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A CONSTRUCTIVE LOOK AT THE COMPLETENESS OF THE SPACE $\mathscr{D}(\mathbb{R})$

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Abstract. We show, within the framework of Bishop's constructive mathematics, that (sequential) completeness of the locally convex space $\mathscr{D}(\mathbb{R})$ of test functions is equivalent to the principle BD-$\mathbb{N}$ which holds in classical mathemtatics, Brouwer's intuitionism and Markov's constructive recursive mathematics, but does not hold in Bishop's constructivism.

§1. Introduction. The space $\mathscr{D}(\mathbb{R})$ of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with compact support together with a locally convex structure defined by the seminorms

$$p_{\alpha,\beta}(f) := \sup_{n} \max_{l \leq \beta(n)} \sup_{|x| \geq n} 2^{\alpha(n)} |f^{(l)}(x)| \quad (\alpha, \beta \in \mathbb{N} \to \mathbb{N})$$

is an important example of a locally convex space. Classically the space $\mathscr{D}(\mathbb{R})$—the space of test functions—is complete, but it has not been known whether the constructive completion of $\mathscr{D}(\mathbb{R})$, whose explicit description was given in [1, Appendix A] and [2, Chapter 7, Notes], coincides with the original space or not. This leads us to a difficulty in developing the theory of distributions in Bishop's constructive mathematics; see [1, Appendix A] and [2, Chapter 7, Notes] for more details.

The aim of our paper is to find a principle which is necessary and sufficient to establish the completeness of $\mathscr{D}(\mathbb{R})$. Although it is formulated in the setting of informal Bishop-style constructive mathematics, the proofs could easily be formalized in a system based on intuitionistic finite-type arithmetics $\boldsymbol{HA}^{\omega}$ [8, Chapter 1], [9, Chapter 9]; see also [5].

A subset A of $\mathbb{N}$ is said to be *pseudobounded* if for each sequence $\{a_n\}_n$ in A,

$$\lim_{n\to\infty} \frac{a_n}{n} = 0.$$

A bounded subset of $\mathbb{N}$ is pseudobounded. The converse for countable sets holds in in classical mathematics, intuitionistic mathematics and constructive recursive mathematics of Markov's school; see [6]. However, a natural recursivisation of the following principle is independent of Heyting arithmetic [4].

BD-$\mathbb{N}$**:** Every countable pseudobounded subset of $\mathbb{N}$ is bounded.

BD-$\mathbb{N}$ has been proved to be equivalent to the following theorems [6, 7, 4]; Banach's inverse mapping theorem; the open mapping theorem; the closed graph theorem;

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the Banach-Steinhaus theorem; the Hellinger-Toeplitz theorem; every sequentially continuous mapping of a separable metric space into a metric space is pointwise continuous; every uniformly sequentially continuous mapping of a separable metric space into a metric space is uniformly continuous. In this paper, we will show that it is also equivalent to the (sequential) completeness of $\mathscr{D}(\mathbb{R})$.

In the rest of the paper, we assume familiarity with the constructive calculus, as found in [1, Chapter 2], [3, Appendix], [2, Chapter 2], or [9, Chapter 6]. In the next section, we shall show that the test function

$$\hat{\varphi}(x) := \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

is well-defined in Bishop's constructive mathematics. In the last section, we shall prove our main result with the completeness of the space $\mathscr{K}(\mathbb{R})$, which is another important example of a locally convex space, of all uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support together with the seminorms

$$q_\alpha(f) := \sup_n \sup_{|x| \geq n} 2^{\alpha(n)} |f(x)| \quad (\alpha \in \mathbb{N} \to \mathbb{N}).$$

Note that since functions differentiable on a compact interval are uniformly continuous on the interval, functions in $\mathscr{D}(\mathbb{R})$ belong to $\mathscr{K}(\mathbb{R})$.

§2. An example of a test function. A function $f : (a, b) \to \mathbb{R}$ is said to *vanish at end points* if for each k there exists m such that for all $x \in (a, b)$,

$$x < a + 2^{-m} \vee b - 2^{-m} < x \Longrightarrow |f(x)| < 2^{-k}.$$

Proposition 1. *Let $f : (a, b) \to \mathbb{R}$ be a function which vanishes at end points and is uniformly continuous on each compact subinterval of (a, b). Then there exists a uniformly continuous function $\hat{f} : \mathbb{R} \to \mathbb{R}$ such that $\hat{f} = f$ on (a, b) and $\hat{f} = 0$ on $(-\infty, a) \cup (b, \infty)$.*

Proof. We first show that f is uniformly continuous on (a, b). To this end, let $k \in \mathbb{N}$. Then there exists m such that for all $x \in (a, b)$,

$$x < a + 2^{-m} \vee b - 2^{-m} < x \Longrightarrow |f(x)| < 2^{-k-1}.$$

Since f is uniformly continuous on each compact subinterval of (a, b), we can find $n > m$ such that for all $x, y \in [a + 2^{-m-2}, b - 2^{-m-2}]$,

$$|x - y| < 2^{-n} \Longrightarrow |f(x) - f(y)| < 2^{-k}.$$

Let $x, y \in (a, b)$ with $|x - y| < 2^{-n}$. Then since $(a, b) = (a, a + 2^{-m-1}) \cup (a + 2^{-m-2}, b - 2^{-m-2}) \cup (b - 2^{-m-1}, b)$, either $x, y \in (a + 2^{-m-2}, b - 2^{-m-2})$, $x \in (a, a + 2^{-m-1}) \cup (b - 2^{-m-1}, b)$, or $y \in (a, a + 2^{-m-1}) \cup (b - 2^{-m-1}, b)$. In the first case, we have $|f(x) - f(y)| < 2^{-k}$. In the second case, if $x \in (a, a + 2^{-m-1})$, then

$$a < y \leq x + |x - y| < a + 2^{-m-1} + 2^{-n} \leq a + 2^{-m},$$

and hence $a < y < a + 2^{-m}$; or else $x \in (b - 2^{-m-1}, b)$, similarly we have $b - 2^{-m} < y < b$. Hence

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| < 2^{-k-1} + 2^{-k-1} = 2^{-k}.$$

In the last case, similarly we have $|f(x) - f(y)| < 2^{-k}$. Therefore f is uniformly continuous on (a, b).

Define the function $F : (-\infty, a) \cup (a, b) \cup (b, \infty) \to \mathbb{R}$ by

$$F(x) := \begin{cases} f(x) & \text{if } a < x < b \\ 0 & \text{if } x < a \text{ or } b < x. \end{cases}$$

We show that F is uniformly continuous on $(-\infty, a) \cup (a, b) \cup (b, \infty)$. Let $k \in \mathbb{N}$. Then there exists n such that for all $x, y \in (a, b)$,

$$|x - y| < 2^{-n} \Longrightarrow |f(x) - f(y)| < 2^{-k},$$
$$x < a + 2^{-n} \vee b - 2^{-n} < x \Longrightarrow |f(x)| < 2^{-k}.$$

Let $x, y \in (-\infty, a) \cup (a, b) \cup (b, \infty)$ with $|x - y| < 2^{-n}$. Then either $x, y \in (a, b)$, $x \in (-\infty, a) \cup (b, \infty)$, or $y \in (-\infty, a) \cup (b, \infty)$. In the first case, we have

$$|F(x) - F(y)| = |f(x) - f(y)| < 2^{-k}.$$

In the second case, if $x \in (-\infty, a)$, then $y \in (-\infty, a) \cup (a, a + 2^{-n})$, and hence

$$|F(x) - F(y)| = |F(y)| < 2^{-k};$$

or else $x \in (b, \infty)$, we have $y \in (b - 2^{-n}, b) \cup (b, \infty)$, and hence $|F(x) - F(y)| < 2^{-k}$. The last case is similar. Thus F is uniformly continuous.

Therefore by [2, Lemma 4.3.7], there exists a uniformly continuous function $\hat{f} : \mathbb{R} \to \mathbb{R}$ such that $\hat{f}(x) = F(x)$ for all $x \in (-\infty, a) \cup (a, b) \cup (b, \infty)$. ⊣

A function f from a subset X of $\mathbb{R}$ into $\mathbb{R}$ is *uniformly differentiable* on X, with a derivative f', if for each k, there exists n such that for all $x, y \in X$,

$$|x - y| < 2^{-n} \Longrightarrow |f'(x)(x - y) - (f(x) - f(y))| < 2^{-k}.$$

We shall use the familiar notation for iterated derivatives: $f^{(0)} := f$, $f^{(l+1)} := (f^{(l)})'$.

Let $f, f' : (a, b) \to \mathbb{R}$ be functions which vanish at end points, and suppose that f is uniformly differentiable on each compact subinterval of (a, b) with a derivative f'. Then by [3, A.1], f and f' are uniformly continuous on each compact subinterval of (a, b), and hence they have the uniformly continuous extensions $\hat{f}$ and $\widehat{f'}$.

Proposition 2. *Let $f, f' : (a, b) \to \mathbb{R}$ be functions which vanish at end points, and suppose that f is uniformly differentiable on each compact subinterval of (a, b) with a derivative f'. Then $\hat{f}$ is uniformly differentiable on $\mathbb{R}$ with a derivative $\widehat{f'}$.*

Proof. We first show that f is uniformly differentiable on (a, b) with a derivative f'. To this end, let $k \in \mathbb{N}$. Then since $\widehat{f'}$ is uniformly continuous, there exists n such that for all $x, y \in (a, b)$,

$$|x - y| < 2^{-n} \Longrightarrow |f'(x) - f'(y)| < 2^{-k}.$$

Let $x, y \in (a, b)$ with $|x - y| < 2^{-n}$, and note that

$$f(w) = \int_y^w f'(t)dt + f(y)$$

on a compact subinterval of (a,b) containing x and y; see [2, Theorem 2.6.8]. Then

$$\begin{aligned}|f'(x)(x-y)-(f(x)-f(y))| &= \left|f'(x)(x-y)-\int_y^x f'(t)dt\right| \\ &= \left|\int_y^x (f'(x)-f'(t))dt\right| \\ &\le 2^{-k}|x-y|.\end{aligned}$$

Therefore f is uniformly differentiable on (a,b) with a derivative f'.

We show that $\hat{f}$ is uniformly differentiable on $\mathbb{R}$ with a derivative $\widehat{f'}$. For given $k \in \mathbb{N}$, there exists n such that for all $x, y \in (a,b)$,

$$|x-y| < 2^{-n} \Longrightarrow |f'(x)(x-y)-(f(x)-f(y))| \le 2^{-k-1}|x-y|,$$
$$x < a+2^{-n} \vee b-2^{-n} < x \Longrightarrow |f'(x)| < 2^{-k-1}.$$

Let $x, y \in \mathbb{R}$ with $|x-y| < 2^{-n}$, and suppose that

$$|\widehat{f'}(x)(x-y)-(\hat{f}(x)-\hat{f}(y))| > 2^{-k}|x-y|.$$

Then there exist $u, v \in (-\infty, a) \cup (a,b) \cup (b,\infty)$ with $|u-v| < 2^{-n}$ and m such that

$$|\widehat{f'}(u)(u-v)-(\hat{f}(u)-\hat{f}(v))| > 2^{-k}|u-v|+2^{-m}.$$

Either $u, v \in (a,b)$, $u \in (-\infty,a)\cup(b,\infty)$, or $v \in (-\infty,a)\cup(b,\infty)$. The first case is absurd. In the second case, if $u \in (-\infty,a)$, then since $v \in (-\infty,a)$ is impossible, $v \in (a, a+2^{-n})$, and hence choosing w with $a < w < v < a+2^{-n}$ so that $|f(w)| < 2^{-m}$, we have

$$\begin{aligned}2^{-k}|u-v|+2^{-m} &< |\widehat{f'}(u)(u-v)-(\hat{f}(u)-\hat{f}(v))| \\ &\le |f'(w)(w-v)-(f(w)-f(v))| \\ &\quad +|f'(w)(w-v)|+|f(w)| \\ &< 2^{-k-1}|w-v|+2^{-k-1}|w-v|+2^{-m} \\ &< 2^{-k}|u-v|+2^{-m},\end{aligned}$$

a contradiction; or else $u \in (b,\infty)$, by a similar argument, we have a contradiction. Similarly the last case is absurd. Therefore

$$|\widehat{f'}(x)(x-y)-(\hat{f}(x)-\hat{f}(y))| \le 2^{-k}|x-y|. \qquad \dashv$$

The function

$$\varphi(x) := \exp\left(-\frac{1}{1-x^2}\right)$$

from $(-1,1)$ to $\mathbb{R}$ is infinitely differentiable on each compact subinterval of $(-1,1)$, and its l-th derivative is

$$\varphi^{(l)}(x) = \frac{P_l(x)}{(1-x^2)^{2l}}\exp\left(-\frac{1}{1-x^2}\right)$$

for some polynomial P_l. Since for each m and k there exists n such that

$$t > 2^n \Longrightarrow \frac{t^m}{\exp(t)} < 2^{-k} \quad (t \in \mathbb{R}),$$

each $\varphi^{(l)}$ vanishes at end points. Hence $\hat{\varphi} = \widehat{\varphi^{(0)}}$ is infinitely differentiable on $\mathbb{R}$, and its l-th derivative $\hat{\varphi}^{(l)}$ is $\widehat{\varphi^{(l)}}$.

§3. Completeness and BD-$\mathbb{N}$.

LEMMA 3. *A subset A of $\mathbb{N}$ is pseudobounded if and only if for each sequence $\{a_n\}$ in A, $a_n < n$ for all sufficiently large n.*

PROOF. The "only if" part is trivial. To prove the converse, let $\{a_n\}$ be a sequence in A, k a positive integer, and construct a binary sequence such that

$$\lambda_n = 0 \Longrightarrow \max\left\{a_m/m : n2^k \leq m < (n+1)2^k\right\} < 2^{-k},$$
$$\lambda_n = 1 \Longrightarrow \max\left\{a_m/m : n2^k \leq m < (n+1)2^k\right\} \geq 2^{-k}.$$

Define a sequence $\{a'_n\}$ in A as follows: if $\lambda_n = 0$, set $a'_n := a_0$; if $\lambda_n = 1$, choose m with $n2^k \leq m < (n+1)2^k$ such that $a_m/m \geq 2^{-k}$ and set $a'_n := a_m$. Then there exists a positive integer N such that $a'_n < n$ for all $n \geq N$. If $\lambda_n = 1$ for some $n \geq N$, then there exists m such that $n2^k \leq m < (n+1)2^k$ and $a'_n/m \geq 2^{-k}$, and hence

$$n \leq m2^{-k} \leq a'_n < n,$$

a contradiction. Thus $\lambda_n = 0$ for all $n \geq N$. ⊣

THEOREM 4. *The following are equivalent.*

1. $\mathscr{K}(\mathbb{R})$ *is (sequentially) complete.*
2. $\mathscr{D}(\mathbb{R})$ *is (sequentially) complete.*
3. *BD*-$\mathbb{N}$.

PROOF. (3) $\Longrightarrow$ (1). Let $\{f_i\}$ be a Cauchy sequence in $\mathscr{K}(\mathbb{R})$. Then taking $\alpha := \lambda n.0$, for each k there exists I such that

$$\sup_{|x|\geq 0} |f_i(x) - f_j(x)| \leq q_\alpha(f_i - f_j) < 2^{-k} \quad (i, j \geq I).$$

By a straightforward modification of the proof of [2, Theorem 2.4.11], $\{f_i\}$ converges uniformly to a uniformly continuous function f. Note that for each $\alpha \in \mathbb{N} \to \mathbb{N}$ and k there exists I such that

$$\forall n \forall x \in \mathbb{R}(|x| \geq n \Longrightarrow 2^{\alpha(n)}|f_i(x) - f(x)| \leq 2^{-k}) \quad (i \geq I).$$

In fact, given $\alpha \in \mathbb{N} \to \mathbb{N}$ and k, there exists I such that $q_\alpha(f_i - f_j) < 2^{-k-1}$ for all $i, j \geq I$. Let $i \geq I$, and suppose that there exists n such that $2^{\alpha(n)}|f_i(x') - f(x')| > 2^{-k}$ for some $x' \in \mathbb{R}$ with $|x'| \geq n$. Then there exists j with $j \geq I$ such that $|f_j(x) - f(x)| < 2^{-\alpha(n)-k-1}$ for all $x \in \mathbb{R}$, and hence

$$\begin{aligned}
2^{-k} &< 2^{\alpha(n)}|f_i(x') - f(x')| \\
&\leq 2^{\alpha(n)}|f_i(x') - f_j(x')| + 2^{\alpha(n)}|f_j(x') - f(x')| \\
&\leq q_\alpha(f_i - f_j) + 2^{-k-1} < 2^{-k},
\end{aligned}$$

a contradiction. We shall show that f has compact support, and hence $\{f_i\}$ converges to f in $\mathscr{K}(\mathbb{R})$. To this end, let

$$A := \{0\} \cup \{n \in \mathbb{N} : \exists m \in \mathbb{N} \exists u \in \mathbb{Q}(|u| \geq n \wedge |f(u)| > 2^{-m})\}.$$

Then A is a countable subset of $\mathbb{N}$. Given sequence $\{a_n\}$ in A, construct a binary sequence $\{\lambda_n\}$ such that $\lambda_0 := 0$ and for $n \geq 1$,

$$\lambda_n = 0 \implies a_n < n,$$
$$\lambda_n = 1 \implies a_n \geq n.$$

Define a sequence $\alpha \in \mathbb{N} \to \mathbb{N}$ as follows: if $\lambda_n = 0$, set $\alpha(n) := 0$; if $\lambda_n = 1$, choose m such that $\exists u \in \mathbb{Q}(|u| \geq a_n \wedge |f(u)| > 2^{-m})$ and set $\alpha(n) := m$. Then there exists I such that

$$\forall n \forall x \in \mathbb{R}(|x| \geq n \Longrightarrow 2^{\alpha(n)}|f_I(x) - f(x)| \leq 1).$$

Choosing N such that $f_I(x) = 0$ for all $x \in \mathbb{R}$ with $|x| \geq N$, consider any integer $n \geq N$. If $\lambda_n = 1$, then there exists $u \in \mathbb{Q}$ such that $|u| \geq a_n \geq n \geq N$ and $|f(u)| > 2^{-\alpha(n)}$, and hence

$$1 < 2^{\alpha(n)}|f(u)| = 2^{\alpha(n)}|f_I(u) - f(u)| \leq 1,$$

a contradiction. Thus $\lambda_n = 0$ for all $n \geq N$. Therefore A is pseudobounded, and so A is bounded, that is f has compact support.

(1) $\Longrightarrow$ (2). Let $\{f_i\}$ be a Cauchy sequence in $\mathscr{D}(\mathbb{R})$. Then for each l, $\alpha \in \mathbb{N} \to \mathbb{N}$ and k, letting $\beta := \lambda n.l$, there exists I such that

$$q_\alpha(f_i^{(l)} - f_j^{(l)}) \leq p_{\alpha,\beta}(f_i - f_j) < 2^{-k} \quad (i, j \geq I).$$

Hence for each l, $\{f_i^{(l)}\}$ is a Cauchy sequence in $\mathscr{K}(\mathbb{R})$, and thus converges to a limit $f^{(l)}$ in $\mathscr{K}(\mathbb{R})$. We show that $f^{(l)}$ is uniformly differentiable on $\mathbb{R}$ with a derivative $f^{(l+1)}$, and so $f := f^{(0)} \in \mathscr{D}(\mathbb{R})$. For given k, since $f^{(l+1)}$ is uniformly continuous, there exists n such that for all $x, y \in \mathbb{R}$,

$$|x - y| < 2^{-n} \Longrightarrow |f^{(l+1)}(x) - f^{(l+1)}(y)| < 2^{-k}.$$

Let $x, y \in \mathbb{R}$ with $|x - y| < 2^{-n}$. Then since $\{f_i^{(l)}\}$ and $\{f_i^{(l+1)}\}$ converge uniformly to $f^{(l)}$ and $f^{(l+1)}$ respectively, we have

$$\begin{aligned} f^{(l)}(x) - f^{(l)}(y) &= \lim_{i\to\infty} \left(f_i^{(l)}(x) - f_i^{(l)}(y)\right) = \lim_{i\to\infty} \int_y^x f_i^{(l+1)}(t)dt \\ &= \int_y^x f^{(l+1)}(t)dt \end{aligned}$$

by [2, Lemma 2.6.9], and hence

$$\begin{aligned} &|f^{(l+1)}(x)(x - y) - (f^{(l)}(x) - f^{(l)}(y))| \\ &\qquad = \left|f^{(l+1)}(x)(x - y) - \int_y^x f^{(l+1)}(t)dt\right| \\ &\qquad = \left|\int_y^x (f^{(l+1)}(x) - f^{(l+1)}(t))dt\right| \\ &\qquad \leq 2^{-k}|x - y|. \end{aligned}$$

We show that $\{f_i\}$ converges to f in $\mathscr{D}(\mathbb{R})$. For given $\alpha, \beta \in \mathbb{N} \to \mathbb{N}$ and $k \in \mathbb{N}$, there exists I such that $p_{\alpha,\beta}(f_i - f_j) < 2^{-k-1}$ for all $i, j \geq I$. Suppose that $p_{\alpha,\beta}(f_i - f) > 2^{-k}$ for some $i \geq I$. Then there exists n and l with $l \leq \beta(n)$ such that $\sup_{|x|\geq n} 2^{\alpha(n)}|f_i^{(l)}(x) - f^{(l)}(x)| > 2^{-k}$. Choosing $j \geq I$ so that $q_\alpha(f_j^{(l)} - f^{(l)}) < 2^{-k-1}$, we have

$$\begin{aligned}
2^{-k} &< \sup_{|x|\geq n} 2^{\alpha(n)}|f_i^{(l)}(x) - f^{(l)}(x)| \\
&\leq \sup_{|x|\geq n} 2^{\alpha(n)}|f_i^{(l)}(x) - f_j^{(l)}(x)| + \sup_{|x|\geq n} 2^{\alpha(n)}|f_j^{(l)}(x) - f^{(l)}(x)| \\
&\leq p_{\alpha,\beta}(f_i - f_j) + q_\alpha(f_j^{(l)} - f^{(l)}) < 2^{-k},
\end{aligned}$$

a contradiction. Therefore $p_{\alpha,\beta}(f_i - f) \leq 2^{-k}$ for all $i \geq I$.

(2) $\Longrightarrow$ (3). Let A be a pseudobounded subset of $\mathbb{N}$ and $\{a_n\}$ an enumeration of A. We may assume that $a_n \geq 1$ for all n. For each m, define the infinitely differentiable function $g_m : \mathbb{R} \to \mathbb{R}$ by

$$g_m(x) := \frac{\hat{\varphi}(2(x - a_m) + 1)}{2^m}.$$

Then

- $0 < g_m(a_m - 1/2)$ for all m,
- $0 < |g_m^{(l)}(x)| \Longrightarrow 0 \leq a_m - 1 \leq x \leq a_m$ for all m and l, and
- for each l and $\varepsilon > 0$ there exists I such that
$$\sum_{m=I}^{\infty} |g_m^{(l)}(x)| < \varepsilon \quad (x \in \mathbb{R}).$$

We shall show that the sequence $\{f_i\} := \{\sum_{m=0}^{i} g_m(x)\}$ in $\mathscr{D}(\mathbb{R})$ is a Cauchy sequence. To this end, we first show that

$$\sup_{|x|\geq n} 2^{\alpha(n)} \sum_{m=0}^{\infty} |g_m^{(l)}(x)|$$

exists for all $\alpha \in \mathbb{N} \to \mathbb{N}$, n and l, and hence

$$s_n^{\alpha,\beta} := \max_{l\leq\beta(n)} \sup_{|x|\geq n} 2^{\alpha(n)} \sum_{m=0}^{\infty} |g_m^{(l)}(x)|$$

exists for all $\alpha, \beta \in \mathbb{N} \to \mathbb{N}$ and n. Fix $\alpha \in \mathbb{N} \to \mathbb{N}$, n and l, and let $a, b \in \mathbb{R}$ with $a < b$. Then there exists I such that

$$\sum_{m=I+1}^{\infty} |g_m^{(l)}(x)| < \frac{b-a}{2^{\alpha(n)+1}} \quad (x \in \mathbb{R}).$$

Either $a < \sup_{|x|\geq n} 2^{\alpha(n)} \sum_{m=0}^{I} |g_m^{(l)}(x)|$ or $\sup_{|x|\geq n} 2^{\alpha(n)} \sum_{m=0}^{I} |g_m^{(l)}(x)| < (a + b)/2$: in the former case, we have

$$a < 2^{\alpha(n)} \sum_{m=0}^{I} |g_m^{(l)}(x')| \leq 2^{\alpha(n)} \sum_{m=0}^{\infty} |g_m^{(l)}(x')|$$

for some $x' \in \mathbb{R}$ with $|x'| \geq n$; in the latter case, we have

$$2^{\alpha(n)} \sum_{m=0}^{\infty} |g_m^{(l)}(x)| \leq 2^{\alpha(n)} \sum_{m=0}^{I} |g_m^{(l)}(x)| + 2^{\alpha(n)} \sum_{m=I+1}^{\infty} |g_m^{(l)}(x)| < b$$

for all $x \in \mathbb{R}$ with $|x| \geq n$. Therefore by the constructive least-upper-bound principle [2, Proposition 2.4.3], the supremum exists.

For given $\alpha, \beta \in \mathbb{N} \to \mathbb{N}$ and k, construct a binary sequence $\{\lambda_n\}$ such that

$$\lambda_n = 0 \implies s_n^{\alpha,\beta} < 2^{-k},$$
$$\lambda_n = 1 \implies s_n^{\alpha,\beta} > 0.$$

Define a sequence $\{a_n'\}$ in A as follows: if $\lambda_n = 0$, set $a_n' := a_0$; if $\lambda_n = 1$, choosing $l \leq \beta(n)$, $x \in \mathbb{R}$ with $|x| \geq n$ and m such that $0 < |g_m^{(l)}(x)|$, we have $n \leq x \leq a_m$, and set $a_n' := a_m$. Then since A is pseudobounded, there exists N such that $a_n' < n$ for all $n \geq N$. If $\lambda_n = 1$ for some $n \geq N$, then $n \leq a_n' < n$, a contradiction. Hence $\lambda_n = 0$ for all $n \geq N$. Letting $M := \max\{\alpha(n) : n < N\}$ and $L := \max\{\beta(n) : n < N\}$, there exists I such that

$$\sum_{m=I}^{\infty} |g_m^{(l)}(x)| < 2^{-M-k} \quad (x \in \mathbb{R}, l \leq L).$$

For each i, j with $j \geq i \geq I$, we have for $n < N$

$$\begin{aligned}\max_{l\leq\beta(n)} \sup_{|x|\geq n} 2^{\alpha(n)} \left|\sum_{m=i}^{j} g_m^{(l)}(x)\right| &\leq \max_{l\leq L} \sup_{|x|\geq n} 2^{M} \sum_{m=i}^{j} |g_m^{(l)}(x)| \\ &\leq \max_{l\leq L} \sup_{|x|\geq n} 2^{M} 2^{-M-k} = 2^{-k},\end{aligned}$$

and for $n \geq N$

$$\begin{aligned}\max_{l\leq\beta(n)} \sup_{|x|\geq n} 2^{\alpha(n)} \left|\sum_{m=i}^{j} g_m^{(l)}(x)\right| &\leq \max_{l\leq\beta(n)} \sup_{|x|\geq n} 2^{\alpha(n)} \sum_{m=i}^{j} |g_m^{(l)}(x)| \\ &\leq s_n^{\alpha,\beta} < 2^{-k}.\end{aligned}$$

Therefore

$$p_{\alpha,\beta}(f_i - f_j) = \sup_{n} \max_{l\leq\beta(n)} \sup_{|x|\geq n} 2^{\alpha(n)} \left|\sum_{m=i+1}^{j} g_m^{(l)}(x)\right| \leq 2^{-k}.$$

Thus $\{f_i\}$ is a Cauchy sequence, and hence has a limit f in $\mathscr{D}(\mathbb{R})$. Let K be a positive integer such that $f(x) = 0$ whenever $|x| \geq K$. If $a_n > K$ for some n, then $K < a_n - 1/2$ and $0 < g_n(a_n - 1/2) \leq f(a_n - 1/2)$, a contradiction. Therefore $a_n \leq K$ for all n. ⊣

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REFERENCES

[1] Errett Bishop, ***Foundations of Constructive Analysis***, McGraw-Hill, New York, 1967.

[2] Errett Bishop and Douglas Bridges, *Constructive analysis*, ***Grundlehren der Mathematischen Wissenschaften***, vol. 279 (1985), Springer-Verlag, Heidelberg.

[3] DOUGLAS BRIDGES, ***Constructive Functional Analysis***, Pitman, London, 1979.

[4] DOUGLAS BRIDGES, HAJIME ISHIHARA, PETER SCHUSTER, and LUMINIŢA VÎŢĂ, ***Strong continuity implies uniform sequential continuity***, 2001, Preprint.

[5] N. GOODMAN and J. MYHILL, *The formalization of Bishop's constructive mathematics*, ***Toposes, Algebraic Geometry and Logic*** (F. Lawvere, editor), Springer, Berlin, 1972, pp. 83–96.

[6] HAJIME ISHIHARA, *Continuity properties in constructive mathematics*, this JOURNAL, vol. 57 (1992), pp. 557–565.

[7] ———, *Sequential continuity in constructive mathematics*, ***Combinatorics, Computability and Logic, Proceedings of the Third International Conference on Combinatorics, Computability and Logic, (DMTCS'01) in Constanţa Romania*** (London) (C.S. Calude, M.J. Dinneen, and S. Sburlan, editors), Springer-Verlag, July 2–6, 2001, pp. 5–12.

[8] A. S. TROELSTRA, ***Metamathematical Investigation of Intuitionistic Arithmetic and analysis***, Springer, Berlin, 1973.

[9] A. S. TROELSTRA and D. VAN DALEN, ***Constructivism in Mathematics***, vol. 1 and 2, North-Holland, Amsterdam, 1988.

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