Uniform Normalisation beyond Orthogonality

Zurab Khasidashvili[1], Mizuhito Ogawa[2], and Vincent van Oostrom[3]

[1] Department of Mathematics and Computer Science, Bar-Ilan University
Ramat-Gan 52900, Israel
khasidz@cs.bu.ac.il

[2] Japan Science and Technology Corporation, PRESTO and
NTT Communication Science Laboratories
3-1 Morinosato-Wakamiya, Atsugi, Kanagawa 243-0198, Japan
mizuhito@theory.brl.ntt.co.jp

[3] Department of Philosophy, Utrecht University
P.O. Box 80089, 3508 TB Utrecht, The Netherlands
oostrom@phil.uu.nl

Abstract. A rewrite system is called uniformly normalising if all its steps are perpetual, i.e. are such that if $s \to t$ and s has an infinite reduction, then t has one too. For such systems termination (SN) is equivalent to normalisation (WN). A well-known fact is uniform normalisation of *orthogonal non-erasing* term rewrite systems, e.g. the λI-calculus. In the present paper both restrictions are analysed. Orthogonality is seen to pertain to the linear part and non-erasingness to the non-linear part of rewrite steps. Based on this analysis, a modular proof method for uniform normalisation is presented which allows to go beyond orthogonality. The method is shown applicable to biclosed first- and second-order term rewrite systems as well as to a λ-calculus with explicit substitutions.

1 Introduction

Two classical results in the study of uniform normalisation are:

- the λI-calculus is uniformly normalising [7, p. 20, 7 XXV], and
- non-erasing steps are perpetual in orthogonal TRSs [14, Thm. II.5.9.6].

In previous work we have put these results and many variations on them in a unifying framework [13]. At the heart of that paper is the result (Thm. 3.16) that a term s not in normal form contains a redex which is external for *any* reduction from s.[1] Since external redexes need not exist in rewrite systems having critical pairs, the result does not apply to these. The method presented here, is based instead on the existence of redexes which are external for all reductions which are permutation equivalent to a *given* reduction. Since this so-called standardisation theorem holds for all left-linear rewrite systems, with or without critical pairs, the resulting framework is more general. It is applied to obtain

[1] According to [11, p. 404], a redex at position p is external to a reduction if in the reduction no redex is contracted above p to which the redex did not contribute.

uniform normalisation results for abstract rewrite systems (ARSs), first-order term rewrite systems (TRSs) and second-order term rewrite systems (P_2RS) in Sect. 2, 3 and 4, respectively. In each section, the proof method is presented for the orthogonal case first, deriving traditional results. We then vary on it, relaxing the orthogonality restriction. This leads to new uniform normalisation results for biclosed rewrite systems (e.g. Cor. 2, 5, 6, and 8). In Sect. 5 uniform normalisation for λx^-, a prototypical λ-calculus with explicit substitutions, is shown to hold, extending earlier work of [6] who only shows it for the explicit substitution part x of the calculus. The proof boils down to an analysis of the (only) critical pair of λx^- and uses a particularly simple proof of preservation of strong normalisation for λx^-, also based on the standardisation theorem.

2 Abstract rewriting

Although trivial, the results in this section and their proofs form the heart of the following sections. Moreover, they are applicable to various concrete (linear) rewrite systems, for instance to interaction nets [16]. The reader is assumed to be familiar with *abstract rewrite systems* (ARSs, [15, Chap. 1] or [1, Chap. 2]).

Definition 1. *Let a be an object of an abstract rewrite system. a is* terminating *(strongly normalising,* SN*) if no infinite reductions are possible from it. We use* ∞ *to denote the complement of* SN*. a is* normalising *(weakly normalising,* WN*) if some reduction to normal form is possible from it.*

Definition 2. *A rewrite step* $s \to t$ *is* critical *if* $s \in \infty$ *and* $t \in \mathsf{SN}$*, and* perpetual *otherwise. A rewrite system is* uniformly normalising *if there are no critical steps.*

First, note that a rewrite system is uniformly normalising iff $\mathsf{WN} \subseteq \mathsf{SN}$ holds. Moreover, uniform normalisation holds for deterministic rewrite systems.

Definition 3. *A* fork *in a rewrite system is pair of steps* $t_1 \leftarrow s \to t_2$*. It is called* trivial *if* $t_1 = t_2$*. A rewrite system is* deterministic *if all forks are trivial, and* non-deterministic *otherwise.*

To analyse uniform normalisation for non-deterministic rewrite systems it thus seems worthwhile to study their non-trivial forks.

Definition 4. *A rewrite system is* linear orthogonal *if every fork* $t_1 \leftarrow s \to t_2$ *is either trivial or* square*, that is,* $t_1 \to s' \leftarrow t_2$ *for some* s' *[1, Exc. 2.33].*

We will show the *fundamental theorem of perpetuality*:

Theorem 1 (FTP). *Steps are perpetual in linear orthogonal rewrite systems.*

Corollary 1. *Linear orthogonal rewrite systems are uniformly normalising.*

In the next section we will show (Lem. 1) that the abstract rewrite system associated to a term rewrite system which is linear and orthogonal, is linear orthogonal. Linear orthogonality is a weakening of the diamond property [1, Def. 2.7.8], and a strengthening of subcommutativity [15, Def. 1.1.(v)] and of the balanced weak Church-Rosser property [25, Def. 3.1], whence:

Proof. (of Thm. 1) Suppose $s \in \infty$ and $s \to t$. We need to show $t \in \infty$. By the first assumption, there exists an infinite reduction $S : s_0 \to s_1 \to s_2 \to \ldots$, with $s_0 = s$. One can build an infinite reduction T from t as follows: let $t_0 = t$ be the first object of T. By orthogonality we can find for every non-trivial fork $s_{i+1} \leftarrow s_i \to t_i$ a next object t_{i+1} of T such that $s_{i+1} \to t_{i+1} \leftarrow t_i$. Consider a maximal reduction T thus constructed. If T is infinite we are done. If T is finite, it has a final object, say t_n, and a fork $s_{n+1} \leftarrow s_n \to t_n$ exists which is trivial, i.e. $s_{n+1} = t_n$. Hence, T and the infinite reduction S from s_{n+1} on can be concatenated. □

FTP can be brought beyond linear orthogonality. Let $\to^{=}$ and $\twoheadrightarrow$ denote the reflexive and reflexive-transitive closure of $\to$, respectively.

Definition 5. *A fork $t_1 \leftarrow s \to t_2$ is* closed *if $t_1 \twoheadrightarrow t_2$. A rewrite system is* linear biclosed *if all forks are either closed or square.*[2]

By replacing the appeal to triviality by an appeal to closedness in the proof of FTP, i.e. by replacing $s_{n+1} = t_n$ by $s_{n+1} \twoheadleftarrow t_n$, we get:

Corollary 2. *Linear biclosed rewrite systems are uniformly normalising.*

3 First-order term rewriting

In this section first the uniform normalisation results of Section 2 are instantiated to linear term rewriting. Next, the *fundamental theorem of perpetuality for first-order term rewrite systems* is established:

Theorem 2 ($\mathrm{F_1TP}$). *Non-erasing steps are perpetual in orthogonal TRSs.*

Corollary 3. *Non-erasing orthogonal TRSs are uniformly normalising.*

The chief purpose of this section is to illustrate our proof method based on standardisation. Except for the results on biclosed systems, the results obtained are not novel (cf. [15, Lem. 8.11.3.2] and [9, Sect. 3.3]). The reader is assumed to be familiar with *first-order term rewrite systems* (TRSs) as can be found in e.g. [15] or [1]. We summarise some aberrations and additional concepts:

Definition 6. – *A term is* linear *if any variable occurs at most once in it. Let $\varrho : l \to r$ be a TRS rule. It is* left-linear *(*right-linear*) if l (r) is linear. It is* linear *if $Var(l) = Var(r)$ and both sides are linear. A TRS is (left-,right) linear if all its rules are.*

[2] Beware of the symmetry: if the fork is not square, then both $t_1 \twoheadrightarrow t_2$ and $t_2 \twoheadrightarrow t_1$.

– *Let* $\varrho : l \to r$ *be a rule. A variable* $x \in Var(l)$ *is* erased *by* ϱ *if it does not occur in* r*. The rule* ϱ *is erasing if it erases some variable. A rewrite step is erasing if the applied rule is. A TRS is erasing if some step is.*
– *Let* $\varrho : l \to r$ *and* $\vartheta : g \to d$ *be rules which have been renamed apart. Let* p *be a non-variable position in* l*.* ϱ *is said to* overlap ϑ at p *if a unifier* σ *of* $l_{|p}$ *and* g *does exist. If* σ *is a most general such unifier, then both* $\langle l[d]^\sigma_p, r^\sigma\rangle$ *and* $\langle r^\sigma, l[d]^\sigma_p\rangle$ *are* critical pairs at p between ϱ and ϑ.[3]
– *If for all such critical pairs* $\langle t_1, t_2\rangle$ *of a left-linear TRS* $\mathcal{R}$ *it holds that:*
  $\exists s'\, t_1 \twoheadrightarrow s' \leftarrow^= t_2$, *then* $\mathcal{R}$ *is* strongly closed *[10, p. 812]*
  $t_1 \twoheadrightarrow t_2$, *then* $\mathcal{R}$ *is* biclosed *[22, p. 70]*
  $t_1 = t_2$, *then* $\mathcal{R}$ *is* weakly orthogonal
  $t_1 = t_2$ *and* $p = \epsilon$, *then* $\mathcal{R}$ *is* almost orthogonal
  $t_1 = t_2$, $p = \epsilon$ *and* $\varrho = \vartheta$, *then* $\mathcal{R}$ *is* orthogonal

Some remarks are in order. First, our critical pairs for a TRS are the critical pairs $\langle s, t\rangle$ of [1, Def. 6.2.1] extended with their opposites ($\langle t, s\rangle$) and the trivial critical pairs between a rule with itself at the head ($\langle r, r\rangle$ for every rule $l \to r$). Next, linearity in our sense implies linearity in the sense of [1, Def. 6.3.1], but not vice versa. Linearity of a step $s = C[l^\sigma] \to C[r^\sigma] = t$ as defined here captures the idea that every symbol in the context-part C or the substitution-part σ in s has a *unique* descendant in t, whereas linearity in the sense of [1, Def. 6.3.1] only guarantees that there is *at most one* descendant in t. Remark:

orth. $\Longrightarrow$ almost orth. $\Longrightarrow$ weakly orth. $\Longrightarrow$ biclosed $\Longrightarrow$ strongly closed

3.1 Linear term rewriting

In this subsection the results of Section 2 for abstract rewriting are instantiated to linear term rewriting. First, remark that linear strongly closed TRSs are confluent (combine Lem. 6.3.2, 6.3.3 and 2.7.4 of [1]). Therefore, a linear TRS satisfying any of the above mentioned critical pair criteria is confluent.

Lemma 1. *If* $\mathcal{R}$ *is a linear orthogonal TRS,* $\to_{\mathcal{R}}$ *is a linear orthogonal ARS.*

Proof. The proof is based on the standard critical pair analysis of a fork $t_1 \leftarrow_{\mathcal{R}} s \to_{\mathcal{R}} t_2$ as in [1, Sect. 6.2]. Actually, it is directly obtained from the proof of [1, Lem. 6.3.3], by noting that:

Case 1 (parallel) establishes that the fork is square (joinable into a diamond),
Case 2.1 (nested) also yields that the fork is square,[4] and
Case 2.2 (overlap) can occur only if the steps in the fork arise by applying the same rule at the same position, by orthogonality, so the fork is trivial. □

From Lem. 1 and Cor. 1 we obtain a special case of Corollary 3.

Corollary 4. *Linear orthogonal TRSs are uniformly normalising.*

[3] Beware of the symmetry (see the next item and cf. Footnote 2).
[4] Note that the case $x \notin Var(r_1)$ cannot happen, due to our notion of linearity.

Lemma 2. *If $\mathcal{R}$ is a linear biclosed TRS, $\to_{\mathcal{R}}$ is a linear biclosed ARS.*

Proof. The analysis in the proof of Lem. 1 needs to be adapted as follows:

Case 2.2 , the instance of a critical pair, is closed by biclosedness of critical pairs and the fact that rewriting is closed under substitution. □

Corollary 5. *Linear biclosed TRSs are uniformly normalising.*

3.2 Non-linear term rewriting

In this subsection the results of the previous subsection are adapted to non-linear TRSs, leading to a proof of F_1TP (Thm. 2). The adaptation is non-trivial, since uniform normalisation may fail for orthogonal non-linear TRSs.

Example 1. The term $e(a)$ in the TRS $\{a \to a, e(x) \to b\}$ witnesses that orthogonal TRSs need not be uniformly normalising.

Non-linearity of a TRS may be caused by non-left-linearity. Although non-left-linearity in itself is not fatal for uniform normalisation of TRSs (see [9, Chap. 3], e.g. Cor. 3.2.9), it will be in case of second-order rewriting (cf. Ex. 2) and our method cannot deal with it. Hence: We assume TRSs to be left-linear.
Under this assumption, non-linearity may only be caused by some symbol having zero or multiple descendants after a step. The problem in Ex. 1 is seen to arise from the fork $e(a) \leftarrow e(a) \to b$ which is not balancedly joinable: it is neither trivial ($e(a) \neq b$) nor square ($\nexists s'\, e(a) \to s' \leftarrow b$). Erasingness is the only problem. To prove F_1TP, we will make use of the apparent asymmetry in the non-linearity

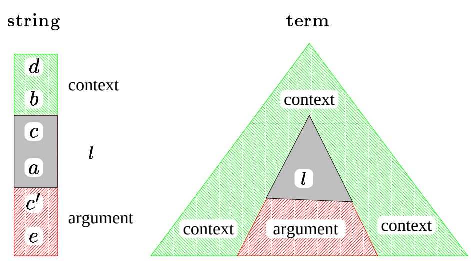

Fig. 1. Split

of term rewrite steps: an occurrence of a left-hand side of a rule $l \to r$ *splits* the surrounding into two parts (see Fig. 1):

- the *context*-part above or parallel to [1, Def. 3.1.3] l, and
- the *argument*-part, below l.

Observe that term rewrite steps in the context-part might replicate the occurrence of the left-hand side l, whereas steps in the argument-part cannot do so. To deal with such replicating steps in the context-part, we will actually prove a strengthening of $\mathrm{F_1TP}$ for parallel steps instead of ordinary steps.

Definition 7. *Let $\varrho : l \to r$ be a TRS rule. s* parallel rewrites *to t* using *ϱ, $s \mathrel{+\!\!\!\!\!\!\;+\!\!\!\!\to}_{\varrho} t$ [10, p. 814],*[5] *if it holds that $s = C[l^{\sigma_1}, \dots, l^{\sigma_k}]$ and $t = C[r^{\sigma_1}, \dots, r^{\sigma_k}]$, for some $k \geq 0$. The step is* erasing *if the rule is. The* context*(*argument*)-part of the step is the part above or parallel to all (below some) occurrences of l.*

To reduce $\mathrm{F_1TP}$ to FTP it suffices to reduce to the case where the infinite reduction does not take place (entirely) in the context-part, since then the steps either have overlap or are in the, linear, argument-part. To that end, we want to transform the infinite reduction into an infinite reduction where the steps in the context-part precede the steps in the argument-part.

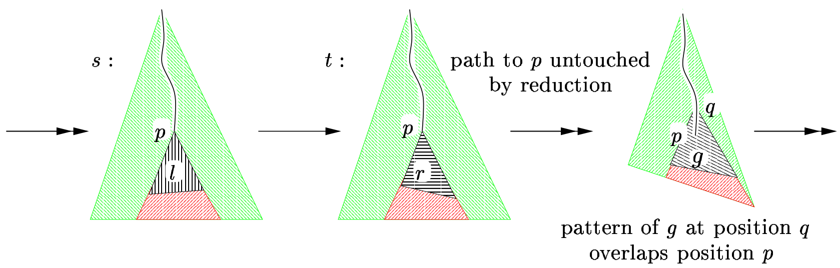

Fig. 2. Standard

Definition 8. *A reduction is* standard *(see Fig. 2) if for any step $C[l^\sigma]_p \to C[r^\sigma]_p$ in the reduction, p is in the pattern of the first step after that step which is above p. That is, if $D[g^\tau]_q$ displays the occurrence of the first redex with $p = qo$, we have that o is a non-variable position in g.*

Theorem 3 (STD). *Any reduction in a TRS can be transformed into a standard one. The transformation preserves infiniteness.*

Proof. The first part of the theorem was shown to hold for orthogonal TRSs in [11, Thm. 3.19] and extended to left-linear TRSs possibly having critical pairs in [8]. That standardisation preserves infiniteness follows from the fact that at some moment along an infinite reduction $S : s_0 \to s_1 \to \dots$ a redex at minimal position p w.r.t. the prefix order $\leq$ [1, Def. 3.1.3] must be contracted. Say this happens the first time in step $s_i \to_p s_{i+1}$. Permute all steps parallel to p in S after this step resulting in $S_0; S_1$, where S_0 contains only steps below p and ends

[5] Actually our notion is a restriction of his, since we allow only one rule.

with a step at position p, and S_1 is infinite. Standardise S_0 into T_0, note that it is non-empty and that concatenating T_0 with *any* standardisation of S_1 will yield a standard reduction by the choice of p. Repeat the process on S_1. □

Proof. (of Thm. 2) Suppose $s \in \infty$ and $s \twoheadrightarrow\!\!\!\!\!\!+\!\!\!\!\!+_{\varrho}^{k} t$ is non-erasing, contracting k redexes w.r.t. rule $\varrho : l \to r$ in parallel. We need to show $t \in \infty$. If $k = 0$, then $t = s \in \infty$. Otherwise, there exists by the first assumption an infinite reduction $S : s_0 \to_{q_0} s_1 \to_{q_1} s_2 \to \ldots$, with $s_0 = s$ and $s_i \to_{q_i} s_{i+1}$ contracting a redex at position q_i w.r.t. rule $\vartheta_i : g_i \to d_i$. By STD S may be assumed standard. Consider the relative positions of the redexes in the fork $s_1 \leftarrow_{q_0} s \twoheadrightarrow\!\!\!\!\!\!+\!\!\!\!\!+_{\varrho} t$.

(context) If g_0 occurs entirely in the context-part of the parallel step, then by the Parallel Moves lemma [1, Lem. 6.4.4] the fork is joinable into $s_1 \twoheadrightarrow\!\!\!\!\!\!+\!\!\!\!\!+_{\varrho}^{k'} t_1 \leftarrow_{q_0} t_0$. Since $t_0 \to t_1$, $s_1 \in \infty$, and $s_1 \twoheadrightarrow\!\!\!\!\!\!+\!\!\!\!\!+_{\varrho} t_1$ is non-erasing, repeating the process will yield an infinite reduction from $t_0 = t$ as desired.

(non-context) Otherwise g_0 must be below one or overlap at least one contracted left-hand side l, say the one at position p. Hence, $s \twoheadrightarrow\!\!\!\!\!\!+\!\!\!\!\!+^{k} t$ can be decomposed as $s \to_p s' \twoheadrightarrow\!\!\!\!\!\!+\!\!\!\!\!+^{k-1} t$. We claim $s' \in \infty$. The proof is as for FTP, employing standardness to exclude replication of the pivotal l-redex. Construct a maximal reduction T as follows. Let $t_0 = s'$ be the first object of T. If g_0 overlaps the l at position p, then T is empty. Otherwise, g_0 must be below that l and we set $o_0 = q_0$.

- Suppose the fork $s_{i+1} \leftarrow_{q_i} s_i \to_p t_i$ is such that the contracted redexes do not have overlap. As an invariant we will use that o_i records the outermost position below l (at p) and above q_0 where a redex was contracted in the reduction S up to step i, hence $p \le o_{i+1} \le o_i \le q_0$. Then $q_i < p$ is not possible, since by the non-overlap assumption g_i would be entirely above p, hence above o_i as well, violating standardness of S. Hence, q_i is parallel to or below l (at p). By another appeal to the Parallel Moves lemma the fork can be joined via $s_{i+1} \to_p t_{i+1} \twoheadleftarrow\!\!\!\!\!\!+\!\!\!\!\!+^{k} t_i$, where $k > 0$ by non-erasingness of $s_i \to t_i$ (†). The invariant is maintained by setting o_{i+1} to q_i if $q_i < o_i$, and to o_i otherwise.

If T is infinite we are done. If T is finite, it has a final object, say t_n, and a fork $s_{n+1} \leftarrow_{q_i,\vartheta_i} s_n \to_p t_n$ such that the redexes have overlap (‡). By the orthogonality assumption we must have $q_n = p$ and $\vartheta_n = \varrho$, hence $s_{n+1} = t_n$. By concatenating T and the infinite reduction S from s_{n+1}, the claim ($s' \in \infty$) is then proven. From the claim, we may repeat the process with an infinite standard reduction from s' and $s' \twoheadrightarrow\!\!\!\!\!\!+\!\!\!\!\!+^{k-1} t$.

Observe that the (context)-case is the only case producing a rewrite step from t, but it must eventually always apply since the other case decreases k by 1. □

By replacing the appeal to orthogonality by an appeal to biclosedness in the proof of $\mathrm{F_{I}TP}$, i.e. by replacing $s_{n+1} = t_n$ by $s_{n+1} \twoheadleftarrow t_n$, we get:

Theorem 4. *Non-erasing steps are perpetual in biclosed TRSs.*

Corollary 6. *Non-erasing biclosed TRSs are uniformly normalising.*

Note that we are beyond orthogonality since biclosed TRSs need not be confluent. The example is as for strongly closed TRSs [10, p. 814], but note that the latter need not be uniformly normalising! Next, we show [15, Lem. 8.11.3.2].

Definition 9. *A step $C[l^\sigma] \to C[r^\sigma]$ is* ∞-erasing, *if it erases all ∞-variables, that is, if $x \in Var(r)$ then $x^\sigma \in \mathsf{SN}$.*

Theorem 5. *Non-∞-erasing rewrite steps are perpetual in biclosed TRSs.*

Proof. Replace in the proof of Thm. 4 everywhere non-erasingness by non-∞-erasingness. The only thing which fails is the statement resulting from (†):

- By another appeal to the Parallel Moves Lemma the fork can be joined via $s_{i+1} \to_p t_{i+1} \leftarrow\!\!\!+\!\!\!\!+^k t_i$, where $k > 0$ by non-∞-erasingness of $s_i \to t_i$.

We split this case into two new ones depending on whether some argument (instance of variable) to l is ∞ or not.

- In the former case, $t_i \in \infty$ follows directly from non-∞-erasingness.
- In the latter case, $s_i \to s_{i+1}$ may take place in an erased argument, and $s_{i+1} \to_p t_{i+1} = t_i$. But since all arguments to l are SN, this can happen only finitely often and eventually the first case applies. □

In [9] a uniform normalisation result not requiring left-linearity, but having a critical pair condition incomparable to biclosedness was presented.

4 Second-order term rewriting

In this section, the *fundamental theorem of perpetuality for second-order term rewrite systems* is established, by generalising the method of Section 3.

Theorem 6 (F_2TP). *Non-erasing steps are perpetual in orthogonal P_2RSs.*

Corollary 7. *Non-erasing orthogonal P_2RSs are uniformly normalising.*

For ERSs and CRSs these results can be found as [12, Thm. 60] and [14, Cor. II.5.9.4], respectively. The reader is assumed to be familiar with *second-order term rewrite systems* be it in the form of combinatory reduction systems (CRSs [14]), expression reduction systems (ERSs [13]), or higher-order pattern rewrite systems (PRSs [17]). We employ PRSs as defined in [17], but will write $x.s$ instead of $\lambda x.s$, thereby freeing the λ for usage as a function symbol.

Definition 10.
- *The* order *of a rewrite rule is the maximal order of the free variables in it. The order of a PRS is the maximal order of the rules in it. P_nRS abbreviates n^{th}-order PRS.*
- *A rule $l \to r$ is* fully-extended *(FE) if for every occurrence $Z(t_1, \ldots, t_n)$ in l of a free variable Z, $t_1, \ldots, t_n$ is the list of variables bound above it.*
- *A rewrite step $s = C[l^\sigma] \to C[r^\sigma] = t$ is* non-erasing *if every symbol from C and σ in s descends [20, Sect. 3.1.1] to some symbol in t.*[6]

third-order	non-fully-extended	non-left-linear
$(\lambda z.M(z))N \to M(N)$	$M(z)\langle z := N\rangle \to M(N)$	$M(x)\langle x := N\rangle \to M(N)$
$fxy.Z(u.x(u), y) \to Z(u.c, \Omega)$	$gxy.Z(y) \to Z(a)$	$g(x.Z(x), x.Z(x)) \to Z(a)$
	$e(x, y) \to c$	$e(x) \to c$
	$f(a) \to f(a)$	$f(a) \to f(a)$

Table 1. Three counterexamples against uniform normalisation of PRSs

The adaptation is non-trivial since uniform normalisation may fail for orthogonal, but third-order or non-left-linear or non-fully-extended systems.

Example 2. (third-order) [13, Ex. 7.1] Consider the 3$^{\text{rd}}$-order PRS in Tab. 1. It is the standard PRS-presentation of the $\lambda\beta$-calculus [17] extended by a rule. $@ : o{\to}o{\to}o$ and $\lambda : (o{\to}o){\to}o$ are the function symbols and $M : o{\to}o$ and $N : o$ are the free-variables of the first (β-)rule. We have made @ an implicit binary infix operation and have written $\lambda x.s$ for $\lambda(x.s)$, for the λ-calculus to take a more familiar form. If Ω abbreviates $(\lambda x.xx)(\lambda x.xx)$, the step $fxy.(\lambda u.x(u))y \to_\beta fxy.x(y)$ is non-erasing but critical.

(non-fully-extended) [13, Ex. 5.9] Consider the non-FE P$_2$RS in Tab. 1. The step $gxy.e(z, x)\langle z := f(y)\rangle \to gxy.e(f(y), x)$ is non-erasing but critical.

(non-left-linear) Consider the non-left-linear P$_2$RS in Tab. 1. The rewrite step $g(y.e(x)\langle x{:=}f(y)\rangle, y.c\langle x{:=}f(y)\rangle) \to g(y.e(f(y)), y.c\langle x{:=}f(y)\rangle)$ from s to t is non-erasing but critical; t is terminating, but we have the infinite reduction

$$s \to g(y.c\langle x := f(y)\rangle, y.c\langle x := f(y)\rangle) \to c\langle x := f(a)\rangle \to \ldots$$

In each item, the second rule causes failure of uniform normalisation.

Hence, for uniform normalisation to hold some restrictions need to be imposed: We assume PRSs to be left-linear and fully-extended P$_2$RSs. For TRSs the fully-extendedness condition is vacuous, hence the assumption reduces to left-linearity as in Sect. 3 The restriction to P$_2$RSs entails no restriction w.r.t. the other formats, since both CRSs and ERSs can be embedded into P$_2$RSs, by coding metavariables in rules as free variables of type $o{\to}\ldots{\to}o{\to}o$ [23]. To adapt the proof of F$_1$TP to P$_2$RSs, we review its two main ingredients. The first one was a notion of simultaneous reduction, extending one-step reduction such that:

- The residual of a non-erasing step after a context-step is non-erasing.

The second ingredient was STD. It guarantees the following property:

- Any redex pattern l which is entirely above a contracted redex is external to the reduction S; in particular, l cannot be replicated along S, it can only be eliminated by contraction of an overlapping redex in S.

Since the residual of a parallel reduction after a step above it is usually not parallel, we switch from $\mathrel{+\!\!\!\!\!\!\;\|\!\!\!\rightarrow}$ to $\mathrel{-\!\!\circ\!\!\rightarrow}$, where the latter is the (one-rule restriction of the) simultaneous reduction relation of [21, Def. 3.4]. The context-part of such a $\mathrel{-\!\!\circ\!\!\rightarrow}$-step is the part above or parallel to all occurrences of l.

[6] A TRS step is non-erasing in this sense iff it is non-erasing in the sense of Def. 6.

Definition 11. *Let $\varrho : l \to r$ be a rewrite rule. Write $s \twoheadrightarrow_\varrho t$ if it holds that $s = C[l^{\sigma_1}, \ldots, l^{\sigma_k}]$ and $t = C[r^{\tau_1}, \ldots, r^{\tau_k}]$, where $\sigma_i \twoheadrightarrow_\varrho \tau_i$ for all $1 \leq i \leq k$.*

Lemma 3 (Finiteness of Developments). *(FD [20, Thm. 3.1.45]) Let $s \twoheadrightarrow t$ by simultaneously contracting redexes at positions in P. Repeated contraction of residuals of redexes in P starting from s terminates and ends in t.*

The second lemma on $\twoheadrightarrow$ is a close relative of [13, Lem. 5.1] and establishes the first ingredient above. It fails for P_3RSs as witnessed by the first item of Ex. 2.

Lemma 4 (Parallel Moves). *Let $\varrho : l \to r$ and $\vartheta : g \to d$ be PRS rules, with ϑ second-order. If $s' \leftarrow_\vartheta s \twoheadrightarrow_\varrho t$ is a fork such that g is in the context-part of the non-erasing simultaneous step, then the fork is joinable into $s' \twoheadrightarrow_\varrho t' \leftarrow_\vartheta t$, with the simultaneous step non-erasing.*

Proof. Joinability follows by FD. It remains to show non-erasingness. ϑ being of order 2, each free variable Z occurs in g as $Z(x_1, \ldots, x_n)$ with $x_i : o$ and $Z : o \to \ldots \to o \to o$ and in d as $Z(t_1, \ldots, t_n)$ with $t_i : o$. Hence, the residuals in s' of redexes of $s \twoheadrightarrow t$ are first-order substitution instances of them. Then, to show preservation of non-erasingness it suffices to show that $Var(s) \subseteq Var(s^\sigma)$ for any first-order substitution σ, which follows by induction on s. □

Left-linearity and fully-extendedness are sufficient conditions for STD to hold.

Theorem 7 (STD). *Any reduction in a P_2RS can be transformed into a standard one. The transformation preserves infiniteness.*

Proof. The proof of the second part of the theorem is as for TRSs. For a proof of the first part for left-linear fully-extended (orthogonal) CRSs see [18, Sect. 7.7.3] ([26]). By the correspondence between CRSs and P_2RSs this suffices for our purposes. (STD even holds for PRSs [22, Cor. 1.5].) □

Proof. (of Thm. 6) Replace in the proof of Thm. 2 everywhere $\twoheadrightarrow\!\!\!\!\!\!\!\!\;+$ by $\twoheadrightarrow$. That the (context)-case eventually applies follows by an appeal to FD. □

The proofs of the results below are obtained by analogous modifications.

Theorem 8. *Non-erasing rewrite steps are perpetual in biclosed P_2RSs.*

F_2TP can be strengthened in various ways. Unlike for TRSs, a critical step in a P_2RS need not erase a term in ∞ as witnessed by $e(f(x))\langle x{:=}a\rangle \to c\langle x{:=}a\rangle$ in the PRS $\{M(x)\langle x := N\rangle \to M(N), e(Z) \to c, f(a) \to f(a)\}$. Note that $f(x) \in \mathsf{SN}$, but by contracting the $\_\langle\_ := \_\rangle$-redex a is substituted for x and $f(a) \in \infty$.

Definition 12. *An occurrence of (the head symbol of) a subterm is* potentially infinite *if some descendant [20] of it along some reduction is in ∞. A step is ∞-erasing if it erases all potentially infinite subterms in its arguments.*

For TRSs this notion of ∞-erasingness coincides with the one of Def. 9.

Corollary 8. *Non-∞-erasing rewrite steps are perpetual in biclosed P_2RSs.*

Many variations of this result are possible. We mention two. First, the motivation for this paper originates with [13, Sect. 6.4], where we failed to obtain:

Theorem 9. *([5]) λ-δ_K-calculus is uniformly normalising.*

Proof. By Cor. 8, since λ-δ_K-calculus is weakly orthogonal. □

Second, we show that non-fully-extended P_2RSs may have uniform normalisation. By the same method, P_2RSs where non-fully-extended steps are terminating and postponable have uniform normalisation.

Theorem 10. *Non-∞-erasing steps are perpetual in $\lambda\beta\eta$-calculus [24, Prop. 27].*

Proof. It suffices to remark that η-steps can be postponed after β-steps in a standard reduction [2, Cor. 15.1.6]. Since η is terminating, an infinite standard reduction must contain infinitely many β-steps, hence may be assumed to consist of β's only and the proof of F_2TP goes through unchanged. □

5 $\lambda \mathrm{x}^-$

In this section familiarity with the nameful λ-calculus with explicit substitutions $\lambda\mathrm{x}^-$ of [4] is assumed. We define it as a *P_2RS* and establish the *fundamental theorem of perpetuality for $\lambda\mathrm{x}^-$*:

Theorem 11 (F_xTP). *Non-erasing steps are perpetual in $\lambda\mathrm{x}^-$.*

Definition 13. *The alphabet of $\lambda\mathrm{x}^-$ [4] consists of the function symbols $@ : o{\to}o{\to}o$, $\lambda : (o{\to}o){\to}o$ and $\_\langle\_ := \_\rangle : (o{\leftarrow}o){\to}o{\to}o$. As above, we make @ an implicit infix operator associating to the left. The rules of $\lambda\mathrm{x}^-$ are (for $x \neq y$):*

$$\begin{array}{rcl} (\lambda x.M(x))N & \to_{\mathrm{Beta}} & M(x)\langle x := N\rangle \\ x\langle x := N\rangle & \to_{=} & N \\ y\langle x := N\rangle & \to_{\neq} & y \\ (\lambda y.M(y,x))\langle x := N\rangle & \to_{\lambda} & \lambda y.M(y,x)\langle x := N\rangle \\ (M(x)L(x))\langle x := N\rangle & \to_{@} & M(x)\langle x := N\rangle L(x)\langle x := N\rangle \end{array}$$

The last four rules are the explicit substitution *rules denoted* x, *generating* $\to_{\mathrm{x}}$.

$\to_{\mathrm{x}}$ is a terminating and orthogonal *P_2RS*, hence the normal form of a term s exists uniquely and is denoted by $s{\downarrow_{\mathrm{x}}}$. Note that $s{\downarrow_{\mathrm{x}}}$ is a *pure* λ-term, i.e. it does not contain *closures* ($\_\langle\_ := \_\rangle$-symbols). $\lambda\mathrm{x}^-$ implements (only) substitution [4]:

Lemma 5. *1. If $s =_{\mathrm{x}} t$, then $s{\downarrow_{\mathrm{x}}} = t{\downarrow_{\mathrm{x}}}$.*
2. If $s \to_{\mathrm{Beta}} t$, then $s{\downarrow_{\mathrm{x}}} \twoheadrightarrow_{\beta} t{\downarrow_{\mathrm{x}}}$.
3. If s is pure and $s \to_{\beta} t$, then $s \to_{\mathrm{Beta}} \cdot \to_{\mathrm{x}}^{+} t$.

Remark that in the second item the number of β-steps might be zero, but is always positive when the Beta-step is not inside a closure. We call $\lambda\mathrm{x}^-$-reductions without steps inside closures *pretty*. $\lambda\mathrm{x}^-$ *preserves strong normalisation* in the sense that any pure term which is β-terminating is $\lambda\mathrm{x}^-$-terminating.

Lemma 6 (PSN). *[4, Thm. 4.19] If s is pure and β-SN, then s is λx^--SN.*

Proof. Suppose $s \in \infty$. Since λx^- is a fully-extended left-linear sub-P$_2$RS[7], we may by STD assume an infinite standard reduction $S : s_0 \to s_1 \to \ldots$ from $s = s_0$. We show that we may choose S to be pretty decent, where a reduction is *decent* [4, Def. 4.16] if for every closure $\langle x := t\rangle$ in any term, $t \in \mathsf{SN}$.

(init) s is decent since it is pure.

(step) Suppose $s_i \in \infty$ and s_i is decent. From the shape of the rules we have that 'brackets are king' [19][8]: if any step takes place in t inside some closure $\langle x := t\rangle$ in a standard reduction, then no step above the closure can be performed later in the reduction. This entails that if t is terminating, S need not perform any step inside t. Hence assume $s_i \to s_{i+1}$ is pretty.

(Beta) Suppose $s_i \to_{\mathrm{Beta}} s_{i+1}$ contracting $(\lambda x.M(x))N$ to $M(x)\langle x := N\rangle$. We may assume that N is terminating since otherwise we could instead perform an infinite reduction on N itself, hence the reduct is decent.

(x) Otherwise, decency is preserved, since x-steps do not create closures.

Since x is terminating S must contain infinitely many Beta-steps. Since S is pretty $S{\downarrow_{\mathrm{x}}}$ is an infinite β-reduction from s by (the remark after) Lem. 5. □

Our method relates to closure-tracking [3] as preventing to curing. Trying to apply it to prove [4, Conj. 6.45], stating that explicification of *redex preserving* CRSs is PSN, led to the following counterexample.

Example 3. Consider the term $s = (\tilde{\lambda}(x.b))a$ in the P$_2$RS $\mathcal{R}$ with rewrite rules $\{(\tilde{\lambda}x.M(x))N \to M(g(N,N)), a \to b, g(a,b) \to g(a,b)\}$. On the one hand s is terminating, since $s \to b[x{:=}g(a,a)] = b$. On the other hand, explicifying $\mathcal{R}$ will make s infinite, since $g(a,a) \to g(a,b) \to g(a,b)$. The PRS is redex preserving in the sense of [4, Def. 6.44] since any redex in the argument $g(N,N)$ to M occurs in N already. So s is a term for which PSN does not hold.

We expect the conjecture to hold for orthogonal CRSs. For our purpose, uniform normalisation, we will need the following corollary to Lem. 6, on preservation of infinity. It is useful in situations where terms are only the same up to the Substitution Lemma [2, Lem. 2.1.16]: $M(x,y)\langle x := N(y)\rangle\langle y := L\rangle{\downarrow_{\mathrm{x}}} = M(x,y)\langle y := P\rangle\langle x := N\langle y := L\rangle\rangle{\downarrow_{\mathrm{x}}}$.

Corollary 9. *If s is decent and $s{\downarrow_{\mathrm{x}}} = t{\downarrow_{\mathrm{x}}}$, then $s \in \infty$ implies $t \in \infty$.*

How should non-erasingness be defined for λx^-? The naïve attempt falters.

Example 4. From the term $s = ((\lambda x.z)(y\omega))\langle y := \omega\rangle$, where $\omega = \lambda x.xx$, we have a unique terminating reduction starting with a 'non-erasing' Beta-step:

$$s \to_{\mathrm{Beta}} z\langle x := y\omega\rangle\langle y := \omega\rangle \to_{\mathrm{x}} z\langle y := \omega\rangle \to_{\mathrm{x}} z$$

On the other hand, developing $\langle y := \omega\rangle$ yields the term $\omega\omega \in \infty$.

[7] It only is a *sub*-P$_2$RS since the y in the $\to_{\neq}$-rule ranges over variables not over terms.

[8] Thinking of terms as trees representing hierarchies of people, creating a redex above (overruling) someone (the ruler) from below (the people) is a revolution. For closures/brackets this is not possible, whence these are king.

Translating the example into $\lambda\beta$-calculus shows that the culprit is the 'non-erasing' Beta-step, which translates into an erasing β-step. Therefore:

Definition 14. *A* $\lambda\mathrm{x}^-$*-step contracting redex* s *to* t *is* erasing *if* $s \to t$ *is*

$$\begin{aligned}(\lambda x.M(x))N &\to_{\mathrm{Beta}} M, \text{ with } x \notin \mathit{Var}(M(x)\downarrow_{\mathrm{x}}), \text{ or}\\ y\langle x := N\rangle &\to_{\neq} \; y\end{aligned}$$

Proof. (of Thm. 11) Since $\lambda\mathrm{x}^-$ is a sub-P$_2$RS, it suffices by the proof of F$_2$TP to consider perpetuality of a step $s \to_{p,\varrho} t$, for some infinite standard reduction $S : s_0 \to_{q_0,\vartheta_0} s_1 \to_{q_1,\vartheta_1} \ldots$ starting from $s = s_0$ such that $s_1 \leftarrow s \to t$ is an overlapping fork (case (‡) on p. 7). $\lambda\mathrm{x}^-$ has only one non-trivial critical pair. It arises by @ and Beta from $s' = ((\lambda x.M(x,y))N(y))\langle y := P\rangle$, so let $s = C[s']$.

(Beta,@) In case $s \to_{\mathrm{Beta}} C[M(x,y)\langle x := N(y)\rangle\langle y := P\rangle] = s_1$, we note that

$$\begin{aligned}s &\to_{p,@} C[(\lambda x.M(x,y))\langle y := P\rangle N(y)\langle y := P\rangle] = t\\ &\to_{\lambda} C[(\lambda x.M(x,y)\langle y := P\rangle)N(y)\langle y := P\rangle]\\ &\to_{\mathrm{Beta}} C[M(x,y)\langle y := P\rangle\langle x := N(y)\langle y := P\rangle\rangle] = t_1\end{aligned}$$

Consider a minimal closure in s_1 (or s_1 itself) which is decent and ∞, say at position o. If o is parallel or properly below p, i.e. inside one of $M(x,y)$, $N(y)$ or P, then obviously $t_1 \in \infty$. Otherwise, o is above p and $t_1 \in \infty$ follows from Corollary 9, since $s_{1|o}\downarrow_{\mathrm{x}} = t_{1|o}\downarrow_{\mathrm{x}}$.

(@,Beta) The case $s \to_{p,\mathrm{Beta}} C[M(x,y)\langle x := N(y)\rangle\langle y := P\rangle] = t$ is more involved. Construct a maximal reduction T as follows. Let $t_0 = t$ be the first term of T and set $o_0 = p$.

- Suppose $s_i \to_{q_i,\vartheta_i} s_{i+1}$ does not contract a redex below o_i. As an invariant we will use that o_i traces the position of @ (initially at p) along S. If q_i is parallel to o_i, then we set $t_i \to_{q_i,\vartheta_i} t_{i+1}$. Otherwise $q_i < o_i$ and by standardness this is only possible in case of an @-step distributing closures over the @ at o_i. Then we set $t_{i+1} = t_i$ and $o_{i+1} = q_i$.

If this process continues, then T is infinite since in case no steps are generated $o_{i+1} < o_i$, hence eventually a step must be generated. If the process stops, say at n, then by construction $s_n = D[u]_{o_n}$ and $t_n = D[v]_{o_n}$, with $u = (\lambda x.M(x,y))\langle \boldsymbol{y} := \boldsymbol{P}\rangle N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle$, $v = M(x,y)\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle$ and $\langle \boldsymbol{y} := \boldsymbol{P}\rangle$ abbreviates a sequence of closures the first of which is $\langle y := P\rangle$. Per construction, $o_n \leq q_n$ for the step $s_n \to_{q_n} s_{n+1}$ and we are in the 'non-replicating' case: by standardness the @ cannot be replicated along S and it can only be eliminated as part of a Beta-step. Consider a maximal part of S not contracting o_n. Remark that if any of $M(x,y)$, $N(y)$ and $\boldsymbol{P}$ is infinite, then $t_n \in \infty$, so we assume them terminating.

(context) If infinitely many steps parallel to o_i take place, then $D \in \infty$, hence $t_n = D[v] \in \infty$.

(left) Suppose infinitely many steps are in $(\lambda x.M(x,y))\langle \boldsymbol{y} := \boldsymbol{P}\rangle$. This implies $M(x,y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle \in \infty$, hence $M(x,y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle\langle x := N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle\rangle \in \infty$, which by Corollary 9 implies $t_n \in \infty$.

(right) Suppose infinitely many steps are in $N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle$. By non-erasingness of $s \to_{\mathrm{Beta}} t$, $x \in M(x,y)\downarrow_{\mathrm{x}}$ hence

$$\begin{aligned} v &\twoheadrightarrow_{\mathrm{x}} M(x,y)\downarrow_{\mathrm{x}}\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle \\ &= E[x,\dots,x]\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle \\ &\twoheadrightarrow_{\mathrm{x}} E^*[x\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle,\dots,x\langle x := N(y)\rangle\langle \boldsymbol{y} := \boldsymbol{P}\rangle] \\ &\twoheadrightarrow_{=} E^*[N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle,\dots,N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle] \in \infty \end{aligned}$$

where E^* arises by pushing $\langle x{:=}N(y)\rangle\langle \boldsymbol{y}{:=}\boldsymbol{P}\rangle$ through E, and $E[\,,\dots,\,]$ is a pure λ-calculus context with at least one hole. Hence $t = D[v] \in \infty$.

(Beta) Suppose o_n is Beta-reduced sometime in S. By standardness steps before Beta can be neither in occurrences of the closures $\langle \boldsymbol{y} := \boldsymbol{P}\rangle$ nor in $M(x,y)$, hence we may assume S proceeds as:

$$\begin{aligned} s_n &\twoheadrightarrow_{\lambda} D[(\lambda x.M(x,y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle)N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle] \\ &\to_{\mathrm{Beta}} D[M(x,y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle\langle x := N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle\rangle] = u' \end{aligned}$$

We proceed as in item (Beta,@), using $u'\downarrow_{\mathrm{x}} = v\downarrow_{\mathrm{x}}$ to conclude $v \in \infty$ by Corollary 9. The only exception to this is an infinite reduction from $N(y)\langle \boldsymbol{y} := \boldsymbol{P}\rangle$, but such a reduction can be simulated from v by non-erasingness of the Beta-step as in item (right). ☐

The proof is structured as before, only di/polluted by explicit substitutions travelling through the pivotal Beta-redex. Again, one can vary on these results. For example, it should not be difficult to show that non-∞-erasing steps are perpetual, where $y\langle x{:=}N\rangle \to_{\neq} y$ is ∞-erasing if $N \in \infty$ and $(\lambda x.M(x))N \to_{\mathrm{Beta}} M$ is ∞-erasing if $x \notin \mathit{Var}(\mathrm{x}(M(x)))$ and N contains a *potentially infinite* subterm.

6 Conclusion

The uniform normalisation proofs in literature are mostly based on particular *perpetual* strategies, that is, strategies performing only perpetual steps. Observing that the non-computable[9] such strategies usually yield standard reductions we have based our proof on standardisation, instead of searching for yet another 'improved' perpetual strategy. This effort was successful and resulted in a flexible proof strategy with a simple invariant easily adaptable to a λ-calculus with explicit substitutions. Nevertheless, our results are still very much orthogonality-bound: the biclosedness results arise by tweaking orthogonality and the $\lambda\mathrm{x}^-$ results by interpretation in the, orthogonal, $\lambda\beta$-calculus. It would be interesting to see what can be done for truly non-orthogonal systems. The fully-extendedness and left-linearity restrictions are serious ones, e.g. in the area of process-calculi (scope extrusion) or even already for $\lambda\mathrm{x}$ [4], so should be ameliorated.

Acknowledgments We would like to thank R. Bloo, E. Bonelli, D. Kesner, P.-A. Melliès, A. Visser and the members of PAM at CWI for feedback.

[9] No computable strategy exists which is both perpetual and standard, since then one could for all terms s, t decide whether $\mathsf{SN}(s) \Rightarrow \mathsf{SN}(t)$ or $\mathsf{SN}(t) \Rightarrow \mathsf{SN}(s)$.

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