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Numerical Methods for Solving Optimal Control Problems Using Chebyshev Polynomials

by

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Abstract

Many computational methods have been proposed to solve optimal control problems. These methods are classified as indirect methods and direct methods. This thesis is based on solving optimal control problems using direct methods in which an optimal control problem is converted into a mathematical programming problem. The direct methods can be employed by using the parameterization technique which can be applied in three different ways: Control parameterization, control-state parameterization and state parameterization. The control parameterization and the control-state parameterization have been used extensively to solve general optimal control problems. However, the use of the state parameterization was limited to very special cases. In this thesis, we solve general optimal control problems by using the state parameterization.

This thesis presents numerical methods to solve unconstrained and constrained optimal control problems. The solution method is based on using the second method of the quasilinearization to replace the nonlinear optimal control problem by a sequence of time-varying linear quadratic optimal control problems. Each of these problems is solved by converting it into quadratic programming problem. To this end, the state parameterization technique is employed by using the Chebyshev polynomials of the first type to approximate the system state variables by a finite length Chebyshev series of unknown parameters.

In addition, in this thesis we describe a method to determine the optimal feedback control of nonlinear optimal control problems. The method is based on applying the parameterization using Chebyshev polynomials. To facilitate the computation of the optimal feedback control law, a new property of Chebyshev polynomials called differentiation operational matrix is derived.

The proposed methods have been applied on several examples and we find that the proposed methods give better or comparable results compared with some other methods. Additionally, to make sure that the proposed methods can handle real complex problems, we applied these methods on two practical problems, F8 fighter aircraft and container crane problems.

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Contents

Chapter 1

Introduction

1.1 Motivations and Goals

The optimal control problem is to find a control function $u^*(t)$ which minimizes a given cost functional (performance index) while satisfying the system state equations and constraints. It has successful applications in many disciplines, economics, environment, management, engineering etc. A particular important class of optimal control problems is the linear quadratic optimal control problem, which has found a wide acceptance because of the possibility of obtaining the feedback optimal control law.

Generally, the solutions of optimal control problems, except for the simplest cases, are usually carried out numerically. Therefore, numerical methods and algorithms for solving optimal control problems have evolved significantly over the past thirty-five years. Most of the early methods were based on finding a solution that either satisfies the Euler-Lagrange equations, which are the necessary conditions of optimality, or satisfies Hamilton-Jacobi-Bellman equation, which is sufficient condition of the optimality. These methods are called indirect methods.

The main drawbacks of the indirect methods are the following: (1) It is very difficult to solve Hamilton-Jacobi-Bellman equation. (2) The introduction of artificial costates. (3) The lack of robustness, i.e. the iterations must start close to a local solution in order to solve the two-point boundary value problem. (4) The user must have a deep insight into the physical and the mathematical nature of the optimal control problem.

To overcome these drawbacks, many researchers proposed the use of the direct methods to solve the optimal control problems. In these methods, the optimal solution is obtained by direct minimization of the performance index subject to constraints. This can be achieved by approximating the dynamic optimal control problem by a finite dimensional nonlinear programming problem. The direct methods can be applied by using the discretization technique or the parameterization technique (control parameterization; control-state parameterization; state parameterization). In this thesis, we use the parameterization technique (state parameterization) to convert the optimal control problem into mathematical programming problem.

For the parameterization method, there are two issues to be addressed. The first issue is: Which variables (state, control) should be parameterized? The second issue is: Which functions can be used to perform the parameterization?

Concerning the first issue, a tremendous proliferation of papers have been published which are based on either control parameterization or control-state parameterization. These two approaches have some drawbacks such as: In control parameterization case, there is a need to integrate the system state equations which is a computationally expensive. While in control-state parameterization case, the optimal control problem is reduced to a large mathematical programming problem, i.e. has a large number of unknown parameters and a large number of equality constraints.

There is a third type of parameterization, the state parameterization. So far the use of this approach has been limited to special cases. For example, linear systems of equal number of state variables and control variables or nonlinear systems of a single input and in controllability canonical form. This approach, in spite of some advantages that can offer, has not been not used extensively because it is difficult to apply it to general optimal control problems. This is because it is not clear which state variables to be parameterized in case of unequal number of state variables and control variables. Therefore the **first goal** of this thesis is to apply the state parameterization to general optimal control problems, linear and nonlinear, constrained and unconstrained.

For the second issue, many functions to perform the parameterization have been used. In particular, the orthogonal functions used to solve some of the optimal control problems. Among the orthogonal functions, Chebyshev polynomials have several advantages compared with other functions. The Chebyshev polynomials have been used, in several papers, to solve linear quadratic optimal control problems. Also Vlassenbroeck and Van Dooren [39, 40] applied the Chebyshev polynomials to solve general unconstrained nonlinear optimal control problem and constrained optimal control problem. However, their method has several disadvantages and drawbacks which will be discussed in the next chapter. Hence the **second goal** of this thesis is to use the Chebyshev polynomials to parameterize the system state variables to solve linear and nonlinear, constrained and unconstrained optimal control problems, and on the same time to avoid the problems of Vlassenbroeck and Van Dooren's method.

In applying the direct methods, most of the researchers, convert the nonlinear optimal control problem into a nonlinear mathematical programming problem and then the new problem is solved by using sequential quadratic programming method. There are two exceptions: The work by Bashen and Inns [55] which converts the optimal control problem into a sequence of quadratic programming problems. However, this method was applied to a particular problem and the discretization method was used. Another approach was proposed by Ma and Levine [57]. In this case an upper bound of the optimal value only was obtained and the discretization method was also used. The **third goal** of this thesis is to solve the optimal control problem by converting it directly, using state parameterization via Chebyshev polynomials, into a sequence of quadratic programming problems. This has two advantages: the first advantage is that the linear and the nonlinear optimal control problems can be solved in uniform way. The second advantage is that guessing nominal trajectories, which we need to convert the nonlinear optimal control problem into a sequence of linear quadratic optimal control problems, is easier than guessing the parameters of these trajectories, which we need in order to solve the nonlinear mathematical programming problem.

The direct methods were used to obtain open loop solution of optimal control problems. But from practical point of view, it is desired to obtain the feedback optimal control solutions because they provide robustness with respect to external disturbances, unmodeled dynamics and variations in the physical parameters of the system to be controlled. Obtaining the optimal feedback control of the nonlinear optimal control problems by using the direct method, state parameterization, is the **fourth goal** of this thesis.

In summary the goals of this thesis are: To propose an efficient numerical methods to solve linear and nonlinear, constrained and unconstrained optimal control problems and to determine the optimal feedback control law.

1.2 Thesis Contribution

In this thesis, we propose numerical methods to solve several optimal control problems. The basic idea of these methods is to use the second method of the quasilinearization and the state parameterization. The quasilinearization replaces the nonlinear optimal control problem by a sequence of linear quadratic optimal control problems. Then each of these problems is approximated by a quadratic programming problem, which can be solved by parameterizing the state variables by Chebyshev series of unknown parameters.

The state parameterization has several advantages compared with other types of parameterizations. The first advantage is that the optimal control problem is converted into a small mathematical programming problem, compared with control-state parameterization, in the sense of the number of unknown parameters and the number of equality constraints . The second advantage is that there is no need for numerical integration of the system equations as in the case of control parameterization. The third advantage is that the state constraints can be approximated directly and not as in the control parameterization case.

The contribution of this thesis can be summarized as follows:

- Presents a new method to solve the linear quadratic optimal control problem by using state parameterization via Chebyshev polynomials. This converts the linear quadratic optimal control problem into a quadratic programming problem which can be solved by matrix-vector multiplication.

- Derives an explicit formula to approximate the quadratic performance index.

- Describes numerical method to solve the nonlinear optimal control problem using the quasilinearization and the state parameterization via Chebyshev polynomials.

- Derives a formula to perform Chebyshev series multiplications.

- Presents a numerical method to solve the nonlinear optimal control problem subject to terminal state constraints and control saturation constraints.

- Provides a numerical method to solve the linear quadratic optimal control problem subject to state and control constraints, terminal state constraints and interior points constraints.

- Introduces and derives a new property of Chebyshev polynomials called differentiation operational matrix. This property simplifies the computations of optimal feedback control law.

- Derives an explicit formula to determine the optimal feedback control law. This feedback control law has some advantages compared with the previous known methods. The first advantage is that the obtained optimal feedback control law can be implemented easier than the optimal feedback control obtained by using power series method. The second advantage is that we do not need to store the open loop optimal state and control trajectories as in the methods that give the neighboring optimal feedback control [3, 19–22]. The third advantage is that the obtained optimal feedback control is a nonlinear one and, although it appears as a linear one, the nonlinear terms of the states are included in the time-varying terms.

1.3 Thesis Organization

The remaining chapters of this thesis are organized as follows:

Chapter 2 gives an overview of the optimal control theory, and reviews some computational methods to solve optimal control problems. In this chapter, we classify the computational methods into direct and indirect methods. The direct methods are, in their turn, also classified into methods based on discretization and methods based on parameterization. Also the indirect methods are classified into methods that give an open loop solution and methods that give a closed loop solution. This chapter shows the place of this thesis compared with other works.

Chapter 3 presents a numerical method to solve the linear quadratic optimal control problem. In this chapter, the concept of the state parameterization using Chebyshev polynomials is introduced. Also it describes the most appropriate way to perform the state parameterization. This chapter also describes a method of approximating the dynamic optimal control problem into a standard quadratic programming problem. In addition, it shows the derivation of two results: The first result is an explicit formula to approximate the quadratic performance index. The second result is the proof that the Hessian of the quadratic programming problem is positive definite. Finally, this chapter shows computational results of a standard example and compares our results with those obtained previously.

Chapter 4 generalizes the method of chapter 3 to solve the nonlinear optimal control problem and as a special case the time-varying linear optimal control problem. In this chapter the concept of quasilinearization is introduced. It also shows the reformulation of the nonlinear optimal control problem into a sequence of quadratic programming problems. A new result is given in this chapter which is a formula to compute the multiplication of two Chebyshev series. In addition to a standard example which is solved for the purpose of comparison, we present the computational results of a practical problem, the automatic flight control problem as an application of this chapter and the previous one.

Chapter 5 describes a numerical method, which is based on the method described in chapter 3, to solve the constrained linear quadratic optimal control problem. The constraints which are considered in this chapter are: state and control constraints, terminal state constraints and interior point constraints. It also shows the reformulation of the constrained optimal control problem into a quadratic programming problem. Moreover it shows the computational results of a constrained optimal control problem.

Chapter 6 presents a numerical method to solve the nonlinear optimal control problem subject to terminal state constraints and control saturation constraints. The difficult constrained nonlinear optimal control problem is converted into a sequence of quadratic programming problems. This method is applied on Van der Pol oscillator problem subject to terminal state constraints and control saturation constraints. In addition, in this chapter, we show the simulation results of a high dimension practical nonlinear optimal control problem subject to terminal state constraints, control saturation constraints and state constraints.

Chapter 7 describes a new method to determine the optimal feedback control law of nonlinear optimal control problems. A new property of Chebyshev polynomials is derived. This property, differentiation operational matrix property, simplifies the computations of the optimal feedback control law. Also this chapter presents an explicit formula to determine the feedback gain matrix. Computational results of an example are also presented.

Chapter 8 contains the conclusions of this thesis and recommendations for future work.

Chapter 2

Optimal Control Problem: A Review

2.1 Introduction

The optimal control problem has been treated in depth in many textbooks [1–7] and in some important survey papers [8–10]. The objective of optimal control is to determine an optimal open loop control $u^*(t)$ or an optimal feedback control $u^*(x,t)$ that forces the system to satisfy the system physical constraints and at the same time minimizes or maximizes a performance index.

The basic optimal control problem consists of the following three elements:

1. A mathematical model of the system to be controlled: The dynamical system to be controlled can be described by state equations which are a set of first order differential equations

$$\dot{x} = f(x(t), u(t), t) \tag{2.1}$$

   where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector, f is continuously differentiable with respect to all its arguments. The control functions u are assumed to be the piecewise continuous functions from $t \in [0, t_f]$ into R^m. Let $\mathcal{U}$ be the class of such control functions.

2. A set of boundary conditions on the state variables which gives the value of the system states at the initial time

$$x(t_0) = x_0 \tag{2.2}$$

   where x_0 is a known vector of initial conditions.

3. A performance index which is to be minimized (maximized). The performance index describes some desired specifications, expressed mathematically in form of a scalar function. The optimal control problem helps the designer to select the "best" control, by minimizing a given performance index. The performance index which we are interested in can be expressed as

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt \tag{2.3}$$

where ϕ and L are scalar functions, continuously differentiable in all arguments. The terminal cost $\phi(x(t_f), t_f)$ and the "loss" function $L(x(t), u(t), t)$ are selected depending on the performance objectives. These functions are generally nonnegative functions of x and u, with $\phi(0, t_f) = 0$ and $L(0, 0, t) = 0$.

2.2 Statement of the Optimal Control Problem

The unconstrained optimal control problem can be stated as follows:
Given f, x_0, t_0 , t_f, ϕ and L, find an optimal open loop control or an optimal feedback control that minimizes the performance index

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt \tag{2.4}$$

subject to the system state equations and the initial conditions

$$\dot{x} = f(x(t), u(t), t) \qquad x(t_0) = x_0 \tag{2.5}$$

This problem, basically, can be solved by one of the following methods:

- Bellman's dynamic programming method which is based on the principle of optimality (Hamilton-Jacobi-Bellman equation).
- Variational method and Pontryagin's minimum principle (Euler-Lagrange equations).
- Direct methods using discretization or parameterization (nonlinear mathematical programming)

These methods will briefly be discussed in the following sections.

In general it is not possible to solve the problem (2.4)-(2.5) analytically. However, an analytical solution is possible for a special case of this problem, the linear quadratic optimal control problem, in which the performance index is quadratic and the system state equations are linear. This problem can be stated as follows: Find the optimal control that minimizes

$$J = x^T(t_f)Sx(t_f) + \int_{t_0}^{t_f} (x^TQx + u^TRu)dt \tag{2.6}$$

subject to

$$\dot{x} = A(t)x + B(t)u \qquad x(t_0) = x_0 \tag{2.7}$$

where S, Q are positive semidefinite matrices and R is a positive definite matrix. For this problem the solution can be expressed in feedback form

$$u^*(x,t) = -R^{-1}B^T(t)P(t)x \tag{2.8}$$

where $P(t)$ is the solution of Riccati equation.

2.3 Dynamic Programming: Hamilton-Jacobi-Bellman Equation

The use of the principle of optimality, usually known as dynamic programming, to derive an equation for solving optimal control problem was first proposed by Bellman [11]. The application of this principle on continuous optimal control problem has led to the invention of the famous Hamilton-Jacobi-Bellman (HJB) equation. It is a nonlinear first order hyperbolic partial differential equation which is used for constructing a nonlinear optimal feedback control law. For the optimal control problem (2.4)-(2.5), the HJB equation is given by

$$\frac{\partial J^*(x(t),t)}{\partial t} = -\min_{u(t)}\{L(x(t),u(t),t) + \frac{\partial J^*(x(t),t)}{\partial x}^T f(x(t),u(t),t)\} \tag{2.9}$$

and the boundary condition is

$$J^*(x(t_f),t_f) = \phi(x(t_f),t_f) \tag{2.10}$$

To obtain a solution to equation (2.9), we proceed in two steps. The first step is to perform the indicated minimization. This leads to a control law of the form

$$u^* = \psi(\frac{\partial J^*}{\partial x}, x, t). \tag{2.11}$$

The second step is to substitute (2.11) back into (2.9) and solve the nonlinear, partial differential equation

$$-\frac{\partial J^*(x,t)}{\partial t} = L(x,\psi,t) + \frac{\partial J^*(x,t)}{\partial x}^T f(x,\psi,t) \tag{2.12}$$

for $J^*(x,t)$, subject to the boundary condition (2.10). Then the gradient of $J^*(x,t)$ with respect to x is computed, and the optimal feedback control law is obtained

$$u^* = \psi(\frac{\partial J^*}{\partial x}, x, t) = \Phi(x,t) \tag{2.13}$$

The derivation of HJB equation can be found in any standard optimal control textbook, see for example [2, 4–6]. This equation is a sufficient condition for optimality. The HJB equation is satisfied for all time-state pairs $(x(t), t)$ by the optimal value function $J^*(x(t), t)$, i.e. if the system starts in state $x(t)$ at time t, the minimum value of the performance index is $J^*(x(t), t)$.

An advantage of using the HJB approach to solve the optimal control problem, is that we obtain optimal feedback control law. However, the HJB equation does not in general, possess classical solution, that is, solutions $J^*(x(t), t)$ which are differentiable with respect to t and x. In recent years, a new notion of solution, called the viscosity solution, has been introduced. For more details of this approach, the reader can consult Ahmed [13] and the relevant references cited therein.

In general it is not possible to solve (2.12) analytically. However, in the case of linear quadratic optimal control problem (2.6)-(2.7), the HJB equation reduces to Riccati differential equation, which is given by

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + Q - P(t)B(t)R^{-1}B^T(t)P(t) \tag{2.14}$$

$$P(t_f) = S \tag{2.15}$$

This result can be obtained if the value $J^* = x^T P(t)x$ is substituted in the HJB equation.

2.4 Necessary Conditions of Optimality

2.4.1 Euler-Lagrange Equations

The necessary conditions can be derived by the methods of Calculus of Variations which are based on the fact that, at each stationary point, the variation in the cost function should vanish for arbitrary variation in the control [3].

To solve the optimal control problem (2.4)-(2.5), we shall use Lagrange multipliers $\lambda(t) \in R^n$ to adjoin the system state equations (2.5), to the performance index (2.4). Therefore, the augmented performance index is given by,

$$J_A = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} [L(x(t), u(t), t) + \lambda(t)^T (f(x(t), u(t), t) - \dot{x})]dt \tag{2.16}$$

Introducing the Hamiltonian function H defined by

$$H(x(t), u(t), \lambda(t), t) = L(x(t), u(t), t) + \lambda^T(t) f(x(t), u(t), t) \tag{2.17}$$

we can rewrite equation(2.16) in the form

$$J_A = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} H(x(t), u(t), \lambda(t), t)dt - \int_{t_0}^{t_f} \lambda(t)^T \dot{x}(t)dt \tag{2.18}$$

The integration of the last term on the right hand side by parts yields,

$$\int_{t_0}^{t_f} \lambda(t)^T \dot{x}(t)dt = \lambda(t_f)^T x(t_f) - \lambda(t_0)^T x(t_0) - \int_{t_0}^{t_f} \dot{\lambda}(t)^T x(t)dt \tag{2.19}$$

and therefore equation (2.18) becomes

$$J_A = \phi(x(t_f), t_f) - \lambda(t_f)^T x(t_f) + \lambda(t_0)^T x(t_0) + \int_{t_0}^{t_f} [H(x(t), u(t), \lambda(t), t) + \dot{\lambda}(t)^T x(t)]dt \tag{2.20}$$

The original problem (2.4)-(2.5) has been converted to the problem of minimizing (2.20) without constraints.

To achieve the stationarity, the first order effect of control variations on the cost function must be zero for $0 \leq t \leq t_f$. Assuming that the initial time t_0 and final time t_f are fixed, then the first variation of J_A due to control variation is

$$\delta J_A = \left[(\frac{\partial \phi}{\partial x} - \lambda^T)\delta x \right]_{t=t_f} + \lambda^T \delta x \mid_{t=t_0} + \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial u}\delta u + (\frac{\partial H}{\partial x} + \dot{\lambda}^T)\delta x \right] dt \tag{2.21}$$

Since $\lambda(t)$ is arbitrary so far, we may set it to be

$$\dot{\lambda}^T(t) = -\frac{\partial H}{\partial x} = -\lambda^T \frac{\partial f}{\partial x} - \frac{\partial L}{\partial x} \tag{2.22}$$

with boundary condition,

$$\lambda^T(t_f) = \frac{\partial \phi}{\partial x} \mid_{t=t_f} \tag{2.23}$$

equation (2.22) is called costate equation and the Lagrange multiplier $\lambda(t)$ is called the costate.

Since the initial condition $x(t_0)$ is fixed, this implies $\delta x(t_0)$ vanishes. Therefore, equation (2.21) reduces to

$$\delta J_A = \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial u}\delta u \right] dt \tag{2.24}$$

For a local minimum, it is necessary that δJ_A vanishes for arbitrary δu, hence it is necessary that

$$\frac{\partial H}{\partial u} = \left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial L}{\partial u}\right)^T = 0 \tag{2.25}$$

for all $t_0 \leq t \leq t_f$.

Equations (2.5), (2.22),(2.23) and (2.25) are necessary conditions to be satisfied by optimal solutions of the problem, when the final time is fixed. These equations are called the Euler-Lagrange equations.

In summary, to find the optimal control $u^*(t)$ that minimizes the performance index (2.4) subject to the system equation (2.5), the following equations must be solved

$$\begin{aligned} \dot{x} &= f(x,u,t) && (2.26) \\ x(t_0) &= x_0 && (2.27) \\ \dot{\lambda} &= -\left(\frac{\partial f}{\partial x}\right)^T \lambda - \left(\frac{\partial L}{\partial x}\right)^T && (2.28) \\ \lambda(t_f) &= \left(\frac{\partial \phi}{\partial x}\right)^T && (2.29) \end{aligned}$$

where $u^*(t)$ is determined by:

$$\left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial L}{\partial u}\right)^T = 0 \qquad (2.30)$$

Thus, the solution of the optimal control problem is determined by a two-point boundary value problem, expressed by the state equation (2.26) with the initial condition (2.27) and the costate equation (2.28) with the final condition (2.29).

Remarks:

(1) If L(x(t),u(t),t) and f(x(t),u(t),t) are not functions of time explicitly , then the Hamiltonian is constant along the optimal path.

(2) In the case of free end time, in which the final time can be chosen to further minimize the cost function, another necessary condition must be provided. This condition can be derived from the first variation of the cost function with respect to the time. Hence the following necessary condition is obtained for optimality with free end time.

$$\left(\frac{\partial \phi}{\partial t} + H\right)_{t=t_f} = 0 \qquad (2.31)$$

From this equation, it is clear that if the terminal cost $\phi(x(t_f), t_f)$ does not depend on the time explicitly, then

$$H \mid_{t=t_f} = 0 \qquad (2.32)$$

Therefore, if H also does not depend explicitly on time, then $H = 0$ for all $0 \le t \le t_f$.

(3) It is assumed in the previous derivations that the final state $x(t_f)$ is free. If the final state is fixed i.e.

$$x(t_f) = x_f \qquad (2.33)$$

then the previous necessary conditions still hold except (2.29) which is replaced by (2.33).

2.4.2 Pontryagin Minimum Principle

In real problems, the control variables are usually bounded, therefore we can not differentiate the Hamiltonian with respect to the control, equation (2.30). Let the bounded control lie in the subset $U \in R^m$. In this case, the necessary conditions are derived from the Minimum Principle which was developed by Pontryagin and his school [12].

Pontryagin Minimum Principle:

Suppose that $u^*(t)$ is the optimal control with corresponding optimal trajectories $x^*(t)$, and let the Hamiltonian be defined by equation (2.17). In order that $u^*(t)$ and $x^*(t)$ be optimal of the problem (2.4)-(2.5), then there must exist a costate vector $\lambda^*(t)$ such that the following conditions hold:

$$\dot{\lambda} = -\frac{\partial H}{\partial x}^T \tag{2.34}$$

$$\lambda(t_f) = \left(\frac{\partial \phi}{\partial x}\right)^T \tag{2.35}$$

$$\tag{2.36}$$

and

$$H(x^*, u^*, \lambda^*, t) \leq H(x^*, u, \lambda^*, t) \tag{2.37}$$

for any $t \in [t_0, t_f]$ and for all controls $u(t) \in U$, which indicates that the optimal control must minimize the Hamiltonian.

Inequality (2.37) is very useful to obtain the optimal control if the control is bounded by inequality constraints. It should be pointed out that Pontryagin's minimum principle is a generalization of the calculus of variations approach. The difference between the calculus of variations approach and the minimum principle is that equation (2.30) is replaced by (2.37).

From the previous discussion, it is clear that the variational approach and the minimum principle lead to a nonlinear two-point boundary value problem which is very difficult to solve analytically.

There is a very large number of methods which have been proposed to obtain a numerical solutions of the HJB equation and the nonlinear two-point boundary value problem. These methods are called indirect methods. There is another class of methods to solve the optimal control problem, called direct methods. The direct methods are based on solving the optimal control problem by transforming it into a nonlinear programming problem. In the following sections, we review these methods. A block diagram which shows these methods is depicted by Figure 2.1

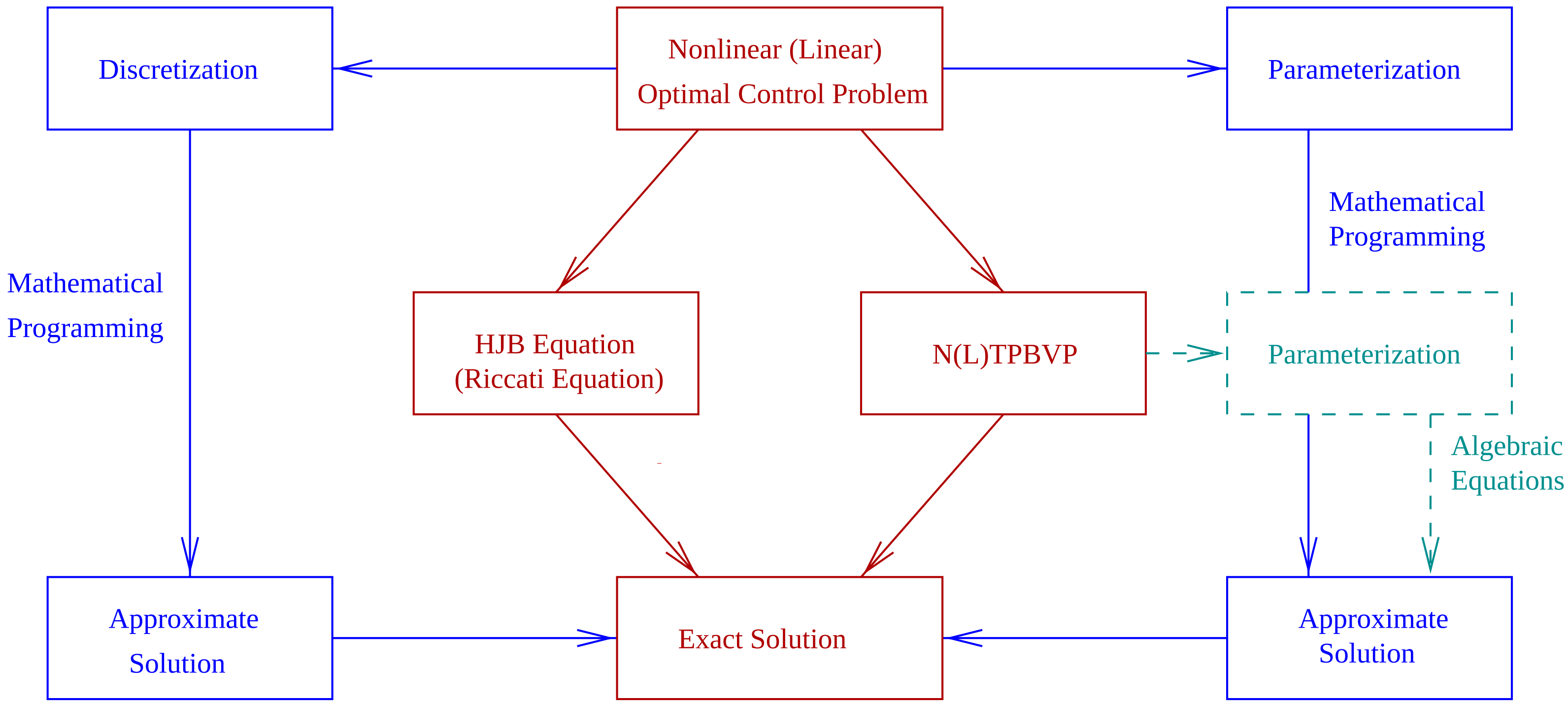

Figure 2.1: Computation methods of optimal control problem

2.5 Indirect methods

These are the methods that based on solving the optimal control problem using HJB equation or the nonlinear two-point boundary value problem. These methods can be divided into two categories: closed loop methods and open loop methods.

2.5.1 Closed Loop Control Methods

Some of the methods which were proposed to obtain the feedback optimal control are summarized as follows

- The first approach to obtain feedback optimal control is based on using the power series expansion to solve either the HJB equation or the nonlinear two-point boundary value problem successively to obtain an approximate optimal feedback control law. This approach has been applied by Lukes [14], to find an approximate solution of HJB equation of the infinite horizon general nonlinear optimal control problem. The solution of HJB equation is reduced to solving successively systems of linear algebraic equation. Using the same idea, Willemstein [15] extended Lukes' work to handle the finite time nonlinear optimal control problem. The work of Lukes has been applied by Garrard and Jordan [16] to control F8 fighter aircraft. The power series technique has also been used by Nishikawa *et al.* [17] to obtain the approximate optimal solution of finite time quadratic performance index subject to the perturbed system given by,

$$\dot{x} = A(t)x + \epsilon f(x,t) + B(t)u \tag{2.38}$$

  This optimal control problem was solved by expanding the costate by a power series with respect to ϵ, and the solution was reduced to solving a sequence of linear partial differential equations. Also, similar idea was applied by Yoshida *et al* [18] to solve the finite and the infinite time quadratic performance indices subject to the system

$$\dot{x} = f(x) + Bu \tag{2.39}$$

  In this case, the lexicographic listing vector $x^{[k]}$ was used to express the function $f(x)$ in a power series about the origin and also to express the costates by a power series of unknown parameters. The solution of the finite time optimal control problem was reduced to solving a Riccati equation and a sequence of ordinary linear differential equations, while the solution of the infinite time optimal control problem was reduced to solving sequence of algebraic equations.

- The second approach to obtain the optimal feedback control is to obtain the neighboring optimal feedback control which can be obtained either by linearizing the necessary conditions of the optimality around the optimal solution or expanding the performance index up to the second order and the constraints up to the first order around the optimal solution [3, 19–22].

- The third approach to find the optimal feedback control law is based on writing the nonlinear state equations into a linear form state equations as follows

$$\dot{x} = f(x,u,t) = A(x,u,t)x + B(x,u,t)u \tag{2.40}$$

and then the quadratic optimal control problem is solved by solving the following Riccati equation

$$\begin{aligned} \dot{P}(x,u,t) &= P(x,u,t)A(x,u,t) + A^T(x,u,t)P(x,u,t) \\ &\quad -P(x,u,t)B(x,u,t)R^{-1}B^T(x,u,t)P(x,u,t) + Q \end{aligned} \tag{2.41}$$

and the optimal control is given by

$$u^*(x,t) = -R^{-1}B^T(x,u,t)P(x,u,t)x(t) \tag{2.42}$$

Thus for a given state x the optimal control is found by simultaneously solving equations (2.41) and (2.42).

This method was developed by Burghart [23], Wernli and Cook [24].

- The fourth approach to find the optimal feedback control solution is to solve the inverse optimal control problem [25–28].

- Some of other approaches can be found in Nedeljkovic [29], Goh [30], and Longmuir and Bohn [31].

2.5.2 Open Loop Control Methods

There is a great number of papers that present numerical methods for finding the optimal open loop control. These methods are based on solving the nonlinear two-point boundary value problem. Some of these methods are: Gradient methods, quasilinearization, penalty function methods, neighboring extremal methods. These are standard methods to solve the optimal control problems, for details of these methods, the reader can refer to [3–5].

2.6 Direct Methods

This is another major class of methods for solving the optimal control problems. These methods offer some advantages when applied to optimal control problems. The first advantage is that the difficult dynamic optimal control problem can be converted into static parameters optimization problem which is easier than the original one; the second advantage is that there are well-developed algorithms to solve the nonlinear programming problems; the third advantage is that it is possible to treat different types of constraints easily.

Due to these attractive features of the direct methods and the drawbacks, mentioned earlier, of the indirect methods, a number of authors has used the direct methods to solve the optimal control problem. The direct methods are based on obtaining the solution through a direct minimization of the performance index, subject to constraints, of the optimal control problem. These methods can be applied by converting the nonlinear optimal control problem into a nonlinear mathematical programming problem [32–34, 37, 39–41, 43, 48, 49, 52, 54, 55, 70, 75–78].

The optimal control problem can be converted into a mathematical programming problem by using either the discretization or the parameterization techniques. The work in this thesis is based on using the parameterization technique to convert the optimal control problem into mathematical programming problem.

2.6.1 Discretization Methods

All discretization approaches divide the time interval into n_s segments

$$t_0 < t_1 < t_2 < \cdots < t_{n_s} = t_f$$

where the time points are referred to as mesh points, grid points or nodes.

One approach to apply this method is to discretize both the state variables and the control variables, therefore we have the following sequence of unknown values of state variables and control variables,

$$z = (x_0, x_1, \cdots, x_{n_s}, u_0, u_1, \cdots, u_{n_s-1})$$

and the system state equations are replaced by a set of algebraic equations which are considered as equality constraints. Hence this problem can be solved using any of the nonlinear programming techniques. One of the disadvantages of this approach is the high dimensionality of the vector z.

Another approach is to discretize the control variables only

$$z = (u_0, u_1, \cdots, u_{n_s-1})$$

and then the system state equations have to be integrated to find the state variables as a function of the control variables. For more details of these approaches the interested reader is referred to [35, 75, 78] and the references cited therein.

2.6.2 Parameterization Methods

The parameterization technique is an essential part of this research, therefore we will explain this approach in some details.

The parameterization technique can be applied in one of the following three forms

1. *Control Parameterization:*

   The control parameterization is based on approximating the control variables by choosing an appropriate structure with finitely many unknown parameters as follows

$$u_l(t) = \sum_{i=0}^{N} b_i^{(l)} \Phi_i(t) \qquad l = 1, 2, \cdots, m \tag{2.43}$$

   where b_i's are unknown parameters, $\Phi_i(t)$ denotes an appropriate set of functions forming a basis of a finite dimensional control space.

   The state variables are obtained as a function of the unknown parameters of the control variables, by integrating the system state equations. And by substituting the approximated control variables and the corresponding state variables into the performance index, the optimal control problem can be converted into a static parameters programming problem, which can be solved easier than the original one.

   Some of the functions that have been used to approximate the control variables are [33]: Piecewise constant functions, piecewise linear functions, piecewise polynomials, splines of a given order, or functions known to be well-suited for practical realization.

   The control parameterization approach is the most widely used parameterization approach. It has been used in many research papers and books, [33, 34, 36–38] and the cited therein references. Applying this technique requires the integration of the system state equations, which is a computationally expensive process [73].

2. *Control-State Parameterization*

   The control-state parameterization approach [39–42, 48, 52, 54, 70, 71] is based on approximating both the state variables and the control variables by a sequence of known functions with unknown parameters as follows

$$x_j(t) = \sum_{i=0}^{N} a_i^{(j)} \Phi_i(t) \qquad j = 1, 2, \cdots, n \tag{2.44}$$

$$u_l(t) = \sum_{i=0}^{N} b_i^{(l)} \Phi_i(t) \qquad l = 1, 2, \cdots, m \tag{2.45}$$

where a_i, b_i are unknown parameters, and $\Phi_i(t)$ is an appropriate set of functions.

Using this method, the optimal control problem can be converted into a nonlinear mathematical programming problem.

The main disadvantages of this approach are: A large number of unknown parameters which have to be determined a_i and b_i; the system state equations have to be replaced by a large number of equality constraints. Therefore, using this approach we end up with a large dimensional nonlinear mathematical programming problem, in the sense of the number of unknown parameters and the number of equality constraints.

3. *State Parameterization*

The idea of the state parameterization is to approximate only the system state variables by a sequence of given functions with unknown parameters

$$x_j(t) = \sum_{i=0}^{N} a_i^{(j)} \Phi_i(t) \qquad j = 1, 2, \cdots, n \tag{2.46}$$

and then the control variables are obtained from the state equations.

In comparison with the previous two approaches, control parameterization and control-state parameterization, this method has some advantages: (1) There is no need to integrate the system state equations as in control parameterization. (2) The number of the unknown parameters is lower than those in control-state parameterization. (3) The system state equations will be satisfied directly and will not be replaced by equality constraints. (4) The state constraints can be handled directly.

In spite of many advantages of this technique, it has not been used extensively compared with the previous two approaches [43, 49, 77, 79]. The main reasons for this are the following:

(a) It is difficult to handle the nonlinear systems using the state parameterization, because it is not always easy to find the control variables as function of the state variables.

(b) There is no systematic way to apply this technique on general optimal control problems of unequal number of state variables and control variables.

In this research, we overcome these difficulties by using the second method of quasi-linearization and by proposing a method to help the user to decide which state variables to prameterize.

In the previous few works [43, 49, 77, 79] concerning the state parameterization, this technique was applied on special cases, for example linear optimal control problem with equal number of state variables and control variables or single input nonlinear systems which can be expressed in the controllability canonical form. Also there is no detailed treatment of this technique for general optimal control problems, linear or nonlinear, constrained or unconstrained. Moreover, there are no details on how to apply this technique. Therefore, the first purpose of this thesis is to clarify this approach showing a systematic way how we can apply it to convert the optimal control problem into mathematical programming problem. Also we will generalize this technique to handle general, linear and nonlinear, constrained and unconstrained, optimal control problems.

As we mentioned earlier, one of the problems of this technique is the difficulty of handling the nonlinear systems. In this work, we overcome this problem by using the second method of the quasilinearization [45]. In this research, all aspects of the state parameterization will be considered. Moreover we will show the most appropriate methods of using this technique.

The state parameterization can be employed using different basis functions [33]. In this work the Chebyshev polynomials will be used to parameterize the system state variables. The Chebyshev polynomials have several advantages. Some of these advantages are fast convergence and minimax property [44]. Vlach [46] stated that, of all ultraspherical polynomials, the Chebyshev polynomials of the first type can uniformly approximate a much broader class of functions. This does not mean at all that we are saying that the Chebyshev polynomials perform better than others in all applications, some other orthogonal polynomials may perform better for certain applications.

The use of the Chebyshev polynomials to solve the optimal control problems is not new. Paraskevopoulos [66], Wang and Nagurka [69], Chou and Horng [62], Liu and Shih [64] used Chebyshev polynomials to solve linear quadratic optimal control problems. On the other hand, Vlassenbroeck and Van Dooren [39,40,70,71] used the Chebyshev polynomials to parameterize the state variables and the control variables to solve the unconstrained nonlinear optimal control problem, and the constrained optimal control problem. In spite of their generalization to solve nonlinear optimal control problems, their method has some severe disadvantages [48]. Some of these disadvantages are: Extremely complicated method of approximation; the optimal control problem is reduced to a large size nonlinear programming problem.

The second purpose of this thesis is to use the Chebyshev polynomials to parameterize the system state variables to solve the unconstrained linear and nonlinear optimal control problems. Moreover, we extend this approach to solve both the constrained linear quadratic optimal control problem subject to all types of constraints and the constrained nonlinear optimal control problems subject to terminal state constraints and control saturation constraints. Some of the advantages of our method are: (1) Easy approximation method, (2) explicit formula to approximate the performance index, (3) small size mathematical programming problems.

In all the direct methods mentioned previously, the nonlinear optimal control problem was converted into a nonlinear mathematical programming problem. One of the methods to solve the nonlinear programming problem is a sequential quadratic programming. There are two exceptions: the work of [55] and the work of [57]. In these papers the nonlinear optimal control problem was converted directly into a sequence of quadratic programming problems using the discretization technique. These two works have some drawbacks as in the first work a specific class of problems was solved, moreover there was a need for special program to handle the control saturation constraints. For the second work, it only gives an upper bound on the optimal value, moreover the states and costates have to be integrated in each iteration.

The third purpose of this thesis is to reduce the nonlinear optimal control problem directly to a sequence of quadratic programming problems using the state parameterization. Our method has the following advantages: (1) It can handle general problems, (2) there is no need for special program to solve it, (3) there is no need to integrate the system states or the costates, (4) the optimal solution can be obtained, (5) due to the use of the state parameterization, each of the quadratic programming problems is a small size problem, in the sense of the number of the unknown parameters and the number of equality constraints.

Although the direct methods give the open loop solution of the optimal control problems, there are few works [61, 62, 64, 66, 67, 69] in which the parameterization technique, state-costate parameterization, was used to obtain the feedback solution of the linear quadratic optimal control problems. The fourth purpose of this thesis is to extend the use of the direct methods to obtain the feedback optimal solution of the nonlinear optimal control problems using the parameterization technique via Chebyshev polynomials.

In short, we can say that this thesis answers unanswered questions in the previous works, completes and extends previous works, and develops new direction of research concerning the computations of optimal feedback control of the nonlinear systems.

Another approach to solve the optimal control problem using the parameterization technique is by applying this technique on the nonlinear two-point boundary value problem, by parameterizing the states and the costates [61, 62, 64, 66, 67, 69]. Hence, the nonlinear two-point boundary problem is reduced to solving a set of algebraic equations.

Chapter 3

Linear Optimal Control Problem

3.1 Introduction

As has been shown in the previous chapter, the linear quadratic optimal control problem is one of the few optimal control problems in which an optimal analytical feedback solution can be obtained [2, 3]. The solution of this problem can be obtained either by solving matrix Riccati equation, which is a nonlinear ordinary differential equation, or by solving linear two-point boundary value problem.

To avoid the difficulties associated with the numerical integration of these methods, there are two approaches: The first approach is to convert the linear quadratic optimal control problem into a quadratic programming problem, Razzaghi and Elnagar [84] used shifted Legendre polynomials to parameterize the derivative of each of the state variables; Frich and Stech [41] used the Walsh functions to parameterize the state variables and the control variables; Elnagar and Razzaghi [86] parameterized the state variables and the control variables in terms of their values at Legendre-Gauss-Lobatto points. The second approach is to solve it by converting the linear two-point boundary value problem into a set of linear algebraic equations by parameterizing the system state variables and costate variables [61, 62, 64, 66, 67, 69, 85].

As has been mentioned in the previous chapter, most of the parameterization methods are based on either control parameterization or control-state parameterization. But these two approaches have some drawbacks. Therefore, throughout this thesis the state parameterization method is employed.

The first purpose of this chapter is to discuss the state parameterization and show how we can apply it in systematic way. The second purpose is to present the reformulation method of the optimal control problem into a quadratic programming problem. The third purpose is to derive an explicit formula to approximate the performance index.

For all of these objectives, in this chapter, we present a new numerical method to solve the simplest optimal control problem, the linear quadratic optimal control problem, by directly converting it into a quadratic programming problem. To this end we employ the state parameterization method by using the Chebyshev polynomials of the first type, therefore the optimal control problem is converted into quadratic programming problem which can be solved in one iteration by performing matrix−vector multiplication. The advantages of this numerical method are: There is no need to integrate the system state or costate equations; the optimal control problem is converted into a small quadratic programming problem.

3.2 Problem Statement

Consider the dynamical system described by the following state equations:

$$\dot{x} = Ax + Bu \tag{3.1}$$

where $x \in R^n$, $u \in R^m$, $m \leq n$; A and B are respectively, $n \times n$ and $n \times m$ real−valued matrices. We have assumed that the process starts from $t = 0$ and ends at the fixed time $t_f > 0$. A process which starts from $t_0 \neq 0$ may be transformed to satisfy this assumption by suitable shifting the time axis.

The initial condition for the state equations (3.1) are:

$$x(0) = x_0 \tag{3.2}$$

where x_0 is a given vector in R^n.

The optimal control problem is to find an optimal control $u^*(t)$ on $0 \leq t \leq t_f$ which minimizes the quadratic performance index,

$$J = \int_0^{t_f} (x^T Q x + u^T R u) dt \tag{3.3}$$

subject to the state equations (3.1) and the initial condition (3.2). Here Q is an $n \times n$ positive semidefinite matrix and R is an $m \times m$ positive definite matrix.

In this chapter, we propose a method to solve this optimal control problem by converting it directly into a quadratic programming problem. This method is based on approximating the system state variables by Chebyshev series of finite length but with unknown parameters. This method will be generalized in the next chapter to solve the nonlinear optimal control problem.

3.3 State Parameterization Using Chebyshev Polynomials

Before we start discussing the state parameterization, some important properties of the Chebyshev polynomials of the first type will be summarized.

3.3.1 Chebyshev Polynomials

Because the Chebyshev polynomials have some advantages, compared with other orthogonal polynomials, such as fast convergence and minimax properties [44], they will be used in this research to perform the state parameterization. To facilitate the presentation of the materials that follows, we present in this section some background on Chebyshev polynomials.

The Chebyshev polynomials of the first type are defined on the interval $\tau \in [-1, 1]$. These polynomials are defined as follows:

$$T_r(\tau) = \cos(r\theta) \qquad \cos\theta = \tau \qquad -1 \leq \tau \leq 1 \tag{3.4}$$

Therefore, the first three Chebyshev polynomials are:

$$\begin{aligned} T_0(\tau) &= 1 \\ T_1(\tau) &= \tau \\ T_2(\tau) &= 2\tau^2 - 1 \end{aligned} \tag{3.5}$$

The remaining Chebyshev polynomials can be obtained from the recurrence relation,

$$T_{r+1}(\tau) = 2\tau T_r(\tau) - T_{r-1}(\tau) \qquad r \geq 1 \tag{3.6}$$

The Chebyshev polynomial $T_n(\tau)$ is a solution of the Chebyshev equation

$$(1-\tau^2)\frac{d^2y}{d\tau^2} - \tau\frac{dy}{d\tau} + n^2 y = 0 \tag{3.7}$$

The polynomials $T_n(\tau)$ and $T_m(\tau)$ are orthogonal in the interval $\tau \in [-1, 1]$ with respect to the weighting function

$$w(\tau) = \frac{1}{(1-\tau^2)^{1/2}} \tag{3.8}$$

and therefore

$$\int_{-1}^{1} \frac{T_n(\tau)T_m(\tau)}{(1-\tau^2)^{1/2}} d\tau = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \neq 0 \\ \pi & n = m = 0 \end{cases} \tag{3.9}$$

The Chebyshev polynomials have some interesting properties which will be used frequently throughout this thesis. Some of these properties are:

- The product relation

$$T_n(\tau)T_m(\tau) = \frac{1}{2}(T_{n+m}(\tau) + T_{|n-m|}(\tau)) \tag{3.10}$$

- The initial and final values

$$T_n(1) = 1 \tag{3.11}$$
$$T_n(-1) = (-1)^n \tag{3.12}$$

- Integration property

$$\int_{-1}^{1} T_n(\tau)d\tau = \begin{cases} 0 & n \text{ odd} \\ \frac{-2}{n^2-1} & n \text{ even} \\ 2 & n = 0 \end{cases} \tag{3.13}$$

A function $x(\tau)$ can be approximated by a Chebyshev series of length N as follows,

$$x(\tau) = \frac{a_0}{2} + \sum_{i=1}^{N} a_i T_i(\tau) \tag{3.14}$$

where

$$a_j = \frac{2}{K}\sum_{i=1}^{K} x(\cos(\theta_i))\cos(j\theta_i) \quad j = 0, 1, \cdots, N \tag{3.15}$$

where $\theta_i = \frac{2i-1}{2K}\pi, i = 1, 2, \cdots, K,$ and $K > N$

The derivative of $x(\tau)$ with respect to τ is given by

$$\dot{x}(\tau) = \frac{b_0}{2} + \sum_{i=1}^{N-1} b_i T_i(\tau) \tag{3.16}$$

where

$$\begin{aligned} b_{N-1} &= 2Na_N \\ b_{N-2} &= 2(N-1)a_{N-1} \\ b_{r-1} &= b_{r+1} + 2ra_r \qquad r = 1, 2, \cdots, N-2 \end{aligned} \tag{3.17}$$

3.3.2 State Parameterization

The state parameterization has several advantages over the other parameterization methods. But so far its use was restricted to special problems. In this section, different aspects of state parameterizations are discussed.

The idea of the state parameterization, using the Chebyshev polynomials of the first type, is to approximate the state variables by a finite length Chebyshev series

$$x_j(\tau) = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{N} a_i^{(j)} T_i(\tau) \qquad j = 1, 2, \cdots, n \tag{3.18}$$

where $T_i(\tau)$ is the i-th order Chebyshev polynomial of the first type and a_i's are the unknown parameters. The control variables are determined from the system state equations as a function of the unknown parameters of the state variables. Therefore, all the system state equations, in most cases, are satisfied directly. By substituting these approximations of the state variables and the control variables into the performance index, it can be converted into a quadratic function of the unknown parameters a_i. The initial conditions are replaced by equality constraints.

In applying the state parameterization, we distinguish two cases:

1. The number of the state variables is equal to the number of control variables i.e. $n = m$.

   If the numbers of the state variables and the control variables are equal, then each state variable will be approximated by a finite length Chebyshev series

$$x_j(\tau) = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{N} a_i^{(j)} T_i(\tau) \quad j = 1, 2, \cdots, n \tag{3.19}$$

   and the control vector can be obtained as a function of these state variables as follows, assuming that the matrix B is nonsingular,

$$u(\tau) = B^{-1}[\frac{2}{t_f}\frac{dx}{d\tau} - Ax(\tau)] \tag{3.20}$$

   which can be expressed in series form as

$$u_l(\tau) = \frac{b_0^{(l)}}{2} + \sum_{i=1}^{N} b_i^{(l)} T_i(\tau) \quad l = 1, 2, \cdots, m = n \tag{3.21}$$

   where $b_0^{(l)}$, $b_1^{(l)}$, $b_2^{(l)}$, $\cdots$, $b_N^{(l)}$ are expressed in terms of $a_0^{(j)}$, $a_1^{(j)}$, $a_2^{(j)}$, $\cdots$, $a_N^{(j)}$.

2. The number of the control variables is less than the number of the state variables $m < n$.

   If the number of the control variables is less than the number of the state variables, then there is no need to approximate all the state variables. This is because if all the state variables are approximated then many of the state equations are replaced by a large number of equality constraints. Therefore, in this case, we choose and directly approximate a set of the state variables which will enable us to find the remaining state variables and the control variables as a function of this set. Assume that this set is $x_1, x_2, \cdots, x_q$ and $q < n$, then this set can be approximated by

$$x_j(\tau) = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{N} a_i^{(j)} T_i(\tau) \quad j = 1, 2, \cdots, q \tag{3.22}$$

and the remaining $n-q$ state variables and the control variables are obtained from the system equations

$$x_j(\tau) = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{N} a_i^{(j)} T_i(\tau) \quad j = q+1, q+2, \cdots, n \tag{3.23}$$

$$u_l(\tau) = \frac{b_0^{(l)}}{2} + \sum_{i=1}^{N} b_i^{(l)} T_i(\tau) \quad l = 1, 2, \cdots, m \tag{3.24}$$

where $a_0^{(j)}, a_1^{(j)}, \cdots, a_N^{(j)}$, $j = q+1, \cdots, n$ and $b_0^{(l)}, b_1^{(l)}, \cdots, b_N^{(l)}$, $l = 1, 2, \cdots, m$ are functions of the parameters $a_0^{(j)}, a_1^{(j)}, \cdots, a_N^{(j)}$, $j = 1, 2, \cdots, q$. The advantage of not approximating all state variables is that the optimal control problem is reduced to a quadratic programming problem with fewer unknown parameters.

For the special case of a single input single output systems expressed in controllability canonical form, we need to approximate only one state variable and all other state variables and the control variable can be found as a function of this state variable. This special case is the main interest of previous works [43, 49, 77]. Also [79] proposed, if the number of control variables is less than the number of state variables, to add $n-m$ new artificial control variables to the system. This technique has two disadvantages: (1) There are a large number of unknown parameters, (2) The original problem is changed.

Remarks:

In some cases, we may face the situation that all the state variables and the control variables are approximated but not all the state equations are satisfied. In this case, the unsatisfied state equations will be converted into equality constraints.

3.4 Which State Variables to Parameterize?

It is clear from the previous section that if the number of the state variables is larger than the number of the control variables, then the set of state variables which can be selected and approximated is not unique. We can choose different sets, each of them can give us the remaining state variables and the control variables as in the following example

$$\dot{x}_1 = x_2 \tag{3.25}$$

$$\dot{x}_2 = x_1 + x_2 + u \tag{3.26}$$

For this simple example, we have two possibilities: The first possibility is to approximate x_1 by a finite length Chebyshev series and x_2 can be found from the first state equation by differentiating x_1, while u can be found from the second state equation as a function of both x_1, x_2. The second possibility is to approximate x_2 and to find x_1 from the first equation by integrating x_2, while u can be found from the second state equation.

To limit the number of the state variables that can be selected and directly approximated, we propose to select the set of the state variables that enables us to express the remaining state variables and the control variables, as a function of this set, by differentiation (i.e. same as first possibility of the previous example) rather than by integration. There are three reasons to justify this proposal: (1) The first reason is that the length of the series will increase at each time we perform the integration, (2) the second reason is that by differentiation we get more accurate results, because in differentiation the unknown parameters are multiplied by an integer and therefore there is no truncation error. However, in integration, the unknown parameters are divided by an integer and hence there is a truncation error, (3) the third reason is that the integration may lead to a very complicated approximation e.g $\dot{x}_1 = x_1 + x_2$. If x_2 is approximated directly, then x_1 will have a complicated form. However, if x_1 is approximated directly, then x_2 will have simple form.

These ideas are clarified by the following example,

Example 1: [40] Minimize

$$J = \int_0^1 (x_1^2 + x_2^2 + 0.005u^2)dt \tag{3.27}$$

subject to

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0 \tag{3.28}$$

$$\dot{x}_2 = -x_2 + u \qquad x_2(0) = -1 \tag{3.29}$$

The exact optimal value of this problem is $J = 0.06936094$. After Changing the time interval into $\tau \in [-1, 1]$, this problem is solved by two approaches:

- The first approach is to approximate x_1 by ninth order Chebyshev series and to calculate x_2 from the first state equation by differentiation. Then x_2 will be of 8th order Chebyshev series. The control variable u can be found from the second state equation. In this case, we obtain the optimal value $\hat{J} = 0.0693689$ and the initial conditions are satisfied exactly.

- The second approach is to approximate x_2 by 9th order Chebyshev series and x_1 can be calculated from the first state equation by integration. Then x_1 will be of 10th order Chebyshev series. The control variable is obtained from the second state equation. In this case, we obtain $\hat{J} = 0.0660667$ and the initial conditions are $x_1(0) = 0.0586702$ and $x_2(0) = -1$. This indicates that $x_1(0)$ is not satisfied accurately.

From this example, it is clear that the first approach gives more accurate results as expected.

3.5 Problem Reformulation

Because the Chebyshev polynomials of the first type are defined on the interval $\tau \in [-1,1]$, the time interval $t \in [0, t_f]$ of the optimal control problem is transformed into the interval $\tau \in [-1,1]$ using the transformation

$$\tau = \frac{2t}{t_f} - 1 \tag{3.30}$$

This transforms the optimal control problem (3.1)-(3.3) into: Find the optimal control $u^*(\tau)$ that minimizes the quadratic performance index

$$J = \frac{t_f}{2}\int_{-1}^{1}(x^T Q x + u^T R u)d\tau \tag{3.31}$$

subject to the state equations

$$\frac{dx}{d\tau} = \frac{t_f}{2}(Ax(\tau) + Bu(\tau)) \tag{3.32}$$

$$x(-1) = x_0 \tag{3.33}$$

To formulate this problem into quadratic programming problem, our method is based on parameterizing the system state variables using Chebyshev polynomials of the first type. From (3.19)-(3.24) of section 3.3.2, the state variables and the control variables can be approximated by

$$x_k(\tau) = \frac{a_0^{(k)}}{2} + \sum_{i=1}^{N} a_i^{(k)} T_i \quad k = 1, 2, \cdots, n \tag{3.34}$$

$$u_l(\tau) = \frac{b_0^{(l)}}{2} + \sum_{i=1}^{N} b_i^{(l)} T_i \quad l = 1, 2, \cdots, m \tag{3.35}$$

where the unknown parameters are $a_0^{(k)}, a_1^{(k)}, \cdots, a_N^{(k)}$, $k = 1, 2, \cdots, q$. The parameters of the remaining state variables and the control variables are function of these unknown parameters.

Equations (3.34) and (3.35) can be written in a matrix form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{a_0^{(1)}}{2} & a_1^{(1)} & \cdots & a_{N-1}^{(1)} & a_N^{(1)} \\ \frac{a_0^{(2)}}{2} & a_1^{(2)} & \cdots & a_{N-1}^{(2)} & a_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{a_0^{(n)}}{2} & a_1^{(n)} & \cdots & a_{N-1}^{(n)} & a_N^{(n)} \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ T_N \end{bmatrix} \tag{3.36}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \frac{b_0^{(1)}}{2} & b_1^{(1)} & \cdots & b_{N-1}^{(1)} & b_N^{(1)} \\ \frac{b_0^{(2)}}{2} & b_1^{(2)} & \cdots & b_{N-1}^{(2)} & b_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{b_0^{(m)}}{2} & b_1^{(m)} & \cdots & b_{N-1}^{(m)} & b_N^{(m)} \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ T_N \end{bmatrix} \tag{3.37}$$

or in compact form

$$x = \alpha T \qquad u = \beta T \tag{3.38}$$

By this approximation, in most of the cases, the system state equations are satisfied directly and replaced by the approximated state variables and control variables. If there are state equations which are still unsatisfied, they will be treated as equality constraints. An example of this case is shown in section 4.5 of the next chapter.

Using the Chebyshev polynomials properties at $\tau = -1$, the initial states can also be approximated as follows

$$\frac{a_0^{(k)}}{2} - a_1^{(k)} + a_2^{(k)} - a_3^{(k)} + \cdots + (-1)^N a_N^{(k)} - x_k(-1) = 0 \quad k = 1, 2, \cdots, n \tag{3.39}$$

These equations will be treated as equality constraints.

The last part of the optimal control problem which also has to be approximated is the performance index. By substituting (3.38) into (3.31) we get,

$$\hat{J} = \frac{t_f}{2} \int_{-1}^{1} (T^T \alpha^T Q \alpha T + T^T \beta^T R \beta T) d\tau \tag{3.40}$$

where $\hat{J}$ is the approximate value of J. Let $\alpha^T Q \alpha = M$ and $\beta^T R \beta = P$, and notice that M and P are symmetric matrices. The first part of $\hat{J}$, namely $T^T M T$ can be written as

$$T^T M T = \sum_{i=1}^{N+1} T_{i-1} \sum_{j=1}^{N+1} m_{ij} T_{j-1} = \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} T_{i-1} m_{ij} T_{j-1} \tag{3.41}$$

which can be expanded into

$$\begin{aligned} T^T M T = m_{11} T_0 T_0 + 2m_{12} T_0 T_1 + 2m_{13} T_0 T_2 + \cdots + 2m_{1,\ N+1} T_0 T_N \\ + m_{22} T_1 T_1 + 2m_{23} T_1 T_2 + \cdots + 2m_{2,\ N+1} T_1 T_N \\ + m_{33} T_2 T_2 + \cdots + 2m_{3,\ N+1} T_2 T_N \\ \ddots \qquad \vdots \\ + m_{N+1,\ N+1} T_N T_N \end{aligned} \tag{3.42}$$

The integration of all terms of (3.42) which contains $T_i T_j$, such that $i + j$ is odd, is zero. By considering the remaining terms, equation (3.42) reduces to

$$\begin{aligned} T^T M T = m_{11} T_0 T_0 + 0 + 2m_{13} T_0 T_2 + 0 + 2m_{15} T_0 T_4 + \cdots \\ + m_{22} T_1 T_1 + 0 + 2m_{24} T_1 T_3 + \cdots \\ + m_{33} T_2 T_2 + 0 + 2m_{35} T_2 T_4 + \cdots \\ \ddots \qquad \vdots \\ + m_{N+1,\ N+1} T_N T_N \end{aligned} \tag{3.43}$$

The integration of these terms can be obtained using the following result,

Theorem 1 *The integration of the term (3.43) can be given by*

$$\int_{-1}^{1} T^T MT d\tau = \frac{1}{2}\Big\{-\frac{2}{k^2-1}\sum_{i=1}^{N+1-k} 2\acute{m}_{i,i+k} - 2\sum_{i=1}^{N+1-k}\frac{1}{(k-2+2i)^2-1}2\acute{m}_{i,i+k}\Big\} \quad (3.44)$$

where $k = 0, 2, 4, \cdots, N$ (N even) or N-1 (N odd), and

$$\acute{m}_{i,i+k} = \begin{cases} m_{i,i+k} & k \neq 0 \\ \frac{m_{ii}}{2} & k = 0 \end{cases} \quad (3.45)$$

Proof: (3.43) can be written as,

$$T^T MT = \sum_{i=1} T_{i-1}m_{ii}T_{i-1} + 2\sum_{i=1}\sum_{j=1} T_{i-1}m_{i,i+2j}T_{i-1+2j} \quad (3.46)$$

$$= 2\sum_{j=0}\sum_{i=1} T_{i-1}\acute{m}_{i,i+2j}T_{i-1+2j} \quad (3.47)$$

where $\acute{m}_{i,i+2j}$ is as defined in the theorem.

To decide the upper limits of the summation, it is clear that the largest possible value for $i-1+2j$ is N, hence, the upper limit of i is $N+1-2j$. On the other hand, $2j$ can not be greater than N, therefore, the upper limit for $2j$ is N if N is an even number, or $N-1$ if N is an odd one. Substituting these limits into equation (3.47) and by using the product property of Chebyshev polynomials, we get

$$\begin{cases} 2\sum_{j=0}^{N/2}\sum_{i=1}^{N+1-2j}\frac{1}{2}\acute{m}_{i,i+2j}(T_{2i-2+2j}+T_{2j}) & N = \text{even} \\ 2\sum_{j=0}^{(N-1)/2}\sum_{i=1}^{N+1-2j}\frac{1}{2}\acute{m}_{i,i+2j}(T_{2i-2+2j}+T_{2j}) & N = \text{odd} \end{cases} \quad (3.48)$$

which can be integrated, using the integration property of Chebyshev polynomials, to give

$$\begin{cases} 2\sum_{j=0}^{N/2}\sum_{i=1}^{N+1-2j}\frac{1}{2}\acute{m}_{i,i+2j}(\frac{-2}{(2i-2+2j)^2-1}+\frac{-2}{(2j)^2-1}) & N = \text{even} \\ 2\sum_{j=0}^{(N-1)/2}\sum_{i=1}^{N+1-2j}\frac{1}{2}\acute{m}_{i,i+2j}(\frac{-2}{(2i-2+2j)^2-1}+\frac{-2}{(2j)^2-1}) & N = \text{odd} \end{cases} \quad (3.49)$$

letting $2j = k$, the previous equation can be written as

$$2\sum_{i=1}^{N+1-k}\frac{1}{2}\acute{m}_{i,i+k}(\frac{-2}{(2i-2+k)^2-1}+\frac{-2}{k^2-1}) \quad (3.50)$$

where k=0, 2, 4, $\cdots, N$ (N even) or $N-1$ (N odd). Equation (3.50) is the required result. □

Following the same procedure, the integration of the second part of the performance index (3.40) can be computed

$$\int_{-1}^{1} T^T PT d\tau = 2\sum_{i=1}^{N+1-k}\frac{1}{2}\acute{p}_{i,i+k}(\frac{-2}{(2i-2+k)^2-1}+\frac{-2}{k^2-1}) \quad (3.51)$$

where

$$\acute{p}_{i,i+k} = \begin{cases} p_{i,i+k} & k \neq 0 \\ \frac{p_{ii}}{2} & k = 0 \end{cases} \tag{3.52}$$

and $k = 0, 2, 4, \cdots$ N (N even) or N-1 (N odd).

The performance index (3.40) can be rewritten as follows

$$\hat{J} = t_f \sum_{i=1}^{N+1-k} \frac{1}{2}\Big(\acute{p}_{i,i+k} + \acute{m}_{i,i+k}\Big)\big(\frac{-2}{(2i-2+k)^2-1} + \frac{-2}{k^2-1}\big) \tag{3.53}$$

which can be expressed as,

$$\hat{J} = \frac{1}{2}\mathbf{a}^T H \mathbf{a} \tag{3.54}$$

because the entries of the matrices M and P are quadratic functions of the unknown parameters $\mathbf{a}$, where $\mathbf{a}^T = [a_0^{(1)}\ a_1^{(1)}\ \cdots\ a_N^{(1)}\ a_0^{(2)}\ \cdots\ a_N^{(2)}\ \cdots\ a_0^{(q)}\ \cdots\ a_N^{(q)}]$. The matrix H can be obtained by finding the Hessian of $\hat{J}$,

$$H = \frac{\partial^2 \hat{J}}{\partial a_i^{(k)} \partial a_j^{(k)}} \tag{3.55}$$

where $i, j = 0, 1, \cdots, N$, and $k = 1, 2, \cdots, q$.

For the special case, $n = m$, the matrix H can be obtained explicitly. In this case $\mathbf{a}^T = [a_0^{(1)}\ a_1^{(1)}\ \cdots\ a_N^{(1)}\ a_0^{(2)}\ \cdots\ a_N^{(2)}\ \cdots\ a_0^{(n)}\ \cdots\ a_N^{(n)}]$. The state variables x can be expressed as

$$x = (I_n \otimes T^T)\mathbf{a} \tag{3.56}$$

where $\otimes$ denotes the Kronecker product, I_n denotes $n \times n$ identity matrix and $T^T = [T_0\ T_1 \cdots\ T_N]$.

Using Chebyshev polynomials' differentiation operational matrix D (see chapter 7 for details), the control variables u can be obtained

$$u = B^{-1}\Big(\frac{2}{t_f}(I_n \otimes T^T D^T)\mathbf{a} - A(I_n \otimes T^T)\mathbf{a}\Big) \tag{3.57}$$

By substituting (3.56) and (3.57) into (3.31), we get

$$\begin{aligned}\hat{J} &= \frac{t_f}{2}\int_{-1}^{1}\Big[\mathbf{a}^T(I_n \otimes T)Q(I_n \otimes T^T)\mathbf{a} \\ &+\mathbf{a}^T\Big(\frac{2}{t_f}(I_n \otimes DT) - (I_n \otimes T)A^T\Big)F\Big(\frac{2}{t_f}(I_n \otimes T^T D^T) - A(I_n \otimes T^T)\Big)\mathbf{a}\Big]d\tau\end{aligned} \tag{3.58}$$

where $F = (B^{-1})^T R B^{-1}$. Equation (3.58) can be simplified further

$$\begin{aligned}\hat{J} &= \frac{1}{2}\mathbf{a}^T\Big[t_f\int_{-1}^{1}\Big((Q \otimes TT^T) + \frac{4}{t_f^2}(F \otimes DTT^T D^T) - \frac{2}{t_f}(FA \otimes DTT^T) \\ &-\frac{2}{t_f}(A^T F \otimes TT^T D^T) + (A^T FA \otimes TT^T)\Big)d\tau\Big]\mathbf{a}\end{aligned} \tag{3.59}$$

Therefore, the matrix H is given by

$$\begin{aligned} H &= t_f \int_{-1}^{1} \Big((Q \otimes TT^T) + \frac{4}{t_f^2}(F \otimes DTT^T D^T) - \frac{2}{t_f}(FA \otimes DTT^T) \\ &\quad - \frac{2}{t_f}(A^T F \otimes TT^T D^T) + (A^T FA \otimes TT^T) \Big) d\tau \end{aligned} \tag{3.60}$$

The optimal control problem (3.1)-(3.3) is converted into parameters optimization problem which is quadratic in the unknown parameters and the new problem can be stated as,

$$\min_{\mathbf{a}} \frac{1}{2} \mathbf{a}^T H \mathbf{a} \tag{3.61}$$

subject to the linear constrains

$$F\mathbf{a} - b = 0 \tag{3.62}$$

where the linear constraints are due to the initial and final conditions, and in some cases may also represent some of the system equations which are not satisfied yet.

The optimal value of the vector $\mathbf{a}^*$ can be obtained from the standard quadratic programming method [34], given that H, which is the Hessian of $\hat{J}$, is a positive definite matrix.

$$\mathbf{a}^* = H^{-1} F^T (F H^{-1} F^T)^{-1} b \tag{3.63}$$

Lemma 1

The matrix H is a positive definite matrix.

Proof: Previously, we wrote $x = \alpha T$, hence x can be written in another way,

$$x = \mathcal{T}\mathbf{a} \tag{3.64}$$

where $\mathbf{a}$ is a $\acute{k} \times 1$ vector, ($\acute{k} = (N+1)q$ is the total number of unknown parameters used in the approximation of all the state variables) and $\mathcal{T}$ is a $n \times \acute{k}$ matrix of Chebyshev polynomials. These Chebyshev polynomials are the coefficients of the unknown parameters in the state variables approximation (3.34). The matrix $\mathcal{T}$ can have two forms: The first form is obtained if all state variables are directly approximated, while the second form is obtained if some of the state variables are directly approximated. In both cases, the rows of $\mathcal{T}$ are linearly independent, and hence its rank is n for all $\tau \in [-1, 1]$.

Similarly, writing u as

$$u = \mathcal{L}\mathbf{a} \tag{3.65}$$

where $\mathcal{L}$ is $m \times \acute{k}$ matrix of Chebyshev polynomials which are obtained as function of state variables such that the system differential equations are satisfied. Therefore, the rank of the matrix $\mathcal{L}$ is m because all of its rows are linearly independent. Hence $\mathcal{T}^T Q \mathcal{T} + \mathcal{L}^T R \mathcal{L}$ will be a positive definite, and

$$H = \int_{-1}^{1} (\mathcal{T}^T Q \mathcal{T} + \mathcal{L}^T R \mathcal{L}) d\tau \tag{3.66}$$

will also be a positive definite. □

The algorithm to solve the optimal control problem can be summarized as follows:

(1) Approximate the state variables by Chebyshev series, after changing the time interval to $\tau \in [-1, 1]$. Usually we do not need to approximate all the state variables.

(2) Find the control variables and the state variables, which are not directly approximated, as a function of the approximated state variables.

(3) Calculate the matrix M from $\alpha^T Q \alpha$, and the matrix P from $\beta^T R \beta$.

(4) Find an expression of $\hat{J}$ from equations (3.50) and (3.51).

(5) Determine the set of equality constraints, due to the initial and final conditions and due to system differential equations which are not yet satisfied, if any.

(6) Find the matrix H, by calculating the Hessian of $\hat{J}$.

(7) Find the optimal parameters from equation (3.63), and substitute these parameters into equations(3.34) and (3.35) to find the approximate optimal trajectories and the approximate optimal control.

3.6 Computational Results

Find $u^*(t)$ that minimizes

$$\int_{0}^{1} (x_1^2 + x_2^2 + 0.005u^2) dt \tag{3.67}$$

subject to

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0 \tag{3.68}$$

$$\dot{x}_2 = -x_2 + u \qquad x_2(0) = -1 \tag{3.69}$$

The first step in solving this problem by the proposed method is to transform the time interval to $\tau \in [-1, 1]$. This will lead to the following problem

$$minimize \quad \frac{1}{2}\int_{-1}^{1}(x_1^2 + x_2^2 + 0.005u^2)d\tau \tag{3.70}$$

subject to

$$\frac{dx_1}{d\tau} = \frac{1}{2}x_2 \qquad x_1(-1) = 0 \tag{3.71}$$

$$\frac{dx_2}{d\tau} = \frac{1}{2}(-x_2 + u) \quad x_2(-1) = -1 \tag{3.72}$$

Then by approximating $x_1(\tau)$ by 5th order Chebyshev series of unknown parameters, we get

$$x_1(\tau) = \frac{a_0^{(1)}}{2} + \sum_{i=1}^{5} a_i^{(1)} T_i(\tau) \tag{3.73}$$

Using the Chebyshev polynomials differentiation property, $\dot{x}_1(\tau)$ is calculated and by substituting $\dot{x}_1(\tau)$ into equation (3.71), $x_2(\tau)$ can be determined,

$$x_2(\tau) = (2a_1^{(1)}+6a_3^{(1)}+10a_5^{(1)})T_0+(8a_2^{(1)}+16a_4^{(1)})T_1+(12a_3^{(1)}+20a_5^{(1)})T_2+16a_4^{(1)}T_3+20a_5^{(1)}T_4 \tag{3.74}$$

$$= \frac{a_0^{(2)}}{2} + \sum_{i=1}^{4} a_i^{(2)} T_i(\tau) \tag{3.75}$$

and by substituting $x_2(\tau)$ and $\dot{x}_2(\tau)$ into (3.72), the control $u(\tau)$ can also be found,

$$u(\tau) = 2\dot{x}_2 + x_2 \tag{3.76}$$

$$\begin{aligned} &= (2a_1^{(1)} + 16a_2^{(1)} + 6a_3^{(1)} + 128a_4^{(1)} + 10a_5^{(1)})T_0 + (8a_2^{(1)} + 96a_3^{(1)} + 16a_4^{(1)} \\ &\quad +480a_5^{(1)})T_1 + (12a_3^{(1)} + 192a_4^{(1)} + 20a_5^{(1)})T_2 + (16a_4^{(1)} + 320a_5^{(1)})T_3 \\ &\quad +20a_5^{(1)}T_4 \end{aligned} \tag{3.77}$$

$$= \frac{b_0^{(1)}}{2} + \sum_{i=1}^{4} b_i^{(1)} T_i(\tau) \tag{3.78}$$

From these approximations of $x_1(\tau)$, $x_2(\tau)$ and $u(\tau)$, the system state equations (3.71) and (3.72) are satisfied directly. This is a clear advantage of using the state parameterizations.

By substituting (3.73), (3.74) and (3.77) into (3.70), and then using the result (3.53), the following expression of $\hat{J}$ can be obtained. In this expression, for simplification, we write $a_i^{(1)}$ as a_i, $\quad i = 0, 1, \cdots, 5$.

$$\begin{aligned} \hat{J} = \; & 0.25a_0^2 + 4.3533a_1^2 - 0.3333a_0a_2 + 0.32a_1a_2 + 23.1867a_2^2 + 7.64a_1a_3 \\ & +2.88a_2a_3 + 71.3217a_3^2 - 0.0667a_0a_4 + 1.28a_1a_4 + 44.1821a_2a_4 \\ & +11.52a_3a_4 + 194.321a_4^2 + 7.9448a_1a_5 + 8.a_2a_5 + 173.359a_3a_5 \\ & +32.a_4a_5 + 505.604a_5^2 \end{aligned} \tag{3.79}$$

From equations (3.73) and (3.74), another two equations representing the initial states are obtained

$$\begin{aligned} \frac{a_0^{(1)}}{2} - a_1^{(1)} + a_2^{(1)} - a_3^{(1)} + a_4^{(1)} - a_5^{(1)} &= 0 \qquad (3.80) \\ 2a_1^{(1)} - 8a_2^{(1)} + 18a_3^{(1)} - 32a_4^{(1)} + 50a_5^{(1)} + 1 &= 0 \qquad (3.81) \end{aligned}$$

These two equations are considered as equality constraints.

The dynamic optimal control problem is approximated by a quadratic programming problem. The new problem is to minimize (3.79) subject to the equality constraints (3.80) and (3.81). The optimal parameters can be obtained using (3.63). And by substituting these optimal parameters into (3.79), the approximate optimal value can be calculated. For this particular case, the optimal value is found to be 0.0759522. The optimal parameters, the optimal value and the execution time, are summarized in Table 3.1. In this table, T_{total} refers to the total execution time (including the time needed to reformulate the optimal control problem into quadratic programming) needed to solve the problem on SUN-SPARC 4/5 workstation. The time T_Q is the time needed to solve the quadratic programming problem.

The previous problem is also solved by expanding $x_1(\tau)$ into 9th order Chebyshev series, and the optimal value is found to be 0.0693689 which is very close to both the exact value 0.06936094 and the result obtained in [40] which is 0.069368 using 9th order Chebyshev series. The method of [40], which is based on control-state parameterization using Chebyshev polynomials, requires the solution of quadratic programming problem of 30 unknown parameters and subject to 22 equality constraints. However, our method requires the solution of quadratic programming problem of 10 unknown parameters and subject to 2 equality constraints. The optimal parameters and optimal value of this case are shown also in Table 3.1.

Note that the Chebyshev coefficients decrease rapidly as N increases. This is one of the very important advantages of the use of Chebyshev series approximation.

The state trajectories and the approximate optimal control of this example, using 5th order and 9th order Chebyshev series, are shown in Figures 3.1 and 3.2.

This example was solved by Hsieh [51] using a modified steepest method and by Neuman and Sen [52] using collocation and approximation by cubic splines, also Vlassenbroack [40] solved this example using control-state parameterization via Chebyshev polynomials. These results, along with our results, are shown in Table 3.2. This table is taken from Vlassenbroeck [40] and is completed by our results.

N=5				N=9			
i	$a_i^{(1)}$	$a_i^{(2)}$	$b_i^{(1)}$	i	$a_i^{(1)}$	$a_i^{(2)}$	$b_i^{(1)}$
0	-0.0513997	-0.146602	1.54476	0	-0.0466305	-0.134355	2.08661
1	-0.00812707	0.294704	-3.6203	1	-0.00809808	0.272406	-3.92827
2	0.0138808	-0.260695	1.9432	2	0.0134297	-0.236317	3.11599
3	-0.0121963	0.183658	-1.64579	3	-0.0114624	0.164968	-2.14518
4	0.0114786	-0.11434	-0.11434	4	0.00699568	-0.098768	1.27391
5	-0.00571701	0	0	5	-0.00369844	0.0530374	-0.67682
				6	0.0017457	-0.0247992	0.287136
				7	-0.000735377	0.0111405	-0.123536
				8	0.000348142	-0.00420865	-0.00420865
				9	-0.000116907	0	0
$\hat{J} = 0.0759522$				$\hat{J} = 0.0693689$			
T_{total}= 0.68333 sec.				T_{total}= 2.53333 sec.			
$T_Q = 0.0166667$ sec.				$T_Q = 0.025$ sec.			

Table 3.1: The Chebyshev parameters of order N=5 and 9

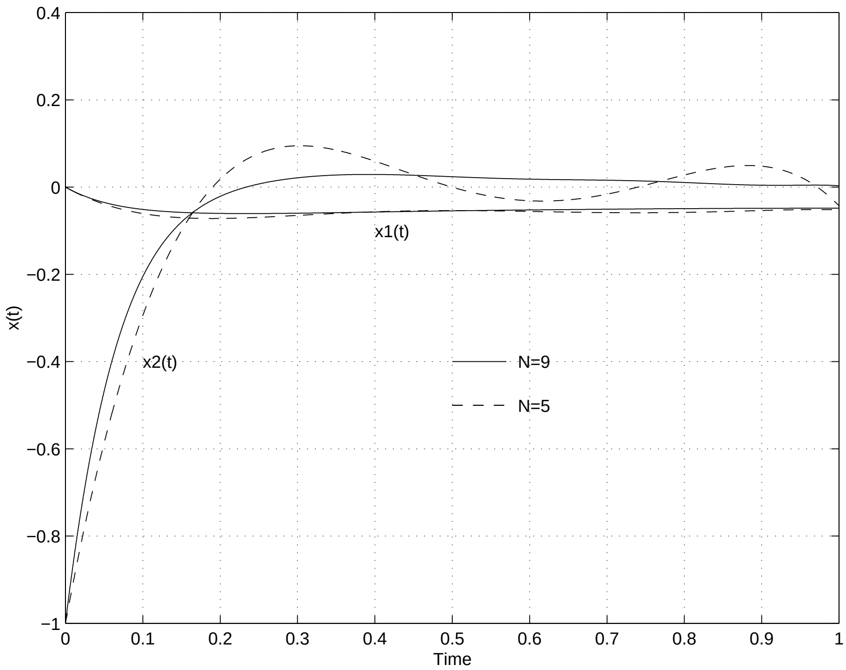

Figure 3.1: State trajectories $x_1(t)$ and $x_2(t)$

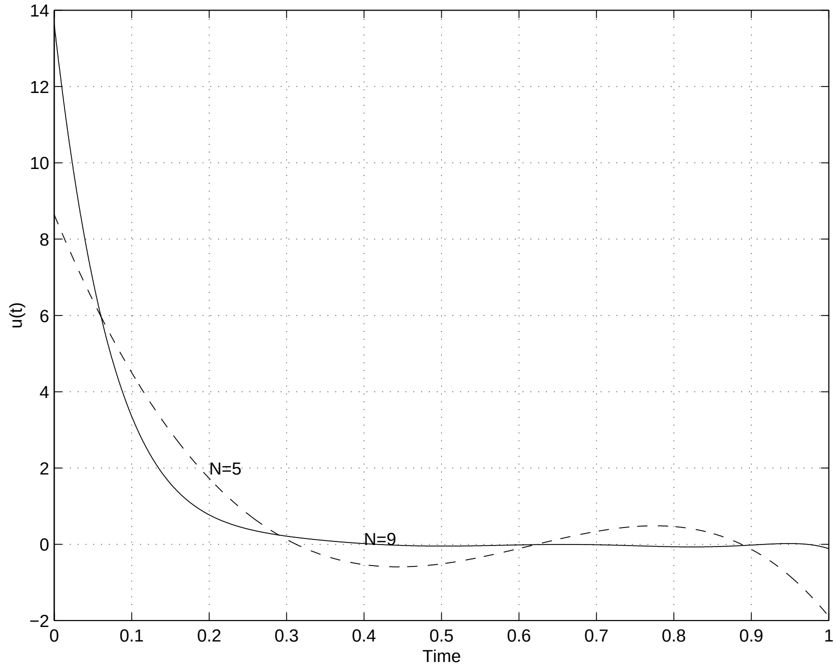

Figure 3.2: Approximated optimal control u(t)

Source	J	Deviation error
Exact value	0.06936094	0
Hsieh [51]	0.0702	8.4×10^{-4}
Neuman and Sen [52]		
N=4	0.0703	9.4×10^{-4}
N=9	0.06989	5.3×10^{-4}
Vlassenbroeck [40]		
N=5	0.07595	6.6×10^{-3}
N=9	0.069368	7.1×10^{-6}
⋮	⋮	⋮
This research		
N=5	0.07595646	6.59×10^{-3}
N=9	0.0693689	7.96×10^{-6}

Table 3.2: Minimum values of J of the example

3.7 Conclusion

In this chapter, we have proposed an effective numerical method to solve linear quadratic optimal control problems. The method is based on parameterizing the system states by Chebyshev series of finite length. Also, we have derived an explicit formula to approximate the performance index. In addition, in this chapter, we have discussed the state parameterization and have showed the most appropriate way to apply this technique.

The main advantages of the proposed numerical method are: The difficult optimal control problem is converted into quadratic programming problem, with a few linear constraints, which can be solved using the standard quadratic programming results. Another important advantage of this approach is that the number of unknown parameters is kept as small as possible.

Many ideas and results of this chapter will be used and extended in the following chapters.

Chapter 4

Nonlinear Optimal Control Problem

4.1 Introduction

In this chapter, we extend the method described in the previous chapter to solve nonlinear optimal control problems, and as a special case the time-varying linear optimal control problems.

One of the methods to solve the unconstrained nonlinear optimal control problem is to convert it into a nonlinear programming problem by using the direct methods. For example, Sirisena [37] proposed a method based on parameterizing the control variables using piecewise polynomials, Frich and Stech [41] proposed to use the Walsh functions to parameterize the state variables and the control variables, also Vlassenbroeck and Van Doreen [39] used the control-state parameterization via Chebyshev polynomials to convert the optimal control problem into a nonlinear mathematical programming problem. Some other methods can be found in [32, 42, 75, 76]. The nonlinear mathematical programming problem, in its turn, can be solved using different methods, in particular the sequential quadratic programming method [78], which replaces the nonlinear mathematical programming problem by a sequence of quadratic programming problems.

In this thesis, the nonlinear optimal control problem is converted directly into a sequence of quadratic programming problems, without converting it into nonlinear programming problem. This approximation can be achieved by employing the second method of the quasilinearization [45], in which the performance index is expanded up to the second order and the state equations are expanded up to the first order around a nominal trajectories and controls. The use of the second method of quasilinearization facilitate the application of the state parameterization technique on nonlinear systems.

Using the quasilinearization, the nonlinear optimal control problem is replaced by a sequence of time-varying linear quadratic optimal control problems and then each of these problems is converted into a quadratic programming problem by using the state parameterization via Chebyshev polynomials. Since the obtained quadratic programming problem is subject to equality constraints only, it can be solved in one iteration by matrix-vector multiplication.

At the end of this chapter, we present the simulation results of practical problem, optimal flight control of F-8 fighter aircraft starting from a large initial conditions.

4.2 Problem Statement

Consider the nonlinear optimal control problem which can be stated as: Find an optimal control $u^*(t)$ on the $0 \leq t \leq t_f$ which minimizes the performance index

$$J = \int_0^{t_f} (x^T Q x + u^T R u) dt \tag{4.1}$$

subject to the system state equations

$$\dot{x}(t) = f(x(t), u(t), t) \quad x(0) = x_0 \tag{4.2}$$

where $x \in R^n$, $u \in R^m$, Q is $n \times n$ positive semidefinite matrix and R is $m \times m$ positive definite matrix; f is continuously differentiable with respect to all its arguments.

As shown in chapter 2, the solution of this problem, by applying the necessary conditions, leads to nonlinear two-point boundary value problem, while applying the sufficient conditions leads to the HJB partial differential equation. In this chapter, we solve this problem without using neither the necessary conditions nor the sufficient conditions.

The idea of the solution is to use the second method of the quasilinearization to replace the nonlinear optimal control problem by a sequence of linear quadratic optimal control problems. And then to use the state parameterization via Chebyshev polynomials to convert each of these problems into a quadratic programming problem. Before we start in reformulating the problem, we discuss briefly the quasilinearization method.

4.3 Quasilinearization

The quasilinearization method was developed by Bellman and Kalaba [45] from an origin in the theory of dynamic programming. The quasilinearization can be applied to optimal control problems in two ways:

The first method, which is widely used, is to linearize the nonlinear two-point boundary value problem around nominal trajectories and controls. As a result of this method, the nonlinear two-point boundary value problem is replaced by a sequence of linear two-point boundary value problems.

Assume that $\frac{\partial H}{\partial u} = 0$ has been solved for $u(t)$ and substituted in both the state equations and in the costate equations, then the nonlinear two-point boundary value problem can be written as follows,

$$\begin{aligned} \dot{x} &= f(x, \lambda, t) && (4.3) \\ \dot{\lambda} &= g(x, \lambda, t) && (4.4) \end{aligned}$$

where $x(t_0) = x_0$, $\lambda(t_f) = \lambda_f$, f and g are nonlinear functions of $x(t)$, $\lambda(t)$ and t. To apply the first method of the quasilinearization, the previous two equations are expanded up to the first order around nominal trajectories $x^k(t)$, $\lambda^k(t)$. Then we obtain

$$\begin{aligned} \dot{x}^{k+1} &= A_{11}(t)x^{k+1} + A_{12}(t)\lambda^{k+1} + E_1(t) && (4.5) \\ \dot{\lambda}^{k+1} &= A_{21}(t)x^{k+1} + A_{22}(t)\lambda^{k+1} + E_2(t) && (4.6) \end{aligned}$$

where $x^{k+1}(t_0) = x_0$, $\lambda^{k+1}(t_f) = \lambda_f$ and

$$\begin{aligned} A_{11}(t) &= \frac{\partial f}{\partial x}(x^k, \lambda^k, t) && (4.7) \\ A_{12}(t) &= \frac{\partial f}{\partial \lambda}(x^k, \lambda^k, t) && (4.8) \\ A_{21}(t) &= \frac{\partial g}{\partial x}(x^k, \lambda^k, t) && (4.9) \\ A_{22}(t) &= \frac{\partial g}{\partial \lambda}(x^k, \lambda^k, t) && (4.10) \\ E_1(t) &= f(x^k, \lambda^k, t) - \frac{\partial f}{\partial x}(x^k, \lambda^k, t)x^k(t) - \frac{\partial f}{\partial \lambda}(x^k, \lambda^k, t)\lambda^k(t) && (4.11) \\ E_2(t) &= g(x^k, \lambda^k, t) - \frac{\partial g}{\partial x}(x^k, \lambda^k, t)x^k(t) - \frac{\partial g}{\partial \lambda}(x^k, \lambda^k, t)\lambda^k(t) && (4.12) \end{aligned}$$

Therefore, the nonlinear two-point boundary value problem, (4.3) and (4.4), is replaced by a sequence of linear two-point boundary value problems (4.5) and (4.6).

The second method of the quasilinearization is to expand the performance index up to the second order and to expand the state equations up to the first order around a nominal trajectories $x^k(t), u^k(t)$. Therefore, the nonlinear optimal control problem is replaced by a sequence of time-varying linear quadratic optimal control problems. Consider the following general nonlinear optimal control problem,

$$J = \phi(x(t_f), t_f) + \int_0^{t_f} L(x, u, t)dt \qquad (4.13)$$

and the state equations given by

$$\dot{x} = f(x, u, t) \qquad x(0) = x_0 \qquad (4.14)$$

Applying the second method of the quasilinearization, we get

$$J^{k+1} = J^k + \{\phi_x \delta x + \frac{1}{2}\delta x^T \phi_{xx} \delta x\}_{t=t_f} + \int_0^{t_f} \Big\{ L_x \delta x + L_u \delta u + \frac{1}{2}\delta x^T L_{xx} \delta x + \frac{1}{2}\delta u^T L_{uu} \delta u + \delta x^T L_{xu} \delta u \Big\} dt \tag{4.15}$$

and the linearized state equations

$$\dot{x}^{k+1} = A(t)x^{k+1} + B(t)u^{k+1} + h^k(t) \qquad x^{k+1}(0) = x_0 \tag{4.16}$$

where $\delta x = x^{k+1} - x^k$, $\delta u = u^{k+1} - u^k$, and

$$A(t) = \frac{\partial f}{\partial x}(x^k, u^k, t) \quad n \times n \ \text{ matrix} \tag{4.17}$$

$$B(t) = \frac{\partial f}{\partial u}(x^k, u^k, t) \quad n \times m \ \text{ matrix} \tag{4.18}$$

$$h^k(t) = f(x^k, u^k, t) - A(t)x^k - B(t)u^k \quad n \times 1 \ \text{ vector} \tag{4.19}$$

(4.15)-(4.16) are sequences of time-varying linear quadratic optimal control problems.

It is known [65] that the Legendre-Clebsch condition and the conjugate point condition, which are sufficient conditions for the existence of a solution, are satisfied for second method of the quasilinearization if the following conditions are satisfied

- the function f is continuously differentiable.
- ϕ_{xx} is positive semi-definite.
- L_{uu} is positive definite.
- $L_{xx} - L_{xu}^T L_{uu}^{-1} L_{ux}$ is positive semi-definite.

4.4 Problem Reformulation

To solve the nonlinear optimal control problem (4.1)- (4.2) using the proposed algorithm, the first step is to apply the second method of quasilinearization, by expanding the state equations (4.2) up to the first order around nominal trajectories $x(t)^{(k)}$,$u(t)^{(k)}$, and by expanding the performance index up to the second order around the same nominal trajectories. Then the optimal control problem is reduced to the following sequence of problems:

Minimize

$$J^{k+1} = \int_0^{t_f} \Big\{ (x^{k+1})^T Q x^{k+1} + (u^{k+1})^T R u^{k+1} \Big\} dt \tag{4.20}$$

subject to

$$\dot{x}^{k+1}(t) = A(t)x^{k+1}(t) + B(t)u^{k+1}(t) + h^k(t) \quad x^{k+1}(0) = x_0 \tag{4.21}$$

$A(t), B(t)$ and $h^k(t)$ are as defined previously. Notice that one of the advantages of using quadratic performance index is that by applying the second method of the quasilinearization, the form the performance index remains the same.

The sequence of linear quadratic optimal control problems (4.20)-(4.21) are solved by converting each problem into a quadratic programming problem using the state parameterization via Chebyshev polynomials of the first type. Therefore, the second step is to transform the time interval $t \in [0, t_f]$ to $\tau \in [-1, 1]$. The optimal control problem (4.20)-(4.21) becomes,

$$J^{k+1} = \frac{t_f}{2}\int_{-1}^{1}\Big\{(x^{k+1})^T Q x^{k+1} + (u^{k+1})^T R u^{k+1}\Big\}d\tau \tag{4.22}$$

subject to

$$\dot{x}^{k+1}(\tau) = \frac{t_f}{2}\Big(A(\tau)x^{k+1}(\tau) + B(\tau)u^{k+1}(\tau) + h^k(\tau)\Big) \quad x^{k+1}(-1) = x_0 \tag{4.23}$$

In order to simplify the computations, we express $A(\tau), B(\tau)$ and $h^k(\tau)$ in terms of Chebyshev polynomials. To this end, let $A_{jl}(\tau) = g(\tau, x^k(\tau), u^k(\tau))$ be the (j,l)th element of the matrix $A(\tau)$ and each of the nominal trajectories $x^k(\tau), u^k(\tau)$ are expressed in terms of Chebyshev series of the previous quasilinearization iteration. By using the Chebyshev polynomials properties, the term $A_{jl}(\tau)$ can be expressed as a Chebyshev series of the form [44]

$$A_{jl}(\tau) = \frac{G_0}{2} + \sum_{i=1}^{M} G_i T_i(\tau) \tag{4.24}$$

where

$$G_j = \frac{2}{K}\sum_{i=1}^{K} cos(j\theta_i) g\Big(\cos\theta_i, x^k(\cos\theta_i), u^k(\cos\theta_i)\Big) \tag{4.25}$$

$j = 0, 1, \cdots, M$, and

$$\theta_i = \frac{2i-1}{2K}\pi \quad i = 1, 2, \cdots, K \quad K > M. \tag{4.26}$$

The same approximation can be done for each element of the matrices $A(\tau), B(\tau)$ and $h^k(\tau)$.

The third step is to perform the state parameterization. This step can be performed using the method of the previous chapter. However there are two differences: The first difference is that, in equation (4.23), $A(\tau)$ and $B(\tau)$ are time-varying matrices expressed as functions of Chebyshev polynomials of the previous quasilinearization steps. In this case, there is a need for an algorithm to multiply Chebyshev series. This algorithm is given by the following lemma.

Lemma 1 *Given two Chebyshev series*

$$X = \sum_{i=0}^{n} x_i T_i \tag{4.27}$$

$$Y = \sum_{j=0}^{m} y_j T_j \tag{4.28}$$

The multiplication of these two Chebyshev series is a Chebyshev series of length $n+m$*, given by*

$$\sum_{k=0}^{n+m} z_k T_k \tag{4.29}$$

where

$$z_k = \frac{1}{2}\sum_{i=0}^{n}[x_i y_{k-i} + x_i y_{i-k} + x_i y_{i+k}] \tag{4.30}$$

$$= \frac{1}{2}\sum_{j=0}^{m}[y_j x_{k-j} + y_j x_{j+k} + y_j x_{j-k}] \tag{4.31}$$

Remark 1:

For k=0, the second or the third term of z_k will be replaced by 0 because of the repetition of the same term for $k = 0$.

Proof: The multiplication of (4.27) and (4.28) can be given by,

$$\sum_{k=0}^{n+m} z_k T_k = \sum_{i=0}^{n}\sum_{j=0}^{m} x_i y_j T_i T_j \tag{4.32}$$

and by using Chebyshev polynomials multiplication property, (4.32) can be written as,

$$\sum_{k=0}^{n+m} z_k T_k = \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m} x_i y_j (T_{i+j} + T_{|i-j|}) \tag{4.33}$$

$$= \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{m} x_i y_j (T_{i+j} + T_{i-j} + T_{j-i}) \tag{4.34}$$

Equating the coefficients of both sides of the same Chebyshev polynomial order of equation (4.34), we get

$$z_k T_k = \frac{1}{2}\sum_{i=0}^{n}[x_i y_{k-i} + x_i y_{i-k} + x_i y_{i+k}] T_k \tag{4.35}$$

or

$$z_k T_k = \frac{1}{2}\sum_{j=0}^{m}[y_j x_{k-j} + y_j x_{j+k} + y_j x_{j-k}] T_k \tag{4.36}$$

which is the required result. $\square$

Remark 2:

In (4.34), we redefined the absolute value of $|i-j|$. It should be clear for this equation that T_{-x} is zero (where x is any number), although the Chebyshev polynomial of a negative order is equal to Chebyshev polynomial of positive order i,e $T_{-x} = T_x$. Hence, one of the second and the third terms will be zero for any i and j. Finally, if $i = j$, then we just need one term of the last two terms of equation (4.34).

The second difference is that the approximation of the state variables and the control variables, using Chebyshev series, have a different form and different length than that of linear systems case. This is due to the fact that equation (4.23) is time-varying system and due to the existence of the term $h^k(\tau)$. Therefore, following the state parameterization procedure of the previous chapter, the approximations of the state variables and the control variables can be expressed generally as follows:

$$x_j^{k+1}(\tau) = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{\mathcal{N}} a_i^{(j)} T_i + v_j(\tau) \quad j = 1, 2, \cdots, n \tag{4.37}$$

$$u_l^{k+1}(\tau) = \frac{b_0^{(l)}}{2} + \sum_{i=1}^{\mathcal{N}} b_i^{(l)} T_i + g_l(\tau) \quad l = 1, 2, \cdots, m \tag{4.38}$$

where $\mathcal{N}$ depends on N, (the order of the Chebyshev series of the directly approximated states), and on $A(\tau)$, $B(\tau)$ of equation (4.23). $v_j(\tau)$ and $g_l(\tau)$ are known functions which appear as a result of the presence of $h^k(\tau)$. The unknown parameters are the coefficients of the directly approximated states.

Equations (4.37) and (4.38) can be written in matrices form as

$$x^{k+1}(\tau) = \alpha T + VT \tag{4.39}$$

$$u^{k+1}(\tau) = \beta T + GT \tag{4.40}$$

where α, β are matrices which contain the unknown parameters and V, G are matrices of known elements. T is a vector of Chebyshev polynomials.

The fourth step is to approximate the performance index. By substituting equations (4.39) and (4.40) in the performance index (4.22), yields

$$\hat{J} = \frac{t_f}{2} \int_{-1}^{1} (T^T(\alpha^T + V^T)Q(\alpha + V)T + T^T(\beta^T + G^T)R(\beta + G)T)d\tau \tag{4.41}$$

The integration of this equation can be done using the method described in the previous chapter. However, in this case, two new terms will appear after the integration: a constant term due to the integration of $T^T V^T QVT + T^T G^T RGT$, and a linear term of the unknown parameters due to the integration of $T^T(\alpha^T QV + V^T Q\alpha + \beta^T RG + G^T R\beta)$T.

The fifth step is to approximate the initial conditions. This is the same as in the previous chapter. The initial conditions are replaced by equality constraints.

The optimal control problem (4.22)-(4.23) is transformed into a quadratic programming problem subject to equality constraints. This new problem can be stated as follows,

$$\min_{\mathbf{a}} \frac{1}{2}\mathbf{a}^T H\mathbf{a} + c^T\mathbf{a} + d \tag{4.42}$$

subject to the linear constrains

$$g(\mathbf{a}) = F\mathbf{a} - b = 0 \tag{4.43}$$

where d is a constant which can be obtained from the integration of $T^T V^T QVT + T^T G^T RGT$; c is a $(N+1)q \times 1$ vector and can be determined by finding the Jacobian of $\hat{J}$

$$c = \left.\frac{\partial \hat{J}}{\partial a_i^{(k)}}\right|_{a_i^{(k)}=0} \quad \text{where} \quad i = 0, 1, \cdots, N, \quad \text{and} \quad k = 1, 2, \cdots, q \tag{4.44}$$

($\acute{k}$ is the number of unknown parameters). H is a $(N+1)q \times (N+1)q$ positive definite matrix as proved in the previous chapter and can be determined by calculating the Hessian of $\hat{J}$. The linear constraints are due to the initial and final conditions, and in some cases may appear to represent some of the system equations. The optimal parameters $\mathbf{a}^*$ can be calculated from the quadratic programming results [34],

$$\mathbf{a}^* = -H^{-1}c + H^{-1}F^T(FH^{-1}F^T)^{-1}(FH^{-1}c + b) \tag{4.45}$$

From these optimal parameters $\mathbf{a}^*$ the approximate optimal trajectories can be obtained, by substituting the optimal parameters in equations (4.39) and (4.40). And then the optimal trajectories and controls have to be used to perform another quasilinearization iteration and so on.

To solve the nonlinear optimal control problem (4.1)-(4.2), we need to solve linear quadratic optimal control problems (4.20)-(4.21) successively until some stopping criteria is satisfied. For example when the difference $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$ is sufficiently small. In our computational experiments, the computations are terminated when $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid \leq \epsilon$. For the first example of this chapter ϵ is taken to be 1×10^{-3} and for the second example ϵ is taken to be 1×10^{-4}.

The previous procedures can be summarized as follows:

1. Apply the second method of quasilinearization, starting from a nominal trajectories and controls.

2. Transform the time interval of the optimal control problem into the interval $\tau \in [-1, 1]$. And express $A(\tau), B(\tau)$ and $h^k(\tau)$ in terms of Chebyshev polynomials.

3. Approximate some or all the state variables by a finite length Chebyshev series of unknown parameters. To decide which state variables to parameterize see the previous chapter.

4. Find the control variables and the state variables, which are not directly approximated, as a function of the directly approximated state variables.

5. Find an expression of $\hat{J}$, using the result of the previous chapter.

6. Determine the set of equality constraints, due to initial conditions, and due to state equations which are not yet satisfied, if any.

7. Find the matrix H and the vector c.

8. Find the optimal parameters from equation (4.45), and substitute these parameters into equations (4.39) and (4.40) to find the approximate optimal trajectories.

9. Repeat the previous procedure, using the obtained trajectories as the new nominal trajectories and control, until the stopping criteria is satisfied.

For the special case $n = m$, the matrix H, the vector c and d can be determined explicitly. For this case, the state variables can be approximated as follows,

$$x^{k+1} = (I_n \otimes T^T)\mathbf{a} \tag{4.46}$$

From equation (4.23), (4.46) and using Chebyshev polynomials' differentiation operational matrix, the control variables u^{k+1} can be determined

$$u^{k+1} = B^{-1}(\tau)\Big[\frac{2}{t_f}(I_n \otimes T^T D^T)\mathbf{a} - A(\tau)(I_n \otimes T^T)\mathbf{a} - h(\tau)\Big] \tag{4.47}$$

Substituting (4.46) and (4.47) into the performance index (4.22), we get

$$\begin{aligned}\hat{J} \;=\; & \frac{t_f}{2}\int_{-1}^{1}\Big[\mathbf{a}^T(I_n \otimes T)Q(I_n \otimes T^T)\mathbf{a} + \Big(\frac{2}{t_f}\mathbf{a}^T(I_n \otimes DT) - \mathbf{a}^T(I_n \otimes T)A^T(\tau) \\ & -h^T(\tau)\Big)F(\tau)\Big(\frac{2}{t_f}(I_n \otimes T^T D^T)\mathbf{a} - A(\tau)(I_n \otimes T^T)\mathbf{a} - h(\tau)\Big)\Big]d\tau \qquad (4.48)\end{aligned}$$

where $F(\tau) = (B^{-1}(\tau))^T R B^{-1}(\tau)$. Equation (4.48) can be simplified

$$\begin{aligned}
\hat{J} &= \frac{t_f}{2}\int_{-1}^{1}\Big[\mathbf{a}^T(Q\otimes TT^T)\mathbf{a}+\frac{4}{t_f^2}\mathbf{a}^T(F(\tau)\otimes DTT^TD^T)\mathbf{a}-\frac{2}{t_f}\mathbf{a}^T(F(\tau)A(\tau)\otimes DTT^T)\mathbf{a} \\
&-\frac{2}{t_f}\mathbf{a}^T(A^T(\tau)F(\tau)\otimes TT^TD^T)\mathbf{a}+\mathbf{a}^T(A^T(\tau)F(\tau)A(\tau)\otimes TT^T)\mathbf{a} \\
&-\frac{2}{t_f}\mathbf{a}^T(F(\tau)h(\tau)\otimes DT)+\mathbf{a}^T(A^T(\tau)F(\tau)h(\tau)\otimes T)-\frac{2}{t_f}(h^T(\tau)F(\tau)\otimes T^TD^T)\mathbf{a} \\
&+(h^T(\tau)F(\tau)A(\tau)\otimes T^T)\mathbf{a}+h^T(\tau)F(\tau)h(\tau)\Big]d\tau \qquad (4.49)
\end{aligned}$$

From this equation H, c^T and d can be obtained

$$\begin{aligned}
H &= t_f\int_{-1}^{1}\Big[(Q\otimes TT^T)+\frac{4}{t_f^2}(F(\tau)\otimes DTT^TD^T)-\frac{2}{t_f}(F(\tau)A(\tau)\otimes DTT^T) \\
&-\frac{2}{t_f}(A^T(\tau)F(\tau)\otimes TT^TD^T)+(A^T(\tau)F(\tau)A(\tau)\otimes TT^T)\Big]d\tau \qquad (4.50) \\
c^T &= 2t_f\int_{-1}^{1}\Big[-\frac{2}{t_f}(h^T(\tau)F(\tau)\otimes T^TD^T)+(h^T(\tau)F(\tau)A(\tau)\otimes T^T)\Big]d\tau \qquad (4.51) \\
d &= t_f\int_{-1}^{1}h^T(\tau)F(\tau)h(\tau)d\tau \qquad (4.52)
\end{aligned}$$

4.5 Computational Results

The numerical method of this chapter is tested on the well known Rayleigh problem to find $u^*(t)$ that minimizes

$$J=\int_0^{2.5}(x_1^2+u^2)dt \qquad (4.53)$$

subject to

$$\begin{aligned}
\dot{x}_1 &= x_2 & x_1(0)&=-5 \qquad (4.54) \\
\dot{x}_2 &= -x_1+1.4x_2-0.14x_2^3+4u & x_2(0)&=-5 \qquad (4.55)
\end{aligned}$$

Attempts by several researchers to solve this problem by means of the second variation method failed [80]. The differential dynamic programming method solved the problem in nine iterations [80]. Also Nedeljkovic [29] solved this problem in three, four and two iterations using three different algorithms which are based on the first order Riccati equation.

To solve this problem by using the proposed algorithm, the second method of the quasilinearization is applied on this problem around nominal trajectories $x_1^{(k)}(t)$ and $x_2^{(k)}(t)$. The expanded performance index is

$$J^{k+1}=\int_0^{2.5}\{(x_1^{k+1})^2+(u^{k+1})^2\}dt \qquad (4.56)$$

Q.L.step(i)	$\hat{J}$	$\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$	T_{total} [sec.]	T_Q [sec.]
1	36.8601	–	3.3	0.05
2	29.4414	7.4187	4.36	0.05
3	29.4101	0.0313	4.45	0.05
4	29.4032	0.0069	4.4	0.05
5	29.4022	0.0010	4.38	0.05

Table 4.1: Approximate optimal value of 5 quasilinearization

and the linearized state equations are

$$\dot{x}_1^{(k+1)} = x_2^{(k+1)} \tag{4.57}$$

$$\dot{x}_2^{(k+1)} = -x_1^{(k+1)} + (1.4 - 0.42(x_2^{(k)})^2)x_2^{(k+1)} + 4u^{(k+1)} + 0.28(x_2^k)^3 \tag{4.58}$$

After changing the time interval $t \in [0, 2.5]$ to the interval $\tau \in [-1, 1]$, $x_1(\tau)$ is approximated by a 9th order Chebyshev series, $x_2(\tau)$ is determined from (4.57) while $u(\tau)$ is determined from (4.58). The linear quadratic optimal control problems (4.56)-(4.58) are solved successively until the difference $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid \leq 1 \times 10^{-3}$. This difference is achieved in five quasilinearization iterations. The approximate optimal value and the difference $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$ of these five quasilinearization iterations, starting from zero nominal trajectories, are summarized in Table 4.1. In this table, T_{total} is the time needed to reformulate the problem into quadratic programming problem and then to solve the resulted quadratic programming problem, while T_Q is the time needed to solve the resulted quadratic programming problem on SUN-SPARC 4/5. The same problem but with different time interval, $t \in [0, 0.5]$ has been solved by Frick and Stech [41] using parallel implementation on Intel eight processor hypercube in 5 iterations each iteration took 3.95 seconds.

Table 4.2 shows a comparison between the optimal value of the fifth quasilinearization iteration of our method with the results obtained by other researchers.

The approximate optimal control and the corresponding state trajectories are shown in Figures 4.1 and 4.2 for the five quasilinearization iterations.

4.6 Practical Application

As a practical application of the proposed method in this chapter and the previous one, the optimal flight control problem of F8 fighter aircraft is considered. This problem was

Source	J
Nedeljkovic [29]	29.419
Sirisena [37]	29.451
This research	29.4022

Table 4.2: Approximate optimal values of the Rayleigh problem

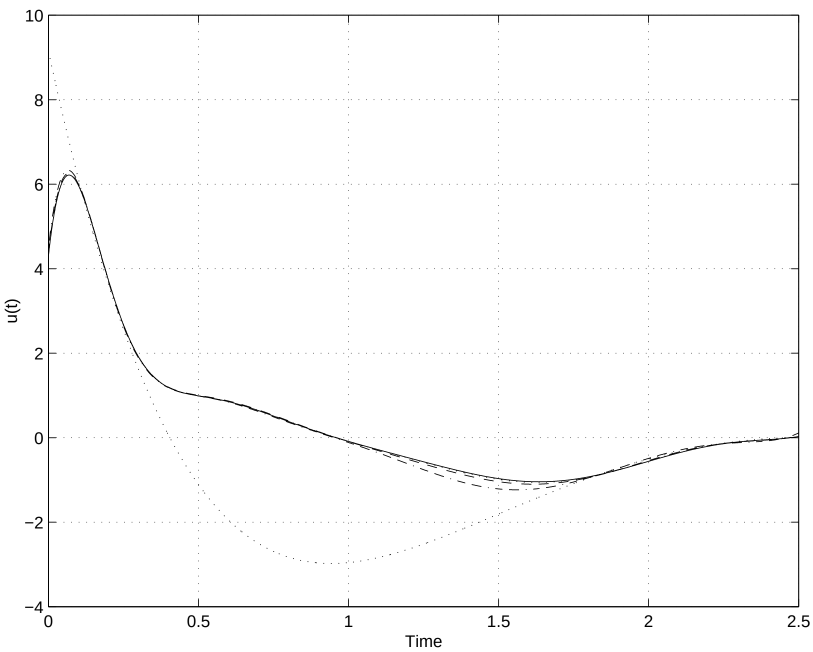

Figure 4.1: Control $u(t)$ of Rayleigh problem for 5 quasilinearization iterations. $(\cdots)$ 1st Q.L., (-.-.) 2ed Q.L., (- - -) 3rd Q. L., $(\cdots)$ 4th Q.L., (—) 5th Q.L.

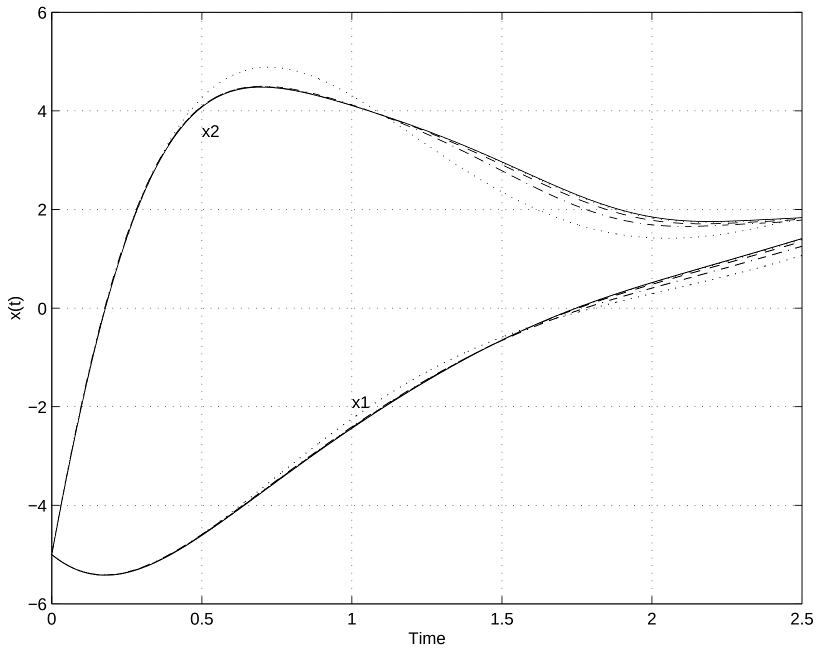

Figure 4.2: States $x(t)$ of Rayleigh problem for 5 quasilinearization iterations. $(\cdots)$ 1st Q.L., (-.-.) 2ed Q.L., (- - -) 3rd Q. L., $(\cdots)$ 4th Q.L., (—) 5th Q.L.

treated by Garrard and Jordan [16], using Lukes' method [14]. The dynamic equations of this aircraft are [16],

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -0.877 & 0 & 1 \\ 0 & 0 & 1 \\ -4.208 & 0 & -0.396 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -x_1^2 x_3 - 0.088 x_1 x_3 - 0.019 x_2^2 + 0.47 x_1^2 + 3.846 x_1^3 \\ 0 \\ -0.47 x_1^2 - 3.564 x_1^3 \end{bmatrix} + \begin{bmatrix} -0.215 \\ 0 \\ -20.967 \end{bmatrix} u \quad (4.59)$$

where x_1 is the wing angle of attack, x_2 is the pitch angle, x_3 is the pitch rate and control input u is the tail deflection angle.

The optimal control problem, which was considered by Garrard and Jordan, was to find the optimal control $u^*(t)$ which minimizes the performance index

$$J = \frac{1}{2} \int_0^{\infty} (x^T Q x + r u^2) dt \quad (4.60)$$

where $\quad Q = \text{diag}\,[0.25, 0.25, 0.25]$ and $r = 1$

Since the proposed method does not deal with infinite time problems, we consider the finite horizon version of this problem. Arbitrary, we select $t_f = 10$ seconds.

Remark 3:

Lee and Bien [47] proved that the infinite time optimal performance index can be approximated by finite time optimal performance index if the states $x^*(t_f)$ and $x^\dagger(t_f)$ are near the origin, where $x^*(t_f)$ is the optimal state of the infinite time problem at time $t = t_f$ and $x^\dagger(t_f)$ is the optimal state of the finite time problem at the end time.

Two cases of this problem are considered, the linearized system of (4.59) around the origin and the nonlinear system (4.59).

In the linearized case and after changing the time interval to $\tau \in [-1, 1]$, x_1 and x_2 are approximated by Chebyshev series and x_3 is calculated from the second state equation while u is calculated from the first state equation. For this problem the third state equation is not satisfied yet, therefore, this equation in addition to the initial conditions represents equality constraints.

Figures 4.3-4.6 show the optimal trajectories for different lengths of Chebyshev series approximations along with the exact trajectories, which are obtained by using the linear feedback control of the linearized system. The optimal trajectories are obtained for the initial conditions $x_1(0) = 30.1^\circ$, $x_2(0) = 0$ and $x_3(0) = 0$. From these Figures, it is clear that the Chebyshev series approximation converges to the correct optimal trajectories as the length of the series increases.

The optimal value of the linearized system of infinite horizon is 0.0222032 while the optimal value of the finite horizon problem, using the Chebyshev series of order 17, is 0.0222109 which gives an error of 7.7×10^{-6}. By using the estimate of Lee and Bien [47], we can calculate the maximum error in the optimal value of the performance index due to approximating the infinite time problem by finite time one. This maximum error is found to be of order 1.66×10^{-6}.

For the nonlinear case, the system equations (4.59) are expanded up to the first order around nominal trajectories. Then after changing the time interval to $\tau \in [-1, 1]$, x_1 and x_2 are approximated by Chebyshev series of order 17, while x_3 is calculated from the second differential equation and the control u is calculated from the first differential equations. In this case, the third differential equation in addition to the initial conditions represents equality constraints.

The termination criteria, $\epsilon = 1 \times 10^{-4}$, for this problem is satisfied after five quasilinearization iterations starting from the zero nominal trajectories. The optimal trajectories are shown in Figures 4.7-4.10. From these Figures, it is clear that the trajectories converge to the optimal ones nearly after the third quasilinearization step. Also Table 4.3 shows the optimal value and the difference $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$ for five quasilinearization iterations.

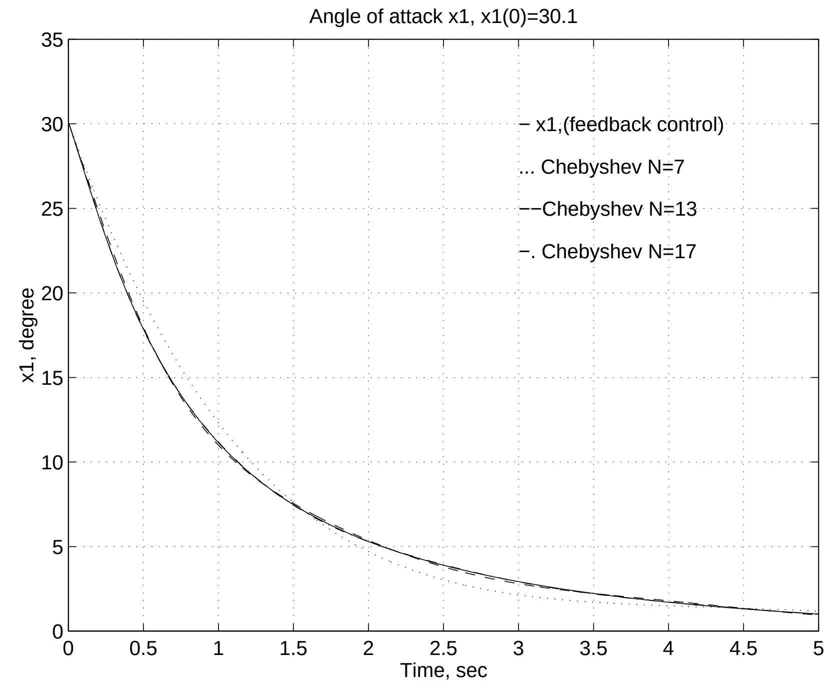

Figure 4.3: State variable $x_1(t)$

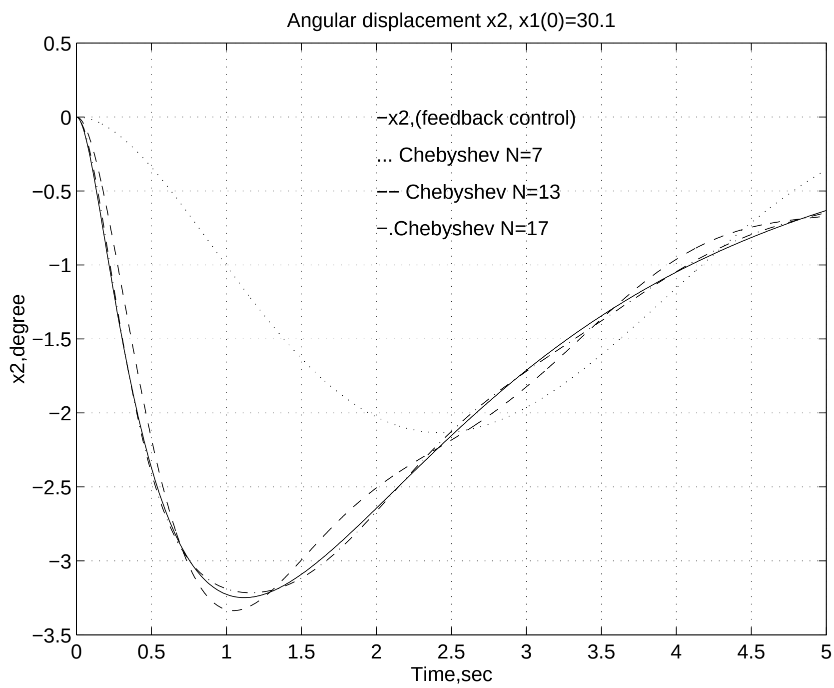

Figure 4.4: State variable $x_2(t)$

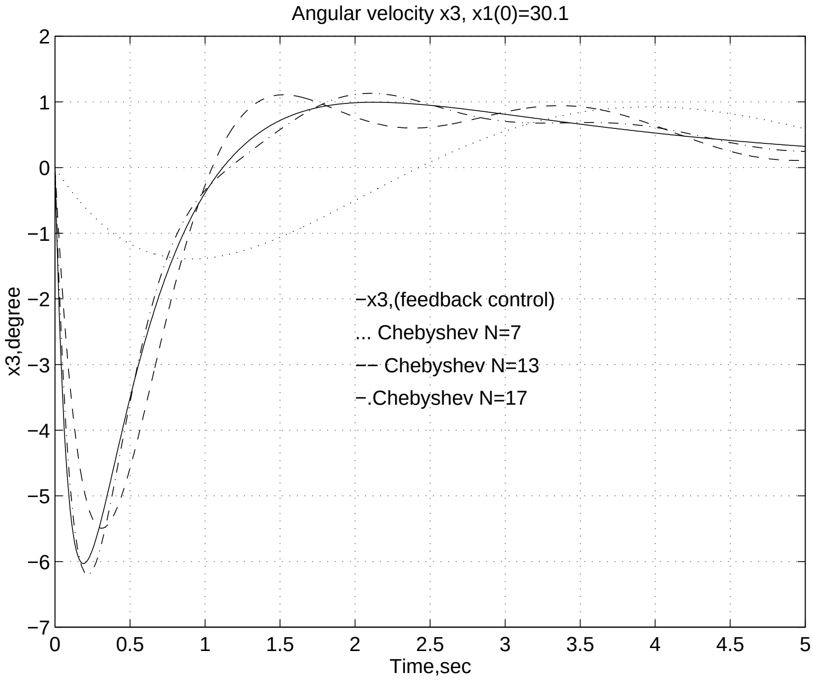

Figure 4.5: State variable $x_3(t)$

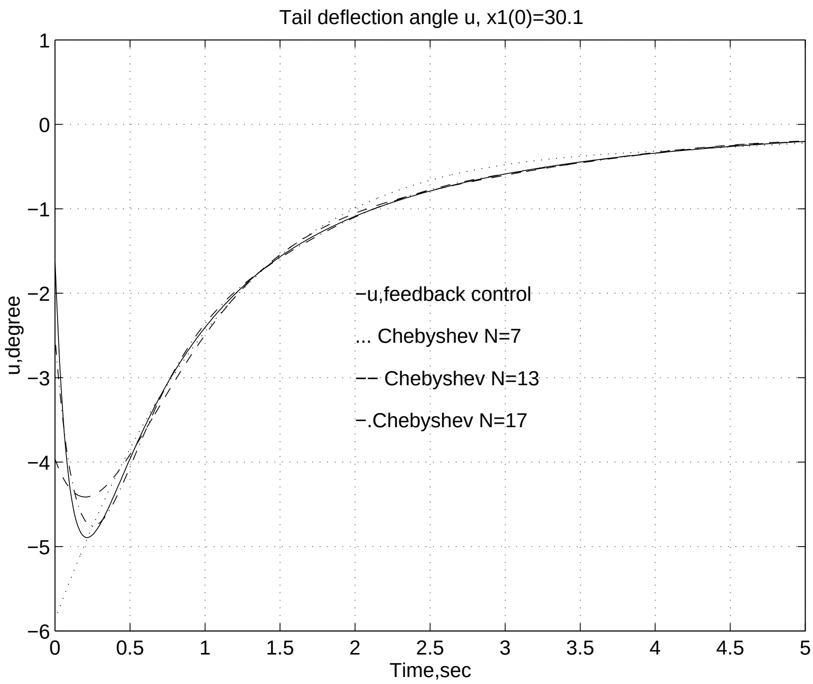

Figure 4.6: Control variable u(t)

Q.L.step(i)	$\hat{J}$	$\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$
1	0.0222109	–
2	0.0823456	0.0601
3	0.0831956	0.00085
4	0.0823616	0.000834
5	0.0823724	0.0000107

Table 4.3: Approximate optimal value of 5 quasilinearization

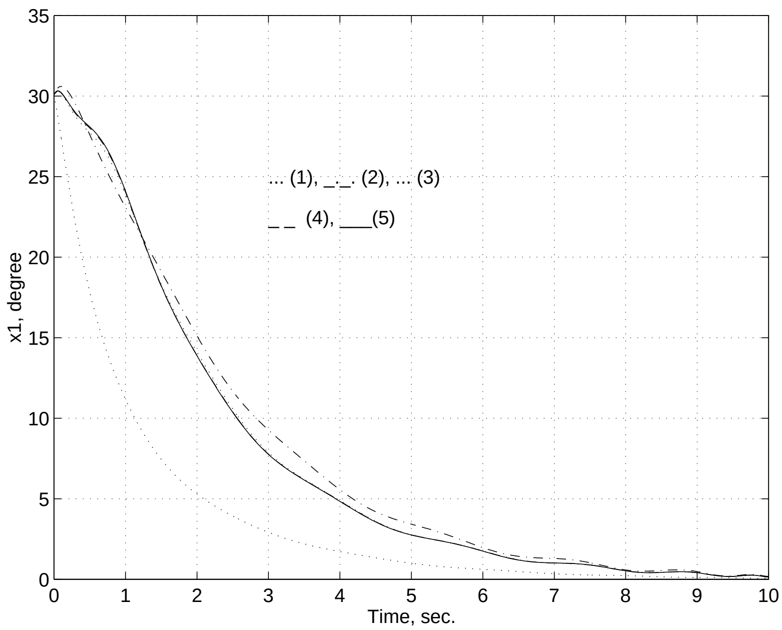

Figure 4.7: $x_1(t)$ for 5 quasilinearization steps

4.7 Conclusion

In this chapter an efficient method is proposed to solve the unconstrained nonlinear optimal control problems. The method is based on using the second method of the quasilinearization and the state parameterization to convert the problem into a sequence of quadratic programming problems which can be solved easily. As it is clear from the simulation, this method gives comparable results compared with other methods.

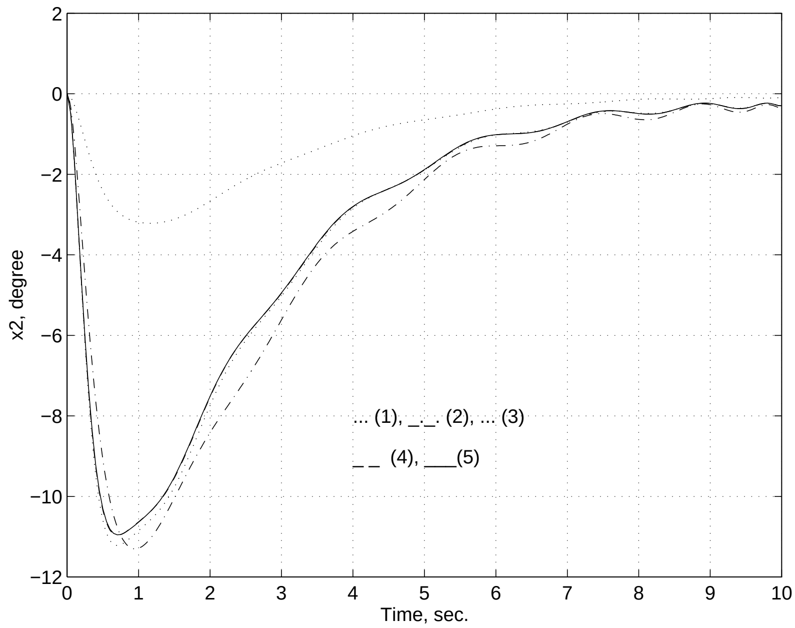

Figure 4.8: $x_2(t)$ for 5 quasilinearization steps

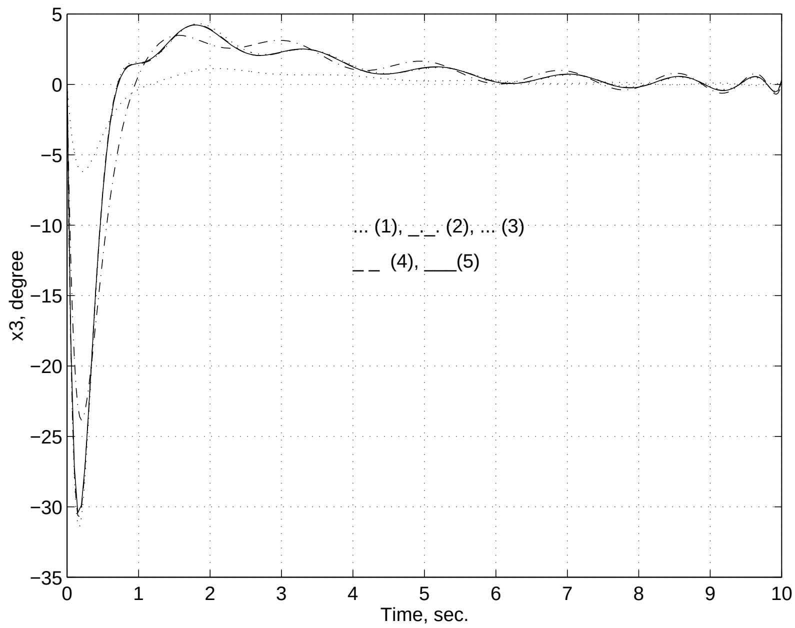

Figure 4.9: $x_3(t)$ for 5 quasilinearization steps

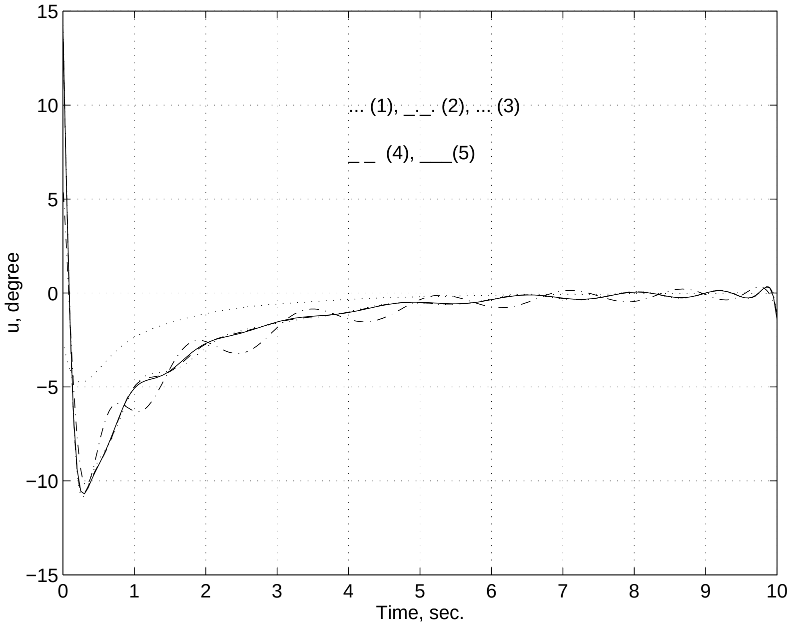

Figure 4.10: u(t) for 5 quasilinearization steps

Chapter 5

Constrained Linear Quadratic Optimal Control Problem

5.1 Introduction

The optimal control problems considered in the last two chapters are assumed to be free of constraints. However, practical optimal control problems usually do have some constraints. Most of the constraints encountered in the practice can be classified as follows:

1. Control saturation constraints

$$U_{min} \leq u(t) \leq U_{max} \tag{5.1}$$

2. Terminal state constraints

$$\Psi(x(t_f), t_f) = 0 \tag{5.2}$$

3. Interior point constraints

$$N(x(t_i), t_i) = 0 \quad 0 < t_i < t_f \tag{5.3}$$

4. Equality constraints on functions of the state and control variables

$$C(x, u, t) = 0 \tag{5.4}$$

5. Inequality constraints on functions of the state and control variables

$$C(x, u, t) \leq 0 \tag{5.5}$$

The presence of these constraints often causes both analytical and computational difficulties. Theoretical treatment of the optimal control problems subject to the constraints can be found in [3,4]. On the other hand, there are many computational techniques and methods proposed by researchers to handle each of the constraints. Most of these methods are reviewed in [36] and the articles cited therein. One of the widely used methods to solve the constrained optimal control problems is to convert them into mathematical programming problem [33,34,40,52,75,78] by using either the discretization or the parameterization.

As has been pointed out previously, the state parameterization method was applied on special cases of unconstrained optimal control problems. And concerning the constrained case, Yen and Nagurka [79] applied Fourier-based state parameterization on special case linear quadratic optimal control problem in which the number of state variables and control variables is equal. Moreover, they treated only the state-control inequality constraints. The authors proposed to add an extra artificial control inputs if the number of control inputs is less than the number of state variables. This technique results in a large number of unknown parameters and in a new problem that may differ from the original one.

In this chapter, we extend the method proposed in the previous chapters to solve linear optimal control problems subject to state and control constraints, terminal state constraints and interior point constraints. The advantages of the proposed method can be summerized as follows:

1. Easy method of approximation.
2. The method can be applied on constrained optimal control problems with unequal number of state variables and control variables.
3. Inequality and equality constraints can be handled.

The presence of the interior point constraints usually complicate severely the optimal control problem. But, as we will see in this chapter, the proposed algorithm can handle these constraints easily.

In this chapter we show the simulation results of a numerical example and compare our results with the results obtained by using control-state parameterization.

5.2 Problem Statement

In this chapter, we consider the following optimal control problem. Find the optimal control $u^*(t)$, which minimizes the performance index

$$J = x(t_f)^T S x(t_f) + \int_0^{t_f} (x^T Q x + u^T R u) dt \tag{5.6}$$

subject to the following constraints:

1. System state equations and initial conditions

$$\dot{x} = A(t)x + B(t)u \qquad x(0) = x_0 \tag{5.7}$$

where $x \in R^n$,$u \in R^m$, $m \leq n$, $A(t)$ and $B(t)$ are respectively $n \times n$ and $n \times m$ real valued matrices defined on $[0, t_f]$.

2. Terminal state constraints

$$\Psi(x(t_f)) \leq 0 \tag{5.8}$$

3. Interior point constraints

$$N[x(t_i), t_i] \leq 0 \tag{5.9}$$

where $0 < t_i < t_f$

4. State and control constraints

$$a(t)x(t) + b(t)u(t) \leq c(t) \tag{5.10}$$

Where Q,S are positive semidefinite matrices and R is a positive definite matrix, $\Psi(x(t_f))$ is linear in $x(t_f)$ and $N[x(t_i), t_i]$ is linear in $x(t_i)$. $a(t)$ is $s \times n$ matrix and $b(t)$ is $s \times m$ matrix, s is the number of state and control inequality constraints.

To solve this problem, we propose to convert it into a quadratic programming problem by parameterizing the system state variables using the first type Chebyshev polynomials.

5.3 State Parameterization Using Chebyshev Polynomials

The Chebyshev polynomials of the first type are defined on the interval $\tau \in [-1, 1]$. To use the Chebyshev polynomials to parameterize the state variables of the stated optimal control problem, the first step is to transform the time interval $t \in [0, t_f]$ into the interval $\tau \in [-1, 1]$. This can be done using the transformation,

$$\tau = \frac{2t}{t_f} - 1 \tag{5.11}$$

This transforms (5.6)-(5.10) into

$$J = x(1)^T S x(1) + \frac{t_f}{2} \int_{-1}^{1} (x^T Q x + u^T R u) d\tau \tag{5.12}$$

subject to the following constraints:

1. System state equations and initial conditions

$$\frac{dx}{d\tau} = \frac{t_f}{2}(A(\tau)x(\tau) + B(\tau)u(\tau)) \qquad x(-1) = x_0 \tag{5.13}$$

2. Terminal state constraints

$$\Psi(x(1)) \leq 0 \tag{5.14}$$

3. Interior point constraints

$$N[x(\tau_i), \tau_i] \leq 0 \tag{5.15}$$

where $\tau_i = \frac{2t_i}{t_f} - 1$

4. State and control constraints

$$a(\tau)x(\tau) + b(\tau)u(\tau) \leq c(\tau) \tag{5.16}$$

The next step, in order to simplify the computations, is to express $A(\tau), B(\tau), a(\tau), b(\tau)$ and $c(\tau)$ in terms of Chebyshev series. To this end let the $(i,j)th$ entry of $A(\tau) = g(\tau)$, then $g(\tau)$ can be approximated by a Chebyshev series of finite length as follows [44],

$$g(\tau) = \frac{G_0}{2} + \sum_{i=1}^{M} G_i T_i(\tau) \tag{5.17}$$

where

$$G_j = \frac{2}{K} \sum_{i=0}^{K} g(\cos\theta_i)\cos(j\theta_i) \tag{5.18}$$

$j = 0, 1, 2, \cdots, M$ and

$$\theta_i = \frac{2i-1}{2K}\pi \quad i = 1, 2, \cdots, K \quad K > M \tag{5.19}$$

The same procedure can be repeated for each entry of $A(\tau), B(\tau), a(\tau), b(\tau)$ and $c(\tau)$.

The third step is to apply the state parameterization approach as described in Chapter 3.

5.4 Optimal Control Problem Reformulation

After performing the state parameterization, we can generally express the state variables and the control variables as,

$$x_j(\tau) = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{\mathcal{N}} a_i^{(j)} T_i(\tau) \quad j = 1, 2, \cdots, n \tag{5.20}$$

$$u_l(\tau) = \frac{b_0^{(l)}}{2} + \sum_{i=1}^{\mathcal{N}} b_i^{(l)} T_i(\tau) \quad l = 1, 2, \cdots, m \tag{5.21}$$

Where $\mathcal{N}$ is the order of the longest approximation of state variables or control variables. Here we should clarify that all the directly approximated states are of length N. The previous two equations can be written in a matrix form as

$$x(\tau) = \alpha T(\tau) \tag{5.22}$$

$$u(\tau) = \beta T(\tau) \tag{5.23}$$

where α is $n \times \mathcal{N}$ matrix of unknown parameters $[a_0^{(j)}, \cdots, a_{\mathcal{N}}^{(j)}]$, β is $m \times \mathcal{N}$ matrix of the unknown parameters $[b_0^{(j)}, \cdots, b_{\mathcal{N}}^{(j)}]$ which are linear function of $[a_0^{(j)}, \cdots, a_{\mathcal{N}}^{(j)}]$. $T(\tau)$ is $\mathcal{N} \times 1$ vector of Chebyshev polynomials. These approximations will be used in reformulating the optimal control problem into quadratic programming problem as follows:

- Initial states approximation:

  Using the Chebyshev polynomials property at $\tau = -1$, the initial states can be approximated by,

$$x_j(-1) = x_0^{(j)} = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{\mathcal{N}} (-1)^i a_i^{(j)} \quad j = 1, 2, \cdots, n \tag{5.24}$$

  which will represent equality constraints in the new problem formulation.

- Terminal state constraints:

  Also, by using Chebyshev polynomials property at $\tau = 1$, we can express the value of the state variables at $t = t_f$ or equivalently $\tau = 1$ as follows,

$$x_j(1) = x(t_f)^{(j)} = \frac{a_0^{(j)}}{2} + \sum_{i=1}^{\mathcal{N}} a_i^{(j)} \quad j = 1, 2, \cdots, n \tag{5.25}$$

  and by substituting this equation in (5.8), we get

$$\Psi(x(1)) \leq 0 \tag{5.26}$$

- Performance index approximation:

  Substituting (5.22) and (5.23) into (5.12), we get

$$\hat{J} = \min_{\mathbf{a}} \left\{ x^T(1) S x(1) + \frac{t_f}{2} \int_{-1}^{1} \Big(T^T(\tau) \alpha^T Q \alpha T(\tau) + T^T(\tau) \beta^T R \beta T(\tau) \Big) d\tau \right\} \tag{5.27}$$

  where $\hat{J}$ is the approximate value of J; $\mathbf{a}$ is $q(N+1) \times 1$ vector of all unknown parameters $\mathbf{a}^T = [a_0^{(1)}\ a_1^{(1)}\ \cdots\ a_N^{(1)}\ a_0^{(2)}\ \cdots\ a_N^{(2)}\ a_0^{(q)}\ \cdots\ a_N^{(q)}]$ and q is the number of directly approximated state variables.

  Using the Chebyshev polynomials property at $\tau = 1$, equation (5.27) can be written as

$$\hat{J} = \min_{\mathbf{a}} \left\{ T^T(1) \alpha^T S \alpha T(1) + \frac{t_f}{2} \int_{-1}^{1} \Big(T^T(\tau) M T(\tau) + T^T(\tau) P T(\tau) \Big) d\tau \right\} \tag{5.28}$$

  where $T(1)$ is $\mathcal{N} \times 1$ vector whose all elements are 1's; $M = \alpha^T Q \alpha$ and $P = \beta R \beta$. This equation can be approximated using the result derived in chapter 3, to get

$$\hat{J} = \min_{\mathbf{a}} \Big\{ T^T(1)\alpha^T S \alpha T(1) + \frac{2t_f}{2} \sum_{i=1}^{\mathcal{N}+1-k} \frac{1}{2} \Big(\acute{m}_{i,i+k} + \acute{p}_{i,i+k} \Big) \cdot \Big(\frac{-2}{(2i-2+k)^2-1} + \frac{-2}{k^2-1} \Big) \Big\} \tag{5.29}$$

where k=0,2,4,$\cdots$,$\mathcal{N}$ ($\mathcal{N}$ even) or $\mathcal{N}-1$ ($\mathcal{N}$ odd), and

$$\acute{m}_{i,i+k} = \begin{cases} m_{i,i+k} & k \neq 0 \\ \frac{m_{ii}}{2} & k = 0 \end{cases} \tag{5.30}$$

$$\acute{p}_{i,i+k} = \begin{cases} p_{i,i+k} & k \neq 0 \\ \frac{p_{ii}}{2} & k = 0 \end{cases} \tag{5.31}$$

m_{ij}, p_{ij} are the $(i,j)th$ entry of matrix M and matrix P, respectively.

Equation (5.29) is a quadratic function of the unknown parameters and can be written as,

$$\hat{J} = \min_{\mathbf{a}} \frac{1}{2} \mathbf{a}^T H \mathbf{a} \tag{5.32}$$

- Interior point constraints:

The interior point constraints can be expressed easily, by substituting the value of τ_i in

$$N[x(\tau_i), \tau_i] \leq 0 \tag{5.33}$$

to get

$$N[\alpha T(\tau_i), \tau_i] \leq 0 \tag{5.34}$$

These are inequality constraints which are a function of the unknown parameters $\mathbf{a}$

- State and control constraints:

The initial states constraints, the terminal states constraints and the interior points constraints are finite dimension constraints. However, the state and control constraints are infinite dimension constraints that have to be satisfied at every time $\tau \in [-1, 1]$. To handle such constraints, we will satisfy them at discrete points, $-1 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_r = 1$. As r approaches infinity, these constraints approach the continuous constraints. Therefore each of these constraints is replaced by $r+1$ constraints. This technique was used by [52, 54]. By substituting (5.22) and (5.23) into (5.16), we get

$$a(\tau)\alpha T(\tau) + b(\tau)\beta T(\tau) \leq c(\tau) \tag{5.35}$$

and this equation can be replaced by $r+1$ finite dimension constraints as,

$$a(\tau_h)\alpha T(\tau_h) + b(\tau_h)\beta T(\tau_h) \leq c(\tau_h) \quad h = 0, 1, \cdots, r \tag{5.36}$$

From the previous reformation, we can express the new problem as

$$\min_{\mathbf{a}} \frac{1}{2}\mathbf{a}^T H\mathbf{a} \tag{5.37}$$

subject to

$$F_1\mathbf{a} - b_1 = 0 \tag{5.38}$$

$$F_2\mathbf{a} - b_2 \leq 0 \tag{5.39}$$

where the equality constraints represent the state and control equality constraints, initial states, terminal state equality constraints and interior points equality constraints . The inequality constraints represent the state and control inequality constraints, terminal state inequality constraints and interior point inequality constraints.

The optimization problem (5.37)-(5.39) is a standard quadratic programming problem which can be solved using the active set method [34].

5.5 Computational Results

In this section, we consider linear quadratic optimal control problem subject to one of three constraints: first order state inequality constraint; second order state inequality constraint or interior point constraint.

Example 1 *Find $u^*(t)$ that minimizes*

$$J = \int_0^1 (x_1^2 + x_2^2 + 0.005u^2)dt \tag{5.40}$$

subject to the system dynamic equations, initial conditions,

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0 \tag{5.41}$$

$$\dot{x}_2 = -x_2 + u \qquad x_2(0) = -1 \tag{5.42}$$

and to the first order state inequality constraint [1].

$$x_2(t) - 8(t-0.5)^2 + 0.5 \leq 0 \quad 0 \leq t \leq 1 \tag{5.43}$$

Example 2 *Find $u^*(t)$ that minimizes (5.40) subject to (5.41), (5.42) and to the second order state inequality constraint given by,*

$$x_1(t) - 8(t-0.5)^2 + 0.5 \leq 0 \quad 0 \leq t \leq 1 \tag{5.44}$$

Example 3 *Find $u^*(t)$ that minimizes (5.40) subject to (5.41), (5.42) and to the following interior point constraint*

$$x_1(0.5) = 0.5 \tag{5.45}$$

[1]The control variable will appear by differentiating the inequality constraint once

The first two examples have been solved by several researchers using different approaches, but the results obtained by Valssenbroeck [40] using control-state parameterization were the best results, therefore we will compare our results with those in [40].

To solve these examples using the proposed algorithm, $x_1(\tau)$ is approximated by 13th order Chebyshev series of unknown parameters, after changing the time interval to $\tau \in [-1, 1]$, then $x_2(\tau), u(\tau)$ are found from the first and second state equations respectively. In each case, the optimal control problem is converted into quadratic programming problem and then it is solved using MATLAB program which is based on the active set method.

In Example 1, we find $\hat{J} = 0.1707848$ in 3.7 seconds on SUN-SPARC 4/5. However, by looking very closely at the inequality constraint, we find that there is a very small violation ($< 3 \times 10^{-4}$) for very short period of time (about 0.06 second). To prevent this violation, we modify the inequality state constraint as proposed in [40].

$$x_2(t) - 8(t - 0.5)^2 + 0.5 + \delta \leq 0 \tag{5.46}$$

By solving this example for $\delta = 0.0005$, we obtain $\hat{J} = 0.17102286$, and there is no violation of the inequality constraint at all, moreover, $x_2(t)$ does not touch the constraint boundary. On the other hand, Vlassenbroeck [40], obtained $J = 0.17185$ in 21.4 seconds, on a CDC Cyber 170/750, by using $\delta = 0.0035$ to get rid of the inequality constraint violation. This means that our algorithm, before adding δ, violates the constraints very much less than his approach.

In Example 2, we find $\hat{J} = 0.7394399$ in 1.67 seconds. Also, in this case, there is a very small violation of the constraint ($< 5 \times 10^{-4}$) for about 0.02 seconds, but by adding $\delta = 0.0005$ we prevent any violation of the constraint and obtain $\hat{J} = 0.7409643$. On the other hand in [40] the author obtained $J = 0.74096$ in 31 seconds and needed to add $\delta = 0.012$ to get rid of the constraint violation, which means that our algorithm solves the original problem with very much less violation than his algorithm.

From the results of Examples 1 and 2, it is clear that our algorithm gives better or comparable results with that of Vlassenbroack algorithm, although, the amount of computations in our method is very much less than in his algorithm.

To have an idea concerning the computations complexity of both methods: In our method we approximate each problem by 14 unknown parameters only, while in Vlassenbroack method each problem was approximated by 56 unknown parameters. Using our algorithm, we need to solve a quadratic programming problem of 14 unknown parameters subject to 2 equality constraints (due to initial conditions) and 41 inequality constraints (due to discretization of the inequality constraints), however by using Vlassenbroack method, there was a need to solve nonlinear programming problem of 56 unknown parameters subject to 44 equality constraints.

In Example 3, we obtained $\hat{J} = 1.0749928$ and the interior point constraint is satisfied exactly.

The optimal control and the optimal states for the three examples are shown in Figures 5.1-5.3

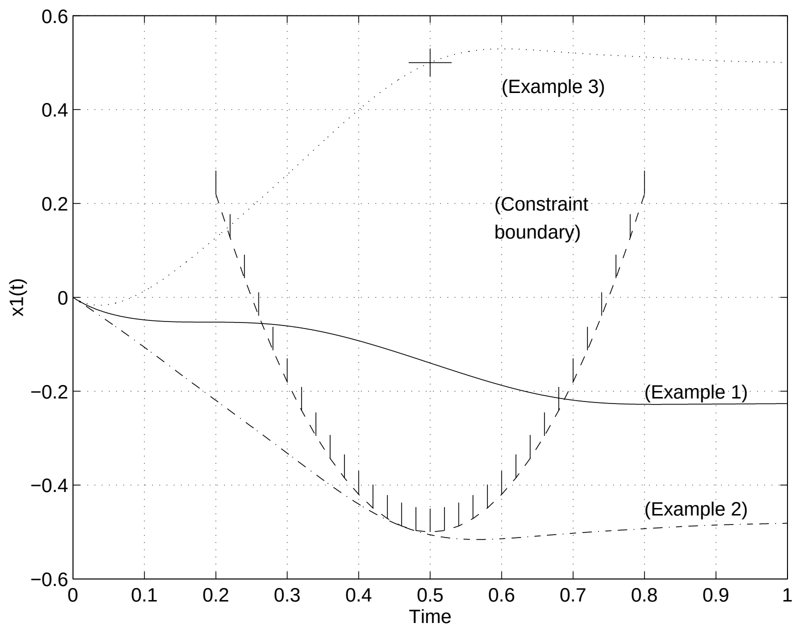

Figure 5.1: $x_1(t)$ for the 3 Examples

5.6 Conclusion

An effective computational method is proposed to solve the linear quadratic optimal control problem subject to terminal state constraints, state and control constraints and interior point constraints. The use of the state parameterization technique enables us to handle the state inequality constraints and interior point constraints easily. The main advantage of this algorithm is that the difficult constrained optimal control problem is transformed into quadratic programming problem.

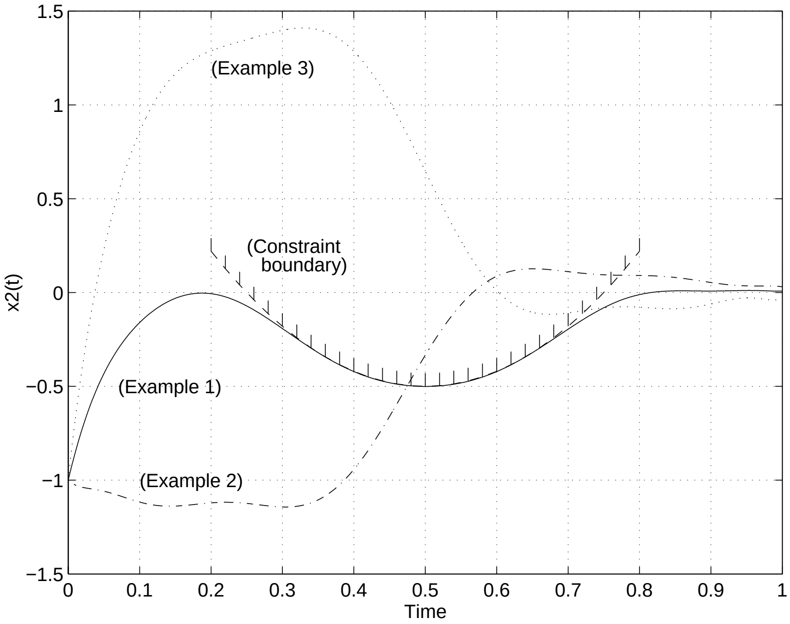

Figure 5.2: $x_2(t)$ for the 3 Examples

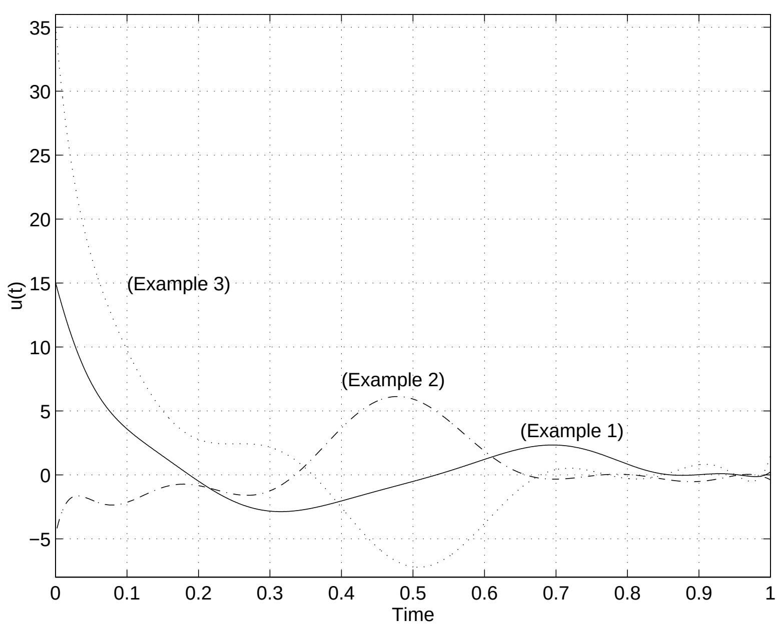

Figure 5.3: $u(t)$ for the 3 Examples

Chapter 6

Constrained Nonlinear Optimal Control Problem

6.1 Introduction

In this chapter, we treat the constrained nonlinear optimal control problem. Basically, the method of this chapter is based on that proposed in the previous chapters. Namely, it is based on using the second method of the quasilinearization and the state parameterization using Chebyshev polynomials.

The direct methods have been applied on different classes of constrained nonlinear optimal control problem. Cullum [72], Kraft [75], Stryk and Bulirsch [73] and Betts [78] have applied the discretization method to convert the problem into a nonlinear mathematical programming problem. On the other hand, using the parameterization technique, Goh and Teo [36], Troch *et al.* [33] applied the control parameterization on general constrained nonlinear optimal control problems. Sirisena and Tan [50] also applied control parameterization using piecewise polynomials on nonlinear optimal control problems subject to terminal state constraints and saturation control constraints. In addition, the control-state parameterization has been applied by Vlassenbroeck [40] using Chebyshev polynomials to solve the constrained nonlinear optimal control problem, also it has been applied by Frick and Stech [42] using Walsh functions to solve nonlinear optimal control problems subject to control saturation constraints.

The use of the state parameterization to solve constrained optimal control problems was only used by [79] to solve linear quadratic control problem, subject to state and control inequality constraints, with equal number of state variables and control variables.

In this chapter, we use the state parameterization to solve nonlinear optimal control problem subject to terminal state constraints and control saturation constraints. Other types of constraints can also be used by modifying the proposed method. At the end of the chapter, we present simulation results of a practical nonlinear optimal control problem subject to terminal state constraints, control saturation constraints and state saturation constraints.

The direct methods have been applied in two ways to solve nonlinear optimal control problems subject to terminal state constraints and control saturation constraints. These two methods are:

1. The first method is to convert the nonlinear optimal control problem into a nonlinear mathematical programming problem, and then to use Han's method [81, 82] to replace the nonlinear mathematical programming problem by a sequence of quadratic programming problems. Lin [83] applied this method on nonlinear optimal control problem subject to control saturation constraints and terminal state constraints, by discretizing the nonlinear optimal control problem, then by replacing the new problem by a sequence of quadratic programming problems.

2. The second method is to replace the nonlinear optimal control problem by a sequence of linear quadratic optimal control problems, each of which is approximated by a quadratic programming problem. This method was used by Bashein and Enns [55] to solve nonlinear optimal control problem subject to terminal state constraints and control saturation constraints. In their approach, the terminal state constraints were satisfied up to the first order and the control saturation constraints were handled by using the bounded variable quadratic programming algorithm. Recently the same method is used by Ma and Levine [56, 57] to replace a nonlinear optimal control problem subject to terminal state constraints and control saturation constraints by a sequence of quadratic programming problems. Their algorithm gives only an upper bound of the optimal value. Moreover, the nonlinear system state equations and the costate equations have to be integrated in each iteration.

In Bashein and Enns [55]; Ma and Levine [56,57]; Lin's [83] methods a large dimension quadratic programming problem, in the sense of the number of unknown parameters and the number of constraints, has to be solved in each iteration. A large dimension problem arises because of the use of the discretization technique.

In this chapter, we present a new method to solve the nonlinear optimal control problem subject to terminal state constraints and control saturation constraints. Our method avoids the problems associated with the previous methods, namely: We do not need a special program to solve the quadratic programming problem as in Bashin and Enns [55]; we do not need to integrate the system state equations or costate equations; the exact optimal value can be obtained. Moreover, we obtain small size quadratic programming problems, which can be solved easily by matrix−vector multiplication.

The method is based on using the second method of the quasilinearization and the state parameterization to convert the constrained nonlinear optimal control problem into a sequence of standard quadratic programming problem.

As an application of the proposed algorithm, at the end of this chapter, we present a real practical nonlinear constrained optimal control problem.

6.2 Problem Statement

In this chapter, we consider the following optimal control problem: Find the optimal control $u^*(t)$, which minimizes the performance index

$$J = x(t_f)^T S x(t_f) + \int_0^{t_f} (x^T Q x + u^T R u) dt \tag{6.1}$$

subject to the following constraints

1. System state equations and initial conditions

$$\dot{x} = f(x(t), u(t), t) \quad x(0) = x_0 \tag{6.2}$$

   where $x \in R^n$, $u \in R^m$ and $m \leq n$.

2. Terminal state constraints

$$\Psi(x(t_f), t_f) = 0 \tag{6.3}$$

3. Saturation control constraints

$$u(t) \leq U_{max} \qquad u(t) \geq U_{min} \tag{6.4}$$

we will assume, for simplification, that each of the saturation constraints is a scalar.

The proposed method to solve the stated optimal control problem consists mainly of two steps:

1. replacing the constrained nonlinear optimal control problem by a sequence of constrained linear optimal control problems by using the quasilinearization technique.

2. solving successively the constrained linear problems until an acceptable convergence is achieved. To accomplish this step, we propose to use the parameterization technique, specifically state variables parameterization using Chebyshev polynomials, to transform the difficult dynamic optimal control problem into static quadratic programming one. These two steps will be discussed in the following section.

6.3 Proposed Method

6.3.1 Quasilinearization

In order to apply the second method of quasilinearization to constrained optimal control problem, we can adjoin all types of constraints to the performance index using suitable penalty functions [4]. In this work, the saturation control constraints will be adjoined to the performance index using Kelly's penalty function [58]. The terminal state constraints will be satisfied to the first order [55], hence the stated problem reduces to: Minimize

$$J_1 = J + \int_0^{t_f} \Big\{ (U_{max} - u)^2 \mathbf{H}[U_{max} - u] + (u - U_{min})^2 \mathbf{H}[u - U_{min}] \Big\} dt \tag{6.5}$$

subject to (6.2) and (6.3). $\mathbf{H}$ is the Heaviside function given by,

$$\mathbf{H}[z] = \begin{cases} 0 & z \geq 0 \\ K_1 & z < 0 \end{cases} \tag{6.6}$$

and K_1 is a positive weighting constant.

Applying the second method of the quasilinearization by expanding J_1 up to the second order and by expanding the state equations and the terminal constraints up to the first order, around nominal trajectories $x^k(t)$ and $u^k(t)$, we get

$$\begin{aligned} J_1^{(k+1)} &= x^{(k+1)^T}(t_f) S x^{(k+1)}(t_f) + \int_0^{t_f} \Big\{ x^{(k+1)^T} Q x^{(k+1)} + u^{(k+1)^T} R u^{(k+1)} \\ &+ \Big(U_{max}^2 - 2U_{max} u^{k+1} + (u^{k+1})^2 \Big) \mathbf{H}[U_{max} - u^{k+1}] \\ &+ \Big((u^{k+1})^2 - 2U_{min} u^{k+1} + U_{min}^2 \Big) \mathbf{H}[u^{k+1} - U_{min}] \Big\} dt \end{aligned} \tag{6.7}$$

subject to

$$\dot{x}^{k+1} = A(t) x^{k+1} + B(t) u^{k+1} + h^k(t) \tag{6.8}$$

$$x^{k+1}(0) = x_0$$

and the linearized terminal state constraints

$$\Psi_x(x^k(t_f), t_f)(x^{k+1}(t_f) - x^k(t_f)) + \Psi(x^k(t_f), t_f) = 0 \tag{6.9}$$

where

$$h^k(t) = f(x^k, u^k, t) - A(t) x^k - B(t) u^k \tag{6.10}$$

$$A(t) = \left. \frac{\partial f(x,u,t)}{\partial x} \right|_{x^k, u^k} \tag{6.11}$$

$$B(t) = \left. \frac{\partial f(x,u,t)}{\partial u} \right|_{x^k, u^k} \tag{6.12}$$

Remark 1: It is possible to handle the general nonlinear performance index and other nonlinear constraints (control and state constraints) by adjoining these constraints to the performance index using penalty functions [4]. Then the augmented performance index is expanded up to the second order and the state equations are expanded up to the first order around nominal trajectories.

6.3.2 State Parameterization

The constrained linear quadratic optimal control problem (6.7)-(6.9) can be solved by converting it into standard quadratic programming problem. To this end, we use the Chebyshev polynomials of the first type as described in the previous chapters to reformulate the optimal control problem as follows:

- System state equations approximation:

  Using the state parameterization method proposed in previous chapters, the state variables and the control variables can be approximated in matrix form as follows,

$$x^{k+1}(\tau) = \alpha T(\tau) + VT(\tau) \tag{6.13}$$

$$u^{k+1}(\tau) = \beta T(\tau) + GT(\tau) \tag{6.14}$$

  where α, β are matrices of unknown parameters; V,G are matrices of known parameters and $T(\tau)$ is a vector of Chebyshev polynomials. The dimension of the vector $T(\tau)$ depends on the longest series of the state variables and the control variables. Assume that its dimension is $\mathcal{N} \times 1$.

- Performance index approximation:

  By substituting (6.13) and (6.14) into the performance index (6.7) and using the Chebyshev polynomials property at $\tau = 1$, we get

$$\begin{aligned}
\hat{J}_1^{(k+1)} = & \Big(T^T(1)(\alpha^T + V^T)S(\alpha + V)T(1)\Big) \\
& + \frac{t_f}{2}\int_{-1}^{1}\Big\{T^T(\tau)(\alpha^T + V^T)Q(\alpha + V)T(\tau) + T^T(\tau)(\beta^T + G^T)R(\beta + G)T(\tau) \\
& + \Big(U_{max}^2 - 2U_{max}(\beta_c + G_c)T(\tau) + ((\beta_c + G_c)T(\tau))^2\Big)\mathbf{H}[U_{max} - u^{k+1}] \\
& + \Big(((\beta_c + G_c)T(\tau))^2 - 2U_{min}(\beta_c + G_c)T(\tau) + U_{min}^2\Big)\mathbf{H}[u^{k+1} - U_{min}]\Big\}d\tau
\end{aligned} \tag{6.15}$$

  where $\hat{J}_1^{(k+1)}$ is the approximate value of $J_1^{(k+1)}$, β_c and G_c are the row of (6.14) which corresponds to the constrained control.

All terms of (6.15) can be integrated using the result of chapter 3, except the constant terms (U_{max}^2, U_{min}^2) and the terms ($2U_{max}(\beta_c + G_c)T(\tau)$, $2U_{min}(\beta_c + G_c)T(\tau)$) which can be integrated easily using Chebyshev polynomials integration property [44]

$$\int_{-1}^{1} T_n(\tau)d\tau = \begin{cases} 0 & n \text{ odd} \\ \frac{-2}{n^2-1} & n \text{ even} \\ 2 & n = 0 \end{cases} \tag{6.16}$$

After performing the integration, the performance index will be reduced to

$$\underset{\mathbf{a}}{minimize} \left(\frac{1}{2}\mathbf{a}^T H\mathbf{a} + c^T\mathbf{a} + d\right) \tag{6.17}$$

where $\mathbf{a}$ is a vector of the unknown parameters which are the coefficients of the directly approximated states, d is a constant and c is a constant vector.

- Initial and terminal state constraints:

  By using Chebyshev polynomials properties [44], the initial conditions can be expressed in general as

$$\begin{aligned} x_j(-1) &= \frac{a_0^j}{2} - a_1^j + a_2^j - \cdots + (-1)^{\mathcal{N}} a_{\mathcal{N}}^j + v_j(-1) \\ &= x_0 \end{aligned} \tag{6.18}$$

  and the final state conditions can be expressed as

$$x_j(1) = x(T) = \frac{a_0^j}{2} + a_1^j + a_2^j + \cdots + a_{\mathcal{N}}^j + v_j(1) \tag{6.19}$$

  where $j = 1, 2, \cdots, n$. The terminal state constraints (6.9), reduce to

$$\Psi_x(x^k(1), 1)(x^{k+1}(1) - x^k(1)) + \Psi(x^k(1), 1) = 0 \tag{6.20}$$

From the previous formulation, the optimal control problem is reduced to quadratic programming problem of the form

$$\underset{\mathbf{a}}{minimize} \left(\frac{1}{2}\mathbf{a}^T H\mathbf{a} + c^T\mathbf{a} + d\right) \tag{6.21}$$

subject to the linear equality constraints

$$F\mathbf{a} - b = 0 \tag{6.22}$$

where the equality constraints represent the initial states (6.18) and the terminal state constraints (6.20). The optimal value of the unknown parameters $\mathbf{a}^*$ can be obtained using the standard quadratic programming results [34].

$$\mathbf{a}^* = -H^{-1}c + H^{-1}F^T(FH^{-1}F^T)^{-1}(FH^{-1}c + b) \tag{6.23}$$

The linear quadratic optimal control problems (6.7)-(6.9) have to be solved successively until a stopping criteria is satisfied. In this research the computations are terminated when the difference $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$ is sufficiently small.

Remark 2: Another method to handle the control saturation constraints

$$u(\tau) \leq U_{max} \qquad u(\tau) \geq U_{min} \tag{6.24}$$

is to satisfy them at discrete points, $-1 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_r = 1$. Therefore each of these constraint will be replaced by $r+1$ finite dimension inequality constraints as follows.

$$\frac{b_0^{(l)}}{2} + \sum_{i=1}^{\mathcal{N}} b_i^{(l)} T_i(\tau_h) + g_l(\tau_h) \leq U_{max} \tag{6.25}$$

$$-\frac{b_0^{(l)}}{2} - \sum_{i=1}^{\mathcal{N}} b_i^{(l)} T_i(\tau_h) - g_l(\tau_h) \leq -U_{min} \tag{6.26}$$

where $h = 0, 1, 2, \cdots, r$. This technique has been used by [52, 54].

Yet another possibility to handle the inequality constraints is by converting them into equality constraints using slack variables as in [39]. However the use of the slack variables has two disadvantages namely it converts a linear problem into a nonlinear one and secondly it increases the number of the unknown parameters.

Generally, from the previous reformulation the constraints can be expressed as

$$F_1\mathbf{a} = b_1 \tag{6.27}$$

$$F_2\mathbf{a} \leq b_2 \tag{6.28}$$

where the equality constraints represent the initial states, terminal state constraints, while the inequality constraints represent the saturation constraints if the conversion into finite dimension scheme is employed. In this case, the active set method can be used to solve the resulted quadratic programming problem.

Remark 3: If the control should appear in bang-bang form, as pointed out by Vlassenbroeck and van Dooren [39], then the time interval has to be divided into sections at the discontinuities, and the original problem can be solved by solving subproblems in each section and taking into account the continuity of states at the point of control discontinuities.

Iteration	$\hat{J}$	$\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$
1	1.6858217	–
2	1.4334359	0.2524
3	1.4334872	5.13×10^{-5}

Table 6.1: Optimal value of the first case

6.4 Computational Results

In this section, we consider Van der Pol oscillator problem. The system state equations are:

$$\dot{x}_1 = x_2 \tag{6.29}$$

$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + u \tag{6.30}$$

The cost function to be minimized, starting from the initial states $x_1(0) = 1$ and $x_2(0) = 0$, is:

$$J = \frac{1}{2}\int_0^5 (x_1^2 + x_2^2 + u^2)dt \tag{6.31}$$

Based on this problem, we consider three cases: unconstrained problem, terminal states constrained problem and terminal states and control constrained problem.

• Free end point and no control constraints: J was found by Bullock and Franklin [59] to be 1.433508 using the second variation method. Based on the quasilinearization and discretization, Bashein and Enns [55] found J starting from $u(t)^{(0)} = 0$ and $u(t)^{(0)} = 1$, in four and five iterations respectively, they obtained $J = 1.4380970$.

By using the proposed algorithm, quasilinearization and state parameterization, and starting from zero nominal trajectories, the stopping criteria ($\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid \leq 1 \times 10^{-3}$) is satisfied after three iterations only. We obtain $\hat{J} = 1.4334872$, which is smaller than both of the results reported earlier. In this problem, we approximate $x_1(\tau)$ by a Chebyshev series of ninth order and $x_2(\tau)$, $u(\tau)$ are found from the system equations. The approximate optimal control and state trajectories are shown in Figure 6.1, while $\hat{J}$ of each iteration with the difference $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$ are shown in Table 6.1

• Terminal state constraint:

$$\Psi(x(t_f)) = 1 - x_2(t_f) + x_1(t_f) = 0 \tag{6.32}$$

Using the second order method, Bullock and Franklin [59] found $J = 1.6857$ and $\Psi(x(t_f)) = -5 \times 10^{-6}$ after seven iterations, while Bashein and Enns [55] found $J =$

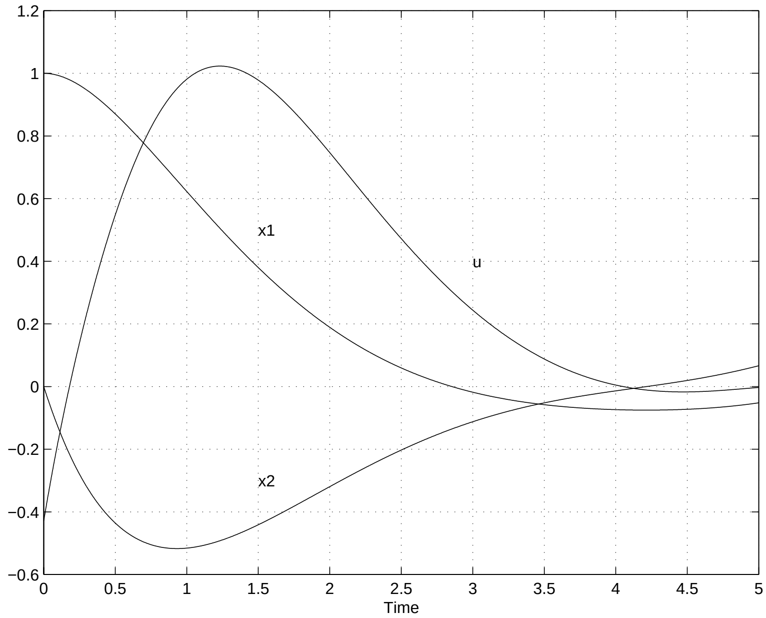

Figure 6.1: Approximate optimal control and state trajectories, case 1

Iteration	$\hat{J}$	$\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$	Ψ
1	1.9246170	–	4.4×10^{-9}
2	1.6856769	0.2389	3.4×10^{-9}
3	1.6857113	3.44×10^{-5}	-5.0×10^{-9}

Table 6.2: Approximate optimal value of the second case

1.6905756 and $\Psi(x(t_f)) = -7.5 \times 10^{-6}$ in five iterations. In this work, $x_1(\tau)$ is approximated by ninth order Chebyshev series and $x_2(\tau)$, $u(\tau)$ are obtained from the system equations. We obtain $\hat{J} = 1.6857113$ and $\Psi(x(t_f)) = -5 \times 10^{-9}$ in three iterations. Figure 6.2 shows the approximate optimal control and state trajectories for this case and Table 6.2 summarizes $\hat{J}$ and the difference $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$ of each iteration.

- Terminal state constraints and saturation constraints on control: The constraints are

$$|u(t)| \leq 0.75 \tag{6.33}$$

$$\Psi_1 = x_1(t_f) + 1 = 0 \tag{6.34}$$

$$\Psi_2 = x_2(t_f) = 0 \tag{6.35}$$

This problem has been solved by Bashein and Enns [55], they obtained $J = 2.1439039$, $\Psi_1 = 1 \times 10^{-7}$ and $\Psi_2 = 7 \times 10^{-7}$ after seven iterations. In this work, $x_1(\tau)$ is approximated by 12th order Chebyshev series and $x_2(\tau)$, $u(\tau)$ are found from the state equations .

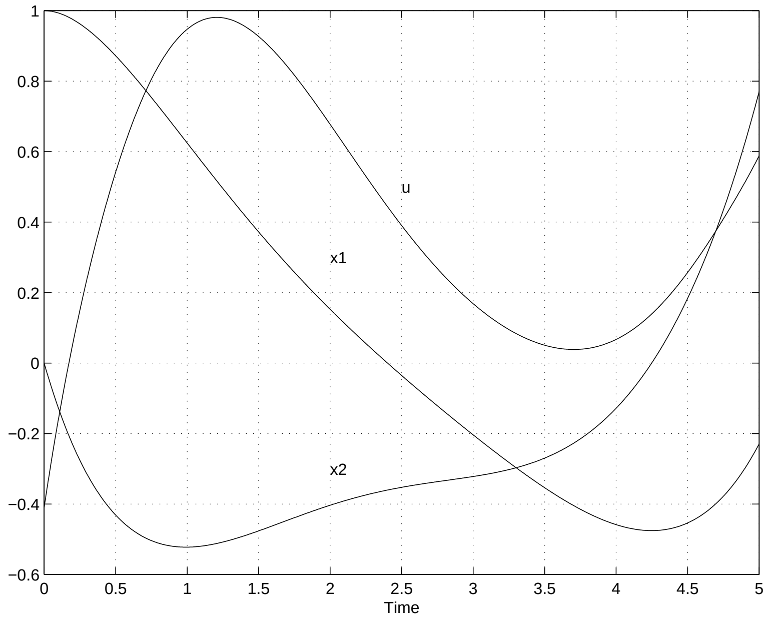

Figure 6.2: Approximate optimal control and state trajectories, case 2

This problem is solved for different values of K_1. The computations are terminated when the difference $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid < 1 \times 10^{-5}$ is satisfied. The obtained results are summarized in Table 6.3, while Figure 6.3 shows the approximate optimal control and the state trajectories of the 7th iteration for $K_1 = 20000$.

To show the violation of the control constraints, Figure 6.4 shows part of the optimal control for different values of K_1.

In addition, the last problem is solved by converting the inequality constraints into finite dimensional constraints using discretization. In this case, $x_1(t)$ is approximated by

K_1	$\hat{J}$	$\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$	Ψ_1	Ψ_2	no. of iter.	max. $\lvert u \rvert$ violation
1	2.0660665	4.8×10^{-7}	0	-2.77×10^{-16}	4	< 0.2
100	2.1358300	5.1×10^{-6}	0	-2.77×10^{-16}	5	< 0.014
600	2.1419409	6.2×10^{-6}	0	2.22×10^{-16}	6	< 0.0036
1500	2.1429523	2.4×10^{-6}	0	-2.77×10^{-16}	7	< 0.002
5000	2.1437176	7.4×10^{-6}	0	-2.77×10^{-17}	7	< 0.001
20000	2.1443893	3.6×10^{-6}	0	3×10^{-16}	7	< 0.0004

Table 6.3: Approximate optimal value in case of using penalty function to handle inequality constraints

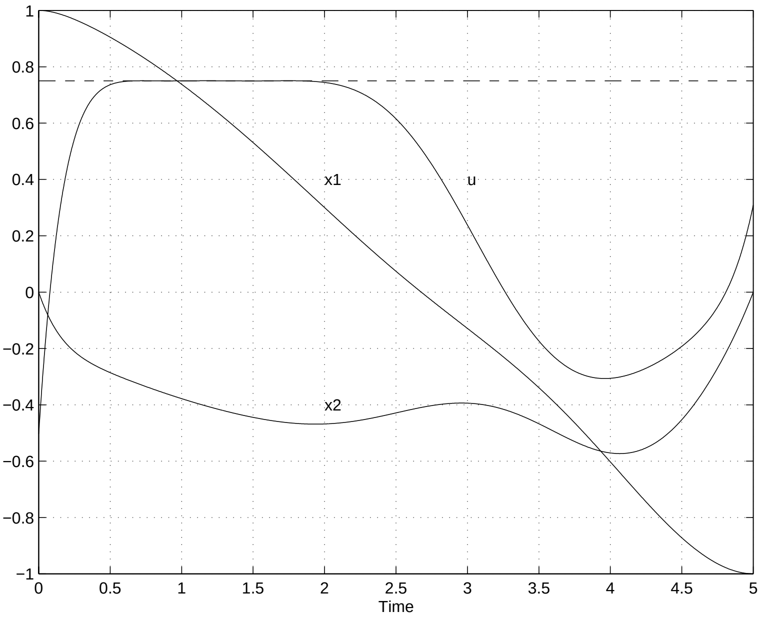

Figure 6.3: Approximate optimal control and state trajectories, case 3

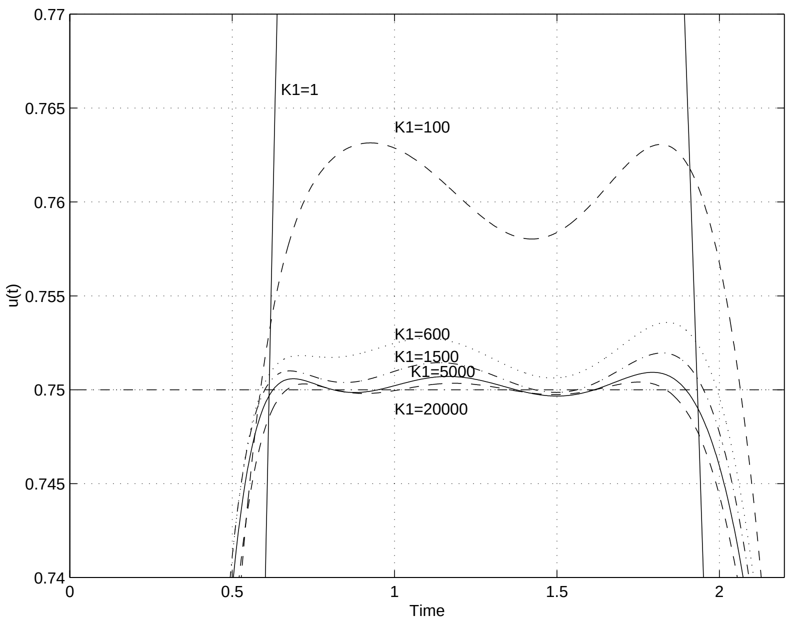

Figure 6.4: Constraints violation

Iteration	$\hat{J}$	$\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$	Ψ_1	Ψ_2
1	3.2428173	–	4.57×10^{-8}	1.4×10^{-8}
2	2.2120969	1.0307	-1.8×10^{-8}	-8.7×10^{-9}
3	2.1439884	0.0681	1.24×10^{-8}	1.27×10^{-9}
4	2.1439199	6.9×10^{-5}	2.56×10^{-8}	8.69×10^{-9}

Table 6.4: Approximate optimal value in case of using discretization of inequality constraints

a 12-th order Chebyshev series and $x_2(t)$, $u(t)$ are found from the state equations . After four iteration, we get $\hat{J} = 2.1439194$, $\Psi_1 = 2.56 \times 10^{-8}$ and $\Psi_2 = 8.7 \times 10^{-9}$. Figure 6.5 shows the state and control trajectories while Table 6.4 shows $\hat{J}$ and the difference $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$.

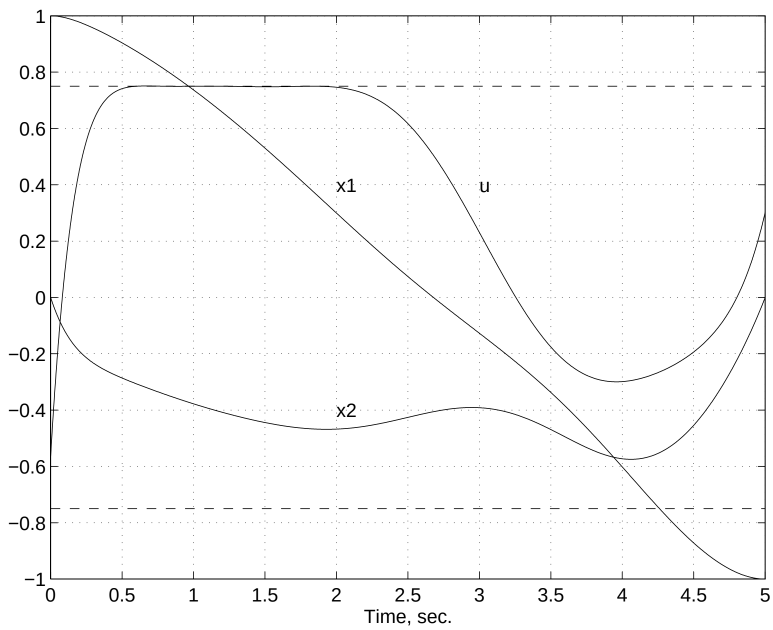

Figure 6.5: Approximate optimal trajectories in case of using discretization of inequality constraints

6.5 Practical Application

In this section, we consider a realistic and complex problem of transferring containers from a ship to a cargo truck at the port of Kobe [74]. The container crane is driven by a hoist motor and a trolley drive motor. For safety reason, the objective is to minimize the swing during and at the end of the transfer.

Without going into the details of the modeling aspect, we shall summarize the problem as follows: Minimize

$$J = \frac{1}{2}\int_0^9 (x_3^2 + x_6^2)dt \tag{6.36}$$

subject to the dynamical equations

$$\dot{x}_1 = x_4 \tag{6.37}$$
$$\dot{x}_2 = x_5 \tag{6.38}$$
$$\dot{x}_3 = x_6 \tag{6.39}$$
$$\dot{x}_4 = u_1 + 17.2656x_3 \tag{6.40}$$
$$\dot{x}_5 = u_2 \tag{6.41}$$
$$\dot{x}_6 = \frac{1}{x_2}\Big(u_1 + 27.0756x_3 + 2x_5x_6\Big) \tag{6.42}$$

where

$$\mathbf{x}(0) = [0, 22, 0, 0, -1, 0]^T \tag{6.43}$$
$$\mathbf{x}(9) = [10, 14, 0, 2.5, 0, 0]^T \tag{6.44}$$

and

$$|u_1(t)| \leq 2.83374 \ \forall t \in [0, 9] \tag{6.45}$$

$$-0.80865 \leq u_2(t) \leq 0.71265 \ \forall t \in [0, 9] \tag{6.46}$$

with continuous state inequality constraints

$$|x_4(t)| \leq 2.5 \ \forall t \in [0, 9] \tag{6.47}$$

$$|x_5(t)| \leq 1 \ \forall t \in [0, 9] \tag{6.48}$$

This problem was solved by Sakawa and Shindo [74], but no optimal value was reported. Also it was solved by Goh and Teo [34] using piecewise constant functions and piecewise linear functions to parameterize the control variables. In the first case, the authors found J to be 0.005361, while in the second case they found $J = 0.005412$ and they concluded that as the controls become smoother the optimal value become larger.

Using our algorithm, we apply the second method of the quasilinearization and then approximate the x_1, x_2, x_5 by 9th order Chebyshev series of unknown parameters. The remaining states and controls are obtained from the system state equations. All the state equations are satisfied directly except the last equation, which is replaced by equality constraints. The problem is solved for three iterations and the $\hat{J}$ is found to be 0.00562.

This optimal value is higher than that reported by Goh and Teo [34], but the control is a smooth function and there is no violation of the constraints at all. The optimal trajectories are shown in Figures 6.5 and 6.6

6.6 Conclusion

In this chapter we have proposed a computational method to solve the constrained nonlinear optimal control problems. This problem is converted into a sequence of quadratic programming problems. The solution method is based on using the second method of quasilinearization and the state parameterization. The inequality constraints are handled either by using penalty functions or by converting them into finite dimensions inequality constraints.

For the sake of comparison, we have applied the proposed method on Van der Pol oscillator problem . Moreover, we have used the proposed method to solve a real practical constrained nonlinear optimal control problem, the container crane problem.

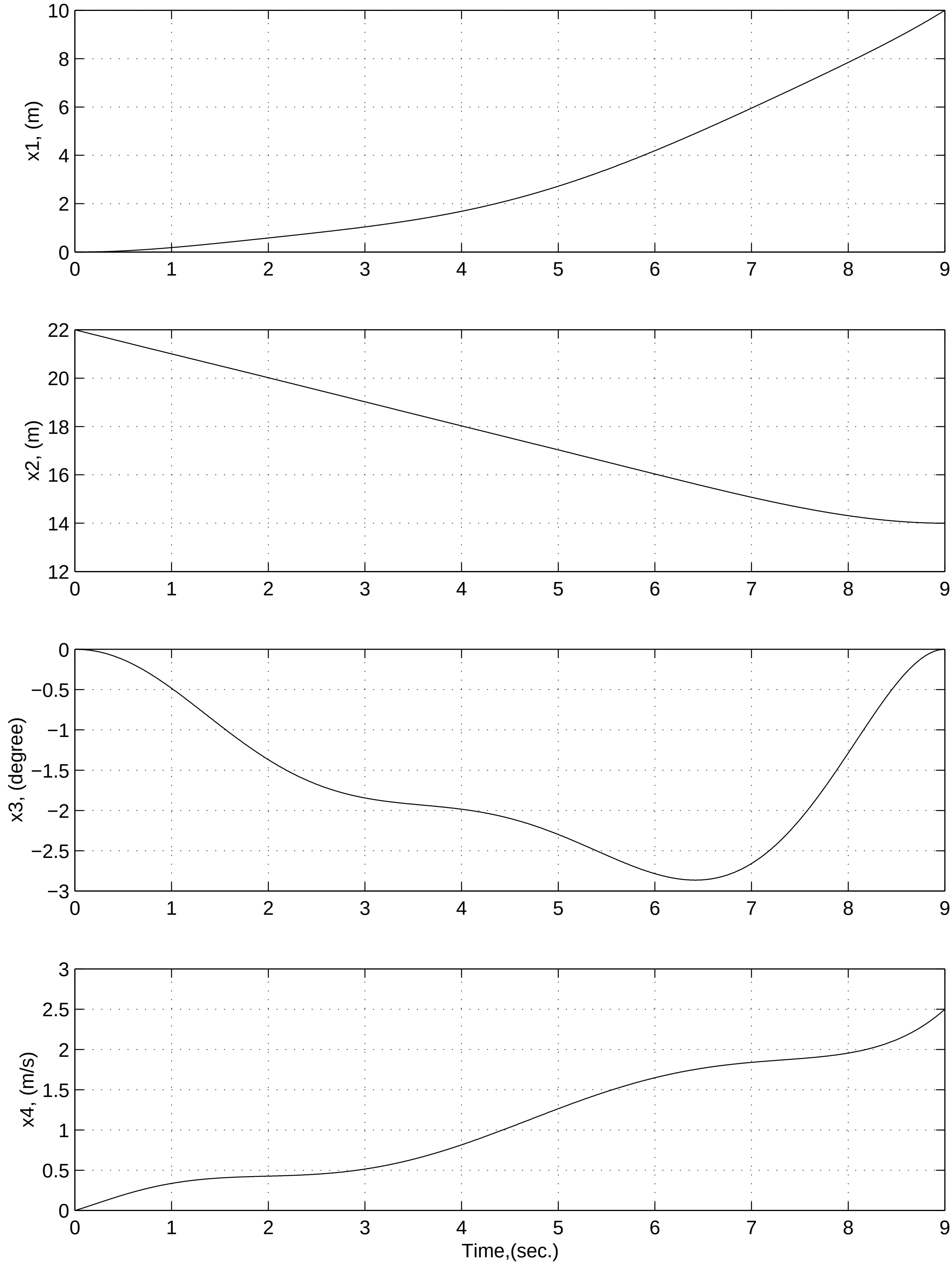

Figure 6.6: x_1, x_2, x_3 and x_4 of the container crane problem

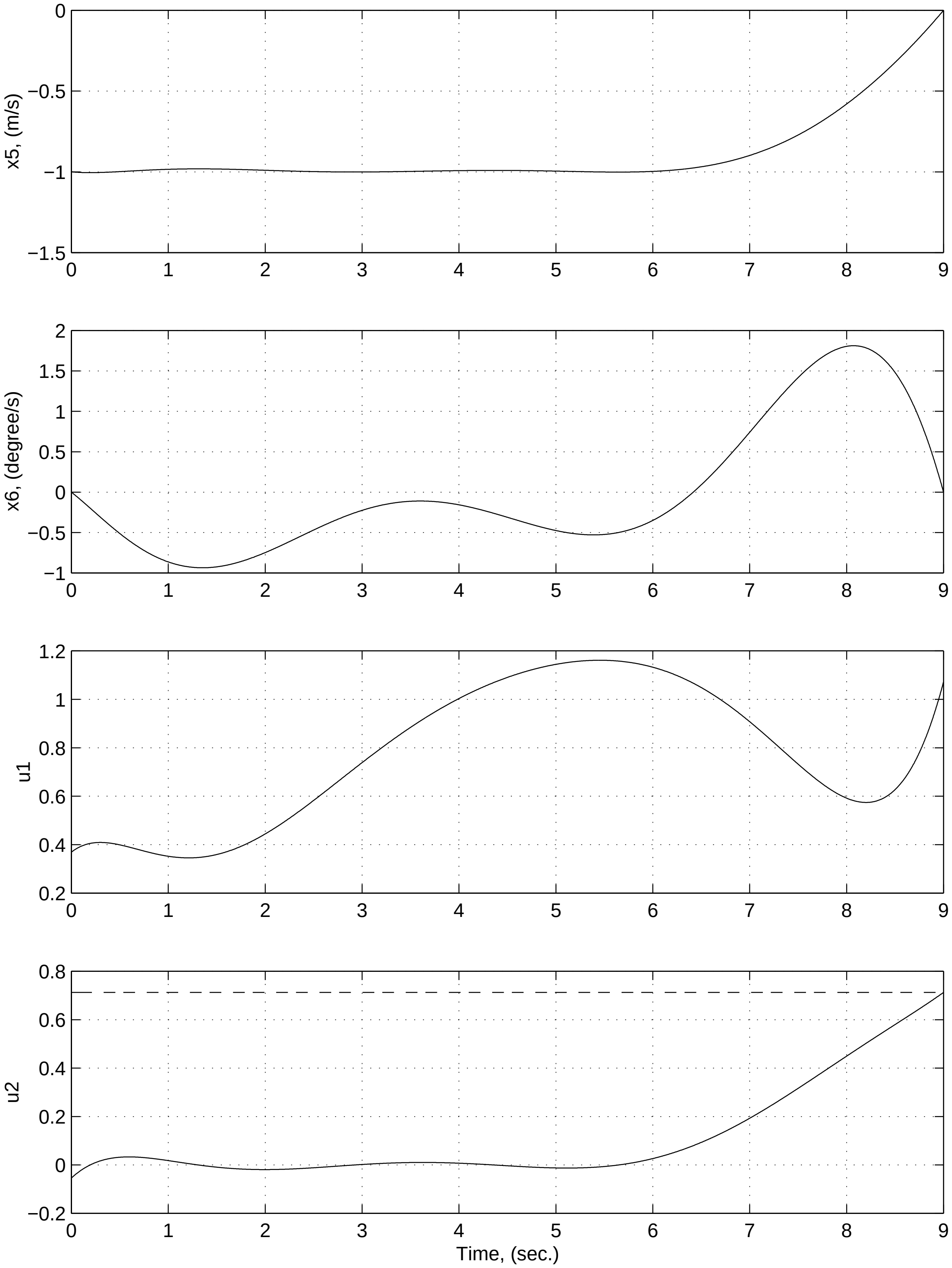

Figure 6.7: x_5, x_6, u_1 and u_2 of the container crane problem

Chapter 7

Construction of Optimal Feedback Control

7.1 Introduction

The optimal feedback control law of linear quadratic optimal control problems can be obtained by solving the matrix Riccati equation, or by determining the transition matrix of the Hamiltonian system [2,3]. However, for general nonlinear optimal control problems, it is not possible to obtain the exact optimal feedback control solution analytically. But nevertheless suboptimal feedback control can be obtained by using either power series expansion method [14, 16–18, 63], or neighboring optimal control method [3, 19, 21]. Also, the optimal feedback control can be obtained using dynamic programming method [60], but this method suffers from the curse of dimensionality.

On the other hand, during the last twenty years, the orthogonal functions have been used extensively for determining the optimal feedback control of the linear quadratic optimal control problems. For example, [61] used the Walsh functions, [66] used the Chebyshev polynomials of the first type, [62], [64], [69] used the Chebyshev polynomials of the second type, [67] used Block pulse functions, [87] used the Fourier series. The solution method in all of the previous works is based on using the forward or backward integration operational matrix, associated with the used orthogonal polynomials, to transform the Hamiltonian system (state and costate differential equations) into algebraic equations.

In the previous chapters, we presented algorithms to solve different classes of optimal control problems. However, the solutions were obtained as a function of the time i.e open loop control. The purpose of this chapter is to use the Chebyshev polynomials of the first type to determine the optimal feedback control law of the nonlinear optimal control problem. Also, in this chapter, the differentiation operational matrix of Chebyshev polynomials of the first type is introduced. This operational matrix can also be defined for each of the other orthogonal polynomials.

Our approach to determine the optimal feedback control law of the nonlinear optimal control problem consists of two steps: The first step is to determine the open loop optimal control and trajectories, by using the quasilinearization and the state variables parameterization via Chebyshev polynomials of the first type. Therefore the nonlinear optimal control problem is replaced by a sequence of small quadratic programming problems which can be solved easily. The second step is to use the results of the last quasilinearization iteration (when the stopping criteria $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid \leq \epsilon$ is satisfied) to obtain the optimal feedback control law. To this end, the matrix Riccati equation and another n linear differential equations are solved using the Chebyshev polynomials of the first type.

The proposed method has some advantages over the power series method [15, 17, 18] and over the methods that give the neighboring optimal feedback control [3, 19–22]. These advantages are:

1. The obtained optimal feedback control can be implemented easier than the control obtained by using the power series method.

2. We do not need to store the optimal open loop state and control trajectories as in the methods of [3, 19–22] .

3. The obtained closed loop control is a nonlinear one and, although it appears as a linear one, the nonlinear terms of the states are included in the time varying terms. While the neighboring optimal control approach gives linear feedback control due to perturbed initial conditions from the optimal open loop solution.

7.2 Differentiation Operational Matrix

To facilitate the computation of optimal feedback control, we derive a new property of Chebyshev polynomials called differentiation operational matrix.

The Chebyshev polynomials can be obtained from the recurrence relation,

$$T_{r+1}(\tau) = 2\tau T_r(\tau) - T_{r-1}(\tau) \quad r = 1, 2, 3 \cdots \tag{7.1}$$

where $T_0(\tau) = 1, T_1(\tau) = \tau$.

Also a function $x(\tau)$ can be approximated by Chebyshev series of length m as follows,

$$x(\tau) = \frac{a_0}{2} + \sum_{i=1}^{m} a_i T_i(\tau) \tag{7.2}$$

where the coefficients $a_n, n = 0, 1, \cdots, m$ can be determined using the following formula [68]

$$a_n = \frac{2}{K}\sum_{i=1}^{K} x(\cos(\theta_i))cos(n\theta_i) \tag{7.3}$$

where $\theta_i = \frac{2i-1}{2K}\pi, i = 1, 2, \cdots, K$, and $K > m$. As m approaches infinity the previous approximation approaches the exact $x(\tau)$. Equation (7.2) can be expressed in vector form as

$$x(\tau) = [\frac{a_0}{2}\ a_1\ \cdots\ a_m]T(\tau) \tag{7.4}$$

where $T(\tau) = [T_0(\tau)\ T_1(\tau)\ \cdots\ T_m(\tau)]^T$. It is easy to prove that the derivative of $x(\tau)$ with respect to τ can be given by

$$\dot{x}(\tau) = [\frac{a_0}{2}\ a_1\ \cdots\ a_m]DT(\tau) \tag{7.5}$$

where the matrix D is the differentiation operational matrix. This matrix can be given as follows

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 3 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 8 & 0 & 8 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 5 & 0 & 10 & 0 & 10 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 12 & 0 & 12 & 0 & 12 & 0 & 0 & \cdots & 0 \\ 7 & 0 & 14 & 0 & 14 & 0 & 14 & 0 & \cdots & 0 \\ 0 & 16 & 0 & 16 & 0 & 16 & 0 & 16 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m & 0 & 2m & 0 & 2m & 0 & 2m & 0 & \cdots & 0 \end{bmatrix} \tag{7.6}$$

In the previous matrix it is assumed that m is odd. However, if m is even then the last row of D becomes

$$[0\ 2m\ 0\ 2m\ 0\ 2m\ \cdots\ 0]$$

Proof:

Differentiating the recurrence relation (7.1) with respect to τ, we get

$$\dot{T}_{r+1}(\tau) = 2T_r(\tau) + 2T_1(\tau)\dot{T}_r(\tau) - \dot{T}_{r-1}(\tau) \tag{7.7}$$

which can be written as

$$\dot{T}_r(\tau) = 2T_{r-1}(\tau) + 2T_1(\tau)\dot{T}_{r-1}(\tau) - \dot{T}_{r-2}(\tau) \tag{7.8}$$

$\dot{T}_{r-1}(\tau)$ is a polynomial of order $r-2$ and $\dot{T}_{r-2}(\tau)$ is also a polynomial of order $r-3$. These can be expressed as

$$\dot{T}_{r-1}(\tau) = \frac{b_0}{2} + b_1 T_1 + b_2 T_2 + \cdots + b_{r-2} T_{r-2} \tag{7.9}$$

$$\dot{T}_{r-2}(\tau) = \frac{c_0}{2} + c_1 T_1 + c_2 T_2 + \cdots + c_{r-3} T_{r-3} \tag{7.10}$$

substituting (7.9) and (7.10) into (7.8), we get

$$\dot{T}_r = 2T_{r-1} + 2T_1\left(\frac{b_0}{2} + \sum_{i=1}^{r-2} b_i T_i\right) - \frac{c_0}{2} - \sum_{j=1}^{r-3} c_j T_j \tag{7.11}$$

By expanding the second term on the right hand side using Chebyshev polynomials product property and collecting the coefficients of similar Chebyshev polynomials, we get

$$\dot{T}_r = (b_0 - \frac{c_0}{2}) + \sum_{i=1}^{r-2}\Big[(b_{i-1} + b_{i+1} - c_i)T_i\Big] + (b_{r-2} + 2)T_{r-1} \tag{7.12}$$

Note that $\dot{T}_0 = 0$ and $\dot{T}_1 = T_0$. Applying equation (7.12) recursively, the matrix D can be obtained.

Also, from (7.5), it can be proved that

$$\frac{d^k}{d\tau^k} T(\tau) = D^k T(\tau) \tag{7.13}$$

The differentiation operational matrix will be used in the next section to convert the differential equations into algebraic equations.

The advantages of the differentiation operational matrix over the integration operational matrix are: all its elements are integers and hence there is no truncation error; easy to construct because it has special structure. This matrix is a lower triangular matrix, also if r is odd then

$$\frac{d}{d\tau} T_r(\tau) = [r \ 0 \ 2r \ 0 \ 2r \ 0 \ 2r \ \cdots \ 0] T(\tau) \tag{7.14}$$

however if r is even, then

$$\frac{d}{d\tau} T_r(\tau) = [0 \ 2r \ 0 \ 2r \ 0 \ 2r \ 0 \cdots \ 0] T(\tau) \tag{7.15}$$

7.3 Solution of Nonlinear Optimal Control Problem

This section reviews the approach that proposed in chapter 4 to solve the nonlinear optimal control problem in open loop form. This method is based on transforming the nonlinear optimal control problem into a sequence of linear quadratic optimal control problems using the second method of quasilinearization. Then, each linear quadratic optimal control problem is converted into a standard quadratic programming problem by employing the state parameterization using Chebyshev polynomials.

The problem we are considering is to find the optimal control $u^*(t)$ that minimizes the performance index

$$J = \frac{1}{2}x(t_f)Sx(t_f) + \frac{1}{2}\int_0^{t_f}(x^TQx + u^TRu)dt \tag{7.16}$$

subject to the system state equations

$$\dot{x} = f(x(t), u(t), t) \quad x(0) = x_0 \tag{7.17}$$

where $x \in R^n, u \in R^m, m \leq n$, S, Q are positive semidefinite matrices and R is a positive definite matrix. The first step to solve this problem, in open loop form, is to apply the second method of the quasilinearization (expand the system state equations up to the first order and the performance index up to the second order around a nominal trajectories $x^k(t), u^k(t)$). The problem will be reduced to: Minimize

$$J^{k+1} = \frac{1}{2}x^{(k+1)^T}(t_f)Sx^{(k+1)}(t_f) + \frac{1}{2}\int_0^{t_f}(x^{(k+1)^T}Qx^{(k+1)} + u^{(k+1)^T}Ru^{(k+1)})dt \tag{7.18}$$

subject to

$$\dot{x}^{k+1} = A(t)x^{k+1} + B(t)u^{k+1} + h^k(t) \quad x^{k+1}(0) = x_0 \tag{7.19}$$

where

$$h^k(t) = F(x^k, u^k, t) - A(t)x^k - B(t)u^k \tag{7.20}$$

$$A(t) = \left.\frac{\partial F(x,u,t)}{\partial x}\right|_{x^k,u^k} \tag{7.21}$$

$$B(t) = \left.\frac{\partial F(x,u,t)}{\partial u}\right|_{x^k,u^k} \tag{7.22}$$

The second step is to use the Chebyshev polynomials of the first type to parameterize the state variables and to convert the linear quadratic optimal control problem (7.18)-(7.19) into quadratic programming problem. The optimal control problems (7.18)-(7.19) have to be solved successively until the difference $\mid \hat{J}^{(i+1)} - \hat{J}^{(i)} \mid$ is sufficiently small. $\hat{J}^i$ is the approximate value of J^i due to the parameterization approximation.

7.4 Determination of Optimal Feedback Gain

The idea to obtain the local optimal feedback control law is to use the results of the last quasilinearization iteration of the previous section, and then to linearize the system state equations and to expand the performance index up to the second order around this solution (i.e. apply the quasilinearization one more time).

Since we are using the Chebyshev polynomials, which are defined on the interval $\tau \in [-1,1]$, the time interval $t \in [0,t_f]$ of the optimal control problem is transformed into the interval $\tau \in [-1,1]$. The problem becomes: Find the optimal feedback control $u^*(x,\tau)$ on $-1 \leq \tau \leq 1$, that minimizes

$$J = \frac{1}{2}x^T(1)Sx(1) + \frac{1}{2}\frac{t_f}{2}\int_{-1}^{1}(x^TQx + u^TRu)d\tau \tag{7.23}$$

subject to

$$\frac{dx}{d\tau} = \frac{t_f}{2}\Big(A(\tau)x(\tau) + B(\tau)u(\tau) + h(\tau)\Big) \quad x(-1) = x_0 \tag{7.24}$$

Note that $A(\tau), B(\tau), h(\tau)$ are expressed in terms of the the optimal trajectories and optimal control determined in the previous section.

The necessary conditions to determine the optimal solution of this problem are the Euler-Lagrange equations given by

$$\dot{x}(\tau) = \frac{t_f}{2}\Big(A(\tau)x(\tau) + B(\tau)u(\tau) + h(\tau)\Big) \tag{7.25}$$

$$\dot{\lambda}(\tau) = \frac{t_f}{2}\Big(-Qx(\tau) - A(\tau)\lambda(\tau)\Big) \tag{7.26}$$

$$u(\tau) = -R^{-1}B^T(\tau)\lambda(\tau) \tag{7.27}$$

where $x(-1) = x_0$ and $\lambda(1) = Sx(1)$. The linear two-point boundary value problem (7.25)-(7.26) can be solved by assuming that $\lambda(\tau)$ has the form [65],

$$\lambda(\tau) = l(\tau) + K(\tau)x(\tau) \tag{7.28}$$

where $l(\tau)$ is an n vector and $K(\tau)$ is an $n \times n$ symmetric matrix. From (7.25), (7.26), (7.27) and (7.28), we can get the solution for $K(\tau)$ and $l(\tau)$ by solving the following differential equations

$$\dot{K}(\tau) = \frac{t_f}{2}\Big(K(\tau)B(\tau)R^{-1}B^T(\tau)K(\tau) - K(t)A(\tau) - A^T(\tau)K(\tau) - Q\Big) \tag{7.29}$$

$$\dot{l}(\tau) = \frac{t_f}{2}\Big(\Big[-A^T(\tau) + K(\tau)B(\tau)R^{-1}B^T(\tau)\Big]l(\tau) - K(\tau)h(\tau)\Big) \tag{7.30}$$

where $K(1) = S$, $l(1) = 0$. Equation (7.29) is the matrix Riccati equation, and equation (7.30) is a vector of n linear differential equations. These two equations will be solved using the Chebyshev polynomials.

The solution of the matrix Riccati equation can be obtained by solving the following system of linear differential equations [6],

$$\begin{bmatrix} \dot{U}(\tau) \\ \dot{V}(\tau) \end{bmatrix} = \frac{t_f}{2} \begin{bmatrix} A(\tau) & -B(\tau)R^{-1}B^T(\tau) \\ -Q & -A^T(\tau) \end{bmatrix} \begin{bmatrix} U(\tau) \\ V(\tau) \end{bmatrix} \tag{7.31}$$

where $U(\tau)$ and $V(\tau)$ are $n \times n$ matrices that satisfy the following boundary conditions

$$\begin{bmatrix} U(1) \\ V(1) \end{bmatrix} = \begin{bmatrix} I_{n\times n} \\ S \end{bmatrix} \tag{7.32}$$

where I is the identity matrix. The solution of the matrix Riccati equation (7.29) is then given by $K(\tau) = V(\tau)U^{-1}(\tau)$.

To solve equation (7.31) subject to (7.32) using Chebyshev polynomials, these equations are rewritten as,

$$\begin{aligned} \dot{Y}(\tau) &= F(\tau)Y(\tau) & (7.33)\\ Y(1) &= Y_f & (7.34) \end{aligned}$$

where

$$\begin{aligned} Y(\tau) &= \begin{bmatrix} U(\tau) \\ V(\tau) \end{bmatrix} \\ F(\tau) &= \frac{t_f}{2} \begin{bmatrix} A(\tau) & -B(\tau)R^{-1}B^T(\tau) \\ -Q & -A^T(\tau) \end{bmatrix} \\ Y_f &= \begin{bmatrix} I_{n\times n} \\ S \end{bmatrix} \end{aligned}$$

$Y(\tau)$ is $2n \times n$ matrix, $F(\tau)$ is $2n \times 2n$ matrix, and Y_f is $2n \times n$ matrix. Then each component of equation (7.33) can be approximated by Chebyshev series of finite length m. The approximation of $F(\tau)$ can be given by

$$F(\tau) = \frac{F_0}{2} + \sum_{i=1}^{m} F_i T_i(\tau) \tag{7.35}$$

which can be written in matrix form as

$$F = \begin{bmatrix} F_0/2 & F_1 & \cdots & F_m \end{bmatrix} T(\tau) \tag{7.36}$$

where F_i, $i = 0, 1, \cdots, m$ is $2n \times 2n$ matrix of known parameters which can be determined from (7.3). Also the matrix $Y(\tau)$ can be approximated by a Chebyshev series as follows,

$$Y(\tau) = T^T(\tau) \begin{bmatrix} Y_0/2 & Y_1 & \cdots & Y_m \end{bmatrix}^T \tag{7.37}$$

where Y_i, $i = 0, 1, \cdots, m$ is $2n \times n$ matrix of unknown parameters.

Using the differentiation operational matrix, $\dot{Y}(\tau)$ can be determined,

$$\dot{Y}(\tau) = T^T(\tau)D^T \left[\begin{array}{cccc} Y_0/2 & Y_1 & \cdots & Y_m \end{array} \right]^T \tag{7.38}$$

Substituting (7.36), (7.37) and (7.38) into (7.33) yields

$$T^T(\tau)D^T[Y_0/2\ Y_1\ \cdots Y_m]^T = [F_0/2\ F_1\ \cdots F_m]T(\tau)T^T(\tau)[Y_0/2\ Y_1\ \cdots\ Y_m]^T \tag{7.39}$$

the right hand side can be simplified using the result of [64], although this result is derived for Chebyshev polynomials of the second type, it will be the same for the Chebyshev polynomials of first type also,

$$[F_0/2\ F_1\ \cdots F_m]T(\tau)T^T(\tau) = T^T(\tau)\tilde{F} \tag{7.40}$$

where

$$\tilde{F} = \begin{bmatrix} F_0/2 & F_1/2 & F_2/2 & \cdots & F_m/2 \\ F_1 & (F_0+F_2)/2 & (F_1+F_3)/2 & \cdots & F_{m-1}/2 \\ F_2 & (F_1+F_3)/2 & (F_0+F_4)/2 & \cdots & F_{m-2}/2 \\ F_3 & (F_2+F_4)/2 & (F_1+F_5)/2 & \cdots & F_{m-3}/2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ F_m & F_{m-1}/2 & F_{m-2}/2 & \cdots & F_0/2 \end{bmatrix} \tag{7.41}$$

Hence equation (7.39) is reduced to

$$T^T(\tau)D^T[Y_0/2\ Y_1 \cdots Y_m]^T = T^T(\tau)\tilde{F}[Y_0/2\ Y_1 \cdots Y_m]^T \tag{7.42}$$

The left hand side of this equation is a polynomial of order $m-1$, because the last row of D^T is zero, while the right hand side is a polynomial of order m. We will equate the coefficients of Chebyshev polynomials up to order $m-1$, after [68], to get

$$\hat{D}^T[Y_0/2\ Y_1\ \cdots\ Y_m]^T = \hat{F}[Y_0/2\ Y_1\ \cdots\ Y_m]^T \tag{7.43}$$

where $\hat{D}^T$ and $\hat{F}$ are D^T and $\tilde{F}$ respectively, but with the last row discarded in both matrices.

The final condition (7.34), can also be expressed using the Chebyshev polynomials property at $\tau = 1$,

$$Y(1) = Y_0/2 + Y_1 + \cdots + Y_m = Y_f \tag{7.44}$$

This condition must also be satisfied to find solution of equation (7.33). Combining this condition with (7.43) gives

$$D_{mod}\begin{bmatrix} Y_0/2 \\ Y_1 \\ \vdots \\ Y_{m-1} \\ Y_m \end{bmatrix} = \tilde{F}_{mod}\begin{bmatrix} Y_0/2 \\ Y_1 \\ \vdots \\ Y_{m-1} \\ Y_m \end{bmatrix} + \begin{bmatrix} 0_{2n\times n} \\ 0_{2n\times n} \\ \vdots \\ 0_{2n\times n} \\ Y_f \end{bmatrix} \tag{7.45}$$

Note that in this equation all the multiplications has to be performed block-wise. To allow element-wise multiplications, the left hand side can be rewritten as follows,

$$(D_{mod}\otimes I_{2n})\begin{bmatrix} Y_0/2 \\ Y_1 \\ \vdots \\ Y_{m-1} \\ Y_m \end{bmatrix} = \tilde{F}_{mod}\begin{bmatrix} Y_0/2 \\ Y_1 \\ \vdots \\ Y_{m-1} \\ Y_m \end{bmatrix} + \begin{bmatrix} 0_{2n\times n} \\ 0_{2n\times n} \\ \vdots \\ 0_{2n\times n} \\ Y_f \end{bmatrix} \tag{7.46}$$

where D_{mod} is D^T but with the last row is replaced by a row of 1's, $\tilde{F}_{mod}$ is $\tilde{F}$ but with the last row is replaced by 0's, each 0 is $2n \times 2n$ matrix with all its elements are zeros, and $\otimes$ denotes the Kronecker product. From equation (7.46), the solution of the unknown parameters can be obtained,

$$\begin{bmatrix} Y_0/2 \\ Y_1 \\ \vdots \\ Y_{m-1} \\ Y_m \end{bmatrix} = \Big((D_{mod}\otimes I_{2n}) - \tilde{F}_{mod}\Big)^{-1}\begin{bmatrix} 0_{2n\times n} \\ 0_{2n\times n} \\ \vdots \\ 0_{2n\times n} \\ Y_f \end{bmatrix} \tag{7.47}$$

and the matrices $U(\tau)$, $V(\tau)$ and $K(\tau)$ can be determined.

Another equation (7.30) has also to be solved for $l(\tau)$,

$$\dot{l}(\tau) = M(\tau)l(\tau) - H(\tau) \quad l(1) = 0 \tag{7.48}$$

where

$$M(\tau) = \frac{t_f}{2}\Big(-A^T(\tau) + K(\tau)B(\tau)R^{-1}B^T(\tau)\Big) \tag{7.49}$$

$$H(\tau) = \frac{t_f}{2}K(\tau)h(\tau) \tag{7.50}$$

$M(\tau)$ and $H(\tau)$ are known because $K(\tau)$ is known after solving (7.47).

To find the solution of (7.48), $M(\tau), l(\tau)$ and $H(\tau)$ are approximated by Chebyshev series of finite length m as follows,

$$\begin{aligned} M(\tau) &= [M_0/2\ M_1\ \cdots\ M_m]T(\tau) & (7.51)\\ H(\tau) &= T^T(\tau)[H_0/2\ H_1\ \cdots H_m]^T & (7.52)\\ l(\tau) &= T^T(\tau)[l_0/2\ l_1 \cdots l_m]^T & (7.53)\end{aligned}$$

where $M_i, i = 0, 1, \cdots, m$ is $n \times n$ matrix of known parameters, $H_i, i = 0, 1, \cdots, m$ is $n \times 1$ vector of known parameters and $l_i, i = 0, 1, \cdots, m$ is also $n \times 1$ vector of unknown parameters. From (7.53), $\dot{l}(\tau)$ can be expressed as

$$\dot{l}(\tau) = T^T(\tau)D^T[l_0/2\ l_1\ \cdots\ l_m]^T \tag{7.54}$$

Substituting (7.51),(7.52), (7.53) and (7.54) into (7.48) and using the fact that

$$[M_0/2\ M_1\ \cdots M_m]T(\tau)T^T(\tau) = T^T(\tau)\tilde{M}$$

gives

$$T^T(\tau)D^T[l_0/2\ l_1\ \cdots\ l_m]^T = T^T(\tau)\tilde{M}[l_0/2\ l_1\ \cdots\ l_m]^T - T^T(\tau)[\frac{H_0}{2}\ H_1\ \cdots H_m]^T \tag{7.55}$$

where $\tilde{M}$ is defined the same way as $\tilde{F}$. The left hand side is a polynomial of length $m-1$ while the right hand side is a polynomial of length m. Equating the coefficients of the first $m-1$ Chebyshev polynomials and taking into account the final conditions of $l(1) = 0$, gives

$$\begin{bmatrix} l_0/2 \\ l_1 \\ \vdots \\ l_m \end{bmatrix} = -\Big((D_{mod} \otimes I_n) - \tilde{M}_{mod}\Big)^{-1} \begin{bmatrix} H_0/2 \\ H_1 \\ \vdots \\ H_{m-1} \\ 0 \end{bmatrix} \tag{7.56}$$

where $\tilde{M}_{mod}$ is $\tilde{M}$ but with the last row is replaced by 0's, each 0 is $n \times n$ matrix with all its elements are zeros.

After obtaining $K(\tau)$ and $l(\tau)$, the optimal feedback control law can be formed from (7.27) and (7.28),

$$u(x,\tau) = -R^{-1}B^T\Big(l(\tau) + K(\tau)x(\tau)\Big) \tag{7.57}$$

which can be expressed in terms of $t \in [0, t_f]$, by the transformation $\tau = \frac{2t}{t_f} - 1$, as follows,

$$u(x,t) = -L(t) - K'(t)x(t) \tag{7.58}$$

This control law is a nonlinear optimal control law of the nonlinear optimal control problem, although it appears as linear in states. The nonlinear parts enter the time varying terms. A block diagram showing the optimal feedback control is depicted in Figure 7.1.

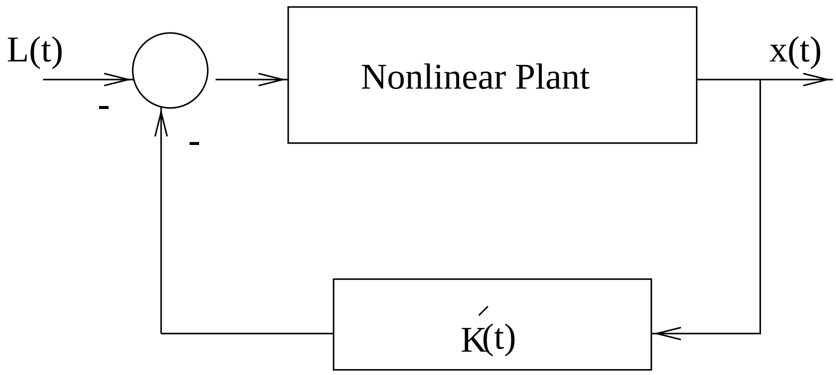

Figure 7.1: Optimal feedback control

7.5 Computational results

Example 1: Time-varying linear quadratic probelm:
Find the optimal feedback control $u*(x,t)$ the minimizes

$$J = \frac{1}{2}\int_0^1 \Big(x^2(t) + u^2(t)\Big)dt \tag{7.59}$$

subject to

$$\dot{x} = tx(t) + u(t) \tag{7.60}$$

After Changing the time interval from $t \in [0,1]$ into $\tau \in [-1,1]$, we used the Chebyshev polynomials of the 3rd order to solve the Riccati equation. The optimal feedback control is

$$u^*(x,\tau) = \frac{0.5179 - 0.5243T_1 + 0.0060T_2 + 0.0003T_3}{0.9187 - 0.04643T_1 + 0.1239T_2 + 0.0037T_3}x(\tau) \tag{7.61}$$

The exact solution of Riccati equation using numerical integration and the approximate solution using the proposed method are shown in Figure 7.2

Example 2: Van der Pol problem
In this section, we consider finding the optimal feedback control law for the Van der Pol oscillator problem. The system state equations are:

$$\dot{x}_1 = x_2 \tag{7.62}$$

$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + u \tag{7.63}$$

The cost function to be minimized, starting from the initial states $x_1(0) = 1$ and $x_2(0) = 0$, is:

$$J = \frac{1}{2}\int_0^5 (x_1^2 + x_2^2 + u^2)dt \tag{7.64}$$

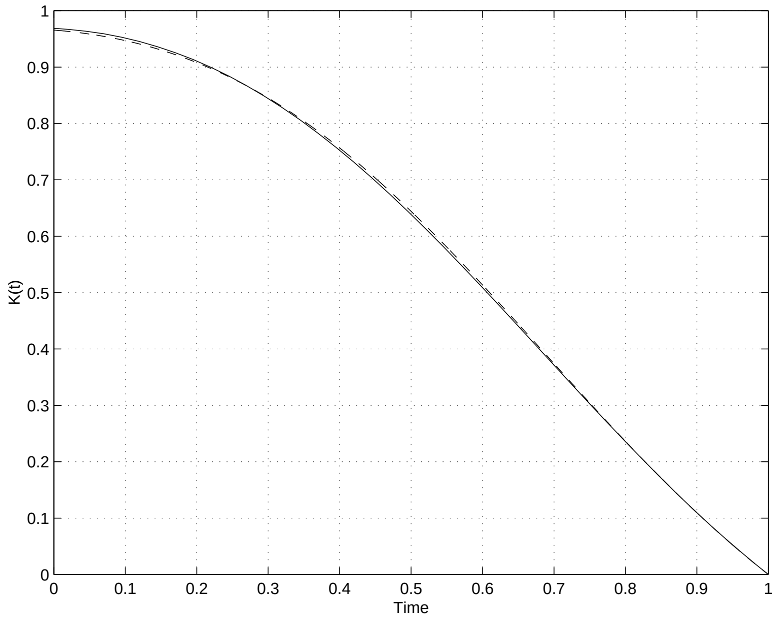

Figure 7.2: Solution of Riccati equation (—) using numerical integration, (– – –) using the proposed algorithm (Example 1)

The first step to solve this problem is to obtain the optimal open loop solution. This was obtained in the last chapter. The open loop optimal trajectories are given by

$$\begin{aligned} x_1^* &= \Big[0.30483 \quad -0.559136 \quad 0.211342 \quad 0.012940 \quad -0.037916 \\ &\quad 0.019442 \quad -0.004024 \quad 0.000828 \quad 0.000168 \quad -0.000011\Big] T(\tau) \\ x_2^* &= \Big[-0.166962 \quad 0.196425 \quad 0.113385 \quad -0.141723 \quad 0.082327 \\ &\quad -0.020392 \quad 0.004559 \quad -0.001078 \quad -0.000077 \quad 0\Big] T(\tau) \end{aligned}$$

The second step is to find the optimal feedback control. To this end, the quasilinearization is applied one more time i.e. to expand the performance index and the state equations , around the optimal trajectories and optimal control, up to the second order and up to the first order respectively. We get the following problem: Find the optimal feedback control $u^*(x,\tau)$ of the system

$$\begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & 2.5 \\ 2.5(-1-2x_1^*x_2^*) & 2.5(1-x_1^{*2}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} u + \begin{bmatrix} 0 \\ 5x_1^{*2}x_2^* \end{bmatrix} \tag{7.65}$$

and the performance index is

$$J = \frac{1}{2}\int_{-1}^{1}\left(\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2.5u^2\right) d\tau \tag{7.66}$$

The feedback control can be obtained for this system by solving the following matrix Riccati equation for $K(\tau)$ and 2 linear differential equations for $l(\tau)$,

$$\dot{K}_{11}(\tau) = 2.5K_{12}K_{21} - K_{12}A_{21} - K_{21}A_{21} - 2.5 \quad (7.67)$$
$$\dot{K}_{12}(\tau) = 2.5K_{12}K_{22} - (K_{11}A_{12} + K_{12}A_{22}) - K_{22}A_{21} \quad (7.68)$$
$$\dot{K}_{21}(\tau) = 2.5K_{21}K_{22} - K_{22}A_{21} - (K_{11}A_{12} + K_{21}A_{22}) \quad (7.69)$$
$$\dot{K}_{22}(\tau) = 2.5K_{22}^2 - (K_{21}A_{12} + K_{22}A_{22}) - (K_{12}A_{12} + K_{22}A_{22}) - 2.5 \quad (7.70)$$

$$\dot{l}_1(\tau) = M_{12}l_2(\tau) - H_{11} \quad (7.71)$$
$$\dot{l}_2(\tau) = M_{21}l_1(\tau) + M_{22}l_2(\tau) - H_{21} \quad (7.72)$$

where

$$\begin{aligned}
A_{12} &= 2.5 \\
A_{21} &= 2.5(-1 - 2x_1^* x_2^*) \\
A_{22} &= 2.5(1 - x_1^{*2}) \\
M_{12} &= -A_{21} + 2.5K_{12}(\tau) \\
M_{21} &= -A_{12} \\
M_{22} &= -A_{22} + 2.5K_{22}(\tau) \\
H_{11} &= 5K_{12}(\tau)x_1^{*2}x_2^* \\
H_{21} &= 5K_{22}(\tau)x_1^{*2}x_2^*
\end{aligned}$$

The final conditions of the Riccati equations are $K_{11}(1) = K_{12}(1) = K_{21}(1) = K_{22}(1) = 0$ and the final conditions of $l(1)$ are $l_1(1) = l_2(1) = 0$.

To determine $K(\tau)$ using the method of this chapter, each of A_{21} and A_{22} is approximated by a Chebyshev polynomial of order 10, and the matrix F is constructed. After solving equation (7.47), the matrices $U(\tau)$ and $V(\tau)$ are extracted from the matrix $Y(\tau)$, then $K(\tau)$ is determined.

The exact solution of equations (7.67)-(7.70) which is obtained by backward numerical integration starting from $K(1) = 0$, and the solution obtained using the algorithm proposed in this chapter are shown in Figure 7.3

To determine $l(\tau)$, each of M_{12}, M_{21}, M_{22}, H_{11} and H_{21} is approximated by a Chebyshev polynomial of order 10. The matrix of the unknown parameters of $l(\tau)$ is determined from equation (7.56). Figure 7.4 shows the exact solution of equations (7.71) and (7.72) using backward numerical integration starting from $l(1) = 0$ and the solution obtained using the proposed method.

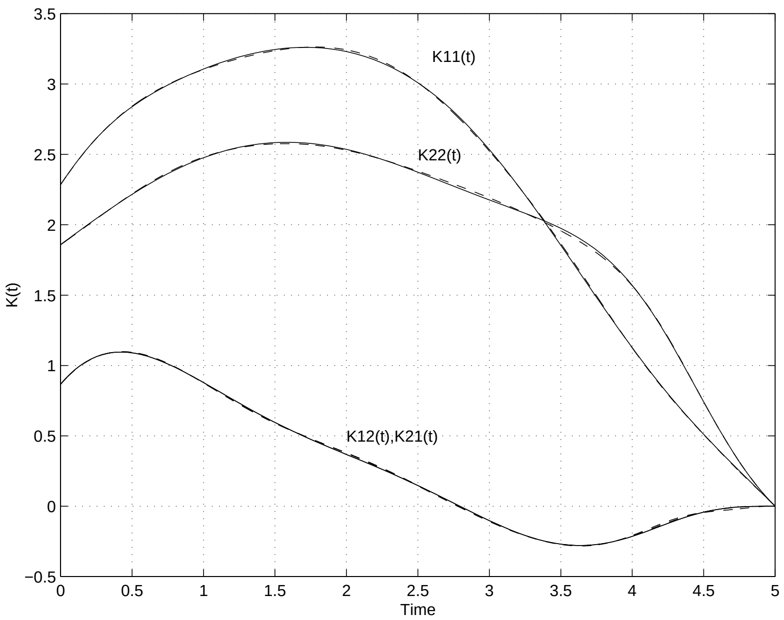

Figure 7.3: Solution of Riccati equation (—) using numerical integration, (− − −) using the proposed algorithm (Example 2)

Using $l(t)$ and $K(t)$, The feedback control can be constructed. Figure 7.5 shows the open loop optimal control and optimal trajectories, it also shows the closed loop control law and the corresponding trajectories. From this figure, it is clear that the optimal open loop control is expressed accurately by the optimal feedback control, and the optimal trajectories of the closed loop system match accurately with those of the open loop system.

To show that the derived feedback control law can stabilize the system starting from different initial conditions, Figure 7.6 compares the feedback control and the state trajectories starting from the initials conditions $[1, 0]$ and $[2, -1]$. It is interesting here to notice that the initial condition $[1, 0]$ is inside the limit cycle of the unforced system u=0, while the initial condition $[2, -1]$ is outside the limit cycle.

7.6 Conclusion

A method is proposed to determine the optimal feedback control law of the nonlinear optimal control problem. Using this method, we do not need to integrate the matrix Riccati equation and the associated n linear differential equations. The simulation results show that the closed loop control and the corresponding state trajectories approximate the open loop optimal control and trajectories. Also using the closed loop control, the system can be stabilized locally starting from different initial conditions.

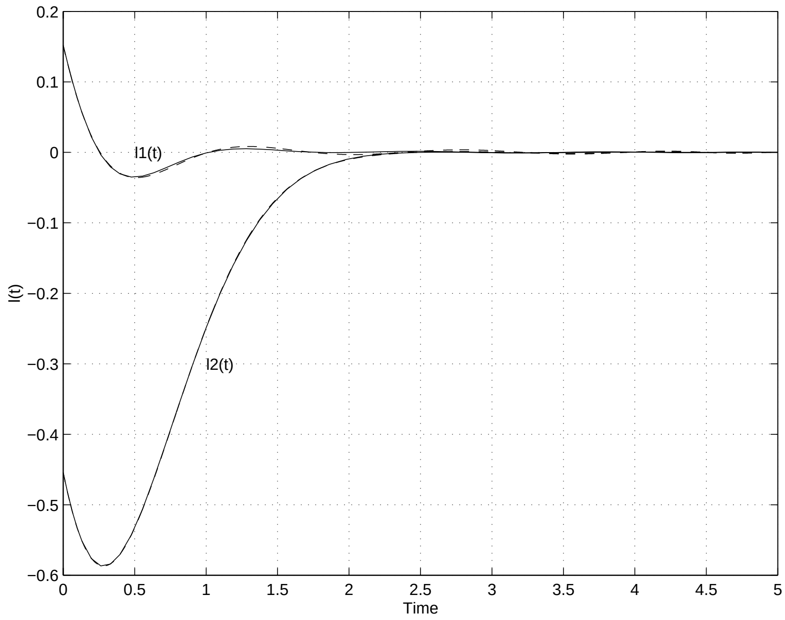

Figure 7.4: Solution of $l(t)$ (—) using numerical integration, $(- - -)$ using the proposed algorithm

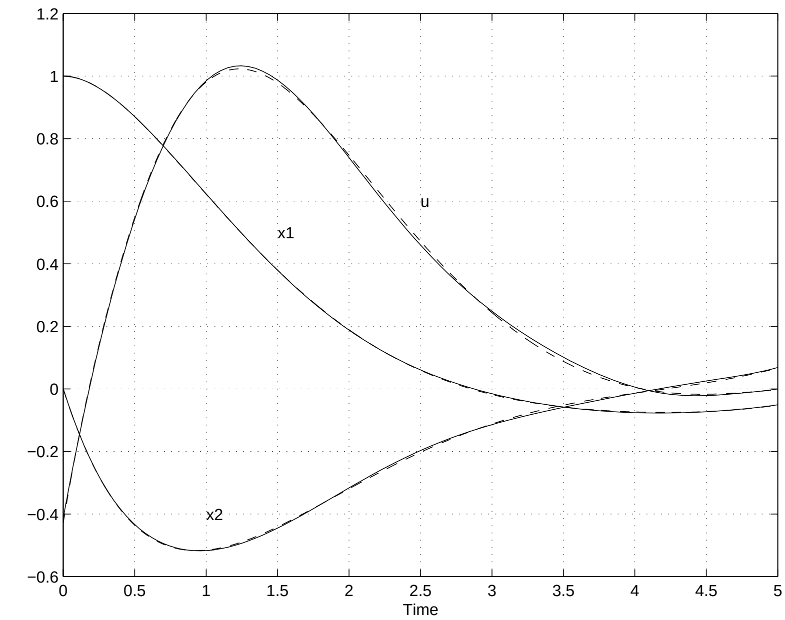

Figure 7.5: (—) open loop control and open loop trajectories, $(- - -)$ closed loop control and trajectories

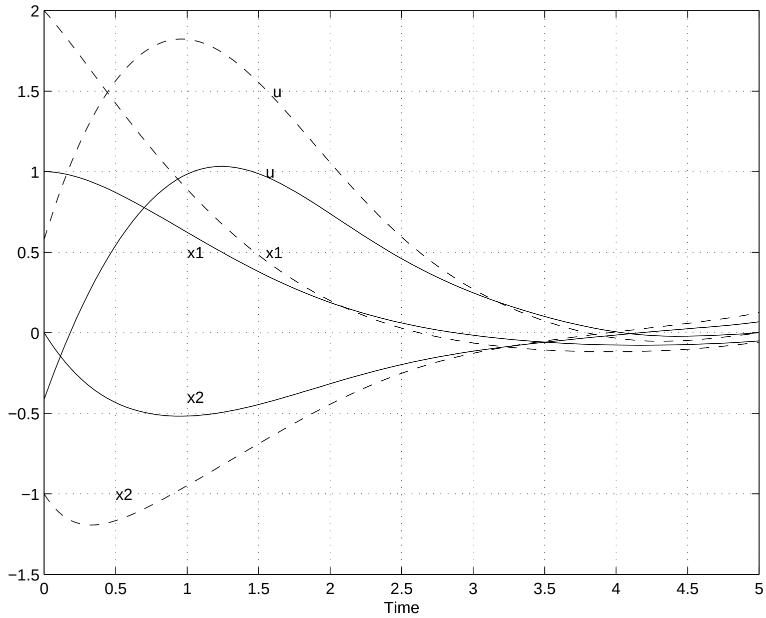

Figure 7.6: (—) closed loop control and trajectories from the initial conditions $[1, 0]$, $(- - -)$ closed loop control and trajectories from the initial conditions $[2, -1]$

Chapter 8

Conclusions and Future Work

8.1 Conclusions

In this thesis, we proposed numerical methods to solve several types of optimal control problems. These methods are based on using the second method of quasilinearization and on parameterizing the system state variables using Chebyshev polynomials of the first type. And then the control variables are obtained from the system state equations. Therefore, the system state equations, in most cases, are satisfied directly and will not be replaced by a large number of equality constraints. The use of the state parameterization is motivated by several advantages it offers compared with control parameterization and control-state parameterization.

Applying the proposed methods, convert the linear optimal control problem into quadratic programming problem and convert the nonlinear optimal control problem into sequence of quadratic optimal control problems.

The numerical methods proposed in this thesis have the following advantages: Easy method of approximation; no integration of the state equations or costate equations is needed; explicit formula is derived to approximate the quadratic performance index; small quadratic programming problems are to be solved.

Although we have no mathematical proof of the convergence of the proposed algorithms, a property that is shared by many other numerical methods to solve the optimal control problem, we applied our methods on several test examples which were solved by other researchers using different methods. Also we applied the proposed methods on two real practical optimal control problems: F8 fighter aircraft and container crane problems. From the computational results obtained for these examples and problems, we can conclude that the proposed algorithms give better or comparable results compared with some other methods.

The solutions of the optimal control problems in Chapters 3, 4, 5 and 6 are open loop solutions. But the feedback solution is desired to obtain because of several advantages it can offer, therefore in Chapter 7, we proposed a method to give an optimal feedback control solution of nonlinear optimal control problems and as a special case the optimal feedback control of time-varying linear optimal control problem. The idea to obtain the optimal feedback control is to solve the problem successively as proposed in Chapter 3 and 4 and then, when acceptable convergence error is achieved, we perform the second method of the quasilinearization once more to obtain the feedback control law. To the best of our knowledge this is the first time the orthogonal polynomials are used in computing the optimal feedback of nonlinear optimal control problems.

To facilitate the computation of the optimal feedback control, we derived a new property of Chebyshev polynomials called differentiation operational matrix. The proposed algorithm to find the optimal feedback control is based on using the differentiation operational matrix.

The differentiation operational matrix can be derived for several other orthogonal polynomials which can be used to solve several problems. Therefore, we believe that the derivation of this new property of Chebyshev polynomials will lead to active research based on this property.

8.2 Future Work

The work of this thesis can be extended in two ways:

- As we mention earlier, still we do not have mathematical proof of the convergence of the proposed method. Therefore one of the problems that can be treated in future studies is the convergence of the proposed algorithm. This problem is not trivial.
- In chapter 7, we proposed a method to find the optimal feedback control of nonlinear optimal control problem without considering any constraints. This can be extended to find the optimal feedback of nonlinear problems subject to constraints.

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