A sequent calculus for Limit Computable Mathematics

Stefano Berardi and Yoriyuki Yamagata

Background : LCM

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- Part of Hayashi's "Proof Animation Project"

LCM and classical logic

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$$EM_1(P) \equiv \forall x(\exists y Pxy \vee \forall y \neg Pxy)$$

P : decidable, is valid in LCM, while

$$EM_2(Q) \equiv \forall x(\exists y \forall z Qxyz \vee \forall y \exists z \neg Qxyz)$$

Q : decidable, is **not** valid.

Strength of LCM

Akama, Berardi, Hayashi, Kohlenbach (2004)

- Known : Implies WKL_0 in higher order setting (with a weak form of Axiom Choice)

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- Known : Implies WKL_0 in higher order setting (with a weak form of Axiom Choice)
- Conjecture : Intuitionism + EM_1

Game semantics of LCM

1-bck. game : Simple extension of Lorenzen/Hintikka game

Theorem. (Berardi, Coquand, Hayashi 2005)
A is valid in LCM $\Leftrightarrow$ Prover ($\mathcal{E}$) is winning in 1-bck. game of A.

Our contribution

Give an infinitary logic $\mathbf{PA}_1$ for LCM.

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Isomorphism Theorem.

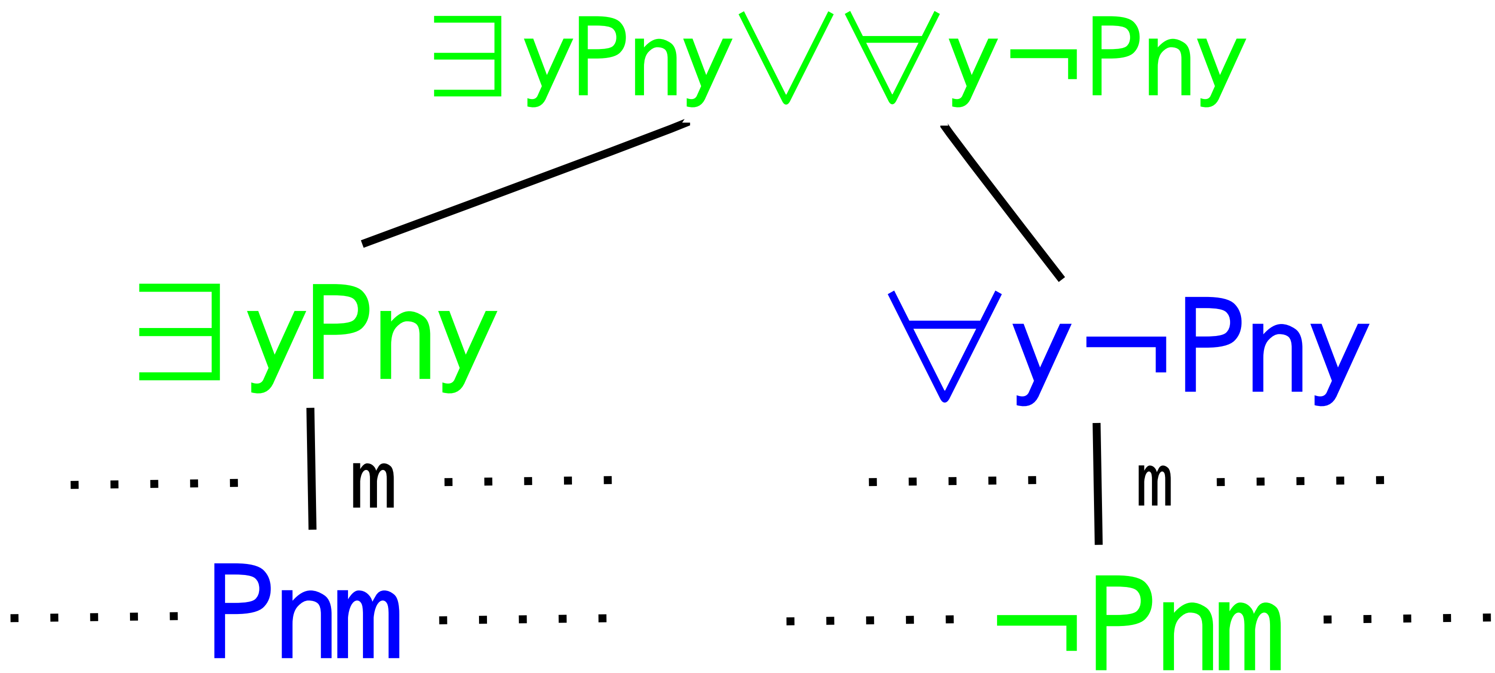
A proof π of formula A in PA_1

$\leftrightarrow_{\text{1:1,tree-iso.}}$

a winning strategy of 1-bck. game of A.

Lorenzen/Hintikka game

2-person game between $\mathcal{E}$ and $\mathcal{A}$. Conjunctions and false atomics are played by $\mathcal{E}$, otherwise positions are played by $\mathcal{A}$. Pnm below is true.



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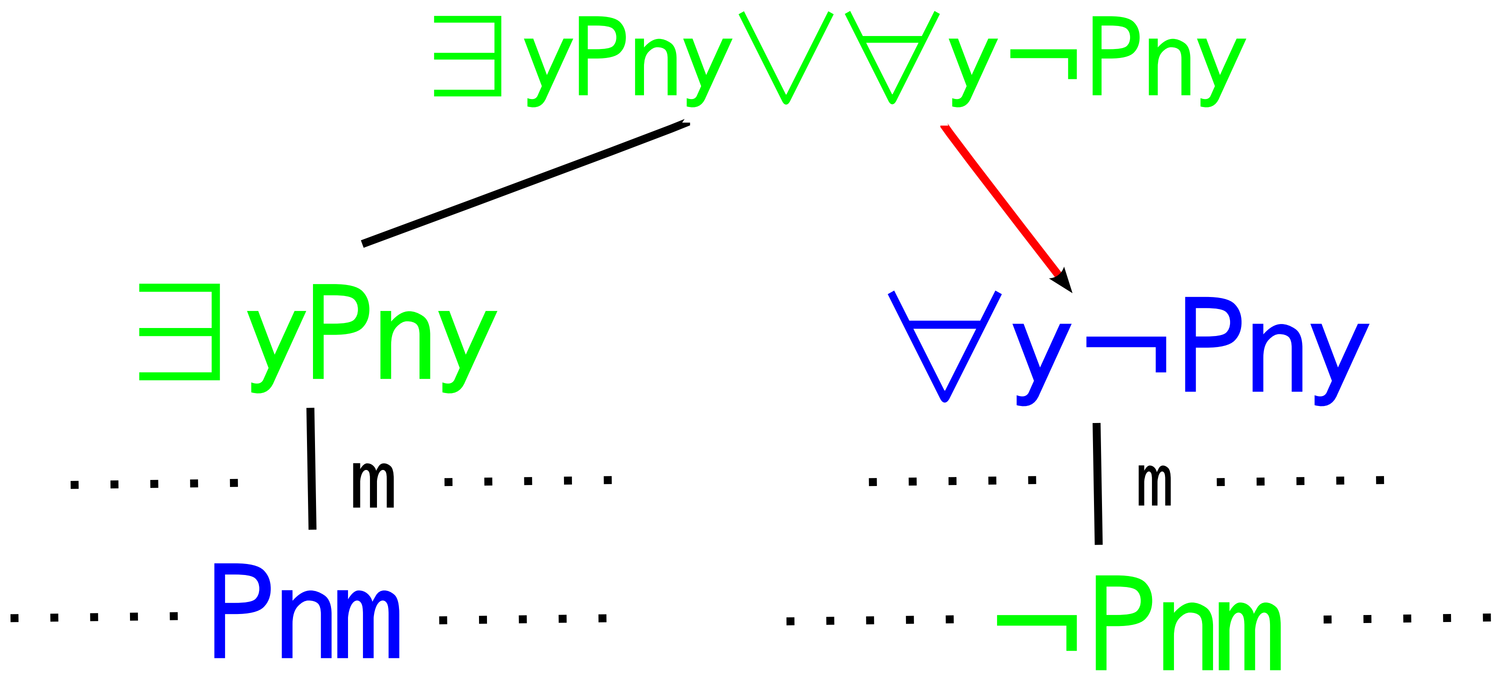

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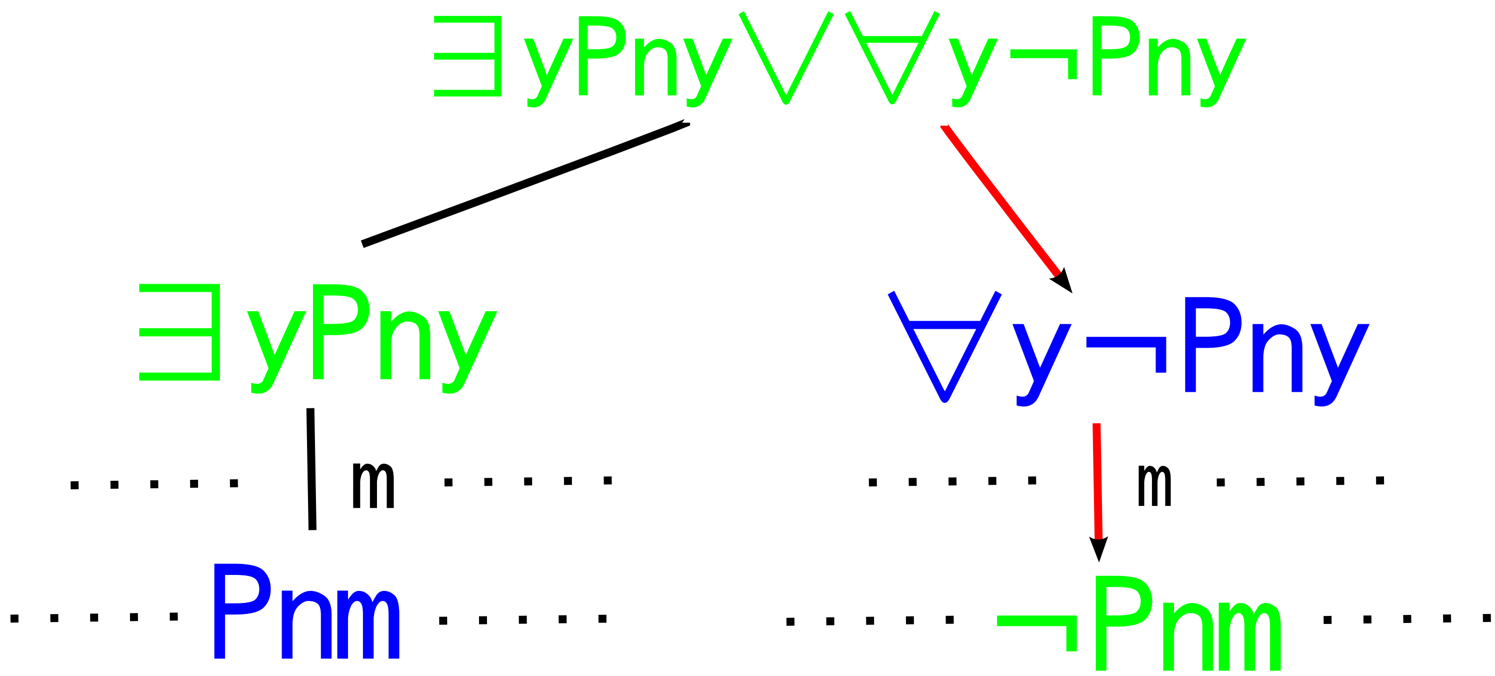
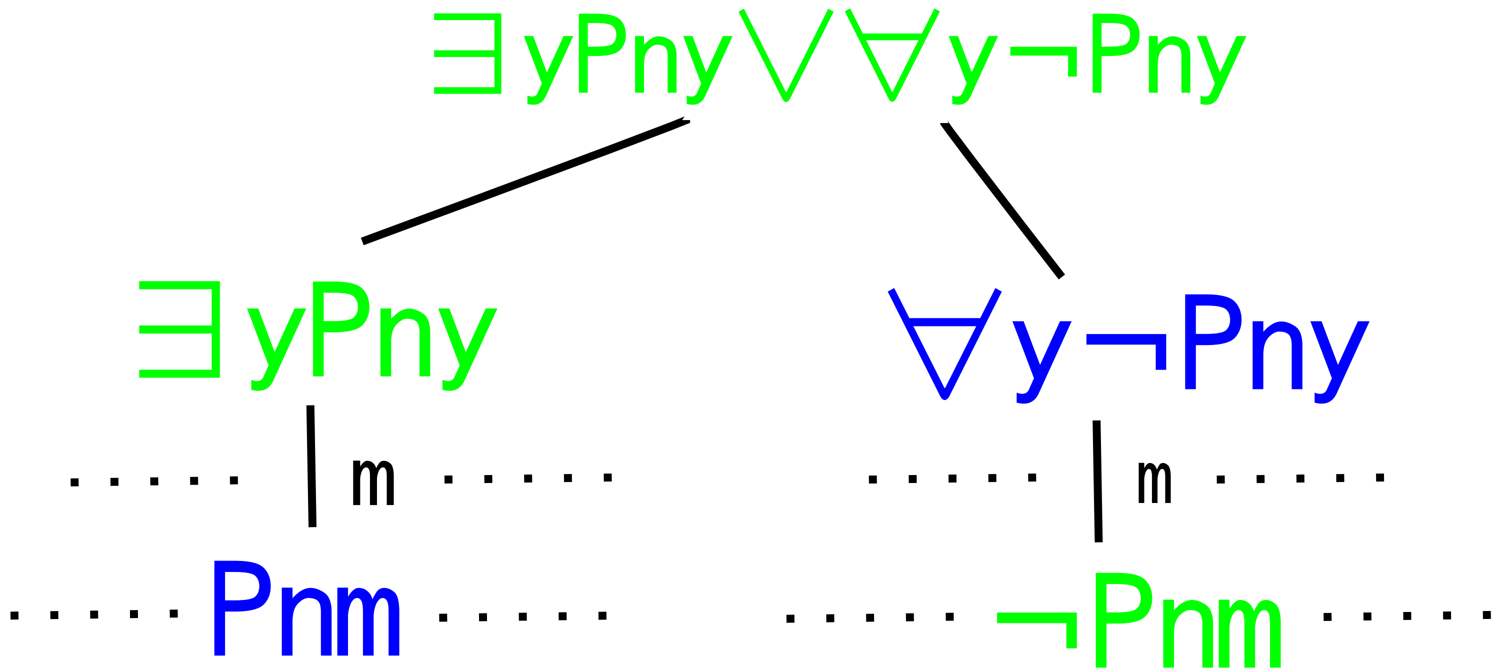


$\mathcal{E}$ looses here.

1-bck. game

Similar to Lorenzen/Hintikka game, but $\mathcal{E}$ can backtrack to the previous position and restart the game from there. The position $\mathcal{E}$ backtracks must be an ancestor of the current position.



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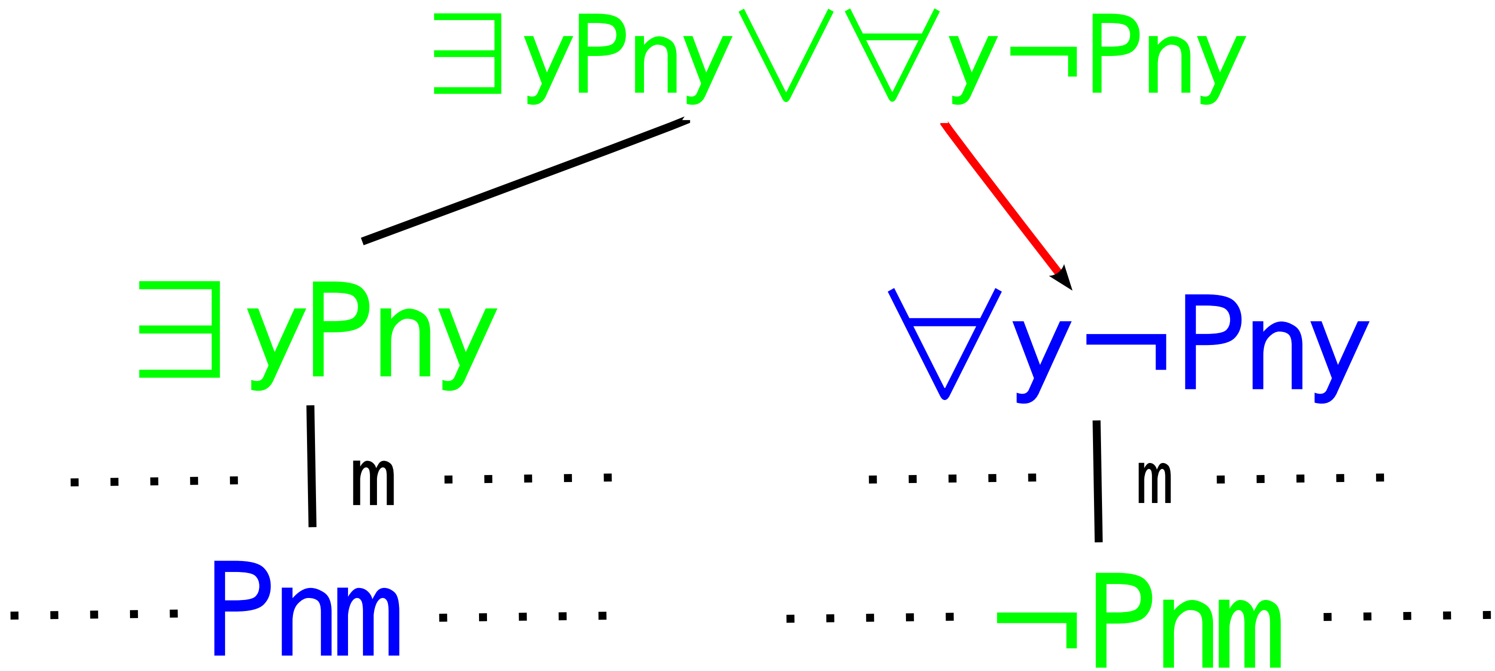

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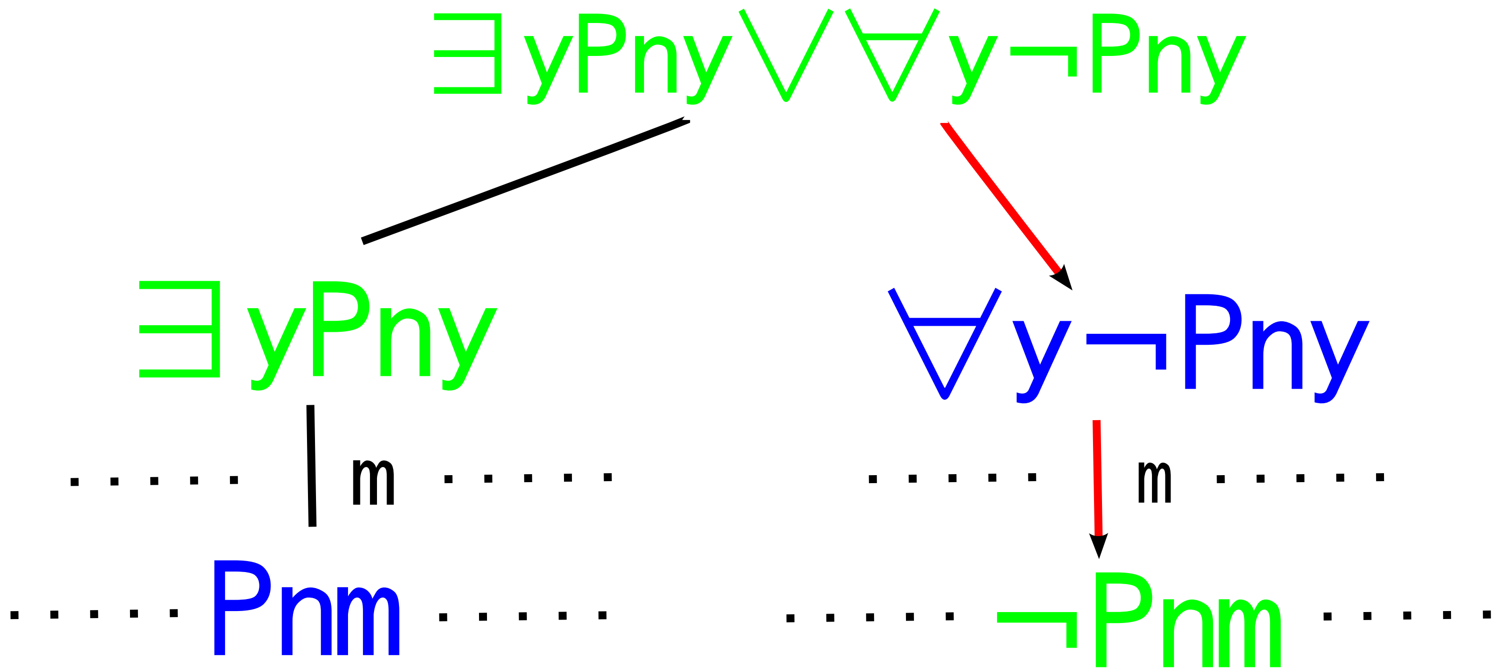

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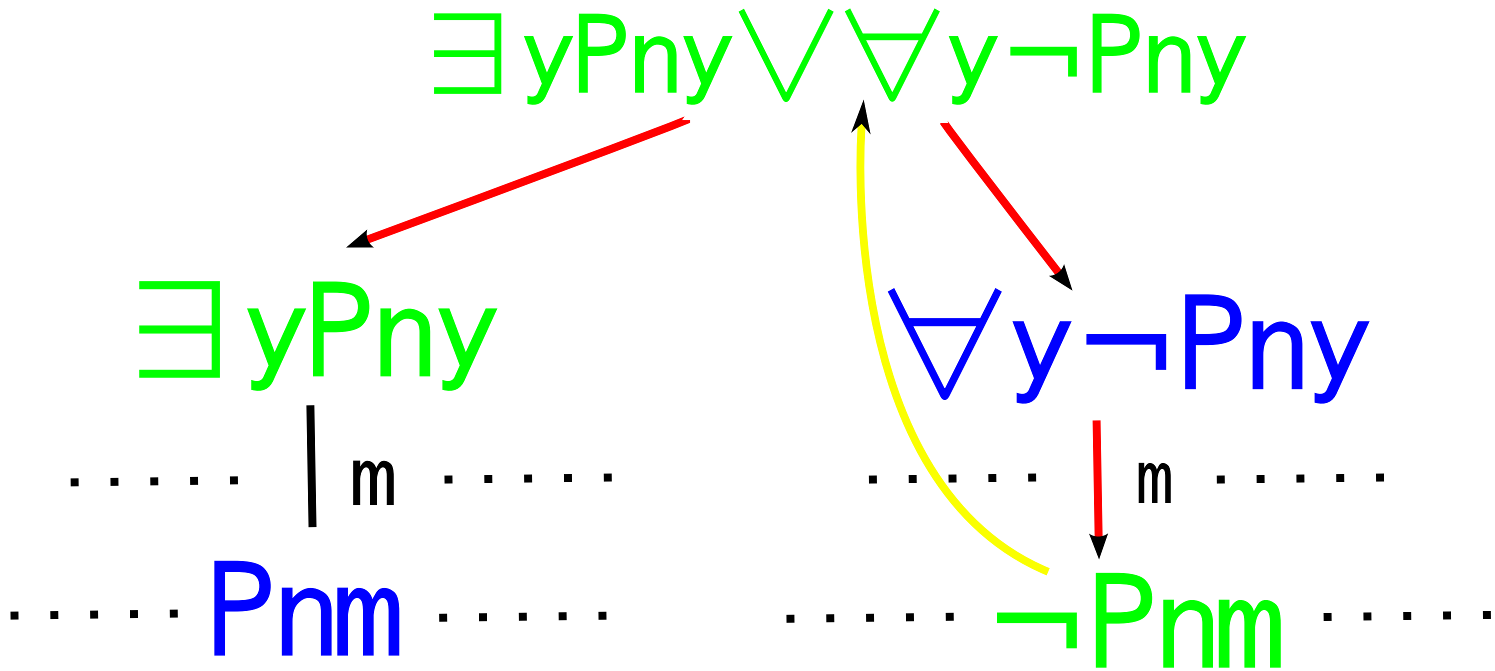

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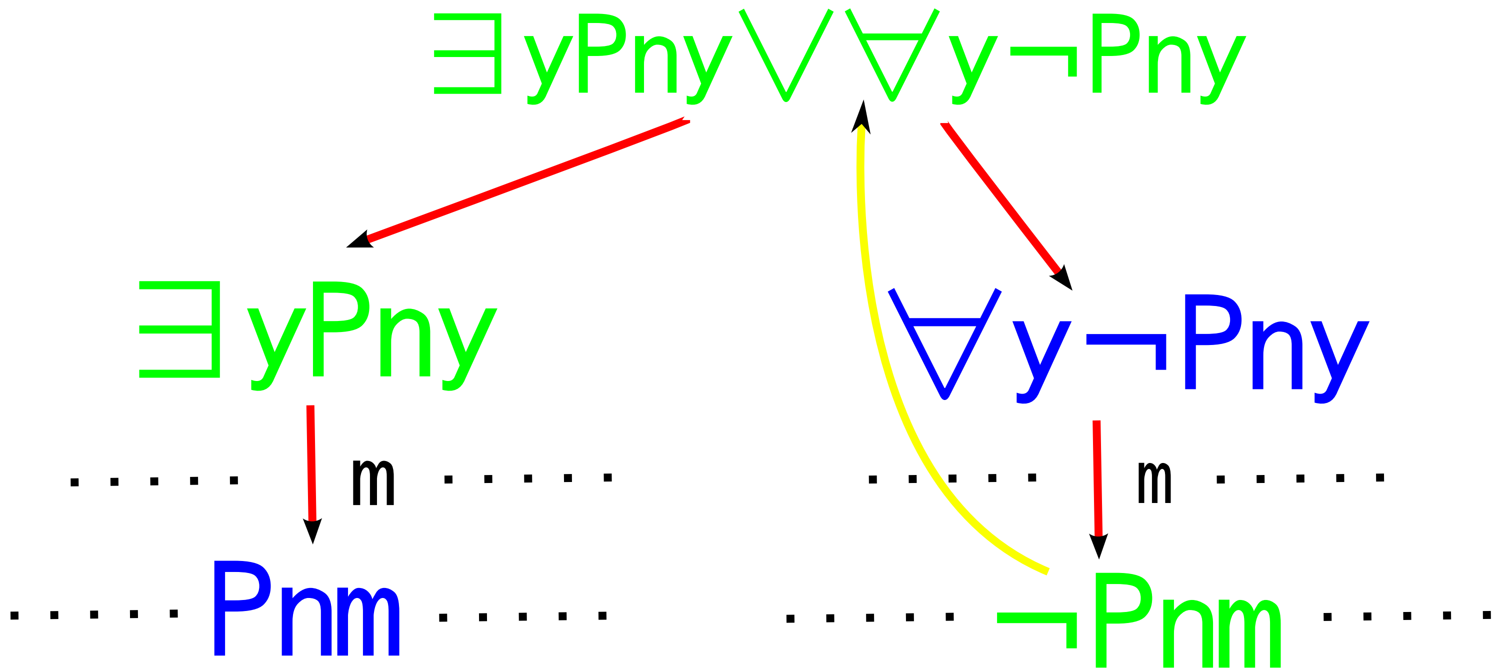

$\mathcal{E}$ wins.

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$\text{Formulas } F ::= Px \mid F \wedge F \mid F \vee F \mid \forall x F \mid \exists x F$

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This interpretation naturally leads to inference rules.

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where p is a true atomic.

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Since $\mathcal{E}$ is going to win in this position, no more need of strategies.

PA_1 : Conjunctions

$\mathcal{A}$ moves at conjunctions

$$\frac{\vdash \Gamma, A_1 \qquad \vdash \Gamma, A_2}{\vdash \Gamma, A_1 \wedge A_2} \wedge$$

$$\frac{\vdash \Gamma, A(0) \qquad \ldots \qquad \vdash \Gamma, A(n) \qquad \ldots}{\vdash \Gamma, \forall x A(x)} \forall$$

$\mathcal{E}$ prepares all possible moves of $\mathcal{A}$.

PA_1 : Disjunction

$$\frac{\vdash \Gamma, A_1 \vee A_2, A_i}{\vdash \Gamma, A_1 \vee A_2, \Delta} \vee$$

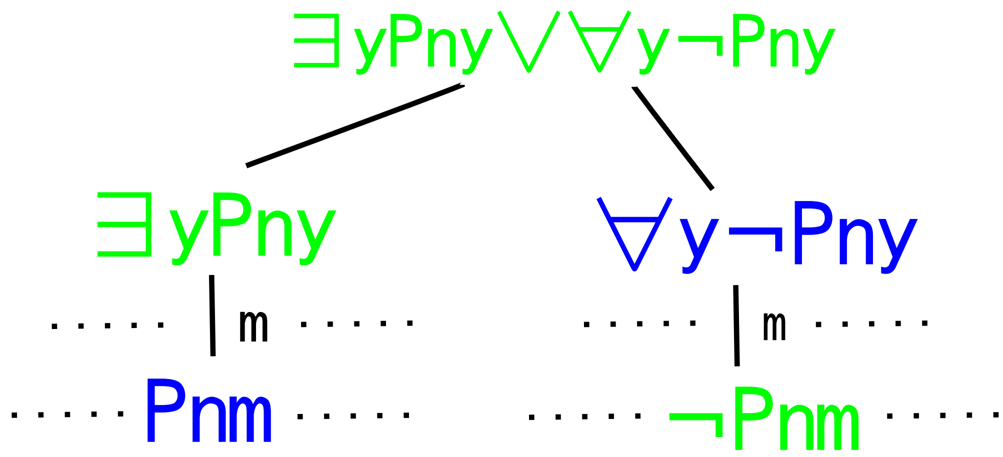
$\mathcal{E}$ retracts all moves in Δ and backtracks to $A_1 \vee A_2$, then chooses a node A_i.

PA_1 : Disjunction

$$\frac{\vdash \Gamma, \exists x A(x), A(n)}{\vdash \Gamma, \exists x A(x), \Delta} \exists$$

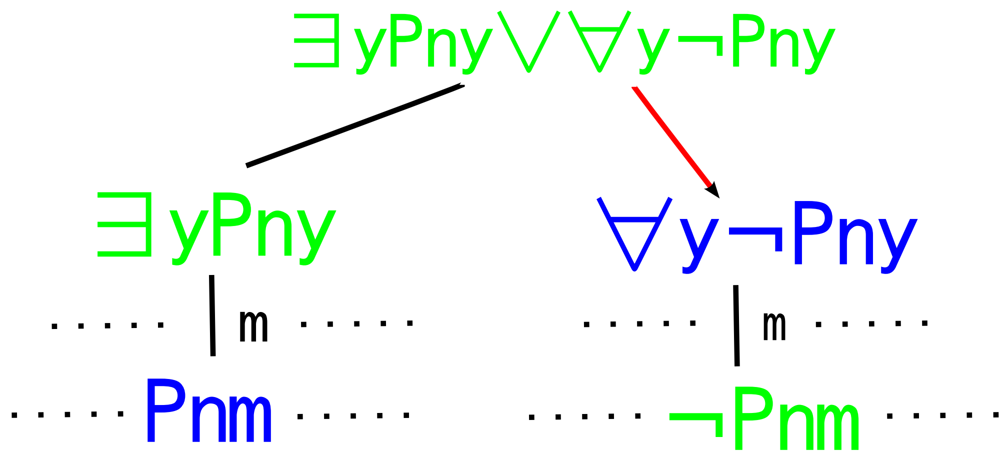
$\mathcal{E}$ retracts all moves in Δ and backtracks to $\exists x A(x)$, then chooses a node $A(n)$.

Proof and Game play



$$\vdash \exists y Pny \vee \forall y \neg Pny$$

Proof and Game play



$$\frac{\vdash \exists y Pny \vee \forall y \neg Pny, \forall y \neg Pny}{\vdash \exists y Pny \vee \forall y \neg Pny} \ \vee$$

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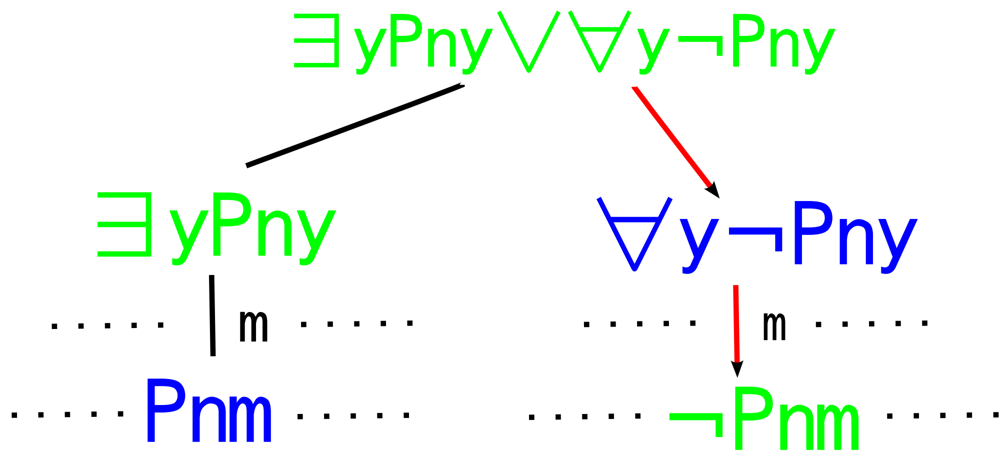

$$\dfrac{\dfrac{\ldots \quad \vdash \exists y Pny \vee \forall y \neg Pny, \neg Pnm \quad \ldots}{\vdash \exists y Pny \vee \forall y \neg Pny, \forall y \neg Pny}\ \forall}{\vdash \exists y Pny \vee \forall y \neg Pny}\ \vee$$

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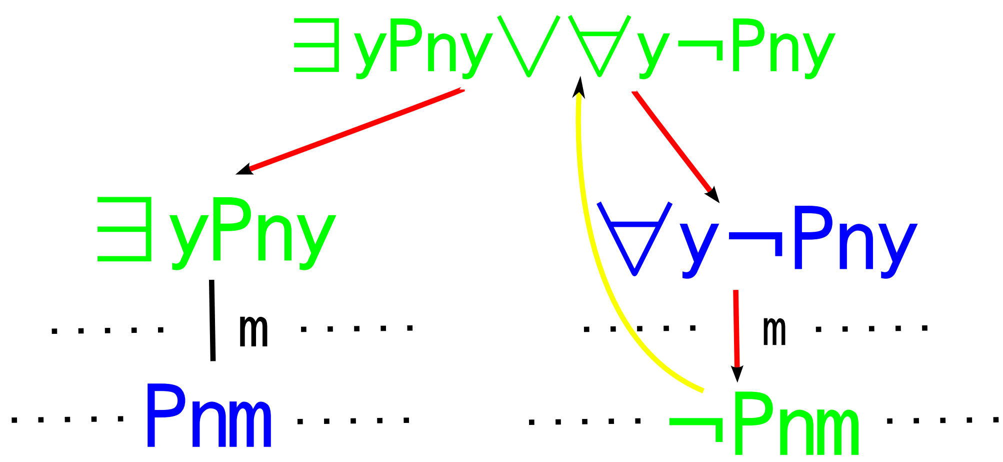

$$\dfrac{\cdots \quad \dfrac{\vdash \exists yPny \vee \forall y\neg Pny, \exists yPny}{\vdash \exists yPny \vee \forall y\neg Pny, \neg Pnm}\vee \quad \cdots}{\dfrac{\vdash \exists yPny \vee \forall y\neg Pny, \forall y\neg Pny}{\vdash \exists yPny \vee \forall y\neg Pny}\vee}\forall$$

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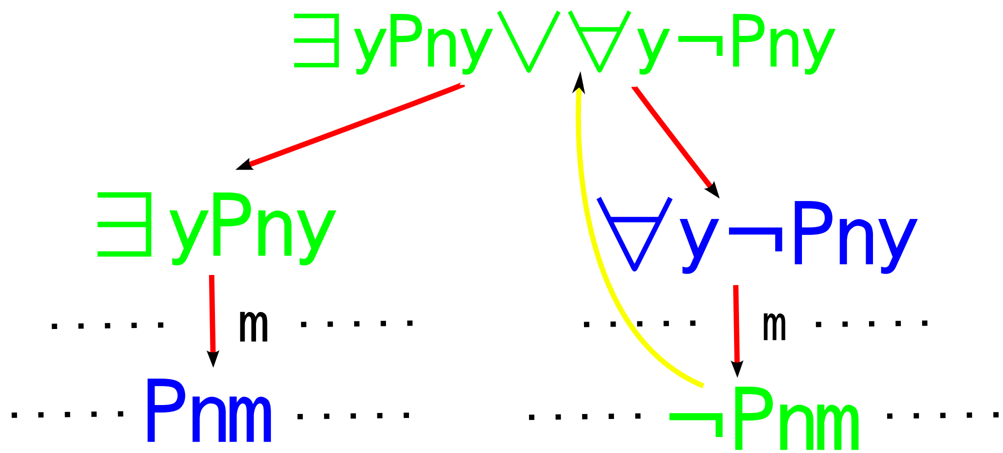

$$
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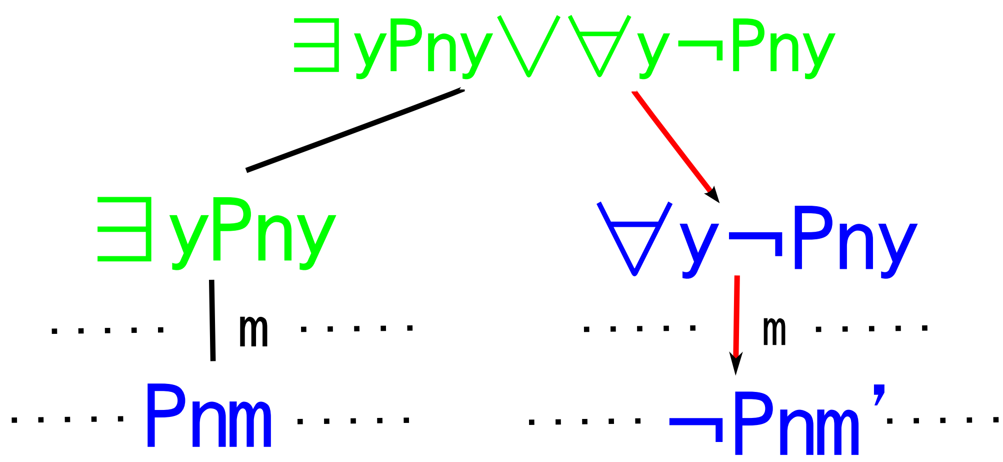

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Conclusion

- We introduce a proof system PA_1, an ω-logic without Exchange
- We show proofs of formula A in PA_1 and winning strategies of 1-bck. games over A has a tree-isomorphism

Future work

- Interpretation of Cut-rule.
- Interpretation of implication and Modus ponens
- Relation to cut-elimination

The End