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A CONSTRUCTIVE LOOK AT THE COMPLETENESS OF THE SPACE $\mathcal{D}(\mathbb{R})$

HAJIME ISHIHARA AND SATORU YOSHIDA

Abstract. We show, within the framework of Bishop’s constructive mathematics, that (sequential) completeness of the locally convex space $\mathcal{D}(\mathbb{R})$ of test functions is equivalent to the principle BD-$\mathbb{N}$ which holds in classical mathematics, Brouwer’s intuitionism and Markov’s constructive recursive mathematics, but does not hold in Bishop’s constructivism.

§1. Introduction. The space $\mathcal{D}(\mathbb{R})$ of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with compact support together with a locally convex structure defined by the seminorms

$$p_{\alpha, \beta}(f) := \sup_n \sup_{l \leq \beta(n)} \sup_{|x| \geq n} 2^{\alpha(n)} |f^{(l)}(x)| \quad (\alpha, \beta \in \mathbb{N} \to \mathbb{N})$$

is an important example of a locally convex space. Classically the space $\mathcal{D}(\mathbb{R})$—the space of test functions—is complete, but it has not been known whether the constructive completion of $\mathcal{D}(\mathbb{R})$, whose explicit description was given in [1, Appendix A] and [2, Chapter 7, Notes], coincides with the original space or not. This leads us to a difficulty in developing the theory of distributions in Bishop’s constructive mathematics; see [1, Appendix A] and [2, Chapter 7, Notes] for more details.

The aim of our paper is to find a principle which is necessary and sufficient to establish the completeness of $\mathcal{D}(\mathbb{R})$. Although it is formulated in the setting of informal Bishop-style constructive mathematics, the proofs could easily be formalized in a system based on intuitionistic finite-type arithmetics $\mathcal{HA}^n$ [8, Chapter 1], [9, Chapter 9]; see also [5].

A subset $A$ of $\mathbb{N}$ is said to be pseudobounded if for each sequence $\{a_n\}$ in $A$,

$$\lim_{n \to \infty} \frac{a_n}{n} = 0.$$

A bounded subset of $\mathbb{N}$ is pseudobounded. The converse for countable sets holds in classical mathematics, intuitionistic mathematics and constructive recursive mathematics of Markov’s school; see [6]. However, a natural recursivisation of the following principle is independent of Heyting arithmetic [4].

BD-$\mathbb{N}$: Every countable pseudobounded subset of $\mathbb{N}$ is bounded. BD-$\mathbb{N}$ has been proved to be equivalent to the following theorems [6, 7, 4]: Banach’s inverse mapping theorem; the open mapping theorem; the closed graph theorem;
the Banach-Steinhaus theorem: the Hellinger-Toeplitz theorem: every sequentially continuous mapping of a separable metric space into a metric space is pointwise continuous; every uniformly sequentially continuous mapping of a separable metric space into a metric space is uniformly continuous. In this paper, we will show that it is also equivalent to the (sequential) completeness of $\mathcal{D}(\mathbb{R})$.

In the rest of the paper, we assume familiarity with the constructive calculus, as found in [1, Chapter 2], [3, Appendix], [2, Chapter 2], or [9, Chapter 6]. In the next section, we shall show that the test function

$$
\hat{\phi}(x) := \begin{cases} 
\exp \left( -\frac{1}{1-x^2} \right) & \text{if } |x| < 1 \\
0 & \text{if } |x| \geq 1
\end{cases}
$$

is well-defined in Bishop’s constructive mathematics. In the last section, we shall prove our main result with the completeness of the space $\mathcal{F}(\mathbb{R})$, which is another important example of a locally convex space, of all uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support together with the seminorms

$$
g_o(f) := \sup_n \sup_{|x| \geq n} 2^{n(a)}|f(x)| \quad (\alpha \in \mathbb{N} \to \mathbb{N}).
$$

Note that since functions differentiable on a compact interval are uniformly continuous on the interval, functions in $\mathcal{D}(\mathbb{R})$ belong to $\mathcal{F}(\mathbb{R})$.

§2. An example of a test function. A function $f : (a, b) \to \mathbb{R}$ is said to vanish at end points if for each $k$ there exists $m$ such that for all $x \in (a, b)$,

$$
x < a + 2^{-m} \lor b - 2^{-m} < x \implies |f(x)| < 2^{-k}.
$$

**Proposition 1.** Let $f : (a, b) \to \mathbb{R}$ be a function which vanishes at end points and is uniformly continuous on each compact subinterval of $(a, b)$. Then there exists a uniformly continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $f = \tilde{f}$ on $(a, b)$ and $f = 0$ on $(-\infty, a) \cup (b, \infty)$.

**Proof.** We first show that $f$ is uniformly continuous on $(a, b)$. To this end, let $k \in \mathbb{N}$. Then there exists $m$ such that for all $x \in (a, b)$,

$$
x < a + 2^{-m} \lor b - 2^{-m} < x \implies |f(x)| < 2^{-k-1}.
$$

Since $f$ is uniformly continuous on each compact subinterval of $(a, b)$, we can find $n > m$ such that for all $x, y \in [a + 2^{-m-2}, b - 2^{-m-2}]$,

$$
|x - y| < 2^{-n} \implies |f(x) - f(y)| < 2^{-k}.
$$

Let $x, y \in (a, b)$ with $|x - y| < 2^{-n}$. Then since $(a, b) = (a, a + 2^{-m-1}) \cup (a + 2^{-m-2}, b - 2^{-m-1}) \cup (b - 2^{-m-1}, b)$, either $x, y \in (a + 2^{-m-2}, b - 2^{-m-2})$, $x \in (a, a + 2^{-m-1}) \cup (b - 2^{-m-1}, b)$, or $y \in (a, a + 2^{-m-1}) \cup (b - 2^{-m-1}, b)$. In the first case, we have $|f(x) - f(y)| < 2^{-k}$. In the second case, if $x \in (a, a + 2^{-m-1})$, then

$$
a < y \leq x + |x - y| < a + 2^{-m-1} + 2^{-n} \leq a + 2^{-m},
$$

and hence $a < y < a + 2^{-m}$, or else $x \in (b - 2^{-m-1}, b)$, similarly we have $b - 2^{-m} < y < b$. Hence

$$
|f(x) - f(y)| \leq |f(x)| + |f(y)| < 2^{-k-1} + 2^{-k-1} = 2^{-k}.
$$
In the last case, similarly we have $|f(x) - f(y)| < 2^{-k}$. Therefore $f$ is uniformly continuous on $(a,b)$.

Define the function $F : (-\infty, a) \cup (a, b) \cup (b, \infty) \to \mathbb{R}$ by

$$F(x) := \begin{cases} f(x) & \text{if } a < x < b \\ 0 & \text{if } x < a \text{ or } b < x. \end{cases}$$

We show that $F$ is uniformly continuous on $(-\infty, a) \cup (a, b) \cup (b, \infty)$. Let $k \in \mathbb{N}$. Then there exists $n$ such that for all $x, y \in (a,b)$,

$$|x - y| < 2^{-n} \implies |f(x) - f(y)| < 2^{-k}.$$  

$x < a + 2^{-n} \lor b - 2^{-n} < x \implies |f(x)| < 2^{-k}$.

Let $x, y \in (-\infty, a) \cup (a, b) \cup (b, \infty)$ with $|x - y| < 2^{-n}$. Then either $x, y \in (a,b)$, $x \in (-\infty, a) \cup (b, \infty)$, or $y \in (-\infty, a) \cup (b, \infty)$. In the first case, we have

$$|F(x) - F(y)| = |f(x) - f(y)| < 2^{-k}.$$  

In the second case, if $x \in (-\infty, a)$, then $y \in (-\infty, a) \cup (a, a + 2^{-n})$, and hence

$$|F(x) - F(y)| = |F(y)| < 2^{-k};$$  

or else $x \in (b, \infty)$, we have $y \in (b - 2^{-n}, b) \cup (b, \infty)$, and hence $|F(x) - F(y)| < 2^{-k}$.

The last case is similar. Thus $F$ is uniformly continuous.

Therefore by [2, Lemma 4.3.7], there exists a uniformly continuous function $\hat{f} : \mathbb{R} \to \mathbb{R}$ such that $\hat{f}(x) = F(x)$ for all $x \in (-\infty, a) \cup (a, b) \cup (b, \infty)$.

A function $f$ from a subset $X$ of $\mathbb{R}$ into $\mathbb{R}$ is uniformly differentiable on $X$, with a derivative $f'$, if for each $k$, there exists $n$ such that for all $x, y \in X$,

$$|x - y| < 2^{-n} \implies |f'(x)(x - y) - (f(x) - f(y))| < 2^{-k}.$$  

We shall use the familiar notation for iterated derivatives: $f^{(0)} := f$, $f^{(l+1)} := f^{(l)}$.

Let $f, f' : (a, b) \to \mathbb{R}$ be functions which vanish at end points, and suppose that $f$ is uniformly differentiable on each compact subinterval of $(a, b)$ with a derivative $f'$. Then by [3, A.1], $f$ and $f'$ are uniformly continuous on each compact subinterval of $(a, b)$, and hence they have the uniformly continuous extensions $\hat{f}$ and $\hat{f}'$.

**Proposition 2.** Let $f, f' : (a, b) \to \mathbb{R}$ be functions which vanish at end points, and suppose that $f$ is uniformly differentiable on each compact subinterval of $(a, b)$ with a derivative $f'$. Then $f$ is uniformly differentiable on $\mathbb{R}$ with a derivative $\hat{f}'$.

**Proof.** We first show that $f$ is uniformly differentiable on $(a,b)$ with a derivative $f'$. To this end, let $k \in \mathbb{N}$. Then since $f^{[k]}$ is uniformly continuous, there exists $n$ such that for all $x, y \in (a,b)$,

$$|x - y| < 2^{-n} \implies |f^{[k]}(x) - f^{[k]}(y)| < 2^{-k}.$$  

Let $x, y \in (a,b)$ with $|x - y| < 2^{-n}$, and note that

$$f(w) = \int_y^w f'(t)dt + f(y)$$
on a compact subinterval of \((a, b)\) containing \(x\) and \(y\); see \cite[Theorem 2.6.8]{2}. Then
\[
|f'(x)(x - y) - (f(x) - f(y))| = |f'(x)(x - y) - \int_y^x f'(t)dt| = \int_y^x |f'(x) - f'(t)|dt \leq 2^{-k}|x - y|.
\]
Therefore \(f\) is uniformly differentiable on \((a, b)\) with a derivative \(f'\).

We show that \(\hat{f}\) is uniformly differentiable on \(\mathbb{R}\) with a derivative \(\hat{f}'\). For given \(k \in \mathbb{N}\), there exists \(n\) such that for all \(x, y \in (a, b)\),
\[
|x - y| < 2^{-n} \implies |f'(x)(x - y) - (f(x) - f(y))| \leq 2^{-k-1}|x - y|,
\]
and hence choosing \(w \in (a, a + 2^{-n})\), then since \(|f(w)| < 2^{-m}\), we have
\[
|\hat{f}'(x)(x - y) - (\hat{f}(x) - \hat{f}(y))| > 2^{-k}|x - y|.
\]
Then there exist \(u, v \in (-\infty, a) \cup (a, b) \cup (b, \infty)\) with \(|u - v| < 2^{-n}\) and \(m\) such that
\[
|\hat{f}'(u)(u - v) - (\hat{f}(u) - \hat{f}(v))| > 2^{-k}|u - v| + 2^{-m}.
\]
Either \(u, v \in (a, b), u \in (-\infty, a) \cup (b, \infty),\) or \(v \in (-\infty, a) \cup (b, \infty)\). The first case is absurd. In the second case, if \(u \in (-\infty, a)\), then since \(v \in (-\infty, a)\) is impossible, \(v \in (a, a + 2^{-n})\), and hence choosing \(w\) with \(a < w < v < a + 2^{-n}\) so that \(|f(w)| < 2^{-m}\), we have
\[
2^{-k}|u - v| + 2^{-m} < |\hat{f}'(u)(u - v) - (\hat{f}(u) - \hat{f}(v))| \leq |f'(w)(w - v) - (f(w) - f(v))| \leq |f'(w)(w - v)| + |f(w)| + 2^{-k-1}|w - v| + 2^{-m} < 2^{-k}|u - v| + 2^{-m},
\]
a contradiction; or else \(u \in (b, \infty)\), by a similar argument, we have a contradiction. Similarly the last case is absurd. Therefore
\[
|\hat{f}'(x)(x - y) - (\hat{f}(x) - \hat{f}(y))| \leq 2^{-k}|x - y|. \tag{1}
\]

The function
\[
\varphi(x) := \exp\left(-\frac{1}{1 - x^2}\right)
\]
from \((-1, 1)\) to \(\mathbb{R}\) is infinitely differentiable on each compact subinterval of \((-1, 1)\), and its \(l\)-th derivative is
\[
\varphi^{(l)}(x) = \frac{P_l(x)}{(1 - x^2)^{l+1}}\exp\left(-\frac{1}{1 - x^2}\right).
\]
Define a sequence \( n \geq n \) hence for some polynomial \( P_t \). Since for each \( m \) and \( k \) there exists \( n \) such that
\[
t > 2^n \implies \frac{t^m}{\exp(t)} < 2^{-k} \quad (t \in \mathbb{R}),
\]
each \( \varphi^{(l)} \) vanishes at end points. Hence \( \hat{\varphi} = \varphi^{(0)} \) is infinitely differentiable on \( \mathbb{R} \), and its \( l \)-th derivative \( \varphi^{(l)} \) is \( \varphi^{(l)}(\cdot) \).

§3. Completeness and BD-\( \mathbb{N} \).

**Lemma 3.** A subset \( A \) of \( \mathbb{N} \) is pseudobounded if and only if for each sequence \( \{a_n\} \in A \), \( a_n < n \) for all sufficiently large \( n \).

**Proof.** The “only if” part is trivial. To prove the converse, let \( \{a_n\} \) be a sequence in \( A \), \( \alpha \) a positive integer, and construct a binary sequence such that
\[
\hat{\lambda}_n = 0 \implies \max \left\{ a_m/m : n2^k \leq m < (n + 1)2^k \right\} < 2^{-k},
\]
\[
\hat{\lambda}_n = 1 \implies \max \left\{ a_m/m : n2^k \leq m < (n + 1)2^k \right\} \geq 2^{-k}.
\]
Define a sequence \( \{a'_n\} \) in \( A \) as follows: if \( \hat{\lambda}_n = 0 \), set \( a'_n := a_0 \); if \( \hat{\lambda}_n = 1 \), choose \( m \) with \( n2^k \leq m < (n + 1)2^k \) such that \( a_m/m \geq 2^{-k} \) and set \( a'_n := a_m \). Then there exists a positive integer \( N \) such that \( a'_n < n \) for all \( n \geq N \). If \( \hat{\lambda}_n = 1 \) for some \( n \geq N \), then there exists \( m \) such that \( n2^k \leq m < (n + 1)2^k \) and \( a'_n/m \geq 2^{-k} \), and hence
\[
n \leq m2^{-k} \leq a'_n < n.
\]
a contradiction. Thus \( \hat{\lambda}_n = 0 \) for all \( n \geq N \).

**Theorem 4.** The following are equivalent.

1. \( \mathcal{F}(\mathbb{R}) \) is (sequentially) complete.
2. \( \mathcal{D}(\mathbb{R}) \) is (sequentially) complete.
3. BD-\( \mathbb{N} \).

**Proof.** (3) \( \implies \) (1). Let \( \{f_i\} \) be a Cauchy sequence in \( \mathcal{F}(\mathbb{R}) \). Then taking \( \alpha := \hat{\lambda}n.0 \), for each \( \beta \) there exists \( I \) such that
\[
\sup_{|x| \geq 0} |f_i(x) - f_j(x)| \leq q_\alpha(f_i - f_j) < 2^{-k} \quad (i, j \geq I).
\]
By a straightforward modification of the proof of [2, Theorem 2.4.11], \( \{f_i\} \) converges uniformly to a uniformly continuous function \( f \). Note that for each \( \alpha \in \mathbb{N} \to \mathbb{N} \) and \( k \) there exists \( I \) such that
\[
\forall n \forall x \in \mathbb{R}(|x| \geq n \implies 2^nq_\alpha(x) - f(x) \leq 2^{-k}) \quad (i \geq I).
\]
In fact, given \( \alpha \in \mathbb{N} \to \mathbb{N} \) and \( k \), there exists \( I \) such that \( q_\alpha(f_i - f_j) < 2^{-k-1} \) for all \( i, j \geq I \). Let \( i \geq I \), and suppose that there exists \( n \) such that \( 2^{n+1}|f_i(x) - f(x)| \leq 2^{-k} \) for some \( x' \in \mathbb{R} \) with \( |x'| \geq n \). Then there exists \( j \) with \( j \geq I \) such that \( |f_j(x) - f(x)| < 2^{-\alpha(n)-k-1} \) for all \( x \in \mathbb{R} \), and hence
\[
2^{-k} \leq 2^nq_\alpha(f_i(x') - f(x')) \leq 2^nq_\alpha(f_i(x') - f_j(x')) + 2^nq_\alpha(f_j(x') - f(x')) \leq q_\alpha(f_i - f_j) + 2^{-k-1} < 2^{-k}.
\]
a contradiction. We shall show that \( f \) has compact support, and hence \( \{f_i\} \) converges to \( f \) in \( \mathcal{S}(\mathbb{R}) \). To this end, let

\[
A := \{0\} \cup \{n \in \mathbb{N} : \exists m \in \mathbb{N} \forall u \in \mathbb{Q}(|u| \geq n \land |f(u)| > 2^{-m})\}.
\]

Then \( A \) is a countable subset of \( \mathbb{N} \). Given sequence \( \{a_n\} \) in \( A \), construct a binary sequence \( \{\lambda_n\} \) such that \( \lambda_0 := 0 \) and for \( n \geq 1 \),

\[
\lambda_n = 0 \implies a_n < n, \\
\lambda_n = 1 \implies a_n \geq n.
\]

Define a sequence \( \alpha \in \mathbb{N} \to \mathbb{N} \) as follows: if \( \lambda_n = 0 \), set \( \alpha(n) := 0 \); if \( \lambda_n = 1 \), choose \( m \) such that \( \exists u \in \mathbb{Q}(|u| \geq a_n \land |f(u)| > 2^{-m}) \) and set \( \alpha(n) := m \). Then there exists \( I \) such that

\[
\forall n \forall x \in \mathbb{R}(|x| \geq n \implies 2^{\alpha(n)}|f_I(x) - f(x)| \leq 1).
\]

Choosing \( N \) such that \( f_I(x) = 0 \) for all \( x \in \mathbb{R} \) with \( |x| \geq N \), consider any integer \( n \geq N \). If \( \lambda_n = 1 \), then there exists \( u \in \mathbb{Q} \) such that \( |u| \geq a_n \geq n \geq N \) and \( |f(u)| > 2^{-\alpha(n)} \), and hence

\[
1 < 2^{\alpha(n)}|f(u)| = 2^{\alpha(n)}|f_I(u) - f(u)| \leq 1,
\]

a contradiction. Thus \( \lambda_n = 0 \) for all \( n \geq N \). Therefore \( A \) is pseudobounded, and so \( A \) is bounded, that is \( f \) has compact support.

\((1) \implies (2)\). Let \( \{f_i\} \) be a Cauchy sequence in \( \mathcal{S}(\mathbb{R}) \). Then for each \( l, \alpha \in \mathbb{N} \to \mathbb{N} \) and \( k \), letting \( \beta := \lambda n l \), there exists \( I \) such that

\[
g_{\alpha}(f_i^l) - f_j^l) \leq g_{\alpha,\beta}(f_i^l - f_j^l) < 2^{-k} \quad (i, j \geq l).
\]

Hence for each \( l \), \( \{f_i^l\} \) is a Cauchy sequence in \( \mathcal{S}(\mathbb{R}) \), and thus converges to a limit \( f^l \) in \( \mathcal{S}(\mathbb{R}) \). We show that \( f^l \) is uniformly differentiable on \( \mathbb{R} \) with a derivative \( f^{l+1} \), and so \( f := f^{(0)} \in \mathcal{S}(\mathbb{R}) \). For given \( k \), since \( f^{l+1} \) is uniformly continuous, there exists \( n \) such that for all \( x, y \in \mathbb{R} \),

\[
|x - y| < 2^{-n} \implies |f^{l+1}(x) - f^{l+1}(y)| < 2^{-k}.
\]

Let \( x, y \in \mathbb{R} \) with \( |x - y| < 2^{-n} \). Then since \( \{f_i^l\} \) and \( \{f_i^{l+1}\} \) converge uniformly to \( f^l \) and \( f^{l+1} \) respectively, we have

\[
f^l(x) - f^l(y) = \lim_{i \to \infty} \left( f_i^l(x) - f_i^l(y) \right) = \lim_{i \to \infty} \int_y^x f_i^{l+1}(t) dt = \int_y^x f^{l+1}(t) dt
\]

by [2, Lemma 2.6.9], and hence

\[
|f^{l+1}(x)(x - y) - (f^l(x) - f^l(y))| \\
= \left| f^{l+1}(x)(x - y) - \int_y^x f^{l+1}(t) dt \right| \\
= \left| \int_y^x f^{l+1}(x) - f^{l+1}(t) dt \right| \\
\leq 2^{-k}|x - y|.
\]
We show that \( \{ f_i \} \) converges to \( f \) in \( \mathcal{D}(\mathbb{R}) \). For given \( \alpha, \beta \in \mathbb{N} \to \mathbb{N} \) and 
\( k \in \mathbb{N} \), there exists \( I \) such that 
\( p_{\alpha, \beta}(f_i - f_j) < 2^{-k-1} \) for all \( i, j \geq I \). Suppose 
that \( p_{\alpha, \beta}(f_i - f_j) > 2^{-k} \) for some \( i \geq I \). Then there exists \( n \) and \( l \) with \( l \leq \beta(n) \) such that 
\( \sup_{|x| \geq n} 2^{\alpha(n)} |f_i^{(l)}(x) - f_j^{(l)}(x)| > 2^{-k} \). Choosing \( j \geq I \) so that 
\( q_{\alpha}(f_j^{(l)} - f_i^{(l)}) < 2^{-k-1} \), we have 
\[
2^{-k} < \sup_{|x| \geq n} 2^{\alpha(n)} |f_i^{(l)}(x) - f_j^{(l)}(x)| \\
\leq \sup_{|x| \geq n} 2^{\alpha(n)} |f_i^{(l)}(x) - f_j^{(l)}(x)| + \sup_{|x| \geq n} 2^{\alpha(n)} |f_j^{(l)}(x) - f_i^{(l)}(x)| \\
\leq p_{\alpha, \beta}(f_i - f_j) + q_{\alpha}(f_i^{(l)} - f_j^{(l)}) < 2^{-k},
\]
a contradiction. Therefore \( p_{\alpha, \beta}(f_i - f_j) \leq 2^{-k} \) for all \( i \geq I \).

(2) \( \implies \) (3). Let \( A \) be a pseudobounded subset of \( \mathbb{N} \) and \( \{ a_n \} \) an enumeration of \( A \). We may assume that \( a_n \geq 1 \) for all \( n \). For each \( m \), define the infinitely differentiable function \( g_m : \mathbb{R} \to \mathbb{R} \) by 
\[
g_m(x) := \frac{\hat{\phi}(2(x - a_m) + 1)}{2^m}.
\]
Then
- \( 0 < g_m(a_m - 1/2) \) for all \( m \),
- \( 0 < |g_m^{(l)}(x)| \implies 0 \leq a_m - 1 \leq x \leq a_m \) for all \( m \) and \( l \), and
- for each \( l \) and \( \varepsilon > 0 \) there exists \( I \) such that 
\[
\sum_{m=I}^{\infty} |g_m^{(l)}(x)| < \varepsilon \quad (x \in \mathbb{R}).
\]
We shall show that the sequence \( \{ f_i \} := \{ \sum_{m=0}^{\infty} g_m(x) \} \) in \( \mathcal{D}(\mathbb{R}) \) is a Cauchy sequence. To this end, we first show that 
\[
\sup_{|x| \geq n} \sum_{m=0}^{\infty} |g_m^{(l)}(x)|
\]
exists for all \( \alpha \in \mathbb{N} \to \mathbb{N} \), \( n \) and \( l \), and hence 
\[
S_n^{\alpha, \beta} := \max_{I \leq \beta(n)} \sup_{|x| \geq n} \sum_{m=0}^{\infty} |g_m^{(l)}(x)|
\]
exists for all \( \alpha, \beta \in \mathbb{N} \to \mathbb{N} \), \( n \) and \( l \). Fix \( \alpha \in \mathbb{N} \to \mathbb{N} \), \( n \) and \( l \), and let \( a, b \in \mathbb{R} \) with \( a < b \). Then there exists \( I \) such that 
\[
\sum_{m=I+1}^{\infty} |g_m^{(l)}(x)| < \frac{b - a}{2^{\alpha(n) + 1}} \quad (x \in \mathbb{R}).
\]
Either \( a < \sup_{|x| \geq n} 2^{\alpha(n)} \sum_{m=0}^{I} |g_m^{(l)}(x)| \) or \( \sup_{|x| \geq n} 2^{\alpha(n)} \sum_{m=0}^{I} |g_m^{(l)}(x)| < (a + b)/2 \): in the former case, we have 
\[
a < 2^{\alpha(n)} \sum_{m=0}^{I} |g_m^{(l)}(x')| \leq 2^{\alpha(n)} \sum_{m=0}^{\infty} |g_m^{(l)}(x')|.
\]
for some \( x' \in \mathbb{R} \) with \( |x'| \geq n \); in the latter case, we have

\[
2^{\alpha(n)} \sum_{m=0}^{\infty} |g_m^{(l)}(x)| \leq 2^{\alpha(n)} \sum_{m=0}^{I} |g_m^{(l)}(x)| + 2^{\alpha(n)} \sum_{m=I+1}^{\infty} |g_m^{(l)}(x)| < b
\]

for all \( x \in \mathbb{R} \) with \( |x| \geq n \). Therefore by the constructive least-upper-bound principle [2, Proposition 2.4.3], the supremum exists.

For given \( \alpha, \beta \in \mathbb{N} \rightarrow \mathbb{N} \) and \( k \), construct a binary sequence \( \{ \lambda_n \} \) such that

\[
\lambda_n = 0 \implies s_n^{\alpha \beta} < 2^{-k},
\]

\[
\lambda_n = 1 \implies s_n^{\alpha \beta} > 0.
\]

Define a sequence \( \{ a_n' \} \) in \( A \) as follows: if \( \lambda_n = 0 \), set \( a_n' := a_0 \); if \( \lambda_n = 1 \), choosing \( l \leq \beta(n) \), \( x \in \mathbb{R} \) with \( |x| \geq n \) and \( m \) such that \( 0 < |g_m^{(l)}(x)| \), we have \( n \leq x \leq a_m \), and set \( a_n' := a_m \). Then since \( A \) is pseudobounded, there exists \( N \) such that \( a_n' < n \) for all \( n \geq N \). If \( \lambda_n = 1 \) for some \( n \geq N \), then \( n \leq a_n' < n \), a contradiction. Hence \( \lambda_n = 0 \) for all \( n \geq N \). Letting \( M := \max(\alpha(n) : n < N) \) and \( L := \max(\beta(n) : n < N) \), there exists \( I \) such that

\[
\sum_{m=I}^{\infty} |g_m^{(l)}(x)| < 2^{-M-k} \quad (x \in \mathbb{R}, l \leq L).
\]

For each \( i, j \) with \( j \geq i \geq I \), we have for \( n < N \)

\[
\max \sup_{l \leq \beta(n) \mid x \geq n} 2^{\alpha(n)} \left| \sum_{m=i}^{j} g_m^{(l)}(x) \right| \leq \max \sup_{l \leq L \mid x \geq n} 2^M \sum_{m=i}^{j} |g_m^{(l)}(x)| \leq \max \sup_{l \leq L \mid x \geq n} 2^M 2^{-M-k} = 2^{-k},
\]

and for \( n \geq N \)

\[
\max \sup_{l \leq \beta(n) \mid x \geq n} 2^{\alpha(n)} \left| \sum_{m=i}^{j} g_m^{(l)}(x) \right| \leq \max \sup_{l \leq L \mid x \geq n} 2^{\alpha(n)} \sum_{m=i}^{j} |g_m^{(l)}(x)| \leq s_n^{\alpha \beta} < 2^{-k}.
\]

Therefore

\[
p_{\alpha, \beta}(f_i - f_j) = \sup_n \max \sup_{l \leq \beta(n) \mid x \geq n} \left| \sum_{m=i}^{j} g_m^{(l)}(x) \right| \leq 2^{-k}.
\]

Thus \( \{ f_i \} \) is a Cauchy sequence, and hence has a limit \( f \) in \( \mathcal{D} (\mathbb{R}) \). Let \( K \) be a positive integer such that \( f(x) = 0 \) whenever \( |x| \geq K \). If \( a_n > K \) for some \( n \), then

\[ K < a_n - 1/2 \quad \text{and} \quad 0 < g_n(a_n - 1/2) \leq f(a_n - 1/2), \]

a contradiction. Therefore \( a_n \leq K \) for all \( n \).

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