

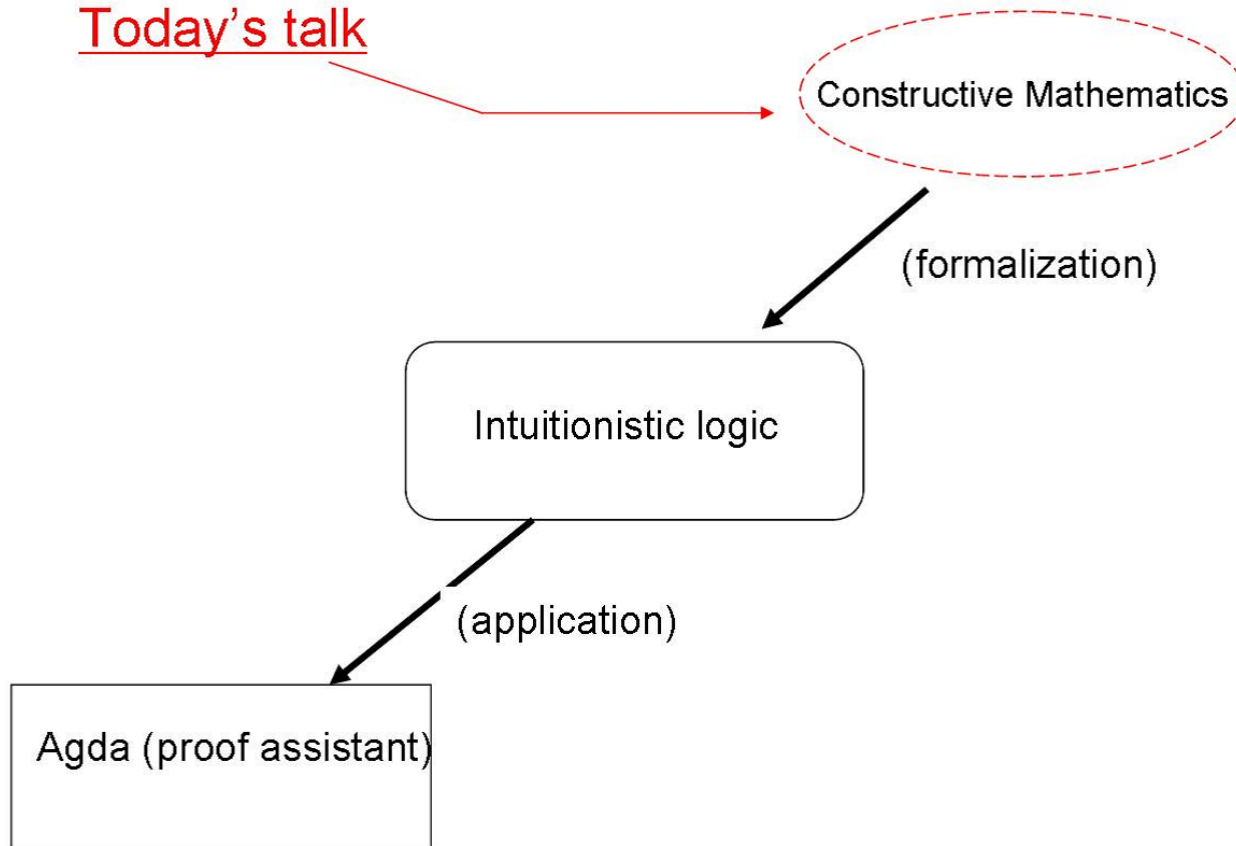
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A note on the weak topology for the
constructive completion of the space $\mathcal{D}(\mathbb{R})$

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Today's talk



Constructive mathematics is a framework of study for computability in mathematics.

Today we consider computability in generalized function theory, via constructive mathematics.

§1 Introduction

The space $\mathcal{D}(\mathbb{R})$ is an important example of a non-metrizable locally convex space.

A *distribution* (or *generalized function*) is a sequentially continuous linear functional on $\mathcal{D}(\mathbb{R})$ (a sequentially continuous linear function from $\mathcal{D}(\mathbb{R})$ into \mathbb{R}).

We had not had the start of constructive theory of generalized functions until the following problems, given by the 1960's monograph of Bishop, were solved:

1. How constructively difficult is the completeness properties of $\mathcal{D}(\mathbb{R})$ and its dual space $\mathcal{D}^*(\mathbb{R})$ (i.e. the space of distributions)?
2. Can we take the natural forms of the constructive completions of $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}^*(\mathbb{R})$ respectively ?

These problems are solved by the following results:

(1a) $\mathcal{D}(\mathbb{R})$ is complete if and only if the principle $BD-\mathbb{N}$ is can be proved [Ishihara and Y, '02].

(2) The constructive completion $\tilde{\mathcal{D}}(\mathbb{R})$ of $\mathcal{D}(\mathbb{R})$ is obtained by generalizing every element (test functions) of $\mathcal{D}(\mathbb{R})$ [Y, '05].

(1b) The dual space $\tilde{\mathcal{D}}^*(\mathbb{R})$ of $\tilde{\mathcal{D}}(\mathbb{R})$ is equal to the dual space $\mathcal{D}^*(\mathbb{R})$ as sets, and the weak completeness of $\mathcal{D}^*(\mathbb{R})$ can be proved [Y, '03].

$$\begin{array}{ccc}
 \text{weakly complete} \rightarrow & \widetilde{\mathcal{D}}^*(\mathbb{R}) & = & \mathcal{D}^*(\mathbb{R}) & (*: \text{ dual spaces}) \\
 & & (1b) & & \\
 & \uparrow & & \uparrow & \\
 (\sim: \text{ completion}) & \widetilde{\mathcal{D}}(\mathbb{R}) & \neq & \mathcal{D}(\mathbb{R}) & \\
 & (2) & (1a) & &
 \end{array}$$

Fig. The spaces $\mathcal{D}(\mathbb{R})$ and $\widetilde{\mathcal{D}}(\mathbb{R})$ in Bishop's constructive mathematics.

Here

1a $\mathcal{D}(\mathbb{R})$ is complete if and only if BD- \mathbb{N} can be proved.

Notes:

A subset A of \mathbb{N} is *pseudobounded* if
 $\forall \{a_n\} \in A^{\mathbb{N}} \exists N \in \mathbb{N} [a_n < n] \quad (n \geq N)$.

A bounded subset of \mathbb{N} is pseudobounded.

BD- \mathbb{N} : every countable *pseudobounded* subset of \mathbb{N} is bounded.

BD- \mathbb{N} can be proved in

classical mathematics,

Brouwer's intuitionistic mathematics and

constructive recursive mathematics of Markov's school,

but **not in Bishop's constructive mathematics.**

2 The constructive completion $\tilde{\mathcal{D}}(\mathbb{R})$ is obtained by generalizing every element (test functions) of $\mathcal{D}(\mathbb{R})$

Note:

- A *test function* is an infinitely differentiable function on \mathbb{R} with *compact support*.
- A element of $\tilde{\mathcal{D}}(\mathbb{R})$ is an infinitely differentiable function on \mathbb{R} with *pseudobounded support*.

1b The weak completeness of the dual space $\mathcal{D}^*(\mathbb{R})$ holds.

This matter follows from [the Banach-Steinhaus theorem for \$\mathcal{D}\(\mathbb{R}\)\$](#) : for a sequence $\{u_k\}$ of distributions (sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$), if the sequence $\{u_k(\phi)\}$ converges in \mathbb{R} for any ϕ in $\mathcal{D}(\mathbb{R})$, then the limit u exists and is a distribution.

The completeness of $\mathcal{D}(\mathbb{R})$ is not necessary for proving the weak completeness of $\mathcal{D}^*(\mathbb{R})$, although many classical proofs require it.

Now this version of Banach-Steinhaus theorem was proved by showing the following properties.

- The Banach-Steinhaus theorem for the completion $\tilde{\mathcal{D}}(\mathbb{R})$.
- every distribution is uniquely extended to $\tilde{\mathcal{D}}(\mathbb{R})$.

In particular, the second implies that $\tilde{\mathcal{D}}^*(\mathbb{R}) = \mathcal{D}^*(\mathbb{R})$ as sets.

We then have a question:

is $\tilde{\mathcal{D}}^*(\mathbb{R})$ topologically equivalent to $\mathcal{D}^*(\mathbb{R})$?

We discuss this problem in this talk.

$$\widetilde{\mathcal{D}}^*(\mathbb{R}) \stackrel{?}{\simeq} \mathcal{D}^*(\mathbb{R}) \quad (*: \text{dual spaces})$$

$$\uparrow$$

$$\uparrow$$

$$(\sim: \text{completion}) \quad \widetilde{\mathcal{D}}(\mathbb{R}) \neq \mathcal{D}(\mathbb{R})$$

Fig. The spaces $\mathcal{D}(\mathbb{R})$ and $\widetilde{\mathcal{D}}(\mathbb{R})$ in Bishop's constructive mathematics.

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§2 Preliminary

(2a) The principle $\text{BD-}\mathbb{N}$ in constructive mathematics

BHK(Brouwer-Heyting-Kolmogorov)-interpretation

	constructive math.	classical math.
$P \vee Q$	we can judge P or Q by finite procedures	$\neg P$ and $\neg Q$ imply a contradiction
$\exists x P(x)$	we can construct c such that $P(c)$ by finite procedures	" $\neg P(x)$ for all x " implies a contradiction

Under this interpretation, **Principle of Excluded Middle (PEM)**

for all proposition P , $P \vee \neg P$
and its negation cannot be proved.

Bishop's constructive mathematics is mathematics with BHK-interpretation.

The three frameworks of constructive mathematics:

- Brouwer's **intuitionistic mathematics**
- **Constructive recursive mathematics** of Markov's school
- Bishop's **Constructive mathematics**

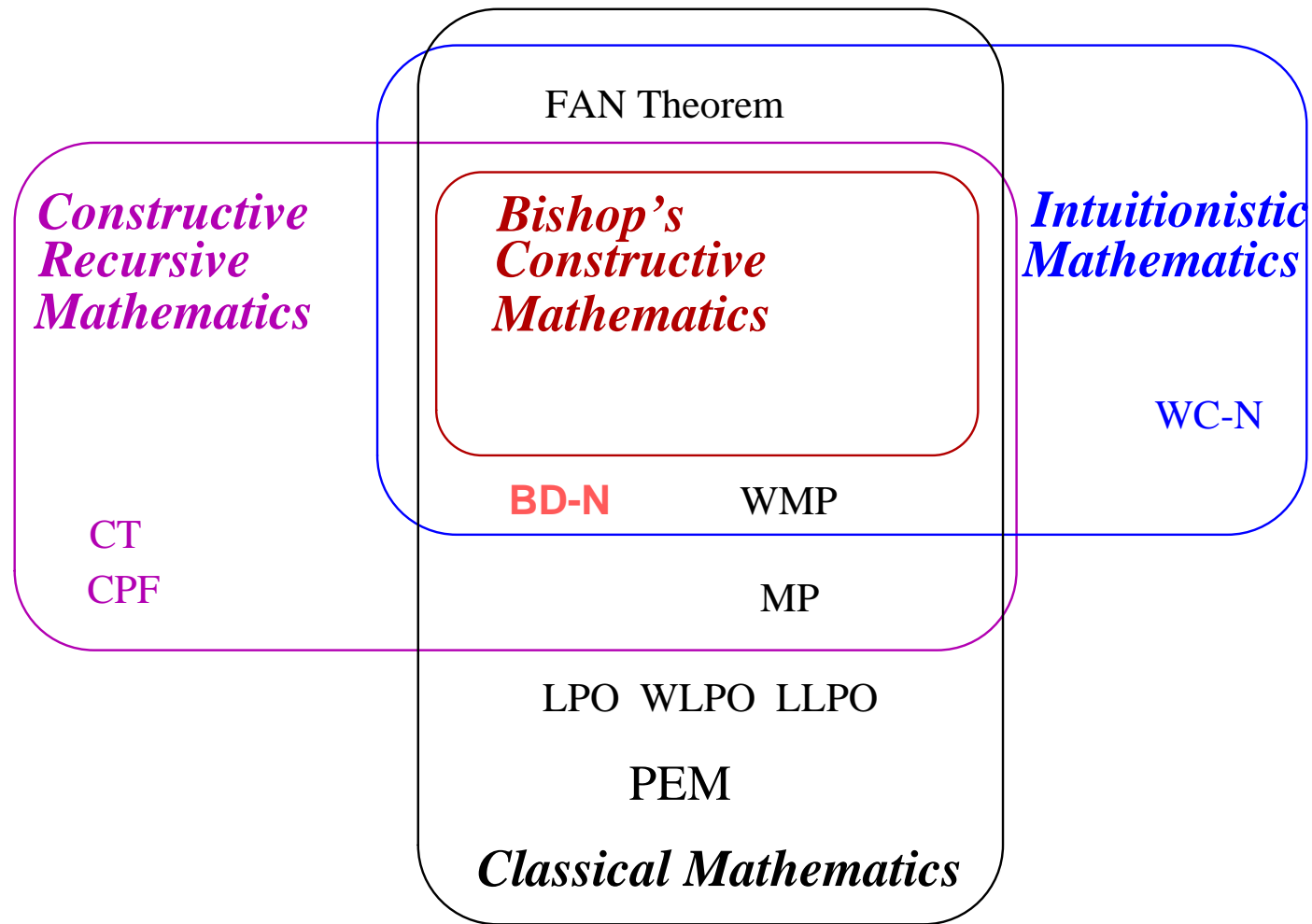


Fig.2.1 The sets of theorems in the four frameworks

(2b) Locally convex spaces

Let X be a vector space over \mathbb{R} .

A mapping $p : X \rightarrow \mathbb{R}^{0+}$ is a **seminorm** on X if it satisfies that for $x, y \in X$ and $\lambda \in \mathbb{R}$, (1) $p(x+y) \leq p(x) + p(y)$ and (2) $p(\lambda x) = |\lambda|p(x)$.

A pair $(X, \{p_i\})$ is **locally convex space** over \mathbb{R} if for all index i and x in X , whenever $p_i(x) = 0$ then $x = 0$.

$\{x_n\}$ **converges** to x in $X \stackrel{\text{def}}{\iff}$
 $\forall k \in \mathbb{N} \forall i \in I \exists N \in \mathbb{N} [n \geq N \implies p_i(x - x_n) < 2^{-k}]$.

Let u be a linear functional on X .

u is **sequentially continuous** on $X \stackrel{\text{def}}{\iff}$
for each sequence $\{x_n\}$ in X and $x \in X$
 $\{x_n\}$ converges to x in X
 \implies the sequence $\{u(x_n)\}$ converges to $u(x)$ in \mathbb{R} .

The **dual space** X^* **with weak topology** of a locally convex space $(X, \{p_i\})$ is a locally convex space of sequentially continuous linear functionals on X , with the seminorms $\{\|\cdot\|_x\}$ defined by $\|u\|_x := |u(x)| \quad (x \in X)$.

§3 The space $\mathcal{D}(\mathbb{R})$ and its completion $\widetilde{\mathcal{D}}(\mathbb{R})$

The set $\text{supp} f$ denotes the closure of the set $\{x \in \mathbb{R} : |f(x)| > 0\}$ in Euclid space \mathbb{R} .

Set $\text{supp}_{\mathbb{N}} f := \{0\} \cup \{n \in \mathbb{N} : \exists q \in \mathbb{Q}[|q| \geq n \wedge |f(q)| > 0]\}$.

Notes that for a continuous function f , $\text{supp} f$ is bounded \iff $\text{supp}_{\mathbb{N}} f$ is bounded.

Example 1 (The bump function).

$$\phi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then $\text{supp}\phi = [-1, 1]$ and $\text{supp}_{\mathbb{N}}\phi = \{0\}$.

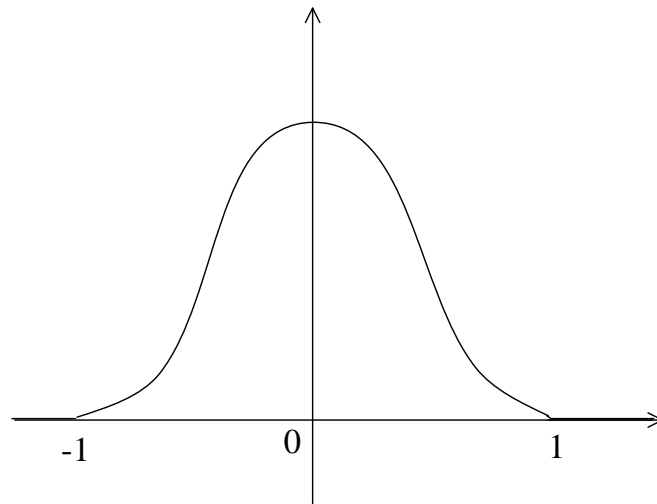


Fig.3.2 An example of a test function: the bump function

f *has compact support* $\stackrel{\text{def}}{\Leftrightarrow}$ $\text{supp } f$ is bounded.

f *has pseudobounded support* $\stackrel{\text{def}}{\Leftrightarrow}$ $\text{supp}_{\mathbb{N}} f$ is pseudobounded.

A *test function* is an infinitely differentiable functions from \mathbb{R} into it itself with compact support.

$\mathcal{D}(\mathbb{R})$ denotes the locally convex space of test functions with the seminorms

$$p_{\alpha, \beta}(\phi) := \sup_n \max_{l \leq \beta(n)} \sup_{|x| \geq n} 2^{\alpha(n)} |\phi^{(l)}(x)| \quad (\phi \in \mathcal{D}(\mathbb{R}), \alpha, \beta \in \mathbb{N} \rightarrow \mathbb{N}).$$

$\tilde{\mathcal{D}}(\mathbb{R})$ denotes the locally convex space of infinitely differentiable functions on \mathbb{R} with pseudobounded support, taken with the seminorms $\{p_{\alpha, \beta}\}$.

§4 The dual spaces $\mathcal{D}^*(\mathbb{R})$ and $\tilde{\mathcal{D}}^*(\mathbb{R})$

Theorem 2 (Y, '03). *Every distribution is uniquely extended to $\tilde{\mathcal{D}}(\mathbb{R})$.*

That is, $\mathcal{D}^*(\mathbb{R}) = \tilde{\mathcal{D}}^*(\mathbb{R})$ as sets.

Theorem 3. *A sequence $\{u_n\}$ of distributions converges to 0 in $\mathcal{D}^*(\mathbb{R})$ if and only if it does in $\tilde{\mathcal{D}}^*(\mathbb{R})$.*

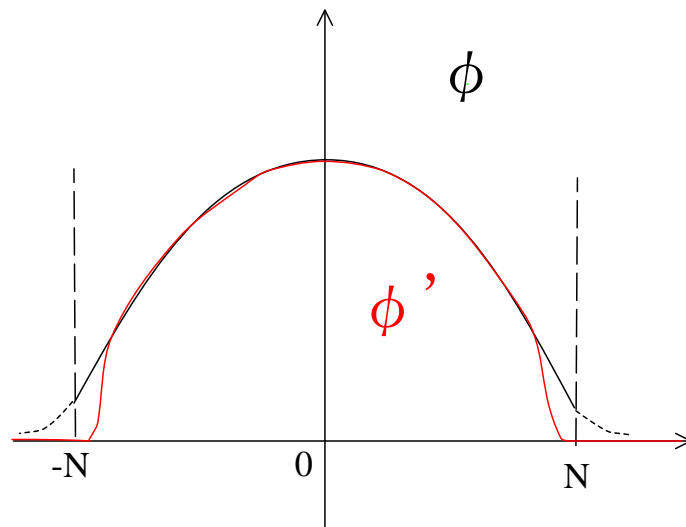
(Proof) We consider the part “only if”.

It is sufficient to show that if $\{u_n\} \rightarrow 0$ in $\mathcal{D}^*(\mathbb{R})$, then $\{u_n\} \rightarrow 0$ in $\tilde{\mathcal{D}}^*(\mathbb{R})$.

That is, we show that $\{u_n(\phi)\} \rightarrow 0$ for all $\phi \in \mathcal{D}^*(\mathbb{R})$, then $\{u_n\} \rightarrow 0$ for all $\phi' \in \tilde{\mathcal{D}}^*(\mathbb{R})$.

Let $\{u_n\}$ be a given sequence of distributions, ϕ be in $\tilde{\mathcal{D}}(\mathbb{R})$, and any k in \mathbb{N} .

We can then construct some ϕ' in $\mathcal{D}^*(\mathbb{R})$ such that for some N in \mathbb{N} , $|u_n(\phi')| < 2^{-(k+1)} \rightarrow |u_n(\phi)| < 2^{-k} \quad (n \geq N)$.



(q.e.d)

We again note that

Given a sequence $\{u_n\}$ of distributions and an element ϕ in $\tilde{\mathcal{D}}(\mathbb{R})$, we can construct a test function ϕ' such that for some N in \mathbb{N} ,

$$\|u_n\|_{\phi'} < 2^{-(k+1)} \rightarrow \|u_n\|_{\phi} < 2^{-k} \quad (n \geq N)$$

That is, $\tilde{\mathcal{D}}^*(\mathbb{R})$ is equal to $\mathcal{D}^*(\mathbb{R})$ w.r.t. **convergence**.

$$\widetilde{\mathcal{D}}^*(\mathbb{R}) \simeq \mathcal{D}^*(\mathbb{R}) \quad (*: \text{ dual spaces})$$

↑

↑

(\sim : completion)

$$\widetilde{\mathcal{D}}(\mathbb{R}) \neq \mathcal{D}(\mathbb{R})$$

Open question:

Is $\tilde{\mathcal{D}}^*(\mathbb{R})$ equal to $\mathcal{D}^*(\mathbb{R})$ w.r.t. neighbourhoods?

Reference

Satoru Yoshida,

A note on weak topology for the constructive completion of the space $\mathcal{D}(\mathbb{R})$,

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