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Description	

A Probability-Based Approach to Comparison of Fuzzy Numbers and Applications to Target-Oriented Decision Making

Van-Nam Huynh, *Member, IEEE*, Yoshiteru Nakamori, *Member, IEEE*, and Jonathan Lawry

***Abstract*—In this paper, we introduce a new comparison relation on fuzzy numbers based on their alpha-cut representation and comparison probabilities of interval values. Basically, this comparison process combines a widely accepted interpretation of fuzzy sets together with the uncertain characteristics inherent in the representation of fuzzy numbers. The proposed comparison relation is then applied to the issue of ranking fuzzy numbers using fuzzy targets in terms of target-based evaluations. Some numerical examples are used to illuminate the proposed ranking technique as well as to compare with previous methods. More interestingly, according to the interpretation of the new comparison relation on fuzzy numbers, we provide a fuzzy target-based decision model as a solution to the problem of decision making under uncertainty, with which an interesting link between the decision maker's different attitudes about target and different risk attitudes in terms of utility functions can be established. Moreover, an application of the proposed comparison relation to the fuzzy target-based decision model for the problem of fuzzy decision making with uncertainty is provided. Numerical examples are also given for illustration.**

***Index Terms*—Decision-making, fuzzy number, fuzzy target, ranking, uncertainty.**

I. Introduction

THE issue of comparison and ranking of fuzzy numbers has been a topic of investigation since the 1970s, mainly related to applications of fuzzy sets in decision analysis [11], [24], [25], [31], [44], [45], [49], [56]. As we know, in practice evaluations for selection and for ranking among alternatives are two closely related and common facets of human decision making activities. Frequently, decision-makers are faced with a lack of precise information when assessing alternatives. In such situations, fuzzy numbers are extensively applied to represent the performance of alternatives and therefore, the ranking or selection of alternatives eventually leads to comparisons of the resulted fuzzy numbers.

Many methods for comparison and ranking of fuzzy numbers have previously been proposed in the literature. Most early ranking methods in the field have been reviewed and analyzed by Bortolan and Degani [7], and more recently by Chen and Hwang [11]. In particular, the collection of cases examined by Bortolan and Degani [7] has been widely used as the benchmark examples for comparative studies of ranking methods. As observed in a recent review by Wang and Kerre [47], ranking methods can be classified into three categories. Methods belonging to the first class aim to define a ranking function mapping a fuzzy number into a real number, and then use a natural order for ranking purpose. In other words, these methods tend to defuzzify an intrinsically fuzzy number into a crisp one and base the comparison of fuzzy numbers on that of real numbers, where a natural order exists. Examples of these methods are given for instance in [8], [18], [20], [33], and [48]. The main criticism of these methods is, as Freeling [19] pointed out, that "by reducing the whole of our analysis to a single number, we are losing much of the information we have purposely been keeping throughout our calculations." The second class consists of methods that compare fuzzy numbers based on their relation(s) to predefined reference set(s), e.g., as given in [13], [24], [25], and [28]. More recently, Yeh and Deng [55] have also presented a new reference-based ranking approach accompanying with a comprehensive discussion of the use of reference sets for ranking fuzzy numbers in the literature. Lastly, methods of the third class tend to construct a fuzzy binary relation on fuzzy numbers representing pairwise comparisons between them and then develop a procedure for obtaining the final ranking based on these pairwise comparisons. For example, methods given in [3], [16], [30], [40], and [45] could be considered as belonging to this class.

Though many methods for ranking fuzzy numbers have been presented in the last decades, none of them is a well accepted "golden choice" for all cases [7], [47]. Main drawbacks found in most methods include: counterintuitive, nondiscriminating, inconsistency, using only local information or restricting the shape of fuzzy numbers to be ranked, and difficult to understand [18], [30]. Recently, Lee-Kwang and Lee [31] have proposed a new method for ranking fuzzy numbers based on the so-called satisfaction function (SF) and a viewpoint-dependent evaluation method. Their method could be viewed as a hybrid of reference set based methods and fuzzy preference relation based methods mentioned above, while taking the overall possibility distribution of fuzzy numbers involved into consideration. More particular, the SF $S(A > B)$ (see Section IV) defined for any two fuzzy numbers A and B is interpreted as "the possibility that A is greater than B." Then the proposed ranking method is based on evaluations of the SF of every fuzzy number involved with

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a predefined viewpoint T, which is also a fuzzy number. Formally, by means of the SF S a comparison relation on fuzzy numbers is established

$$A \geq B \text{ iff } S(A > T) \geq S(B > T).$$

Note that the formulation of the SF is different from the possibility theory based approach proposed by Dubois and Prade [16], though semantic interpretations of them are somehow similar. It is interesting here to observe that if fuzzy numbers involving in a ranking could be considered as the fuzzy performance assessments of alternatives, a predefined viewpoint T in Lee-Kwang and Lee's method could be seen as the decision-maker's fuzzy target [22]. Then, obeying the optimizing principle, the decision maker should choose an alternative that maximizes the possibility of "meeting his target" represented by the SF as showed above. This view can be considered as one of underlying motivations for ranking methods based on viewpoint-dependent evaluations. Naturally, it also suggests a thinking of a probability-based comparison relation in a similar manner, supported by a probability-based representation of fuzzy sets as discussed, e.g., in [17].

Furthermore, our other motivation comes from the desire to bring fuzzy targets within the reach of the target-based decision model [4], [9]. More concretely, in decision analysis with uncertainty, a classical problem is to rank a set of acts defined on a state space $\mathcal{S}$ accompanying with a probability distribution $P_{\mathcal{S}}$, where, due to the uncertainty in the state of nature, each act a may lead to different outcomes taking from a set of outcomes D, usually associated with a random outcome $X_a{:}\mathcal{S} \to D$. The decision maker (DM) must then use some ranking procedure over acts for making decisions. The most commonly used ranking procedure is based on the expected utility model, which suggests that the ranking be obtained by using the value function

$$v(a) = EU(X_a) = \sum_{s \in \mathcal{S}} U(X_a(s)) P_{\mathcal{S}}(s)$$

where U is a utility function over D. In the target-based model, instead the DM could assess some random variable T as his uncertain target (or *benchmark*) and then rank an act a by the probability $P(X_a \succeq T)$ that it meets the target T (or, it outperforms the benchmark), provided that the target T is stochastically independent of the random outcomes to be evaluated. Namely, the target-based model suggests using the value function

$$v(a) = P(X_a \succeq T) = \sum_{s \in \mathcal{S}} P(X_a(s) \succeq T) P_{\mathcal{S}}(s).$$

Interestingly enough, as proved in [4], this target-based decision model satisfies the Savage axioms [42] serving as an axiomatic foundation for rational decision making under uncertainty, while maintaining the appealing features from the target-based approach as thinking about targets is very natural in many practical situations of decision making. Therefore, it would be interesting to study of the target-based decision model using fuzzy targets, instead of random ones, because in many contexts, defining fuzzy targets is much easier and intuitively natural than directly defining random targets.

Motivated by the above observations, we propose in this paper a new comparison relation on fuzzy numbers, viewed as the SF in Lee-Kwang and Lee's work, based on a probabilistic approach. Obviously, it is straightforward to apply the proposed comparison relation to the issue of ranking fuzzy numbers using fuzzy targets in terms of target-based evaluations. This method of ranking fuzzy numbers basically works in a similar way to Lee-Kwang and Lee's method, i.e., consisting of two steps: evaluation and ordering, but with the new comparison relation interpreted as the probability of "meeting the target." According to the interpretation of the proposed comparison relation, we then introduce a target-based formulation for solving the problem of decision making under uncertainty (DMUU) using fuzzy targets. It is shown that the proposed approach can transform fuzzy targets so as to allow the application of the target-based decision model extensively discussed in the decision analysis with uncertainty literature, e.g., [1], [4], [6], [9], [10], and [32]. Furthermore, as will be discussed in Section VI, the fuzzy target-based approach can provide a unified way for solving the problem of fuzzy decision making with uncertainty about the state of nature and imprecision about payoffs. It is of interest noting that by this approach to fuzzy decision analysis, we can discuss an interesting relation between different attitudes about target and different attitudes towards risk in terms of utility functions.

The organization of this paper is as follows. In Section II, the basic notions of fuzzy numbers and the α-cut representations are briefly presented. Section III introduces a new comparison relation on fuzzy numbers based on the α-cut representation and the comparison probabilities of interval values. In Section IV, we provide a method for ranking fuzzy numbers based on the proposed comparison relation and a target-based evaluation method. Section V explores a fuzzy target-based model for the problem of DMUU using the proposed comparison relation. Section VI then extends the application to the problem of fuzzy DMUU. Finally, some concluding remarks and further work are presented in Section VII.

II. Fuzzy Numbers and the α-Cut Representation

A fuzzy number A is defined as a fuzzy subset with the membership function $\mu_A(x)$ of the set $\mathbb{R}$ of all real numbers that satisfies the following properties [27], [56]:

- A is a normal fuzzy set, i.e., $\sup_{x \in \mathbb{R}} \mu_A(x) = 1$;
- A is a convex fuzzy set, i.e.,

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

  for $\forall x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$;
- the support of A, i.e., the set $\text{supp}(A) = \{x \in \mathbb{R} | \mu_A(x) > 0\}$, is bounded.

For $\alpha \in (0,1]$, the α-cut A_α of A is a crisp set defined as

$$A_\alpha = \{x \in \mathbb{R} | \mu_A(x) \geq \alpha\}.$$

According to [15] and [29], a fuzzy number A can be conveniently represented by the canonical form

$$\mu_A(x) = \begin{cases} f_A(x), & a \le x \le b \\ 1, & b \le x \le c \\ g_A(x), & c \le x \le d \\ 0, & \text{otherwise} \end{cases}.$$

where $f_A(x)$ is a real-valued function that is monotonically increasing and $g_A(x)$ is a real-valued function that is monotonically decreasing. In addition, as in most applications, we assume that functions f_A and g_A are continuous. If $f_A(x)$ and $g_A(x)$ are linear functions, then A is called a trapezoidal fuzzy number and denoted by $[a, b, c, d]$. In particular, $[a, b, c, d]$ becomes a triangular fuzzy number if $b = c$.

For any fuzzy number A expressed in the canonical form, its α-cuts are expressed for all $\alpha \in (0, 1]$ by the formula [29]

$$A_\alpha = \begin{cases} \left[f_A^{-1}(\alpha), g_A^{-1}(\alpha)\right], & \text{when } \alpha \in (0,1) \\ [b, c], & \text{when } \alpha = 1 \end{cases} \tag{1}$$

where f_A^{-1} and g_A^{-1} are the inverse functions of f_A and g_A, respectively. In the case that A degenerates into a crisp interval, i.e., $A = [a, b]$, we define $A_\alpha = A$ for all $\alpha \in (0, 1]$.

It should be noted that in fuzzy set theory, the concept of α-cuts plays an important role in establishing the relationship between fuzzy sets and crisp sets. Intuitively, each α-cut A_α of a fuzzy set A can be viewed as a crisp approximation of A at the level $\alpha \in (0, 1]$. In the area of fuzzy arithmetic, the α-cut representation plays an essential role in implementing arithmetic operations on fuzzy numbers, with help from the extension principle [35] and the interval arithmetic [34].

In the case where a fuzzy set A has a discrete membership function, i.e.,

$$A = \{(x_k, \mu_A(x_k))\}, \quad \text{for } x_k \in \mathbb{R} \text{ and } k = 1, \ldots, N$$

with N being a finite positive integer, Dubois and Prade [14] pointed out that the family of its α-cuts forms a nested family of focal elements in terms of Dempster–Shafer theory [43]. In particular, assuming the range of the membership function μ_A, denoted by $\text{rng}(\mu_A)$, is $\text{rng}(\mu_A) = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i > \alpha_{i+1} > 0$, for $i = 1, \ldots, n-1$, then the so-called *body of evidence* induced from A is defined as the collection of pairs

$$\mathcal{F}_A = \{(A_{\alpha_i}, \alpha_i - \alpha_{i+1}) \,|i = 1, \ldots, n\}$$

with $\alpha_{n+1} = 0$ by convention. Then the membership function μ_A can be expressed by

$$\mu_A(x_k) = \sum_{x_k \in F_{\alpha_i}} m_i \tag{2}$$

where $m_i = (\alpha_i - \alpha_{i+1})$ can be viewed as the probability that A_{α_i} stands as a crisp representative of the fuzzy set A [17], and so $\mathcal{F}_A$ is referred to as a consonant random set. Note that the normalization assumption of A insures the body of evidence does not contain the empty set. This view of fuzzy sets has been also used by Baldwin [2] to introduce the so-called mass assignment of a fuzzy set, with relaxing of the normalization assumption of fuzzy sets.

In the case of a fuzzy number A that possesses a continuous membership function, as discussed in Dubois and Prade [17], the family $\{A_\alpha | \alpha \in (0, 1]\}$ can be viewed as a uniformly distributed random set, consisting of the Lebesgue probability measure on [0,1] and the set-valued mapping $\alpha \mapsto A_\alpha$. Then the membership function μ_A is expressed as an integral

$$\mu_A(x) = \int_0^1 \mu_{A_\alpha}(x) d\alpha \tag{3}$$

where μ_{A_α} is the characteristic function of crisp set A_α.

In computer applications, a fuzzy number A can be usually approximated by sampling the membership function along the membership axis. That is, assuming uniform sampling and that the sample values are taken at membership grades $\alpha_1 = 1 > \alpha_2 > \ldots > \alpha_{n-1} > \alpha_n > 0$, then, from the perspective of the above interpretation of fuzzy sets, we can approximately represent A as

$$\mathcal{F}_A = \{(A_{\alpha_i}, \alpha_i - \alpha_{i+1}) \,|i = 1, \ldots, n\} \tag{4}$$

and then membership degrees can be approximately computed via (2), the discrete version of (3). The approximation becomes better when the sample of membership grades is finer. Interestingly, regarding the issue of ranking fuzzy numbers, this approximate representation of fuzzy numbers has been either implicitly or explicitly used by many authors previously, for instance, in [12], [18], [39], and [48].

III. A Probability-Based Comparison Relation

In this section, we propose a new comparison relation on fuzzy numbers based on the α-cut representation. The section first begins with the case of intervals and then generalizes to the case of fuzzy numbers. Finally, an extension to the case of nonconvex and subnormal fuzzy sets is also discussed.

A. Intervals Case

Let us consider two interval values denoted by $X = [x_1, x_2]$ and $Y = [y_1, y_2]$. In [40], the authors proposed a ranking procedure for intervals based on the Hurwicz criterion as X ranks over Y if and only if

$$\delta x_1 + (1 - \delta)x_2 > \delta y_1 + (1 - \delta)y_2$$

where $\delta \in [0, 1]$ is a parameter reflecting the strategy that is adopted by the decision maker. Roughly speaking, interval values are first mapped into real numbers taking the decision maker's attitude expressed by the Hurwicz criterion into account, and then a ranking is based on the natural order of resulted real numbers.

Here we utilize an approach to comparing intervals motivated by a probabilistic view of the underlying uncertainty, instead. More formally, motivated by our later developments, we aim at defining a probability-based comparison relation over intervals, denoted by $P(X \succeq Y)$. To this end, we consider intervals X and Y as uncertain values having uniform distributions $p_X(x)$ and $p_Y(y)$ over $[x_1, x_2]$ and $[y_1, y_2]$, respectively. Then, based on the probability theory, we can work out the probability of the ordering of uncertain values X and Y taking into account

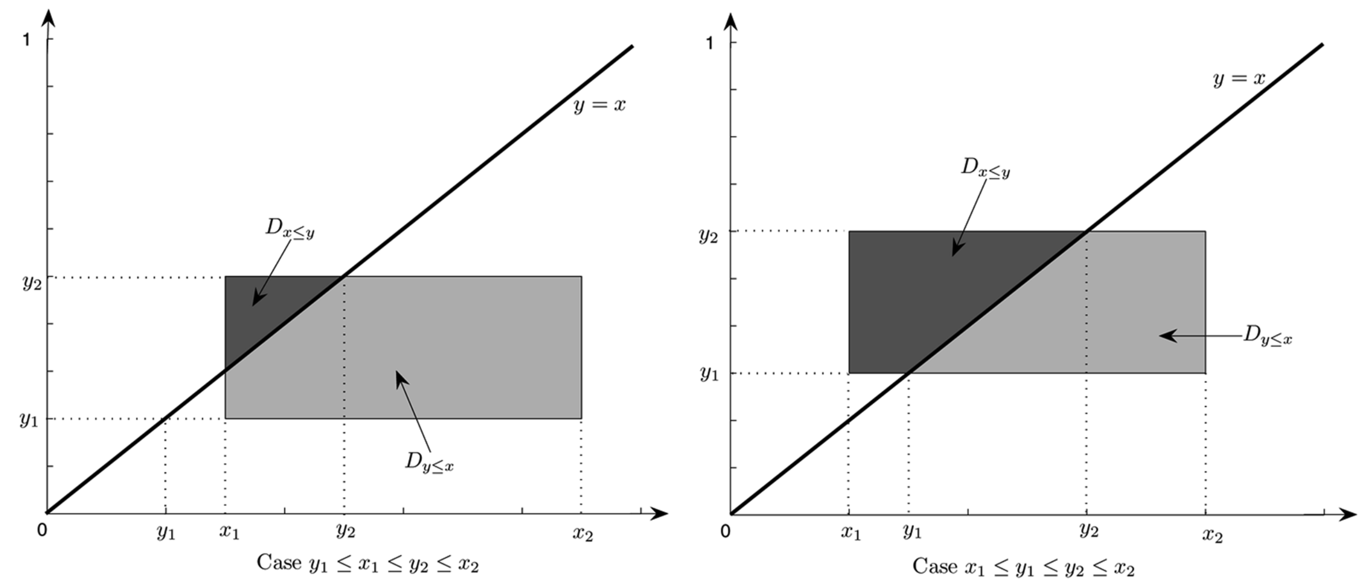


Fig. 1. Comparison probability of two intervals.

associated probability distributions $p_X(x)$ and $p_Y(y)$. Namely, we define

$$P(X \succeq Y) = \int_{-\infty}^{\infty} p_X(x) \left[\int_{-\infty}^{x} p_Y(y) dy \right] dx. \qquad (5)$$

Recall that

$$p_X(x) = \begin{cases} \frac{1}{x_2 - x_1}, & \text{if } x_1 \leq x \leq x_2 \\ 0, & \text{otherwise} \end{cases}$$
$$p_Y(y) = \begin{cases} \frac{1}{y_2 - y_1}, & \text{if } y_1 \leq y \leq y_2 \\ 0, & \text{otherwise} \end{cases}$$

Obviously, the result of computation for (5) depends on the relative position of x_1 and x_2 with respect to y_1 and y_2. By a direct computation, we easily obtain the result of (5) for all cases where at least one of "$x_1 < x_2$" or "$y_1 < y_2$" holds as follows.

1) If $x_2 \leq y_1$, $P(X \succeq Y) = 0$.
2) If $x_1 \geq y_2, P(X \succeq Y) = 1$.
3) If $x_1 \leq y_1 \leq x_2 \leq y_2, x_1 < x_2$, and $y_1 < y_2$, we have

$$P(X \succeq Y) = \frac{1}{(x_2 - x_1)(y_2 - y_1)} \int_{y_1}^{x_2} \left[\int_{y_1}^{x} 1 dy \right] dx = \frac{(x_2 - y_1)^2}{2(x_2 - x_1)(y_2 - y_1)}.$$

4) If $y_1 \leq x_1 \leq y_2 \leq x_2, x_1 < x_2$ and $y_1 < y_2$, similar to case 3), we obtain

$$P(X \succeq Y) = 1 - \frac{(y_2 - x_1)^2}{2(x_2 - x_1)(y_2 - y_1)}.$$

Intuitively, this case is illustrated as in Fig. 1 (left), where the area where Y is smaller than X is denoted by $D_{y \leq x}$ and $P(X \succeq Y)$ is the ratio of $D_{y \leq x}$ to the whole rectangle, i.e., $D_{y \leq x} + D_{x \leq y}$.

5) If $x_1 \leq y_1 \leq y_2 \leq x_2$, and $x_1 < x_2$, we have

$$P(X \succeq Y) = \frac{1}{(x_2 - x_1)(y_2 - y_1)} \int_{y_1}^{y_2} \left[\int_{y_1}^{x} 1 dy \right] dx + \frac{1}{(x_2 - x_1)} \int_{y_2}^{x_2} 1 dx = \frac{x_2 - 0.5(y_1 + y_2)}{(x_2 - x_1)}$$

Intuitively, this case is graphically illustrated as in Fig. 1 (right).

6) If $y_1 \leq x_1 \leq x_2 \leq y_2$ and $y_1 < y_2$, similar to case 5), we obtain

$$P(X \succeq Y) = \frac{0.5(x_1 + x_2) - y_1}{(y_2 - y_1)}.$$

In the case where both intervals X and Y degenerate into scalar numbers, i.e., $x_1 = x_2$ and $y_1 = y_2$, we define by convention

$$P(X \succeq Y) = \begin{cases} 1, & \text{if } x_1 > y_1 \\ \frac{1}{2}, & \text{if } x_1 = y_1 \\ 0, & \text{if } x_1 < y_1 \end{cases}. \qquad (6)$$

Note that this definition of the degenerate case has been suggested in [52] and motivated by the fact that if we define the order relation $\geq_I$ over intervals as $X >_I Y$ iff $P(X \succeq Y) > P(Y \succeq X)$ and $X =_I Y$ iff $P(X \succeq Y) = P(Y \succeq X)$, then the definition of $P(X \succeq Y)$ in case of crisp numbers leads to the natural ordering of numbers with the ordering procedure defined by $\geq_I$.

As a consequence of the above computational results and (6), we get the following.

Proposition 1: We have the following.

1) $P(X \succeq Y) = 1 - P(Y \succeq X)$.
2) If $(x_1 + x_2) = (y_1 + y_2)$, $P(X \succeq Y) = 0.5$.

Remark 1: In [52], the authors provide an indirect way to obtain $P(X \succeq Y)$ for intervals X and Y, equivalently, by computing $P(X - Y \geq 0)$, where the probability distribution $p_Z(z)$

of uncertain value $Z = X - Y$ is defined as the convolution of $p_X(x)$ and $p_Y(y)$ [37]. Namely

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(z+y)p_Y(y)dy \tag{7}$$

and then

$$P(X \succeq Y) = P(Z \geq 0) = \int_0^{\infty} p_Z(z)dz. \tag{8}$$

However, in our opinion, this method of obtaining $P(X \succeq Y)$ is more complicated and difficult to figure out geometrically than the direct method as presented above. In addition, as we will see later in Section IV, the formulation of (5) also allows us to provide a probabilistic interpretation for the SF proposed in [31], which is clearly more intuitive than a possibilistic interpretation as suggested by the authors.

B. Fuzzy Numbers Case

Now let us turn to the case of fuzzy numbers. Consider two fuzzy numbers A and B whose membership functions are expressed in the canonical form by

$$\mu_A(x) = \begin{cases} f_A(x), & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ g_A(x), & a_3 \leq x \leq a_4 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_B(x) = \begin{cases} f_B(x), & b_1 \leq x \leq b_2 \\ 1, & b_2 \leq x \leq b_3 \\ g_B(x), & b_3 \leq x \leq b_4 \\ 0, & \text{otherwise} \end{cases}$$

respectively. According to (1), we obtain for all $\alpha \in (0,1]$

$$A_\alpha \triangleq [a_l(\alpha), a_r(\alpha)] = \begin{cases} \left[f_A^{-1}(\alpha), g_A^{-1}(\alpha)\right], & \text{when } \alpha \in (0,1) \\ [a_2, a_3], & \text{when } \alpha = 1 \end{cases} \tag{9}$$

$$B_\alpha \triangleq [b_l(\alpha), b_r(\alpha)] = \begin{cases} \left[f_B^{-1}(\alpha), g_B^{-1}(\alpha)\right], & \text{when } \alpha \in (0,1) \\ [b_2, b_3], & \text{when } \alpha = 1. \end{cases} \tag{10}$$

Based on the comparison relation on intervals defined in the preceding section and the α-cut representations of fuzzy numbers, we now define a comparison relation on fuzzy numbers, denoted by $P(A \succeq B)$, as follows:

$$P(A \succeq B) = \int_0^1 P(A_\alpha \succeq B_\alpha)d\alpha. \tag{11}$$

Fig. 2 graphically illustrates the idea of the comparison of two triangular fuzzy numbers.

Remark 2: Due to the continuity and monotonicity of functions f_A, f_B and g_A, g_B, it follows from the computational results of cases 1)–6) in the preceding section that the function $f(\alpha) = P(A_\alpha \succeq B_\alpha)$ is a piecewise continuous function on [0,1], which makes the definition of $P(A \succeq B)$ via (11) eligible.

As a direct consequence of Proposition 1 and (11), we obtain the following.

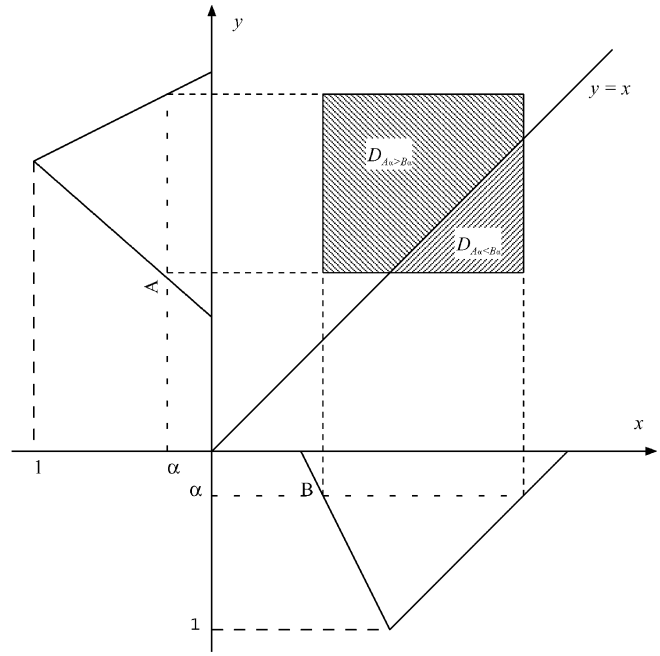


Fig. 2. Comparison of two triangular fuzzy numbers.

Proposition 2: For any fuzzy numbers A and B, we have the following.
1) $P(A \succeq B) = 1 - P(B \succeq A)$.
2) If $(a_l(\alpha) + a_r(\alpha)) = (b_l(\alpha) + b_r(\alpha))$, for all $\alpha \in (0,1], P(A \succeq B) = 0.5$.

Regarding the interpretation of $P(A \succeq B)$, let us express (11) by

$$P(A \succeq B) = \int_0^1 P(A_\alpha \succeq B_\alpha)dF(\alpha)$$

where $F(\alpha) = \alpha$ is the cumulative probability distribution of a random variable having the uniform distribution on $[0,1]$. Then according to the probability-based representations of A and B (again, see Dubois and Prade [17]), that view $\{A_\alpha | \alpha \in (0,1]\}$ and $\{B_\alpha | \alpha \in (0,1]\}$ as uniformly distributed random intervals, we can view $P(A \succeq B)$ as expected probability of A dominating B.

C. Extension to Nonconvex and Subnormal Fuzzy Numbers

Considering now two nonconvex fuzzy numbers A and B, then for $\alpha \in (0,1]$, we can express α-cuts A_α and B_α, respectively, as unions of distinct intervals [48]

$$A_\alpha = \bigcup_{i=1}^{n_\alpha} \left[a_l^i(\alpha), a_r^i(\alpha)\right] \tag{12}$$

$$B_\alpha = \bigcup_{j=1}^{m_\alpha} \left[b_l^j(\alpha), b_r^j(\alpha)\right]. \tag{13}$$

Here we still assume that A and B are normal. Intuitively, recall that the probability $P(X \succeq Y)$ of the ordering of two intervals X and Y is defined by the ratio of the area where Y is smaller than X, i.e., $D_{y\leq x}$, to the whole area determined by the rectangle $X \times Y$ (graphically, see, for example, Fig. 1). Keeping this in mind, we can define $P(A_\alpha \succeq B_\alpha)$ as

$$P(A_\alpha \succeq B_\alpha) = \frac{\sum_{i=1}^{n_\alpha}\sum_{j=1}^{m_\alpha} P\left(A_\alpha^j \succeq B_\alpha^i\right)\cdot M_{ij}}{\sum_{i=1}^{n_\alpha}\sum_{j=1}^{m_\alpha} M_{ij}} \quad (14)$$

where $A_\alpha^i = [a_l^i(\alpha), a_r^i(\alpha)]$, $B_\alpha^j = [b_l^j(\alpha), b_r^j(\alpha)]$, and M_{ij} is the area determined by the rectangle $[b_l^j(\alpha), b_r^j(\alpha)] \times [a_l^i(\alpha), a_r^i(\alpha)]$. Note that in this case we also have

$$P(A_\alpha \succeq B_\alpha) = 1 - P(B_\alpha \succeq A_\alpha).$$

Further, once having defined $P(A_\alpha \succeq B_\alpha)$ by (14), we can also obtain $P(A \succeq B)$ as defined in (11).

Now let us consider the case of subnormal fuzzy numbers A and B. Denote by $\text{hgt}(A)$ and $\text{hgt}(B)$ the heights of fuzzy sets A and B, respectively. Assuming that A and B are nonempty, i.e., $\text{hgt}(A) > 0$ and $\text{hgt}(B) > 0$, let

$$\beta = \min(\text{hgt}(A), \text{hgt}(B)).$$

Then the relation established in Proposition 2 suggests to define $P(A \succeq B)$ as

$$P(A \succeq B) = \frac{1}{\beta}\int_0^\beta P(A_\alpha \succeq B_\alpha)d\alpha. \quad (15)$$

IV. Application to Ranking Fuzzy Numbers

In this section, we propose a ranking procedure of fuzzy numbers based on the comparison relation on fuzzy numbers introduced in the preceding section.

A. Ranking Procedure

Given fuzzy numbers A and B, as discussed previously, $P(A \succeq B)$ could be interpreted as the expected probability of the relation "A dominates B." From a perspective of decision making, assuming that A and B are considered as fuzzy performance assessments of two alternatives $\mathbf{A}$ and $\mathbf{B}$, respectively, then $P(A \succeq B)$ can be also interpreted as the probability that $\mathbf{A}$ outperforms $\mathbf{B}$. Under such an interpretation and motivated by the target-based approach to decision making [4], [9], a procedure is proposed in the following, which ranks fuzzy numbers by the probability that they outperform some prespecified target or benchmark, which itself is also fuzzy.

Assume that $\mathcal{S} = \{A_1, \ldots, A_N\}$ is a finite set of fuzzy numbers that need to be ranked. Then by a fuzzy target involving in the ranking problem, we mean a fuzzy set T over $\mathbb{R}$ having the membership function μ_T:$\mathbb{R} \to [0,1]$ satisfying the following.

1) μ_T is a piecewise continuous function having a bounded support.
2) For any i, supp $(A_i) \subseteq$ supp (T).
3) T is not empty, i.e., $\int_{-\infty}^{+\infty} \mu_T(x)dx > 0$.

Once having specified target T, the ranking procedure is simply carried out as follows.

1) Evaluate $E_T(A_i) = P(A_i \succeq T), i = 1, \ldots, N$.
2) Rank fuzzy numbers in $\mathcal{S}$ according to their evaluation values $E_T(A_i), i = 1, \ldots, N$.

Similar to [31], we also define the so-called *relative index* of a fuzzy number A_i in $\mathcal{S}$ with respect to a prespecified target T as

$$R_T(A_i) = \frac{E_T(A_i)}{\max_{A_j\in\mathcal{S}}\{E_T(A_j)\}}.$$

Though the relative index $R_T(A_i)$ does the same as the index $E_T(A_i)$ in ranking fuzzy numbers, it provides, however, the information that shows how close A_i is to the best one according to the target (or viewpoint [31]) T.

Let us denote supp $(T) = [x_{\min}, x_{\max}]$. In the case of triangular and trapezoidal fuzzy numbers, we have the following.

Proposition 3: Assuming T_{neut} is the neutral target, i.e.,

$$\mu_{T_{\text{neut}}}(x) = \begin{cases} 1, & \text{if } x_{\min} \leq x \leq x_{\max} \\ 0, & \text{otherwise} \end{cases}$$

we have the following.

1) If $A = [a, b, c]$

$$P(A \succeq T_{\text{neut}}) = \frac{\frac{1}{2}\left(\frac{(a+c)}{2} + b\right) - x_{\min}}{x_{\max} - x_{\min}}. \quad (16)$$

2) If $A = [a, b, c, d]$

$$P(A \succeq T_{\text{neut}}) = \frac{\frac{1}{4}(a+b+c+d) - x_{\min}}{x_{\max} - x_{\min}}. \quad (17)$$

Informally, Proposition 3 means that if the decision maker has a neutral behavior on the target, triangular and trapezoidal fuzzy numbers are ranked according to the (weighted) average of their crucial points, where for the case of triangular fuzzy numbers the modal value is weighted double compared to left and right spreads.

It should be noted that this ranking procedure is similar to that proposed by Lee-Kwang and Lee in [31]; however, as discussed above, our motivation here is somehow different. Furthermore, their ranking procedure is based on the SF defined as

$$S(A > T) = \frac{\int_{-\infty}^{\infty}\int_y^{\infty} \mu_A(x)\odot\mu_T(y)dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu_A(x)\odot\mu_T(y)dxdy}$$

where $\odot$ is a T-norm and $S(A > T)$ is interpreted as the possibility that A is greater than T (or the evaluation of A in the local viewpoint of T). That is, in their ranking procedure, the evaluation value of fuzzy number A with respect to a target T is defined by

$$E_T(A) = S(A > T) \quad (18)$$

where the multiplication operator is selected as T-norm $\odot$ in the SF. The following proposition is due to Lee-Kwang and Lee [31].

Proposition 4: If the multiplication operator is selected as T-norm $\odot$ in the SF and the given target is T_{neut}, then fuzzy numbers are ranked according to their centroids. Namely

$$V_{T_{\text{neut}}}(A) = S(A > T_{\text{neut}}) = \frac{C(A) - x_{\min}}{x_{\max} - x_{\min}}$$

where $C(A)$ is the centroid of fuzzy number A.

Consequently, by a simple calculation, we have the following.

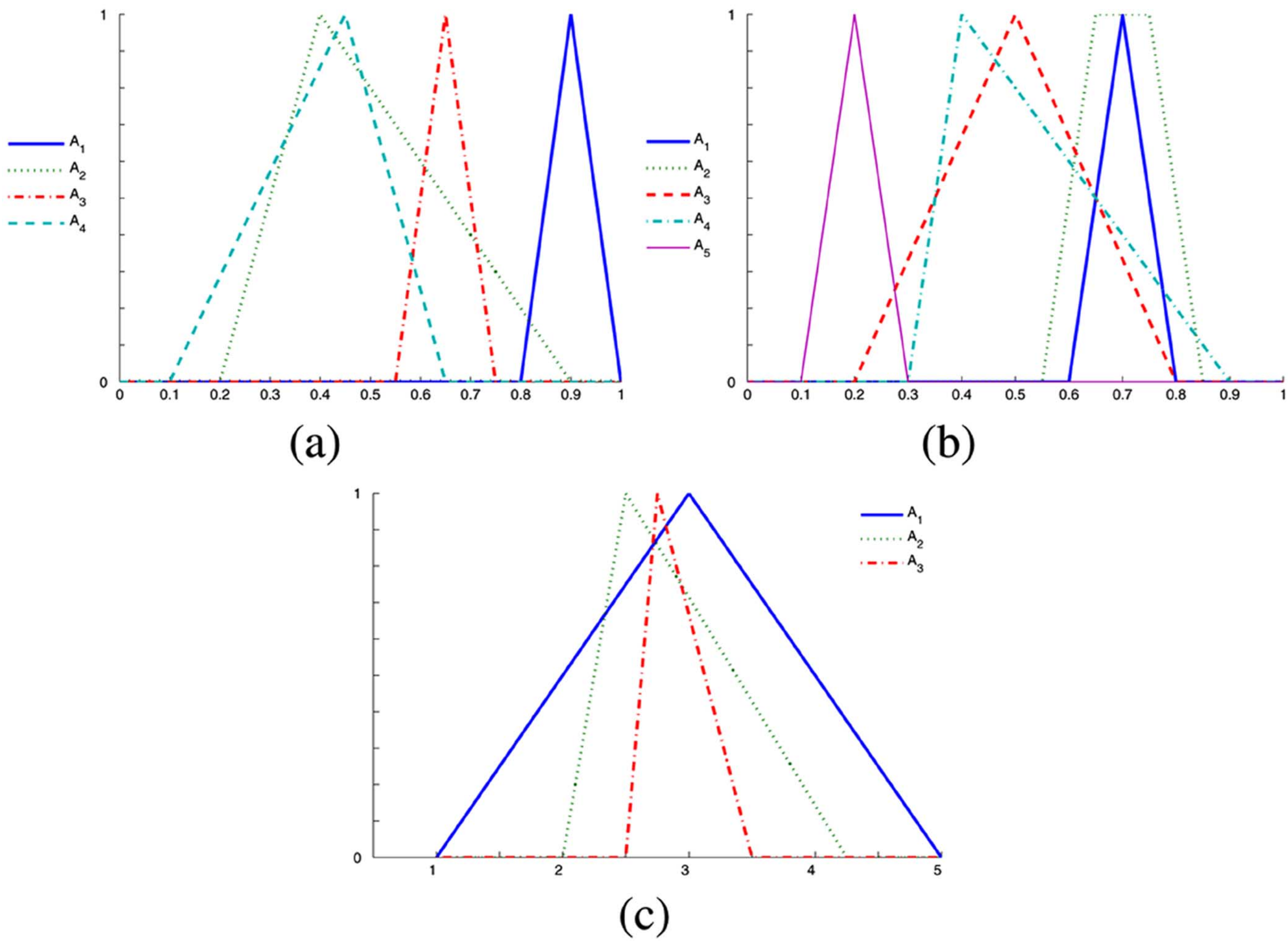


Fig. 3. Fuzzy numbers in (a) Example 1, (b) Example 2, and (c) Example 3.

1) If $A = [a, b, c]$

$$S(A > T_{\text{neut}}) = \frac{\frac{1}{3}(a+b+c) - x_{\min}}{x_{\max} - x_{\min}}. \quad (19)$$

2) If $A = [a, b, c, d]$

$$S(A > T_{\text{neut}}) = \frac{\frac{1}{3}\left[\frac{(d^2+c^2+dc)-(a^2+b^2+ab)}{(d+c)-(a+b)}\right] - x_{\min}}{x_{\max} - x_{\min}}. \quad (20)$$

Remark 3: In our opinion, $S(A > B)$ should have a probabilistic interpretation rather than a possibility interpretation as originally provided by Lee-Hwang and Lee [31]. Particularly, let us consider possibility distributions $\mu_A(x)$ and $\mu_B(y)$ of fuzzy numbers A and B, respectively. Using Yager's method [53] of converting possibility distributions into probability distributions via a simple normalization, we obtain associated probability distributions of A and B as follows:

$$P_A(x) = \frac{\mu_A(x)}{\int_{-\infty}^{\infty} \mu_A(x)dx}$$
$$P_B(y) = \frac{\mu_B(y)}{\int_{-\infty}^{\infty} \mu_B(y)dy}.$$

Having considered A and B as random variables with associated probability distributions $P_A(x)$ and $P_B(y)$, respectively, we can define the probability of the ordering of random variables A and B taking into account distributions $P_A(x)$ and $P_B(y)$ as

$$\begin{aligned} Q(A \succeq B) &= \int_{-\infty}^{\infty} P_B(y)\left[\int_y^{\infty} P_A(x)dx\right]dy \\ &= \int_{-\infty}^{\infty}\int_y^{\infty} P_A(x)P_B(y)dxdy \end{aligned} \quad (21)$$

which clearly turns out to be the SF $S(A > B)$ defined by Lee-Hwang and Lee [31] with T-norm $\odot$ selected as the multiplication operator.

B. Examples

In order to illustrate the proposed ranking method and to see how different targets affect the ranking results, we now examine following numeric examples.

Example 1: Let us first consider an example taken from [31]. Assume that we have four fuzzy numbers as depicted in Fig. 3(a). Let us consider three prototypical targets that are *pessimist, optimist,* and *neutral,* as depicted in Fig. 4.

The probabilities that given fuzzy numbers meet various targets and the corresponding ranking results are shown in Table I. From the table, we see that the ranking order is the same for all

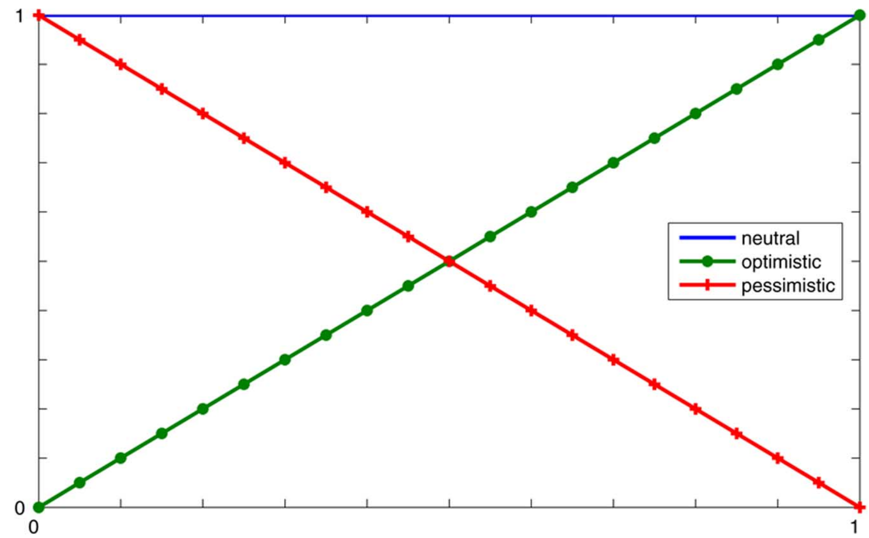


Fig. 4. Fuzzy targets.

TABLE I
RESULTS OF THE EXAMPLE 1

Targets		A_1	A_2	A_3	A_4	Ranking Order
Optimistic	$E_{T_{opt}}(A_i)$	0.67	0.17	0.28	0.10	$A_1 \succ A_3 \succ A_2 \succ A_4$
	$R_{T_{opt}}(A_i)$	1	0.25	0.42	0.14	
Neutral	$E_{T_{neut}}(A_i)$	0.90	0.47	0.65	0.41	$A_1 \succ A_3 \succ A_2 \succ A_4$
	$R_{T_{neut}}(A_i)$	1	0.53	0.72	0.46	
Pessimistic	$E_{T_{pess}}(A_i)$	0.99	0.84	0.93	0.76	$A_1 \succ A_3 \succ A_2 \succ A_4$
	$R_{T_{pess}}(A_i)$	1	0.85	0.94	0.77	

TABLE II
RESULTS OF EXAMPLE 2

Targets		A_1	A_2	A_3	A_4	A_5	Ranking Order
Optimistic	$E_{T_{opt}}(A_i)$	0.339	0.342	0.186	0.190	0.023	$A_2 \succ A_1 \succ A_4 \succ A_3 \succ A_5$
	$R_{T_{opt}}(A_i)$	0.991	1	0.545	0.556	0.066	
Neutral	$E_{T_{neut}}(A_i)$	0.70	0.70	0.50	0.50	0.20	$\{A_1, A_2\} \succ \{A_3, A_4\} \succ A_5$
	$R_{T_{neut}}(A_i)$	1	1	0.714	0.714	0.286	
Pessimistic	$E_{T_{pess}}(A_i)$	0.949	0.946	0.873	0.867	0.522	$A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5$
	$R_{T_{pess}}(A_i)$	1	0.997	0.920	0.915	0.550	

three targets. Intuitively, it is clear that A_1 dominates A_3, and A_3 dominates both A_2 and A_4. In addition, while the modal value of A_2 is less than that of A_4 with a small differentiation, the area where A_2 dominates A_4 is much larger than that where A_2 is dominated by A_4. Thus, it is intuitively reasonable to order A_2 over A_4. On the other hand, it can also be seen that the evaluation value of each fuzzy number as well as its relative index vary considerably according to selected target. Particularly, let us compare with the case of the neutral target, which indicates a uniform preference distribution on the domain. While the evaluation values of A_2, A_3, and A_4 and, consequently, their relative indexes are much improved in relation to those of the best A_1 according to the pessimistic target, they are considerably decreased in relation to those of the best A_1 according to the optimistic one.

Example 2: Given five fuzzy numbers on [0,1] as shown in Fig. 3(b), we also consider three prototypical targets that are *pessimist, optimist,* and *neutral* as in Example 1. Table II shows the evaluation values, relative indexes of given fuzzy numbers according to various targets, and the corresponding ranking results. In this example, we obtain different rankings among fuzzy numbers according to different targets. If the neutral target is selected, the corresponding result makes no distinction between A_1 and A_2 as well as between A_3 and A_4. However, if an optimistic target is selected, A_2 is ranked over A_1 and A_4 is ranked over A_3, while a reverse result holds for the case of pessimistic target.

Example 3: This is a more complex example. Assume that we are given three fuzzy numbers on [1,5] as shown in Fig. 3(c). Intuitively, it is not obvious to us what the ranking order among given fuzzy numbers should be. However, having interpreted $E_T(A)$ as the probability of fuzzy number A meeting target T, the decision maker can establish a target that reflects his attitude of preference and then rank the fuzzy numbers according

TABLE III
RESULTS OF EXAMPLE 3

Targets		A_1	A_2	A_3	Ranking Order
Optimistic	$E_{T_{opt}}(A_i)$	0.176	0.153	0.143	$A_1 \succ A_2 \succ A_3$
	$R_{T_{opt}}(A_i)$	1	0.869	0.815	
Neutral	$E_{T_{neut}}(A_i)$	0.50	0.453	0.469	$A_1 \succ A_3 \succ A_2$
	$R_{T_{neut}}(A_i)$	1	0.906	0.937	
Pessimistic	$E_{T_{pess}}(A_i)$	0.824	0.828	0.832	$A_3 \succ A_2 \succ A_1$
	$R_{T_{pess}}(A_i)$	0.99	0.995	1	

to their probabilities of meeting the target. In this example, pessimistic, optimistic, and neutral targets are represented by triangular fuzzy numbers [1,1,5], [1,5,5], and the interval [1,5], respectively. As we have seen from the ranking result shown in Table III, different targets lead to different ranking orders of fuzzy numbers. This is a reasonable consequence since a change in the target corresponds to a change in the decision maker's attitude of preference in the decision-making process.

C. Comparison With Previous Methods

Now we examine the proposed ranking method in comparison with several previous methods. In particular, for the purpose of comparative study, we select the following methods: Lee-Kwang and Lee [31], Baldwin and Guild [3], Jain [25], Liou and Wang [33], Kim and Park [28], and Peneva and Popchev [36], all of which allow a change in the evaluation strategy that reflects the attitude of the decision maker. Note here that targets pessimistic, neutral, and optimistic correspond to viewpoints V_1, V_2, and V_3 of Lee-Kwang and Lee's method.

The comparative study is performed on eight cases, all of which are reproduced from [7] and [31]. The results are shown in Tables IV and V. From these results, we can see that in some cases the last five methods either are not discriminative [Baldwin and Guild's method in case b), Liou and Wang's and Kim and Park's methods in case e) with $k = 0$] or provide counterintuitive results [Kim and Park's method in case c); Jain's method in case c) with $k = 0.5, 1$ and Peneva and Popchev's method in case e) with $k = 0$]. It is of interest to see that, though our method and Lee-Kwang-Lee's method provide different results, they are consistent in ranking involved fuzzy numbers with respect to corresponding fuzzy targets. Except for the case of neutral target [refer to (16) and (17) and (19) and (20)], where our method is indifferent in between A and B of example d) but A slightly dominates B according to Lee-Kwang-Lee's method, conversely, Lee-Kwang and Lee's method is indifferent in between B and C of example f), while our method ranks C over B. Detailed discussions on the results can be found in [31], from which Tables IV and V show that in all cases both the methods produce reasonable and almost consistent results. This is an understandable consequence as both methods work in a similar manner with only difference is different representations of fuzzy numbers to be used in each method; i.e., while Lee-Kwang and Lee's method uses the possibility distribution (or, membership function) representation, our method uses the random set representation of fuzzy numbers. Furthermore, it should be emphasized here that though all considered ranking methods allow a change in the evaluation strategy, it is difficult to see clearly how the change of parameters in the last five methods reflects the decision maker's corresponding attitude of evaluation. In Lee-Kwang and Lee's and our methods fuzzy targets have a clear semantics associated with a well-interpreted evaluation strategy, and hence, the change in target reflects clearly and directly the corresponding change in attitude of the decision maker.

V. DECISION MAKING UNDER UNCERTAINTY USING FUZZY TARGETS

In this section, we aim to apply a target-based language to the problem of decision making in the face of uncertainty, with the help of the new comparison relation proposed above. The fundamental framework of DMUU can be most effectively described using the decision matrix shown in Table VI (see, e.g., [50]). In this matrix, $A_i (i = 1, \ldots, n)$ represents the alternatives (or actions) available to a decision maker (DM), one of which must be selected. The elements $S_j (j = 1, \ldots, m)$ correspond to the possible values/states associated with the so-called state of nature S. Each element c_{ij} of the matrix is the payoff the DM receives if alternative A_i is selected and state S_j occurs. The uncertainty associated with this problem is generally a result of the fact that the value of S is unknown before the DM must choose an alternative A_i. Let us consider the decision problem as described in Table VI, assuming a probability distribution $P_{\mathcal{S}}$ over $\mathcal{S}$. Here, we restrict ourselves to a bounded domain of the payoff variable that $D = [c_{\min}, c_{\max}]$, i.e., $c_{\min} \le c_{ij} \le c_{\max}$.

A. Target-Based Model of the Expected Value

As is well known, the most commonly used method for valuating alternatives A_i to solve the DMUU problem described by Table VI is to use the expected payoff value

$$v(A_i) \triangleq EV_i = \sum_{j=1}^{m} p_j c_{ij}. \tag{22}$$

On the other hand, each alternative A_i can be formally considered as a random payoff having the probability distribution P_i defined, with an abuse of notation, as follows:

$$P_i(A_i = x) = P_{\mathcal{S}}(\{S_j : c_{ij} = x\}). \tag{23}$$

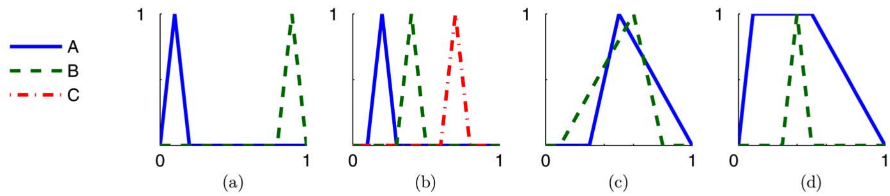
Then, similar to Bordley and LiCalzi's result [4], we now define a random target T that has a uniform distribution, denoted by P_T, on D and is defined by

$$P_T(x) = \begin{cases} \frac{1}{c_{\max} - c_{\min}}, & c_{\min} \le x \le c_{\max} \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Under the assumption that the random target T is stochastically independent of any random payoffs A_i [4], we have

$$\begin{aligned} v(A_i) &\triangleq P(A_i \succeq T) \\ &= \sum_x P(x \succeq T) P_i(A_i = x) \\ &= \sum_x \left[\int_{-\infty}^{x} P_T(t)dt \right] P_i(A_i = x) \end{aligned} \tag{25}$$

TABLE IV
COMPARATIVE EXAMPLE 1



Methods		(a) A	(a) B	(b) A	(b) B	(b) C	(c) A	(c) B	(d) A	(d) B
Huynh-Nakamori-Lawry	*Pessimistic*	0.34	0.99	0.53	0.77	0.95	0.90	0.84	0.74	0.77
	Neutral	0.10	0.90	0.20	0.40	0.70	0.57	0.53	0.40	0.40
	Optimistic	0.01	0.66	0.02	0.09	0.33	0.23	0.16	0.16	0.09
Lee-Kwang-Lee	*Pessimistic*	0.19	0.99	0.36	0.64	0.91	0.82	0.73	0.61	0.64
	Neutral	0.10	0.90	0.20	0.40	0.70	0.60	0.50	0.41	0.40
	Optimistic	0.01	0.81	0.04	0.16	0.49	0.38	0.27	0.22	0.16
Baldwin-Guild	$k = 0.5$	0.00	0.69	0.00	0.00	0.30	0.34	0.24	0.33	0.46
	$k = 1$	0.00	0.82	0.00	0.00	0.44	0.42	0.33	0.43	0.42
	$k = 2$	0.00	0.82	0.00	0.00	0.40	0.44	0.37	0.47	0.22
Jain	$k = 0.5$	0.40	0.95	0.55	0.72	0.94	0.78	0.81	0.78	0.66
	$k = 1$	0.18	0.90	0.32	0.55	0.89	0.66	0.69	0.76	0.45
	$k = 2$	0.03	0.84	0.12	0.33	0.80	0.53	0.51	0.54	0.23
Liou-Wang	$k = 0$	0.05	0.85	0.15	0.35	0.65	0.40	0.35	0.05	0.35
	$k = 0.5$	0.10	0.90	0.20	0.40	0.70	0.58	0.53	0.40	0.40
	$k = 1$	0.15	0.95	0.25	0.45	0.75	0.75	0.70	0.75	0.45
Kim-Park	$k = 0$	0.09	0.82	0.13	0.34	0.75	0.35	0.37	0.09	0.33
	$k = 0.5$	0.14	0.86	0.29	0.44	0.81	0.49	0.51	0.38	0.39
	$k = 1$	0.18	0.91	0.25	0.50	0.88	0.63	0.65	0.67	0.45
Peneva-Popchev	$k = 0$	0.10	0.90	0.20	0.40	0.70	0.67	0.49	0.30	0.40
	$k = 0.5$	0.10	0.90	0.20	0.40	0.70	0.63	0.47	0.46	0.40
	$k = 1$	0.10	0.90	0.20	0.40	0.70	0.61	0.44	0.63	0.40

where

$$P(x \succeq T) = \int_{-\infty}^{x} P_T(t)dt$$

is the cumulative distribution function of the target T. It is of interest to note here that, in a different but similar context, a similar idea has been used in [21] to develop the so-called satisfactory-oriented decision model for multiple-expert decision making with linguistic assessments.

Due to (23) and (24) and the additive property of the probability measure, from (25) we easily obtain

$$v(A_i) = \sum_{j=1}^{m} \left[\int_{-\infty}^{c_{ij}} P_T(t)dt \right] P_S(S = S_j) = \sum_{j=1}^{m} \frac{c_{ij} - c_{\min}}{c_{\max} - c_{\min}} p_j. \tag{26}$$

From (22) and (26), we easily see that there is no way to tell if the DM selects an alternative by maximizing the expected value or by maximizing the probability of meeting the uncertain target T. In other words, the target-based decision model with decision function $v(A_i)$ in (26) above is equivalent to the expected value model defined by (22).

Intuitively, in the target-based model of the expected value above, we can think of T as an interval target represented as a membership function $T(x) = 1$ for $c_{\min} \leq x \leq c_{\max}$, and $T(x) = 0$ otherwise. Then it is interesting to extend target-based decision models with the use of fuzzy targets as in the following.

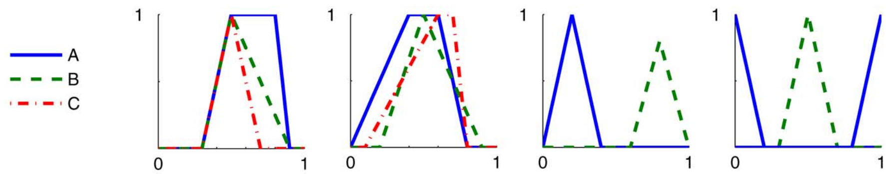
B. Fuzzy Target-Based Model of DMUU

In this section, by a fuzzy target, we mean a possibility variable T over the payoff domain D represented by a possibility distribution $\mu_T{:}D \to [0,1]$. For simplicity, we also assume further that T is a piecewise continuous function having supp $(T) = [c_{\min}, c_{\max}]$.

In the target-based decision model, assume now that the DM assesses a fuzzy target T that reflects his attitude. Then, according to the optimizing principle, after assessing the target the DM would select an alternative as the best that maximizes the expected probability of meeting the target defined by

$$v(A_i) = \sum_{j=1}^{m} p_j \mathbb{P}(c_{ij} \succeq T) \tag{27}$$

TABLE V
COMPARATIVE EXAMPLE 2



(e) (f) (g) (h)

Methods		(e)			(f)			(g)		(h)	
		A	B	C	A	B	C	A	B	A	B
Huynh-Nakamori-Lawry	*Pessimistic*	0.90	0.89	0.85	0.78	0.86	0.84	0.53	0.97	0.55	0.84
	Neutral	0.62	0.55	0.50	0.42	0.52	0.55	0.20	0.80	0.50	0.50
	Optimistic	0.26	0.21	0.15	0.13	0.18	0.17	0.02	0.58	0.45	0.16
Lee-Kwang-Lee	*Pessimistic*	0.84	0.80	0.74	0.66	0.76	0.75	0.35	0.95	0.56	0.74
	Neutral	0.62	0.57	0.50	0.44	0.53	0.53	0.20	0.80	0.50	0.50
	Optimistic	0.41	0.34	0.26	0.22	0.31	0.30	0.05	0.65	0.44	0.26
Baldwin-Guild	$k = 0.5$	0.31	0.28	0.21	0.30	0.34	0.32	0.00	0.57	0.38	0.74
	$k = 1$	0.45	0.37	0.27	0.40	0.42	0.42	0.00	0.68	0.58	0.58
	$k = 2$	0.53	0.40	0.28	0.40	0.42	0.44	0.00	0.66	0.78	0.38
Jain	$k = 0.5$	0.94	0.80	0.77	0.84	0.80	0.89	0.54	0.80	1.00	0.74
	$k = 1$	0.90	0.69	0.64	0.73	0.69	0.80	0.33	0.80	1.00	0.58
	$k = 2$	0.82	0.56	0.45	0.58	0.56	0.67	0.13	0.68	1.00	0.38
Liou-Wang	$k = 0$	0.40	0.40	0.40	0.20	0.35	0.35	0.10	0.70	–	–
	$k = 0.5$	0.63	0.55	0.50	0.45	0.53	0.55	0.20	0.80	–	–
	$k = 1$	0.85	0.70	0.60	0.70	0.70	0.75	0.30	0.90	–	–
Kim-Park	$k = 0$	0.22	0.22	0.22	0.31	0.42	0.43	–	–	–	–
	$k = 0.5$	0.50	0.40	0.33	0.52	0.55	0.61	–	–	–	–
	$k = 1$	0.78	0.58	0.44	0.73	0.69	0.80	–	–	–	–
Peneva-Popchev	$k = 0$	0.46	0.61	0.50	0.34	0.56	0.44	–	–	–	–
	$k = 0.5$	0.60	0.59	0.50	0.41	0.54	0.47	–	–	–	–
	$k = 1$	0.75	0.58	0.50	0.49	0.52	0.50	–	–	–	–

TABLE VI
THE GENERAL DECISION MATRIX

Alternatives	The State of Nature S			
	S_1	S_2	$\cdots$	S_m
A_1	c_{11}	c_{12}	$\cdots$	c_{1m}
A_2	c_{21}	c_{22}	$\cdots$	c_{2m}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_n	c_{n1}	c_{n2}	$\cdots$	c_{nm}

where $\mathbb{P}(c_{ij} \succeq T)$ is a formal notation indicating the *probability of meeting the target* of value c_{ij}.

At this juncture, by using Yager's method of converting a possibility distribution into an associated probability distribution via the simple normalization as mentioned above, we have a direct way to define $\mathbb{P}(c_{ij} \succeq T)$ as the cumulative distribution function (cdf)

$$\mathbb{P}(c_{ij} \succeq T) \triangleq P_C(c_{ij} \succeq T) = \int_{c_{\min}}^{c_{ij}} P_T(t)dt \tag{28}$$

where

$$P_T(t) = \frac{\mu_T(t)}{\int_{c_{\min}}^{c_{\max}} \mu_T(t)dt}.$$

It should be noted that this definition of $\mathbb{P}(c_{ij} \succeq T)$ is also formally used but without a probabilistic interpretation, for the SF $S(T < c_{ij})$ in [31] for the comparison between a fuzzy number T with a crisp number c_{ij}.

On the other hand, based on the discussion presented in Section III, we can also define

$$\mathbb{P}(c_{ij} \succeq T) \triangleq P(c_{ij} \succeq T) = \int_0^1 P(c_{ij} \succeq T_\alpha)d\alpha \tag{29}$$

and call this the probabilistic comparison function (pcf). Note that in the case of $T = [c_{\min}, c_{\max}]$, we have $T_\alpha = T$ for all $\alpha \in (0, 1]$, which immediately implies

$$P(c_{ij} \succeq T) = \frac{c_{ij} - c_{\min}}{c_{\max} - c_{\min}}.$$

Thus the value function (27) for a fuzzy target with $\mathbb{P}(c_{ij} \succeq T)$ defined by (29) is also an extension of the value function (26) for an interval target.

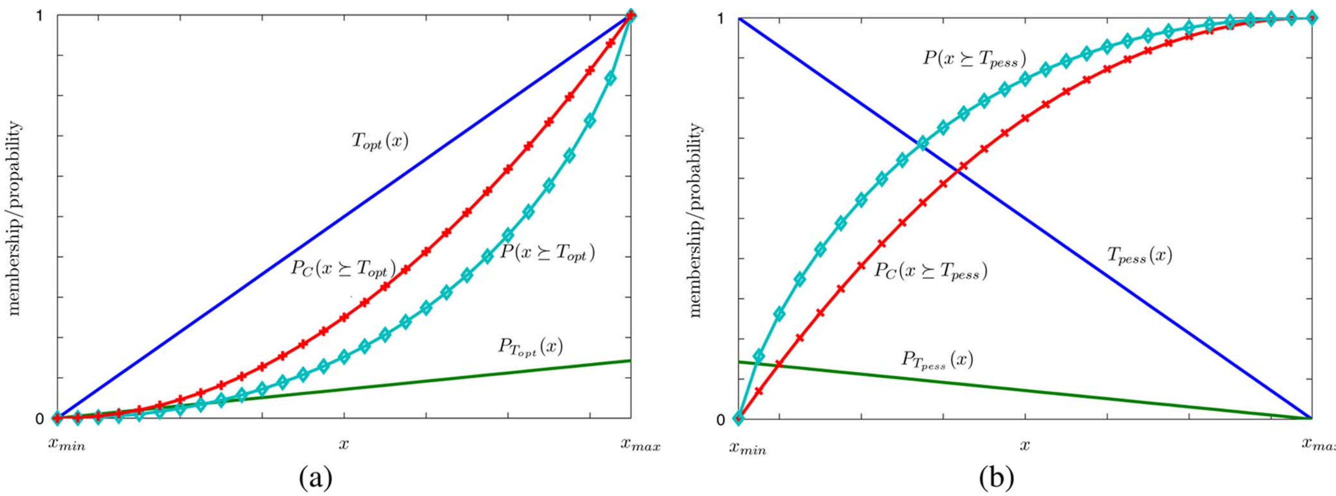


Fig. 5. Cumulative distribution versus proposed comparison probability: optimistic and pessimistic cases.

Importantly, note here that in the utility-based language of decision theory, the probability $\mathbb{P}(c_{ij} \succeq T)$ could be considered as the formulation of a utility function $U(c_{ij})$ and then (27) turns out to be an expected utility model. A formal connection between the utility-based approach and the target-based approach in decision analysis with uncertainty has been established and intensively discussed in, e.g., [4]–[6], [9], [10], and [32]. In particular, see Castagnoli and LiCalzi [9] for the target-based interpretation of Von Neumann and Morgenstern's expected utility model [46] and Bordley and LiCalzi [4] for the target-based interpretation of Savage's expected utility model [42]. Here we have also been showing that the procedure suggested in Yager [53] and that proposed in Section III both can be used to bring fuzzy targets within the reach of the target-based decision model.

Let us now consider three prototypical fuzzy targets. The first is called the *optimistic target*. This target would be set by a DM who has an aspiration towards the maximal payoff. Formally, the optimistic fuzzy target, denoted by T_{opt}, is defined as follows:

$$T_{\text{opt}}(x) = \begin{cases} \frac{x - c_{\min}}{c_{\max} - c_{\min}}, & \text{if } c_{\min} \leq x \leq c_{\max} \\ 0, & \text{otherwise.} \end{cases}$$

Fig. 5(a) graphically depicts the membership function $T_{\text{opt}}(x)$, the associated probability distribution $P_{T_{\text{opt}}}(x)$, the cdf $P_C(x \succeq T_{\text{opt}})$, and the pcf $P(x \succeq T_{\text{opt}})$ corresponding to this target. The second target is called the *pessimistic target*. This target is characterized by a DM who believes bad things may happen and has a conservative assessment of the target, which correspond to ascribing high possibility to the uncertain target being a low payoff. The membership function of this target is defined by

$$T_{\text{pess}}(x) = \begin{cases} \frac{c_{\max} - x}{c_{\max} - c_{\min}}, & \text{if } c_{\min} \leq x \leq c_{\max} \\ 0, & \text{otherwise.} \end{cases}$$

The portraits of related functions corresponding to the pessimistic target are shown in Fig. 5(b). Consider now the third target linguistically represented as "*about* c_0" whose membership function is defined by

$$T_{\tilde{c_0}}(x) = \begin{cases} \frac{x - c_{\min}}{c_0 - c_{\min}}, & c_{\min} \leq x \leq c_0 \\ \frac{c_{\max} - x}{c_{\max} - c_0}, & c_0 \leq x \leq c_{\max} \\ 0, & \text{otherwise} \end{cases}$$

where $c_{\min} < c_0 < c_{\max}$. This fuzzy target characterizes the situation at which the DM establishes a modal value c_0 as the most likely target and assesses the possibilistic uncertain target as distributed around it. We call this target the *unimodal*. Fig. 6 graphically illustrates this situation.

Looking at Figs. 5 and 6, we see that the portraits of the cdf $P_C(x \succeq T)$ and the pcf $P(x \succeq T)$ have similar shapes for each corresponding target. However, the behavior of the pcf $P(x \succeq T)$ is steeper towards the modal value of the corresponding targets than that of the cdf $P_C(x \succeq T)$. This practically implies that the value function $v(\cdot)$ defined with the pcf $P(x \succeq T)$ reflects a stronger decision attitude towards the target than that defined with the cdf $P_C(x \succeq T)$ as shown in the example below.

As we have seen from Fig. 5(a), the optimistic target T_{opt} leads to the convex pcf $P_C(x \succeq T_{\text{opt}})$, which is equivalent to a convex utility function and, therefore, exhibits a risk-seeking behavior. This is because, having an aspiration towards the maximal payoff, the DM always feels loss over the whole domain except the maximum, which would produce more risk-seeking behavior globally. By contrast, Fig. 5(b) shows that the pessimistic target induces the concave pcf $P_C(x \succeq T_{\text{opt}})$ and thus equivalently corresponds to global risk-aversion behavior. More interestingly, as we see from Fig. 6, the unimodal target induces the S-shape pcf $P_C(x \succeq T_{\tilde{c}_0})$ that is equivalent to the S-shape utility function of Kahneman and Tversky's prospect theory [26], according to which people tend to be risk averse over gains and risk seeking over losses. In the fuzzy target-based language, as the DM assesses his uncertain target as distributed around the modal value, he feels loss (respectively, gain) over payoff values that are coded as negative (respectively, positive) changes with respect to the modal value. This would lead to the behavior consistent with that described in the prospect theory.

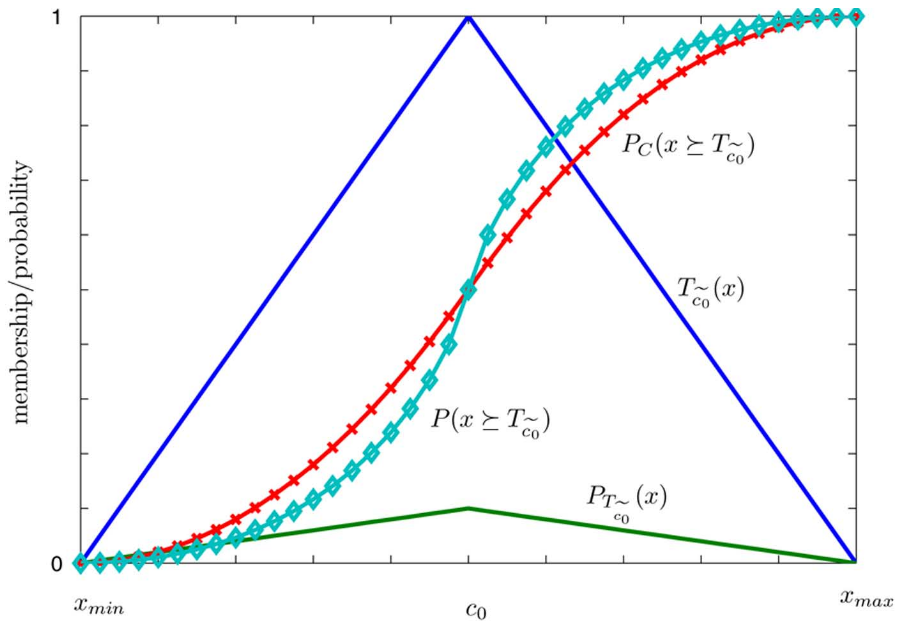


Fig. 6. Cumulative distribution versus proposed comparison probability: unimodal case.

TABLE VII
THE PAYOFF MATRIX

Acts	States			
	1	2	3	4
A_1	400	320	540	600
A_2	250	350	700	550
A_3	600	280	150	400

Let us consider the following example from Samson [41] to illustrate the point discussed above.

Example 4: In this example, payoffs are shown in thousands of dollars for a problem with three acts and four states as described in Table VII. A proper prior over the four possible states of $p_1 = 0.2, p_2 = 0.4, p_3 = 0.3, p_4 = 0.1$ is also assumed [41].

Table VIII shows the computational results of two value functions with different fuzzy targets for acts, where

$$v_1(A_i) = \sum_{j=1}^{m} p_j P_C(c_{ij} \succeq T)$$

and

$$v_2(A_i) = \sum_{j=1}^{m} p_j P(c_{ij} \succeq T).$$

From the result shown in Table VIII, we see that both value functions $v_1(\cdot)$ and $v_2(\cdot)$ suggest almost the same solution for the selection problem. That is, the act A_2 is the preferred choice according to a DM who has a neutral (equivalently, who abides by the expected value) or optimistic-oriented behavior about targets, a DM having pessimistic-oriented behavior about targets selects A_1 as his preferred choice. Especially, in the case of symmetrical unimodal target $4\tilde{2}5$, the acts A_1 and A_2 are almost indifferent to a DM who use $v_1(\cdot)$, while A_1 slightly dominates A_2 if using $v_2(\cdot)$. In addition, though the act A_3 is not selected in all cases, its value is much improved with respect to a pessimistic-oriented decision maker. However, the computational results of these two functions are different except, obviously, for the case of the neutral target. Especially, it is of interest to see that the spread of the difference of the value function $v_2(\cdot)$ between opposite-oriented targets is much larger than that of the value function $v_1(\cdot)$. This illustrates that the target-based decision model using the pcf $P(x \succeq T)$ reflects a stronger decision attitude towards the target than that using the cdf $P_C(x \succeq T)$.

TABLE VIII
THE TARGET-BASED VALUE MATRIX

Targets	The Value Functions					
	$v_1(A_1)$	$v_1(A_2)$	$v_1(A_3)$	$v_2(A_1)$	$v_2(A_2)$	$v_2(A_3)$
Neutral	0.51	**0.55**	0.30	0.51	**0.55**	0.30
Optimist	0.3	**0.41**	0.18	0.20	**0.37**	0.13
Pessimist	**0.72**	0.7	0.43	**0.82**	0.79	0.51
$\widetilde{300}$	**0.62**	0.59	0.33	**0.69**	0.63	0.33
$\widetilde{425}$	**0.50**	**0.50**	0.27	**0.48**	0.47	0.25
$\widetilde{550}$	0.40	**0.45**	0.23	0.35	**0.42**	0.22

VI. APPLICATION TO FUZZY DECISION ANALYSIS

A. Target-Based Decision Procedure

As discussed above, the fuzzy target-based method of uncertain decision making is formally equivalent to a procedure that, once having designed a target T, consists of the following two steps.

1) For each alternative A_i and state S_j, we define

$$p_{ij} = P(c_{ij} \succeq T)$$

TABLE IX
THE DERIVED DECISION MATRIX

Alternatives	The State of Nature S			
	S_1	S_2	$\cdots$	S_m
A_1	p_{11}	p_{12}	$\cdots$	p_{1m}
A_2	p_{21}	p_{22}	$\cdots$	p_{2m}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_n	p_{n1}	p_{n2}	$\cdots$	p_{nm}

and then form a "probability of meeting the target" table described in Table IX from the payoff table (i.e., Table VI).

2) Define the value function as the expected probability of meeting the target

$$v^{\dagger}(A_i) = \sum_{j=1}^{m} p_{ij} p_j. \tag{30}$$

We now consider the problem of decision making under uncertainty where payoffs may be given imprecisely. Let us turn back to the general decision matrix shown in Table VI, where c_{ij} can be a crisp number, an interval value, or a fuzzy number. Clearly in this case, we have an inhomogeneous decision matrix, and traditional methods cannot be applied directly. One of the methods to deal with this decision problem is to use fuzzy set based techniques with help of the extension principle and many procedures of ranking fuzzy numbers developed in the literature. In the following, we provide a fuzzy target-based procedure for solving this problem.

First, using the preceding mechanism, once having assessed a fuzzy target T, we need to transform the payoff table into one of the probabilities of meeting the target. For each alternative A_i and state S_j, the probability of payoff value c_{ij} meeting the target is defined by

$$p_{ij} = P(c_{ij} \succeq T).$$

If c_{ij} is a crisp number or interval, as previously discussed, we have

$$p_{ij} = \int_0^1 P(c_{ij} \succeq T_\alpha) d\alpha.$$

If c_{ij} is a fuzzy number, we get

$$p_{ij} = \int_0^1 P\left(c_{ij_\alpha} \succeq T_\alpha\right) d\alpha.$$

As such, we have transformed an inhomogeneous decision matrix into the derived decision matrix described by Table IX, where each element p_{ij} of the derived decision matrix can be uniformly interpreted as the probability of payoff c_{ij} meeting the target T. From this derived decision matrix, we can then use the value function (30) for ranking alternatives and making decisions. It is worth emphasizing that as an important characteristic of this target-based approach, it allows for including the DM's attitude, which is expressed in assessing his target, into the formulation of decision functions. Consequently, different attitudes about the target may lead to different results of the selection.

Note that in the fuzzy set method [38], we first apply the extension principle to obtain the fuzzy expected payoff for each alternative and then utilize either a defuzzification method or a ranking procedure for fuzzy numbers for the purpose of making the decision. Therefore, we may also get different results if different methods of ranking fuzzy numbers or defuzzification are used. However, this difference of results caused by using different ranking methods does not reflect the influence of the DM's attitude. Furthermore, a bunch of methods for ranking fuzzy numbers developed in the literature may also make it difficult for people choosing the most suitable method for each particular problem.

B. A Numerical Example

For illustration, let us consider the following application example adapted from [38].

LuxElectro is a manufacturer of electroutensils, and currently the market demand for its products is higher than the output. Therefore, the management is confronted with the problem of making a decision on possible expansion of the production capacity. Possible alternatives for the selection are as following:

- A_1 enlargement of the actual manufacturing establishment with an increase in capacity of 25%;
- A_2 construction of a new plant with an increase in total capacity of 50%;
- A_3 construction of a new plant with an increase in total capacity of 100%;
- A_4 renunciation of an enlargement of the capacity, the status quo.

The profit earned with the different alternatives depends upon the demand, which is not known with certainty. Due to the amount of information, the management estimates three states of nature corresponding to high, average, and low demand with associated prior probabilities of 0.3, 0.5 and 0.2, respectively. Then the prior matrix of fuzzy profits $\tilde{U}_{ij}$ (measured in millions of euros) is given in Table X, where fuzzy profits are represented parametrically by triangular and trapezoidal fuzzy numbers.

Using the extension principle in fuzzy set theory, we obtain the expected profits of alternatives as shown in Table XI, where risk neutrality is assumed. Then to make a decision, one can apply one of the ranking methods developed in the literature on these fuzzy profits. Looking at the membership functions of the expected profits depicted in Fig. 7, we can intuitively see that the alternatives A_4 and A_1 are much worse than the alternatives A_3 and A_2. However, it is not so easy to say which alternative is dominated by the other among these two better alternatives. Here, if using, for example, the centroid of fuzzy numbers as the ranking criterion, we get the ranking order as $A_2 \succ A_3 \succ A_1 \succ A_4$.

To apply the target-based procedure suggested above for solving this problem, according to the information given by this problem, we define the domain of profits as $D = [-90, 230]$. Assume, for instance, that a fuzzy optimistic target T_{opt} has been estimated based upon the optimistic attitude of the management, where

$$T_{\text{opt}}(x) = \frac{x + 90}{320}.$$

TABLE X
FUZZY PROFIT MATRIX $\tilde{U}_{ij} = \tilde{U}(A_i, S_j)$

Alternatives	States		
	$S_1 : 0.3$	$S_2 : 0.5$	$S_3 : 0.2$
A_1	$(80; 90; 100; 110)$	$(75; 85; 90; 100)$	$(50; 60; 70)$
A_2	$(135; 145; 150; 165)$	$(120; 130; 140)$	$(-40; -30; -20)$
A_3	$(170; 190; 210; 230)$	$(100; 110; 125)$	$(-90; -80; -70; -60)$
A_4	70	70	70

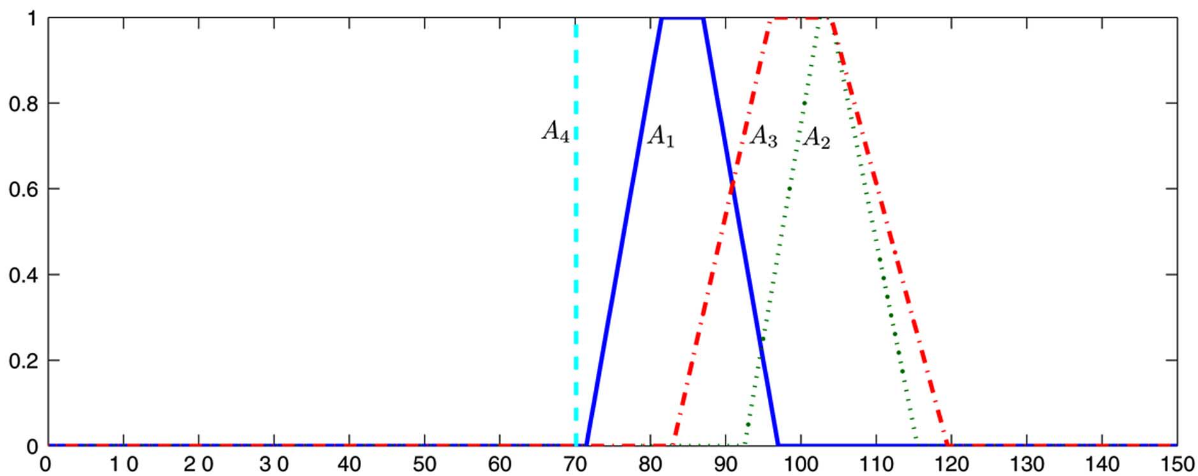


Fig. 7. Membership functions of expected profits.

TABLE XI
EXPECTED FUZZY PROFITS VIA EXTENSION PRINCIPLE

Alternatives	Expected Fuzzy Profit	Centroid Value
A_1	$(71.5; 81.5; 87; 97)$	84.25
A_2	$(92.5; 102.5; 104; 115.5)$	103.73
A_3	$(83; 96; 104; 119.5)$	100.76
A_4	70	70

TABLE XII
DERIVED DECISION MATRIX $p_{ij} = P(\tilde{U}_{ij} \succeq T_{\rm opt})$

Alternatives	States		
	S_1	S_2	S_3
A_1	0.2141	0.1943	0.1326
A_2	0.3984	0.3237	0.0188
A_3	0.6863	0.2619	0.0015
A_4	0.1532	0.1532	0.1532

TABLE XIII
DERIVED DECISION MATRIX $p_{ij} = P(\tilde{U}_{ij} \succeq T_{\rm neut})$

Alternatives	States		
	S_1	S_2	S_3
A_1	0.5781	0.5547	0.4688
A_2	0.7461	0.6875	0.1875
A_3	0.9063	0.6289	0.0469
A_4	0.5	0.5	0.5

TABLE XIV
DERIVED DECISION MATRIX $p_{ij} = P(\tilde{U}_{ij} \succeq T_{\rm pess})$

Alternatives	States		
	S_1	S_2	S_3
A_1	0.8948	0.8816	0.8241
A_2	0.9653	0.9451	0.5017
A_3	0.9914	0.9217	0.1898
A_4	0.8468	0.8468	0.8468

Then with this optimistic target, using the above procedure we obtain the derived decision matrix as shown in Table XII.

In the same way, we also obtain the derived decision matrices corresponding to neutral and pessimistic targets, denoted, respectively, by $T_{\rm neut}$ and $T_{\rm pess}$, as shown in Tables XIII and XIV. After assessing a target and obtaining the derived decision matrix accordingly, the value function (30) is then applied for making the decision. Table XV shows the results of the value function for three above targets and the corresponding ranking orders of alternatives.

From Table XV, we see that the result reflects very well the behavior of the DM which is expressed in assessing the target. In particular, the ranking order of alternatives corresponding to the neutral target is the same as that obtained by using the fuzzy expected profits with centroid-based ranking criterion, where the

TABLE XV
THE RANKING RESULT USING DIFFERENT TARGETS

Targets	Value Function				Ranking Order
	A_1	A_2	A_3	A_4	
Optimistic	0.1879	0.2851	0.3371	0.1532	$A_3 \succ A_2 \succ A_1 \succ A_4$
Pessimistic	0.8741	0.8625	0.7962	0.8468	$A_1 \succ A_2 \succ A_4 \succ A_3$
Neutral	0.5445	0.6051	0.5957	0.50	$A_2 \succ A_3 \succ A_1 \succ A_4$

risk neutrality is assumed. As shown in Section V, the neutral target T_{neut} induces a linear utility function $U(x) = P(x \succeq T_{\text{neut}})$, which is also equivalent to risk neutrality behavior. For the case of optimistic target T_{opt}, it provides a convex utility function $U(x) = P(x \succeq T_{\text{opt}})$ [refer to Fig. 5(a)] that is equivalent to a risk-seeking behavior. In this case, the DM wishes to have profit as big as possible, accepting a risk that if the desirable state will not occur, he may get a big loss. This attitude leads to the selection of alternative A_3 that has the biggest profit in case of a high demand occurs. In contrast, the pessimistic target T_{pess} yields a concave utility function $U(x) = P(x \succeq T_{\text{pess}})$, which corresponds to a risk-aversion behavior [refer to Fig. 5(b)]. In this case, we see that A_1 is selected and, in addition, the alternative A_3 becomes the worst. This reflects the situation that the DM is somewhat looking for certainty of gaining profit. It should be noted here that we have defined membership degrees for T_{pess} linearly decrease over the profit domain, which exhibits a neutral-pessimistic attitude, and consequently in this case the DM is not risk averse enough to rank A_4 over A_2. However, other types of membership function can be used to express a more or less pessimistic attitude depending on the behavior of the DM.

VII. CONCLUSION

The issue of comparison and ranking of fuzzy numbers plays an important role in many applications of fuzzy set theory to decision analysis. Though there are many methods proposed for ranking fuzzy numbers, many of them are difficult to understand and may produce counterintuitive results, as pointed out in the literature. In this paper, we have proposed a new comparison relation on fuzzy numbers based on the alpha-cut representation and comparison probabilities of interval values. Inspired by the target-based ranking procedure in decision theory under uncertainty, we applied the proposed comparison relation to the issue of ranking fuzzy numbers using fuzzy targets in terms of target-based evaluations. This also suggested to us to provide a better understanding of Lee-Kwang and Lee's method of ranking fuzzy numbers with a probability-based interpretation of the SF. More interestingly, we have applied the proposed comparison relation to bring fuzzy targets within the reach of DMUU paradigm on which an interesting link between different attitudes about target and different risk attitudes in terms of utility functions has been established. Furthermore, it has been also shown that the fuzzy target-based decision model provides a unified way for fuzzy decision making with uncertainty.

It is also worth noting that although the proposed ranking method also reduces the comparison of fuzzy numbers into that of real numbers, it differs from defuzzification-based ranking methods in that single comparison values in the proposed method are associated with a probabilistic semantics in terms of target/benchmark-based evaluations. However, this consequently restricts the application scope of the proposed ranking method to the paradigm of target-oriented decision analysis as well.

By the consideration of a fuzzy target-based approach to DMUU in this paper, we think that it suggests an interesting perspective for further studies on various different decision problems. The first problem of constructing target-based decision functions for attitudinal decision making [50] as well as for intelligent decision making with fuzzy modelling techniques [51], [54] is worth study. Also, it would be interesting to study whether and how a fuzzy target-based approach can be applied to developing decision models for multiple-attribute decision making as well as group decision making.

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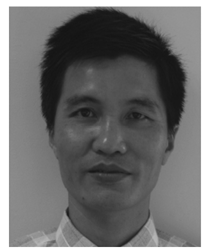

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