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A CONTINUITY PRINCIPLE, A VERSION OF BAIRE'S THEOREM AND A BOUNDEDNESS PRINCIPLE

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Abstract. We deal with a restricted form WC-N′ of the weak continuity principle, a version BT′ of Baire's theorem, and a boundedness principle BD-N. We show, in the spirit of constructive reverse mathematics, that WC-N′, BT′ + ¬LPO and BD-N + ¬LPO are equivalent in a constructive system, where LPO is the limited principle of omniscience.

§1. Introduction. The Baire space **B** is the set $\mathbf{N}^{\mathbf{N}}$ with the product topology obtained from the discrete topology on **N** which is metrizable with a complete metric. The principles characteristic of Brouwer's intuitionistic mathematics, that is, the continuity of mappings from **B** into **N** together with an appropriate choice principle, lead to the following scheme of weak continuity for numbers [17, 4.6.3]:

WC-N: $\forall\alpha\, \exists n\, A(\alpha, n) \Longrightarrow \forall\alpha\, \exists n\, \exists m\, \forall\beta \in \overline{\alpha}(m)\, A(\beta, n)$.

Here and in the following, we follow the notational conventions in [17]: m, n, i, j, k are supposed to range over **N**, a, b, c over the set $\mathbf{N}^*$ of finite sequences of **N**, and $\alpha, \beta, \gamma, \delta$ over $\mathbf{N}^{\mathbf{N}}$; $|a|$ denotes the length of a finite sequence a, $a*b$ the concatenation of two finite sequences a and b, and $a \preceq b$ that a is an initial segment of b, that is, $\exists c\, (a * c = b)$; $\overline{\alpha}(n)$ denotes the finite initial segment $(\alpha(0), \dots, \alpha(n-1))$ of α with length n, and $\alpha \in a$ that α has initial segment a, that is, $\overline{\alpha}(|a|) = a$.

When we set $X_n = \{\alpha \in \mathbf{B} \mid A(\alpha, n)\}$, WC-N expresses that if the sequence $(X_n)_n$ covers **B**, then the sequence $(\mathrm{Int}(X_n))_n$ of the open interiors again covers **B**; see [17, 7.2.6].

Furthermore assuming that the X_n's are closed in **B**, the schema:

BT: $\forall\alpha\, \exists n\, A(\alpha, n) \Longrightarrow \exists n\, \exists a\, \forall\beta \in a\, A(\beta, n)$,

which is far weaker than WC-N, formalizes Baire's theorem for **B**, that is, if the sequence $(X_n)_n$ covers **B**, then X_n has inhabited interior for some n. Note that a form of Baire's theorem has a constructive proof [1, Theorem 4.4], [5, 2.2] (see also [15, Theorem 8.11] for the formal reals), but its classical equivalent, the form above, which is used in the standard argument to prove important theorems in functional analysis, such as the Banach-Steinhaus theorem, the open mapping theorem and the closed graph theorem, has no constructive proof; see [9], and [4] and [6, 6.6] for a constructively provable variants of the above form of Baire's theorem.

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In this paper, we shall deal with the schemata WC-N and BT, in Bishop's constructive mathematics [1, 2, 5, 6], for an increasing sequence $(X_n)_n$ which consists of *closed* subsets of $\mathbf{B}$ (or subsets in the class Π^0_1) in the sense of Veldman [18], that is, there exists $\sigma : \mathbf{N}^* \times \mathbf{N} \to \mathbf{N}$ such that

$$\alpha \in X_n \iff A(\alpha, n) \iff \forall k\, (\sigma(\overline{\alpha}(k), n) = 0)$$

and

$$\sigma(a, m) = 0 \wedge m \leq n \Longrightarrow \sigma(a, n) = 0. \tag{1}$$

We call these restricted schemata WC-N$'$ and BT$'$, respectively.

A subset S of $\mathbf{N}$ is *pseudobounded* if

$$\lim_{n\to\infty} \frac{s_n}{n} = 0$$

for each sequence $(s_n)_n$ in S, or equivalently

$$s_n < n \text{ for all sufficiently large } n$$

for each sequence $(s_n)_n$ in S; see [12, Lemma 3], and also Richman [16] for pseudobounded sets. We shall also deal with a boundedness principle:

BD-N: Every countable pseudobounded subset of $\mathbf{N}$ is bounded,

which is derivable in classical, intuitionistic and constructive recursive mathematics. Lietz [13] proved that BD-N is not derivable in E-$\mathbf{HA}^\omega$ + $\mathrm{AC}_{\sigma,\tau}$ which is a natural formal framework for Bishop's constructive mathematics. The principle BD-N is equivalent to certain continuity principles [8, 3], and to theorems in analysis including the theorems in functional analysis mentioned above [9, 12].

The *limited principle of omniscience* is a nonconstructive logical principle:

LPO: $\forall\gamma\, [\exists k\, (\gamma(k) \neq 0) \vee \forall k\, (\gamma(k) = 0)]$,

which is an instance of the principle of excluded middle and refutable both in intuitionistic mathematics and in constructive recursive mathematics.

In the following, we shall show, in the spirit of constructive reverse mathematics [10, 14, 19, 11], that WC-N$'$ implies BT$'$ + $\neg$LPO, BT$'$ implies BD-N, and BD-N + $\neg$LPO implies WC-N$'$, and hence that WC-N$'$, BT$'$ + $\neg$LPO and BD-N + $\neg$LPO are all equivalent. In particular, since BD-N is not derivable in E-$\mathbf{HA}^\omega$ + $\mathrm{AC}_{\sigma,\tau}$, BT$'$ is not constructively provable, and, since BD-N + $\neg$LPO is derivable from the extended Church's thesis (ECT_0) and Markov's principle (MP) [8, Proposition 4], [17, 4.3.4] (see [17, 4.4.8, 4.5] for ECT_0 and MP), WC-N$'$ is not only intuitionistically valid, but also holds in constructive recursive mathematics. Since BT$'$ is also classically true, BT$'$ is another principle like BD-N that holds in classical, intuitionistic, and constructive recursive mathematics.

Although the results are formulated in the setting of Bishop's constructive mathematics, the proofs can be formalized in a system based on intuitionistic analysis **EL** [17, 3.6] or intuitionistic finite-type arithmetic $\mathbf{HA}^\omega$ [17, 9.1] with the countable choice Π^0_1-AC_{00}, that is the number-number choice

AC_{00}: $\forall m\, \exists n\, A(m, n) \Longrightarrow \exists\alpha\, \forall m\, A(m, \alpha(m))$,

for $A(m, n)$ of the form $\forall k\, (\tau(m, n, k) = 0)$. Note that the quantifier-free axiom of choice QF-AC_{00} [17, 3.6.2], that is, AC_{00} for $A(m, n)$ of the form $\tau(m, n) = 0$, is a consequence of Π^0_1-AC_{00}.

Like in **EL** and $\mathbf{HA}^{\omega}$, we assume that our universe of functions (sequences) of natural numbers contains primitive recursive (or elementary) functions and is closed under composition and bounded minimization. Hence, characteristic functions of, for example, a relation $\delta(n)n \wedge s_{\delta(n)} \geq \delta(n)$ on $\mathbf{N}$ where $(s_n)_n$ is a sequence of natural numbers, a relation $\forall b \preceq a\,(\sigma(b,n) = 0)$ on $\mathbf{N}^* \times \mathbf{N}$ and so on, exist without invoking any choice principle [17, 3.1].

§2. The main results.

Proposition 1. WC-N′ *implies* BT′ + ¬LPO.

Proof. Trivially WC-N′ implies BT′. Suppose that LPO holds, and define $\sigma\colon \mathbf{N}^* \times \mathbf{N} \to \mathbf{N}$ by

$$\sigma(a,n) = 0 \iff [|a| \geq n+2 \to \exists i < |a|-1\,(a(i) \neq 0) \vee a(|a|-1) = 0].$$

Then σ satisfies the condition (1). For each α either $\alpha(n) \neq 0$ for some n or $\alpha(n) = 0$ for all n, by LPO. In the former case, $\forall k \geq n+2\,\exists i < k-1\,(\alpha(i) \neq 0)$ for some n, and in the latter case, $\forall k \geq n+2\,(\alpha(k-1) = 0)$ for all, and hence for some n. Hence $\forall k \geq n+2\,[\exists i < k-1\,(\alpha(i) \neq 0) \vee \alpha(k-1) = 0]$ for some n, and therefore $\exists n\,\forall k\,(\sigma(\overline{\alpha}(k), n) = 0)$. By applying WC-N′ and taking $\alpha = (0,0,\dots)$, there exist m and n such that

$$\forall \beta \in \overline{\alpha}(m)\,\forall k\,(\sigma(\overline{\beta}(k), n) = 0).$$

Letting $k = \max\{m+1, n+2\}$ and $\beta = (\overbrace{0,\dots,0,1}^{k}, 0, \dots)$, we have $\beta \in \overline{\alpha}(m)$ and $\sigma(\overline{\beta}(k), n) \neq 0$, a contradiction. ⊣

Proposition 2. BT′ *implies* BD-N.

Proof. Let $S = \{s_n \mid n \in \mathbf{N}\}$ be a countable pseudobounded subset of $\mathbf{N}$, and define $\sigma\colon \mathbf{N}^* \times \mathbf{N} \to \mathbf{N}$ by

$$\sigma(a,n) = 0 \iff [|a| \geq n+1 \to s_{a(|a|-1)} < |a|-1].$$

Then σ satisfies the condition (1). For each α, since $(s_{\alpha(n)})_n$ is a sequence in S, $s_{\alpha(n)} < n$ for all sufficiently large n, and hence $\forall k \geq n+1\,(s_{\alpha(k-1)} < k-1)$ for some n. Therefore $\exists n\,\forall k\,(\sigma(\overline{\alpha}(k), n) = 0)$. By applying BT′, there exist a and n such that

$$\forall \beta \in a\,\forall k\,(\sigma(\overline{\beta}(k), n) = 0).$$

Without loss of generality, we may assume that $|a| \geq n$. For each m, taking $\beta = a * (m, 0, 0, \dots)$, we have $\beta \in a$, and hence $\sigma(\overline{\beta}(|a|+1), n) = 0$. Thus $s_m < |a|$ for all m, which is to say that S is bounded. ⊣

Next, we show that BD-N + ¬LPO implies WC-N′. To this end, suppose that $\forall \alpha\,\exists n\,A(\alpha,n)$, and $A(\alpha,n)$ is of the form $\forall k\,(\sigma(\overline{\alpha}(k), n) = 0)$ for some $\sigma\colon \mathbf{N}^* \times \mathbf{N} \to \mathbf{N}$ satisfying (1). Let T be the subset of $\mathbf{N}^*$ defined by

$$T = \{a \in \mathbf{N}^* \mid \exists b \succeq a\,(\sigma(b, |a|) \neq 0)\}. \tag{2}$$

Then we say that T *has an infinite path* if there exists α such that $\overline{\alpha}(k) \in T$ for all k. Note that if $a \in T$ and $b \preceq a$, then $b \in T$.

In the following, we assume that there exist an encoding function $f\colon \mathbf{N}^* \to \mathbf{N}$ and a decoding function $g\colon \mathbf{N} \to \mathbf{N}^*$ such that $g \circ f = \mathrm{id}_{\mathbf{N}^*}$.

PROPOSITION 3. *If T has an infinite path, then* LPO *holds.*

PROOF. Let α be an infinite path in T. Then, since $\forall n\, \exists b\, (b \succeq \overline{\alpha}(n) \wedge \sigma(b,n) \neq 0)$, we have $\forall n\, \exists b\, (g(f(b)) \succeq \overline{\alpha}(n) \wedge \sigma(g(f(b)),n) \neq 0)$, and hence

$$\forall n\, \exists m\, (g(m) \succeq \overline{\alpha}(n) \wedge \sigma(g(m),n) \neq 0).$$

By QF-AC$_{00}$, there exists δ such that

$$\forall n\, (g(\delta(n)) \succeq \overline{\alpha}(n) \wedge \sigma(g(\delta(n)),n) \neq 0).$$

Let $b_n = g(\delta(n))$. For given γ, define a sequence $(\beta_n)_n$ in $\mathbf{B}$ as follows: if $\forall i \leq n\, (\gamma(i) = 0)$, set $\beta_n = \alpha$; if $\exists i \leq n\, (\gamma(i) \neq 0)$, set $\beta_n = b_m * (0,0,\dots)$, where $m = \min_{j \leq n}[\gamma(j) \neq 0]$. Then $(\beta_n)_n$ is a Cauchy sequence: in fact, since $b_n \succeq \overline{\alpha}(n)$ for each n, we have $\beta_j \in \overline{\beta_n}(n)$ whenever $j \geq n$. Let β be the limit of $(\beta_n)_n$ in $\mathbf{B}$. Choose n such that $\forall k\, (\sigma(\overline{\beta}(k),n) = 0)$. If $\forall i < m\, (\gamma(i) = 0) \wedge \gamma(m) \neq 0$ for some $m \geq n$, then $\beta = b_m * (0,0,\dots)$, and therefore, since $\sigma(b_m,m) \neq 0$ and since $\sigma(b_m,m) \neq 0$ implies $\sigma(b_m,n) \neq 0$, we have $\sigma(\overline{\beta}(|b_m|),n) = \sigma(b_m,n) \neq 0$, a contradiction. Now either $\gamma(i) = 0$ for every $i < n$, and thus for all i, or else $\gamma(i) \neq 0$ for some $i < n$. ⊣

For given α, define a subset S_α of $\mathbf{N}$ by

$$S_\alpha = \{s \in \mathbf{N} \mid s = 0 \vee \overline{\alpha}(s) \in T\}. \tag{3}$$

Then, since $S_\alpha = \{s \in \mathbf{N} \mid \exists b \in \mathbf{N}^*\, (b \succeq \overline{\alpha}(s) \wedge \sigma(b,s) \neq 0)\} \cup \{0\}$, S_α is a countable subset of $\mathbf{N}$. In fact, with a coding function j of pairs of natural numbers and its inverses j_1, j_2 satisfying $j_1(j(m,n)) = m$ and $j_2(j(m,n)) = n$, there is a surjection $h\colon \mathbf{N} \to S_\alpha$ such that

$$h(n) = \begin{cases} j_2(n) & \text{if } g(j_1(n)) \succeq \overline{\alpha}(j_2(n)) \wedge \sigma(g(j_1(n)), j_2(n)) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We need the following lemmas whose proofs employ techniques from [7].

LEMMA 4. *Let $(s_k)_k$ be a sequence in S_α and let n be a natural number. Then either $s_k < n + k + 1$ for all k or $s_k \geq n + k + 1$ for some k.*

PROOF. Since $\forall k\, [s_k = 0 \vee \exists b\, (b \succeq \overline{\alpha}(s_k) \wedge \sigma(b,s_k) \neq 0)]$, we have $\forall k\, \exists b\, [s_k = 0 \vee (g(f(b)) \succeq \overline{\alpha}(s_k) \wedge \sigma(g(f(b)),s_k) \neq 0)]$, and hence

$$\forall k\, \exists m\, [s_k = 0 \vee (g(m) \succeq \overline{\alpha}(s_k) \wedge \sigma(g(m),s_k) \neq 0)].$$

By QF-AC$_{00}$, there exists δ such that

$$\forall k\, [s_k = 0 \vee (g(\delta(k)) \succeq \overline{\alpha}(s_k) \wedge \sigma(g(\delta(k)),s_k) \neq 0)].$$

Let $b_k = g(\delta(k))$, and define a sequence $(\beta_k)_k$ in $\mathbf{B}$ as follows: if $\forall i \leq k\, (s_i < n + i + 1)$, set $\beta_k = \alpha$; if $\exists i \leq k\, (s_i \geq n + i + 1)$, set $\beta_k = b_l * (0,0,\dots)$, where $l = \min_{j \leq k}[s_j \geq n + j + 1]$, and note that, in this case, since $s_l \geq n + l + 1 > l \geq 0$, we have $b_l \succeq \overline{\alpha}(s_l) \succeq \overline{\alpha}(l)$ and $\sigma(b_l, s_l) \neq 0$. Then $(\beta_k)_k$ is a Cauchy sequence: in fact $\beta_j \in \overline{\beta_k}(k)$ whenever $j \geq k$. So $(\beta_k)_k$ converges to a limit β in $\mathbf{B}$. Choose m such that $\forall k\, (\sigma(\overline{\beta}(k),m) = 0)$. If $\forall i < l\, (s_i < n + i + 1) \wedge s_l \geq n + l + 1$ for some $l \geq m$, then $\beta = b_l * (0,0,\dots)$ and $\sigma(b_l, s_l) \neq 0$, and therefore, since $s_l \geq n + l + 1 > m$, we have $\sigma(\overline{\beta}(|b_l|),m) = \sigma(b_l,m) \neq 0$, a contradiction. Now either $s_k < n + k + 1$ for every $k < m$, and thus for all k, or else $s_k \geq n + k + 1$ for some $k < m$. ⊣

LEMMA 5. *Let $(s_n)_n$ be a sequence in S_α. Then either $s_n < n$ for all sufficiently large n or $s_n \geq n$ for infinitely many n.*

PROOF. First note that, as in the proof of Lemma 4, there exists a sequence $(b_k)_k$ such that

$$\forall k\, [s_k = 0 \vee (b_k \succeq \overline{\alpha}(s_k) \wedge \sigma(b_k, s_k) \neq 0)],$$

by QF-AC_{00}.

By applying Lemma 4 to the subsequence $(s_{n+k+1})_k$ and the natural number n, we have $\exists k > n\, (s_k \geq k) \vee \forall k > n\, (s_k < k)$, and hence

$$\forall n\, \exists k'\, \forall k\, [(k' > n \wedge s_{k'} \geq k') \vee (k > n \rightarrow s_k < k)].$$

Therefore, by Π^0_1-AC_{00}, there exists δ such that

$$\forall n\, \forall k\, [(\delta(n) > n \wedge s_{\delta(n)} \geq \delta(n)) \vee (k > n \rightarrow s_k < k)].$$

Let $(\lambda_n)_n$ be a binary sequence such that $\lambda_n = 0 \leftrightarrow \delta(n) > n \wedge s_{\delta(n)} \geq \delta(n)$. Note that if $\lambda_n = 0$, then $s_{\delta(n)} \geq \delta(n) > n \geq 0$, and hence $b_{\delta(n)} \succeq \overline{\alpha}(s_{\delta(n)}) \succeq \overline{\alpha}(\delta(n)) \succeq \overline{\alpha}(n)$ and $\sigma(b_{\delta(n)}, s_{\delta(n)}) \neq 0$, and that if $\lambda_n = 1$, then $\forall k > n\, (s_k < k)$. We may assume that $\lambda_0 = 0$. Define a sequence $(\beta_n)_n$ in $\mathbf{B}$ as follows: if $\forall i \leq n\, (\lambda_i = 0)$, set $\beta_n = b_{\delta(n)} * (0, 0, \dots)$; if $\exists i \leq n\, (\lambda_i = 1)$, set $\beta_n = \beta_{n-1}$. Then $(\beta_n)_n$ is a Cauchy sequence: in fact, $\beta_n \in \overline{\alpha}(n)$ whenever $\forall i \leq n\, (\lambda_i = 0)$. Let β be the limit of $(\beta_n)_n$ in $\mathbf{B}$. Choose m such that $\forall k\, (\sigma(\overline{\beta}(k), m) = 0)$. If $\forall i \leq n\, (\lambda_n = 0) \wedge \lambda_{n+1} = 1$ for some $n \geq m$, then $\beta = b_{\delta(n)} * (0, 0, \dots)$, and therefore, since $\sigma(b_{\delta(n)}, s_{\delta(n)}) \neq 0$ and $s_{\delta(n)} \geq \delta(n) > n \geq m$, we have $\sigma(\overline{\beta}(|b_{\delta(n)}|), m) = \sigma(b_{\delta(n)}, m) \neq 0$, a contradiction. Now either $\lambda_n = 0$ for every $n \leq m$, and thus for all n, or else $\lambda_n = 1$ for some $n \leq m$. ⊣

PROPOSITION 6. *If T has no infinite path, then S_α is a pseudobounded subset of $\mathbf{N}$ for each α.*

PROOF. Suppose that T has no infinite path. For given α, let $(s_n)_n$ be a sequence in S_α. Then, by Lemma 5, either $s_n < n$ for all sufficiently large n or $s_n \geq n$ for infinitely many n. In the latter case, for each positive integer n there exists $k \geq n$ such that $s_k \geq k$, and therefore, since $\overline{\alpha}(n) \preceq \overline{\alpha}(k) \preceq \overline{\alpha}(s_k) \in T$, we have $\overline{\alpha}(n) \in T$. Thus T has an infinite path α, a contradiction, and so the former must be the case. ⊣

From the results obtained so far, we have the following theorem.

THEOREM 7. BD-N + $\neg$LPO *implies* WC-N$'$.

PROOF. Suppose that $\forall \alpha\, \exists n\, A(\alpha, n)$, and $A(\alpha, n)$ is of the form $\forall k (\sigma(\overline{\alpha}(k), n) = 0)$ for $\sigma \colon \mathbf{N}^* \times \mathbf{N} \rightarrow \mathbf{N}$ satisfying (1). By replacing σ by σ' such that $\sigma'(a, n) = 0 \iff \forall b \preceq a\, (\sigma(b, n) = 0)$, if necessary, we have $\forall k\, (\sigma'(\overline{\alpha}(k), n) = 0) \iff \forall k\, (\sigma(\overline{\alpha}(k), n) = 0)$, and hence we may further assume, without loss of generality, that if $\sigma(a, n) = 0$ and $b \preceq a$, then $\sigma(b, n) = 0$. Define a subset T of $\mathbf{N}^*$ by (2). Then, by Proposition 3 and $\neg$LPO, T has no infinite path. For each α the set S_α, constructed by (3), is a countable pseudobounded subset of $\mathbf{N}$, by Proposition 6, and hence S_α is bounded, by BD-N. Therefore there exists a positive integer n such that $\forall k\, (\sigma(\overline{\alpha}(k), n) = 0)$ and $n \notin S_\alpha$. For any $\beta \in \overline{\alpha}(n)$, since $\overline{\alpha}(n) \notin T$, we have $\forall k \geq n\, (\sigma(\overline{\beta}(k), n) = 0)$, and hence $\forall k\, (\sigma(\overline{\beta}(k), n) = 0)$. Thus $\forall \beta \in \overline{\alpha}(n)\, A(\beta, n)$. ⊣

Now, Proposition 1, Proposition 2 and Theorem 7 culminate in the following theorem.

THEOREM 8. *The following are equivalent.*

1. WC-N$'$.
2. BT$'$ + $\neg$LPO.
3. BD-N + $\neg$LPO.

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