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Author(s)	Huynh, Van Nam; Murai, Tetsuya; Nakamori, Yoshiteru
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An Algebraic Foundation for Linguistic Reasoning

Van Nam Huynh [C]
School of Knowledge Science
Japan Advanced Institute of Science and Technology
1-1 Asahidai, Nomi, Ishikawa 923-1292, Japan
Email: huynh@jaist.ac.jp

Yoshiteru Nakamori
School of Knowledge Science
Japan Advanced Institute of Science and Technology
1-1 Asahidai, Nomi, Ishikawa 923-1292, Japan

Tetsuya Murai
Graduate School of Information Science and Technology
Hokkaido University
Kita 14, Nishi 9, Kita-ku, Sapporo 060-0814, Japan
murahiko@main.eng.hokudai.ac.jp

Abstract. It is well known that algebraization has been successfully applied to classical and non-classical logics (Rasiowa and Sikorski, 1968). Following this direction, an ordered-based approach to the problem of finding out a tool to describe algebraic semantics of Zadeh's fuzzy logic has been introduced and developed by Nguyen Cat-Ho and colleagues during the last decades. In this line of research, RH_algebra has been introduced in [20] as a unified algebraic approach to the natural structure of linguistic domains of linguistic variables. It was shown that every RH_algebra of a linguistic variable with a chain of the primary terms is a distributive lattice. In this paper we will examine algebraic structures of RH_algebras corresponding to linguistic domains having exactly two distinct primary terms, one being an antonym of the other, called symmetrical RH_algebras. Computational results for the relatively pseudo-complement operation in these algebras will be given.

Keywords: Linguistic reasoning, fuzzy logic, linguistic variable, hedge algebra, RH-algebra, distributive lattice.

1. Introduction

The notion of linguistic variables was introduced and investigated by Zadeh in 1975 [31], and it has been playing an important role in investigations on fuzzy logics and approximate reasoning [2, 30, 32].

[C]Corresponding author

Formally, a linguistic variable is characterized by a 5-tuple $(X, T(X), U, R, M)$, where X is the name of variable such as age variable Age, truth variable $Truth$ etc., $T(X)$ denotes the **term-set** of X, that is, the set of linguistic values of the linguistic variable; U is a universe of discourse of the base variable; R is a syntactic rule for generating linguistic terms of $T(X)$, and M is a semantic rule assigning to each linguistic term a fuzzy set on U. Under such a formalization, the values of a linguistic variable are generated from primary terms (e.g. $young$ and old in the case of linguistic variable Age), by various hedges (e.g. $very$, *more or less*, etc.) and connectives (e.g. *AND, OR*).

In fuzzy set theoretic based investigations of linguistic variables (e.g. [31]), Zadeh has always emphasized two most important characteristics of linguistic variables. The first is the context-independent meaning of hedges and connectives, whereas the meaning of the primary terms is context-dependent. The second is the universality of their structure. That is most linguistic variables possess the same basic structure in the sense that their respective linguistic values have the same expressions except for the primary terms. Therefore, a set of linguistic hedges (or hedges, for short) under consideration may be applied to many different linguistic variables, where the meanings of hedges are interpreted by operators on fuzzy sets [12, 29, 30, 31]. From another point of view [21], these characteristics of linguistic variables and the meaning of hedges in natural language permit us to consider each domain of a linguistic variable as an algebraic structure called *hedge algebra*, say $AX = (X, G, H, \leq)$, where X is a set of values of a linguistic variable (regarded as a poset), G is the set of the primary terms of the linguistic variable and H is a set of unary operations representing linguistic hedges.

Mathematically, it is well known that algebraization has been applied to classical and non-classical logics. We know that every deductive system in classical or non-classical logic always determines an algebra of a certain class of universal algebras of the same category of the corresponding algebra of truth values (see, e.g., [23, 24]). By this means many characteristics of a logical system, which a deductive system bases on, can be determined by the algebraic structure of the set of truth values of the corresponding logic. Motivated by this direction, the notion of hedge algebra has been introduced by Nguyen Cat-Ho and W. Wechler [21] as an algebraic approach to the structure of linguistic domains (also called term-sets) of linguistic variables. Since then the theory of hedge algebras and its applications have been intensively investigated, e.g. [15, 16, 17, 18, 19, 22]. On the other hand, since the axiomatization of hedge algebras is constructed based on semantic properties of linguistic hedges, they have many properties that reflect interesting semantic characteristics of linguistic terms in natural language. This makes them to be even useful in the fuzzy set theoretic based construction of linguistic domains in fuzzy logic and approximate reasoning [5, 6, 10].

In [20], a unified algebraic approach to modeling of linguistic domains of linguistic variables has been established. Particularly, authors have introduced the notion of refined hedge algebra (RH_algebra, for short) that gives a unification of notions of hedge algebra as well as its extensions. Many fundamental properties of RH_algebras have been also examined. In this paper we will examine algebraic structures of RH_algebras that correspond to linguistic domains of linguistic variables having exactly two distinct primary terms, one being an antonym of the other, called symmetrical RH_algebras. It is shown that these RH_algebras describe an algebraically rich structure of linguistic domains of most linguistic variables in practice. This may consequently permit us to believe in pursuing the development of the axiomatic approach to linguistic-valued logics for linguistically approximate reasoning.

Further, it should be also noticed that the research on fuzzy relational equations is one of the most active and fruitful research topics in fuzzy set theory both from mathematical point of view as well as system modeling point of view. One of the fundamental forms of fuzzy relational equations is sup-T

equations [3]. Solving these equations in [0,1] or in complete Brouwerian lattices has been investigated by De Baets, Kerre [3, 4], and Wang [27, 28] among others. In solving sup-T equations in a bounded poset, the computation of the relatively pseudo-complement operation is essential. Under such an observation, in this paper we also investigate some computational results for the relatively pseudo-complement operation in finite symmetrical RH_algebras, which would play an important role in developing methods of linguistic reasoning.

To proceed, it is necessary to recall the basic notions from lattice theory and the algebraic approach to modeling linguistic variables in terms of hedge algebra and its extensions. These are undertaken in Section 2 and followed in Section 3 by an introduction of RH_algebras and their main properties. Section 4 briefly introduces symmetrical RH_algebra which models linguistic domains having exactly two distinct primary terms with one being an antonym of the other. Section 5 then examines algebraic versions of finitely symmetrical RH_algebras and Section 6 provides computational results for the relatively pseudo-complement operation in these algebras. Finally, some concluding remarks are presented in Section 7.

2. Preliminaries

2.1. Basic notions from lattice theory

In this subsection, we briefly recall some necessary notions and notations from lattice theory used in the paper (see e.g. in [1]).

Definition 2.1. Let $\mathcal{P}$ be a partially ordered set (poset, for short). An element a is said to cover an element b in a poset $\mathcal{P}$, if $a > b$ and there is no $x \in \mathcal{P}$ such that $a > x > b$.

Denote by $l(\mathcal{P})$ the length of a poset $\mathcal{P}$. For a given poset $\mathcal{P}$ of finite length with the least element denoted by $\mathbf{0}$, the height of an element $x \in \mathcal{P}$ is, by definition, the least upper bound of the length of the chains $\mathbf{0} = x_0 < x_1 < \ldots < x_n = x$ between $\mathbf{0}$ and x, and it is denoted by $height(x)$. If $\mathcal{P}$ has the greatest element, denoted by $\mathbf{1}$, then clearly $height(\mathbf{1}) = l(\mathcal{P})$. Clearly also that $height(x) = 1$ iff x covers $\mathbf{0}$.

Definition 2.2. A poset $\mathcal{P}$ is said to be graded if there exists a function g from $\mathcal{P}$ into the set $\mathbb{Z}$ of all integers with the natural ordering such that :

$G1$. $x > y$ implies $g(x) > g(y)$.

$G2$. If x covers y then $g(x) = g(y) + 1$.

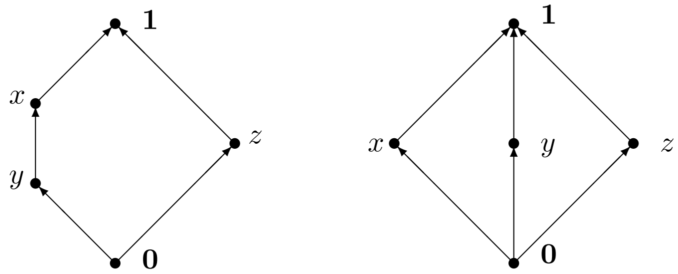
Such a function g is called a graded function of $\mathcal{P}$. It is known [1] that any modular lattice of finite length is graded by its height function $height(x)$.

Let $\mathcal{L}$ be a modular lattice of finite length, we can define a relation R on $\mathcal{L}$ as follows:

$$\forall x, y \in \mathcal{L}, (x, y) \in R \text{ iff } height(x) = height(y).$$

It is easily shown that R is an equivalence relation and then we have

$$\mathcal{L} = \bigcup_{i=0}^{l(\mathcal{L})} L_i,$$



a) N_5 $\qquad$ $b)$ M_5

Figure 1. Lattices N_5 and M_5

where $L_i = \{x \in \mathcal{L} : height(x) = i\}$, for $i = 0, \dots, l(\mathcal{L})$, are the equivalence classes of the relation R. Clearly, $L_0 = \{\mathbf{0}\}$ and $L_{l(\mathcal{L})} = \{\mathbf{1}\}$, where $\mathbf{0}$ and $\mathbf{1}$ are the zero-element and the unit-element in $\mathcal{L}$, respectively.

Motivated by a practical property of linguistic hedges, we introduce the following condition:

(C_0) *For any* $x \in L_i, y \in L_j$ *and* $i \neq j$, *we have either* $x > y$ *or* $x < y$.

As an illustration, we can easily see that in Figure 1 the lattice M_5 satisfies the condition (C_0) while the lattice N_5 does not. Graded classes of M_5 are $L_1 = \{\mathbf{0}\}, L_2 = \{x, y, z\}$ and $L_3 = \{\mathbf{1}\}$.

It is not difficult to see that the following holds.

Proposition 2.1. Let $\mathcal{L}$ be a modular lattice of finite length satisfying (C_0). Then the following statement holds:
If $|L_i| > 1$ for an index $i \in \{1, \dots, l(\mathcal{L}) - 1\}$ then $|L_{i-1}| = |L_{i+1}| = 1$, where $|A|$ denotes the cardinality of the set A. Moreover, if we denote $e(L_{i+1})$ and $e(L_{i-1})$ the single element of L_{i+1} and L_{i-1}, respectively, then $e(L_{i+1}) = \vee_{x \in L_i} x$ and $e(L_{i-1}) = \wedge_{x \in L_i} x$, where $\vee$ and $\wedge$ stand for the join and meet in $\mathcal{L}$, respectively.

Recall that a lattice $\mathcal{L}$ is called Brouwerian if for any $x, y \in \mathcal{L}$, the set $\{z \in \mathcal{L} | x \wedge z \leq y\}$ has its greatest element denoted by $x \rightarrow y$. It is known in [1] that in a finite lattice, the distributive character is equivalent to Brouwerian character. The operation $\rightarrow$ in a Brouwerian lattice is called *relatively pseudo-complement operation* and, $x \rightarrow y$ the pseudo-complement of x relative to y. By definition, we have the following proposition.

Proposition 2.2. Let $\mathcal{L}$ be a Brouwerian lattice. For any $x, y \in \mathcal{L}$ we have

(1) $x \leq y$ if and only if $x \rightarrow y = \mathbf{1}$,

(2) $x \rightarrow y \geq y$,

(3) if $x > y$ and $x \rightarrow y > y$ then x and $x \rightarrow y$ are incomparable and $x \wedge (x \rightarrow y) = y$,

(4) if x and y are incomparable then so are x and $x \rightarrow y$, and $x \wedge (x \rightarrow y) < y$.

2.2. Term-sets as abstract algebras

In the algebraic approach to modeling linguistic variables, each linguistic domain is considered as an abstract algebra, say $AX = (X, G, \mathcal{O}, \leq)$, where X is a underlying set, G is the set of generators (or constants, i.e. zero-argument operations), $\mathcal{O}$ is a set of one-argument operations and $\leq$ is a partially ordering relation over X. Under such an abstraction, X is intended to be interpreted as a term-set, G as a set of primary terms and special constants, $\mathcal{O}$ as a set of hedges or modifiers and $\leq$ as semantically ordering relation. Note that the set G may contain special constants such as $\mathbf{1}, \mathbf{0}$ and $\mathbf{W}$ which are different from the primary terms and, for example, understood as "*absolutely true*", "*absolutely false*" and the "*neutral*", respectively. These constants can be characterized by the conditions that $hc = c$ for all $h \in \mathcal{O}, c \in \{\mathbf{1}, \mathbf{W}, \mathbf{0}\}$ and $\mathbf{1} > \mathbf{W} > \mathbf{0}$. Since every h in $\mathcal{O}$ can be considered as a mapping from X into X and several operations can be used in concatenation, for convenience the image of an element x in X under h will be denoted by hx instead of $h(x)$. And therefore the result of the applying operations $h_1, h_2, \ldots h_n \in \mathcal{O}$ to an element $x \in X$ in concatenation can be written as $h_n \ldots h_1 x$.

Stimulating the semantic properties of terms-sets structure, the following assumptions are assumed in the approach.

1. Each element $h \in \mathcal{O}$ is *an ordering operation*, i.e. the statement $(\forall x \in X)$[either $hx \geq x$ or $hx \leq x$] holds for every h.

2. $\mathcal{O}$ is decomposed into two non-empty subsets $\mathcal{O}^+$ and $\mathcal{O}^-$ such that for any $h \in \mathcal{O}^+$ and $k \in \mathcal{O}^-$, h and k are *converse*.

3. Let I be the identity of X, i.e. $\forall x \in X, Ix = x$. The sets $\mathcal{O}^+ + I$ and $\mathcal{O}^- + I$ are lattices with unit-elements V and L, respectively, and zero-element I. Because $X, \mathcal{O}^+$ and $\mathcal{O}^-$ are disjoint, there is no confusion to assume for simplicity that the partially ordering relations on each sets $X, \mathcal{O}^+$ and $\mathcal{O}^-$ will be denoted by the same notation $\leq$.

Let $h, k \in \mathcal{O}$, h and k are said to be *converse* (or h is said to be converse to k and vice versa) if $\forall x \in X, x \leq hx$ iff $x \geq kx$. And if $(\forall x \in X)[x \leq hx$ iff $x \leq kx]$ holds, h and k are said to be *compatible*. For any $h, k \in \mathcal{O}, h$ is said to be *positive* (or *negative*, resp.) with respect to k if the statement $(x \in X)[$ either $kx \geq x$ implies $hkx \geq kx (hkx \leq kx,$ resp.) or $kx \leq x$ implies $hkx \leq kx (hkx \geq kx,$ resp.)] holds.

As discussed in [20], the partially ordering relations on each sets $X, \mathcal{O}^+$ and $\mathcal{O}^-$ have a close semantic relationship, which will be formulated in the following definition. In order to simplify notations and formulations, we use in the sequent superscript ^{c} to denote either the superscript $^+$ or $^-$.

Definition 2.3. Let $AX = (X, G, \mathcal{O}, \leq)$ be an arbitrary abstract algebra. As it is assumed above, the set $\mathcal{O}$ is decomposed into two disjoint subsets $\mathcal{O}^+$ and $\mathcal{O}^-$ such that $\mathcal{O}^+ + I$ and $\mathcal{O}^- + I$ are finite lattices with the zero-element I. Then, X and $\mathcal{O}$ are said to be *semantically consistent* if the following conditions hold:

(1) X is generated from the generators by means of hedges in $\mathcal{O}$, i.e. elements of X is of the form $h_n \ldots h_1 a$, for $h_i \in \mathcal{O}, i = 1, \ldots, n$, and $a \in G$.

(2) For any $h, k \in \mathcal{O}^c + I, h < k$ in $\mathcal{O}^c + I$ iff $(\forall x \in X)[(hx > x$ or $kx > x$ implies $hx < kx)$ and $(hx < x$ or $kx < x$ implies $hx > kx)]$. And, h, k are incomparable in $\mathcal{O}^c + I$ iff $(\forall x \in X)[hx \neq x$ or $kx \neq x$ implies that hx and kx are incomparable].

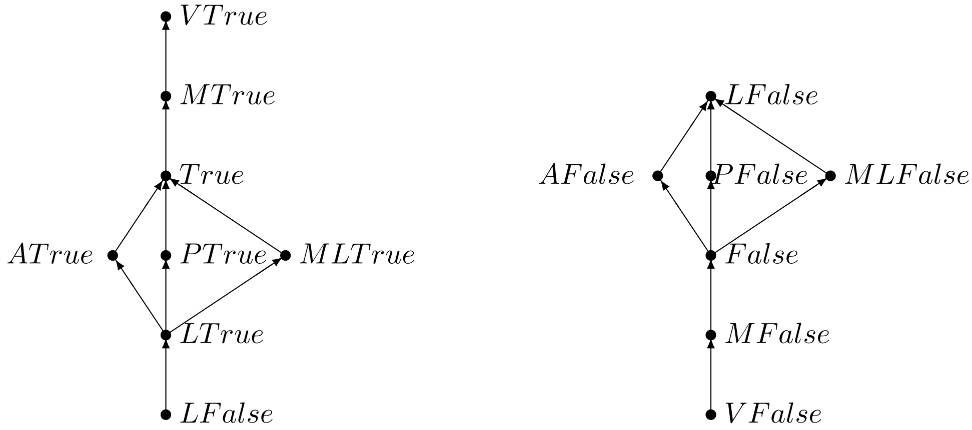


Figure 2. A poset of values of the linguistic variable $Truth$

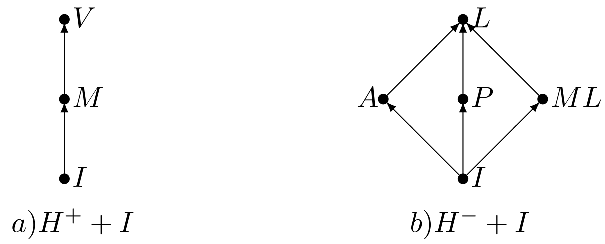


Figure 3. Lattices of hedges

Example 2.1. Let X be a poset of values of the linguistic variable $Truth$ as represented in Figure 2, where $H = \{V, M, A, P, ML, L\}$ is a set of linguistic hedges with V, M, A, P, ML, L standing for $very$, $more$, $approximately$, $possibly$, $more\ or\ less$, $less$ (or $little$). Intuitively, it can be seen that $H^+ = \{V, M\}$ and $H^- = \{L, A, P, ML\}$, $H^+ + I$ and $H^- + I$ are lattices given in Figure 3. Such an X can be considered as an abstract algebra $AX = (X, G, H, \leq)$, in which $G = \{True, False\}$, $\leq$ on X is the partially ordering relation represented as the graph given in Figure 2 and $\leq$ on $H^c + I$ is given in Figure 3. The result of applying any operation h to an element x can be understood as follows: $hTrue$ and $hFalse$ are defined to be the elements given in Figure 2; and $khx = hx$, for all $h, k \in H$ and $x \in X$. It can easily be seen that X and H are semantically consistent.

2.3. General construction of abstract algebras for term-sets

Let us consider a term-set X_0 and H be the set of all hedges occurring in X_0. In general, not all terms in X_0 can be written in an expression of the form $h_n \dots h_1 u$ of an abstract algebra considered above, where $h_n, \dots, h_1 \in H$, e.g. the term '*little app false OR little poss false*'. Formally, this term can be rewritten as $little(app \vee poss)false$, where $\vee$ is an operation on the set of hedges. The expression $(app \vee poss)$ can be regarded as a new, artificial hedge and hence this term can be considered as being expressed in

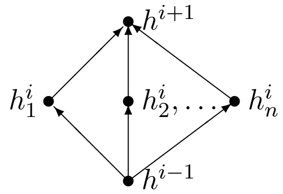


Figure 4. $H_{i-1}^+ < H_i^+ < H_{i+1}^+$

the above form. It suggested the authors in [20] to extend the sets H^c into distributive lattices, denoted by LH^c, respectively. Then, they have tried to show, as in the next section, that the abstract algebra $AX = (X, G, LH, \leq)$ will model the "structure" of X_0 and that '*little app false*' $\cup$ '*little poss false*' $=$ '*little*$(app \vee poss)$*false*', where $\cup$ is the join operation in the lattice AX. The extension of H^c is carried out as follows.

Let H be a set of linguistic hedges such that $H^+ + I$ and $H^- + I$ are finite lattices. We will denote by N^+ and N^- the lengths of $H^+ + I$ and $H^- + I$, respectively. Suppose that g^+ and g^- are the graded functions of $H^+ + I$ and $H^- + I$, respectively.

Unless stated otherwise, we shall always adopt in the sequel the assumption that $H^+ + I$ and $H^- + I$ are finite modular lattices satisfying the condition (C_0). From now on, V and L stand for the unit-operations in $H^+ + I$ and $H^- + I$, respectively. Hence, we have $g^+(V) = N^+, g^-(L) = N^-$ and

$$H^+ + I = \bigcup_{i=0}^{N^+} H_i^+, \text{ where } H_i^+ = \{h \in H^+ + I / g^+(h) = i\},$$

$$H^- + I = \bigcup_{i=0}^{N^-} H_i^-, \text{ where } H_i^- = \{h \in H^- + I / g^-(h) = i\}.$$

Now, we are going to construct the lattices, which can be seen as being "freely" generated from $H^+ + I$ and $H^- + I$.

Let us consider $H^+ + I$. Assume that for some index $i \in \{1, \ldots, N^+\}, |H_i^+| > 1$ and $H_i^+ = \{h_1^i, \ldots, h_n^i\}$. By Proposition 2.1, the sets $H_{i+1}^+ = \{h^{i+1}\}$ and $H_{i-1}^+ = \{h^{i-1}\}$ are single-element sets. For such an i, the ordering relationships between the elements of $H_{i-1}^+, H_i^+, H_{i+1}^+$ can be expressed as Figure 4. Note that there exists a natural ordering relation between graded classes H_i^+: $H_i^+ < H_j^+$ iff $i < j$; where $H_i^+ < H_j^+$ means that $h < k$ for every $h \in H_i^+$ and $k \in H_j^+$. By $LH_i^+ = (L(H_i^+), \vee, \wedge)$ we denote the free distributive lattice[1] generated from the incomparable elements $h_1^i, \ldots, h_n^i$ of H_i^+. Particularly, for an index i such that $|H_i^+| = 1$ we have $LH_i^+ = H_i^+$. Put $LH^+ = \bigcup_{i=1}^{N^+} LH_i^+$ and $LH^+ + I = LH^+ \cup \{I\} = \bigcup_{i=0}^{N^+} LH_i^+$. Then, $LH^+ + I$ becomes a distributive lattice under the ordering relation induced by the ordering relations on the lattices LH_i^+ and the one defined between classes LH_i^+ (that is we have $LH_i^+ \leq LH_j^+$, for any i, j such that $i \leq j$). The classes LH_i^+ are called also graded classes of LH^+, for convenience. Figure 5 shows a segment of the lattice $LH^+ + I$, where $|H_i^+| > 1$. By an analogous way, we can construct the lattice $LH^- + I$ generated from $H^- + I$. Here, there is no confusion, because H^+ and H^- are assumed to be disjoint and hence, so are LH^+ and LH^-, where $LH^+ = LH^+ + I \setminus \{I\}$ and $LH^- = LH^- + I \setminus \{I\}$. Thus, we have the following

[1]see Birkhoff [1]

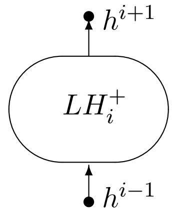


Figure 5. $LH^+_{i-1} < LH^+_i < LH^+_{i+1}$

Proposition 2.3. $(LH^+ + I, \wedge, \vee, I, V, \leq)$ and $(LH^- + I, \wedge, \vee, I, L, \leq)$ are finite distributive lattices with the unit-elements V and L, respectively, and the zero-element I.

Example 2.2. Let us consider a set of hedges as in Example 2.1. Clearly, $H^+ + I$ and $H^- + I$ are finite modular lattices and satisfy condition (C_0). By the way of constructing as above, the obtained distributive lattices $LH^+ + I$ and $LH^- + I$ generated from $H^+ + I$ and $H^- + I$, respectively, can be represented as in Figure 6.

Now, let us turn back to the previous consideration of the given term-set X_0 and the set H of hedges. Regard H as a set of unary operations and construct LH as above. Let $AX = (X, G, LH, \leq)$ be an abstract algebra satisfying the following conditions: (i) G is the set of the primary terms occurring in X_0 and the additional special constants $\mathbf{0}, \mathbf{W}$ and $\mathbf{1}$; (ii) $X = LH(G)$; (iii) $\leq$ is a partially ordering relation on X such that X and LH are semantically consistent. We treat it as an abstract algebra for the term set X_0.

Obviously, it is easy to find an abstract algebra AX fulfilling (i) and (ii). Furthermore, as shown in [20], there exists a partially ordering relation $\leq$ on X such that (iii) is satisfied. Therefore, given X_0 we can always define a corresponding abstract algebra AX for X_0. The sets X and X_0 are not identical, e.g. the composed term containing connectives AND, OR, NOT in X_0 like '*little app false OR little poss false*' do not occur in X. However, it has been shown in [20] that they are very similar.

From now on we always denote H the set of primary hedges and LH the set of composed hedges constructed as above. However, the elements in H or LH are called simply hedges.

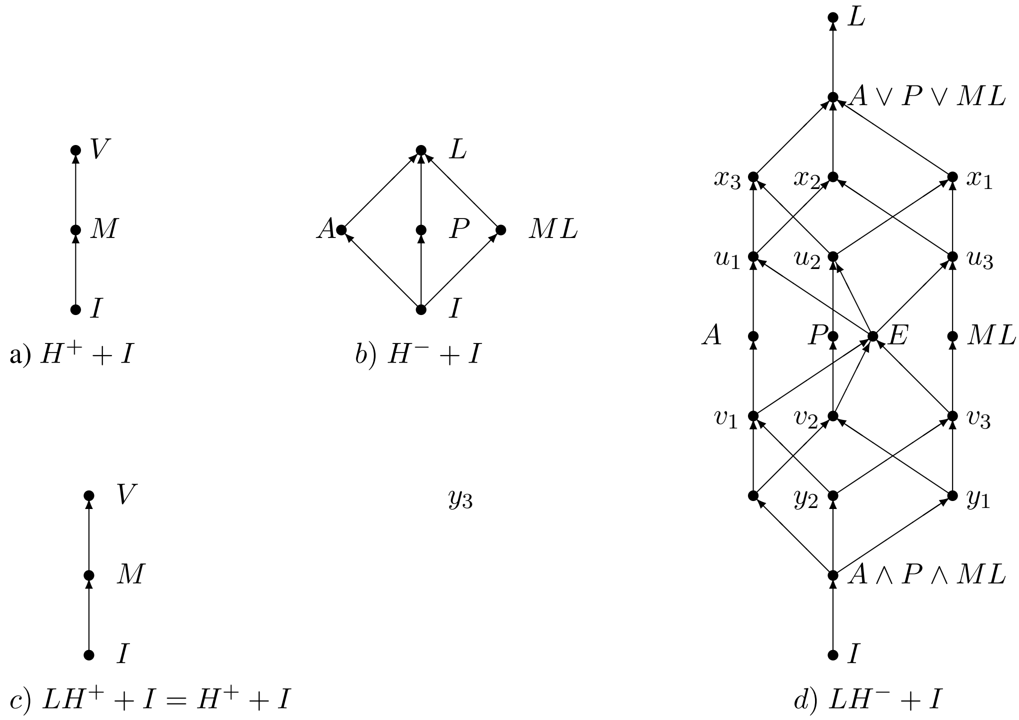
3. RH_algebras: Definition and Properties

In this section we will recall the axiomatization of RH_algebras and most important properties of these algebras which are essential to understand the results obtained in this paper. More detailed discussions, results as well as motivations of related notions could be referred to [20].

First we need the following notion.

Definition 3.1. Let $AX = (X, G, LH, \leq)$ be an arbitrary abstract algebra and V be the unit operation in $LH^+ + I$. The set H of primary hedges is said to have *PN-homogeneous property* provided that for any graded class H^c_i, if V is positive (or negative, resp.) w.r.t. a certain operation h in H^c_i, then V is also positive (or negative, resp.) w.r.t. any other ones in H^c_i.

Suppose that $LH^+ + I$ and $LH^- + I$ are distributive lattices, which are generated from $H^+ + I$ and $H^- + I$, respectively, as presented in the previous section.



where

$$
\begin{array}{lll}
x_1 = P \vee ML; & x_2 = ML \vee A; & x_3 = A \vee P; \\
u_1 = x_2 \wedge x_3; & u_2 = x_3 \wedge x_1; & u_3 = x_1 \wedge x_2; \\
y_1 = P \wedge ML; & y_2 = ML \wedge A; & y_3 = A \wedge P; \\
v_1 = y_2 \vee y_3; & v_2 = y_3 \vee y_1; & v_3 = y_1 \vee y_2;
\end{array}
$$

and $E = (A \vee P) \wedge (P \vee ML) \wedge (ML \vee A) = (A \wedge P) \vee (P \wedge ML) \vee (ML \wedge A)$.

Figure 6. Lattices of hedges $H^c + I$ and 'freely' generated lattices $LH^c + I$.

Let $I^+ = \{0, 1, \ldots, N^+\}, I^- = \{0, 1, \ldots, N^-\}$ and $SI^+ = \{i \in I^+ : |H_i^+| > 1\}$, $SI^- = \{i \in I^- : |H_i^-| > 1\}$. That is the set SI^c consists of the indexes i which are not *single-element* classes.

Recall that by ^{c} we mean either $^+$ or $^-$. For instance, given the term "LH_i^c for some $i \in SI^c$", the statement presents two instances to be obtained by substituting ^{c} in turn by $^+$ and $^-$. Let us denote by UOS the set of two unit-elements V and L of $LH^+ + I$ and $LH^- + I$, respectively. Further, we denote by $\mathbb{N}$ the set of all non-negative integers.

Set $LH = LH^+ \cup LH^-$ and $LH + I = LH^+ \cup LH^- \cup \{I\}$.

Consider an algebra $AX = (X, G, LH, \leq)$, where G is a set of constants or zero-argument operations, LH is a set of one-argument operations.

For every $x \in X, LH(x)$ called the term-set of x denotes the set of all elements generated from x by means of operations in LH, i.e. elements of $LH(x)$ are of the form $h_n \ldots h_1 x$, where $h_i \in LH, i = 1, \ldots, n$. More generally, for any $Y \subset X$ and $H' \subset LH, H'(Y)$ denotes the subset of X generated from the elements in Y by means of the operations in H'. As usual, LH^* denotes the set of all strings of hedges in LH. However, by $H'[Y]$ we denote the set $\{hx : h \in H' \text{ and } x \in Y\}$.

Remark 3.1. From the way the lattices $LH^+ + I$ and $LH^- + I$ have been constructed, it can be seen that lattices $LH^+ + I$ and $LH^- + I$ also satisfy condition (C_0), in which the notations L_i and L_j are replaced with LH_i^c and LH_j^c, respectively.

In [21] and [22], two systems of axioms for hedge algebras and extended hedge algebras were introduced respectively. In [17, 18] an extension of hedge algebras called the refinement structure of hedge algebras was defined by the assumption that they firstly are hedge algebras and, secondly, must fulfil certain additional axioms. All these notions of hedge algebras have been unified into the uniquely notion of RH_algebras in [20] which is defined as follows.

Definition 3.2. An algebra $AX = (X, G, LH, \leq)$ is said to be a refined hedge algebra (abbr. RH_algebra), if X and LH are semantically consistent and the following conditions hold (where $h, k \in LH$) :

$(A1)$ Every operation in LH^+ is converse to each operation in LH^-.

$(A2)$ The unit operation V of $H^+ + I$ is either positive or negative w.r.t. any operation in H. In addition, H should satisfy the *PN-homogeneous* property.

$(A3)$ If u and v are independent, i.e. $u \notin LH(v)$ and $v \notin LH(u)$, then $x \notin LH(v)$ for any $x \in LH(u)$ and vice-versa. If $x \neq hx$ then $x \notin LH(hx)$. Further, if $hx \neq kx$ then hx and kx are independent.

$(A4)$ If hx and kx are incomparable, then so are any elements $u \in LH(hx)$ and $v \in LH(kx)$. Especially, if $a, b \in G$ and $a < b$ then $LH(a) < LH(b)$. And if $hx < kx$ then
(i) In the case that $h, k \in LH_i^c$, for some $i \in SI^c$, the following statements hold:

- $\delta hx < \delta kx$, for any $\delta \in LH^*$.
- δhx and y are incomparable, for any $y \in LH(kx)$ such that $y \ngeq \delta kx$.
- δkx and z are incomparable, for any $z \in LH(hx)$ such that $z \nleq \delta hx$.

(ii) If $\{h, k\} \not\subset LH_i^c$ for every $i \in SI^c$ or $hx = kx$, then $h'hx \leq k'kx$, for any $h', k' \in UOS$.

$(A5)$ Let us consider $u \in LH(x)$ and suppose that $u \notin LH(LH_i^c[x]) = \bigcup_{h \in LH_i^c} LH(hx)$, for some $i \in I^c$. If there exists $v \in LH(hx)$, for some $h \in LH_i^c$ such that $u \geq v$ (or $u \leq v$), then $u \geq h'v$ (or $u \leq h'v$, respectively), for any $h' \in UOS$.

Example 3.1. Let us consider an algebraic structure $AX = (X_1, G, LH, \leq)$, where, as considered in previous examples, H is the set $\{V, M, L, A, P, ML\}$, $G = \{True, False\}$, but X is the set consisting of the elements $X_1 = \{ha : h \in LH + I, a \in G\}$, which are ordered as represented in Figure 7. Recall that $L(A, P, ML)$ denotes the lattice generated from the incomparable A, P and ML and $L(A, P, ML)[a]$ denotes the set $\{ha : h \in L(A, P, ML)\}$. Here, hx is defined as follows: for every hedge operation h in $LH, hTrue$ and $hFalse$ are defined as the elements given in Figure 7; for $x \neq True$ and $x \neq False$, we define $hx = x$. It can easily be seen that the operations are well defined and AX satisfies the axioms in Definition 3.2.

We now recall the following notion given in [21].

Definition 3.3. Let x and u be two elements of an RH_algebra AX. The expression $h_n \dots h_1 u$ is said to be a *canonical representation* of x w.r.t. u in AX if (i) $x = h_n \dots h_1 u$ and (ii) $h_i \dots h_1 u \neq h_{i-1} \dots h_1 u$ for every $i \leq n$.

Note that in an RH_algebra $AX = (X, G, LH, \leq)$, the operations in LH^c are compatible and, moreover, if $x \in X$ is a fixed point of an operation h in LH, i.e. $hx = x$, then it is also a fixed point of any other k in LH. In addition, similar as proved in [21], if $h \neq k$ and $hx = kx$ then x is a fixed point.

Theorem 3.1. For any $h \in LH$, there exist two unit operations h^- and h^+ such that h^- is negative and h^+ is positive w.r.t. h and for any $h_1, \dots, h_n \in LH, x \in X$,

$$V^n h^- hx \leq h_n \dots h_1 hx \leq V^n h^+ hx, \text{ if } hx \geq x,$$

$$V^n h^- hx \geq h_n \dots h_1 hx \geq V^n h^+ hx, \text{ if } hx \leq x.$$

Corollary 3.1. (i) Suppose that $hx < kx$. If $\{h, k\} \not\subset LH_i^c$ for every $i \in SI^c$, then for any two strings of hedges δ and δ', the inequality $\delta hx < \delta' kx$ holds.
(ii) Let u be an arbitrary element in X and $x \in LH(u)$. Then, there exist always elements $y, z \in UOS(u)$, i.e. z and y are generated from u by means of the unit operations, such that $y \geq x \geq z$. Furthermore, either one of the equalities $u \leq x \leq V^n hu$ and $u \geq x \geq V^n hu$ holds, for a suitably chosen $h \in LH$ and for a sufficiently great number $n \in \mathbb{N}$.

The following theorem establishes criteria for determining the ordering relationship between elements of an RH_algebra. Here, the notation $x_{<j}$ is defined as follows: if $x = h_n \dots h_1 u$, then x_j denotes the expression $h_{j-1} \dots h_1 u$, for $1 \leq j \leq n+1$, with a convention that $x_1 = u$.

Theorem 3.2. Let $x = h_n \dots h_1 u$ and $y = k_m \dots k_1 u$ be two arbitrary canonical representations of x and y w.r.t. u, respectively. Then

1. $x = y$ iff $m = n$ and $h_j = k_j$ for all $j \leq n$.

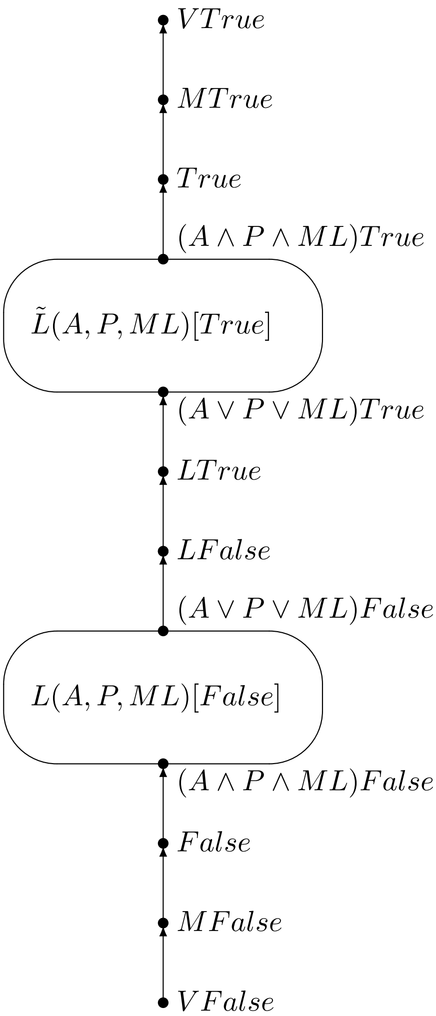


Figure 7. The poset of Example 3.1, where $\tilde{L}(A, P, ML)[True]$ denotes the dual of $L(A, P, ML)[True]$.

2. If $x \neq y$ then there exists an index $j \leq min\{m,n\}+1$(here as a convention it is understood that if $j = min\{m,n\}+1$, then either $h_j = I$ for $j = n+1 \leq m$ or $k_j = I$ for $j = m+1 \leq n$) such that $h_{j'} = k_{j'}$, for all $j' < j$ and

   (a) $x < y$ iff one of the following conditions holds

   - $h_j x_j < k_j x_j$ and $\delta k_j x_j \leq \delta' k_j x_j$ or $\delta h_j x_j \leq \delta' h_j x_j$, if $h_j, k_j \in LH_i^c$ for some $i \in SI^c$(and hence $h_j \neq I$ and $k_j \neq I$), where $\delta = h_n \ldots h_{j+1}, \delta' = k_m \ldots k_{j+1}$.
   - $h_j x_j < k_j x_j$, if otherwise (i.e. either $j \leq min\{m,n\}$ and, for every $i \in SI^c$, $\{h_j, k_j\} \not\subset LH_i^c$ or $j = min\{m,n\}+1$ and one of h_j, k_j is the identity I).

   (b) x and y are incomparable iff there exists $i \in SI^c$ such that both h_j and k_j together belong to LH_i^c and one of the following conditions holds

   - $h_j x_j$ and $k_j x_j$ are incomparable,
   - $h_j x_j < k_j x_j$ and $\delta k_j x_j \not\leq \delta' k_j x_j$,
   - $h_j x_j > k_j x_j$ and $\delta' h_j x_j \not\leq \delta h_j x_j$.

Note that, as a consequence of Theorem 3.2, if x is not a fixed point and u is an arbitrary element in X, then the canonical representation of x w.r.t. u is unique, if it exists.

Recall that the set of hedge operations in an RH_algebra is constructed from a given set of primary hedges satisfying the PN-homogeneous assumption. Naturally, one may ask whether the PN-homogeneous property of $LH^+ + I$ (but not of $H^+ + I$) still holds if we replace H_i^c with LH_i^c in Definition 3.1. The following proposition gives the answer to this question.

Proposition 3.1. If the unit operation V in $LH^c + I$ is positive (negative, resp.) w.r.t. a certain h in H_i^c, for some $i \in SI^c$, then V is also positive (negative, resp.) w.r.t. any operations in LH_i^c.

Proposition 3.2. For any $h, k \in LH_i^c$, with $i \in SI^c$, and for any $x \in X$. The following statements hold:
(i) $\delta hx > x$ $(\delta hx < x)$ iff $\delta kx > x$ $(\delta kx < x)$, for any $\delta \in LH^*$.
(ii) If $hx \neq kx$, then δhx and $\delta' hx$ are incomparable iff δkx and $\delta' kx$ are incomparable, for any $\delta, \delta' \in LH^*$.
(iii) $\delta hx > \delta' hx$ iff $\delta kx > \delta' kx$, for any $\delta, \delta' \in LH^*$.

The following theorem shows that RH_algebras with a chain of generators are distributive lattices and provides recursive formulae for computing the meet and the join of any two elements in these algebras.

Theorem 3.3. Let $AX = (X, G, LH, \leq)$ be an RH_algebra and G be a chain of generators. Then AX is a distributive lattice. Moreover, if x and y are incomparable, then they can be represented in the form $x = \delta hw$ and $y = \gamma kw$, where $h, k \in LH_i^c$, for some $i \in SI^c$, and $\delta, \gamma \in LH^*$, and we have

$$x \cup y (= \delta hw \cup \gamma kw) = \delta w' \cup \gamma w' \text{ and } x \cap y (= \delta hw \cap \gamma kw) = \delta z' \cap \gamma z',$$

where $w' = (h \vee k)w$ and $z' = (h \wedge k)w$ if $hw > w$; $w' = (h \wedge k)w$ and $z' = (h \vee k)w$ if $hw < w$ and $\cup, \cap$ stand for join, meet in AX, while $\vee, \wedge$ stand for join and meet in $LH^c + I$.

Consequently, we easily see that $LH(x)$ is a sublattice of AX and in addition, the following proposition holds.

Proposition 3.3. Let $AX = (X, G, LH, \leq)$ be an RH_algebra and G is a chain. Then, for any $h, k \in LH_i^c$, where $i \in SI^c$, and for any $x \in X$ such that $hx \neq kx$, there exists a lattice isomorphism f from $LH(hx)$ onto $LH(kx)$ defined as follows: $f(\delta hx) = \delta kx$.

4. Symmetrical RH_algebras

Intuitively, we observe that in practice there are many linguistic variables having only two distinct primary terms. These terms have intuitive contradictory meaning such as '*true*' and '*false*', '*old*' and '*young*', '*large*' and '*small*', '*tall*' and '*short*', etc. This suggested the authors of [22] to investigate extended hedge algebras with exactly two primary generators, one of which is called positive generator, denoted by t, and the other is called negative generator, denoted by f. The positive and negative generators are characterized by $Vt \geq t, Vf \leq f$ and $t > f$. Under such a normalization, it seems reasonable to consider '*true*', '*old*', '*large*' and '*tall*' as positive generators and '*false*', '*young*', '*small*' and '*short*' as negative ones. Therefore, in this section we shall examine RH_algebras $AX = (X, G, LH, \leq)$ with exactly one positive t, one negative f, the special constants $\mathbf{0}, \mathbf{1}$ and the neutral $\mathbf{W}$, i.e. $G = \{\mathbf{1}, t, \mathbf{W}, f, \mathbf{0}\}$.

For every x in X, the notion of the so-called contradictory element of x is defined as follows.

Assume that $x = h_n \ldots h_1 a$, where $a \in \{t, f\}$, is a representation of x with respect to a. An element y is said to be a contradictory element of x if it can be represented as $h_n \ldots h_1 a'$, with $a' \in \{t, f\}$ and $a' \neq a$. The contradictory element of $\mathbf{1}$ is $\mathbf{0}$ and, conversely, the contradictory element of $\mathbf{0}$ is $\mathbf{1}$. In the case where $x = \mathbf{W}$, we define contradictory element of $\mathbf{W}$ to be just itself. For example, $y =$'*very very false*' is a contradictory element of $x =$'*very very true*'; $v =$'*very little bad*' is a contradictory element of $u =$ '*very little good*'. By the definition, it is obvious that the positive generator is a contradictory element of the negative one and vice-versa, and if y is a contradictory element of x then x is a contradictory element of y.

Definition 4.1. An RH_algebra $AX = (X, G, LH, \leq)$, where G is defined as above, is said to be a symmetrical RH_algebra provided every element x in X has a uniquely contradictory element in X, denoted by x^-.

The following theorem gives a characterization of symmetrical RH_algebras.

Theorem 4.1. An RH_algebra $AX = (X, G, LH, \leq)$ with $G = \{\mathbf{1}, t, \mathbf{W}, f, \mathbf{0}\}$ as defined above is symmetrical iff AX satisfies the following condition:
(SYM) For every element $x \in X, x$ is a fixed point iff x^- is a fixed point.

Obviously, by virtue of Theorem 3.3, every symmetrical RH_algebra $AX = (X, G, LH, \leq)$ is a distributive lattice.

5. Finite symmetrical RH_algebras

5.1. The construction of finite symmetrical RH_algebras

Let us consider a symmetrical RH_algebra $AX = (X, G, LH, \leq)$ with $G = \{\mathbf{1}, a^+, \mathbf{W}, a^-, \mathbf{0}\}$ where a^+ is the positive generator and a^- is the negative one, and the underlying set X is defined as follows.

First, we define $LH_n[G]$, for $n \geq 0$, by the following procedure:

$$LH_0[G] = G, LH_1[G] = LH[G] = \bigcup_{a \in G} \{ha : h \in LH + I\},$$

$$LH_{n+1}[G] = LH[LH_n[G]].$$

It is easily seen that

$$G \subset LH[G] \subset LH_2[G] \subset \ldots \subset LH_n[G] \subset \ldots$$

In general, this chain is infinite. However, in applications, we use only a bounded number of hedges in concatenation and, hence, this chain of inclusions should be stationary. Thus, let p be a fixed positive integer and assume that for any $x \in LH_p[G]$ and $x \notin LH_{p-1}[G]$, $hx = x$ holds, for every $h \in LH$ and so, we have $G \subset LH[G] \subset LH_2[G] \subset \ldots \subset LH_p[G]$. Let $X = LH_p[G]$. Clearly, $AX = (LH_p[G], G, LH, \leq)$ is well-defined. Obviously, this algebra AX is a complete distributive lattice.

It is well-known that to model logical operations, in investigations of $[0,1]$-valued fuzzy logics (e.g. [7, 8, 9, 11]) ones have extended respective Boolean logical operations to the unit interval $[0,1]$ mainly by using t-norms and t-conorms. For example, one way of extending the classical binary implication to the interval $[0,1]$ by using a t-norm T is to define the *residuation*

$$R_T(x,y) = \sup\{z \in [0,1] : T(x,z) \leq y\}$$

Another extension of the implication is to take advantage of the equivalence between statements "NOT A OR B" and "IF A THEN B" in Boolean logic to define the so-called S-implication

$$I_T(x,y) = S(1-x,y) = 1 - T(x, 1-y)$$

where T is a t-norm and S is its dual t-conorm.

Several $[0,1]$-valued propositional logics such as Łukasiewicz logic, Gödel logic, and Product logic can be axiomatized and their algebraic versions are algebraic structures of the interval $[0,1]$ such as MV_algebra, Heyting algebra and Product algebra, respectively, (cf. [11]). It should be also emphasized that in dealing with formalized mathematical theories, ones have discovered the close relation between logics and abstract algebras (e.g., [23, 24]).

Motivation by such a view, in the sequent we shall examine some algebraic structures of finite symmetrical RH_algebras. It is shown that in these algebras we are able to define operations, which, according to their properties, may be used to model logical operations in a linguistic-valued fuzzy logic.

5.2. Kleen algebraic structure of finite symmetrical RH_algebras

As observed in [22], the negation of vague concept may often be the concept having the opposite meaning, if it exists. For example, '*good*' and '*true*' are vague concepts and they involve an intuitively intended meaning. Refuting this meaning, one may often think of the meaning of the concepts '*bad*' and '*false*', that have the opposite meaning (the antonym) to '*good*' and '*true*' and vice-versa. This interpretation was adopted in many investigations of fuzzy reasoning (see, e.g., [13, 26, 30]). Certainly, it may still be possible to discuss how to refute statements containing vague concepts which are not primary concepts such as '*Very little true*'. However, it is natural to regard the negation of '*Very little true*' as to be a concept of '*false*' and it may most probably be the concept '*Very little false*', which has the opposite meaning to the concept '*Very little true*'. This gives us a way to define the logical negation. However, the important thing is to show that the negation defined in this way has sufficient properties to develop a linguistic-valued fuzzy logic.

Therefore, analogous to the paper [22], the negation of an element x in AX is defined to be its contradictory element, i.e. $\neg x = x^-$. This operation $\neg$ is called *concept-negation* operation, because the elements of AX can be considered as linguistic terms, i.e. vague concepts. The *concept-implication* operation in this algebra, denoted by $\Rightarrow$, is defined in a regular way by means of the negation and the join operations as follows:

$$x \Rightarrow y = \neg x \cup y, \qquad \text{for any } x \text{ and } y \text{ of } AX$$

Let $AX = (X, G, LH, \leq)$, with $G = \{\mathbf{1}, a^+, \mathbf{W}, a^-, \mathbf{0}\}$ and underlying set X defined as above, be a finite symmetrical RH_algebra. As examined above, the operations $\cup, \cap, \neg, \Rightarrow$ can be derived in AX and so, we can write

$$AX = (X, G, LH, \leq, \neg, \cup, \cap, \Rightarrow, \mathbf{0}, \mathbf{W}, \mathbf{1})$$

We are now ready to establish some elementary properties of the negation operation and the implication operation.

Theorem 5.1. Let AX be a finite symmetrical RH_algebra. Then

(1) $\neg(hx) = h\neg x$, for every $h \in LH$ and $x \in X$.

(2) $\neg(\neg x) = x$, for all $x \in X$.

(3) $\neg(x \cup y) = \neg x \cap \neg y$ and $\neg(x \cap y) = \neg x \cup \neg y$, for all $x, y \in X$.

(4) $x \cap \neg x \leq y \cup \neg y$, for all $x, y \in X$.

(5) $x \cap \neg x \leq \mathbf{W} \leq x \cup \neg x$, for all $x \in X$.

(6) $\neg \mathbf{1} = \mathbf{0}$, $\neg \mathbf{0} = \mathbf{1}$ and $\neg \mathbf{W} = \mathbf{W}$.

(7) $x > y$ iff $\neg x < \neg y$, for all $x, y \in X$.

It is worth to mention that the statements $(2) - (4)$ of Theorem 5.1 show that the algebra AX is a Kleene algebra in the sense of Skala [25] and (6) shows that this algebra includes the 3-valued Łukasiewicz algebra $\{\mathbf{0}, \mathbf{W}, \mathbf{1}\}$ as its subalgebra. At the same time, the statements $(2) - (3)$ show that the triple $(\cap, \cup, \neg)$ is a De Morgan system and AX becomes a De Morgan algebra[2] in the sense of Negoita and Ralescu [14].

As a consequence of the definition of the concept-implication operation and Theorem 5.1, we have the following.

Theorem 5.2. Let $AX = (X, G, LH, \leq, \neg, \cup, \cap, \Rightarrow, \mathbf{0}, \mathbf{W}, \mathbf{1})$ be a finite symmetrical RH_algebra. Then,

[2] Also named *Soft algebra*

(1) $x \Rightarrow y = \neg y \Rightarrow \neg x$,

(2) $x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z)$,

(3) $x \Rightarrow y \geq x' \Rightarrow y'$ if $x \leq x'$ and/or $y \geq y'$,

(4) $x \Rightarrow y = \mathbf{1}$ iff $x = \mathbf{0}$ or $y = \mathbf{1}$,

(5) $\mathbf{1} \Rightarrow x = x$ and $x \Rightarrow \mathbf{1} = \mathbf{1}; \mathbf{0} \Rightarrow x = \mathbf{1}$ and $x \Rightarrow \mathbf{0} = \neg x$,

(6) $x \Rightarrow y \geq \mathbf{W}$ iff either $x \leq \mathbf{W}$ or $y \geq \mathbf{W}$, and $x \Rightarrow y \leq \mathbf{W}$ iff $x \geq \mathbf{W}$ and $y \leq \mathbf{W}$.

The statement (4) of Theorem 5.2 shows that the concept-implication operation $\Rightarrow$ is an extension of the implication operation in the two-element Boolean algebra $\{\mathbf{0}, \mathbf{1}\}$.

5.3. Heyting algebraic structure of finite symmetrical RH_algebras

On the other hand, since finite symmetrical RH_algebra $AX = (X, G, LH, \leq)$ is a distributive lattice, it is known [1, 24] that AX is a relatively pseudo-complement lattice. That is, for any $x, y \in X$, the pseudo-complement of x relative to y, denoted by $x \rightarrow y$, always exists, i.e. $x \rightarrow y$ is the greatest element of the set of elements z in X such that $x \cap z \leq y$. We now formulate fundamental properties of the operation $\rightarrow$ in finite symmetrical RH_algebras in the following theorem.

Theorem 5.3. Let AX be a finite symmetrical RH_algebra. Then,

(1) $x \rightarrow y = \mathbf{1}$ iff $x \leq y$,

(2) $\mathbf{1} \rightarrow y = y; \mathbf{0} \rightarrow y = \mathbf{1}$,

(3) $x \cap (x \rightarrow y) \leq y$,

(4) If $x_1 \leq x_2$ then $x_2 \rightarrow y \leq x_1 \rightarrow y$,

(5) If $y_1 \leq y_2$ then $x \rightarrow y_1 \leq x \rightarrow y_2$,

(6) $x \cap (x \rightarrow y) = x \cap y$,

(7) $(x \rightarrow y) \cap y = y$,

(8) $(x \rightarrow y) \cap (x \rightarrow z) = x \rightarrow (y \cap z))$,

(9) $(x \rightarrow z) \cap (y \rightarrow z) = (x \cup y) \rightarrow z$,

(10) $x \rightarrow (y \rightarrow z) = (x \cap y) \rightarrow z = y \rightarrow (x \rightarrow z)$,

(11) $z \rightarrow x \leq (z \rightarrow (x \rightarrow y)) \rightarrow (z \rightarrow y)$,

(12) $(x \rightarrow y) \cap (y \rightarrow z) \leq x \rightarrow z$,

(13) $x \leq y \rightarrow (x \cap y)$,

(14) $x \to (y \to z) \leq (x \to y) \to (x \to z)$,

(15) $z \cap ((z \cap x) \to (z \cap y)) = z \cap (x \to y)$.

It is interesting to note that the statements 1) and 4)-7) of Theorem 5.3 show that the algebra $AX = (X, G, LH, \leq, \cup, \cap, \to, \mathbf{0}, \mathbf{1})$ is a Heyting (pseudo-Boolean) algebra. Furthermore, we are able to define another negation operation, denoted by $\sim$, via $\cap$-complement operation as follows

$$\sim x = x \to \mathbf{0}, \text{ for any } x \text{ in } X.$$

Consequently, by definition, we get

$$\sim x = \begin{cases} \mathbf{1} & \text{if } x = \mathbf{0}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The fundamental properties of the negation operation $\sim$ in finite symmetrical RH_algebras are given in the following theorem.

Theorem 5.4. In every finite symmetrical RH_algebra AX,

(1) If $x \leq y$ then $\sim y \leq \sim x$,

(2) $\sim \mathbf{1} = \mathbf{0}, \sim \mathbf{0} = \mathbf{1}$,

(3) $x \cap \sim x = \mathbf{0}, \sim (x \cap \sim x) = \mathbf{1}$,

(4) $\sim (x \cup y) = \sim x \cap \sim y, \sim x \cup \sim y \leq \sim (x \cap y)$,

(5) $\sim x \cup y \leq x \to y, x \to y \leq \sim y \to \sim x$,

(6) $x \to \sim y = \sim (x \cap y) = y \to \sim x$,

(7) $\sim\sim (x \to y) \leq x \to \sim\sim y$.

6. Computational results for the relatively pseudo-complement operation

It has been known that the research on fuzzy relational equations is one of the most active and fruitful research topics in fuzzy set theory both from mathematical point of view and from system modeling point of view. One of the fundamental forms of fuzzy relational equations is sup-T equations [3]. Solving these equations in [0,1] or in complete Brouwerian lattices has been investigated by De Baets, Kerre [3, 4], and Wang [27, 28] among others (see references therein). In solving sup-T equations in a bounded poset $\mathcal{P}$, the computation of the following binary operation φ in $\mathcal{P}$ plays an important role:

$$\forall x, y \in \mathcal{P}, \varphi(x, y) = \sup\{z \in \mathcal{P} | T(x, z) \leq y\}$$

where T is a t-norm defined on $\mathcal{P}$.

Under such an observation, in this section we investigate some computational results for the relatively pseudo-complement operation in finite symmetrical RH_algebras. Let us consider a finite symmetrical RH_algebra with the structure of Heyting algebra

$$AX = (X, G, LH, \leq, \cup, \cap, \rightarrow, \mathbf{0}, \mathbf{W}, \mathbf{1})$$

From now on let us denote $\varphi_{\mathcal{L}}, \mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}$ respectively for the relatively pseudo-complement operation, the least element, the greatest element in a complete Brouwerian lattice $\mathcal{L}$, whilst φ stands for the relatively pseudo-complement operation $\rightarrow$ in AX. That is,

$$\varphi_{\mathcal{L}}(x, y) = \sup\{z \in \mathcal{L} | x \cap_{\mathcal{L}} z \leq y\}$$

where $\cap_{\mathcal{L}}$ denotes the join operation in $\mathcal{L}$.

Firstly, from the axiom (A4) of Definition 3.2 and AX is a complete Brouwerian lattice, it easily follows the following.

Proposition 6.1. For any $x \in LH(a), y \in LH(b)$ in AX such that $a, b \in \{\mathbf{1}, a^+, \mathbf{W}, a^-, \mathbf{0}\}$ and $a \neq b$, we have

$$\varphi(x, y) = \begin{cases} \mathbf{1}, \text{ if } a < b \\ y, \text{ if } a > b \end{cases}$$

Assume that $\mathcal{L}$ is a lattice, we then denote $\overline{\mathcal{L}}$ its dual lattice [1]. We now give a recursive formula for computing the pseudo-complement of an element x relative to other element y in sublattices of AX.

Lemma 6.1. Assume that $x = h_n \dots h_1 a$ and $y = k_m \dots k_1 a$ are two canonical representations of x and y w.r.t. $a \in \{a^+, a^-\}$, and $x \not\leq y$. Then there exists an index $j \leq \min(n, m) + 1$ such that $h_j \neq k_j$ and $h_i = k_i$ for any $i < j$, and

$$\varphi_{LH(x_j)}(x, y) = \begin{cases} y, \text{ if } \nexists i \in SI^c \text{ such that } h_j, k_j \in LH_i^c \\ \varphi_{LH(hx_j)}(\delta_x h x_j, \delta_y h x_j), \text{ if } \exists i \in SI^c : h_j, k_j \in LH_i^c \\ \qquad\qquad \text{and } h_j x_j > x_j \\ \varphi_{LH(\hat{h}x_j)}(\delta_x \hat{h} x_j, \delta_y \hat{h} x_j), \text{ if } \exists i \in SI^c : h_j, k_j \in LH_i^c \\ \qquad\qquad \text{and } h_j x_j < x_j \end{cases}$$

where $x_j = h_{j-1} \dots h_1 a, \delta_x = h_n \dots h_{j+1}, \delta_y = k_m \dots k_{j+1}$ and

$$h = \begin{cases} \varphi_{LH_i^+}(h_j, k_j), \text{ if } i \in SI^+ \\ \varphi_{\overline{LH_i^-}}(h_j, k_j), \text{ if } i \in SI^- \end{cases} \text{ and } \hat{h} = \begin{cases} \varphi_{\overline{LH_i^+}}(h_j, k_j), \text{ if } i \in SI^+ \\ \varphi_{LH_i^-}(h_j, k_j), \text{ if } i \in SI^- \end{cases}$$

Note that LH_i^c is a sublattice of $LH^c + I$ and its dual $\overline{LH_i^c}$ is a sublattice of $\overline{LH^c + I}$.

Proof:
Since $x \not\leq y$, it follows by Theorem 3.2 that two canonical representations of x and y w.r.t. a are different. Then we choose the index j as the least index satisfying the constraint $h_j \neq k_j$. Clearly, $j \leq \min(n, m) + 1$.

If there is no index $i \in SI^c$ such that both h_j and k_j belong to the same sublattice LH_i^c, by Theorem 3.2 we have x and y are comparable. Consequently, we have $x > y$ and $h_j x_j > k_j x_j$. Assume that $h_j \in LH_{i_0}^c$ for some $i_0 \in I^c$. We now show that $\varphi_{LH(x_j)}(x, y) = y$.

Indeed, if $\varphi_{LH(x_j)}(x, y) > y$ then it directly follows by Proposition 2.2 that x and $\varphi_{LH(x_j)}(x, y)$ are incomparable. Thus, by Theorem 3.2 there exists $h' \in LH_{i_0}^c$ such that $\varphi_{LH(x_j)}(x, y) \in LH(h'x_j)$. Then, by Theorem 3.3, we have $(x \cap \varphi_{LH(x_j)}(x, y)) \in LH(h''x_j)$, for some $h'' \in LH_{i_0}^c$. On the other hand, since $k_j \notin LH_{i_0}^c$ and $h_j x_j > k_j x_j$ we have $h'' x_j > k_j x_j$ and then $x \cap \varphi_{LH(x_j)}(x, y) > y$, which is impossible.

Now assume that there exists an index $i \in SI^c$ such that both h_j and k_j belong to the same sublattice LH_i^c. We will prove the formula for the case where $h_j x_j > x_j$. The remaining case is proved in an analogous way.

We first prove that $\varphi_{LH(x_j)}(x, y) \in LH(h'x_j)$ for some $h' \in LH_i^c$. Assume the contrary, i.e. $\varphi_{LH(x_j)}(x, y) \notin LH(h'x_j)$ for all $h' \in LH_i^c$, then it easily follows from Proposition 2.2 that $\varphi_{LH(x_j)}(x, y) > y$. On the other hand, since $\varphi_{LH(x_j)}(x, y) \in LH(x_j)$, there exists a canonical representation of $\varphi_{LH(x_j)}(x, y)$ w.r.t. x_j, say $\varphi_{LH(x_j)}(x, y) = l \ldots l'x_j$, where $l' \notin LH_i^c$. Then, by Definition 2.3 and Theorem 3.2, we have $l'x_j > \{k_j x_j, h_j x_j\}$. Again, by Theorem 3.2 we get $\varphi_{LH(x_j)}(x, y) > x$. Consequently, we obtain $x \cap \varphi_{LH(x_j)}(x, y) = x \not\leq y$, a contradiction. Thus, $\varphi_{LH(x_j)}(x, y) \in LH(h'x_j)$ for some $h' \in LH_i^c$.

Suppose that $\varphi_{LH(x_j)}(x, y) = \delta h' x_j$, for some $\delta \in LH^*$. Then we have

$$\delta h' x_j \cap x \leq y, \text{ i.e. } \delta h' x_j \cap h_n \ldots h_j x_j \leq k_m \ldots k_j x_j$$

It follows by Theorem 3.3 that

$$\delta(h' \wedge h_j)x_j \cap h_n \ldots h_{j+1}(h' \wedge h_j)x_j \leq k_m \ldots k_j x_j \tag{1}$$

Then, we infer from Theorem 3.2 that $(h' \wedge h_j)x_j \leq k_j x_j$. Further, it also implies by Theorem 3.3 that $h'x_j \cap h_j x_j \leq k_j x_j$.

We now set

$$h = \begin{cases} \varphi_{LH_i^+}(h_j, k_j), & \text{if } i \in SI^+ \\ \varphi_{\overline{LH_i^-}}(h_j, k_j), & \text{if } i \in SI^- \end{cases}$$

Then it follows from the last inequality that $h'x_j \leq hx_j$. On the other hand we also have

$$\delta h x_j \cap h_n \ldots h_j x_j = \delta(h \wedge h_j)x_j \cap h_n \ldots h_{j+1}(h \wedge h_j)x_j \tag{2}$$

Proposition 3.3 and the inequality (1) imply

$$\delta k_j x_j \cap h_n \ldots h_{j+1} k_j x_j \leq k_m \ldots k_j x_j \tag{3}$$

By the definition of h we easily obtain $(h \wedge h_j)x_j \leq k_j x_j$. Then by Proposition 3.3 and the inequality (3) we have

$$\delta(h \wedge h_j)x_j \cap h_n \ldots h_{j+1}(h \wedge h_j)x_j \leq k_m \ldots k_{j+1}(h \wedge h_j)x_j \leq k_m \ldots k_{j+1} k_j x_j$$

From the last inequalities and (2) we get $\delta h x_j \cap x \leq y$, and so $\delta h x_j \leq \delta h' x_j$. Hence $h x_j \leq h' x_j$. Since x_j is not a fixed point, it follows from the inequality $h' x_j \leq h x_j$ that $h = h'$.

Furthermore, it also follows from Proposition 3.3 and the inequality (3) that

$$\delta h x_j \cap h_n \ldots h_{j+1} h x_j \leq k_m \ldots k_{j+1} h x_j$$

Thus,

$$\varphi_{LH(x_j)}(x, y) = \delta h x_j \leq \varphi_{LH(x_j)}(h_n \ldots h_{j+1} h x_j, k_m \ldots k_{j+1} h x_j) \tag{4}$$

Assume that $\varphi_{LH(x_j)}(h_n \ldots h_{j+1} h x_j, k_m \ldots k_{j+1} h x_j) = \delta' h x_j$, for some $\delta' \in LH^*$. Since

$$\delta' h x_j \cap h_n \ldots h_{j+1} h x_j \leq k_m \ldots k_{j+1} h x_j$$

it follows from Proposition 3.3 that

$$\delta'(h \wedge h_j) x_j \cap h_n \ldots h_{j+1}(h \wedge h_j) x_j \leq k_m \ldots k_{j+1}(h \wedge h_j) x_j$$

Hence from Theorem 3.3 we obtain

$$\delta' h x_j \cap h_n \ldots h_{j+1} h x_j \leq k_m \ldots k_{j+1}(h \wedge h_j) x_j \leq k_m \ldots k_{j+1} h x_j$$

i.e. $\delta' h x_j \cap x \leq y$. So

$$\delta' h x_j \leq \delta h x_j = \varphi_{LH(x_j)}(x, y) \tag{5}$$

From inequalities (4) and (5) we finally obtain

$$\varphi_{LH(x_j)}(x, y) = \varphi_{LH(x_j)}(h_n \ldots h_{j+1} h x_j, k_m \ldots k_{j+1} h x_j)$$

As the proof of the remaining case is similar, the lemma is completely proved. □

Now we are ready to give a recursive formula for computing the pseudo-complement of an element x relative to other element y in the complete Brouwerian lattice AX.

Theorem 6.1. Under the same assumption and notation as in Lemma 6.1, we have

$$\varphi(x,y) = \begin{cases} \varphi_{LH(x_j)}(x,y), \text{ if } h_i \in \bigcup\limits_{\substack{s \in (I^c \setminus SI^c) \\ c \in \{+,-\}}} LH_s^c \text{ for any } i \leq j-1 \\ \varphi_{LH(hx_t)}(\delta_x h_{j-1} \ldots h_{t+1} h x_t, \delta_y h_{j-1} \ldots h_{t+1} h x_t), \text{ if there exists the} \\ \qquad \text{least index } t : h_t \in LH_{i_t}^c \text{ for } i_t \in SI^c \\ \qquad \text{and } h_t x_t > x_t \\ \varphi_{LH(\hat{h}x_t)}(\delta_x h_{j-1} \ldots h_{t+1} \hat{h} x_t, \delta_y h_{j-1} \ldots h_{t+1} \hat{h} x_t), \text{ if there exists the} \\ \qquad \text{least index } t : h_t \in LH_{i_t}^c \text{ for } i_t \in SI^c \\ \qquad \text{and } h_t x_t > x_t \end{cases}$$

where $x_t = h_{t-1} \ldots h_1 a, \delta_x = h_n \ldots h_j, \delta_y = k_m \ldots k_j$ and

$$h = \begin{cases} \mathbf{1}_{LH_{i_t}^+}, \text{ if } i_t \in SI^+ \\ \mathbf{0}_{LH_{i_t}^-}, \text{ if } i_t \in SI^- \end{cases} \text{ and } \hat{h} = \begin{cases} \mathbf{0}_{LH_{i_t}^+}, \text{ if } i_t \in SI^+ \\ \mathbf{1}_{LH_{i_t}^-}, \text{ if } i_t \in SI^- \end{cases}$$

Proof:
By Lemma 6.1 there exists an index $j \leq \min(n, m) + 1$ such that $h_j \neq k_j$, and for any $j' \leq j - 1, h_{j'} = k_{j'}$. First we prove the theorem for the case where

$$h_i \in \bigcup_{\substack{s \in (I^c \setminus SI^c) \\ c \in \{+,-\}}} LH_s^c$$

for any $i \leq j - 1$.

By definition we have $\varphi_{LH(x_j)}(x, y) \leq \varphi(x, y)$. Further, by Proposition 6.1 it is easy to see that $\varphi(x, y) \in LH(a)$. Assume that $\varphi(x, y) = l_q \ldots l_1 a$ is the canonical representation of $\varphi(x, y)$ w.r.t. a. If $\varphi(x, y) \notin LH(x_j)$ then there exists the least index t satisfying $t < j - 1$ such that $l_t \neq h_t$ and for any $t' \leq t - 1, l_{t'} = h_{t'}$. Then, since $\varphi(x, y) \notin LH(x_j)$ and $\varphi(x, y) \geq y$, we infer $\varphi(x, y) > y$. Consequently, we have $l_t x_t > h_t x_t$, by Theorem 3.3. By virtue of the above assumption on h_i and Corollary 3.1, we infer $\varphi(x, y) > x$, and so $\varphi(x, y) \cap x = x \not\leq y$, a contradiction. Hence, $\varphi(x, y) \in LH(x_j)$, and then we obtain $\varphi(x, y) = \varphi_{LH(x_j)}(x, y)$.

Now we prove the theorem for the reverse case, i.e. there exists at least an index $i \leq j - 1$ such that $h_i \in LH_s^c$ for some $s \in SI^c$. Assume that t is the least index among such i's and $h_t \in LH_{i_t}^c$ for some $i_t \in SI^c$.

Let us first consider the case where $h_t x_t > x_t$. Then we have $hx_t > h_t x_t$, by definition of h. Further, by virtue of the assumption $x \not\leq y$ and Proposition 3.3, it follows

$$h_n \ldots h_{t+1} hx_t \leq k_m \ldots k_{t+1} hx_t$$

where $h_i = k_i$ for any i satisfying the constraint $t + 1 \leq i \leq j - 1$. Put

$$\varphi_{LH(hx_t)}(h_n \ldots h_{t+1} hx_t, k_m \ldots k_{t+1} hx_t) = \delta hx_t$$

By Theorem 3.3 and the assumption $h_t x_t > x_t$ we have $x \cap \delta hx_t = x \cap \delta h_t x_t$. In addition, since

$$h_n \ldots h_{t+1} hx_t \cap \delta hx_t \leq k_m \ldots k_{t+1} hx_t$$

we infer from Proposition 3.3 that $x \cap \delta h_t x_t \leq y$. So it follows from the last equality that $x \cap \delta hx_t \leq y$. Thus, we infer $\varphi(x, y) \geq \delta hx_t$.

On the other hand, by the case we have proved above, it is easily seen that $\varphi(x, y) \in LH(kx_t)$ for some $k \in LH_{i_t}^c$. Assume that $\varphi(x, y) = \delta' kx_t$. From the last inequality and the definition of h, we easily infer $h = k$. Further, by definition, we have $x \cap \delta' hx_t \leq y$. Thus, $x \cap \delta' h_t x_t \leq y$. Then it follows by Proposition 3.3 that

$$h_n \ldots h_{t+1} hx_t \cap \delta' hx_t \leq k_m \ldots k_{t+1} hx_t$$

Hence, by definition, we get

$$\delta' hx_t \leq \varphi_{LH(hx_t)}(h_n \ldots h_{t+1} hx_t, k_m \ldots k_{t+1} hx_t) = \delta hx_t$$

Combining with the inequality $\varphi(x, y) \geq \delta hx_t$ shown above, we obtain

$$\varphi(x, y) = \varphi_{LH(hx_t)}(h_n \ldots h_{t+1} hx_t, k_m \ldots k_{t+1} hx_t)$$

which is the desired equality.

Since the proof for the case where $h_t x_t < x_t$ is similar, the theorem is completely proved. □

7. Conclusions

In this paper we have given an overview on various algebraic structures of symmetrical RH_algebras. These showed that symmetrical RH_algebras describe an algebraically rich structure of linguistic domains of most linguistic variables in practice, and, therefore, they can be taken as an algebraic foundation for some kinds of linguistic-valued fuzzy logic. These results may also allow us to believe in pursuing the development of the axiomatic approach to linguistic-valued logics for linguistically approximate reasoning. It should be emphasized that various algebraic versions of, for example, the unit interval [0,1] also support respectively various systems of [0,1]-based fuzzy logic.

As mentioned in [20], the axioms of RH_algebras express natural properties of linguistic hedges and linguistic terms that can be formulated in terms of the so-called semantically ordering relation. Thus, these axioms can be considered as an axiomatization of linguistic domains of linguistic variables. As such, if we agree that symmetrical RH_algebras can be taken as an algebraic foundation for some kinds of linguistic-valued fuzzy logic, then the important result of the approach is that these logics underlie methods of linguistic reasoning merely based on semantic properties of hedges and vague concepts in natural language.

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