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The Game of Synchronized Cutcake

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Abstract— **In synchronized games the players make their moves simultaneously and, as a consequence, the concept of turn does not exist. Synchronized Cutcake is the synchronized version of Cutcake, a classical two-player combinatorial game. Even though to determine the solution of Cutcake is trivial, solving Synchronized Cutcake is challenging because of the calculation of the game's value. We present the solution for small board size and some general results for a board of arbitrary size.**

Keywords: Combinatorial Games, Synchronized Cutcake

I. INTRODUCTION

Cutcake is a classical two-player combinatorial game introduced in [1], [2]. Every instance of this game is defined as a set of rectangles of integer side-lengths with edges parallel to the x- and y- axes. The two players are usually called Left and Right. A legal move for Left is to divide one of the rectangles into two rectangles of integer side-length by means of a single cut parallel to the x-axis and a legal move for Right is to divide one of the rectangles into two rectangles of integer side-length by means of a single cut parallel to the y- axis. Players take turns making legal moves until one of them cannot move. Then that player leaves the game and the remaining player is deemed the winner.

We use $[L, R]$ to indicate a L by R rectangle and we indicate a left move by

$$[L, R] \rightarrow [L1, R] + [L2, R]$$

and a right move by

$$[L, R] \rightarrow [L, R1] + [L, R2]$$

where $L1+L2 = L$, $R1+R2 = R$, and $L1, L2, R1, R2 > 0$.

We recall that in the game of Cutcake the outcome for a L by R rectangle depends on the dimension of L and R as shown in Table I. We recall that the floor of a real number is defined as the largest integer less than or equal to x and it is also denoted by $\lfloor x \rfloor$. It is interesting to observe that:

- if $\lfloor \log_2 L \rfloor > \lfloor \log_2 R \rfloor$ then Left has a winning strategy,
- if $\lfloor \log_2 L \rfloor < \lfloor \log_2 R \rfloor$ then Right has a winning strategy, and
- if $\lfloor \log_2 L \rfloor = \lfloor \log_2 R \rfloor$ then the game is a zero-game, i.e., the player that makes the first move is the loser.

For example, in the game $[8, 7]$ Left has a winning strategy and in the game $[3, 4]$ Right has a winning strategy but $[7, 4]$ is still a zero-game.

II. SYNCHRONIZED GAMES

The idea of synchronized games has been introduced in [3] and it has been applied to the game of Tic-Tac-Toe in order to revive this solved game. Following this, the same idea has been applied to the game of Hex [4] in order to increase the interestingness of this game using the concept of late chance or outcome uncertainty. In synchronized games, both players play simultaneously, therefore it does not exist any unfair advantage due to the turn to move. In this work, we apply the same idea to the game of Cutcake in order to study the effects of synchronism on a typical combinatorial game. In the game of Synchronized Cutcake a general instance and the legal moves for Left and Right are defined exactly in the same way as defined for the game of Cutcake. There is only one difference: Left and Right make their legal moves simultaneously, therefore if they choose to move in the same rectangle then this rectangle will be divided in four rectangles because the two cuts are performed simultaneously, i.e.,

$$[L, R] \rightarrow [L1, R1] + [L1, R2] + [L2, R1] + [L2, R2].$$

If Left and Right choose to move in two different rectangles then each of these rectangles will be divided in two rectangles as usual.

In combinatorial game theory we can classify all games into 4 outcome classes, which specify who has the winning strategy when Left starts and who has the winning strategy when Right starts. If G is a game then we have:

- $G > 0$ (*positive game*) if Left has a winning strategy,
- $G < 0$ (*negative game*) if Right has a winning strategy,
- $G = 0$ (*zero game*) if the player that makes the second move has a winning strategy, and
- $G \parallel 0$ (*fuzzy game*) if the player that makes the first move has a winning strategy.

In synchronized games, both players move simultaneously and there exists the possibility to get a draw, therefore for each player we have three possible cases:

- the player has a *winning strategy* (WS) independently by the opponent's strategy,
- the player has a *drawing strategy* (DS), i.e., he/she can always get a draw in the worst case, and
- the player has a *losing strategy* (LS), i.e., he/she has neither a winning nor a drawing strategy.

Table II shows all the possible cases. It is clear that if one player has a winning strategy then the other one cannot have either a winning strategy or drawing strategy therefore

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TABLE I

VALUE FOR RECTANGLES IN CUTCAKE

$L\backslash R$	1	2	3	4	5	6	7	8
1	0	-1	-2	-3	-4	-5	-6	-7
2	1	0	0	-1	-1	-2	-2	-3
3	2	0	0	-1	-1	-2	-2	-3
4	3	1	1	0	0	0	0	-1
5	4	1	1	0	0	0	0	-1
6	5	2	2	0	0	0	0	-1
7	6	2	2	0	0	0	0	-1
8	7	3	3	1	1	1	1	0

TABLE II

OUTCOME CLASSES IN SYNCHRONIZED GAMES

$L\backslash R$	LS	DS	WS
LS	$G \lesseqgtr 0$	$G \leq 0$	$G < 0$
DS	$G \geq 0$	$G = 0$	-
WS	$G > 0$	-	-

the cases WS-WS, WS-DS, and DS-WS never happen. As consequence, if G is a synchronized game then we have 6 possible legal cases:

- $G > 0$ if Left has a winning strategy,
- $G = 0$ if both player have a drawing strategy and the game will always end in a draw under perfect play,
- $G < 0$ if Right has a winning strategy,
- $G \geq 0$ if Left can always get a draw in the worst case but he/she could be able to win if Right makes a wrong move,
- $G \leq 0$ if Right can always get a draw in the worst case but he/she could be able to win if Left makes a wrong move,
- $G \lesseqgtr 0$ if both players have a losing strategy and the outcome is unpredictable.

III. SOLVING SYNCHRONIZED CUTCAKE

Table III shows the outcome for a L by R rectangle of Synchronized Cutcake with $L, R < 9$. Here, we give some results for a general L by R rectangle.

Lemma 1: Let $G = [L, R]$ be a general rectangle of Synchronized Cutcake. If $L = R$ then either $G = 0$ or $G \lesseqgtr 0$.

Proof: Because of the symmetry of the board, we have three possible cases:

1) both players have a winning strategy,
2) both players have a drawing strategy, or
3) both players have a losing strategy.

According to the Table II, the first case never happens, therefore either $G = 0$ or $G \lesseqgtr 0$. ■

Lemma 2: Let K be a positive integer and let G be a general instance of Synchronized Cutcake where every rectangle or pair of rectangles belongs to one of the following classes:

1) $[L, L]$,
2) $[L + K, L]$,
3) $[L, R]$ and $[R, L]$,
4) $[L, R]$ and $[R + K, L]$,

where $L, R > 0$. If there exists at least one rectangle belonging to the second class or a pair of rectangles belonging to the fourth class then Right has not a strategy either to win or to draw in the game G under perfect play.

Proof: We have four possible cases:

1) if Right moves in $[L, L]$ then Left can make the symmetrical move obtaining

$$[L1, L1] + [L1, L2] + [L2, L1] + [L2, L2],$$

2) if Right moves in $[L + K, L]$ then we have

$$[L1, L1] + [L1, L2] + [L2 + K, L1] + [L2 + K, L2],$$

3) if Right moves in $[L, R]$ then Left can move in $[R, L]$ obtaining

$$[L, R1] + [L, R2] + [R1, L] + [R2, L],$$

4) if Right moves in $[L, R]$ then Left can move in $[R + K, L]$ obtaining

$$[L, R1] + [L, R2] + [R1, L] + [R2 + K, L].$$

We observe that in each of these cases the new rectangles belong to one of the four classes mentioned in the hypothesis and at least one rectangle belongs to the second class or a pair of rectangles belongs to the fourth class therefore by the inductive hypothesis Right has not a strategy either to win or to draw in G. ■

TABLE III

OUTCOME FOR RECTANGLES IN SYNCHRONIZED CUTCAKE

$L\backslash R$	1	2	3	4	5	6	7	8
1	$=$	$<$	$<$	$<$	$<$	$<$	$<$	$<$
2	$>$	$=$	$<$	$<$	$<$	$<$	$<$	$<$
3	$>$	$>$	$=$	$<$	$<$	$<$	$<$	$<$
4	$>$	$>$	$>$	$=$	$<$	$<$	$<$	$<$
5	$>$	$>$	$>$	$>$	$=$	$<$	$<$	$<$
6	$>$	$>$	$>$	$>$	$>$	$=$	$<$	$<$
7	$>$	$>$	$>$	$>$	$>$	$>$	$=$	$<$
8	$>$	$>$	$>$	$>$	$>$	$>$	$>$	$=$

Lemma 3: Let K be a positive integer and let G be a general instance of Synchronized Cutcake where every rectangle or pair of rectangles belongs to one of the following classes:

1) $[R, R]$,
2) $[R, R + K]$,
3) $[L, R]$ and $[R, L]$,
4) $[L, R]$ and $[R, L + K]$,

where $L, R > 0$. If there exists at least one rectangle belonging to the second class or a pair of rectangles belonging to the fourth class then Left has not a strategy either to win or to draw in the game G under perfect play.

Proof: Analogous to the Lemma 2. ■

Lemma 4: Let G be a general instance of Synchronized Cutcake. If for every rectangle $[L, R]$ we have $\lfloor \log_2 L \rfloor \geq \lfloor \log_2 R \rfloor$ and there exists at least one rectangle $[A, B]$ such that $\lfloor \log_2 A \rfloor > \lfloor \log_2 B \rfloor$ then Left has a winning strategy.

Proof: We observe that if Left makes the following move

$$[A, B] \rightarrow [A1, B] + [A2, B]$$

where $A1 = \lfloor A/2 \rfloor$ and $A2 = \lceil A/2 \rceil$ then we have two possible cases:

1) If Right moves in $[A, B]$ we have

$$[A, B] \rightarrow [A1, B1] + [A1, B2] + [A2, B1] + [A2, B2].$$

Assuming $B1 \geq B2$, we have
   a) $\lfloor \log_2 A1 \rfloor \geq \lfloor \log_2 B1 \rfloor$,
   b) $\lfloor \log_2 A1 \rfloor > \lfloor \log_2 B2 \rfloor$,
   c) $\lfloor \log_2 A2 \rfloor \geq \lfloor \log_2 B1 \rfloor$,
   d) $\lfloor \log_2 A2 \rfloor > \lfloor \log_2 B2 \rfloor$.

2) If Right moves in another rectangle

$$[L, R] \rightarrow [L, R1] + [L, R2]$$

then we have, assuming $R1 \geq R2$,
   a) $\lfloor \log_2 A1 \rfloor \geq \lfloor \log_2 B \rfloor$,
   b) $\lfloor \log_2 A2 \rfloor \geq \lfloor \log_2 B \rfloor$,
   c) $\lfloor \log_2 L \rfloor \geq \lfloor \log_2 R1 \rfloor$,
   d) $\lfloor \log_2 L \rfloor > \lfloor \log_2 R2 \rfloor$.

Therefore, in both cases and by the inductive hypothesis Left has a winning strategy. ■

Lemma 5: Let G be a general instance of Synchronized Cutcake. If for every rectangle $[L, R]$ we have $\lfloor \log_2 R \rfloor \geq \lfloor \log_2 L \rfloor$ and there exists at least one rectangle $[A, B]$ such that $\lfloor \log_2 B \rfloor > \lfloor \log_2 A \rfloor$ then Right has a winning strategy.

Proof: Analogous to the Lemma 4. ■

Theorem 1: Let $G = [L, R]$ be a rectangle of Synchronized Cutcake. We can distinguish five different cases:

1) if $\lfloor \log_2 L \rfloor > \lfloor \log_2 R \rfloor$ then Left has a winning strategy,
2) if $\lfloor \log_2 L \rfloor = \lfloor \log_2 R \rfloor$ and $L > R$ then Right has not a strategy either to win or to draw,
3) if $L = R$ then either $G = 0$ or $G \lesseqgtr 0$,
4) if $\lfloor \log_2 L \rfloor = \lfloor \log_2 R \rfloor$ and $L < R$ then Left has not a strategy either to win or to draw,
5) if $\lfloor \log_2 L \rfloor < \lfloor \log_2 R \rfloor$ then Right has a winning strategy.

Proof: By the previous lemmas. ■

Conjecture 1: Let $G = [L, R]$ be a rectangle of Synchronized Cutcake. We can distinguish three different cases:

1) if $L = R$ then $G = 0$, i.e., the game ends in a draw,
2) if $L > R$ then $G > 0$, i.e., Left has a winning strategy,
3) if $L < R$ then $G < 0$, i.e., Right has a winning strategy.

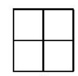
We observe that if we prove 1) then we can easily prove 2) and 3). For example, in the game $[L + K, L]$ with $K > 0$ if Left applies his/her drawing strategy in the sub-rectangle $[L, L]$ then he/she will have at least KL moves of advantage at the end of the game. The conjecture 1 is supported by the previous theorem and the results for small rectangles shown in Table III but further efforts are necessary for a formal proof.

IV. Values of Rectangles in Synchronized Cutcake

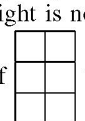
In order to establish the winning strategy for a general rectangle and for a general instance of Synchronized Cutcake, it is necessary to define a function v which represents the value of the game, i.e., the advantage of one player, in terms of moves, with respect to the opponent. We observe that:

- $v([1, 1]) = 0$ because the game

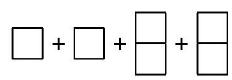
  is a draw.
- Analogously, the game



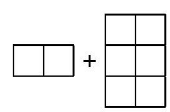
  ends in a draw therefore $v([2, 2]) = 0$.
- $v([L, 1]) = L - 1$ because Left can make $L - 1$ moves respect to Right.
- Analogously, $v([1, R]) = -R + 1$, assuming that the advantage for Right is negative.

Which is the value of  ?

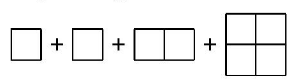
After the first synchronized move, the instance becomes



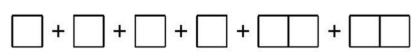
and Left has two moves of advantage therefore $v([3, 2])$ must be positive. In the game



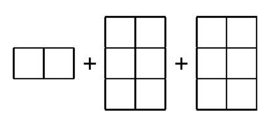
Right has a winning strategy because we have



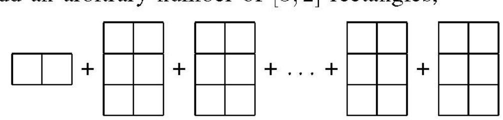
and successively,



therefore $v([3, 2])$ must be less than 1. You can check easily that in the game



Right has still a winning strategy therefore $v([3, 2])$ must be less than $\frac{1}{2}$. Actually, Right has a winning strategy even if we add an arbitrary number of $[3, 2]$ rectangles,



therefore $v([3, 2])$ is an infinitesimal number because it must be less than all the positive fractions. We denote it by ε.

TABLE IV

VALUE FOR RECTANGLES IN SYNCHRONIZED CUTCAKE

	2	3	4	5	6	7
2	0	$-\varepsilon$	-1	$-1-\varepsilon$	-2	$-2-\varepsilon$
3	ε	0	$-1+2\varepsilon$	$-1+\varepsilon$	$-2+3\varepsilon$	$-2+2\varepsilon$
4	1	$1-2\varepsilon$	0	$-\varepsilon^3$	$-\varepsilon$	-3ε
5	$1+\varepsilon$	$1-\varepsilon$	ε^3	0	$-\varepsilon^2$	$-2\varepsilon+\varepsilon^3$
6	2	$2-3\varepsilon$	ε	ε^2	0	$-\varepsilon^3$
7	$2+\varepsilon$	$2-2\varepsilon$	3ε	$2\varepsilon-\varepsilon^3$	ε^3	0

The game $[6,5]$ is really amazing because in the game $[2,3]+[6,5]$ Right has a winning strategy, therefore $v([6,5])$ must be less than ε. You can check easily that in the game $[2,3]+[6,5]+[6,5]$ Right has still a winning strategy, therefore $v([6,5])$ must be less than $\frac{\varepsilon}{2}$. Actually, Right has a winning strategy even if we add an arbitrary number of $[6,5]$ rectangles therefore $v([6,5])$ must be smaller than $\frac{\varepsilon}{n}$ for any n and we denote it by ε^2. Analogously, we denote $v([5,4])$ by ε^3 being infinitesimally smaller than $v([6,5])$. Using the same reasoning we can calculate the values of the other rectangles as shown in Table IV.

The following theorems hold.

Theorem 2: Let $[n,2]$ be a rectangle of Synchronized Cutcake with $n \geq 4$. We have

$$v([n,2]) = \begin{cases} \frac{n-2}{2} & \text{if } n \text{ is even} \\ \frac{n-3}{2}+\varepsilon & \text{if } n \text{ is odd} \end{cases}$$

Proof: We can distinguish 4 different cases.

1) $n \equiv 0 \pmod 4$.

$$\begin{aligned} v([n,2]) &= v\left(\left[\frac{n}{2},2\right]\right)+v\left(\left[\frac{n}{2},2\right]\right)+1 \\ &= \frac{n-4}{4}+\frac{n-4}{4}+1 \\ &= \frac{n-2}{2} \end{aligned}$$

2) $n \equiv 1 \pmod 4$.

$$\begin{aligned} v([n,2]) &= v\left(\left[\frac{n+1}{2},2\right]\right)+v\left(\left[\frac{n-1}{2},2\right]\right)+1 \\ &= \frac{n-5}{4}+\varepsilon+\frac{n-5}{4}+1 \\ &= \frac{n-3}{2}+\varepsilon \end{aligned}$$

3) $n \equiv 2 \pmod 4$.

$$\begin{aligned} v([n,2]) &= v\left(\left[\frac{n+2}{2},2\right]\right)+v\left(\left[\frac{n-2}{2},2\right]\right)+1 \\ &= \frac{n-2}{4}+\frac{n-6}{4}+1 \\ &= \frac{n-2}{2} \end{aligned}$$

4) $n \equiv 3 \pmod 4$.

$$\begin{aligned} v([n,2]) &= v\left(\left[\frac{n+1}{2},2\right]\right)+v\left(\left[\frac{n-1}{2},2\right]\right)+1 \\ &= \frac{n-3}{4}+\frac{n-7}{4}+\varepsilon+1 \\ &= \frac{n-3}{2}+\varepsilon \end{aligned}$$

In each case the middle equality follows from the inductive hypothesis. ∎

Theorem 3: Let $[n,3]$ be a rectangle of Synchronized Cutcake with $n \geq 4$. We have

$$v([n,3]) = \begin{cases} \frac{n-2}{2}-\frac{n}{2}\varepsilon & \text{if } n \text{ is even} \\ \frac{n-3}{2}-\frac{n-3}{2}\varepsilon & \text{if } n \text{ is odd} \end{cases}$$

Proof: We can distinguish 4 different cases.

1) $n \equiv 0 \pmod 4$.

$$\begin{aligned} v([n,3]) &= v\left(\left[\frac{n}{2},3\right]\right)+v\left(\left[\frac{n}{2},3\right]\right)+1 \\ &= \frac{n-4}{4}-\frac{n}{4}\varepsilon+\frac{n-4}{4}-\frac{n}{4}\varepsilon+1 \\ &= \frac{n-2}{2}-\frac{n}{2}\varepsilon \end{aligned}$$

2) $n \equiv 1 \pmod 4$.

$$\begin{aligned} v([n,3]) &= v\left(\left[\frac{n+1}{2},3\right]\right)+v\left(\left[\frac{n-1}{2},3\right]\right)+1 \\ &= \frac{n-5}{4}-\frac{n-5}{4}\varepsilon+\frac{n-5}{4}-\frac{n-1}{4}\varepsilon+1 \\ &= \frac{n-3}{2}-\frac{n-3}{2}\varepsilon \end{aligned}$$

3) $n \equiv 2 \pmod 4$.

$$\begin{aligned} v([n,3]) &= v\left(\left[\frac{n+2}{2},3\right]\right)+v\left(\left[\frac{n-2}{2},3\right]\right)+1 \\ &= \frac{n-2}{4}-\frac{n+2}{4}\varepsilon+\frac{n-6}{4}+\frac{n-2}{4}\varepsilon+1 \\ &= \frac{n-2}{2}-\frac{n}{2}\varepsilon \end{aligned}$$

4) $n \equiv 3 \pmod 4$.

$$\begin{aligned} v([n,3]) &= v\left(\left[\frac{n+1}{2},3\right]\right)+v\left(\left[\frac{n-1}{2},3\right]\right)+1 \\ &= \frac{n-3}{4}-\frac{n+1}{4}\varepsilon+\frac{n-7}{4}+\frac{n-7}{4}\varepsilon+1 \\ &= \frac{n-3}{2}-\frac{n-3}{2}\varepsilon \end{aligned}$$

In each case the middle equality follows from the inductive hypothesis. ∎

Theorem 4: Let $[n,4]$ be a rectangle of Synchronized Cutcake with $n \geq 8$. We have

$$v([n,4]) = \begin{cases} \frac{n-4}{4} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n-5}{4}+\varepsilon^3 & \text{if } n \equiv 1 \pmod 4 \\ \frac{n-6}{4}+\varepsilon & \text{if } n \equiv 2 \pmod 4 \\ \frac{n-7}{4}+3\varepsilon & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Proof: We can distinguish 8 different cases.

1) $n \equiv 0 \pmod 8$.

$$\begin{aligned} v([n,4]) &= v\left(\left[\frac{n}{2},4\right]\right) + v\left(\left[\frac{n}{2},4\right]\right) + 1 \\ &= \frac{n-8}{8} + \frac{n-8}{8} + 1 \\ &= \frac{n-4}{4} \end{aligned}$$

2) $n \equiv 1 \pmod 8$.

$$\begin{aligned} v([n,4]) &= v\left(\left[\frac{n+1}{2},4\right]\right) + v\left(\left[\frac{n-1}{2},4\right]\right) + 1 \\ &= \frac{n-9}{8} + \varepsilon^3 + \frac{n-9}{8} + 1 \\ &= \frac{n-5}{4} + \varepsilon^3 \end{aligned}$$

3) $n \equiv 2 \pmod 8$.

$$\begin{aligned} v([n,4]) &= v\left(\left[\frac{n+2}{2},4\right]\right) + v\left(\left[\frac{n-2}{2},4\right]\right) + 1 \\ &= \frac{n-10}{8} + \varepsilon + \frac{n-10}{8} + 1 \\ &= \frac{n-6}{4} + \varepsilon \end{aligned}$$

4) $n \equiv 3 \pmod 8$.

$$\begin{aligned} v([n,4]) &= v\left(\left[\frac{n+3}{2},4\right]\right) + v\left(\left[\frac{n-3}{2},4\right]\right) + 1 \\ &= \frac{n-11}{8} + 3\varepsilon + \frac{n-11}{8} + 1 \\ &= \frac{n-7}{4} + 3\varepsilon \end{aligned}$$

5) $n \equiv 4 \pmod 8$.

$$\begin{aligned} v([n,4]) &= v\left(\left[\frac{n+4}{2},4\right]\right) + v\left(\left[\frac{n-4}{2},4\right]\right) + 1 \\ &= \frac{n-4}{8} + \frac{n-12}{8} + 1 \\ &= \frac{n-4}{4} \end{aligned}$$

6) $n \equiv 5 \pmod 8$.

$$\begin{aligned} v([n,4]) &= v\left(\left[\frac{n+3}{2},4\right]\right) + v\left(\left[\frac{n-3}{2},4\right]\right) + 1 \\ &= \frac{n-5}{8} + \frac{n-13}{8} + \varepsilon^3 + 1 \\ &= \frac{n-5}{4} + \varepsilon^3 \end{aligned}$$

7) $n \equiv 6 \pmod 8$.

$$\begin{aligned} v([n,4]) &= v\left(\left[\frac{n+2}{2},4\right]\right) + v\left(\left[\frac{n-2}{2},4\right]\right) + 1 \\ &= \frac{n-6}{8} + \frac{n-14}{8} + \varepsilon + 1 \\ &= \frac{n-6}{4} + \varepsilon \end{aligned}$$

8) $n \equiv 7 \pmod 8$.

$$\begin{aligned} v([n,4]) &= v\left(\left[\frac{n+1}{2},4\right]\right) + v\left(\left[\frac{n-1}{2},4\right]\right) + 1 \\ &= \frac{n-7}{8} + \frac{n-15}{8} + 3\varepsilon + 1 \\ &= \frac{n-7}{4} + 3\varepsilon \end{aligned}$$

In each case the middle equality follows from the inductive hypothesis. ■

Theorem 5: Let $[n,5]$ be a rectangle of Synchronized Cutcake with $n \geq 8$. We have

$$v([n,5]) = \begin{cases} \frac{n-4}{4} - \frac{n}{4}\varepsilon^3 & \text{if } n \equiv 0 \pmod 4 \\ \frac{n-5}{4} - \frac{n-5}{4}\varepsilon^3 & \text{if } n \equiv 1 \pmod 4 \\ \frac{n-6}{4} + \varepsilon^2 - \frac{n-6}{4}\varepsilon^3 & \text{if } n \equiv 2 \pmod 4 \\ \frac{n-7}{4} + 2\varepsilon - \frac{n-3}{4}\varepsilon^3 & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Proof: We can distinguish 8 different cases.

1) $n \equiv 0 \pmod 8$.

$$\begin{aligned} v([n,5]) &= v\left(\left[\frac{n}{2},5\right]\right) + v\left(\left[\frac{n}{2},5\right]\right) + 1 \\ &= \frac{n-8}{8} - \frac{n}{8}\varepsilon^3 + \frac{n-8}{8} - \frac{n}{8}\varepsilon^3 + 1 \\ &= \frac{n-4}{4} - \frac{n}{4}\varepsilon^3 \end{aligned}$$

2) $n \equiv 1 \pmod 8$.

$$\begin{aligned} v([n,5]) &= v\left(\left[\frac{n+1}{2},5\right]\right) + v\left(\left[\frac{n-1}{2},5\right]\right) + 1 \\ &= \frac{n-9}{8} - \frac{n-9}{8}\varepsilon^3 + \\ &\quad \frac{n-9}{8} - \frac{n-1}{8}\varepsilon^3 + 1 \\ &= \frac{n-5}{4} - \frac{n-5}{4}\varepsilon^3 \end{aligned}$$

3) $n \equiv 2 \pmod 8$.

$$\begin{aligned} v([n,5]) &= v\left(\left[\frac{n+2}{2},5\right]\right) + v\left(\left[\frac{n-2}{2},5\right]\right) + 1 \\ &= \frac{n-10}{8} + \varepsilon^2 - \frac{n-10}{8}\varepsilon^3 + \\ &\quad \frac{n-10}{8} - \frac{n-2}{8}\varepsilon^3 + 1 \\ &= \frac{n-6}{4} + \varepsilon^2 - \frac{n-6}{4}\varepsilon^3 \end{aligned}$$

4) $n \equiv 3 \pmod 8$.

$$\begin{aligned} v([n,5]) &= v\left(\left[\frac{n+3}{2},5\right]\right) + v\left(\left[\frac{n-3}{2},5\right]\right) + 1 \\ &= \frac{n-11}{8} + 2\varepsilon - \frac{n-3}{8}\varepsilon^3 + \\ &\quad \frac{n-11}{8} - \frac{n-3}{8}\varepsilon^3 + 1 \\ &= \frac{n-7}{4} + 2\varepsilon - \frac{n-3}{4}\varepsilon^3 \end{aligned}$$

5) $n \equiv 4 \pmod 8$.

$$\begin{aligned} v([n,5]) &= v\left(\left[\frac{n+4}{2},5\right]\right) + v\left(\left[\frac{n-4}{2},5\right]\right) + 1 \\ &= \frac{n-4}{8} - \frac{n+4}{8}\varepsilon^3 + \\ &\quad \frac{n-12}{8} - \frac{n-4}{8}\varepsilon^3 + 1 \\ &= \frac{n-4}{4} - \frac{n}{4}\varepsilon^3 \end{aligned}$$

6) $n \equiv 5 \pmod 8$.

$$\begin{aligned} v([n,5]) &= v\left(\left[\frac{n+3}{2},5\right]\right) + v\left(\left[\frac{n-3}{2},5\right]\right) + 1 \\ &= \frac{n-5}{8} - \frac{n+3}{8}\varepsilon^3 + \\ &\quad \frac{n-13}{8} - \frac{n-13}{8}\varepsilon^3 + 1 \\ &= \frac{n-5}{4} + \frac{n-5}{4}\varepsilon^3 \end{aligned}$$

7) $n \equiv 6 \pmod 8$.

$$\begin{aligned} v([n,5]) &= v\left(\left[\frac{n+2}{2},5\right]\right) + v\left(\left[\frac{n-2}{2},5\right]\right) + 1 \\ &= \frac{n-6}{8} - \frac{n+2}{8}\varepsilon^3 + \\ &\quad \frac{n-14}{8} + \varepsilon^2 - \frac{n-14}{8}\varepsilon^3 + 1 \\ &= \frac{n-6}{4} + \varepsilon^2 - \frac{n-6}{4}\varepsilon^3 \end{aligned}$$

8) $n \equiv 7 \pmod 8$.

$$\begin{aligned} v([n,5]) &= v\left(\left[\frac{n+1}{2},5\right]\right) + v\left(\left[\frac{n-1}{2},5\right]\right) + 1 \\ &= \frac{n-7}{8} - \frac{n+1}{8}\varepsilon^3 + \\ &\quad \frac{n-15}{8} + 2\varepsilon - \frac{n-7}{8}\varepsilon^3 + 1 \\ &= \frac{n-7}{4} + 2\varepsilon - \frac{n-3}{4}\varepsilon^3 \end{aligned}$$

In each case the middle equality follows from the inductive hypothesis. ■

The following theorems can be proven in the same way.

Theorem 6: Let $[n, 6]$ be a rectangle of Synchronized Cutcake with $n \geq 8$. We have

$$v([n,6]) = \begin{cases} \frac{n-4}{4} - \frac{n}{4}\varepsilon & \text{if } n \equiv 0 \pmod 4 \\ \frac{n-5}{4} - \frac{n-5}{4}\varepsilon - \varepsilon^2 & \text{if } n \equiv 1 \pmod 4 \\ \frac{n-6}{4} - \frac{n-6}{4}\varepsilon & \text{if } n \equiv 2 \pmod 4 \\ \frac{n-7}{4} - \frac{n-7}{4}\varepsilon - \varepsilon^3 & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Theorem 7: Let $[n, 7]$ be a rectangle of Synchronized Cutcake with $n \geq 8$. We have

$$v([n,7]) = \begin{cases} \frac{n-4}{4} - \frac{3n}{4}\varepsilon & \text{if } n \equiv 0 \pmod 4 \\ \frac{n-5}{4} - \frac{3n-7}{4}\varepsilon + \varepsilon^3 & \text{if } n \equiv 1 \pmod 4 \\ \frac{n-6}{4} - \frac{3n-18}{4}\varepsilon - \varepsilon^3 & \text{if } n \equiv 2 \pmod 4 \\ \frac{n-7}{4} - \frac{3n-21}{4}\varepsilon & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

V. CONCLUSIONS

In conclusion, introducing synchronism in the game of Cutcake has two remarkable effects on this game. On the one hand, there exist no more zero-games, i.e, games where the winner depends exclusively on the player that makes the second move; on the other hand, there exists the possibility to get a draw which is impossible in a typical combinatorial game. To establish the value of a general L by R rectangle is much more difficult than Cutcake because of synchronism and further efforts are necessary to solve completely this game. Future works concern the resolution of the following open problems:

- to prove the Conjecture 1,
- to determine the value of an arbitrary L by R rectangle,
- to analyze other games in order to establish a general mathematical theory about synchronized combinatorial games.

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