$ the model yields the values *Absolutely true* and *Absolutely false*, respectively. Consequently, *Absolutely true* and *Absolutely false* are identified by the following membership functions (see Fig. 1):

$$\mu_{Abs.true}(u) = 1, \text{ for any } u \in [0, 1];$$

$$\mu_{Abs.false}(u) = 0, \text{ for any } u \in [0, 1];$$

which are designed to interpret as the truth values *unknown* and *undefined*, respectively, in [20].

It should be emphasized that in this model, it is impossible to define the

special value *unknown* (also called *undecided*) which has been considered to be important in fuzzy logic [1,20]. To overcome these drawbacks while still maintaining interesting semantic properties of linguistic hedges, an alternative model for the representation of linguistic values of the Boolean linguistic truth variable is introduced in the next section.

3 A new model for the representation of the linguistic truth values

In this section we first define two families of parametric membership functions of linguistic truth values generated from two primary terms *true* and *false*, respectively. Then we examine an algebraic structure of the obtained linguistic truth space via the so-called semantically ordering relation. Also, we introduce a concept of the converse of a give linguistic hedge based on the specific relation introduced in [8]. As we will see in Section 4, this concept can be used in defining another kind of negation in a fuzzy logic.

3.1 Parametric membership function of linguistic truth values

In our model, each linguistic truth value is represented by a parametric membership function defined on the interval $[0, 1]$. This parameter depends on the primary term from which the linguistic truth value is generated by applying linguistic hedges.

Consider the Boolean linguistic truth variable *Truth* with two primary terms *true* and *false*. Let us denote by σ a linguistic hedge or a string of linguistic hedges. We now define the membership function of a linguistic value $\sigma true$ as follows

$$\begin{aligned} \mu_{\sigma true} : [0, 1] &\longrightarrow [0, 1] \\ u &\longmapsto \mu_{\sigma true}(u) = \max(0, (1 - n)^{-1}(u - n)) \end{aligned} \tag{1}$$

for $n \in (-\infty, 1)$. Similarly, we further define the membership function of a linguistic value $\sigma false$ by

$$\begin{aligned} \mu_{\sigma false} : [0, 1] &\longrightarrow [0, 1] \\ u &\longmapsto \mu_{\sigma false}(u) = \max(0, m^{-1}(m - u)) \end{aligned} \tag{2}$$

for $m \in (0, \infty)$.

It is of interest that with these definitions, we obtain the membership functions of some special linguistic truth values as follows (see Fig. 4)

Fig. 4. A space of parametric membership functions

- *true*, with $\mu_{true}(u) = u$, when $n = 0$;
- *Absolutely true*, when $n \rightarrow 1$;
- *false*, with $\mu_{false}(u) = 1 - u$, when $m = 1$;
- *Absolutely false* when $m \rightarrow 0$;
- *unknown* when $n \rightarrow -\infty$, and $m \rightarrow \infty$,

which are the same as considered in [1,2,4,8,20].

Formulation (1) (respectively, (2)) states that an infinite number of hedges can be generated for the linguistic truth value *true* (respectively, *false*) by a parametric family of membership functions. Let us denote by V the set of all linguistic truth values generated from (1) and (2) including the limited elements *Absolutely true*, *unknown*, *Absolutely false*.

3.2 An algebraic structure of the linguistic truth space

To analyse the meaning characteristic of the linguistic truth space, we consider the specific relationship between linguistic truth values as considered in [8]. We note that in our model, when $0 < n < 1$ (respectively, $-\infty < n < 0$), the linguistic value $\sigma true$ is more (respectively, less) specific than the truth value *true*. This is because of when $0 < n < 1$, $\mu_{\sigma true}(u) < \mu_{true}(u)$, and when $-\infty < n < 0$, $\mu_{\sigma true}(u) > \mu_{true}(u)$. Similarly, when $0 < m < 1$ (respectively, $1 < m < \infty$), the linguistic value $\sigma false$ is more (respectively, less) specific than the truth value *false*. It can be seen that when n approaches 1 (respectively, $-\infty$), the linguistic value $\sigma true$ is the most (respectively, the least) specific case with respect to the truth value *true*. A similar situation is also for the parameter m . That is, the more truth (falsity) a linguistic value

Fig. 5. The ordered relation in the linguistic truth space

is, the more specific a linguistic value becomes.

This specific relation can be determined through the areas under the membership functions defined as follows:

$$S_{\sigma true} = \int_0^1 \mu_{\sigma true}(u) du, \text{ and } S_{\sigma false} = \int_0^1 \mu_{\sigma false}(u) du.$$

Then, $\sigma true$ (respectively, $\sigma false$) is more specific than $\sigma' true$ (respectively, $\sigma' false$) if $S_{\sigma true} < S_{\sigma' true}$ (respectively, $S_{\sigma false} < S_{\sigma' false}$).

It should be worthwhile now to note that the specific relation defines an ordered relation, denoted by \leq_s , on the linguistic truth space, which is completely compatible with the so-called semantically ordering relation defined in [13], as follows

$$\begin{aligned} \sigma' true \leq_s \sigma true &\Leftrightarrow S_{\sigma true} \leq S_{\sigma' true}, \\ \sigma' false \leq_s \sigma false &\Leftrightarrow S_{\sigma' false} \leq S_{\sigma false}. \end{aligned}$$

We also note that due to the semantic characteristic of *true* and *false*, we define $\sigma' false \leq_s \sigma true$ for any σ and σ' . Particularly, this order is fully characterized by the natural order defined on the spaces of parameters as depicted in Fig. 5.

At this point, it is easily seen that the following holds.

Theorem 1 *The structure (V, \leq_s) is a completely distributive lattice with $Abs.true$ and $Abs.false$ as the unit and zero elements, respectively.*

Let us denote by \vee , and \wedge the operations *join* and *meet*, respectively, in this lattice, and write $\mathcal{V} = (V, \vee, \wedge, \leq_s)$.

For special cases, we have $S_{true} = S_{false} = 0.5$, $S_{Abs.true} = S_{Abs.false} = 0$, and

$S_{unknown} = 1$. Moreover, for the linguistic values generated from *true*, we have

$$S_{\sigma true} = \begin{cases} \frac{1-n}{2} & \text{for } 0 < n < 1 \\ 1 - \frac{1}{2(1-n)} & \text{for } -\infty < n < 0 \end{cases} \quad (3)$$

With the same calculation for the linguistic values generated from *false*, we obtain

$$S_{\sigma false} = \begin{cases} \frac{m}{2} & \text{for } 0 < m < 1 \\ 1 - \frac{1}{2m} & \text{for } 1 < m < \infty \end{cases} \quad (4)$$

It should be emphasized that Nafarieh and Keller in [8] also proposed a similar calculation but they defined the parameter n as an order of the power of a linguistic truth value *true*.

Now we discuss the problem of how to define the parameter of the antonymous label of a given linguistic truth value in our model. Without loss of generality, consider a linguistic truth value $\sigma true$ with its parameter $n_{\sigma true}$. The antonymous label of $\sigma true$ is the value $\sigma false$, which is called the contradictory element in [16], and the parameter $m_{\sigma false}$ may be defined such that the following holds

$$S_{\sigma true} = S_{\sigma false} \quad (5)$$

Under such a condition, it follows directly from (3) and (4) that

$$n_{\sigma true} = 1 - m_{\sigma false} \quad (6)$$

That is, we have an interesting one-to-one correspondence between the parameter of a linguistic truth value with that of its antonym. Consequently, we have

$$\mu_{\sigma false}(u) = \mu_{\sigma true}(1 - u) \quad (6')$$

which may be suitable to intuitive meaning of an antonymous label. For example, let us define² $n_{true} = 0$, $n_{very true} = 0.5$, and $n_{fairly true} = -1$. Then we obtain $m_{false} = 1$, $m_{very false} = 0.5$, and $m_{fairly false} = 2$, and the membership functions of these linguistic truth values are illustrated in Fig. 6.

We now define a negation operation, denoted by \neg , in V via (6) and (6'). This means that the negation of a linguistic truth value is defined by its antonymous linguistic truth value. This negation operation can be derived in \mathcal{V} , and so we write

$$\mathcal{V} = (V, \vee, \wedge, \neg, \leq_s).$$

Some fundamental properties of this operation is listed in the following proposition. The proof is easily followed.

² The hedge *fairly* is considered to have the same meaning as *more or less* in [1]

Fig. 6. A linguistic truth space

Proposition 2 *The following statements hold in \mathcal{V} .*

- (i) $\neg\neg x = x$, for any $x \in V$;
- (ii) $x \leq_s y$ iff $\neg y \leq_s \neg x$, for any $x, y \in V$;
- (iii) $\neg true = false$, $\neg false = true$;
- (iv) $\neg unknown = unknown$;
- (v) $\neg Abs.true = Abs.false$, $\neg Abs.false = Abs.true$.

Furthermore, we have the following

Theorem 3 *\mathcal{V} is a De Morgan algebra³.*

Proof. By Theorem 1 and (i) of Proposition 2, it is sufficient to prove the triple (\vee, \wedge, \neg) forms a De Morgan triple [9]. Indeed, for any $x, y \in V$, we have the following possibilities:

- (a) both x and y are generated from *true*, with the associated parameters n_x and n_y , respectively.
- (b) both x and y are generated from *false*, with the associated parameters m_x and m_y , respectively.
- (c) x is generated from *true* and y is generated from *false*, with the associated parameters n_x and m_y , respectively.
- (d) x is generated from *false* and y is generated from *true*, with the associated parameters m_x and n_y , respectively.

For the case (a), we have

$$m_{\neg(x \vee y)} = 1 - \max(n_x, n_y) = \min(1 - n_x, 1 - n_y) \quad (7)$$

³ Also named *Soft algebra* [9]

On the other hand, we also have

$$m_{(\neg x \wedge \neg y)} = \min(m_{\neg x}, m_{\neg y}) = \min(1 - n_x, 1 - n_y) \quad (7')$$

It implies by (7) and (7') that $m_{\neg(x \vee y)} = m_{(\neg x \wedge \neg y)}$, and hence,

$$\neg(x \vee y) = (\neg x \wedge \neg y) \quad (8)$$

that we desire. By an analogous argument, we also obtain the equality (8) for the case (b). The remain cases follows directly from the definitions of the relation \leq_s and the negation \neg .

By duality, we also obtain the equality

$$\neg(x \wedge y) = (\neg x \vee \neg y) \quad (9)$$

The qualities (8) and (9) mean that the triple (\vee, \wedge, \neg) is a De Morgan triple. This completes the proof. \square

It is worth to mention that the algebra \mathcal{V} includes the 3-valued Łukasiewicz algebra $\{Abs.false, unknown, Abs.true\}$ as its subalgebra.

3.3 A concept of converse of linguistic hedges

Firstly, we recall that in [1,8] linguistic hedges are identified by orders of powers of the primary linguistic truth value. For example,

$$\begin{aligned} \mu_{(very)^k true}(u) &= [\mu_{true}(u)]^{2^k} \\ \mu_{(very)^k false}(u) &= [\mu_{false}(u)]^{2^k} \\ \mu_{(fairly)^k true}(u) &= [\mu_{true}(u)]^{\frac{1}{2^k}} \\ \mu_{(fairly)^k false}(u) &= [\mu_{false}(u)]^{\frac{1}{2^k}} \end{aligned} \quad (10)$$

for any $k = 0, 1, \dots, \infty$.

Although it was not presented explicitly in [1], we easily see that the following hold

$$\begin{aligned} S_{(very)^k true} &= S_{(very)^k false} \\ S_{(fairly)^k true} &= S_{(fairly)^k false} \\ S_{(very)^k true} &= 1 - S_{(fairly)^k true} \\ S_{(very)^k false} &= 1 - S_{(fairly)^k false} \end{aligned} \quad (11)$$

for any $k = 0, 1, \dots, \infty$. The first two equations in (11) are consistent with (5) that is used to define the parameter of the antonymous label of a given linguistic truth value. By (10) we mean that there is an one-to-one correspondence between values $(very)^k true$ and $(fairly)^k true$ (and also, $(very)^k false$ and $(fairly)^k false$) as that between parameters 2^k and $\frac{1}{2^k}$, as well as equations in (11) are satisfied.

Under such an observation, we now introduce a concept of the converse of a given linguistic hedge via the specific relation mentioned above.

Given a linguistic hedge σ , and σX_p is a linguistic truth value generated from X_p by means of σ , where X_p is *true* or *false*. Then another linguistic hedge σ' is said to be *converse* to σ and vice versa if and only if the following holds

$$S_{\sigma X_p} = 1 - S_{\sigma' X_p} \quad (12)$$

For example, in Baldwin's model [1], $(very)^k$ is converse to $(fairly)^k$ and vice versa, for $k = 1, 2, \dots$. We also note that this concept of converse is a special case of that introduced by Nguyen and Wechler in [15].

It should be of interest that the relationship defined by (12) gives an intuitive meaning of the concepts of *positive* and *negative* [6,15,16] of linguistic hedges with respect to a linguistic truth value to which they are applied directly. For example, *very* strengthens the *positive meaning* of *true*, while *fairly* weakens its *positive meaning*.

We are now ready to establish one-to-one correspondences between parameters of linguistic truth values exhibited the above property of hedges. For this purpose, we define the following mappings

$$\begin{aligned} \psi : (-\infty, 1) &\longrightarrow (-\infty, 1) \\ n &\longmapsto \psi(n) = \frac{n}{n-1} \end{aligned}$$

$$\begin{aligned} \chi : (0, \infty) &\longrightarrow (0, \infty) \\ m &\longmapsto \chi(m) = \frac{1}{m} \end{aligned}$$

which establish, respectively, one-to-one correspondences between $(0, 1)$ and $(-\infty, 0)$ (for parameter n), and between $(0, 1)$ and $(1, \infty)$ (for parameter m).

With this notation, we easily obtain

$$S_{E[n]} = 1 - S_{E[\psi(n)]}, \text{ and } S_{E[m]} = 1 - S_{E[\chi(m)]} \quad (12')$$

where $E[p]$ stands for the linguistic truth value associated with parameter p .

Table 1

Different values of parameters n , m and respective linguistic truth values

n	Linguistic value	m	Linguistic value
1	<i>Absolutely true</i>	0	<i>Absolutely false</i>
$\frac{3}{4}$	<i>very very true</i>	$\frac{1}{4}$	<i>very very false</i>
$\frac{1}{2}$	<i>very true</i>	$\frac{1}{2}$	<i>very false</i>
0	<i>true</i>	1	<i>false</i>
-1	<i>fairly true</i>	2	<i>fairly false</i>
-3	<i>fairly fairly true</i>	4	<i>fairly fairly false</i>
$-\infty$	<i>unknown</i>	∞	<i>unknown</i>

As an illustration, let us define

$$n_{(very)^k true} = \sum_{i=1}^k \frac{1}{2^i} = 1 - \frac{1}{2^k}, \text{ and } m_{(very)^k false} = 1 - n_{(very)^k true} = \frac{1}{2^k}.$$

Then we have the parameters associated respectively with linguistic truth values $(fairly)^k true$ and $(fairly)^k false$ as follows

$$n_{(fairly)^k true} = \frac{\sum_{i=1}^k \frac{1}{2^i}}{\sum_{i=1}^k \frac{1}{2^i} - 1} = 1 - 2^k, \text{ and } m_{(fairly)^k false} = 2^k.$$

It follows by formulae (3) and (4) that $S_{(very)^k true} = 1 - S_{(fairly)^k true}$ and $S_{(very)^k false} = 1 - S_{(fairly)^k false}$. Thus, $(very)^k$ is converse to $(fairly)^k$ and vice versa, for $k = 1, 2, \dots$. Table 1 shows some special cases for different values of n and m for truth values generated from *true* and *false*, respectively, as well as the accepted linguistic translations of these parametric values.

In the next section, we utilize this reverse property of linguistic hedges in defining another kind of negation in a fuzzy logic.

4 A fuzzy logic based on the parametric representation of linguistic truth values

In this section we introduce a fuzzy logic based on the parametric representation of linguistic truth values proposed in the preceding section.

For simplicity of notation, let us denote $N = (-\infty, 1)$, and $M = (0, \infty)$, which are designed as domains of parameters n and m , respectively. Denote

V_t (respectively, V_f) the set of linguistic truth values generated from *true* (respectively, *false*) by means of linguistic hedges.

Let us define logical operations in the linguistic truth space. If we let $\mathbf{v}(P)$ represent the linguistic truth value of a proposition P then for propositions P and Q , the definitions of conjunction, disjunction, negation, implication are given by

$$\mathbf{v}(P \text{ and } Q) = \begin{cases} E[m_{\mathbf{v}(Q)}] & \text{if } \mathbf{v}(P) \in V_t \text{ and } \mathbf{v}(Q) \in V_f \\ E[m_{\mathbf{v}(P)}] & \text{if } \mathbf{v}(Q) \in V_t \text{ and } \mathbf{v}(P) \in V_f \\ E[\min(n_{\mathbf{v}(P)}, n_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_t \\ E[\min(m_{\mathbf{v}(P)}, m_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_f \end{cases} \quad (13)$$

$$\mathbf{v}(P \text{ or } Q) = \begin{cases} E[n_{\mathbf{v}(P)}] & \text{if } \mathbf{v}(P) \in V_t \text{ and } \mathbf{v}(Q) \in V_f \\ E[n_{\mathbf{v}(Q)}] & \text{if } \mathbf{v}(Q) \in V_t \text{ and } \mathbf{v}(P) \in V_f \\ E[\max(n_{\mathbf{v}(P)}, n_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_t \\ E[\max(m_{\mathbf{v}(P)}, m_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_f \end{cases} \quad (14)$$

$$\mathbf{v}(\text{not } P) = \begin{cases} E[1 - n_{\mathbf{v}(P)}] & \text{if } \mathbf{v}(P) \in V_t \\ E[1 - m_{\mathbf{v}(P)}] & \text{if } \mathbf{v}(P) \in V_f \end{cases} \quad (15)$$

$$\mathbf{v}(P \rightarrow Q) = \begin{cases} E[\max(m_{\mathbf{v}(\text{not } P)}, m_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(P) \in V_t \text{ and } \mathbf{v}(Q) \in V_f \\ E[\max(n_{\mathbf{v}(\text{not } P)}, n_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(Q) \in V_t \text{ and } \mathbf{v}(P) \in V_f \\ E[n_{\mathbf{v}(Q)}] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_t \\ E[n_{\mathbf{v}(\text{not } P)}] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_f \end{cases} \quad (16)$$

We note that the parameters $1 - n_{\mathbf{v}(P)}$ and $1 - m_{\mathbf{v}(P)}$ in equation (15) are strictly in relation with (6), i.e. that $1 - n_{\mathbf{v}(P)} \in M$, and $1 - m_{\mathbf{v}(P)} \in N$. As a consequence of above definitions of logical connectives and operators defined in the algebra \mathcal{V} , we have the following.

Theorem 4 *The operators \wedge, \vee , and \neg in \mathcal{V} model exactly logical connectives conjunction, disjunction, and negation, respectively, in the fuzzy logic defined above. More particularly,*

$$\mathbf{v}(P \text{ and } Q) = \mathbf{v}(P) \wedge \mathbf{v}(Q), \quad \mathbf{v}(P \text{ or } Q) = \mathbf{v}(P) \vee \mathbf{v}(Q), \quad \mathbf{v}(\text{not } P) = \neg \mathbf{v}(P).$$

For example, let $\mathbf{v}(P) = \text{very true}$, $n_{\mathbf{v}(P)} = 0.5$, and $\mathbf{v}(Q) = \text{fairly false}$, $m_{\mathbf{v}(Q)} = 2$ as defined in the previous section, moreover, by (6) we obtain

$m_{\mathbf{v}(\text{not } P)} = 0.5$, and $n_{\mathbf{v}(\text{not } Q)} = -1$. Then, we have

$$\mathbf{v}(P \text{ and } Q) = E[m_{\mathbf{v}(Q)}] = \text{fairly false, by (13)}$$

$$\mathbf{v}(P \text{ or } Q) = E[n_{\mathbf{v}(P)}] = \text{very true, by (14)}$$

$$\mathbf{v}(\text{not } P) = E[1 - n_{\mathbf{v}(P)}] = E[m_{\mathbf{v}(\text{not } P)}] = \text{very false, by (15) and (6)}$$

$$\mathbf{v}(\text{not } Q) = E[1 - m_{\mathbf{v}(Q)}] = E[n_{\mathbf{v}(\text{not } Q)}] = \text{fairly true, by (15) and (6)}$$

$$\mathbf{v}(P \rightarrow Q) = E[\max(m_{\mathbf{v}(\text{not } P)}, m_{\mathbf{v}(Q)})] = E[m_{\mathbf{v}(Q)}] = \text{fairly false, by (16)}$$

$$\mathbf{v}(\text{not } P \rightarrow Q) = E[n_{\mathbf{v}(\text{not not } P)}]$$

$$= E[1 - m_{\mathbf{v}(\text{not } P)}] = E[n_{\mathbf{v}(P)}] = \text{very true, by (6) and (16)}.$$

This example shows the same result as those obtained in [6]. Now, to compare with Baldwin's model proposed in [1], as in previous section, let us define

$$n_{(\text{very})^k \text{true}} = \sum_{i=1}^k \frac{1}{2^i} = 1 - \frac{1}{2^k}, \text{ and } m_{(\text{very})^k \text{false}} = 1 - n_{(\text{very})^k \text{true}} = \frac{1}{2^k},$$

for $k = 1, 2, \dots$. By correspondences ψ and χ at the end of Section 3, we obtain the parameters associated respectively with linguistic truth values $(\text{fairly})^k \text{true}$ and $(\text{fairly})^k \text{false}$ as follows

$$n_{(\text{fairly})^k \text{true}} = 1 - 2^k, \text{ and } m_{(\text{fairly})^k \text{false}} = 2^k.$$

Hence, it follows that

$$n_{(\text{very})^k \text{true}} = 1 - \frac{1}{2^k} \longrightarrow 1 \quad \text{as } k \longrightarrow \infty$$

$$m_{(\text{very})^k \text{false}} = \frac{1}{2^k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty$$

$$n_{(\text{fairly})^k \text{true}} = 1 - 2^k \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty$$

$$m_{(\text{fairly})^k \text{false}} = 2^k \longrightarrow \infty \quad \text{as } k \longrightarrow \infty$$

Consequently, we obtain

$$(\text{very})^k \text{true} \longrightarrow \text{Abs. true} \quad \text{as } k \longrightarrow \infty$$

$$(\text{very})^k \text{false} \longrightarrow \text{Abs. false} \quad \text{as } k \longrightarrow \infty$$

$$(\text{fairly})^k \text{true} \longrightarrow \text{unknown} \quad \text{as } k \longrightarrow \infty$$

$$(\text{fairly})^k \text{false} \longrightarrow \text{unknown} \quad \text{as } k \longrightarrow \infty$$

Further, Table 2 is followed easily by using the definitions of conjunction and disjunction, and is easily extended to include other linguistic truth values. The above limited expressions and Table 2 show that our model is compatible with that proposed by Baldwin in [1].

Table 2

A reduced linguistic truth table for conjunction and disjunction

$\mathbf{v}(P)$	$\mathbf{v}(Q)$	$\mathbf{v}(P \text{ and } Q)$	$\mathbf{v}(P \text{ or } Q)$
<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>
<i>false</i>	<i>true</i>	<i>false</i>	<i>true</i>
<i>true</i>	<i>false</i>	<i>false</i>	<i>true</i>
<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>
<i>unknown</i>	<i>true</i>	<i>unknown</i>	<i>true</i>
<i>unknown</i>	<i>false</i>	<i>false</i>	<i>unknown</i>
<i>unknown</i>	<i>unknown</i>	<i>unknown</i>	<i>unknown</i>
<i>unknown</i>	<i>Abs. false</i>	<i>Abs. false</i>	<i>unknown</i>
<i>unknown</i>	<i>Abs. true</i>	<i>unknown</i>	<i>Abs. true</i>
<i>true</i>	<i>very true</i>	<i>true</i>	<i>very true</i>
<i>true</i>	<i>fairly true</i>	<i>fairly true</i>	<i>true</i>
<i>false</i>	<i>very true</i>	<i>false</i>	<i>very true</i>
<i>false</i>	<i>fairly true</i>	<i>false</i>	<i>fairly true</i>
<i>Abs. true</i>	<i>false</i>	<i>false</i>	<i>Abs. true</i>
<i>Abs. false</i>	<i>true</i>	<i>Abs. false</i>	<i>true</i>
<i>Abs. true</i>	<i>Abs. false</i>	<i>Abs. false</i>	<i>Abs. true</i>

We now establish basic linguistic truth expressions associated with respective parameters as follows.

(i) $n_{(very)^k true} = \sum_{i=1}^k \frac{1}{2^i} = 1 - \frac{1}{2^k}$, for $k = 1, \dots, \infty$.

k	1	2	3	4
Truth value	<i>very true</i>	$(very)^2 true$	$(very)^3 true$	$(very)^4 true$
$n_{(very)^k true}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$

(ii) $m_{(very)^k false} = \frac{1}{2^k}$, for $k = 1, \dots, \infty$.

k	1	2	3	4
Truth value	<i>very false</i>	$(very)^2 false$	$(very)^3 false$	$(very)^4 false$
$m_{(very)^k false}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

(iii) $n_{(fairly)^k true} = 1 - 2^k$, for $k = 1, \dots, \infty$.

k	1	2	3	4
Truth value	<i>fairly true</i>	$(fairly)^2 true$	$(fairly)^3 true$	$(fairly)^4 true$
$n_{(fairly)^k true}$	-1	-3	-7	-15

(vi) $m_{(fairly)^k false} = 2^k$, for $k = 1, \dots, \infty$.

k	1	2	3	4
Truth value	<i>fairly false</i>	$(fairly)^2 false$	$(fairly)^3 false$	$(fairly)^4 false$
$m_{(fairly)^k false}$	2	4	8	16

To close this section, as mentioned in the previous section, we now discuss how to use the relationship established by (12) to define a further operation, denoted by \sim , via correspondences ψ and χ . Let $\mathbf{v}(P)$ be the linguistic truth value of a proposition P , we define

$$\sim \mathbf{v}(P) = \begin{cases} E[\chi(m_{\neg \mathbf{v}(P)})] & \text{if } \mathbf{v}(P) \in V_t \\ E[\psi(n_{\neg \mathbf{v}(P)})] & \text{if } \mathbf{v}(P) \in V_f \end{cases} \quad (17)$$

With this definition, we have

$$\sim \sim \mathbf{v}(P) = \mathbf{v}(P); \quad \sim true = false; \quad \sim false = true.$$

Recall that in the conventional approach to fuzzy logic [1,22], there are also possible two forms of negation. Particularly, the truth value of the proposition (not P) is defined by

$$\mu_{\mathbf{v}(\text{not } P)}(u) = \mu_{\mathbf{v}(P)}(1 - u), \text{ for any } u \in [0, 1] \quad (18)$$

while the truth value ($\text{not } \mathbf{v}(P)$) is given by

$$\mu_{\text{not } \mathbf{v}(P)}(u) = 1 - \mu_{\mathbf{v}(P)}(u), \text{ for any } u \in [0, 1] \quad (19)$$

It is easily seen that the operator \neg in our model is fully compatible with that defined by (18). We now show that the operator \sim defined by (17) gives the same result as that computed by (19) in [1]. As a simple illustration, using computed results in (i – vi) and the definition of mappings ψ, χ , we easily establish the result as shown in Table 3.

Comparison the obtained result in Table 3 with that given in [1, Table 4] may allow us to use \sim as another kind of negation in our model. Note that the

Table 3

A reduced linguistic truth table for \neg and \sim

$\mathbf{v}(P)$	$\neg \mathbf{v}(P)$	$\sim \mathbf{v}(P)$
<i>true</i>	<i>false</i>	<i>false</i>
<i>fairly true</i>	<i>fairly false</i>	<i>very false</i>
<i>very true</i>	<i>very false</i>	<i>fairly false</i>
<i>(very)² true</i>	<i>(very)² false</i>	<i>(fairly)² false</i>
<i>Abs. true</i>	<i>Abs. false</i>	
<i>Abs. false</i>	<i>Abs. true</i>	
<i>unknown</i>	<i>unknown</i>	
<i>false</i>	<i>true</i>	<i>true</i>
<i>fairly false</i>	<i>fairly true</i>	<i>very true</i>
<i>very false</i>	<i>very true</i>	<i>fairly true</i>
<i>(very)² false</i>	<i>(very)² true</i>	<i>(fairly)² true</i>

computed result in [1] is only obtained after a step of linguistic approximation, while our model gives directly the result without any step of linguistic approximation.

5 Conclusions

A new model for parametric representation of linguistic truth values has been proposed in this paper. It has been shown that our model is superior to the existing models under several intuitive criteria both algebraically and computationally. We know that every deductive system in classical or non-classical logic always determines an algebra in a certain class of abstract algebras of the same category of the corresponding algebra of truth values [17]. An interesting point is that the proposed model not only yields an efficient method for computing linguistic truth expressions without a step of linguistic approximation, but also accompanies with a rich algebraic structure of the linguistic truth domain, namely De Morgan algebra. This may allow us to examine some characteristics of fuzzy linguistic logic through the algebraic structure of the linguistic truth domain. Furthermore, the model proposed in this paper can be also extended to an arbitrary linguistic variable with the shape of triangular and trapezoidal membership functions of primary fuzzy sets. These problems as well as a method of approximate reasoning based on this approach are being the subject of our further work.

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