Title	同一結合子を持つ非古典論理とその代数的特徴付け
Author(s)	石井，忠夫
Citation	
Issue Date	2000-03
Type	Thesis or Dissertation
Text version	author
URL	http://hdl.handle.net/10119/898
Rights	
Description	Supervisor:小野 寛晰，情報科学研究科，博士

Nonclassical logics with identity connective and their algebraic characterization

by

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submitted to
Japan Advanced Institute of Science and Technology
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

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March 2000

Abstract

In this thesis, we investigate various kinds of nonclassical logics by the property of identity connective. Around 1970, R. Suszko proposed the sentential calculus with identity (**SCI** for short) to realize some philosophical ideas of L. Wittgenstein's *Tractatus*. In **SCI**, besides the logical value, he also formalized the *referent* of sentences by using identity connective. Inspired by his idea, we introduce a weak system, i.e., propositional calculus with identity (**PCI** for short), which is obtained from **SCI** by deleting two axioms which express the reflexivity and transitivity of identity. As an extension of the *simulation* property of **SCI**, we reconstruct various kinds of nonclassical logics on **PCI**, including two types of logics, namely classical logics with additional operators and weak logics with various kinds of weak implications, e.g., strict/relevance/linear implication. In fact, in this thesis we show that the following logics can be translated to some extensions of **PCI**; classical modal logics **K**, **KT**, **KB**, **K4**, **KD**, **K5**, **S4** and **S5** with necessary operator $\Box$, Angell's analytic containment logic **AC** with relevance entailment $\rightsquigarrow$, Corsi's weak logic **F** with strict implication $\rightharpoonup$ and Girard's classical linear logic **GL** with linear implication $\supset$. In particular, the modal logic **K** is shown to be translated into an extension $\mathbf{PCI}_{\mathrm{K}}$ of **PCI**. Then we will focus on the algebraic property of $\mathbf{PCI}_{\mathrm{K}}$-algebras, which offer the algebraic semantics of extensions of $\mathbf{PCI}_{\mathrm{K}}$. We will give a necessary and sufficient condition for a subvariety of $\mathbf{PCI}_{\mathrm{K}}$-algebras to have *equationally definable principal congruences* (EDPC for short) property.

Keywords : EDPC, identity connective, nonclassical logic, non-Fregean logic, SCI, Suszko, PCI

Acknowledgments

First of all, I would like to express my thanks to my principal advisor Professor Hiroakira Ono for his advice, suggestions and constant encouragements during this work. In fact, Professor Hiroakira Ono gave me many advice in our OIL (Ono-Ishihara laborartory) seminar, and he also gave me the first direction of the development of $\mathbf{PCI}_{\mathrm{K}}$ logics. I would also like to thank my advisor Associate Professor Hajime Ishihara for his helpful discussions and suggestions. In our OIL seminar, his keenly advice were vary useful to understand logics. Moreover, I am grateful to Dr. Ryo Kashima, Dr. Piotr Łukowski and Dr. Tomasz Kowalski. Dr. Ryo Kashima gave me useful comments and suggestions in several seminar of JAIST. Dr. Piotr Łukowski gave me constant encouragements and he was also a pioneer of **SCI** logics for me. Dr. Tomasz Kowalski taught me the algebraic method for investigating $\mathbf{PCI}_{\mathrm{K}}$ logics and he suggested also me the idea of proving that $\mathbf{PCI}_{\mathrm{K}}$ algebras have EDPC property. Finally, I am also grateful to all presentators in AL and OIL seminar in JAIST for their interesting presentations.

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Chapter 1

Introduction

In this chapter we will first explain the motivation and short historical background of the thesis. Nowadays the main current of logic is a mathematical logic which is evolved from the Hilbert's formalism. In particular, the nonclassical logic is the best subject in logical field, which is opposite to the classical logic, in which it assumes that all propositions must have *true* or *false* logical values (called the law of *excluded middle*), and most of mathematics admit this logic to construct their proofs. It is included in the nonclassical logic that intuitionistic logic, modal logic, temporal logic, many-valued logic, relevance logic, quantum logic, knowledge and belief logic, and so on. The classification of above logics depends on the difference between objects that each logic deals with. In general, since the logic can be seen as the subject of formalization, there exist many logics which depend on the formalization method. Our main interest in this thesis is to construct the primary logic which is the fundamentals of the above all logics. Namely, in general, the construction of logic will be obtained from something knowledge acquisition (perception), and usually we can consider two methods of the formalization, in which *identity* and *distinction* (that is a dual notion of identity) are assumed as the primary perception. The first method is also called *Leibniz's principle of identity*. Our research is mainly concerned with the former approach of the above formalization. At first, we will survey briefly Frege, Wittgenstein and Suszko's results as the former approaches (Section 1). Then, in Section 2, we will give an outline of our main results and an overview of the thesis.

1.1 Motivation and history

In [29], G. Frege analyzed the distinction between *sense* (or *Sinn*) and *reference* (or *Bedeutung*) of names, by using his famous morning/evening-star example. Frege claimed that all logically true (and , similarly, all false) sentences describe the same thing, namely, have a common referent, while it is possible that two names with different senses have a common referent. His theory is based on Leibniz's principle of identity with respect to sense of names, and he thought that the identity sentence $A \equiv B$ is logically true and also meaningful, different from the trivial case $A \equiv A$, only because of the above assumption.

Against the above Frege's principle of logical two-valuedness, L. Wittgenstein proposed in *Tractatus* the *picture theory of meaning* based on logical atomism (or *Sachverhalt*) and its composition, i.e., facts (or *Tatsache*), under an insight that the true property of language is like a mirror to reflect the world. It appears the following theses in *Tractatus* (see [74] and [77]).

1.1 The world is the totality of facts, not of things.

1.2 The world divides into facts.

Here Thesis 1.1 proposes an *ontology of facts*, and Thesis 1.2 proposes a variant of it, known as *logical atomism*. How to get composite facts from logical atoms was really constructed by using only Sheffer's stroke function in *Tractatus*. By the inspiration of L. Wittgenstein's *Tractatus* that facts are constructed by *states of affairs* (or *situations*), R. Suszko formalized an ontology of facts in *Tractatus* on the basis of Fregean scheme below, and called it *non-Fregean logic* (see [63], [64] and [67]).

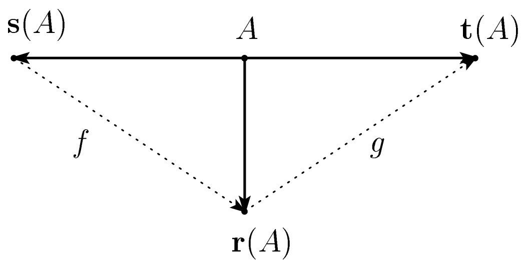

Figure 1.1: Fregean scheme

For any sentence A, let $\mathbf{r}(A)$ be the *referent* of A, i.e., what is given by A, $\mathbf{s}(A)$ the *sense* of A, i.e., the way $\mathbf{r}(A)$ is given by A, and $\mathbf{t}(A)$ the *logical value* of A, i.e., $(\{\mathrm{t}, \mathrm{f}\})$. Then, it is assumed that assignments $\mathbf{s}$, $\mathbf{r}$ and $\mathbf{t}$ are related as follows: for any sentences A and B,

(1) $\mathbf{s}(A) = \mathbf{s}(B)$ implies $\mathbf{r}(A) = \mathbf{r}(B)$,

(2) $\mathbf{r}(A) = \mathbf{r}(B)$ implies $\mathbf{t}(A) = \mathbf{t}(B)$.

Here the converse of (2) means that sentences with the same logical value have a common referent, and slightly weakens the original Frege's claim. Suszko introduced the identity connective $\equiv$ to represent the sameness of referent for sentences, i.e., $\mathbf{t}(A \equiv B) = \mathrm{t}$ if and only if $\mathbf{r}(A) = \mathbf{r}(B)$, while the matreial equivalence connective $\leftrightarrow$ represents the identification of logical value. Moreover, as followed Frege's treatment of identity, Suszko assumed that identity should satisfy Leibniz's principle with respect to referent of sentences below:

$A1$ $\mathbf{t}(A \equiv A) = \mathrm{t}$,

A2 $\mathbf{t}(A \equiv B) = \mathrm{t}$ implies $\mathbf{t}(G[A/p] \equiv G[B/p]) = \mathrm{t}$,
where $G[A/p]$ means the formula obtained from G by replacing each occurrence of p by A.

Then, we have also the following axiom from the assumption (2) above:

A3 $\mathbf{t}(A \equiv B) = \mathrm{t}$ implies $\mathbf{t}(A \leftrightarrow B) = \mathrm{t}$.

For example, if A="I was in Rome." and B="I was in the capital of Italy.", then we have $A \equiv B$ becuase that "Rome" and "the capital of Italy" have a common referent.

In the result, the language of his most simple system (i.e., not include any quantifier formator) consists of $\mathcal{L}_{\mathrm{S}} = \langle \mathrm{L}_{\mathrm{S}}, \neg, \wedge, \vee, \rightarrow, \equiv, \bot, \top \rangle$, and the system on $\mathcal{L}_{\mathrm{S}}$ is called the *sentential calculus with identity* (**SCI** for short). Suszko devoted much of his interest to *theories of situation*, which constructed through adding a certain system of axioms to **SCI**. Then, a typical feature of **SCI** is the ability that some kinds of nonclassical logics can be reconstructed on it. We call them the *simulation property* of **SCI**. In fact, R. Suszko showed that some extensions of **SCI** really correspond to modal systems **S4** and **S5**, if we interpret $A \equiv B$ as $\Box(A \leftrightarrow B)$ (see [65]), and moreover, **SCI** itself also correspond to the three-valued Łukasiewicz logic $\mathbf{Ł_3}$ if we interpret $A \equiv B$ as $(A \Leftrightarrow B)$, where $\Leftrightarrow$ is a three-valued Łukasiewicz equivalence (see [68] and [69]). So inspired by his idea, we first introduce a weak system, i.e., propositional calculus with identity (**PCI** for short), which is obtained from **SCI** by deleting two identity axioms of reflexivity and transitivity. Then, our main purposes of this thesis are to simulate uniformly various kinds of nonclassical logics as the extensions of **PCI** logics, and furthermore, we will investigate what properties of the identity connective we really need to reconstruct each kind of nonclassical logics on **PCI**.

1.2 Main results of the thesis and overview

In general, nonclassical logics are divided into two types according to the construction, i.e., (i): classical logics with additional operators and (ii): weak logics with various kinds of weak implication, e.g., strict/relevance/linear implication. In fact, in this thesis we will consider classical modal logics with necessary operator $\Box$ (see [15]) as the former type, and Angell's analytic containment logic **AC** with relevance entailment (see [2] and [3]), Corsi's weak logic **F** with strict implication (see [16]) and Girard's classical linear logic **GL** with linear implication (see [31]) as the latter type. Then, we will give the following results on some extensions of **PCI** by defining precisely the simulation property as the syntactical equivalence between two logics (see Definition 3.4.1).

(1) Classical modal logics with necessary operator $\Box$:
We define $\mathbf{PCI}_{\mathrm{K}}$ logic as an extension of **PCI** in order to interpret the sameness of modal necessitation $\Box$ by identity $\equiv$. Here we add two identity axioms (WIA1) and

(WIA2), and one inference rule (G) into the original **PCI**, to satisfy the following conditions:

(R3) $\overrightarrow{\Box\alpha} \longmapsto (\overrightarrow{\alpha} \equiv \top)$,

(R4) $\overleftarrow{A \equiv B} \longmapsto \Box(\overleftarrow{A} \leftrightarrow \overleftarrow{B})$,

where $\overrightarrow{\alpha}$ and $\overleftarrow{A}$, $\overleftarrow{B}$ denote the results of translations from **K** to $\mathbf{PCI}_{\mathrm{K}}$, and its converse, respectively.

(WIA1) $((A \to B) \equiv (B \to A)) \to (A \equiv B)$

(WIA2) $((A \to B) \equiv (B \to A)) \to ((A \to B) \equiv \top) \wedge ((B \to A) \equiv \top)$

(G) $\dfrac{A \quad B}{A \equiv B}$

Then, by using above translations, we show that they are syntactically equivalent in a sense of Definition 3.4.1. Moreover, we introduce Kripke type semantics for $\mathbf{PCI}_{\mathrm{K}}$ in the similar way to the modal Kripke type semantics, and by invoking the completeness result of modal logic, we give a completeness theorem of $\mathbf{PCI}_{\mathrm{K}}$ relative to Kripke type semantics. Furthermore, we define $\mathbf{PCI}_{\mathrm{K}}$-algebras which provide an algebraic semantics for $\mathbf{PCI}_{\mathrm{K}}$ logic, and show the representation theorem of $\mathbf{PCI}_{\mathrm{K}}$-algebras in the similar way to the case of modal algebras, and also give an alternative completeness result of $\mathbf{PCI}_{\mathrm{K}}$ logic by using this representation theorem. At the end, we disccuss that all results mentioned so far, can also be extended to various extensions (e.g., **KT**, **KB**, **K4**, **KD**, **K5**, **S4** and **S5**) of modal logics.

(2) Angell's analytic containment logic with relevance entailment $\rightsquigarrow$:
In the similar way to the previous case, we define $\mathbf{PCI}_{\mathrm{W}}$ logic as an extension of **PCI** in order to interpret Angell's analytic containment $\approx$ by identity $\equiv$. Here we add two identity axioms (IT) and (IR) into the original **PCI**, to satisfy the following conditions:

(R1) $\overrightarrow{\alpha \rightsquigarrow \beta} \longmapsto \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta}$,

(R2) $\overleftarrow{A \equiv B} \longmapsto (\overleftarrow{A} \approx \overleftarrow{B})$,

where $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$ and $\overleftarrow{A}$, $\overleftarrow{B}$ denote the results of translations from **AC** to $\mathbf{PCI}_{\mathrm{W}}$, and its converse, respectively.

(IT) $(A \equiv B) \wedge (C \equiv D) \to (A \equiv C) \equiv (B \equiv D)$

(IR) $(A \equiv B) \to (A \leftrightarrow B)$

Then, the extension $\mathbf{PCI}_{\mathrm{W}}$ of **PCI** is nothing but the non-Fregean logic **SCI** mentioned in Section 1.1. We show that **AC** and $\mathbf{PCI}_{\mathrm{W}}$ are syntactically equivalent in a sense of Definition 3.4.1, by defining translations between them, which satisfy the above requirements (R1) and (R2).

(3) Corsi's weak logic with strict implication $\rightharpoonup$:
In this case in order to interpret the strict implication $\rightharpoonup$ by identity $\equiv$, we need the following conditions in **PCI**:

(R5) $\overrightarrow{\alpha \rightharpoonup \beta} \longmapsto \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta}$,

(R6) $\overleftarrow{A \equiv B} \longmapsto (\overleftarrow{A} \rightleftharpoons \overleftarrow{B})$,

where $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$ and $\overleftarrow{A}$, $\overleftarrow{B}$ denote the results of translations from **F** to $\mathbf{PCI}_{\mathrm{K}}$, and its converse, respectively.

Since we can rewrite the second requirement (R6) by $(\overleftarrow{A} \rightleftharpoons \overleftarrow{B})$ iff $\Box(\overleftarrow{A} \leftrightarrow \overleftarrow{B})$ in the sight of both Kripke model between **F** and **K**, the above requirements (R5) and (R6) are reduced to (R3) and (R4). Therefore, $\mathbf{PCI}_{\mathrm{K}}$ logic, introduced in the case (1) above, can also use to this case. We show that every formulas in **F**-language can be tanslated into $\mathbf{PCI}_{\mathrm{K}}$ with keeping logical validity by introducing an auxiliary language with a material implication $\rightarrow$ to restore the balance of both languages of **F** and $\mathbf{PCI}_{\mathrm{K}}$.

(4) Girard's classical linear logic with linear implication $\supset$:
In the similar way to the previous cases (2) and (3), we define $\mathbf{PCI}_{\mathrm{GL}}$ logic as an extension of **PCI** in order to interpret the classical linear implication $\supset$ by identity $\equiv$. Here we add the identity axioms (LT), (LE), (L*1), (L*2) and (LDN), which corresponded to axioms (L2), (L3), (10), (L11) and (L18) in **GL** (see Section 5.3.1), respectively, under the system $\mathbf{PCI}_{\mathrm{K}}$ which is also defined by adding two identity axioms (WIA1) and (WIA2), and one inference rule (G) to the original **PCI** logic.

(R7) $\overrightarrow{\alpha \supset \beta} \longmapsto \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta}$,

(R8) $\overleftarrow{A \equiv B} \longmapsto (\overleftarrow{A} \supset\subset \overleftarrow{B})$,

where $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$ and $\overleftarrow{A}$, $\overleftarrow{B}$ denote the results of translations from **GL** to $\mathbf{PCI}_{\mathrm{GL}}$, and its converse, respectively.

(WIA1) $((A \rightarrow B) \equiv (B \rightarrow A)) \rightarrow (A \equiv B)$

(WIA2) $((A \rightarrow B) \equiv (B \rightarrow A)) \rightarrow ((A \rightarrow B) \equiv \top) \wedge ((B \rightarrow A) \equiv \top)$

(LT) $(A > B) > ((B > C) > (A > C))$

(LE) $(A > (B > C)) \rightarrow (B > (A > C))$

(L*1) $A > (B > A \circ B)$

(L*2) $(A > (B > C)) \rightarrow (A \circ B > C)$

(LDN) $\smile\smile A \rightarrow A$

(G) $\dfrac{A \quad B}{A \equiv B}$

Here above each connectives $>$, $\smile$ and $\circ$ are abbreviations in $\mathbf{PCI}_{\mathrm{GL}}$ as : $A > B := (A \equiv A \wedge B)$, $\smile A := A > \neg(A \equiv A)$ and $A \circ B :=\smile (A >\smile B)$.

We show in the similar way to the case (3) that every formulas in **GL**-language can be tanslated into $\mathbf{PCI}_{\mathrm{GL}}$ with keeping logical validity by introducing an auxiliary language with a material implication $\rightarrow$ to restore the balance of both languages of **GL** and $\mathbf{PCI}_{\mathrm{GL}}$. All results mentioned above, can also be extended to various extensions (e.g., $\mathbf{GL}_{\mathrm{c}}$, $\mathbf{GL}_{\mathrm{w}}$ and $\mathbf{GL}_{\mathrm{cw}}$) of Girard' classical linear logic.

In the last part of our thesis, we develop an algebraic study of extensions of $\mathbf{PCI}_{\mathrm{K}}$ logic. We show that $\mathbf{PCI}_{\mathrm{K}}$-algebras form a variety, and a necessary and sufficient condition for a subvariety of $\mathbf{PCI}_{\mathrm{K}}$-algebras to have *equationally definable principal congruences* (EDPC for short) property. Here, EDPC property is closely connected with the deduction theorem of a logic. Because of an isomorphism between the lattice of filters of $\mathbf{PCI}_{\mathrm{K}}$-algebras and the lattice of congruences of $\mathbf{PCI}_{\mathrm{K}}$-algebras, EDPC property of $\mathbf{PCI}_{\mathrm{K}}$-algebras can be restated by that principal filters of $\mathbf{PCI}_{\mathrm{K}}$-algebras are equationally definable. Then we can show that a necessary and sufficient condition for $\mathbf{PCI}_{\mathrm{K}}$-algebras to have EDPC by introducing a unary operator r on $\mathbf{PCI}_{\mathrm{K}}$-algebra:

$$r(x) = (x\Delta \mathrm{t}) \cap x,$$

where,

$$r^0(x) = x,$$
$$r^{n+1}(x) = r(r^n(x)).$$

The thesis is organized as follows. In Chapter 2, we explain basic concepts of the formal background in our investigations in this thesis. It contains basic notions of deductive systems (Section 2.1), and the axiomatic deductive system of **SCI** and basic results on **SCI** that is used in this thesis (Section 2.2). At the end we give a note which includes historical remarks and biblographical informations (Section 2.3).

In Chapter 3, we introduce the axiomatic deductive system of **PCI** and its related results in order to simulate various kinds of nonclassical logics. In Section 3.1, at first, we explain the system **PCI** and its fundamental properties. In Section 3.2, we survey briefly Angell's analytic containment logic **AC**. Then in Section 3.3, we define $\mathbf{PCI}_{\mathrm{W}}$ logic by adding two identity axioms (IT) and (IR) to the original system **PCI** in order to interpret Angell's analytic containment $\approx$ by identity $\equiv$. In Section 3.4, we investigate a general method of showing syntactical equivalence between various logics. After this, in Section 3.5, we give translations between **AC** and $\mathbf{PCI}_{\mathrm{W}}$, and hence prove that they are syntactically equivalent. Finally in Section 3.6, we also give further information on related results shown in this chapter.

In Chapter 4, we investigate how classical modal logics are simulated by **PCI** logic which have been introdued in the previous Chapter 3. In Section 4.1, we give a brief survey of classical modal logics, particularly basic normal modal logic **K** and its axiomatic extensions **KT**, **KB**, **K4**, **KD**, **K5**, **S4** and **S5**, in syntactical and semantical points of view (see [15], [52]). Then in Section 4.2, we define $\mathbf{PCI}_{\mathrm{K}}$ logic by adding two identity axioms (WIA1) and (WIA2), and one inference rule (G) to the original system **PCI** in order to interpret the necessary operator $\Box$ by identity $\equiv$. After this, in Section 4.3, we give translations between **K** and $\mathbf{PCI}_{\mathrm{K}}$, and hence prove that they are syntactically equivalent in a sense of Definition 3.4.1. In Section 4.4, we also introduce Kripke type semantics for $\mathbf{PCI}_{\mathrm{K}}$ logic by exchanging the validity of modal formulas in modal Kripke type semantics with new validity of identity formulas. Then we can show that $\mathbf{PCI}_{\mathrm{K}}$ and **K** are semantically equivalent relative to the same Kripke frame. So by invoking the completeness of modal logic, we give a completeness theorem of $\mathbf{PCI}_{\mathrm{K}}$ relative to Kripke type semantics. In Section 4.5, we define $\mathbf{PCI}_{\mathrm{K}}$-algebras and give a representation theorem (Theorem 4.5.8) of this algebras. Furthermore, we give an alternative completeness result of $\mathbf{PCI}_{\mathrm{K}}$ logic by using the above representation theorem. So far mentioned results concern with relationships between **K** and $\mathbf{PCI}_{\mathrm{K}}$. But we can successfully extend their results to various extensions of modal logics. In Section 4.6, we define several extensions of $\mathbf{PCI}_{\mathrm{K}}$ which are counterparts of modal extensions of **K**. Then as the similar way to $\mathbf{PCI}_{\mathrm{K}}$, we can also consider translations between **K** extensions and $\mathbf{PCI}_{\mathrm{K}}$ extensions (Section 4.7), and moreover, Kripke type semantics for $\mathbf{PCI}_{\mathrm{K}}$ extensions (Section 4.8). Finally, we also give further information on related results shown in this chapter (Section 4.9).

In Chapter 5, we investigate how weak logics with two kind of weak implications, e.g., strict/linear implication, are simulated by **PCI** logic introduced in Chapter 3. In fact, we consider both systems of Corsi's weak logic with strict implication (see [16]) and Girard's classical linear logic with linear implication (see [31]). In Section 5.1, we briefly survey Corsi's weak logic **F** and its axiomatic extensions in syntactical and semantical points of view. Then we know that $\mathbf{PCI}_{\mathrm{K}}$ logic introduced in Section 4.2 can also use to interpret the strict implication $\rightharpoonup$ by identity $\equiv$. In Section 5.2, we investigate translations between **F** and $\mathbf{PCI}_{\mathrm{K}}$. Since **F**-language $\mathcal{L}_{\mathrm{F}}$ lacks a material implication $\to$, we define an auxiliary language $\mathcal{L}_{\mathrm{F}'}$ by adding $\to$ to restore the balance between both **PCI** (i.e., **SCI**) and **F** languages. Then, for an auxiliary system $\mathbf{F}'$ of this language, we give translations between $\mathbf{F}'$ and $\mathbf{PCI}_{\mathrm{K}}$, and hence prove that they are syntactically equivalent in a sense of Definition 3.4.1. Moreover, we show that every formulas in **F**-language can be tanslated into $\mathbf{PCI}_{\mathrm{K}}$ with keeping logical validity, since $\mathbf{F}'$ is a conservative extension of **F**. Next as another weak logic, in Section 5.3, we give a brief survey of Girard's classical linear logic and its axiomatic extensions in syntactical and semantical points of view. Then in Section 5.4, we define $\mathbf{PCI}_{\mathrm{GL}}$ logic by adding identity axioms (WIA1), (WIA2), (LT), (LE), (L*1), (L*2) and (LDN), and one inference rule (G) to the original system **PCI** in order to interpret correctly the classical linear implication $\supset$ by identity $\equiv$. After this, in

Section 5.5, we show that every formulas in **GL**-language can be tanslated into $\mathbf{PCI}_{\mathrm{GL}}$ with keeping logical validity by applying the similar discussion with the case of Corsi's weak logic **F**. Finally in Section 5.6, we also give further information on related results shown in this chapter.

In Chapter 6, we investigate algebraic properties of **PCI** logics. In Section 6.1, we first survey broad informations of various methods for the algebraization of deductive systems. The most famous method to algebraize a logic is to construct a Lindenbaum-Tarski algebra by factoring the algebras of formulas by the congruence relative to theories of the logic. Furthermore, we explain equivalential algebras and congruence operators, which also contribute to algebraize a logic. At the end of this section, we consider the case of **PCI** logics introduced so far. In Section 6.2, we show that the class of **PCI**-algebras, defined by the above algebraization, forms a variety. In fact, we only consider a class of $\mathbf{PCI}_{\mathrm{K}}$-algebas whether this class forms a variety or not. In Section 6.3, we also check a variety of $\mathbf{PCI}_{\mathrm{K}}$-algebras to have EDPC property, and show a necessary and sufficient condition to have EDPC property. Finally in Section 6.4, we also give further information on related results shown in this chapter.

Finally in Chapter 7, we summerize achievements in this thesis, and discuss also some remaining problems and several further subjects.

Chapter 2

Preliminaries

In this chapter we will explain basic concepts of the formal background in our investigations in this thesis. It contains basic notions of deductive systems (Section 1), and the axiomatic deductive system of **SCI** and basic results on **SCI** that will be used in this thesis (Section 2). At the end we will give a note which includes historical remarks and biblographical informations (Section 3).

2.1 Methodology of deductive systems

In this section we will introduce several basic notions in the methodology of deductive systems, e.g., mainly the notion of consequence operators and logical matrices. These subjects will appear in many "Polish style" books or papers. In order to explain these notions, we will mainly refer to [19], [76], [17] and [75]. According to the notion of deductive systems we will view a logic not as a fixed set of logical theorems, derivable in some logical calculus, but rather as the *deducibility relation*, or *consequence operator* generated by the given logical axioms and rules of inference. On the other hand, logical matrices are considered as one of the most powerful tools for studying interpretations of logical constants in a logic, i.e., logical connectives, and give matrix semantics for logics under an appropriate valuation function between each logic and its logical matrix.

In this section we assume that $\mathcal{L}$ is a fixed, but arbitrary sentential language and L is the set of all $\mathcal{L}$ formulas. Then endomorphisms of L are usually called *substitutions* in $\mathcal{L}$ and let $\mathrm{Sb}(L)$ be the set of all such substitutions in $\mathcal{L}$. Then a set X of $\mathcal{L}$ formulas is called *invariant* if X is closed under substitution, i.e., $\mathrm{Sb}(X) = X$.

2.1.1 Consequence operators

The original meaning of logical consequence is roughly expressed in such a way that A is a *consequence* of a set X of formulas if under all possible interpretations of non-logical terms in $X \cup \{A\}$, A is true whenever all formulas in X are true. This notion is also captured in terms of rules of inference. Namely, A is said to be a *consequence* of X if and only if it is derivable from X by means of some accepted logical rules. Then having these original meaning on our mind, we can give a precise definition of consequence operators in the following way (see [76]).

Definition 2.1.1 (i) *An unary operator C defined on sets of formulas of $\mathcal{L}$ is called a* consequence operator *if for all $X, Y \subseteq L$, it satisfies the following conditions:*

(C1) $X \subseteq C(X)$,

(C2) $C(C(X)) \subseteq C(X)$,

(C3) *if* $X \subseteq Y$ *then* $C(X) \subseteq C(Y)$.

(ii) *Moreover, a consequence operator C on L is called* structural *if for all substitution e of $\mathcal{L}$ and all $X \subseteq L$, it satisfies in addition to* (C1)-(C3), *also*

(C4) $eC(X) \subseteq C(eX)$, *where eX denotes the set of all $e(A)$ for A in X.*

The notations of $A \in C(X)$ and $X \subseteq C(Y)$ are to be read that a formula A *C-follows* from X, and everything in X *C-follows* from Y, respectively. If $X \subseteq L$ and $A, B, \ldots, G$ are in L then, instead of $C(X \cup \{A, B, \ldots, G\})$ and $C(\{A, B, \ldots, G\})$, we write briefly $C(X; A, B, \ldots, G)$ and $C(A, B, \ldots, G)$, respectively. Let C_1 and C_2 be two consequence operators on sets of formulas of $\mathcal{L}$. Then we say that C_1 is a *subconsequence* of C_2, or C_2 is a *superconsequence* of C_1, in symbols, $C_1 \leq C_2$ if $C_1(X) \subseteq C_2(X)$ for all $X \subseteq L$. A theory is an arbitrary set of formulas of $\mathcal{L}$. If X is closed under a consequence operator C, i.e., $X = C(X)$, then X is called a *C-theory.* Moreover, $C(X)$ is also called a *deductive system* or, simply, a *system* of C. Here $C(X)$ is the least C-theory containing X, $C(\emptyset)$ is the system of all logically provable or valid formulas, namely *tautologies* of C, and $\mathrm{Th}(C)$ is the set of all theories of C.

By a logic we will mean either a couple $\mathfrak{L} = (\mathcal{L}, C)$ where $\mathcal{L}$ is a fixed language and C is a structural consequence operator on $\mathcal{L}$, or C itself as a logic. A couple $\mathfrak{T} = (C, A_X)$ is called a *theory* in the language $\mathcal{L}$ if C is a consequence operator on L and $A_X \subseteq L$. For a given theory $\mathfrak{T} = (C, A_X)$, C is called the logic underlying $\mathfrak{T}$, the set A_X is called an axiom set of $\mathfrak{T}$ and the set $C(A_X)$ is called the set of all theorems of $\mathfrak{T}$. Given two theories $\mathfrak{T}_1 = (C_1, A_1)$ and $\mathfrak{T}_2 = (C_2, A_2)$, we say that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are *equivalent* if $C_1 = C_2$ and $C_1(A_1) = C_2(A_2)$. If only $C_1(A_1) = C_2(A_2)$ holds then the theories $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are called *pseudo-equivalent.*

A set $X \subseteq L$ is *consistent* relative to C if $C(X) \neq L$; otherwise X is called *inconsistent*, or the consequence C on L is called *universal.* Moreover, a consistent set X is *complete* if every consistent set Y which includes X satisfies $C(X) = C(Y)$. Here these two notions are much more general characterization than the classical one since no knowledge on the concept of negation is needed in this cases. Two sets of formulas X and Y are called *C-equivalent*, or simply *equivalent* with respect to C, if their sets of consequence C coincide, i.e., $C(X) = C(Y)$. A set X of formulas is called *finitely axiomatizable* if there exists a finite set which is equivalent to X with respect to C. Then the following basic results are well-known (see [11]).

Proposition 2.1.2 (i) *Any consistent set is contained in a maximal consistent set.*

(ii) *Any maximal consistent set is a theory.*

(iii) *If $A \notin C(X)$, then there exists a maximal consistent superset Y of X which does not contain A.*

The particular consequence operators which we deal with in this thesis are defined as *deducibility* or *derivability* relations. We select a set A_X of logical axioms and a family $\mathbb{R}$ of finitary rules of inference which allow us to draw conclusions from finitely many premisses. A set of all derivations which are produced by the above A_X and $\mathbb{R}$ is called *finite* $(A_X, \mathbb{R})$ *derivations.* Then, we will define the consequence operator C generated through finite $(A_X, \mathbb{R})$ derivations as follows.

Definition 2.1.3 *For any $X \cup \{A\} \subseteq L$, $A \in C(X)$ if and only if A is derived from $A_X \cup X$ in finitely many steps by succesive application of rules in $\mathbb{R}$.*

The finite sequence of formulas which appears in this procedure ends with A, and is called a $(A_X, \mathbb{R})$ *derivation* of A from X, and we will write $X \vdash_{(A_X, \mathbb{R})} A$. Here every consequence operator C defined in this way is finite, namely satisfies the following condition:

($*$) $A \in C(X)$ if and only if either $A \in C(\emptyset)$ or there exist formulas $A_1, \ldots, A_n \in X$ such that $A \in C(A_1, \ldots, A_n)$.

Conversely, let $\vdash_C$ denotes the set of all $(A_X, \mathbb{R})$ derivations with $X \vdash_{(A_X, \mathbb{R})} A$ such that $A \in C(X)$ for an arbitrary consequence operator C. Then the relation $\vdash_C$ between X and A is called the *consequence relation* corresponding to C and satisfies the following equivalence (see [76]).

Proposition 2.1.4 *For any $X \cup \{A\} \subseteq L$, $X \vdash_C A$ if and only if $A \in C(X)$.*

2.1.2 Logical matrices

A logical matrix $\mathfrak{M}$ for the language $\mathcal{L}$ is a couple $\mathfrak{M} = (\mathcal{A}, D)$ where $\mathcal{A}$ is an algebra similar to $\mathcal{L}$ and D is a subset of A, where A is the underlying set of $\mathcal{A}$. Elements of D are called *designated elements* of $\mathfrak{M}$.

Definition 2.1.5 *Let $\mathfrak{M} = (\mathcal{A}, D)$ be a matrix for $\mathcal{L}$. A homomorphism h from L to A is called a* valuation *of $\mathcal{A}$ for formulas of $\mathcal{L}$ in $\mathfrak{M}$. Then for any formula $B \in L$, we have the following definitions:*

(i) h satisfies B *in* $(\mathcal{A}, D)$*, in symbols,* $B \in \mathrm{Sat_h}(\mathcal{A}, \mathrm{D})$*, if* $h(B) \in D$*,*

(ii) B *is* true *in* $(\mathcal{A}, D)$*, in symbols,* $B \in \mathrm{TR}(\mathcal{A}, \mathrm{D})$*, if* $B \in \mathrm{Sat_h}(\mathcal{A}, \mathrm{D})$ *for every valuation* h *of* $\mathcal{A}$*,*

(iii) B *is* valid *in* $\mathcal{A}$*, in symbols,* $\mathcal{A} \models B$*, if* $B \in \mathrm{TR}(\mathcal{A}, \mathrm{D})$ *for every nonempty designated subset* D *of* $\mathcal{A}$*,*

(iv) B *is* valid *if* $\mathcal{A} \models B$ *for every algebra* $\mathcal{A}$*.*

Two matrices $\mathfrak{M} = (\mathcal{A}, D)$ and $\mathfrak{N} = (\mathcal{B}, E)$ are *similar* if the algebras $\mathcal{A}$ and $\mathcal{B}$ have the same similarity type. A matrix $\mathfrak{M} = (\mathcal{A}, D)$ is *appropriate* for $\mathcal{L}$, or simply $\mathfrak{M}$ is a *matrix* for $\mathcal{L}$, if the algebra $\mathcal{A}$ is similar to $\mathcal{L}$. Any class M of matrices for $\mathcal{L}$ is called a *matrix semantics* for $\mathcal{L}$. Here every matrix semantics M for $\mathcal{L}$ induces the consequence operator C_{M} as the following way.

Definition 2.1.6 *For any* $X \cup \{B\} \subseteq L$*,* $B \in C_{\mathrm{M}}(X)$ *if and only if for any matrix* $\mathfrak{M} = (\mathcal{A}, D)$ *in* M *and any valuation* h *of* $\mathcal{L}$ *in* $\mathfrak{M}$*,* $h(B) \in D$ *whenever* $h(X) \subseteq D$*.*

Here if $\mathrm{M} = \{\mathcal{M}\}$ we write $C_{\mathcal{M}}$ instead of C_{M}. Given a logic $\mathfrak{L} = (\mathcal{L}, C)$, we will say that $\mathfrak{L}$ is *strongly complete* with respect to a matrix semantics M for $\mathcal{L}$ if $C = C_{\mathrm{M}}$. A matrix semantics M is *strongly adequate* for a logic $\mathfrak{L} = (\mathcal{L}, C)$ if $\mathfrak{L}$ is strongly complete with respect to M. Moreover, matrices of the form $\mathfrak{L}_X = (\mathcal{L}, C(X))$ are called *Lindenbaum matrix* for C and the class $\mathrm{L_C} = \{\mathfrak{L}_\mathrm{X}; X \subseteq L\}$ is called the *Lindenbaum bundle* for C. A matrix $\mathfrak{M}$ for $\mathcal{L}$ is called a C-matrix if $C \leq C_{\mathfrak{M}}$ and let Matr(C) be the class of all C-matrices. Then, since $\mathrm{L_C} \subseteq$ Matr(C) for a structural C, we can easily verify that Matr(C) is strongly adequate for $\mathfrak{L} = (\mathcal{L}, C)$. Consequently, Matr(C) is the greatest matrix semantics strongly adequate for C.

Let $\mathfrak{M} = (\mathcal{A}, D)$ and $\mathfrak{N} = (\mathcal{B}, E)$ be similar matrices. A mapping $h : \mathcal{A} \to \mathcal{B}$ is called a *strong homomorphism* (or *matrix homomorphism*) from $\mathfrak{M}$ to $\mathfrak{N}$ if h is an algebraic homomorphism from $\mathcal{A}$ to $\mathcal{B}$ and $h^{-1}(E) = D$. 1-1 matrix homomorphisms are called *isomorphic embeddings*. Moreover, if an isomorphic embedding h is also onto then h is called an *isomorphism*. If h is a strong homomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$ then $\mathfrak{N}$ is called a *strong homomorphic image* of $\mathfrak{M}$. We write $\mathfrak{M} \cong \mathfrak{N}$ when matrices $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic. Then we can easily varify the following equivalence (see [17]).

Proposition 2.1.7 *If* $\mathfrak{M} \cong \mathfrak{N}$ *then we have* $C_{\mathfrak{M}} = C_{\mathfrak{N}}$*.*

For given an algebra $\mathcal{A}$ and a congruence $\equiv_\Theta$ on $\mathcal{A}$, let us $|a|_\Theta$ denotes the equivalence class $\{b; a \equiv b \pmod{\Theta}\}$ of a on $\mathcal{A}$, and $\mathcal{A}/\Theta = \{|a|_\Theta; a \in A\}$ the quotient of $\mathcal{A}$ by Θ. Then we can extend the notion of a congruence on an algebra $\mathcal{A}$ to that on a matrix as follows.

Definition 2.1.8 *Let $\mathfrak{M} = (\mathcal{A}, D)$ be a matrix for $\mathcal{L}$. A congruence $\equiv_\Theta$ on $\mathcal{A}$ is called a* congruence on $\mathfrak{M}$ (*or a* matrix congruence) *if $|a|_\Theta \subseteq D$ for all $a \in D$.*

Let $\mathrm{Co}(\mathfrak{M})$ be the set of all congruences in a matrix $\mathfrak{M} = (\mathcal{A}, D)$. Then $\mathrm{Co}(\mathfrak{M})$ is nonempty and obviously $\mathrm{Co}(\mathfrak{M}) \subseteq \mathrm{Co}(\mathcal{A})$. Moreover, the set $(\mathrm{Co}(\mathcal{A}), \subseteq)$ is a complete lattice with respect to the set inclusion $\subseteq$. If $\Theta \in \mathrm{Co}(\mathfrak{M})$ then $\mathfrak{M}/\Theta = (\mathcal{A}/\Theta, D/\Theta)$ is called the *quotient matrix* determined by Θ, where $\mathcal{A}/\Theta$ is the quotient algebra and $D/\Theta = \{|a|_\Theta; a \in D\}$. A congruence $\Theta \in \mathrm{Co}(\mathfrak{M})$ is called *compatible* with a subset D of A if $a \in D$ and $a \equiv b \pmod{\Theta}$ imply $b \in D$ for all $a, b \in A$. For any $\Theta \in \mathrm{Co}(\mathfrak{M})$, the *canonical* (or *natural*) mapping k_Θ from $\mathfrak{M}$ to $\mathfrak{M}/\Theta$ is given by the term $k_\Theta(a) = |a|_\Theta$. Given a matrix $\mathfrak{M} = (\mathcal{A}, D)$ and a strong homomorphism h from $\mathfrak{M}$ to $\mathfrak{N}$, we denote by Θ_h the kernel of h, i.e., for any $a, b \in A$, $a \equiv b \pmod{\Theta_h}$ if and only if $ha = hb$. Let us state the following simple facts without proofs (see [17]).

Proposition 2.1.9 (i) *For any strong homomorphism h from $\mathfrak{M}$ to $\mathfrak{N}$, $\Theta_h \in \mathrm{Co}(\mathfrak{M})$.*

(ii) *For any congruence $\Theta \in \mathrm{Co}(\mathfrak{M})$, k_Θ is a strong homomorphism from $\mathfrak{M}$ to $\mathfrak{M}/\Theta$. Moreover, $\Theta = \Theta_{k_\Theta}$.*

(iii) *If a strong homomorphism h from $\mathfrak{M}$ to $\mathfrak{N}$ is onto, then $\mathfrak{M}/\Theta_h \cong \mathfrak{N}$.*

2.2 SCI and its basic results

In this section we will make a survey of the axiomatic deductive system of **SCI** and its basic results that will serve as a preparation for the investigations in the sequal. The **SCI** system was firstly proposed by R. Suszko to realize some philosophical ideas of L. Wittgenstein's *Tractatus* (see [63], [64], [65], [11] and [67]). It is obtained from the classical sentential calculus by adding a new identity $\equiv$. In **SCI** $A \equiv B$ means that both formulas A and B have a common referent (or same situation), while $A \leftrightarrow B$ means the sameness of both logical values. A typical feature of **SCI** is the ability of representing various nonclassical logics on it. In fact, R. Suszko showed that some axiomatic extensions of **SCI** really correspond to modal systems **S4** and **S5**, by interpreting $A \equiv B$ as $\Box(A \leftrightarrow B)$ (see [65]), and moreover, **SCI** itself also correspond to the three-valued Łukasiewicz logic **Ł₃** by interpreting $A \equiv B$ as $(A \Leftrightarrow B)$, where $\Leftrightarrow$ is a three-valued Łukasiewicz equivalence (see [68] and [69]). Then, the main purposes of this thesis is to realize the above idea of Suszko's **SCI** for various kinds of nonclassical logics.

2.2.1 SCI-language and its axiomatic deductive system

Let $\mathcal{L}_\mathrm{S} = \langle \mathrm{L}_\mathrm{S}, \neg, \wedge, \vee, \rightarrow, \equiv, \bot, \top \rangle$ be the **SCI**-language consisting of an infinite denumerable set VAR_S of sentential variables, constants;$\bot$ (false) and $\top$ (true), and the standard truth functional (TF for short) connectives; $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\to$ (material implication) as well as a new binary connective $\equiv$, called the identity. Formulas $\mathrm{L_S}$ of a given **SCI**-language $\mathcal{L}_{\mathrm{S}}$ are defined in the usual way. The formula $A \equiv B$ means intuitively that the situation that A is the same as the situation that B (i.e., both A and B have a common referent). This formula is called an *equation* because the **SCI**-language was originally designed for two sorted one in which the same symbol $\equiv$ standed for the identity predicate and the identity connective (see [65]). Letters $p, q, r, p_1, \ldots$ will be used to denote sentential variables; $A, B, C, \ldots$ will denote formulas of a **SCI**-language $\mathcal{L}_{\mathrm{S}}$; $X, Y, Z, \ldots$ will denote sets of formulas; $G[A/p]$ will denote the formula obtained from G by replacing each occurrence of p by A. The sentential constants $\top$ (and $\bot$) and other TF-connective $\leftrightarrow$ (material equivalence) are used as the usual abbreviation: $\top := A \vee \neg A$, $\bot := \neg\top := A \wedge \neg A$ and $A \leftrightarrow B := (A \to B) \wedge (B \to A)$. Also we will sometime omit parentheses, following the assumption that the priority of each connective is weak as $\neg, \wedge, \vee, \equiv, \to, \leftrightarrow$ in order.

The logical axioms for **SCI**-language $\mathcal{L}_{\mathrm{S}}$ consist of two sets of schemata TFA (truth functional axioms), i.e., from (A1) to (A10), and IDA (identity axioms), i.e., from (E1) to (E3), from (C1) to (C5) and (SI) below:

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $A \wedge B \to A$

(A4) $A \wedge B \to B$

(A5) $A \to (B \to (A \wedge B))$

(A6) $A \to A \vee B$

(A7) $B \to A \vee B$

(A8) $(A \to C) \to ((B \to C) \to (A \vee B \to C))$

(A9) $A \to (\neg A \to B)$

(A10) $\neg\neg A \to A$

(E1) $A \equiv A$

(E2) $(A \equiv B) \to (B \equiv A)$

(E3) $(A \equiv B) \wedge (B \equiv C) \to (A \equiv C)$

(C1) $(A \equiv B) \to (\neg A \equiv \neg B)$

(C2) $(A \equiv B) \wedge (C \equiv D) \to (A \wedge C) \equiv (B \wedge D)$

(C3) $(A \equiv B) \wedge (C \equiv D) \to (A \vee C) \equiv (B \vee D)$

(C4) $(A \equiv B) \wedge (C \equiv D) \to (A \to C) \equiv (B \to D)$

(C5) $(A \equiv B) \wedge (C \equiv D) \to (A \equiv C) \equiv (B \equiv D)$

(SI) $(A \equiv B) \to (A \to B)$

Also the rule of inference for $\mathcal{L}_{\mathrm{S}}$ is only modus ponens:

(Mp) $\dfrac{A \quad A \to B}{B}$

Here the axioms in TFA with modus ponens as the single rule will give all classical truth functional tautologies (TFT for short). The axioms IDA mean that identity connective $\equiv$ is not only an equivalence relation but also a congruence relation on $\mathcal{L}_{\mathrm{S}}$, and at least as strong as material equivalence $\leftrightarrow$. Then the axiomatic deductive system $C(X)$ for the **SCI**-language $\mathcal{L}_{\mathrm{S}}$ and any $X \subseteq \mathrm{L}_{\mathrm{S}}$ (or **SCI** $= (\mathcal{L}_{\mathrm{S}}, C)$) is defined as the following way.

Definition 2.2.1 (i) *For any $X \subseteq \mathrm{L}_{\mathrm{S}}$, $C(X)$ is the smallest set of formulas closed under the rule* (Mp)*, which contains* TFA, IDA *and X.*

(ii) *The element of $C(\emptyset)$ is called the logical theorem of* **SCI**.

Then it is easily verified that C is a consequence operator and also satisfies the following two propositions (see [11] and [65]).

Proposition 2.2.2 *For any $X \cup \{A, B\} \subseteq \mathrm{L}_{\mathrm{S}}$, it holds the following equivalences:*

(i) $B \in C(X; A)$ *if and only if* $A \to B \in C(X)$, (*Deduction Theorem*)

(ii) $\neg A \in C(X)$ *if and only if* $\bot \in C(X; A)$,

(iii) $A \in C(X)$ *if and only if there exist some finite subset Y of X such that $A \in C(Y)$.* (*Compactness*)

Proposition 2.2.3 *The following are logical theorems of* **SCI**.

(i) $A \equiv A$

(ii) $A \equiv B \leftrightarrow B \equiv A$

(iii) $\neg(A \equiv \neg A)$

(iv) $A \equiv B \to ((A \to B) \equiv (B \to A))$

(v) $A \equiv B \to (G[A/p] \equiv G[B/p])$ (*Replacement Law*)

(vi) $A \equiv \top \to A$, $A \equiv \bot \to \neg A$

(vii) $(A \equiv B) \equiv \top \to A \equiv B$, $(A \equiv B) \equiv \bot \to \neg(A \equiv B)$

Let C_0 be the consequence operator defined only from the rule (Mp) and the axioms TFA. Then C_0 is exactly the consequence operator of the classical logic **CL**. If any formulas $X \cup \{A\}$ of $\mathcal{L}_S$ do not contain any occurrences of the identity connective $\equiv$, then $A \in C(X)$ if and only if $A \in C_0(X)$. Hence in this sense, **SCI** is a conservative extension of the classical logic **CL**. If the following formula will be added to **SCI** as an additional axiom schema:

(FA) $(A \leftrightarrow B) \to (A \equiv B)$,

then both identity $\equiv$ and material equivalence $\leftrightarrow$ are indistinguishable and therefore we will get **CL** as the result. Since the formula (FA) indicates the Frege's idea that all logically true (and similarly false) formulas have to describe the same thing, namely have a common reference, it was called *Fregean Axion*. Moreover, because (FA) is not a logical theorem of **SCI**, we will call **SCI** as a *non-Fregean* logic, while **CL** is called a *Fregean* logic. The following is an essential property of non-Fregean logic (i.e., **SCI**).

Proposition 2.2.4 (The natural postulate) *Any equation which is a logical theorem of* **SCI** *is only a trivial one (i.e.,* $A \equiv A$*).*

This means that logical theorems (tautologies) of **SCI** are cognitively empty, or in other words that **SCI** cannot tell us any non-trivial equation. Hence the following non-trivial equations are not logical theorems of **SCI** as the results (see [65]).

(1) $\neg\top \equiv \bot$

(2) $(A \vee \neg A) \equiv (B \vee \neg B)$

(3) $(A \equiv B) \equiv (B \equiv A)$

(4) $(A \equiv A) \equiv (B \equiv B)$

(5) $(A \to A) \equiv (A \equiv A)$

(6) $(A \to B) \equiv (\neg A \vee B)$

(7) $\neg\neg A \equiv A$

(8) $(A \leftrightarrow B) \equiv (A \equiv B)$

2.2.2 Well-known extensions of SCI

In view of the natural postulate mentioned in the previous subsection, non-Fregean logics (i.e., **SCI**) are very weak. But of course, we can consider the syntactical extension of **SCI** which be able to strengthen up to the classical logic **CL**, and then divide them into two classes, namely elementary and non-elementary extensions of **SCI**. The former extensions are defined as **SCI** with an additional set of axiom schemata added to the logical axiom TFA $\cup$ IDA. On the other hand, the latter extensions are defined as **SCI** with some additional rules of inference, besides (Mp). Now let us consider the following additional axiom schemata, for example:

(TA1) $A \equiv B$, whenever $A, B \in C_0(\emptyset)$,

(TA2) $A \equiv B$, whenever $A, B \in C(\emptyset)$, (WIA) $((A \to B) \equiv (B \to A)) \to (A \equiv B)$,

(SIA) $((A \to B) \equiv (B \to A)) \equiv (A \equiv B)$,

(BIA) $((A \equiv B) \equiv \top) \vee ((A \equiv B) \equiv \bot)$,

(FA1) $(A \equiv B) \vee (A \equiv C) \vee (B \equiv C)$,

(FA2) $(A \equiv \top) \vee (A \equiv \bot)$.

In some literatures (e.g., [65], [64] and [67]), some elementary extensions $\mathbf{W}_{\mathrm{B}}$, $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_{\mathrm{T}}$, $\mathbf{W}_{\mathrm{H}}$ and $\mathbf{W}_{\mathrm{F}}$ of **SCI** which can be defined below, are discussed. Relations between these extensions are also shown in Fig 2.1. A system is located above another one if it is stronger than the other.

Definition 2.2.5 *Let* $\mathbf{SCI} = (\mathcal{L}_{\mathrm{S}}, C)$ *and* $X \subseteq \mathrm{L}_{\mathrm{S}}$. *Then each elementary extension of* **SCI** *is defined as follows:*

(i) $\mathbf{W}_{\mathrm{B}} = (\mathcal{L}_{\mathrm{S}}, C_{\mathrm{B}})$ *is the elementary extension of* **SCI**, *where* C_{B} *is a superconsequence of* C *defined by* $C_{\mathrm{B}}(X) = C(X; \mathrm{TA1}, \mathrm{WIA})$,

(ii) $\mathbf{W}_1 = (\mathcal{L}_{\mathrm{S}}, C_1)$ *is the elementary extension of* **SCI**, *where* C_1 *is a superconsequence of* C *defined by* $C_1(X) = C(X; \mathrm{TA2})$,

(iii) $\mathbf{W}_2 = (\mathcal{L}_{\mathrm{S}}, C_2)$ *is the elementary extension of* **SCI**, *where* C_2 *is a superconsequence of* C *defined by* $C_2(X) = C(X; \mathrm{TA2}, \mathrm{WIA})$,

(iv) $\mathbf{W}_{\mathrm{T}} = (\mathcal{L}_{\mathrm{S}}, C_{\mathrm{T}})$ *is the elementary extension of* **SCI**, *where* C_{T} *is a superconsequence of* C *defined by* $C_{\mathrm{T}}(X) = C(X; \mathrm{TA2}, \mathrm{SIA})$,

(v) $\mathbf{W}_{\mathrm{H}} = (\mathcal{L}_{\mathrm{S}}, C_{\mathrm{H}})$ *is the elementary extension of* **SCI**, *where* C_{H} *is a superconsequence of* C *defined by* $C_{\mathrm{H}}(X) = C(X; \mathrm{TA2}, \mathrm{SIA}, \mathrm{BIA})$,

(vi) $\mathbf{W}_{\mathrm{F}} = (\mathcal{L}_{\mathrm{S}}, C_{\mathrm{F}})$ *is the elementary extension of* **SCI**, *where* C_{F} *is a superconsequence of* C *defined by* $C_{\mathrm{F}}(X) = C(X; \mathrm{FA})$ (*or* $C_{\mathrm{F}}(X) = C(X; \mathrm{FA1})$, *or* $C_{\mathrm{F}}(X) = C(X; \mathrm{FA2})$).

Three axioms (FA), (FA1) and (FA2) are mutually equivalent and mean that there exist at most two situations. So Fregean axiom can be seen as a numerical condition imposed on the universe of situations. As the following proposition shows, $\mathbf{W}_{\mathrm{F}}$ is the only consistent invariant theory of C_{F}, i.e., $\mathrm{Sb}(C_{\mathrm{F}}(X)) = C_{\mathrm{F}}(X)$ for any consistent subset $X \subseteq \mathrm{L}_{\mathrm{S}}$. Therefore, since $\mathbf{W}_{\mathrm{F}}$ has no proper consistent extension it is Post complete. Hence C_{F} is an elementary extension of C with a maximality (see [65]).

Proposition 2.2.6 *There exist exactly two Fregean theories of situations, the set of all* C_{F}*-tautologies* $C_{\mathrm{F}}(\emptyset)$ *and the inconsistent theory* L_{S}.

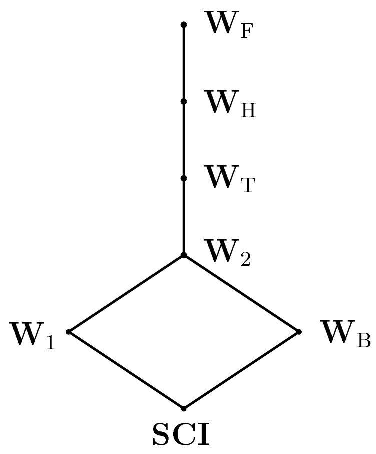

Figure 2.1: Relations between extensions of **SCI**

The combination of tautology axiom (TA1) and weak implication axiom (WIA) yields the Boolean extension $\mathbf{W}_{\mathrm{B}}$ of **SCI** (see [67]), in which we can prove the familiar equational Boolean laws; commutative, distributive and absorption laws, De Morgan laws and the double negation law. Furthermore, since we have $(A \vee \neg A) \equiv (B \vee \neg B)$ and $(A \wedge \neg A) \equiv (B \wedge \neg B)$ as logical theorems of $\mathbf{W}_{\mathrm{B}}$, we can introduce two sentential constants $\top$ and $\bot$ by the definitions; $\top \equiv (A \vee \neg A)$ and $\bot \equiv (A \wedge \neg A)$. Then $\top$ and $\bot$ are the unit and zero of the Boolean algebra of situations, respectively. Also $A \equiv \top$ means, by the definition, exactly that the situation of A is the unit of the Boolean algebra of situations. The modal operator $\Box$ is usually called *intensional* in the sense that if we read formulas of the form $\Box A$ as "it is necessary that A" then we must necessarily purify the meaning of the word "necessity" from any intensional shadows. But if we will interpret modal operator $\Box$ as $\Box A := (A \equiv \top)$ in the Boolean extension $\mathbf{W}_{\mathrm{B}}$, then $\Box$ is here *extensional* in view of the laws of Boolean theories $\mathbf{W}_{\mathrm{B}}$ because it exactly stands for the unit of $\mathbf{W}_{\mathrm{B}}$.

Obviously, every theorem of $\mathbf{W}_{\mathrm{T}}$ is a theorem of $\mathbf{W}_{\mathrm{B}}$ but not conversely, and contains the following formulas as examples:

(1) $(\top \equiv \top) \equiv \top$,

(2) $((A \equiv \top) \equiv \top) \equiv (A \equiv \top)$,

(3) $((A \wedge B) \equiv \top) \equiv ((A \equiv \top) \wedge (B \equiv \top))$,

(4) $((A \equiv \top) \rightarrow A) \equiv \top$.

Hence if we define an interior operator I on the Boolean theory $\mathbf{W}_{\mathrm{T}}$ by $I(A) := (A \equiv \top)$ for any formula A in L_{S}, then we can see that the operator I represents in $\mathbf{W}_{\mathrm{T}}$ a kind of topological interior operator on the Boolean algebra of situations (see [65]). Notice that two operators $\Box$ and $\equiv$ are interdefinable in $\mathbf{W}_{\mathrm{T}}$ as $\Box A := (A \equiv \top)$ and $A \equiv B := \Box(A \leftrightarrow B)$. Moreover, if we add the additional bi-valent axiom (BIA) to $\mathbf{W}_{\mathrm{T}}$ then we get much stronger theory $\mathbf{W}_{\mathrm{H}}$ than $\mathbf{W}_{\mathrm{T}}$. Then the formula $((A \equiv \top) \equiv \top) \vee ((A \equiv \top) \equiv \bot)$ is a logical theorem of $\mathbf{W}_{\mathrm{H}}$. So in $\mathbf{W}_{\mathrm{H}}$ the topological Boolean algebra of situations is a Hanle algebra, which is a topological Boolean algebra of situations with only two open elements $\top$ and $\bot$. It was proved, by using the matrix semantics method, that the theories $\mathbf{W}_{\mathrm{T}}$ and $\mathbf{W}_{\mathrm{H}}$ correspond to Lewis's modal systems **S4** and **S5**, respectively, on account of results of McKinsey and Tarski (see [46]).

Next we will consider non-elementary extensions of **SCI**. The consequence operator C^{R} is called a *non-elementary superconsequence* of C relative to a rule R of inference if it is closed under two rules (Mp) and (R) of inference, and contains the same logical axioms TFA $\cup$ IDA as C. A theory T is called a R-*theory* if it is closed under the rule (R) of inference. Let us consider three types of non-elementary superconsequences $C^{\mathrm{G}}, C^{\mathrm{QF}}$ and C^{I} of C, where the corresponding three inference rules are as follows (see [65]):

(G) $\dfrac{A \quad B}{A \equiv B}$,

(QF) $\dfrac{A \leftrightarrow B}{A \equiv B}$,

(I) $\dfrac{(A \to B) \equiv (B \to A)}{A \equiv B}$.

Then it is easily see that $C^{\mathrm{G}}, C^{\mathrm{QF}}$ and C^{I} are structural. The set of logical theorems of $\mathbf{W}_{\mathrm{T}}$ is closed under two rules (G) and (QF) of inference. Then we have the following proposition (see [65]).

Proposition 2.2.7 (i) *Let* T *be a theory. Then* T *is a* QF-*theory if and only if* T *is a* G-*theory and* I-*theory.*

(ii) *The theory* $\mathbf{W}_{\mathrm{T}}$ *is the least* QF-*theory on* **SCI**, *i.e.,* $\mathbf{W}_{\mathrm{T}} = C^{\mathrm{QF}}(\emptyset)$.

(iii) *The theory* $\mathbf{W}_{\mathrm{T}}$ *is the least* G-*theory on* $\mathbf{W}_{\mathrm{B}}$, *i.e.,* $\mathbf{W}_{\mathrm{T}} = C^{\mathrm{G}}_{\mathrm{B}}(\emptyset)$.

For any theory T, let E(T) be the set of all equations in T. Then a theory T is called EA-*theory* if it has only equational axioms, i.e., E(T) = T. Furthermore, the theory $C(\mathrm{E(T)})$ is called the *equational kernel* of T, denoted by Ker(T). Obviously, a theory T is an EA-theory if and only if T = Ker(T). Then we have the following two propositions (see [65]).

Proposition 2.2.8 *Let* T *be a* G-*theory. Then we have:*

(i) T *is an* EA-*theory, i.e.,* T = Ker(T),

(ii) *all tautology axioms* (TA1) *are theorems of* T,

(iii) *all formulas* $(A \equiv B) \leftrightarrow ((A \equiv B) \equiv \top)$ *are in* T,

(iv) $A \equiv \top$ *is in* T *whenever* $(A \rightarrow B) \equiv \top$ *and* $A \equiv \top$ *are in* T.

Proposition 2.2.9 *Let* $\mathrm{T} = \mathrm{C(D)}$ *where* $\mathrm{D} = \mathrm{E(D)}$, *namely* T *is an* EA-*theory. If* $A \equiv \top$ *is in* T *for every logical axiom* A *in* $\mathrm{TFA} \cup \mathrm{IDA}$ *and all formulas* $(A \rightarrow B) \equiv \top \rightarrow ((A \equiv \top) \rightarrow (B \equiv \top))$ *are in* T, *then* T *is a* G-*theory.*

2.2.3 Semantics of SCI

We will interpret the **SCI**-language $\mathcal{L}_{\mathrm{S}}$ by using the matrix semantics. An **SCI**-algebra $\mathcal{A} = \langle A, -, \cap, \cup, \supset, \supset\subset, \Delta, \mathrm{f}, \mathrm{t}\rangle$ is an algebra of type $\langle 1, 2, 2, 2, 2, 2\rangle$ such that A is a non-empty set, $-$ (complemant), $\cap$ (meet), $\cup$ (join), Δ (delta), and $a \supset b = -a \cup b$ and $a \supset\subset b = (a \supset b) \cap (b \supset a)$ for any $a, b \in A$ (see [11] and [67]). The class of **SCI**-algebra is very board, and it includes, in particular Boolean algebras (with an additional binary operator Δ). Here $\mathrm{f} = a \cap -a$ and $\mathrm{t} = a \cup -a$ are zero and unit of Boolean Algebra, respectively. Given an **SCI**-algebra $\mathcal{A}$, assume its universe A is divided into two non-empty subsets. Denote one of them by F and suppose that F is related to the operations in $\mathcal{A}$ as follows: for any $a, b \in A$,

(F1) $-a \in F$ if and only if a is not in F,

(F2) $a \cap b \in F$ if and only if both a and b are in F,

(F3) $a \cup b \in F$ if and only if either a or b is in F,

(F4) $a \supset b \in F$ if and only if $-a \cup b \in F$,

(F5) $a \supset\subset b \in F$ if and only if $(a \supset b) \cap (b \supset a) \in F$,

(F6) $a \Delta b \in F$ if and only if $a = b$.

Then we will call F as a *filter* of $\mathcal{A}$. Furthermore, a couple $\mathfrak{M} = (\mathcal{A}, F)$ consisting of an **SCI**-algebra $\mathcal{A}$ and a filter F in A, is called an **SCI**-*model* based on algebra $\mathcal{A}$. A formula B is *valid* in $\mathcal{A}$, in symbol, $\mathcal{A} \models B$ if for any valuation v of $\mathcal{A}$ and any filter F, $v(B) \in F$. Notice that $\mathbf{SCI} = (\mathcal{L}_{\mathrm{S}}, C(X))$ is an **SCI**-model whenever X is a consistent set of formulas. Then for any valuation v of $\mathcal{A}$, we can define the consequence operator $C_{\mathfrak{M}}$ relative to an **SCI**-model $\mathfrak{M}$ as follows.

Definition 2.2.10 *For any* $X \cup \{B\} \subseteq \mathrm{L_S}$, $B \in C_{\mathfrak{M}}(X)$ *if and only if for every* **SCI**-*model* $\mathfrak{M} = (\mathcal{A}, F)$ *and every valuation* v *of* $\mathcal{L}_{\mathrm{S}}$ *in* $\mathfrak{M}$, $v(B) \in F$ *whenever* $v(X) \subseteq F$.

In fact, S.L.Bloom showed the right class of algebraic structure for **SCI** by proving the following strong completeness of **SCI** (see [10]).

Theorem 2.2.11 **SCI** *is strongly complete with respect to an* **SCI**-*model, i.e.,* $C = C_{\mathfrak{M}}$.

2.3 Notes

In this section we will give a note which includes historical remarks and biblographical informations. The methodology of deductive system was invented by Alfred Tarski in 1930's. His papers from 1923 to 1938 appear in the book [71]. Also his academic achievements are summarized in the Journal of Symbolic Logic, Volume 51, Number 4, December 1986. Tarski analysed firstly the notion of logical consequence in languages with the classical implication $\to$ and negation $\neg$, and gave the following axiomatization of consequence operator C: for any X, Y and $\{A, B\} \subseteq L$,

(T1) $X \subseteq C(X)$,

(T2) $C(C(X)) \subseteq C(X)$,

(T3) $C(X) = \bigcup\{C(Y); Y \subseteq X$ and Y is finite$\}$,

(T4) $C(A) = L$ for some $A \in L$,

(T5) If $X \subseteq Y$ then $C(X) \subseteq C(Y)$,

$(\to)$ $B \in C(X; A)$ if and only if $A \to B \in C(X)$,

$(\neg)'$ $C(A) \cap C(\neg A) = C(\emptyset)$ and $C(A, \neg A) = L$.

Here the condition (T5) is derivable from (T3). Also under axioms $(\to)$ and (T3), $(\neg)'$ is equivalent to the following:

$(\neg)$ $C(X; \neg A) = L$ if and only if $A \in C(X)$.

Nowadays, an arbitrary function $C : \wp(L) \to \wp(L)$, satisfying merely (T1), (T2) and (T5), is called a *consequence operator*. Moreover, if C satisfies additionally (T3), then C is called *finitary*. As a weakening of (T4), we get that $C(X) = L$ implies $C(Y) = L$ for some finite $Y \subseteq X$, and then C is called *logically compact*. In [43] (also see [44]), G. Malinowski provided a generalization of Tarski's concept of consequence operator which have related to the idea that the rejection and acceptance need not be complementary. According to Malinowski's terminology, $\mathfrak{M} = (\mathcal{A}, D, \overline{D})$ is called a *q-matrix* for $\mathcal{L}$, where D and $\overline{D}$ denote to *accepted* and *rejected* designated elements of $\mathfrak{M}$, respectively. Given a q-matrix $\mathfrak{M}$, we can define the operator $W_{\mathfrak{M}} : \wp(L) \to \wp(L)$ as follows: for any $X \cup \{B\} \subseteq L$,

$$B \in W_{\mathfrak{M}}(X) \text{ if and only if } h(X) \cap \overline{D} = \emptyset \text{ implies } h(B) \in D \text{ for any } h \in \mathrm{HOM}(\mathcal{L}, \mathfrak{M}),$$

where $\mathrm{HOM}(\mathcal{L}, \mathfrak{M})$ is a class of all homomorphisms of $\mathcal{L}$ into $\mathfrak{M}$. Notice that if $\overline{D} \cup D = A$ then $W_{\mathfrak{M}}$ coincides with the consequence operator $C_{\mathfrak{N}}$ determined by the matrix $\mathfrak{N} = (\mathcal{A}, D)$. A syntactical counterpart of the above notion can also be defined as the following: any operator $W : \wp(L) \to \wp(L)$ is called a *q-consequence* on $\mathcal{L}$ if for any $X, Y \subseteq L$, it satisfies the following conditions:

(W1) if $X \subseteq Y$ then $W(X) \subseteq W(Y)$,

(W2) $W(X \cup W(X)) = W(X)$.

Moreover, W is called *structural* if for all substitution e of $\mathcal{L}$ and any $X \subseteq L$,

(S) $eW(X) \subseteq W(eX)$.

As alternative approaches, G. Gentzen (see [30]) studied, for the case of classical and intuitionistic logics, a relation between finite sets of formulas (premisses) and a single formula (conclusion) by expressing that the conjunction of premisses has the disjunction of logical conclusions. On the other hand, D. Scott (see [59]) provided a general framework for studing of the relation between finite sets of formulas, and moreover, D.J. Shoesmith and T.J. Smiley (see [61]) extended the framework onto the case of arbitrary sets. The relation $\vdash \subseteq \wp(L) \times \wp(L)$ is called *entailment relation* or *multiple-conclusion consequence* if for any subsets X, Y, Z of L, it satisfies the following conditions:

(R) if $X \cap Y \neq \emptyset$, then $X \vdash Y$, (Reflexivity)

(M) if $X \vdash Y$ and $X \subseteq X', Y \subseteq Y'$, then $X' \vdash Y'$, (Monotonicity)

(C) for all $Z \subseteq L$, $Z \cup X \vdash Y \cup (L - Z)$, then $X \vdash Y$. (Cut)

Moreover, $\vdash$ is called *structural* if for all substitution e of $\mathcal{L}$ and all $X, Y \subseteq L$,

(S) if $X \vdash Y$, then $eX \vdash eY$.

And $\vdash$ is called *finitary* if

(F) if $X \vdash Y$, then $X' \vdash Y'$ for some finite subsets X', Y' of X, Y respectively.

Here if $\vdash$ is finitary and satisfies (M), then $\vdash$ is closed under (C) if and only if $\vdash$ is closed under (C_f) below:

(C_f) for all $X, Y, \{A\} \subseteq L$, $X \vdash Y, A$ and $X, A \vdash Y$ implies $X \vdash Y$.

We can find in [78] one of intensive investigation for the concept of multiple-conclusion consequence.

R. Suszko invented his non-Fregean logics in the latter of 1960's. His academic achievements are summarized in the special issue of Studia Logic, Volume 43, Number 4, 1984. Moreover, both of the XXXth (Cracow, October 19–21, 1984) and XLVth (Kraków, October 26–27, 1999) History of Logic Conferences dedicated to him (see [54]). Although the research field of non-Fregean logics is not so popular, there exist several results which will be mentioned below. At first, various fragments of **SCI** were investigated in order to contrast identity connective with truth functional connectives. In [66], Suszko studied the relationship between equational logic based on identity predicate (which is well known in universal algebras) and equational logic based on identity connective. A. Michaels studied continuously the fragment EN-logic of **SCI** which only dealt with identity connective $\equiv$ and the truth functional connective of negation $\neg$ (see [47]). Moreover, W. Kielak studied the fragment ENE-logic of **SCI** which only dealt with identity connective $\equiv$ and the truth functional connectives of negation $\neg$ and equivalence $\leftrightarrow$ (see [38]). In [58], M.G. Rogava showed the Cut-elemination theorem of **SCI**. In [73], A. Wasilewska gave a new proof of the decidability theorem of **SCI**. G. Malinowski and M. Michalczyk gave two interpolation theorems of **SCI** (see [45]). Futhermore, M. Omyła studied **SCI** with quantifiers (see [49] and [50]), and P. Łukowski studied intuitionistic sentential calculus with identity (**ISCI** for short) (see [40] and [41]). Finally, as more philosophical point of view, B. Wolniewicz studied the ontology of Wittgenstein's *Tractatus* (see [77]), which was an underlying idea of **SCI**.

Chapter 3

PCI logics and $\mathbf{PCI}_{\mathrm{W}}$ extension for non-Fregean logic

In this chapter we will introduce the axiomatic deductive system of **PCI** and its related results in order to simulate various kinds of nonclassical logics. In general, nonclassical logics are divided into two types according to the construction, i.e., (i): classical logics with additional operators and (ii): weak logics with various kinds of weak implications, e.g., strict/relevance/linear implication. For example, in this thesis we will consider classical modal logics with necessary operator $\Box$ (see [15]) as the former type, and Angell's analytic containment logic **AC** with relevance entailment (see [2] and [3]), Corsi's weak logic **F** with strict implication (see [16]) and Girard's classical linear logic **GL** with linear implication (see [31]) as the latter type. Here the *simulation* of logics means syntactical translations between two logics, which satisfies the syntactic equivalence condition. As an example of simulation, this chapter devotes to demonstrate how Angell's analytic containment logic **AC** is simulated by **PCI** logic. Similarly, we will discuss the case of a classical modal logic **K**, and the cases of Corsi's weak logic **F** and Girard's classical linear logic **GL** in forthcoming two Chapters 4 and 5, respectively. At first we will explain the system **PCI** and its fundamental properties in Section 1. In Section 2, we will survey Angell's analytic containment logic **AC** briefly. Then in Section 3, we will define $\mathbf{PCI}_{\mathrm{W}}$ logic by adding two identity axioms (IT) and (IR) to the original system **PCI** in order to interpret Angell's analytic containment $\approx$ by identity $\equiv$. In Section 4, we will investigate a general method of showing syntactical equivalence between various logics. After this, in Section 5, we will give translations between **AC** and $\mathbf{PCI}_{\mathrm{W}}$, and hence prove that they are syntactically equivalent. Finally we will also give further information on related results shown in this chapter (Section 6).

In this thesis, we will assume that basic language underlying **PCI** and **K** mentioned above, is a propositional language $\mathcal{L} = \langle \mathrm{L}, \neg, \wedge, \vee, \rightarrow, \bot, \top \rangle$ consisting of an infinite denumerable set VAR of propositional variables, constants; $\bot$ (false) and $\top$ (true), and the standard truth functional connectives; $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (material implication). Formulas L of a given language $\mathcal{L}$ are defined in the usual way. The letters $p, q, r, p_1, p_2, p_3, \ldots$ will be used to denote propositional variables. We will use letters $A, B, C, \ldots$ to denote **PCI**'s formulas, and $X, Y, Z, \ldots$ to denote sets of formulas, while letters $\alpha, \beta, \gamma, \ldots$ to denote formulas in various kinds of nonclassical systems like **AC**, **K**, **F** and **GL**, and $\Gamma, \Delta, \Sigma, \ldots$ to denote sets of their formulas. The propositional constants $\bot, \top$ and another TF-connective $\leftrightarrow$ (material equivalence) are to be constructed as the usual abbreviation. For example in case of **PCI**, we have: $\bot := A \land \neg A$, $\top := \neg\bot := A \lor \neg A$ and $A \leftrightarrow B := (A \to B) \land (B \to A)$, and also we will use $G[A/p]$ to denote the formula obtained from G by replacing each occurrence of p by A. Moreover, we will sometime omit parentheses when no confusion will occur, following the assumption that the priority of each connective is weak as $\neg, \land, \lor, \to, \leftrightarrow$ in order.

3.1 Axiomatic deductive system of PCI

In this section we will introduce the axiomatic deductive system of **PCI** and explain its fundamental properties. The language of **PCI** is the same as the **SCI**-language $\mathcal{L}_\mathrm{S}$ in Section 2.2. The system **PCI** is obtained from **SCI** by deleting two axioms (C5) and (SI). Therefore the system **PCI** has the following identity axioms IDA and the rule of modus ponens (Mp) besides TFA in Section 2.2.

(E1) $A \equiv A$

(E2) $(A \equiv B) \to (B \equiv A)$

(E3) $(A \equiv B) \land (B \equiv C) \to (A \equiv C)$

(C1) $(A \equiv B) \to (\neg A \equiv \neg B)$

(C2) $(A \equiv B) \land (C \equiv D) \to (A \land C) \equiv (B \land D)$

(C3) $(A \equiv B) \land (C \equiv D) \to (A \lor C) \equiv (B \lor D)$

(C4) $(A \equiv B) \land (C \equiv D) \to (A \to C) \equiv (B \to D)$

(Mp) $\dfrac{A \quad A \to B}{B}$

Here, in contrast to **SCI**, we can not assume in **PCI** that identity connective $\equiv$ is a congruence relation on $\mathcal{L}_\mathrm{S}$, and also at least as strong as material equivalence $\leftrightarrow$. But in **PCI**, the identity axioms IDA imply that identity connective $\equiv$ is an equivalence relation on $\mathcal{L}_\mathrm{S}$ while it is not a congruence relation. Both material equivalence $\leftrightarrow$ and identity $\equiv$ preserve TF-connectives $(\neg, \land, \lor, \to)$ but they are mutually independent. Then the axiomatic deductive system $C(X)$ for $\mathbf{PCI} = (\mathcal{L}_\mathrm{S}, C)$ is defined as follows.

Definition 3.1.1 (i) *For any $X \subseteq \mathrm{L}_\mathrm{S}$, $C(X)$ is the smallest set of formulas closed under the rule* (Mp)*, which contains* TFA, IDA *and X.*

(ii) *The element of* $C(\emptyset)$ *is called the logical theorem of* **PCI**.

It is easily verified that C is a consequence operator. By the similarity to Proposition 2.2.2, we have the following.

Proposition 3.1.2 *For any* $X \cup \{A, B\} \subseteq \mathrm{L_P}$, *it holds the following equivalences:*

(i) $B \in C(X; A)$ *if and only if* $A \to B \in C(X)$, (*Deduction Theorem*)

(ii) $\neg A \in C(X)$ *if and only if* $\bot \in C(X; A)$,

(iii) $A \in C(X)$ *if and only if there exist some finite subset* Y *of* X *such that* $A \in C(Y)$. (*Compactness*)

Proposition 3.1.3 *The following are logical theorems and derived rules of* **PCI**.

(i) $(A \equiv B) \leftrightarrow (B \equiv A)$

(ii) $A \equiv B \to ((A \to B) \equiv (B \to A))$

(iii) $\dfrac{A \equiv B}{A \leftrightarrow B}$

(iv) $\dfrac{A \quad A \equiv B}{B}$

Proof. (i), (ii) are straightforward. (iii): The identity connective $\equiv$ is a equivalence relation by (E1)–(E3), and moreover preserves all TF-connectives $(\neg, \wedge, \vee, \to)$. So if $A \equiv B$ then $A \leftrightarrow B$. (iv): this is clear by (iii) and (Mp).

■

Here note that the replacement law ,i.e., $A \equiv B \to (G[A/p] \equiv G[B/p])$, does not hold in **PCI**, different from **SCI**. Let C_0 be the consequence operation defined only from the rule (Mp) and the axioms TFA. Then C_0 corresponds to the classical logic **CL**, and by the same reason as **SCI**, we can show that **PCI** is also a concervative extension of **CL**.

3.2 Angell's analytic containment logic AC

In this section we will give a brief survey of Angell's analytic containment logic **AC**. In [2, 3], R. B. Angell proposed the logic **AC** to treat entailment in relevant logic by using the concept of containment (or the sameness of meanings), in Kant's sense of analytic containment, which means that α entails β only if the meaning of β is contained in the meaning of α. Also he compared three systems of similar approach, i.e., Anderson and Belnap's **E** (for entailment) and Parry's **AI** (for analytic implication), and his own **AC** (for analytic containment) in the view of the syntactic conditions of entailment in the sense of containment, described in the following: where $\approx$ and $\leftrightarrow$ denote the sameness of meaning and material equivalence, respectively.

(Ia) If $(\alpha \approx \beta)$ is a theorem, then $(\alpha \leftrightarrow \beta)$ is a theorem of classical logic.

(Ib) If $(\alpha \approx \beta)$ is a theorem, then α and β must share at least one variable.

(Ic) If $(\alpha \approx \beta)$ is a theorem, then α and β contain all and only the same variables.

(Id) If $(\alpha \approx \beta)$ is a theorem, then a variable occurs *positively* (*negatively*) in β if and only if it occurs *positively* (*negatively*) in α.

(Ie) If $(\alpha \approx \beta)$ is a theorem, then a tautology or inconsistency is *implicit* in β if and only if it is *implicit* in α.

(If) If $(\alpha \approx \beta)$ is a theorem, then α and β have identical *maximal ordered normal forms.*

Here above notions of *positively (negatively)* occurrence, *implicit* and *maximal ordered normal form* appear precisely in [3].

Then it was shown that only the system **AC** admits above all conditions. Following Angell, we will introduce the axiomatic deductive system of **AC** in the following. Let $\mathcal{L}_{\mathrm{A}} = \langle \mathrm{L}_{\mathrm{A}}, \sim, \wedge, \approx \rangle$ be the **AC**-language consisting of an infinite denumerable set VAR of propositional variables and primitive connectives; $\sim$ (negation), $\wedge$ (conjunction) and $\approx$ (synonymity). Formulas L_{A} of a given **AC**-language $\mathcal{L}_{\mathrm{A}}$ are defined in the usual way. The letters $p, q, r, p_1, p_2, p_3, \ldots$ will be used to denote propositional variables; $\alpha, \beta, \gamma, \ldots$ will denote *TF-formulas of a* **AC**-*language* $\mathcal{L}_{\mathrm{A}}$ *which contains only TF-connectives* (and φ, ψ, χ denote formulas including those containing $\approx$); Γ, Δ, Σ will denote sets of formulas. The connectives $\vee$ (disjunction), $\rightarrow$ (material implication), $\rightsquigarrow$ (entailment) are to be constructed as the abbreviation: $\alpha \vee \beta :=\sim (\sim \alpha \wedge \sim \beta)$, $\alpha \rightarrow \beta :=\sim \alpha \vee \beta$ and $\alpha \rightsquigarrow \beta := (\alpha \approx \alpha \wedge \beta)$. Here the formula $(\alpha \rightsquigarrow \beta)$ may be interpreted as α *entails* β in the sense of α *analytically contains* β. Also we will sometime omit parentheses, following the assumption that the priority of each connective is weak as $\sim, \wedge, \vee, \approx, \rightsquigarrow, \rightarrow$ in order.

AC is axiomatized for first degree entailments, that is only treats formulas without nestings of $\rightsquigarrow$. The logical axioms and rules of inference for **AC**-language $\mathcal{L}_{\mathrm{A}}$ consist of a set of schemata from (a1) to (a5) and substitution (Sb), adjunction (Ad) and material implication (Im) as rules of inference below:

(a1) $\alpha \approx \sim\sim \alpha$

(a2) $\alpha \approx (\alpha \wedge \alpha)$

(a3) $(\alpha \wedge \beta) \approx (\beta \wedge \alpha)$

(a4) $(\alpha \wedge (\beta \wedge \gamma)) \approx ((\alpha \wedge \beta) \wedge \gamma)$

(a5) $(\alpha \vee (\beta \wedge \gamma)) \approx ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

(Sb) $\dfrac{\alpha \approx \beta \quad \varphi}{\varphi[\alpha/\beta]}$,where $\varphi[\alpha/\beta]$ means the result of replacing some β in φ by α.

(Ad) $\dfrac{\alpha \quad \beta}{\alpha \wedge \beta}$

(Im) $\dfrac{\alpha \rightsquigarrow \beta}{\alpha \to \beta}$

Then the axiomatic deductive system $AC(\Gamma)$ for $\mathbf{AC} = (\mathcal{L}_{\mathrm{A}}, AC)$ is defined as follows.

Definition 3.2.1 (i) *For any* $\Gamma \subseteq \mathrm{L}_{\mathrm{A}}$*,* $AC(\Gamma)$ *is the smallest set of formulas closed under rules of* (Sb)*,* (Ad) *and* (Im) *which contains from* (a1) *to* (a5) *and* Γ*.*

(ii) *The element of* $AC(\emptyset)$ *is called the logical theorem of* **AC***.*

It should be noticed that (Sb) rule is not the usual substitution, but a restricted substitution, i.e., substitution only for first degree entailments. And we also have the following remark (see [3]).

Remark 3.2.2 *For any formula* $\varphi \in \mathrm{L}_{\mathrm{A}}$*, if* $\varphi \in \mathbf{AC}$ *holds, then either* φ *does not contain* $\approx$ *connective at all or* φ *is of the form of* $\alpha \approx \beta$ *for some TF-formulas* $\alpha, \beta \in \mathrm{L}_{\mathrm{A}}$*.*

3.3 $\mathbf{PCI}_{\mathrm{W}}$ logic with identity as relevance entailment

In this section we will define $\mathbf{PCI}_{\mathrm{W}}$ logic as an extension of **PCI** in order to interpret Angell's analytic containment $\approx$ by identity $\equiv$ (see [34] and [35]). Then we need the following conditions in **PCI**:

(R1) $\overrightarrow{\alpha \rightsquigarrow \beta} \longmapsto \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta}$,

(R2) $\overleftarrow{A \equiv B} \longmapsto (\overleftarrow{A} \approx \overleftarrow{B})$,

where $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$ and $\overleftarrow{A}$, $\overleftarrow{B}$ denote the results of translations from **AC** to $\mathbf{PCI}_{\mathrm{W}}$, and its converse, respectively.

Since the identity connective $\equiv$ has to satisfy both rules (Sb) and (Im) of inference from requirement (R1), we need to add the following two identity axioms (IT) and (IR) in **PCI**.

(IT) $(A \equiv B) \wedge (C \equiv D) \to (A \equiv C) \equiv (B \equiv D)$

(IR) $(A \equiv B) \to (A \leftrightarrow B)$

Then we will get the extension $\mathbf{PCI}_{\mathrm{W}}$ of **PCI** below, which is nothing but non-Fregean logic **SCI** in 2.2. Here we can also consider each extensions of $\mathbf{PCI}_{\mathrm{W}}$ as the same way to extensions of **SCI**.

Definition 3.3.1 *Let* $\mathbf{PCI} = (\mathcal{L}_{\mathrm{S}}, C)$ *and* $X \subseteq \mathrm{L}_{\mathrm{S}}$*. Then* $\mathbf{PCI}_{\mathrm{W}} = (\mathcal{L}_{\mathrm{S}}, C_{\mathrm{W}})$ *is the elementary extension of* **PCI***, where* C_{W} *is a superconsequence of* C *defined by* $C_{\mathrm{W}}(X) = C(X; \mathrm{IT}, \mathrm{IR})$*.*

3.4 General method of identifing various logics

In this section we will investigate a general method of showing syntactical equivalence between various logics owing to mainly K. Segerberg's book [60]. For two logics which are formulated in very different object languages, we can intuitively say that these logics are the same or at least equivalent if they are equally strong, or they come to the same thing. We can also say this fact if the languages in which they are formulated are intertranslatable, namely if what can be also expressed in one language can be expressed in other one. And moreover, whenever a formula in one logic is valid, then its counterpart in the other is also valid. We will define the above notion of equivalent of logics more precisely in the following.

Suppose that $\mathbf{L_1}$ and $\mathbf{L_2}$ are two logics in the language $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathbf{L_1} = (\mathcal{L}_1, C_1)$ and $\mathbf{L_2} = (\mathcal{L}_2, C_2)$ where C_1 and C_2 are structural consequence operators on $\mathcal{L}_1$ and $\mathcal{L}_2$, and the sets of formulas of which are L_1 and L_2, respectively. Furthermore assume that the languages $\mathcal{L}_1$ and $\mathcal{L}_2$ have equivalence connectives $\leftrightarrow_1$ and $\leftrightarrow_2$, respectively. Then we define syntactically equivalent of two logics $\mathbf{L_1}$ and $\mathbf{L_2}$ as follows.

Definition 3.4.1 (i) $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are* syntactically equivalent with respect to t_1 and t_2 *if and only if* $t_1 : \mathrm{L}_1 \to \mathrm{L}_2$ *and* $t_2 : \mathrm{L}_2 \to \mathrm{L}_1$ *are functions such that the following conditions are satisfied:*

(1) *For all* $\alpha \in \mathrm{L}_1$, $(t_2(t_1(\alpha)) \leftrightarrow_1 \alpha) \in \mathbf{L_1}$,

(2) *For all* $A \in \mathrm{L}_2$, $(t_1(t_2(A)) \leftrightarrow_2 A) \in \mathbf{L_2}$,

(3) *For all* $\alpha \in \mathrm{L}_1$, $\alpha \in \mathbf{L_1}$ iff $t_1(\alpha) \in \mathbf{L_2}$,

(4) *For all* $A \in \mathrm{L}_2$, $A \in \mathbf{L_2}$ iff $t_2(A) \in \mathbf{L_1}$.

(ii) $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are called* syntactically equivalent *if there exist functions* t_1 *and* t_2 *with respect to which they are syntactically equivalent.*

The definition of the above syntactic equivalence can be understood intuitively as follows (see also Fig 3.1). Two functions t_1 and t_2 are to be understood as translations of one language into the other. Conditions (1) and (2) are to denote a way of checking that two translations do their jobs that at least they are inverse operations of one another. Conditions (3) and (4) are meant to guarantee that both translations preserve logical relationships.

Theorem 3.4.2 *Syntactic equivalene is an equivalence relation in the class of logics* $\mathbf{L} = (\mathcal{L}, C)$, *where* $\mathcal{L}$ *is a fixed language and* C *is a structural consequence operation on* $\mathcal{L}$.

Since syntactic equivalence is an equivalence relation, it partitions the class of all logics. Then in general a logic can simply be seen one of those equivalence classes.

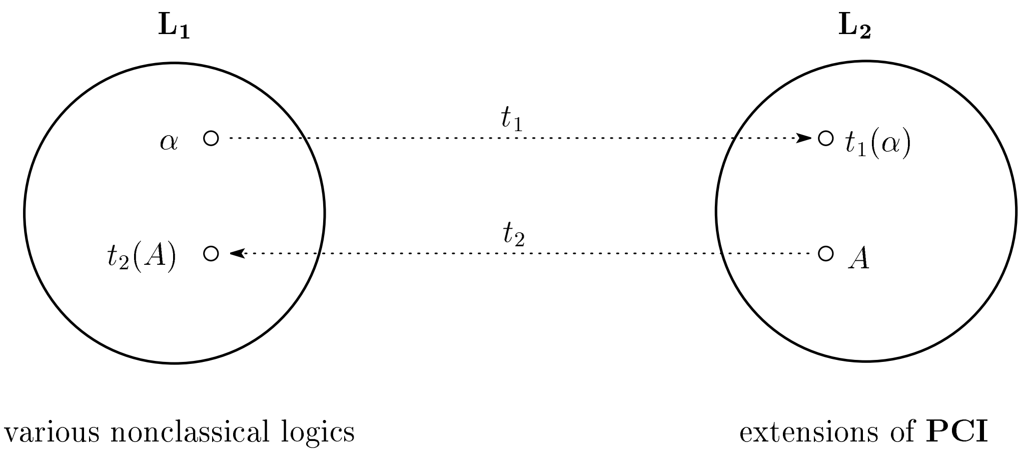

Figure 3.1: Syntactical equivalence of logics

We can use the word 'extension' as refer to either languages or logics. Suppose that $\mathcal{L}_1 = \langle \mathrm{VAR}_1, \mathrm{BOP}_1, \mathrm{AdOP}_1, \mathrm{RNK}_1 \rangle$ and $\mathcal{L}_2 = \langle \mathrm{VAR}_2, \mathrm{BOP}_2, \mathrm{AdOP}_2, \mathrm{RNK}_2 \rangle$ are languages, where VAR_1 and VAR_2 are denumerably infinite variables, BOP_1 and BOP_2 Boolean operators, AdOP_1 and AdOP_2 additional non-Boolean operators, and RNK_1 and RNK_2 ranks, respectively. Then we have the following definitions.

Definition 3.4.3 (i) *$\mathcal{L}_1$ is a* sublanguage *of $\mathcal{L}_2$ or $\mathcal{L}_2$ is an* extension *of $\mathcal{L}_1$ if the following conditions are satisfied:*

(1) $\mathrm{VAR}_1 \subseteq \mathrm{VAR}_2$,

(2) $\mathrm{BOP}_1 \subseteq \mathrm{BOP}_2$,

(3) $\mathrm{AdOP}_1 \subseteq \mathrm{AdOP}_2$,

(4) RNK_1 *and* RNK_2 *agree on* $\mathrm{BOP}_1 \cup \mathrm{AdOP}_1$.

(ii) *If $\mathbf{L_1} = (\mathcal{L}_1, C_1)$ and $\mathbf{L_2} = (\mathcal{L}_2, C_2)$ are logics on $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, and in addition to* (1)-(4), *also*

(5) $C_1 \subseteq C_2$,

then we say that $\mathbf{L_1}$ is a sublogic *of $\mathbf{L_2}$ or that $\mathbf{L_2}$ is an* extension *of $\mathbf{L_1}$.*

(iii) *Furthermore, an extension $\mathbf{L_2}$ of $\mathbf{L_1}$ is* conservative over $\mathbf{L_1}$ *if*

(6) $\mathbf{L_1} = \mathbf{L_2} \cap (\wp(\mathrm{L}_1) \times \wp(\mathrm{L}_1))$.

(iv) *An extension $\mathbf{L_2}$ of $\mathbf{L_1}$ is* definitional over $\mathbf{L_1}$ *if it is satisfied in addition to* (1)-(6), *also*

(7) $\mathrm{VAR}_1 = \mathrm{VAR}_2$.

Theorem 3.4.4 *If* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are logics such that* $\mathbf{L_2}$ *is a conservative definitional extension of* $\mathbf{L_1}$*, then* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are syntactically equivalent.*

Corollary 3.4.5 *Two logics are syntactically equivalent if there is a logic that is a conservative definitional extension of both.*

3.5 Translations between AC and $\mathbf{PCI}_{\mathrm{W}}$

The Angell's analytic containment language and its axiomatic deductive system **AC** were already introduced in Section 3.2. At first we will define two translations t_{A} and t_{P} between **AC**-language $\mathcal{L}_{\mathrm{A}}$ and **SCI**-language $\mathcal{L}_{\mathrm{S}}$ in order to show two logics **AC** and $\mathbf{PCI}_{\mathrm{W}}$ are syntactically equivalent with respect to these maps in the sense of Definition 3.4.1.

Definition 3.5.1 *Let* $\mathrm{L}^1_{\mathrm{A}}$ *be the set of* **AC** *formulas which contains only first degree entailments. Then the mapping* $t_{\mathrm{A}} : \mathrm{L}^1_{\mathrm{A}} \to \mathrm{L}_{\mathrm{S}}$*, called a* AC -translation*, is defined inductively as follows:*

(i) $t_{\mathrm{A}}(p) := p, \ \ p \in \mathrm{VAR},$

(ii) $t_{\mathrm{A}}(\sim \alpha) := (t_{\mathrm{A}}(\alpha) \to \bot),$

(iii) $t_{\mathrm{A}}(\alpha \wedge \beta) := (t_{\mathrm{A}}(\alpha) \wedge t_{\mathrm{A}}(\beta)),$

(iv) $t_{\mathrm{A}}(\alpha \approx \beta) := (t_{\mathrm{A}}(\alpha) \equiv t_{\mathrm{A}}(\beta)).$

Definition 3.5.2 *The mapping* $t_{\mathrm{P}} : \mathrm{L}_{\mathrm{S}} \to \mathrm{L}_{\mathrm{A}}$*, called a* PCI-translation*, is defined inductively as follows:*

(i) $t_{\mathrm{P}}(p) := p, \ \ p \in \mathrm{VAR}$

(ii) $t_{\mathrm{P}}(\bot) :=\sim (t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(A))$

(iii) $t_{\mathrm{P}}(\neg A) :=\sim t_{\mathrm{P}}(A)$

(iv) $t_{\mathrm{P}}(A \vee B) :=\sim (\sim t_{\mathrm{P}}(A) \wedge \sim t_{\mathrm{P}}(B))$

(v) $t_{\mathrm{P}}(A \wedge B) := (t_{\mathrm{P}}(A) \wedge t_{\mathrm{P}}(B))$

(vi) $t_{\mathrm{P}}(A \to B) := (t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(B))$

(vii) $t_{\mathrm{P}}(A \equiv B) := (t_{\mathrm{P}}(A) \approx t_{\mathrm{P}}(B))$

For two maps t_{A} and t_{P} we can prove the following two propositions.

Proposition 3.5.3 *For any formula* φ *in* $\mathrm{L}^1_{\mathrm{A}}$*,* $\varphi \in \mathbf{AC}$ *implies* $t_{\mathrm{A}}(\varphi) \in \mathbf{PCI}_{\mathrm{W}}$*.*

Proof. By induction on the length of derivation in **AC**.

(i) Base step: We have to check the provability of each axioms of **AC** in $\mathbf{PCI}_{\mathrm{W}}$ after a t_{A}-translation. But this is a routine work and will be omitted.

(ii) Induction step: We have to check the admissibility of (Sb),(Ad) and (Im) in $\mathbf{PCI}_{\mathrm{W}}$ after a t_{A}-translation. (Sb): Assume that $\alpha \approx \beta$, φ are provable in **AC**. Then by I.H. $t_{\mathrm{A}}(\alpha \approx \beta), t_{\mathrm{A}}(\varphi)$, i.e., $t_{\mathrm{A}}(\alpha) \equiv t_{\mathrm{A}}(\beta), t_{\mathrm{A}}(\varphi)$ hold in $\mathbf{PCI}_{\mathrm{W}}$. Here $\mathbf{PCI}_{\mathrm{W}}$ is closed with respect to $\equiv$ because of the additional axiom (IT). Hence we have the derivation of $\varphi[\alpha/\beta]$ in $\mathbf{PCI}_{\mathrm{W}}$. (Ad): This case is trivial, since $\mathbf{PCI}_{\mathrm{W}}$ is based on classical logic. (Im): This case is also trivial because of the additional axiom (IR) in $\mathbf{PCI}_{\mathrm{W}}$.

■

Proposition 3.5.4 *For any formula A in $\mathrm{L_S}$ such that A has no nestings of $\equiv$, $A \in \mathbf{PCI}_{\mathrm{W}}$ implies $t_{\mathrm{P}}(A) \in$* **AC**.

Proof. By induction on the length of derivation in $\mathbf{PCI}_{\mathrm{W}}$.

(i) Base step: By noticing that we have also schemata which are obtained by replacing $\approx$ as $\leftrightarrow$ in (a1)–(a5), this case is trivial and will be omitted.

(ii) Induction step: We have to check the admissibility of (Mp) in **AC** after a t_{P}-translation. (Mp): Assume that A, $A \to B$ are provable in $\mathbf{PCI}_{\mathrm{W}}$. Then by I.H. $t_{\mathrm{P}}(A), t_{\mathrm{P}}(A \to B)$, i.e., $t_{\mathrm{P}}(A), t_{\mathrm{P}}(A) \leftrightarrow (t_{\mathrm{P}}(A) \wedge t_{\mathrm{P}}(B))$ hold in **AC**. Since **AC** is closed under substitution for logical equivalence, we can derive $t_{\mathrm{P}}(A) \wedge t_{\mathrm{P}}(B)$. So we have also the derivation of $t_{\mathrm{P}}(B)$ in **AC**.

■

Therefore we can prove the following two theorems.

Theorem 3.5.5 (i) *For any formula φ in $\mathrm{L}^1_{\mathrm{A}}$, $t_{\mathrm{P}}(t_{\mathrm{A}}(\varphi)) \leftrightarrow \varphi \in$* **AC**

(ii) *For any formula A in $\mathrm{L_S}$ such that A has no nestings of $\equiv$, $t_{\mathrm{A}}(t_{\mathrm{P}}(A)) \leftrightarrow A \in \mathbf{PCI}_{\mathrm{W}}$*

Proof. Both cases are almost trivial and will be omitted. ■

Theorem 3.5.6 (i) *For any formula φ in $\mathrm{L}^1_{\mathrm{A}}$, $\varphi \in$ **AC** if and only if $t_{\mathrm{A}}(\varphi) \in \mathbf{PCI}_{\mathrm{W}}$.*

(ii) *For any formula A in $\mathrm{L_S}$ such that A has no nestings of $\equiv$, $A \in \mathbf{PCI}_{\mathrm{W}}$ if and only if $t_{\mathrm{P}}(A) \in$ **AC**.*

Proof. By using Proposition 3.5.3, 3.5.4 and Theorem 3.5.5. ■

Hence we may conclude that two logics **AC** and $\mathbf{PCI}_{\mathrm{W}}$ are syntactically equivalent by Definition 3.4.1, Theorem 3.5.5 and Theorem 3.5.6.

3.6 Notes

In Section 3.1, we introduced the system **PCI** by deleting two axioms (C5) and (SI) from **SCI** in order to simulate a classical modal logic **K**. The reason why is that (C5) and (SI) correspond to axioms (4) and (T) of modal logics, respectively, whenever we will attempt to interpret modal necessitation $\Box$ by identity $\equiv$ (see Section 4.6). As a result, **PCI** is no longer a non-Fregean logic in the sense of R. Suszko. However, **PCI** is very interesting as a logical system because that various nonclassical logics can be simulated on some extensions of **PCI**. Then, we will postpone to interpret **PCI** philosophically in further research, after using this system in many directions.

In Section 3.4, we introduced the definition of syntactically equivalent between two logics. In fact, if two logics are syntactically equivalent, then moreover, we can translate any proofs of both logics mutually.

In Section 3.2, we mentioned three relevance systems **E**, **AI** and **AC**. Extensive studies of the first two systems appear in [1] and [53], respectively. The concepts of entailment in **E** and **AI** are often connected with containment and deducibility, respectively. All three systems reject coincidentally the *paradoxes of strict implication*, e.g., $\alpha \rightsquigarrow (\beta \vee \sim \beta)$ and $(\alpha \wedge \sim \alpha) \rightsquigarrow \beta$, since these formulas express neither relations of containment nor of deducibility. Here the first degree entailment theorems of **E** include all first degree entailment theorems of **AC**, and also **AI** contains **AC**. Let $\mathbf{L}^1$ be the first degree entailment theorems for any system **L**. Then we have the following relationships between **E**, **AI** and **AC** (see [2]):

$$
\begin{aligned}
(\mathbf{E}^1 \cap \mathbf{AI}^1) &= (\mathbf{AC} \oplus (\alpha \rightsquigarrow (\alpha \vee \sim \alpha)))^1 \\
\mathbf{E}^1 &= (\mathbf{AC} \oplus (\alpha \rightsquigarrow (\alpha \vee \beta)))^1 \\
\mathbf{AI}^1 &= (\mathbf{AC} \oplus ((\alpha \vee \beta) \rightsquigarrow (\alpha \vee \sim \alpha)) \oplus ((\alpha \vee (\beta \wedge \sim \beta)) \rightsquigarrow \alpha))^1 \\
(\mathbf{E}^1 \cup \mathbf{AI}^1) &= (\mathbf{AC} \oplus ((\alpha \vee \beta) \rightsquigarrow (\alpha \vee \sim \alpha)) \oplus ((\alpha \vee (\beta \wedge \sim \beta)) \approx \alpha))^1
\end{aligned}
$$

Chapter 4

$\mathbf{PCI}_{\mathrm{K}}$ extension for classical normal modal logic

In this chapter we will investigate how classical modal logics are simulated by **PCI** logic which have been introdued in the previous Chapter 3. R. Suszko showed already that modal systems **S4** and **S5** can be simulated on some extensions of **SCI**. Here we will concentrate on the weaker modal system **K**. In Section 1, we will give a brief survey of classical modal logics, particularly basic normal modal logic **K** and its axiomatic extensions **KT**, **KB**, **K4**, **KD**, **K5**, **S4** and **S5**, in syntactical and semantical points of view (see [15] and [52]). Then in Section 2, we will define $\mathbf{PCI}_{\mathrm{K}}$ logic by adding two identity axioms (WIA1) and (WIA2), and one inference rule (G) to the original system **PCI** in order to interpret correctly the necessary operator $\Box$ by identity $\equiv$. After this, in Section 3, we will give translations between **K** and $\mathbf{PCI}_{\mathrm{K}}$, and hence prove that they are syntactically equivalent in a sense of Definition 3.4.1. In Section 4, we will also introduce Kripke type semantics for $\mathbf{PCI}_{\mathrm{K}}$ logic by exchanging the validity of modal formulas in modal Kripke type semantics with new validity of identity formulas. Then we can show that **K** and $\mathbf{PCI}_{\mathrm{K}}$ are semantically equivalent relative to the same Kripke frame. So by invoking the completeness of modal logic, we will give a completeness theorem of $\mathbf{PCI}_{\mathrm{K}}$ relative to Kripke type semantics. In Section 5, we will define $\mathbf{PCI}_{\mathrm{K}}$-algebras, which is an algebraic counterpart of $\mathbf{PCI}_{\mathrm{K}}$ logics, and give a representation theorem (Theorem 4.5.8) of this algebras. Furthermore, we will give an alternative completeness result of $\mathbf{PCI}_{\mathrm{K}}$ logic by using the above representation theorem. So far mentioned results concern with relationships between **K** and $\mathbf{PCI}_{\mathrm{K}}$. But we can successfully extend their results to various extensions of modal logics. In Section 6, we will define several extensions of $\mathbf{PCI}_{\mathrm{K}}$ which are counterparts of modal extensions of **K**. Then as the similar way to $\mathbf{PCI}_{\mathrm{K}}$, we can consider translations between **K** extensions and $\mathbf{PCI}_{\mathrm{K}}$ extensions (Section 7), and moreover, Kripke type semantics for $\mathbf{PCI}_{\mathrm{K}}$ extensions (Section 8). Finally we will also give further information on related results shown in this chapter (Section 9).

4.1 Classical normal modal logics

In this section we will briefly survey classical modal logics in syntactical and semantical points of view. In particular, we will explain basic normal modal logic **K** and various axiomatic extensions of **K**.

4.1.1 Basic normal modal logic K and its axiomatic extensions

Let $\mathcal{L}_{\mathrm{K}} = \langle \mathrm{L}_{\mathrm{K}}, \neg, \wedge, \vee, \rightarrow, \Box, \bot, \top \rangle$ be the classical normal modal language, where $\langle \mathrm{L}, \neg, \wedge, \vee, \rightarrow, \bot, \top \rangle$ is an underlying propositional language and $\Box$ (necessary) is a unary operator. The logical axioms and rules of inference for **K**-language $\mathcal{L}_{\mathrm{K}}$ consist of sets of schemata TFA, which are same as from (A1) to (A10) in **SCI**, the additional axiom schema (K), and the necessitation (Ns) rule besides modus ponens below (see [15]):

(K) $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$

(Ns) $\frac{\alpha}{\Box\alpha}$

Then the axiomatic deductive system $K(\Gamma)$ for $\mathbf{K} = (\mathcal{L}_{\mathrm{K}}, K)$ is defined as follows.

Definition 4.1.1 (i) *For any* $\Gamma \subseteq \mathrm{L}_{\mathrm{K}}$*,* $K(\Gamma)$ *is the smallest set of formulas closed under the rules of* (Mp) *and* (Ns)*, which contains* TFA*,* (K) *and* Γ*.*

(ii) *The element of* $K(\emptyset)$ *is called the logical theorem of* **K***.*

Then it is easily verified that K is a consequence operator. By the similarity to Proposition 2.2.2, we have the following.

Theorem 4.1.2 *For any* $\Gamma \cup \{\alpha, \beta, \gamma\} \subseteq \mathrm{L}_{\mathrm{K}}$*, it holds the following equivalences:*

(i) $\neg\alpha \in K(\Gamma)$ *if and only if* $\bot \in K(\Gamma; \alpha)$

(ii) $\alpha \in K(\Gamma)$ *if and only if there exist some finite subset* Σ *of* Γ *such that* $\alpha \in K(\Sigma)$*.* (*Compactness*)

(iii) *For any* $p \in \mathrm{VAR}$*,* $(\alpha \equiv \beta) \rightarrow (\gamma[\alpha/p] \equiv \gamma[\beta/p])$ *is a logical theorem of* **K***.* (*Replacement Law*)

The elementary extension of **K** with an additional axiom α will be denoted by $\mathbf{K} \oplus \alpha$. Moreover, $\Diamond$ (possible) operator is the abbreviation of $\neg\Box\neg$. Then it is well-known in some literatures (see [15] and [52]) that there exist the following extensions of **K**:

(1) $\mathbf{KT} = \mathbf{K} \oplus (\Box\alpha \rightarrow \alpha)$

(2) $\mathbf{KB} = \mathbf{K} \oplus (\alpha \rightarrow \Box\Diamond\alpha)$

(3) $\mathbf{K4} = \mathbf{K} \oplus (\Box\alpha \to \Box\Box\alpha)$

(4) $\mathbf{KD} = \mathbf{K} \oplus (\Box\alpha \to \Diamond\alpha)$

(5) $\mathbf{K5} = \mathbf{K} \oplus (\Diamond\alpha \to \Box\Diamond\alpha)$

(6) $\mathbf{S4} = \mathbf{KT4} = \mathbf{K} \oplus (\Box\alpha \to \alpha) \oplus (\Box\alpha \to \Box\Box\alpha)$

(7) $\mathbf{S5} = \mathbf{KT5} = \mathbf{K} \oplus (\Box\alpha \to \alpha) \oplus (\Diamond\alpha \to \Box\Diamond\alpha)$

4.1.2 Kripke type semantics for normal modal logics

Let $\mathcal{F} = (W, R)$ be a modal Kripke frame for $\mathcal{L}_{\mathrm{K}}$, where W is a non-empty set and R a binary relation on W. Moreover, $\mathcal{M} = (W, R, V)$ is a modal Kripke model for $\mathcal{L}_{\mathrm{K}}$, where $\mathcal{F} = (W, R)$ is a modal Kripke frame and V a valuation on $\mathcal{F}$ which is a map from VAR to 2^W such that $V(p) \subseteq W$ for any $p \in \mathrm{VAR}$, $V(\bot) = \emptyset$ and $V(\top) = W$. Then for any point $a \in W$, we can extend V to the valuation of modal formulas $\models_{\mathrm{K}} : \mathrm{FOR}_{\mathrm{K}} \to 2^W$ as the following way.

Definition 4.1.3 *Given a modal Kripke model $\mathcal{M} = (W, R, V)$, the notion of validity of modal formulas at any point $a \in W$ is defined inductively as follows:*

(i) $\mathcal{M}, a \models_{\mathrm{K}} p$ *if and only if* $a \in V(p)$ *for any variable* $p \in \mathrm{VAR}$*,*

(ii) $\mathcal{M}, a \not\models_{\mathrm{K}} \bot$ *and* $\mathcal{M}, a \models_{\mathrm{K}} \top$*,*

(iii) $\mathcal{M}, a \models_{\mathrm{K}} \alpha \wedge \beta$ *if and only if* $\mathcal{M}, a \models_{\mathrm{K}} \alpha$ *and* $\mathcal{M}, a \models_{\mathrm{K}} \beta$*,*

(iv) $\mathcal{M}, a \models_{\mathrm{K}} \alpha \vee \beta$ *if and only if* $\mathcal{M}, a \models_{\mathrm{K}} \alpha$ *or* $\mathcal{M}, a \models_{\mathrm{K}} \beta$*,*

(v) $\mathcal{M}, a \models_{\mathrm{K}} \alpha \to \beta$ *if and only if* $\mathcal{M}, a \models_{\mathrm{K}} \alpha$ *implies* $\mathcal{M}, a \models_{\mathrm{K}} \beta$*,*

(vi) $\mathcal{M}, a \models_{\mathrm{K}} \Box\alpha$ *if and only if for all b with aRb,* $\mathcal{M}, b \models_{\mathrm{K}} \alpha$*.*

For any Kripke frame $\mathcal{F} = (W, R)$, a formula α is *valid* on $\mathcal{F}$, in symbols, $\mathcal{F} \models_{\mathrm{K}} \alpha$ if $\mathcal{M}, a \models_{\mathrm{K}} \alpha$ for any $a \in W$ and any valuation $\models_{\mathrm{K}}$. Then the logics recalled so far are well-known to be sound and complete with respect to natural classes of modal Kripke frames below (see [15] and [52]).

Theorem 4.1.4 (Modal completeness) *For any formula $\varphi \in \mathrm{L}_{\mathrm{K}}$ and any modal Kripke frame $\mathcal{F} = (W, R)$, it holds the following equivalence:*

(i) $\alpha \in \mathbf{K}$ *if and only if* $\mathcal{F} \models_{\mathrm{K}} \alpha$ *for all* $\mathcal{F}$*.*

(ii) $\alpha \in \mathbf{KT}$ *if and only if* $\mathcal{F} \models_{\mathrm{K}} \alpha$ *for all $\mathcal{F}$ such that R is reflexive.*

(iii) $\alpha \in \mathbf{KB}$ *if and only if* $\mathcal{F} \models_{\mathrm{K}} \alpha$ *for all $\mathcal{F}$ such that R is symmetric.*

(iv) $\alpha \in$ **K4** *if and only if* $\mathcal{F} \models_{\mathrm{K}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is transitive.*

(v) $\alpha \in$ **KD** *if and only if* $\mathcal{F} \models_{\mathrm{K}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is serial.*

(vi) $\alpha \in$ **K5** *if and only if* $\mathcal{F} \models_{\mathrm{K}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is Euclidean.*

(vii) $\alpha \in$ **S4** *if and only if* $\mathcal{F} \models_{\mathrm{K}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is quasi-ordered.*

(viii) $\alpha \in$ **S5** *if and only if* $\mathcal{F} \models_{\mathrm{K}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is equivalence.*

4.2 $\mathbf{PCI}_{\mathrm{K}}$ logic with identity as modality

In this section we will define $\mathbf{PCI}_{\mathrm{K}}$ logic as an extension of **PCI** in order to interpret the sameness of modal necessitation $\Box$ by identity $\equiv$ (see [34] and [35]). Then we need the following conditions in **PCI** (see also Fig 4.1):

(R3) $\overrightarrow{\Box\alpha} \longmapsto (\overrightarrow{\alpha} \equiv \top)$,

(R4) $\overleftarrow{A \equiv B} \longmapsto \Box(\overleftarrow{A} \leftrightarrow \overleftarrow{B})$,

where $\overrightarrow{\alpha}$ and $\overleftarrow{A}$, $\overleftarrow{B}$ denote the results of translations from **K** to $\mathbf{PCI}_{\mathrm{K}}$, and its converse, respectively.

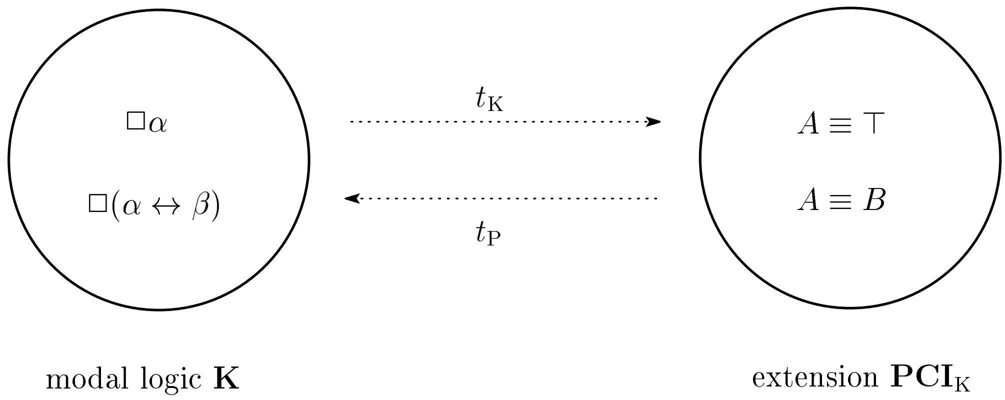

Figure 4.1: Requirements of simulation of **K**

Here we notice that $A \equiv B \to ((A \to B) \equiv (B \to A))$ and $(A \to B) \equiv \top \wedge (B \to A) \equiv \top \to (A \to B) \equiv (B \to A)$ are theorems of **PCI** by Theorem 3.1.3 (ii) and axiom (E3), respectively. Hence in order to satisfy (R3) under (R4), we need to add the following two identity axioms (WIA1) and (WIA2) in **PCI**. Moreover, we also need to satisfy (G) rule in **PCI** from (Ns) rule in **K**.

(WIA1) $((A \to B) \equiv (B \to A)) \to (A \equiv B)$

(WIA2) $((A \to B) \equiv (B \to A)) \to ((A \to B) \equiv \top) \wedge ((B \to A) \equiv \top)$

(G) $\dfrac{A \quad B}{A \equiv B}$

Then we will get the extension $\mathbf{PCI}_{\mathrm{K}}$ of **PCI** below, in which we can show to satisfy the counterpart of axiom (K) in **K** (see Theorem 4.2.2 (xiv)).

Definition 4.2.1 *Let* $\mathbf{PCI} = (\mathcal{L}_{\mathrm{S}}, C)$ *be* **PCI** *logic,* C^{G} *a G-theory of* C *and* $X \subseteq \mathrm{L}_{\mathrm{S}}$*. Then* $\mathbf{PCI}_{\mathrm{K}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{K}})$ *is a non-elementary extension of* **PCI***, where* $C^{\mathrm{G}}_{\mathrm{K}}$ *is a superconsequence of* C *defined by* $C^{\mathrm{G}}_{\mathrm{K}}(X) = C^{\mathrm{G}}(X; \mathrm{WIA1}, \mathrm{WIA2})$.

Then two axioms (E1) and (C1) in **PCI** are derivable in $\mathbf{PCI}_{\mathrm{K}}$. Also we have the following theorems in $\mathbf{PCI}_{\mathrm{K}}$.

Theorem 4.2.2 *The following are derived rules and logical theorems of* $\mathbf{PCI}_{\mathrm{K}}$*.*

(i) $\dfrac{(A \to B) \equiv (B \to A)}{A \equiv B}$ (WI)

(ii) $\dfrac{A \leftrightarrow B}{A \equiv B}$ (QF)

(iii) $A \equiv A$ (E1)

(iv) $(A \equiv B) \to (\neg A \equiv \neg B)$ (C1)

(v) $(A \equiv B) \leftrightarrow (B \equiv A)$ *and* $(A \equiv B) \equiv (B \equiv A)$

(vi) $((A \to B) \equiv (B \to A)) \leftrightarrow (A \equiv B)$ *and* $((A \to B) \equiv (B \to A)) \equiv (A \equiv B)$ (SIA)

(vii) $\neg\neg A \equiv A$

(viii) $(\neg A \equiv B) \leftrightarrow (A \equiv \neg B)$ *and* $(\neg A \equiv B) \equiv (A \equiv \neg B)$

(ix) $(\neg A \equiv \neg B) \leftrightarrow (A \equiv B)$ *and* $(\neg A \equiv \neg B) \equiv (A \equiv B)$

(x) $\neg\bot \equiv \top$ *and* $\bot \equiv \neg\top$

(xi) $\top \equiv (A \vee \neg A)$ *and* $\bot \equiv (A \wedge \neg A)$ *and* $\top \equiv (A \equiv A)$

(xii) $((A \wedge B) \equiv A) \leftrightarrow ((A \vee B) \equiv B)$ *and* $((A \wedge B) \equiv A) \equiv ((A \vee B) \equiv B)$

(xiii) $((A \to B) \equiv \top) \leftrightarrow ((A \wedge \neg B) \equiv \bot)$ *and* $((A \to B) \equiv \top) \equiv ((A \wedge \neg B) \equiv \bot)$

(xiv) $((A \to B) \equiv \top) \to ((A \equiv \top) \to (B \equiv \top))$

(xv) $((A \wedge B) \equiv \top) \leftrightarrow ((A \equiv \top) \wedge (B \equiv \top))$ *and* $((A \wedge B) \equiv \top) \equiv ((A \equiv \top) \wedge (B \equiv \top))$

(xvi) $(A \equiv B) \leftrightarrow ((A \leftrightarrow B) \equiv \top)$ *and* $(A \equiv B) \equiv ((A \leftrightarrow B) \equiv \top)$ (R1)

(xvii) $(A \equiv B) \leftrightarrow ((A \equiv A \wedge B) \wedge (B \equiv B \wedge A))$

Proof. All the proof of (iii)–(v) and (vii)–(xiii) are straightforward and will be omitted. (i): This is derived directly from (WIA1). (ii): Suppose $A \leftrightarrow B$. Then both $A \to B$ and $B \to A$ hold. So we can apply (G) rule to these and get $(A \to B) \equiv (B \to A)$. Then (WI) rule yields the desired result. (vi): $((A \to B) \equiv (B \to A)) \to (A \equiv B)$ is a (WIA1) itself. To show the other direction note that $(A \equiv B) \to ((A \equiv B) \wedge (B \equiv A)) \to ((A \to B) \equiv (B \to A))$ hold by (E2),(C4). The second part follows immediately from (QF) rule. (xiv): Notice that $(\top \to B) \leftrightarrow B$ is a theorem and also (1): $(\top \to B) \equiv B$ by (QF) rule. Moreover, by (E1),(C4) we have $((A \equiv \top) \wedge (B \equiv B)) \to ((A \to B) \equiv (\top \to B))$ and apply (E3) and (2) to this we get (2): $(A \equiv \top) \to (A \to B) \equiv B$. Since we have also (3): $((A \to B) \equiv B) \to (((A \to B) \equiv \top) \to (B \equiv \top))$ by (A5) ,(E3) combining the results (2) and (3) yield $(A \equiv \top) \to (((A \to B) \equiv \top) \to (B \equiv \top))$. Note finally that TFA permits the exchange of premise we get the desired result. (xv): In order to prove $((A \wedge B) \equiv \top) \to ((A \equiv \top) \wedge (B \equiv \top))$ take the axiom (A3): $(A \wedge B) \to A$. Then by (G) rule and (xiv) we get $(((A \wedge B) \to A) \equiv \top) \to (((A \wedge B) \equiv \top) \to (A \equiv \top))$ and similarlly $(((A \wedge B) \to B) \equiv \top) \to (((A \wedge B) \equiv \top) \to (B \equiv \top))$. Therefore by (A5) we get the desired result. To prove the converse direction assume $(A \equiv \top) \wedge (B \equiv \top)$ and apply (C2), then we get the result from (QF) rule. (xvi): At first by (vi),(WIA2) and (xv) we can show $(A \equiv B) \leftrightarrow ((A \leftrightarrow B) \equiv \top)$. Then by (QF) rule the result follows. (xvii):

$$\dfrac{\dfrac{A \equiv B \to A \equiv B \quad \dfrac{\begin{matrix}(\mathrm{E1})\\ \vdots \\ A \equiv A\end{matrix}}{A \equiv B \to A \equiv A}\,(\mathrm{A1}),(\mathrm{Mp})}{A \equiv B \to A \equiv B \wedge A \equiv A}\,(\mathrm{A5}),(\mathrm{Mp}) \qquad \begin{matrix}(\mathrm{C2}), \mathrm{A} \equiv \mathrm{A} \wedge \mathrm{A}\\ \vdots \\ A \equiv B \wedge A \equiv A \to A \equiv A \wedge B\end{matrix}}{A \equiv B \to A \equiv A \wedge B}\,(\mathrm{A2}),(\mathrm{Mp}).$$

Moreover, by the similar way, we get also $A \equiv B \to B \equiv B \wedge A$. Hence by (A5),(Mp) we get $A \equiv B \to (A \equiv A \wedge B) \wedge (B \equiv B \wedge A)$. The converse is a trivial by (E2),(E3).

■

4.3 Translations between K and PCI$_{\mathrm{K}}$

In this section we will give translations between **K** and **PCI**$_{\mathrm{K}}$, and hence prove that they are syntactically equivalent. How to show the syntactically equivalent of two logics follows the previous discipline in Section 3.4. At first we will define two translations t_{K} and t_{P} between **K**-language $\mathcal{L}_{\mathrm{K}}$ and **SCI**-language $\mathcal{L}_{\mathrm{S}}$ in order to show two logics **K** and **PCI**$_{\mathrm{K}}$ are syntactically equivalent with respect to these maps.

Definition 4.3.1 *The mapping* $t_{\mathrm{K}} : \mathrm{L}_{\mathrm{K}} \to \mathrm{L}_{\mathrm{S}}$*, called a* K-translation*, is defined inductively as follows:*

(i) $t_{\mathrm{K}}(p) := p, \;\; p \in \mathrm{VAR},$

(ii) $t_K(\bot) := \bot$,

(iii) $t_K(\neg\alpha) := \neg t_K(\alpha)$,

(iv) $t_K(\alpha \wedge \beta) := (t_K(\alpha) \wedge t_K(\beta))$,

(v) $t_K(\alpha \vee \beta) := (t_K(\alpha) \vee t_K(\beta))$,

(vi) $t_K(\alpha \to \beta) := (t_K(\alpha) \to t_K(\beta))$,

(vii) $t_K(\Box\alpha) := (t_K(\alpha) \equiv \top)$.

Definition 4.3.2 *The mapping* $t_P : \mathrm{L_S} \to \mathrm{L_K}$*, called a* PCI-translation*, is defined inductively as follows:*

(i) $t_P(p) := p, \ \ p \in \mathrm{VAR}$,

(ii) $t_P(\bot) := \bot$,

(iii) $t_P(\neg A) := \neg t_P(A)$,

(iv) $t_P(A \wedge B) := (t_P(A) \wedge t_P(B))$,

(v) $t_P(A \vee B) := (t_P(A) \vee t_P(B))$,

(vi) $t_P(A \to B) := (t_P(A) \to t_P(B))$,

(vii) $t_P(A \equiv B) := \Box(t_P(A) \leftrightarrow t_P(B))$.

For two maps t_K and t_P we can prove the following propositions.

Proposition 4.3.3 *For any formula* α *in* $\mathrm{L_K}$*,* $\alpha \in \mathbf{K}$ *implies* $t_K(\alpha) \in \mathbf{PCI}_K$.

Proof. By induction on the length of derivation in $\mathbf{K}$.

(i) Base step: We have to check the provability of each axioms of $\mathbf{K}$ in $\mathbf{PCI}_K$ after a t_K-translation. The case of TFA is trivial since every t_K-translation preserves the structure of TF-connectives and also $\mathbf{PCI}_K$ has TFA axioms. So we only consider the case of (K).

$$t_K(\alpha) = t_K(\Box(\alpha_1 \to \alpha_2) \to (\Box\alpha_1 \to \Box\alpha_2))$$
$$= ((t_K(\alpha_1) \to t_K(\alpha_2)) \equiv \top) \to ((t_K(\alpha_1) \equiv \top) \to (t_K(\alpha_2) \equiv \top))$$

Then, this is a theorem of $\mathbf{PCI}_K$ because of Theorem 4.2.2 (xiv). Hence, $t_K(\alpha) \in \mathbf{PCI}_K$.

(ii) Induction step: Assume that we have established the theorem for some step, and consider a new derivation from these by applying inference rules of $\mathbf{K}$.
(Mp): This case is trivial since every t_K-translation preserves the structure of TF-connectives and $\mathbf{PCI}_K$ also has (Mp) rule.
(Ns): Assume that $t_K(\alpha_1) \in \mathbf{PCI}_K$ holds by I.H. Moreover, $\top \in \mathbf{PCI}_K$ holds. Hence, by (G) rule we get $(t_K(\alpha_1) \equiv \top) \in \mathbf{PCI}_K$. But $t_K(\alpha) = t_K(\Box\alpha_1) = (t_K(\alpha_1) \equiv \top)$ by the definition, so $t_K(\alpha) \in \mathbf{PCI}_K$.

Thus the t_{K}-translation of any formula provable in **K** is also provable in $\mathbf{PCI}_{\mathrm{K}}$.

■

Proposition 4.3.4 *For any formula A in $\mathrm{L_S}$, $A \in \mathbf{PCI}_{\mathrm{K}}$ implies $t_{\mathrm{P}}(A) \in \mathbf{K}$.*

Proof. By induction on the length of derivation in $\mathbf{PCI}_{\mathrm{K}}$.

(i) Base step: We have to check the provability of each axioms of $\mathbf{PCI}_{\mathrm{K}}$ in **K** after a t_{P}-translation. The case of TFA is trivial because of the similar reason in the above Proposition 4.3.3. Next we will consider IDA and show the typical cases of them below.

1) $A = (\mathrm{C4})$

$$\begin{aligned} t_{\mathrm{P}}(A) &= t_{\mathrm{P}}((A_1 \equiv B_1) \wedge (C_1 \equiv D_1) \to (A_1 \to C_1) \equiv (B_1 \to D_1)) \\ &= \Box(t_{\mathrm{P}}(A_1) \leftrightarrow t_{\mathrm{P}}(B_1)) \wedge \Box(t_{\mathrm{P}}(C_1) \leftrightarrow t_{\mathrm{P}}(D_1)) \\ &\quad \to \Box((t_{\mathrm{P}}(A_1) \to t_{\mathrm{P}}(C_1)) \leftrightarrow (t_{\mathrm{P}}(B_1) \to t_{\mathrm{P}}(D_1))) \end{aligned}$$

Then, this is clearly a theorem of **K**, so $t_{\mathrm{P}}(A) \in \mathbf{K}$.

2) $A = (\mathrm{WIA1})$

$$\begin{aligned} t_{\mathrm{P}}(A) &= t_{\mathrm{P}}(((A_1 \to B_1) \equiv (B_1 \to A_1)) \to (A_1 \equiv B_1)) \\ &= \Box((t_{\mathrm{P}}(A_1) \to t_{\mathrm{P}}(B_1)) \leftrightarrow (t_{\mathrm{P}}(B_1) \to t_{\mathrm{P}}(A_1))) \to \Box(t_{\mathrm{P}}(A_1) \leftrightarrow t_{\mathrm{P}}(B_1)) \end{aligned}$$

Then, this is clearly a theorem of **K**, so $t_{\mathrm{P}}(A) \in \mathbf{K}$.

3) $A = (\mathrm{WIA2})$

$$\begin{aligned} t_{\mathrm{P}}(A) &= t_{\mathrm{P}}(((A_1 \to B_1) \equiv (B_1 \to A_1)) \to ((A_1 \to B_1) \equiv \top) \wedge ((B_1 \to A_1) \equiv \top) \\ &= \Box((t_{\mathrm{P}}(A_1) \to t_{\mathrm{P}}(B_1)) \leftrightarrow (t_{\mathrm{P}}(B_1) \to t_{\mathrm{P}}(A_1))) \\ &\quad \to \Box((t_{\mathrm{P}}(A_1) \to t_{\mathrm{P}}(B_1)) \leftrightarrow t_{\mathrm{P}}(\top)) \wedge \Box((t_{\mathrm{P}}(B_1) \to t_{\mathrm{P}}(A_1)) \leftrightarrow t_{\mathrm{P}}(\top)) \end{aligned}$$

Then, this is clearly a theorem of **K**, so $t_{\mathrm{P}}(A) \in \mathbf{K}$.

(ii) Induction step: Assume that we have established the theorem for some step, and consider a new derivation from these by applying inference rules of $\mathbf{PCI}_{\mathrm{K}}$.

(Mp): This case is trivial because of the similar reason in the above Proposition 4.3.3.

(G): Assume that both $t_{\mathrm{P}}(A_1)$ and $t_{\mathrm{P}}(B_1)$ are theorem of **K** by I.H. Then, it is possible to derive the following proof in **K**.

$$\dfrac{\dfrac{\dfrac{P(A_1)}{t_{\mathrm{P}}(B_1) \to t_{\mathrm{P}}(A_1)} \quad \dfrac{P(B_1)}{t_{\mathrm{P}}(A_1) \to t_{\mathrm{P}}(B_1)}\ (\mathrm{A1})}{t_{\mathrm{P}}(A_1) \leftrightarrow t_{\mathrm{P}}(B_1)}\ (\mathrm{A5})}{\Box(t_{\mathrm{P}}(A_1) \leftrightarrow t_{\mathrm{P}}(B_1))}\ (\mathrm{Ns})$$

Hence, by the definition we get $\Box(t_{\mathrm{P}}(A_1) \leftrightarrow t_{\mathrm{P}}(B_1)) \in \mathbf{K}$. But $t_{\mathrm{P}}(A) = t_{\mathrm{P}}(A_1 \equiv B_1) = \Box(t_{\mathrm{P}}(A_1) \leftrightarrow t_{\mathrm{P}}(B_1))$ by the definition, so $t_{\mathrm{P}}(A) \in \mathbf{K}$.

Thus the t_{P}-translation of any formula provable in $\mathbf{PCI}_{\mathrm{K}}$ is also provable in **K**.

■

Moreover, we can show the following.

Theorem 4.3.5 (i) *For any formula α in $\mathrm{L_K}$, $t_\mathrm{P}(t_\mathrm{K}(\alpha)) \leftrightarrow \alpha \in \mathbf{K}$.*

(ii) *For any formula A in $\mathrm{L_S}$, $t_\mathrm{K}(t_\mathrm{P}(A)) \leftrightarrow A \in \mathbf{PCI}_\mathrm{K}$.*

Proof. (i): By induction on the length of the formula α. It is clear that base step and induction steps for TF connectives $(\neg, \wedge, \vee, \rightarrow)$ hold, so we will only attention to $\Box$ operator. Assume that $t_\mathrm{P}(t_\mathrm{K}(\alpha_1)) \leftrightarrow \alpha_1 \in \mathbf{K}$ holds and consider the formula $\alpha = \Box\alpha_1$. Then in $\mathbf{K}$, the following equivalences hold.

$$
\begin{aligned}
t_\mathrm{P}(t_\mathrm{K}(\Box\alpha_1)) &\leftrightarrow t_\mathrm{P}(t_\mathrm{K}(\alpha_1) \equiv \top) && \text{(Def.4.3.1(vii))}\\
&\leftrightarrow \Box(t_\mathrm{P}(t_\mathrm{K}(\alpha_1)) \leftrightarrow t_\mathrm{P}(\top)) && \text{(Def.4.3.2(vii))}\\
&\leftrightarrow \Box(\alpha_1 \leftrightarrow \top) && \text{(I.H.,}t_\mathrm{P}(\top) \leftrightarrow \top)\\
&\leftrightarrow \Box\alpha_1 && ((\alpha_1 \leftrightarrow \top) \leftrightarrow (\alpha_1 \wedge \top) \leftrightarrow \alpha_1)
\end{aligned}
$$

(ii): By induction on the length of the formula A. For the same reasons of (i) we will only attention to identity connective. Assume that both $t_\mathrm{K}(t_\mathrm{P}(A_1)) \leftrightarrow A_1$ and $t_\mathrm{K}(t_\mathrm{P}(B_1)) \leftrightarrow B_1$ are provable in $\mathbf{PCI}_\mathrm{K}$ by I.H., and consider the formula $A = (A_1 \equiv B_1)$. Then in $\mathbf{PCI}_\mathrm{K}$, the following equivalences hold.

$$
\begin{aligned}
t_\mathrm{K}(t_\mathrm{P}(A_1 \equiv B_1)) &\leftrightarrow t_\mathrm{K}(\Box(t_\mathrm{P}(A_1) \leftrightarrow t_\mathrm{P}(B_1))) && \text{(Def.4.3.2(vii))}\\
&\leftrightarrow t_\mathrm{K}(t_\mathrm{P}(A_1) \leftrightarrow t_\mathrm{P}(B_1)) \equiv \top && \text{(Def.4.3.1(vii))}\\
&\leftrightarrow (t_\mathrm{K}(t_\mathrm{P}(A_1)) \leftrightarrow t_\mathrm{K}(t_\mathrm{P}(B_1))) \equiv \top && \text{(Def.4.3.1(vi),(iv))}\\
&\leftrightarrow (A_1 \leftrightarrow B_1) \equiv \top && \text{(I.H.)}\\
&\leftrightarrow (A_1 \equiv B_1) && \text{(Th.4.2.2(xvi))}
\end{aligned}
$$

■

Theorem 4.3.6 (i) *For any formula α in $\mathrm{L_K}$, $\alpha \in \mathbf{K}$ if and only if $t_\mathrm{K}(\alpha) \in \mathbf{PCI}_\mathrm{K}$.*

(ii) *For any formula A in $\mathrm{L_S}$, $A \in \mathbf{PCI}_\mathrm{K}$ if and only if $t_\mathrm{P}(A) \in \mathbf{K}$.*

Proof. (i): The only-if-part obtains from Proposition 4.3.3. Also other direction can easily be proved as follows:

$$
\begin{aligned}
t_\mathrm{K}(\alpha) \in \mathbf{PCI}_\mathrm{K} &\implies t_\mathrm{P}(t_\mathrm{K}(\alpha)) \in \mathbf{K} && \text{(Prop.4.3.4)}\\
&\implies \alpha \in \mathbf{K} && \text{(Th.4.3.5(i))}
\end{aligned}
$$

(ii): The only-if-part obtains from Proposition 4.3.4. Also if-part is as follows:

$$
\begin{aligned}
t_\mathrm{P}(A) \in \mathbf{K} &\implies t_\mathrm{K}(t_\mathrm{P}(A)) \in \mathbf{PCI}_\mathrm{K} && \text{(Prop.4.3.3)}\\
&\implies A \in \mathbf{PCI}_\mathrm{K} && \text{(Th.4.3.5(ii))}
\end{aligned}
$$

■

Hence we can conclude that two logics $\mathbf{K}$ and $\mathbf{PCI}_\mathrm{K}$ are syntactically equivalent by Definition 3.4.1, Theorem 4.3.5 and Theorem 4.3.6.

4.4 Kripke type semantics for $\mathbf{PCI}_{\mathrm{K}}$ logics

In this section we will introduce Kripke type semantics for $\mathbf{PCI}_{\mathrm{K}}$ logics, and then show a completeness of $\mathbf{PCI}_{\mathrm{K}}$ logic with respect to this semantics. Since $(A \equiv B) \leftrightarrow \Box(A \leftrightarrow B)$ is a theorem of $\mathbf{PCI}_{\mathrm{K}}$ under the definition $\Box A \leftrightarrow (A \equiv \top)$, we can define Kripke models for $\mathbf{PCI}_{\mathrm{K}}$, in the similar way as the case of normal modal logic $\mathbf{K}$, by exchanging the validity of modal formulas in modal Kripke model with new validity of identity formulas according to the above equivalence. A $\mathbf{PCI}_{\mathrm{K}}$ Kripke frame $\mathcal{F}$ for $\mathcal{L}_{\mathrm{S}}$ is a pairs (W, R), which is the same as a modal Kripke frame (see Section 4.1). The only difference between both $\mathbf{PCI}_{\mathrm{K}}$ and $\mathbf{K}$ Kripke model is the definition of validity of formulas. Let $\mathcal{M} = (W, R, V)$ be a $\mathbf{PCI}_{\mathrm{K}}$ Kripke model for $\mathcal{L}_{\mathrm{S}}$, where $\mathcal{F} = (W, R)$ is a $\mathbf{PCI}_{\mathrm{K}}$ Kripke frame and V a valuation on $\mathcal{F}$ which is a map from VAR to 2^W such that $V(p) \subseteq W$ for any $p \in \mathrm{VAR}$, $V(\bot) = \emptyset$ and $V(\top) = W$. Then for any point $a \in W$, we can extend V to the valuation of $\mathbf{PCI}_{\mathrm{K}}$ formulas $\models_{\mathrm{P}} : \mathrm{L}_{\mathrm{S}} \to 2^W$, in the similar way as the case of $\mathbf{K}$, by the following way.

Definition 4.4.1 *Given a $\mathbf{PCI}_{\mathrm{K}}$ Kripke model $\mathcal{M} = (W, R, V)$, the notion of validity of $\mathbf{PCI}_{\mathrm{K}}$ formulas at any point $a \in W$ is defined inductively as follows:*

(i) *$\mathcal{M}, a \models_{\mathrm{P}} p$ if and only if $a \in V(p)$ for any variable $p \in \mathrm{VAR}$,*

(ii) *$\mathcal{M}, a \not\models_{\mathrm{P}} \bot$ and $\mathcal{M}, a \models_{\mathrm{P}} \top$,*

(iii) *$\mathcal{M}, a \models_{\mathrm{P}} A \wedge B$ if and only if $\mathcal{M}, a \models_{\mathrm{P}} A$ and $\mathcal{M}, a \models_{\mathrm{P}} B$,*

(iv) *$\mathcal{M}, a \models_{\mathrm{P}} A \vee B$ if and only if $\mathcal{M}, a \models_{\mathrm{P}} A$ or $\mathcal{M}, a \models_{\mathrm{P}} B$,*

(v) *$\mathcal{M}, a \models_{\mathrm{P}} A \to B$ if and only if $\mathcal{M}, a \models_{\mathrm{P}} A$ implies $\mathcal{M}, a \models_{\mathrm{P}} B$,*

(vi) *$\mathcal{M}, a \models_{\mathrm{P}} A \equiv B$ if and only if for all b with aRb, $\mathcal{M}, b \models_{\mathrm{P}} A \iff \mathcal{M}, b \models_{\mathrm{P}} B$.*

Here the validity of classical parts in both $\mathbf{PCI}_{\mathrm{K}}$ and $\mathbf{K}$ Kripke model is the same. For any Kripke frame $\mathcal{F} = (W, R)$, a formula A is *valid* on $\mathcal{F}$, in symbols, $\mathcal{F} \models_{\mathrm{P}} A$ if $\mathcal{M}, a \models_{\mathrm{P}} A$ for any $a \in W$ and any valuation $\models_{\mathrm{P}}$. Next we will show the semantical equivalence between $\mathbf{PCI}_{\mathrm{K}}$ and $\mathbf{K}$ with respect to the same Kripke frame by using translations t_{K} and t_{P}.

Theorem 4.4.2 *Let $\mathcal{F} = (W, R)$ be a modal Kripke frame. Then $\mathcal{F}$ can be regarded also as a $\mathbf{PCI}_{\mathrm{K}}$ Kripke frame. Let $t_{\mathrm{K}} : \mathrm{L}_{\mathrm{K}} \to \mathrm{L}_{\mathrm{S}}$ be a K-translation and $t_{\mathrm{P}} : \mathrm{L}_{\mathrm{S}} \to \mathrm{L}_{\mathrm{K}}$ a PCI-translation. Then the following equivalences are satisfied:*

(i) *For any formula α in L_{K}, $\mathcal{F} \models_{\mathrm{K}} \alpha$ if and only if $\mathcal{F} \models_{\mathrm{P}} t_{\mathrm{K}}(\alpha)$,*

(ii) *For any formula A in L_{S}, $\mathcal{F} \models_{\mathrm{P}} A$ if and only if $\mathcal{F} \models_{\mathrm{K}} t_{\mathrm{P}}(A)$.*

Proof. First, we note that each valuation V on $\mathcal{F}$ as a modal Kripke frame can be regarded also as a valuation on $\mathcal{F}$ as a $\mathbf{PCI}_{\mathrm{K}}$ Kripke frame, and vice versa. To show (i) of our Theorem, it suffices to show by induction on the length of a formula α in $\mathrm{L_K}$ that in a given $\mathcal{M} = (W, R, V)$, for any $a \in W$ $\mathcal{M}, a \models_{\mathrm{K}} \alpha$ if and only if $\mathcal{M}, a \models_{\mathrm{P}} t_{\mathrm{K}}(\alpha)$. Assume that for any $a \in W$ $\mathcal{M}, a \models_{\mathrm{K}} \alpha_1$ iff $\mathcal{M}, a \models_{\mathrm{P}} t_{\mathrm{K}}(\alpha_1)$. Then we have:

$\mathcal{M}, a \models_{\mathrm{K}} \Box\alpha_1$ iff $\forall b(aRb \Longrightarrow (\mathcal{M}, b \models_{\mathrm{K}} \alpha_1))$ (Def.4.1.3(vi),hypothesis)

iff $\forall b(aRb \Longrightarrow (\mathcal{M}, b \models_{\mathrm{P}} t_{\mathrm{K}}(\alpha_1)))$ (hypothesis,I.H.)

iff $\forall b(aRb \Longrightarrow (\mathcal{M}, b \models_{\mathrm{P}} t_{\mathrm{K}}(\alpha_1) \Longleftrightarrow \mathcal{M}, b \models_{\mathrm{P}} \top))$

iff $\mathcal{M}, a \models_{\mathrm{P}} t_{\mathrm{K}}(\alpha_1) \equiv \top$ (Def.4.4.1(vi))

iff $\mathcal{M}, a \models_{\mathrm{P}} t_{\mathrm{K}}(\Box\alpha_1)$ (Def.4.3.1(vii))

To show (ii) of our Theorem, it suffices to show by induction on the length of a formula A in $\mathrm{L_S}$ that in a given $\mathcal{M} = (W, R, V)$, for any $a \in W$ $\mathcal{M}, a \models_{\mathrm{P}} A$ if and only if $\mathcal{M}, a \models_{\mathrm{K}} t_{\mathrm{P}}(A)$. Assume that for any $a \in W$ $\mathcal{M}, a \models_{\mathrm{P}} A_1$ iff $\mathcal{M}, a \models_{\mathrm{K}} t_{\mathrm{P}}(A_1)$ and $\mathcal{M}, a \models_{\mathrm{P}} B_1$ iff $\mathcal{M}, a \models_{\mathrm{K}} t_{\mathrm{P}}(B_1)$. Then we have:

$\mathcal{M}, a \models_{\mathrm{P}} A_1 \equiv B_1$ iff $\forall b(aRb \Longrightarrow (\mathcal{M}, b \models_{\mathrm{P}} A_1 \Longleftrightarrow \mathcal{M}, b \models_{\mathrm{P}} B_1))$ (Def.4.4.1(vi),hypothesis)

iff $\forall b(aRb \Longrightarrow (\mathcal{M}, b \models_{\mathrm{K}} t_{\mathrm{P}}(A_1) \Longleftrightarrow \mathcal{M}, b \models_{\mathrm{K}} t_{\mathrm{P}}(B_1)))$ (hypothesis,I.H.)

iff $\forall b(aRb \Longrightarrow \mathcal{M}, b \models_{\mathrm{K}} (t_{\mathrm{P}}(A_1) \leftrightarrow t_{\mathrm{P}}(B_1)))$

iff $\mathcal{M}, a \models_{\mathrm{K}} \Box(t_{\mathrm{P}}(A_1) \leftrightarrow t_{\mathrm{P}}(B_1))$ (Def.4.1.3(vi))

iff $\mathcal{M}, a \models_{\mathrm{K}} t_{\mathrm{P}}(A_1 \equiv B_1)$ (Def.4.3.2(vii))

■

Then by invoking the completeness of normal modal logic **K**, we can give a completeness theorem of $\mathbf{PCI}_{\mathrm{K}}$ relative to Kripke type semantics.

Theorem 4.4.3 ($\mathbf{PCI}_{\mathrm{K}}$ completeness) *For any $A \in \mathrm{L_S}$, $A \in \mathbf{PCI}_{\mathrm{K}}$ if and only if $\mathcal{F} \models_{\mathrm{P}} A$ for every $\mathbf{PCI}_{\mathrm{K}}$ Kripke frame $\mathcal{F}$.*

Proof. By Theorem 4.3.6 (ii), we have $A \in \mathbf{PCI}_{\mathrm{K}}$ iff $t_{\mathrm{P}}(A) \in \mathbf{K}$. By the completeness of the modal logic **K** (Theorem 4.1.4), we also have $t_{\mathrm{P}}(A) \in \mathbf{K}$ iff $\mathcal{F} \models_{\mathrm{K}} t_{\mathrm{P}}(A)$ for any modal Kripke frame $\mathcal{F}$. By Theorem 4.4.2 (ii), we have $\mathcal{F} \models_{\mathrm{K}} t_{\mathrm{P}}(A)$ for any modal Kripke frame $\mathcal{F}$ iff $\mathcal{F} \models_{\mathrm{P}} A$ for any $\mathbf{PCI}_{\mathrm{K}}$ Kripke frame $\mathcal{F}$. Thus, we have our theorem.

■

4.5 $\mathbf{PCI}_{\mathrm{K}}$ algebras and its representation theorem

In this section we will introduce $\mathbf{PCI}_{\mathrm{K}}$-algebra which provides an algebraic semantics for $\mathbf{PCI}_{\mathrm{K}}$ logic which is introdued in Section 4.2, and show the representation theorem of $\mathbf{PCI}_{\mathrm{K}}$-algebras in the similar way to the case of modal algebras. Let $\mathcal{A}_0 = \langle A, -, \cap, \cup, \supset, \mathrm{f}, \mathrm{t} \rangle$ be a Boolean algebra with a carrier set A, complement $-$, meet $\cap$, join $\cup$, inclusion $\supset$, zero (f) and unit (t). Here if $a \cap b = a$ or $a \cup b = b$ holds, we write $a \supset b$ and mean that

a is contained in b. Then we will define a $\mathbf{PCI}_\mathrm{K}$-algebra $\mathcal{A}_\mathrm{K} = \langle \mathcal{A}_0, \Delta \rangle$ as a Boolean algebra $\mathcal{A}_0$ with an additional binary operation Δ which satisfies the following conditions: for every $x, y, r, z \in A$,

(1) $x \Delta x = \mathrm{t}$,

(2) $x \Delta y \supset y \Delta x$,

(3) $(x \Delta y) \cap (y \Delta z) \supset x \Delta z$,

(4) $x \Delta y \supset (-x \Delta - y)$,

(5) $(x \Delta y) \cap (r \Delta z) \supset (x \star r) \Delta (y \star z)$, where $\star \in \{\cap, \cup, \supset\}$,

(6) $(x \supset y) \Delta (y \supset x) \supset x \Delta y$,

(7) $(x \supset y) \Delta (y \supset x) \supset ((x \supset y) \Delta \mathrm{t}) \cap ((y \supset x) \Delta \mathrm{t})$.

Here we omitted extra parentheses, following the assumption that the priority of each operation is weak as $-$, $\cap$, $\cup$, Δ, $\supset$ in order. The next lemma is an algebraic translation of Theorem 4.2.2 (xiv), (xv) and (xvi).

Lemma 4.5.1 *For any $\mathbf{PCI}_\mathrm{K}$-algebra $\mathcal{A}_\mathrm{K} = \langle \mathcal{A}_0, \Delta \rangle$, we have the following equations:*

(i) $(x \supset y) \Delta \mathrm{t} \supset (x \Delta \mathrm{t} \supset y \Delta \mathrm{t})$,

(ii) $(x \cap y) \Delta \mathrm{t} = (x \Delta \mathrm{t}) \cap (y \Delta \mathrm{t})$,

(iii) $x \Delta y = (x \supset\subset y) \Delta \mathrm{t}$, *where* $x \supset\subset y = (x \supset y) \cap (y \supset x)$.

Definition 4.5.2 (*see [14] and [22]*) *Let $\mathcal{A}_0 = \langle A, -, \cap, \cup, \supset, \mathrm{f}, \mathrm{t} \rangle$ be a Boolean algebra. Then we define:*

(i) *A subset F of A is called a* filter *of $\mathcal{A}_0$ if F satisfies the following conditions:*

  (1) $\mathrm{t} \in F$,

  (2) $a \in F$ *and* $a \supset b$ *implies* $b \in F$,

  (3) $a, b \in F$ *implies* $a \cap b \in F$.

(ii) *Moreover, a filter F of $\mathcal{A}_0$ is a* maximal filter (*or* ultrafilter) *if F is maximal with respect to the property that* $\mathrm{f} \notin F$.

(iii) *A filter F of $\mathcal{A}_0$ is* proper *if* $\mathrm{f} \notin F$.

Lemma 4.5.3 *Let F be a filter of a Boolean algebra $\mathcal{A}_0$. Then we have:*

(i) *F is an ultrafilter of $\mathcal{A}_0$ if and only if exactly either of a or $-a$ belongs to F for any $a \in A$.*

(ii) *F is an ultrafilter of $\mathcal{A}_0$ if and only if it satisfies both* (1) $\mathrm{f} \notin F$ *and* (2) *for any $a, b \in A$ $a \cup b \in F$ if and only if $a \in F$ or $b \in F$.*

Proof. (i): Suppose F is a filter of $\mathcal{A}_0$. To show only-if-part, assume that F is an ultrafilter. Then we have $\mathcal{A}_0/F \cong \mathbf{2}$ since the interval $[F, \nabla]$ of $\mathrm{Co}(\mathcal{A}_0)$ has exactly two elements and so $\mathcal{A}_0/F$ is simple. Let $k_F : \mathcal{A}_0 \to \mathcal{A}_0/F$ be the natural homomorphism. Then for any $a \in A$, $k_F(-a) = -k_F(a)$ and so $k_F(a) = 1/F$ or $k_F(-a) = 1/F$ as $\mathcal{A}_0/F \cong \mathbf{2}$. Hence $a \in F$ or $-a \in F$. Therefore if we are given $a \in A$ then exactly one of $a, -a$ is in F as $a \cap -a = \mathrm{f} \notin F$. To show the converse assume that exactly one of $a, -a$ is in F for any $a \in A$. Then if G is another filter of $\mathcal{A}_0$ with $F \subseteq G$ and $F \neq G$, let $a \in G - F$. As $-a \in F$ we have $\mathrm{f} = a \cap -a \in G$. Hence $G = A$. Thus F is an ultrafilter. (ii): To show only-if-part, assume that F is an ultrafilter with $a \cup b \in F$. Then as $(a \cup b) \cap (-a \cap -b) = \mathrm{f} \notin F$ we have $-a \cap -b \notin F$. Hence $-a \notin F$ or $-b \notin F$. By the above (i) we have either $a \in F$ or $b \in F$. Since $\mathrm{t} \in F$, for given $a \in A$ we have $\mathrm{t} = a \cup -a \in F$. Hence $a \in F$ or $-a \in F$. But both $a, -a$ can not belong to F as $a \cap -a = \mathrm{f} \notin F$.

■

Let M(A) be the set of all maximal filters of a Boolean algebra $\mathcal{A}_0$. Then it is well-known, in [22], that $(\wp(\mathrm{M(A)}), \subseteq)$ yields also a Boolean algebra, and the following representation theorem (Th. 4.5.6) of $\mathcal{A}_0$ holds. We say that a subset M has a finite intersection property if for any finite subset $\{c_1, \cdots, c_n\}$ of M, the infimum $c_1 \cap \cdots \cap c_n \neq \mathrm{f}$ if and only if the filter $[M)(= \{b \in A; m_1 \cap \cdots \cap m_n \leq b \text{ for some } m_i \in M\})$ generated by M is proper (see e.g. [22]). We will show two lemmas which are essential in proving the representation theorem.

Lemma 4.5.4 *For any subset M of A, M has a finite intersection property if and only if there exists an ultrafilter F of $\mathcal{A}_0$ with $M \subseteq F$.*

Proof. The only-if-part is trivial since M has clearly finite intersection property when there exists an ultrafilter F with $M \subseteq F$. Conversely if M has finite intersection property, then a filter generated by M is proper by the definition. Let $P = \{G \subseteq A; G \text{ is a proper filter of } \mathcal{A}_0 \text{ with } M \subseteq G\}$ and consider the partial order set $(P, \subseteq)$. Clearly, P is nonempty since $[M) \in P$. Moreover, for any chain $K = \{G_i; i \in I\}$ of $(P, \subseteq)$, $\bigcup_{i \in I} G_i$ is the supremum of $(P, \subseteq)$. Therefore by Kuratowski-Zorn Lemma, $(P, \subseteq)$ has a maximal element F. Moreover, F is clearly a maximal filter with $M \subseteq F$.

■

Lemma 4.5.5 *For any homomorphism $h : A \to B$, $h(a) \neq \mathrm{f}$ for any $a \in A$ with $a \neq \mathrm{f}$ if and only if h is an injection.*

Proof. It is sufficient to show that there exists $a \in A$ with $a \neq \mathrm{f}$ and $h(a) = \mathrm{f}$, when h is not an injection. By our assumption there exist x and y in A such that $x \neq y$ and $h(x) = h(y)$. Without loss of generality, assume $x \not\leq y$. Let $a = x \cap -y$. Then we get $a \neq f$ and $h(a) = h(x) \cap -h(y) = f$.

■

Theorem 4.5.6 *Let $\mathcal{A}_0$ be a Boolean algebra and* $\mathrm{M(A)}$ *the set of all maximal filters of* $\mathcal{A}_0$. *Then the map*

$$s : a \longmapsto \{F \in \mathrm{M(A)}; a \in F\}$$

is an isomorphism of A into $\wp(\mathrm{M(A)})$.

Proof. At first we will show that s is a homomorphism. By the definition of s, we have (1): $a \in F$ if and only if $F \in s(a)$. Since $a \cap b \in F$ if and only if $a \in F$ and $b \in F$, we have $s(a \cap b) = s(a) \cap s(b)$ where $\cap$ on the right side denotes the set theoretical intersection. Furthermore, since every filter $F \in \mathrm{M(A)}$ is maximal, by Lemma 4.5.3 (i) and (ii), we infer that $a \in F$ if and only if $-a \notin F$, and $a \cup b \in F$ if and only if $a \in F$ or $b \in F$, respectively. They imply by (1) that $s(-a) = -s(a)$ and $s(a \cup b) = s(a) \cup s(b)$ where $-$ and $\cup$ on the right side of each equation denote the set theoretical complement relative to $\mathrm{M(A)}$ and the set theoretical sum, respectively. Thus s is a homomorphism of A into $\wp(\mathrm{M(A)})$. Next for any $a \in A$ with $a \neq f$, there exists an ultrafilter F with $a \in F$ by Lemma 4.5.4. So $s(a) \neq \emptyset$. Hence by Lemma 4.5.5 we get s is an injection.

■

Next we will consider the case of $\mathbf{PCI}_{\mathrm{K}}$-algebras $\mathcal{A}_{\mathrm{K}} = \langle \mathcal{A}_0, \Delta \rangle$. A subset F of A is called a $\mathbf{PCI}_{\mathrm{K}}$-*filter* of $\mathcal{A}_{\mathrm{K}}$ if F satisfies the following conditions:

(F1) F is a lattice filter of $\mathcal{A}_0$,

(F2) for any $a \in A$, $a \in F$ implies $a\Delta \mathrm{t} \in F$.

Then for any $\mathbf{PCI}_{\mathrm{K}}$-algebras $\mathcal{A}_{\mathrm{K}}$ and any $\mathbf{PCI}_{\mathrm{K}}$-filter F, $\mathfrak{M} = (\mathcal{A}_{\mathrm{K}}, F)$ is called a $\mathbf{PCI}_{\mathrm{K}}$-*model*. For any $\mathbf{PCI}_{\mathrm{K}}$-algebras $\mathcal{A}_{\mathrm{K}}$, a formula B is *valid* in $\mathcal{A}_{\mathrm{K}}$, in symbols, $\mathcal{A}_{\mathrm{K}} \models B$, if $h(B) \in F$ for any valuation h of $\mathcal{A}_{\mathrm{K}}$ and any $\mathbf{PCI}_{\mathrm{K}}$-filter F. Now we will prove the representation theorem of $\mathbf{PCI}_{\mathrm{K}}$-algebra by using the following definitions of duality of frame and algebra (see also Fig 4.2).

Definition 4.5.7 *Let $\mathcal{A}_{\mathrm{K}} = \langle \mathcal{A}_0, \Delta \rangle$ be a $\mathbf{PCI}_{\mathrm{K}}$-algebra and $\mathcal{F} = (W, R)$ a $\mathbf{PCI}_{\mathrm{K}}$ Kripke frame . Then we define:*

(i) $\mathcal{A}_{\mathrm{K}+} = (\mathrm{M(A)}, R)$ *is called the* dual frame *of $\mathcal{A}_{\mathrm{K}}$ if* $\mathrm{M(A)}$ *is the set of all maximal filters of a Boolean algebra $\mathcal{A}_0$ and for any $F, G \in \mathrm{M(A)}$, FRG if and only if $F_\Delta \subseteq G$, where $F_\Delta = \{x \leftrightarrow y; x\Delta y \in F\}$,*

(ii) *moreover,* $\mathcal{F}^+ = \langle \wp(W), \Delta^* \rangle$ *is called the* dual algebra *of* $\mathcal{F}$ *if for any* $X, Y \subseteq W$, $X\Delta^*Y = \{F;$ *for any* $G \in W$ *such that* $FRG(G \in X \iff G \in Y)\}$.

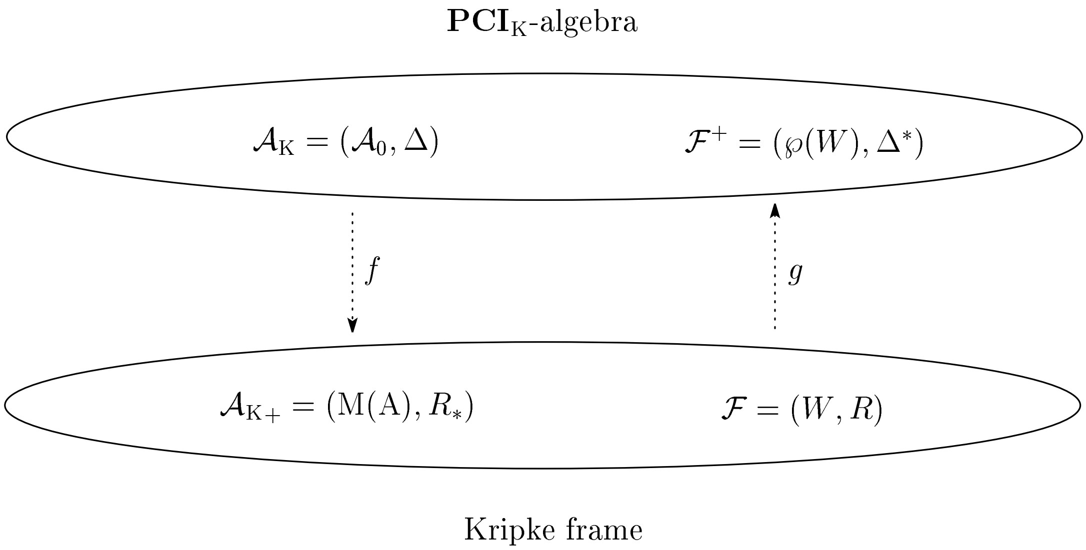

Figure 4.2: Dual algebra/frame of $\mathbf{PCI}_{\mathrm{K}}$-algebras

Theorem 4.5.8 *Let* $\mathcal{A}_{\mathrm{K}} = \langle \mathcal{A}_0, \Delta \rangle$ *be a* $\mathbf{PCI}_{\mathrm{K}}$*-algebra and* $\mathrm{M(A)}$ *the set of all maximal filters of* $\mathcal{A}_0$*. Then the map*

$$h : a \longmapsto \{F \in \mathrm{M(A)}; a \in F\}$$

is an isomorphism of $\mathcal{A}_{\mathrm{K}} = \langle \mathcal{A}_0, \Delta \rangle$ *into* $(\mathcal{A}_{\mathrm{K}+})^+ = \langle \wp(\mathrm{M(A)}), \Delta^* \rangle$.

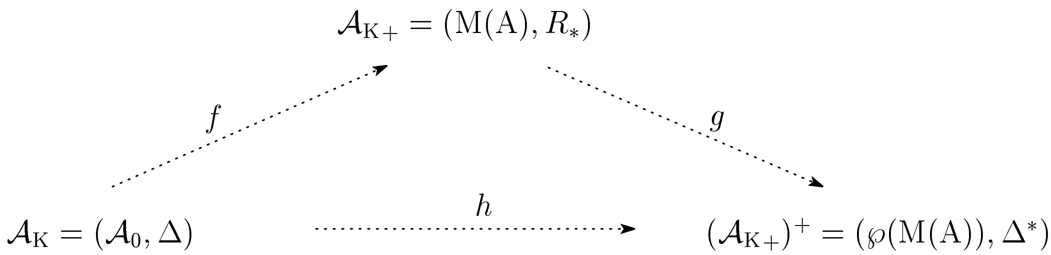

Figure 4.3: Embedding of $\mathbf{PCI}_{\mathrm{K}}$-algebras

Proof. By Theorem 4.5.6 it is clear that s is a homomorphism for operations $(\cap, \cup, -)$ and an injection. So it is sufficient to show $s(x\Delta y) = s(x)\Delta^* s(y)$ for any $x, y \in A$. This implies by (1) in the proof of Theorem 4.5.6 that $x\Delta y \in F$ if and only if for any $G \in \mathrm{M(A)}$ such that $FRG(G \in s(x) \iff G \in s(y))$ if and only if for any $G \in \mathrm{M(A)}$ such that

$F_\Delta \subseteq G(x \in G \iff y \in G)$. So we will show $x\Delta y \in F$ if and only if for any $G \in \mathrm{M(A)}$ such that $F_\Delta \subseteq G(x \in G \iff y \in G)$. The only-if-part: $x\Delta y \in F \Longrightarrow x \leftrightarrow y \in F_\Delta \subseteq G$ $\Longrightarrow x \leftrightarrow y \in G \Longrightarrow (x \in G \iff y \in G)$. The if-part: assume $x\Delta y \notin F$. Then either $y \notin [F_\Delta \cup \{x\})$ or $x \notin [F_\Delta \cup \{y\})$ hold. For the former case, let $X = \{H$ is a proper filter; $[F_\Delta \cup \{x\}) \subseteq H$ and $y \notin H\}$. Let $H_0 = [F_\Delta \cup \{x\})$ in the former case. Hence $H_0 \in X$ and so $X \neq \emptyset$. Furthermore for any chain $K = \{H_i; i \in I\}$ of $(X, \subseteq)$, $\bigcup_{i\in I} H_i$ is a supremum of $(X, \subseteq)$. Therefore by Kuratowski-Zorn Lemma, $(X, \subseteq)$ has a maximal element H^*. Moreover, H^* is clearly a maximal filter. Hence we get that there exists $H^* \in \mathrm{M(A)}$ such that $F_\Delta \subseteq H^*(x \in H^* \not\Longleftrightarrow y \in H^*)$. We can show this also in the latter case.

■

Finally from the above representation theorem, we can prove an alternative completeness theorem of $\mathbf{PCI}_{\mathrm{K}}$ logic with respect to Kripke type semantics as the following way.

Lemma 4.5.9 *For any* $\mathbf{PCI}_{\mathrm{K}}$ *Kripke frame* $\mathcal{F}$ *and any* $A \in \mathrm{L_S}$*, the following conditions are equivalent:*

(i) $\mathcal{F} \models_{\mathrm{P}} A$ *for a* $\mathbf{PCI}_{\mathrm{K}}$ *Kripke frame* $\mathcal{F} = (W, R)$,

(ii) $\mathcal{F}^+ \models A$ *for a* $\mathbf{PCI}_{\mathrm{K}}$*-algebra* $\mathcal{F}^+ = (\wp(W), \Delta^*)$.

Proof. By definitions of validity for $\mathbf{PCI}_{\mathrm{K}}$ Kripke frame and $\mathbf{PCI}_{\mathrm{K}}$-algebras, for any $p \in \mathrm{VAR}$ and any $w \in W$, if we define a valuation $v \in \mathrm{HOM}(\mathrm{L_S}, \wp(W))$ such that $w \models_{\mathrm{P}} p \iff w \in v(p)$, then we get $w \models_{\mathrm{P}} A \iff w \in v(A)$ for any $A \in \mathrm{L_S}$. Therefore, we have $v(A) = W$ (i.e., $= \mathrm{t}_{\mathcal{F}^+}$) $\iff w \models_{\mathrm{P}} A$ for any $w \in W$.

■

Theorem 4.5.10 *For any* $A \in \mathrm{L_S}$ *and any* $\mathbf{PCI}_{\mathrm{K}}$ *logic, the following conditions are equivalent:*

(i) $A \in \mathbf{PCI}_{\mathrm{K}}$,

(ii) $\mathcal{F} \models_{\mathrm{P}} A$ *for any* $\mathbf{PCI}_{\mathrm{K}}$ *Kripke frame* $\mathcal{F}$,

(iii) $\mathcal{A}_{\mathrm{K}} \models A$ *for any* $\mathbf{PCI}_{\mathrm{K}}$*-algebra* $\mathcal{A}_{\mathrm{K}}$.

Proof. (i) $\Longrightarrow$ (ii): soundness of Theorem 4.4.3. (iii) $\Longrightarrow$ (i): usual construction of Lindenbaum-Tarski algebra. (ii) $\Longrightarrow$ (iii): Assume that $v(A) < \mathrm{t}_{\mathcal{B}}$ for some algebra $\mathcal{B}$ and some $v \in \mathrm{HOM}(\mathrm{L_S}, B)$. Then since $\mathcal{B}$ can be embedded into $(\mathcal{B}_+)^+$ by the representation theorem (see Theorem 4.5.8), the above valuation v can also be seen a valuation of $(\mathcal{B}_+)^+$. Therefore, we have not $(\mathcal{B}_+)^+ \models A$. Hence, we have not $\mathcal{B}_+ \models_{\mathrm{P}} A$ by Lemma 4.5.9.

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4.6 Several extensions of $\mathbf{PCI}_{\mathrm{K}}$

We can successfully extend all results so far gotten to various extensions of modal logics. In this section, we will introduce several elementary extensions of $\mathbf{PCI}_{\mathrm{K}}$ which are counterparts of modal extensions of $\mathbf{K}$. So let us first consider the following additional axiom schemata of $\mathbf{PCI}_{\mathrm{K}}$ logic.

	$\mathbf{PCI}_{\mathrm{K}}$ logic	modal logic
(IR)	$(A \equiv B) \rightarrow (A \leftrightarrow B)$	$\mathrm{T} : \Box\alpha \rightarrow \alpha$
(IS)	$(A \leftrightarrow B) \wedge (C \leftrightarrow D) \rightarrow (A \equiv \neg C) \equiv (B \equiv \neg D)$	$\mathrm{B} : \alpha \rightarrow \Box\Diamond\alpha$
(IT)	$(A \equiv B) \wedge (C \equiv D) \rightarrow (A \equiv C) \equiv (B \equiv D)$	$4 : \Box\alpha \rightarrow \Box\Box\alpha$
(IL)	$(A \equiv \neg B) \rightarrow \neg(A \equiv B)$	$\mathrm{D} : \Box\alpha \rightarrow \Diamond\alpha$
(IE)	$\neg((A \equiv C) \equiv (B \equiv D)) \rightarrow (A \equiv B) \vee (C \equiv D)$	$5 : \Diamond\alpha \rightarrow \Box\Diamond\alpha$
(IO)	$(A \leftrightarrow B) \rightarrow (A \equiv B)$	$\mathrm{Z} : \alpha \rightarrow \Box\alpha$

Then we have the following extensions $\mathbf{PCI}_{\mathrm{KT}}$, $\mathbf{PCI}_{\mathrm{KB}}$, $\mathbf{PCI}_{\mathrm{K4}}$, $\mathbf{PCI}_{\mathrm{KD}}$, $\mathbf{PCI}_{\mathrm{K5}}$, $\mathbf{PCI}_{\mathrm{KZ}}$, $\mathbf{PCI}_{\mathrm{S4}}$, $\mathbf{PCI}_{\mathrm{S5}}$ and $\mathbf{PCI}_{\mathrm{KTZ}}$ of $\mathbf{PCI}_{\mathrm{K}}$, which can be defined below.

Definition 4.6.1 *Let* $\mathbf{PCI}_{\mathrm{K}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{K}})$ *and* $X \subseteq \mathrm{L}_{\mathrm{S}}$. *Then elementary extensions of* $\mathbf{PCI}_{\mathrm{K}}$ *are defined as follow:*

(i) $\mathbf{PCI}_{\mathrm{KT}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{KT}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{K}}$*, where* $C^{\mathrm{G}}_{\mathrm{KT}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{K}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{KT}}(X) = C^{\mathrm{G}}_{\mathrm{K}}(X; \mathrm{IR})$.

(ii) $\mathbf{PCI}_{\mathrm{KB}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{KB}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{K}}$*, where* $C^{\mathrm{G}}_{\mathrm{KB}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{K}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{KB}}(X) = C^{\mathrm{G}}_{\mathrm{K}}(X; \mathrm{IS})$.

(iii) $\mathbf{PCI}_{\mathrm{K4}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{K4}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{K}}$*, where* $C^{\mathrm{G}}_{\mathrm{K4}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{K}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{K4}}(X) = C^{\mathrm{G}}_{\mathrm{K}}(X; \mathrm{IT})$.

(iv) $\mathbf{PCI}_{\mathrm{KD}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{KD}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{K}}$*, where* $C^{\mathrm{G}}_{\mathrm{KD}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{K}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{KD}}(X) = C^{\mathrm{G}}_{\mathrm{K}}(X; \mathrm{IL})$.

(v) $\mathbf{PCI}_{\mathrm{K5}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{K5}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{K}}$*, where* $C^{\mathrm{G}}_{\mathrm{K5}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{K}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{K5}}(X) = C^{\mathrm{G}}_{\mathrm{K}}(X; \mathrm{IE})$.

(vi) $\mathbf{PCI}_{\mathrm{KZ}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{KZ}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{K}}$*, where* $C^{\mathrm{G}}_{\mathrm{KZ}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{K}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{KZ}}(X) = C^{\mathrm{G}}_{\mathrm{K}}(X; \mathrm{IO})$.

(vii) $\mathbf{PCI}_{\mathrm{S4}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{S4}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{K}}$*, where* $C^{\mathrm{G}}_{\mathrm{S4}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{K}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{S4}}(X) = C^{\mathrm{G}}_{\mathrm{K}}(X; \mathrm{IR}, \mathrm{IT})$.

(viii) $\mathbf{PCI}_{\mathrm{S5}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{S5}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{K}}$*, where* $C^{\mathrm{G}}_{\mathrm{S5}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{K}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{S5}}(X) = C^{\mathrm{G}}_{\mathrm{K}}(X; \mathrm{IR}, \mathrm{IE})$.

(ix) $\mathbf{PCI}_{\mathrm{KTZ}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{KTZ}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{K}}$*, where* $C^{\mathrm{G}}_{\mathrm{KTZ}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{K}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{KTZ}}(X) = C^{\mathrm{G}}_{\mathrm{K}}(X; \mathrm{IR}, \mathrm{IO})$.

Next we will show logical theorems of each extension of $\mathbf{PCI}_{\mathrm{K}}$.

Theorem 4.6.2 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{KT}}$.

(i) $\neg(A \equiv \neg A)$ *and* $\neg(\top \equiv \bot)$

(ii) $(A \equiv \top) \to A$ *and* $\neg A \to \neg(A \equiv \top)$

(iii) $((A \equiv B) \equiv \top) \to (A \equiv B)$ *and* $\neg(A \equiv B) \to \neg((A \equiv B) \equiv \top)$

(iv) $(A \equiv \bot) \to \neg A$ *and* $A \to \neg(A \equiv \bot)$

(v) $((A \equiv B) \equiv \bot) \to \neg(A \equiv B)$ *and* $\neg(A \equiv B) \to \neg(\neg(A \equiv B) \equiv \bot)$

(vi) $\bot \equiv (A \equiv \neg A)$ *and* $\bot \equiv (\bot \equiv \top)$

(vii) $(A \leftrightarrow B) \to \neg(A \equiv \neg B)$ (IR)*

Proof. (i): By (IR),$(A \equiv \neg A) \to (A \leftrightarrow \neg A)$ is a theorem. Hence by contraposition and (Mp) rule we get the desired result. This yields also $\neg(\top \equiv \bot)$. (ii): By (IR), $(A \equiv \top) \to (A \leftrightarrow \top) \to A$, and also $\neg A \to \neg(A \equiv \top)$. (iii)–(v): Similar to (ii). (vi): By (i) and (G) rule $\neg(A \equiv \neg A) \equiv \top$ is a theorem. Moreover, by (C1) $\neg(A \equiv \neg A) \equiv \top \to \neg\neg(A \equiv \neg A) \equiv \neg\top \to (A \equiv \neg A) \equiv \bot$. So (Mp) rule yields the desired result. (vii): By (IR) $(A \equiv \neg B) \to (A \leftrightarrow \neg B) \to \neg(A \leftrightarrow B)$ holds, hence also $(A \leftrightarrow B) \to \neg(A \equiv \neg B)$. ■

Theorem 4.6.3 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{KB}}$.

(i) $A \to \neg(\neg A \equiv \top) \equiv \top$

(ii) $((A \equiv \bot) \equiv \neg(A \equiv \top)) \equiv ((\neg A \equiv \top) \equiv \neg(\neg A \equiv \bot))$

(iii) $\neg((A \equiv \neg C) \equiv (B \equiv \neg D)) \to (A \leftrightarrow B) \vee (C \leftrightarrow D)$ (IS)*

Proof. (i): By (IS), $A \leftrightarrow (A \leftrightarrow \top) \rightarrow (\neg A \leftrightarrow \bot) \rightarrow (\neg A \leftrightarrow \bot) \wedge (\bot \leftrightarrow \bot) \rightarrow (\neg A \equiv \neg\bot) \equiv (\bot \equiv \neg\bot) \rightarrow (\neg A \equiv \top) \equiv \bot \rightarrow \neg(\neg A \equiv \top) \equiv \top$. (ii): By (C1), $(A \equiv \bot) \leftrightarrow (\neg A \equiv \top)$ and $(A \equiv \top) \leftrightarrow (\neg A \equiv \bot)$ hold and also by (IS) $((A \equiv \bot) \leftrightarrow (\neg A \equiv \top)) \wedge ((A \equiv \top) \leftrightarrow (\neg A \equiv \bot)) \rightarrow ((A \equiv \bot) \equiv \neg(A \equiv \top)) \equiv ((\neg A \equiv \top) \equiv \neg(\neg A \equiv \bot))$. Hence by (Mp) rule we get the desired result. (iii): By (IS), $(A \leftrightarrow \neg B) \wedge (C \leftrightarrow \neg D) \rightarrow (A \equiv \neg C) \equiv (\neg B \equiv \neg\neg D)$ and equivalently $\neg(A \leftrightarrow B) \wedge \neg(C \leftrightarrow D) \rightarrow (A \equiv \neg C) \equiv (B \equiv \neg D)$ hold. So we get the result by law of contraposition.

■

Theorem 4.6.4 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{K4}}$.

(i) $(A \equiv B) \rightarrow ((A \equiv A) \equiv (A \equiv B))$, $(A \equiv B) \rightarrow ((A \equiv B) \equiv \top)$

(ii) $(A \equiv B) \wedge (B \equiv C) \rightarrow ((A \equiv A) \equiv (A \equiv B)) \wedge ((A \equiv B) \equiv (A \equiv C))$

(iii) $\neg((A \equiv C) \equiv (B \equiv D)) \rightarrow \neg(A \equiv \neg B) \vee \neg(C \equiv \neg D)$ (IT)*

Proof. (i): Since $A \equiv A$ is a theorem $(A \equiv B) \rightarrow (A \equiv B) \wedge (A \equiv A) \rightarrow (A \equiv A) \equiv (B \equiv A)$ hold by (IT). Moreover, by $(A \equiv A) \equiv \top$, (E3) we get the second theorem. (ii): Similar to (i). (iii): By (IT) $(A \equiv \neg B) \wedge (C \equiv \neg D) \rightarrow (A \equiv C) \equiv (\neg B \equiv \neg D) \rightarrow (A \equiv C) \equiv (B \equiv D)$. So by contraposition we get the desired result.

■

Theorem 4.6.5 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{KD}}$.

(i) $\neg(A \equiv \neg A)$ *and* $\neg(\top \equiv \bot)$

(ii) $(A \equiv \top) \rightarrow \neg(A \equiv \bot)$ *and* $(A \equiv \bot) \rightarrow \neg(A \equiv \top)$

(iii) $((A \equiv B) \equiv \top) \rightarrow \neg((A \equiv B) \equiv \bot)$ *and* $((A \equiv B) \equiv \bot) \rightarrow \neg((A \equiv B) \equiv \top)$

(iv) $(A \equiv B) \rightarrow \neg(A \equiv \neg B)$

(v) $\neg((A \equiv \bot) \equiv \neg(\neg A \equiv \top))$ *and* $\neg((A \equiv \top) \equiv \neg(\neg A \equiv \bot))$

(vi) $\neg((A \wedge \neg A) \equiv \top)$ *and* $\neg((A \vee \neg A) \equiv \bot)$

(vii) $\neg((A \equiv B) \equiv \neg(B \equiv A))$ *and* $\neg((\bot \equiv \bot) \equiv \neg(\top \equiv \top))$

Proof. (i)–(iv) and (vii): Straightforward. (v): By Theorem 4.2.2 (ix) $(A \equiv \bot) \equiv (\neg A \equiv \top)$ holds. Moreover, by (IL) $(A \equiv \bot) \equiv (\neg A \equiv \top) \neg((A \equiv \bot) \equiv \neg(\neg A \equiv \top))$ holds. Hence by (Mp) rule we get the desired result. The proof of second theorem is similar to first one. (vi): It hold that $(A \wedge \neg A) \equiv \bot$ by Theorem 4.2.2 (xi) and $(A \wedge \neg A) \equiv \bot \rightarrow$

$\neg((A \wedge \neg A) \equiv \neg\bot)$ by (IL). So the result is clear by (Mp) rule. The second is similar to the first.

■

Theorem 4.6.6 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{K5}}$.

(i) $\neg(A \equiv B) \wedge \neg(C \equiv D) \to (A \equiv C) \equiv (B \equiv D)$ (IE)*

(ii) $\neg((A \equiv \neg C) \equiv (B \equiv \neg D)) \to (A \equiv B) \vee (C \equiv D)$

(iii) $\neg(A \equiv \neg B) \wedge \neg(C \equiv \neg D) \to (A \equiv C) \equiv (B \equiv D)$

Proof. (i): By contraposition of (IE) we get the result. (ii): By (IE) $\neg((A \equiv \neg C) \equiv (B \equiv \neg D)) \to (A \equiv B) \vee (\neg C \equiv \neg D) \to (A \equiv B) \vee (C \equiv D)$ holds. (iii): By above (i) $\neg(A \equiv \neg B) \wedge \neg(C \equiv \neg D) \to (A \equiv C) \equiv (\neg B \equiv \neg D) \to (A \equiv C) \equiv (B \equiv D)$ holds.

■

Theorem 4.6.7 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{S4}}$.

(i) $(A \equiv B) \leftrightarrow ((A \equiv B) \equiv \top)$

(ii) $((A \equiv C) \equiv (B \equiv D)) \leftrightarrow (A \equiv B) \wedge (C \equiv D)$

(iii) $\neg(A \equiv \neg B) \vee \neg(C \equiv \neg D) \leftrightarrow \neg((A \equiv C) \equiv (B \equiv D))$

Proof. (i): By Theorem 4.6.2 (iii) and Theorem 4.6.4 (i), we get the desired result. (ii): The left direction is due to (IT). The converse also can be shown as follows: By applying (IR),(IT) $(A \equiv C) \equiv (B \equiv D) \to (A \equiv C) \leftrightarrow (B \equiv D) \to (A \equiv C) \wedge (B \equiv D)$ $\to (A \equiv B) \equiv (C \equiv D)$ holds. Then by (IR) we get $(A \equiv B) \equiv (C \equiv D) \to ((A \equiv B) \leftrightarrow (C \equiv D)) \to (A \equiv B) \wedge (C \equiv D)$. (iii): The right direction is due to Theorem 4.6.4 (iii). The converse is as follows: By above (i) $(A \equiv C) \equiv (B \equiv D) \to (A \equiv C) \equiv (\neg B \equiv \neg D)$ $\to (A \equiv \neg B) \wedge (C \equiv \neg D)$ holds. So by contraposition we get the desired result.

■

Theorem 4.6.8 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{S5}}$.

(i) $(A \equiv \neg B) \to \neg(A \equiv B)$ (IL)

(ii) $(A \equiv B) \wedge (C \equiv D) \to (A \equiv C) \equiv (B \equiv D)$ (IT)

(iii) $(A \leftrightarrow B) \wedge (C \leftrightarrow D) \to (A \equiv \neg C) \equiv (B \equiv \neg D)$ (IS)

(iv) $(A \equiv B) \vee (C \equiv D) \leftrightarrow \neg((A \equiv C) \equiv (B \equiv D))$

(v) $((A \equiv C) \equiv (B \equiv D)) \leftrightarrow \neg(A \equiv B) \wedge \neg(C \equiv D)$

(vi) $\neg(A \equiv B) \leftrightarrow ((A \equiv B) \equiv \bot)$ *and* $\neg(A \equiv B) \equiv ((A \equiv B) \equiv \bot)$

(vii) $((A \equiv B) \equiv \top) \vee ((A \equiv B) \equiv \bot)$

(viii) $\neg(A \equiv \top) \equiv ((A \equiv \top) \equiv \bot)$

(ix) $((A \equiv \top) \equiv \top) \vee ((A \equiv \top) \equiv \bot)$

Proof. (i): By (IR),(IR)* we get $(A \equiv B) \to \neg(A \equiv \neg B)$, so the result by contraposition. (ii): By above (i) we get $(A \equiv B) \wedge (C \equiv D) \to \neg(A \equiv \neg B) \wedge \neg(C \equiv \neg D)$. So by Theorem 4.6.6 (iii) we get the desired result. (iii): It is clear that $(C \leftrightarrow D) \leftrightarrow (\neg C \leftrightarrow \neg D)$ holds. So by above (i) we get $(A \leftrightarrow B) \wedge (C \leftrightarrow D) \to (A \leftrightarrow B) \wedge (\neg C \leftrightarrow \neg D) \to \neg(A \equiv \neg B) \wedge \neg(\neg C \equiv \neg\neg D)$. Moreover, by Theorem 4.6.6 (iii) we get $\neg(A \equiv \neg B) \wedge \neg(\neg C \equiv \neg\neg D) \to (A \equiv \neg C) \equiv (B \equiv \neg D)$. (iv): The left direction is due to (IE). The converse is as follows: By applying (IR),(IT) $(A \equiv C) \equiv (B \equiv D) \to (A \equiv C) \leftrightarrow (B \equiv D) \to (A \equiv C) \wedge (B \equiv D) \to (A \equiv B) \equiv (C \equiv D)$ holds. Then by (IR) we get $(A \equiv B) \equiv (C \equiv D) \to (A \equiv B) \leftrightarrow (C \equiv D) \to \neg(A \equiv B) \leftrightarrow \neg(C \equiv D) \to \neg(A \equiv B) \wedge \neg(C \equiv D)$. Hence by contraposition we get the desired result. (v): Same as (iv). (vi): The left direction is due to Theorem 4.6.2 (v). The converse is as follows: by Theorem 4.6.2 (i),4.6.6 (i) $\neg((A \equiv B) \equiv \bot) \wedge \neg(\top \equiv \bot) \to ((A \equiv B) \equiv \top) \equiv (\bot \equiv \bot) \to ((A \equiv B) \equiv \top) \equiv \top$ holds. Then we get the result by using two times of Theorem 4.6.2 (iii), moreover second by (QF) rule. (vii): Theorem 4.6.7 (i) and above (vi) yield the result. (viii): Due to above (vi). (ix): Due to above (vii).

■

Theorem 4.6.9 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{KZ}}$.

(i) $A \to (A \equiv \top)$

(ii) $\neg(A \equiv \neg B) \to (A \leftrightarrow B)$ (IO)*

Proof. (ii): By (IO) $A \to (A \leftrightarrow \top) \to (A \equiv \top)$. (ii): By (IO) $(A \leftrightarrow \neg B) \to (A \equiv \neg B)$ holds, so by contraposition we get the desired result.

■

Theorem 4.6.10 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{KTZ}}$.

(i) $(A \equiv B) \leftrightarrow (A \leftrightarrow B)$ *and* $(A \equiv B) \equiv (A \leftrightarrow B)$

(ii) $\neg A \leftrightarrow (A \equiv \bot)$ *and* $\neg A \equiv (A \equiv \bot)$

(iii) $(A \equiv B) \equiv \neg(A \equiv \neg B)$ *and* $\neg(A \equiv B) \equiv (A \equiv \neg B)$

(iv) $\neg((A \equiv C) \equiv (B \equiv D)) \rightarrow (A \equiv B) \vee (C \equiv D)$ (IE)

(v) $(A \equiv B) \vee (A \equiv C) \vee (B \equiv C)$ *and* $(A \equiv \top) \vee (A \equiv \bot)$

Proof. (i): (IR),(IO) yield $(A \equiv B) \leftrightarrow (A \leftrightarrow B)$. Second is by (QF) rule. (ii): The left direction is due to Theorem 4.6.2 (iv). The converse is as follows: By above (i) $\neg(A \equiv \bot) \rightarrow \neg(A \leftrightarrow \bot) \rightarrow \neg\neg(A \leftrightarrow \top) \rightarrow (A \leftrightarrow \top) \rightarrow A$ holds. So by contraposition we get the result. (iii): It is clear that $(A \equiv B) \leftrightarrow \neg(A \equiv \neg B)$ holds by (IR),(IR)*, so (QF) rule yields the result. Second is same as first. (iv): By above (iii) and (IR) $\neg(A \equiv B) \wedge \neg(C \equiv D) \rightarrow (A \equiv \neg B) \wedge (C \equiv \neg D) \rightarrow (A \leftrightarrow \neg B) \wedge (C \leftrightarrow \neg D) \rightarrow$ $(A \leftrightarrow \neg B) \leftrightarrow (C \leftrightarrow \neg D) \rightarrow (A \leftrightarrow C) \leftrightarrow (\neg B \leftrightarrow \neg D)$ hold, and by above (i) we get $(A \leftrightarrow C) \leftrightarrow (\neg B \leftrightarrow \neg D) \rightarrow (A \leftrightarrow C) \leftrightarrow (B \leftrightarrow D) \rightarrow (A \equiv C) \equiv (B \equiv D)$. (v): Theorem 4.6.9 (i) and above (ii) yield the result.

■

Corollary 4.6.11 $\mathbf{PCI}_{\mathrm{KTZ}}$ *is identical to classical propositional logic* **CL**.

Proof. By the above Theorem 4.6.10 (i), both connectives $\equiv$ and $\leftrightarrow$ are identical. So $\mathbf{PCI}_{\mathrm{KTZ}}$ collapses to **CL**.

■

4.7 Translations between K extensions and $\mathbf{PCI}_{\mathrm{K}}$ extensions

In this section we will consider translations between **K** extensions and $\mathbf{PCI}_{\mathrm{K}}$ extensions in the same manner as the case of $\mathbf{PCI}_{\mathrm{K}}$. Let **L** be any modal extensions **KT**, **KB**, **K4**, **KD**, **K5**, **S4** and **S5** of **K**. Then we have the similar results to the case of **K** above (see Section 4.3).

Proposition 4.7.1 *For any formula α in* L_{K}*, $\alpha \in \mathbf{L}$ implies $t_{\mathrm{K}}(\alpha) \in \mathbf{PCI}_{\mathrm{L}}$.*

Proof. The proof is similar to Proposition 4.3.3 except that we have to consider the following cases in addition:

(T) $\Box\alpha \rightarrow \alpha$

(B) $\alpha \rightarrow \Box\Diamond\alpha$

(4) $\Box\alpha \rightarrow \Box\Box\alpha$

(D) $\Box\alpha \to \Diamond\alpha$

(5) $\Diamond\alpha \to \Box\Diamond\alpha$

(T): By Definition 4.3.1, we have $t_{\mathrm{K}}(\Box\alpha \to \alpha) = t_{\mathrm{K}}(\alpha) \equiv \top \to t_{\mathrm{K}}(\alpha)$. Then, this is a theorem of $\mathbf{PCI}_{\mathrm{KT}}$ because of Theorem 4.6.2 (ii). (B): By Definition 4.3.1, we have $t_{\mathrm{K}}(\alpha \to \Box\Diamond\alpha) = t_{\mathrm{K}}(\alpha) \to \neg(\neg t_{\mathrm{K}}(\alpha) \equiv \top) \equiv \top$. Then, this is a theorem of $\mathbf{PCI}_{\mathrm{KB}}$ because of Theorem 4.6.3 (i). (4): By Definition 4.3.1, we have $t_{\mathrm{K}}(\Box\alpha \to \Box\Box\alpha) = t_{\mathrm{K}}(\alpha) \equiv \top \to (t_{\mathrm{K}}(\alpha) \equiv \top) \equiv \top$. Then, this is a theorem of $\mathbf{PCI}_{\mathrm{K4}}$ because of Theorem 4.6.4 (i). (D): By Definition 4.3.1, we have $t_{\mathrm{K}}(\Box\alpha \to \Diamond\alpha) = t_{\mathrm{K}}(\alpha) \equiv \top \to \neg(\neg t_{\mathrm{K}}(\alpha) \equiv \top)$. Then, this is a theorem of $\mathbf{PCI}_{\mathrm{KD}}$ because of Theorem 4.6.5 (iv). (5): By Definition 4.3.1, we have $t_{\mathrm{K}}(\Diamond\alpha \to \Box\Diamond\alpha) = \neg(t_{\mathrm{K}}(\alpha) \equiv \top) \to ((t_{\mathrm{K}}(\alpha) \equiv \bot) \equiv \bot)$. Then, this is a theorem of $\mathbf{PCI}_{\mathrm{K5}}$ because of Theorem 4.6.6 (i).

■

Proposition 4.7.2 *For any formula A in $\mathrm{L_S}$, $A \in \mathbf{PCI}_{\mathrm{L}}$ implies $t_{\mathrm{P}}(A) \in \mathbf{L}$.*

Proof. The proof is similar to Proposition 4.3.4 except that we have to consider the following cases in addition:

(IR) $(A \equiv B) \to (A \leftrightarrow B)$

(IS) $(A \leftrightarrow B) \wedge (C \leftrightarrow D) \to (A \equiv \neg C) \equiv (B \equiv \neg D)$

(IT) $(A \equiv B) \wedge (C \equiv D) \to (A \equiv C) \equiv (B \equiv D)$

(IL) $(A \equiv \neg B) \to \neg(A \equiv B)$

(IE) $\neg((A \equiv C) \equiv (B \equiv D)) \to (A \equiv B) \vee (C \equiv D)$

(IR): By Definition 4.3.2, we have $t_{\mathrm{P}}((A \equiv B) \to (A \leftrightarrow B)) = \Box(t_{\mathrm{P}}(A) \leftrightarrow t_{\mathrm{P}}(B)) \to (t_{\mathrm{P}}(A) \leftrightarrow t_{\mathrm{P}}(B))$. Then, this is a theorem of **KT** because of axiom (T). (IS): By Definition 4.3.2, we have $t_{\mathrm{P}}((A \leftrightarrow B) \wedge (C \leftrightarrow D) \to (A \equiv \neg C) \equiv (B \equiv \neg D)) = (t_{\mathrm{P}}(A) \leftrightarrow t_{\mathrm{P}}(B)) \wedge (t_{\mathrm{P}}(C) \leftrightarrow t_{\mathrm{P}}(D)) \to \Box(\Box(t_{\mathrm{P}}(A) \leftrightarrow \neg t_{\mathrm{P}}(C)) \leftrightarrow \Box(t_{\mathrm{P}}(B) \leftrightarrow \neg t_{\mathrm{P}}(D)))$. Then, this is a theorem of **KB** because of axiom (B). (IT): By Definition 4.3.2, we have $t_{\mathrm{P}}((A \equiv B) \wedge (C \equiv D) \to (A \equiv C) \equiv (B \equiv D)) = \Box(t_{\mathrm{P}}(A) \leftrightarrow t_{\mathrm{P}}(B)) \wedge \Box(t_{\mathrm{P}}(C) \leftrightarrow t_{\mathrm{P}}(D)) \to \Box(\Box(t_{\mathrm{P}}(A) \leftrightarrow t_{\mathrm{P}}(C)) \leftrightarrow \Box(t_{\mathrm{P}}(B) \leftrightarrow t_{\mathrm{P}}(D)))$. Then, this is a theorem of **K4** because of axiom (4). (IL): By Definition 4.3.2, we have $t_{\mathrm{P}}((A \equiv \neg B) \to \neg(A \equiv B)) = \Box(t_{\mathrm{P}}(A) \leftrightarrow \neg t_{\mathrm{P}}(B)) \to \neg\Box(t_{\mathrm{P}}(A) \leftrightarrow t_{\mathrm{P}}(B))$. Then, this is a theorem of **KD** because of axiom (D). (IE): By Definition 4.3.2, we have $t_{\mathrm{P}}(\neg((A \equiv C) \equiv (B \equiv D)) \to (A \equiv B) \vee (C \equiv D)) = \neg(\Box(\Box(t_{\mathrm{P}}(A) \leftrightarrow t_{\mathrm{P}}(C)) \leftrightarrow \Box(t_{\mathrm{P}}(B) \leftrightarrow t_{\mathrm{P}}(D)))) \to \Box(t_{\mathrm{P}}(A) \leftrightarrow t_{\mathrm{P}}(B)) \vee \Box(t_{\mathrm{P}}(C) \leftrightarrow t_{\mathrm{P}}(D))$. Then, this is a theorem of **K5** because of axiom (5).

■

Theorem 4.7.3 (i) *For any formula* α *in* $\mathrm{L_K}$*,* $\alpha \in \mathbf{L}$ *if and only if* $t_{\mathrm{K}}(\alpha) \in \mathbf{PCI}_{\mathrm{L}}$.

(ii) *For any formula* A *in* $\mathrm{L_S}$*,* $A \in \mathbf{PCI}_{\mathrm{L}}$ *if and only if* $t_{\mathrm{P}}(A) \in \mathbf{L}$.

Proof. The proof is similar to Theorem 4.3.6.

■

Hence we can conclude that two logics $\mathbf{L}$ and $\mathbf{PCI}_{\mathrm{L}}$ are syntactically equivalent by Definition 3.4.1, Theorem 4.3.5 and Theorem 4.7.3.

4.8 Kripke type semantics for $\mathbf{PCI}_{\mathrm{K}}$ extensions

In this section we will define Kripke type semantics for each extension of $\mathbf{PCI}_{\mathrm{K}}$, which have been introduced in Section 4.6. At first, we get the following properties of Kripke frame for validating each additional axioms of $\mathbf{PCI}_{\mathrm{K}}$ in Section 4.6.

Theorem 4.8.1 *For any* $\mathbf{PCI}_{\mathrm{K}}$ *frame* (W, R) *and any valuation* $\models_{\mathrm{P}}$*, the following hold:*

(i) $(W, R)\models_{\mathrm{P}}\mathrm{IR}$ *if and only if* R *is reflexive,*

(ii) $(W, R)\models_{\mathrm{P}}\mathrm{IS}$ *if and only if* R *is symmetric,*

(iii) $(W, R)\models_{\mathrm{P}}\mathrm{IT}$ *if and only if* R *is transitive,*

(iv) $(W, R)\models_{\mathrm{P}}\mathrm{IL}$ *if and only if* R *is serial,*

(v) $(W, R)\models_{\mathrm{P}}\mathrm{IE}$ *if and only if* R *is Euclidean,*

(vi) $(W, R)\models_{\mathrm{P}}\mathrm{IO}$ *if and only if* R *is isolated.*

Proof. We will only show two cases (i) and (iii).

(i): Assume that R is not reflexive. Then it is not aRa for some a in W. Let p and q be distinct variables. We will define a valuation $\models_{\mathrm{P}}$ by (1) $x\models_{\mathrm{P}}p$ and (2) $(x\models_{\mathrm{P}}q \iff x \neq a)$ for any $x \in W$. Then we get $aRy(y\models_{\mathrm{P}}p \iff y\models_{\mathrm{P}}q)$ for any $y \in W$. Therefore, $a\models_{\mathrm{P}}p \equiv q$. On the other hand, we have $a\models_{\mathrm{P}}p$ but $a \not\models_{\mathrm{P}} q$ by the definition. So, $a \not\models_{\mathrm{P}} p \leftrightarrow q$. Hence for some instance of IR, we get $(W, R) \not\models_{\mathrm{P}} ((p \equiv q) \rightarrow (p \leftrightarrow q))$.

Conversely assume that (W, R) such that R is reflexive. Assume $(W, R)\models_{\mathrm{P}}(A \equiv B)$. Then for any $a \in W$, $a\models_{\mathrm{P}}(A \equiv B)$ iff for any $b \in W$, $aR_{\mathrm{P}}b(b\models_{\mathrm{P}}A \iff b\models_{\mathrm{P}}B)$ iff for any $b \in W$, $aR_{\mathrm{P}}b(b\models_{\mathrm{P}}A \leftrightarrow B)$. Now assume that $a\models_{\mathrm{P}}(A \equiv B)$. As R is reflexive we get $a\models_{\mathrm{P}}A \leftrightarrow B$ since aRa. Hence $a\models_{\mathrm{P}}A \equiv B \Longrightarrow a\models_{\mathrm{P}}A \leftrightarrow B$. So we get $(W, R)\models_{\mathrm{P}}(A \equiv B) \rightarrow (A \leftrightarrow B)$.

(iii): Assume that R is not transitive. Namely, there exist aRb and bRc but not aRc for some a, b, c in W. Let p, q, r, s be distinct variables. We will define a valuation $\models_{\mathrm{P}}$ by (1) $x\models_{\mathrm{P}}p, x\models_{\mathrm{P}}q, x\models_{\mathrm{P}}r$ and (2) $(x\models_{\mathrm{P}}s \iff aRx)$ for any $x \in W$. Then we get $a\models_{\mathrm{P}}p \equiv q$ and

$a \models_{\mathrm{P}} r \equiv s$. Therefore, $a \models_{\mathrm{P}} (p \equiv q) \wedge (r \equiv s)$. On the other hand, we have for any $y \in W$, $y \models_{\mathrm{P}} p \equiv r$, and $b \not\models_{\mathrm{P}} q \equiv s$ by the definition. So, $a \not\models_{\mathrm{P}} (p \equiv r) \equiv (q \equiv s)$ since aRb. Hence for some instance of IT, we get $(W, R) \not\models_{\mathrm{P}} ((p \equiv q) \wedge (r \equiv s) \rightarrow (p \equiv r) \equiv (q \equiv s))$.

Conversely assume that (W, R) such that R is transitive. Assume $(W, R) \models_{\mathrm{P}} (A \equiv B) \wedge (C \equiv D)$. Then (a): for any $a \in W$, $a \models_{\mathrm{P}} (A \equiv B) \wedge (C \equiv D)$ iff for any $d \in W$, $aRd(d \models_{\mathrm{P}} A \iff d \models_{\mathrm{P}} B) \wedge (d \models_{\mathrm{P}} C \iff d \models_{\mathrm{P}} D)$. Assume that $b \models_{\mathrm{P}} A \equiv C$. Namely, (b): $(x \models_{\mathrm{P}} A \iff x \models_{\mathrm{P}} C)$ for any $x \in W$ with bRx. On the other hand, we get (c): $(x \models_{\mathrm{P}} A \iff x \models_{\mathrm{P}} B)$ and $(x \models_{\mathrm{P}} C \iff x \models_{\mathrm{P}} D)$ since R is transitive and (a). From (b) and (c), we get (d): $(x \models_{\mathrm{P}} B \iff x \models_{\mathrm{P}} D)$. Therefore, $b \models_{\mathrm{P}} B \equiv D$. Conversely, if we assume that $b \models_{\mathrm{P}} B \equiv D$, then similarly we get $b \models_{\mathrm{P}} A \equiv C$. Hence for any $b \in W$ with aRb, we have $(b \models_{\mathrm{P}} A \equiv C \iff b \models_{\mathrm{P}} B \equiv D)$. So, $a \models_{\mathrm{P}} (A \equiv C) \equiv (B \equiv D)$.

■

Definition 4.8.2 *For any* $\mathbf{PCI}_{\mathrm{K}}$ *frame* (W, R)*, we define several restricted* $\mathbf{PCI}_{\mathrm{K}}$ *frames as the following way:*

(i) *A frame* (W, R) *is called* $\mathbf{PCI}_{\mathrm{KT}}$ Kripke frame *if* R *is reflexive,*

(ii) *A frame* (W, R) *is called* $\mathbf{PCI}_{\mathrm{KB}}$ Kripke frame *if* R *is symmetric,*

(iii) *A frame* (W, R) *is called* $\mathbf{PCI}_{\mathrm{K4}}$ Kripke frame *if* R *is transitive,*

(iv) *A frame* (W, R) *is called* $\mathbf{PCI}_{\mathrm{KD}}$ Kripke frame *if* R *is serial,*

(v) *A frame* (W, R) *is called* $\mathbf{PCI}_{\mathrm{K5}}$ Kripke frame *if* R *is Euclidean,*

(vi) *A frame* (W, R) *is called* $\mathbf{PCI}_{\mathrm{S4}}$ Kripke frame *if* R *is reflexive and transitive,*

(vii) *A frame* (W, R) *is called* $\mathbf{PCI}_{\mathrm{S5}}$ Kripke frame *if* R *is reflexive and Euclidean.*

Finally, let **L** be any modal extensions **KT**, **KB**, **K4**, **KD**, **K5**, **S4** and **S5** of **K**. Then, by the similarity to Theorem 4.4.3, we can give an alternative proof of the completeness theorem for $\mathbf{PCI}_{\mathrm{L}}$.

4.9 Notes

By the requirement conditions (R3) and (R4) in Section 4.2, we have the equivalence $A \equiv B = \Box(A \leftrightarrow B)$ in $\mathbf{PCI}_{\mathrm{K}}$. Then the identity denotes to the necessitation of each material equivalence formula, namely the sameness of all possible worlds (or situations) that each material equivalence formula can be accessible. Moreover, (G) rule say that if A and B are theorems, then they must to have the same possible worlds that they can be accessible in a form of material equivalence $A \leftrightarrow B$, because $A \leftrightarrow B$ holds always with respect to $\top$.

It is possible that **SCI** simulates stronger modal systems, e.g., **S4** and **S5**. We can find in [67] more serious discussion about the relationship between **SCI** and modal logics. According to this literature, Suszko regarded **SCI** as not a kind of modal logic but the new foundations of logic , while many logician had initially the impression that **SCI** merely is a kind of modal logic. We agreed with his views by the results that **SCI** (and also **PCI**) can simulate various logics, besides modal logics.

In Section 4.4, we discussed the representation theorem of $\mathbf{PCI}_{\mathrm{K}}$-algebras. For modal algebras, the similar result is well-known. Let $\mathcal{A}_{\mathrm{MK}} = \langle \mathcal{A}_0, \mathrm{I} \rangle$ be a modal algebra, where $\mathcal{A}_0$ is a Boolean algebra and I is an interior operator such that the following conditions hold: for every $x, y \in A$,

(1) $\mathrm{It} = \mathrm{t}$,

(2) $\mathrm{I}(x \cap y) = \mathrm{I}x \cap \mathrm{I}y$.

Then by using the following definitions of duality of frame and algebra, we get the representation theorem of modal algebras.

Definition 4.9.1 *Let $\mathcal{A}_{\mathrm{MK}} = \langle \mathcal{A}_0, \mathrm{I} \rangle$ be a modal algebra and $\mathcal{F} = (W, R)$ a modal Kripke frame . Then we define:*

(i) $\mathcal{A}_{\mathrm{MK}+} = (\mathrm{M(A)}, R)$ *is called the* dual frame *of* $\mathcal{A}_{\mathrm{MK}}$ *if* $\mathrm{M(A)}$ *is the set of all maximal filters of a Boolean algebra* $\mathcal{A}_0$ *and for any* $F, G \in \mathrm{M(A)}$, FRG *if and only if* $F_{\mathrm{I}} \subseteq G$, *where* $F_{\mathrm{I}} = \{x; \mathrm{I}x \in F\}$,

(ii) *moreover,* $\mathcal{F}^+ = \langle \wp(W), \mathrm{I}^* \rangle$ *is called the* dual algebra *of* $\mathcal{F}$ *if for any* $X \subseteq W$, $\mathrm{I}^*X = \{F; \textit{for any } G \in W \textit{ such that } FRG \textit{ and } G \in X\}$.

Theorem 4.9.2 *Let $\mathcal{A}_{\mathrm{MK}} = \langle \mathcal{A}_0, \mathrm{I} \rangle$ be a modal algebra and $\mathrm{M(A)}$ the set of all maximal filters of $\mathcal{A}_0$. Then the map*

$$h : a \longmapsto \{F \in \mathrm{M(A)}; a \in F\}$$

is an isomorphism of $\mathcal{A}_{\mathrm{MK}} = \langle \mathcal{A}_0, \mathrm{I} \rangle$ into $(\mathcal{A}_{\mathrm{MK}+})^+ = \langle \wp(\mathrm{M(A)}), \mathrm{I}^* \rangle$.

Chapter 5

Corsi's weak logic F and PCI$_{\text{GL}}$ extension for classical substructural logic

In this chapter we will investigate how weak logics with two kinds of weak implications, e.g., strict/linear implication, are simulated by **PCI** logic introduced in Chapter 3. In fact, we will consider both systems of Corsi's weak logic with strict implication (see [16]) and Girard's classical linear logic with linear implication (see [31]). In Section 1, we will briefly survey Corsi's weak logic **F** and its axiomatic extensions in syntactical and semantical points of view. Then we know that **PCI**$_{\text{K}}$ logic introduced in Section 4.2 can also use to interpret the strict implication $\rightharpoonup$ by identity $\equiv$. In Section 2, we will investigate translations between **F** and **PCI**$_{\text{K}}$. Since **F**-language $\mathcal{L}_{\text{F}}$ lacks a material implication $\rightarrow$, we will define an auxiliary language $\mathcal{L}_{\text{F}'}$ by adding $\rightarrow$ to $\mathcal{L}_{\text{F}}$ to restore the balance between both **PCI** (i.e., **SCI**) and **F** languages. Then, for an auxiliary system **F**$'$ of this language, we will give translations between **F**$'$ and **PCI**$_{\text{K}}$, and hence prove that they are syntactically equivalent in a sense of Definition 3.4.1. Moreover, we will show that every formulas in **F**-language can be tanslated into **PCI**$_{\text{K}}$ formulas with keeping logical validity, since **F**$'$ is a conservative extension of **F**. Next as another weak logic, in Section 3, we will give a brief survey of Girard's classical linear logic and its axiomatic extensions in syntactical and semantical points of view. Then in Section 4, we will define **PCI**$_{\text{GL}}$ logic by adding identity axioms (WIA1), (WIA2), (LT), (LE), (L*1), (L*2) and (LDN), and one inference rule (G) to the original system **PCI** in order to interpret correctly the classical linear implication $\supset$ by identity $\equiv$. After this, in Section 5, we will show that every formulas in **GL**-language can be tanslated into **PCI**$_{\text{GL}}$ with keeping logical validity by applying the similar discussion with the case of Corsi's weak logic **F**. Finally we will also give further information on related results shown in this chapter (Section 6).

5.1 Corsi's weak logic F

In this section we will briefly survey Corsi's weak logic **F** in syntactical and semantical points of view. In [16], G. Corsi investigated sublogics of intuitionistic propositional logic which are characterized by classes of transitive Kripke models, in which logical connectives are interpreted in a standard way like intuitionistic logic but heredity of truth is not assumed. Therefore, Corsi's system has strict implication and strict negation, so that next we will introduce the axiomatic deductive system of **F** to consider the interpretation of strict implication by identity connective $\equiv$ in our system **PCI**.

Let $\mathcal{L}_{\mathrm{F}} = \langle \mathrm{L}_{\mathrm{F}}, \wedge, \vee, \rightharpoonup, \bot, \top \rangle$ be the **F**-language containing of an infinite denumerable set VAR of propositional variables, constant; $\bot$ (false), and intuitionistic connectives; $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightharpoonup$ (strict implication). Formulas L_{F} of a given **F**-language $\mathcal{L}_{\mathrm{F}}$ are defined in the usual way. The propositional constant; $\top$ (true) and other connectives; $\sim$ (strict negation), $\rightleftharpoons$ (strict equivalence) are to be constructed as the usual abbreviation: $\sim \alpha := \alpha \rightharpoonup \bot$, $\top :=\sim \bot := \bot \rightharpoonup \bot$ and $\alpha \rightleftharpoons \beta := (\alpha \rightharpoonup \beta) \wedge (\beta \rightharpoonup \alpha)$. Also we will sometime omit parentheses when no confusion will occur, following the assumption that the priority of each connective is weak as $\sim, \wedge, \vee, \rightharpoonup, \rightleftharpoons$ in order.

The logical axioms and rules of inference for **F**-language $\mathcal{L}_{\mathrm{F}}$ consist of a set of schemata from (a1) to (a10) and modus ponens (FMp) and a fortiori (FAf) as rules of inference below:

(a1) $\alpha \rightharpoonup \alpha$

(a2) $(\alpha \rightharpoonup \beta) \wedge (\beta \rightharpoonup \gamma) \rightharpoonup (\alpha \rightharpoonup \gamma)$

(a3) $(\alpha \wedge \beta) \rightharpoonup \alpha$

(a4) $(\alpha \wedge \beta) \rightharpoonup \beta$

(a5) $(\alpha \rightharpoonup \beta) \wedge (\alpha \rightharpoonup \gamma) \rightharpoonup (\alpha \rightharpoonup \beta \wedge \gamma)$

(a6) $\alpha \rightharpoonup (\alpha \vee \beta)$

(a7) $\beta \rightharpoonup (\alpha \vee \beta)$

(a8) $(\alpha \rightharpoonup \beta) \wedge (\gamma \rightharpoonup \beta) \rightharpoonup (\alpha \vee \gamma \rightharpoonup \beta)$

(a9) $\alpha \wedge (\beta \vee \gamma) \rightharpoonup (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$

(a10) $\bot \rightharpoonup \alpha$

(FMp) $\dfrac{\alpha_1, \ldots, \alpha_n \quad \alpha_1 \wedge \cdots \wedge \alpha_n \rightharpoonup \beta}{\beta}$

(FAf) $\dfrac{\alpha}{\beta \rightharpoonup \alpha}$

Then the axiomatic deductive system $F(\Gamma)$ for $\mathbf{F} = (\mathcal{L}_{\mathrm{F}}, F)$ is defined as follows.

Definition 5.1.1 (i) *For any* $\Gamma \subseteq \mathrm{L_F}$, $F(\Gamma)$ *is the smallest set of formulas closed under rules of* (FMp) *and* (FAf), *which contains from* (a1) *to* (a10) *and* Γ.

(ii) *The element of* $F(\emptyset)$ *is called the logical theorem of* $\mathbf{F}$.

Then it is easily verified that F is a consequence operator. By the similarity to Proposition 2.2.2, we have the following.

Theorem 5.1.2 *For any* $\Gamma \cup \{\alpha, \beta, \gamma\} \subseteq \mathrm{L_F}$, *it holds the following equivalences:*

(i) $\neg\alpha \in F(\Gamma)$ *if and only if* $\bot \in F(\Gamma; \alpha)$

(ii) $\alpha \in F(\Gamma)$ *if and only if there exist some finite subset* Σ *of* Γ *such that* $\alpha \in F(\Sigma)$. (*Compactness*)

(iii) *For any* $p \in \mathrm{VAR}$, $(\alpha \equiv \beta) \to (\gamma[\alpha/p] \equiv \gamma[\beta/p])$ *is a logical theorem of* $\mathbf{F}$. (*Replacement Law*)

The elementary extension of $\mathbf{F}$ with an additional axiom α will be denoted by $\mathbf{F} \oplus \alpha$. Then the following extensions of $\mathbf{F}$ are discussed in [16]:

(1) $\mathbf{FD} = \mathbf{F} \oplus \sim\sim \top$

(2) $\mathbf{FR} = \mathbf{F} \oplus (\alpha \wedge (\alpha \rightharpoonup \beta) \rightharpoonup \beta)$

(3) $\mathbf{FT} = \mathbf{F} \oplus ((\alpha \rightharpoonup \beta) \rightharpoonup (\gamma \rightharpoonup (\alpha \rightharpoonup \beta)))$

(4) $\mathbf{FS} = \mathbf{F} \oplus (\alpha \rightharpoonup (\beta \vee \sim (\alpha \rightharpoonup \beta)))$

(5) $\mathbf{FC} = \mathbf{F} \oplus ((\gamma \wedge (\alpha \rightharpoonup \beta)) \rightharpoonup \delta) \vee ((\alpha \wedge (\gamma \rightharpoonup \delta)) \rightharpoonup \beta)$

(6) $\mathbf{FZ} = \mathbf{F} \oplus (\alpha \rightharpoonup (\beta \rightharpoonup \alpha)) \wedge (\alpha \vee \sim \alpha)$

Let $\mathcal{F} = (W, R)$ be $\mathbf{F}$ Kripke frame for $\mathcal{L}_\mathrm{F}$, which is the same as a modal Kripke frame (see Section 4.1). The only difference between both $\mathbf{F}$ and $\mathbf{K}$ Kripke model is the definition of validity of formulas. Let $\mathcal{M} = (W, R, V)$ be $\mathbf{F}$ Kripke model for $\mathcal{L}_\mathrm{F}$, where $\mathcal{F} = (W, R)$ is $\mathbf{F}$ Kripke frame and V a valuation on $\mathcal{F}_\mathrm{F}$ which is a map from VAR to 2^W such that $V(p) \subseteq W$ for any $p \in \mathrm{VAR}$, $V(\bot) = \emptyset$ and $V(\top) = W$. Then for any point $a \in W$, we can extend V to the valuation of $\mathbf{F}$ formulas $\models_\mathrm{F} : \mathrm{L_F} \to 2^W$, in the similar way as the case of $\mathbf{K}$, by the following way.

Definition 5.1.3 *Given* $\mathbf{F}$ *Kripke model* $\mathcal{M} = (W, R, V)$, *the notion of validity of* $\mathbf{F}$ *formulas at any point* $a \in W$ *is defined inductively as follows:*

(i) $\mathcal{M}, a \models_\mathrm{F} p$ *if and only if* $a \in V(p)$ *for any variable* $p \in \mathrm{VAR}$,

(ii) $\mathcal{M}, a \not\models_\mathrm{F} \bot$ *and* $\mathcal{M}, a \models_\mathrm{F} \top$,

(iii) $\mathcal{M}, a \models_{\mathrm{F}} \alpha \wedge \beta$ *if and only if* $\mathcal{M}, a \models_{\mathrm{F}} \alpha$ *and* $\mathcal{M}, a \models_{\mathrm{F}} \beta$,

(iv) $\mathcal{M}, a \models_{\mathrm{F}} \alpha \vee \beta$ *if and only if* $\mathcal{M}, a \models_{\mathrm{F}} \alpha$ *or* $\mathcal{M}, a \models_{\mathrm{F}} \beta$ *and*

(v) $\mathcal{M}, a \models_{\mathrm{F}} \alpha \rightharpoonup \beta$ *if and only if for all* b *with* aRb, $\mathcal{M}, b \models_{\mathrm{F}} \alpha$ *implies* $\mathcal{M}, b \models_{\mathrm{F}} \beta$.

Here the validity of classical parts in both **F** and **K** Kripke model is the same. For any Kripke frame $\mathcal{F} = (W, R)$, a formula α is *valid* on $\mathcal{F}$, in symbols, $\mathcal{F} \models_{\mathrm{F}} \alpha$ if $\mathcal{M}, a \models_{\mathrm{F}} \alpha$ for any $a \in W$ and any valuation $\models_{\mathrm{F}}$. Then every extensions of **F**, recalled so far, are well-known to be sound and complete with respect to natural classes of **F** Kripke frames (see [16]).

Theorem 5.1.4 (Corsi's weak logic completeness) *For any formula* $\alpha \in \mathrm{L}_{\mathrm{F}}$*, any* **F** *Kripke frame* $\mathcal{F} = (W, R)$ *and any valuation* V *on* $\mathcal{F}$*, it holds the following equivalence:*

(i) $\alpha \in$ **F** *if and only if* $\mathcal{F} \models_{\mathrm{F}} \alpha$ *for all* $\mathcal{F}$.

(ii) $\alpha \in$ **FD** *if and only if* $\mathcal{F} \models_{\mathrm{F}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is serial.*

(iii) $\alpha \in$ **FR** *if and only if* $\mathcal{F} \models_{\mathrm{F}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is reflexive.*

(iv) $\alpha \in$ **FT** *if and only if* $\mathcal{F} \models_{\mathrm{F}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is transitive.*

(v) $\alpha \in$ **FS** *if and only if* $\mathcal{F} \models_{\mathrm{F}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is symmetric.*

(vi) $\alpha \in$ **FC** *if and only if* $\mathcal{F} \models_{\mathrm{F}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is connected.*

(vii) $\alpha \in$ **FZ** *if and only if* $\mathcal{F} \models_{\mathrm{F}} \alpha$ *for all* $\mathcal{F}$ *such that* R *is isolated.*

5.2 Translation of F into $\mathbf{PCI}_{\mathrm{K}}$

In this section we will consider an extension of **PCI** in order to interpret the strict implication $\rightharpoonup$ by identity $\equiv$ (see [34] and [35]). Then we need the following conditions to hold in **PCI**:

(R5) $\overrightarrow{\alpha \rightharpoonup \beta} \longmapsto \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta}$,

(R6) $\overleftarrow{A \equiv B} \longmapsto (\overleftarrow{A} \rightleftharpoons \overleftarrow{B})$,

where $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$, $\overleftarrow{A}$ and $\overleftarrow{B}$ denote the results of translations from **F** to $\mathbf{PCI}_{\mathrm{K}}$, and its converse, respectively. Since we can rewrite the second requirement (R6) by $(\overleftarrow{A} \rightleftharpoons \overleftarrow{B})$ iff $\Box(\overleftarrow{A} \leftrightarrow \overleftarrow{B})$ in the sight of both Kripke model between **F** and **K**, the above requirements (R5) and (R6) are reduced to (R3) and (R4). Therefore, $\mathbf{PCI}_{\mathrm{K}}$ logic introduced in Section 4.2 can also use to do our jobs. But at first to restore the balance between both **F** and **PCI** languages, we need to extend the **F**-language by adding one more material implication $\rightarrow$ (see Fig 5.1 below).

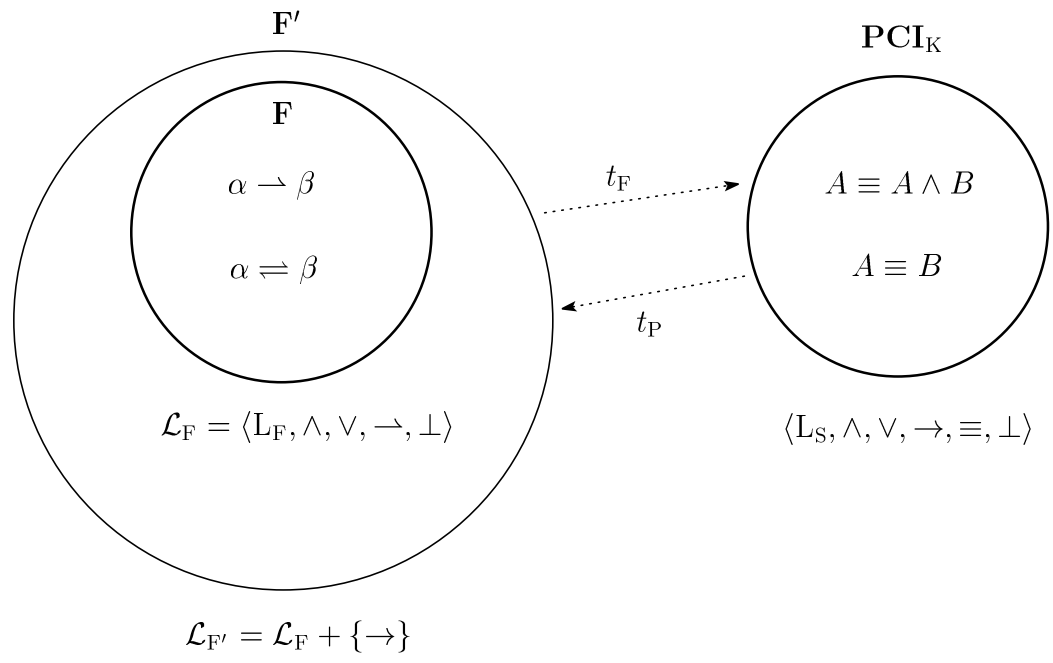

Figure 5.1: Requirements of simulation of **F**

Let this auxiliary language be $\mathcal{L}_{\mathrm{F}'} = \langle \mathrm{L}_{\mathrm{F}'}, \neg, \sim, \wedge, \vee, \rightarrow, \rightharpoonup, \leftrightarrow, \rightleftharpoons, \bot, \top \rangle$. The logical axioms and rules of inference for **F**′-language $\mathcal{L}_{\mathrm{F}'}$ are obtained from a set of schemata from (a1) to (a10) and two rules of inference, modus ponens (FMp) and a fortiori (FAf), by adding the additional axiom schemata TFA, which are same as from (A1) to (A10) in **SCI** for TF-connective part $(\neg, \wedge, \vee, \rightarrow)$, (FW1), (FW2) and the modus ponens (Mp) rule for $\rightarrow$ below:

(FW1) $((\alpha \rightarrow \beta) \rightleftharpoons (\beta \rightarrow \alpha)) \rightarrow (\alpha \rightleftharpoons \beta)$,

(FW2) $((\alpha \rightarrow \beta) \rightleftharpoons (\beta \rightarrow \alpha)) \rightarrow ((\alpha \rightarrow \beta) \rightleftharpoons \top) \wedge ((\beta \rightarrow \alpha) \rightleftharpoons \top)$,

(Mp) $\dfrac{\alpha \quad \alpha \rightarrow \beta}{\beta}$.

Then the axiomatic deductive system $F'(\Gamma)$ for $\mathbf{F}' = (\mathcal{L}_{\mathrm{F}'}, F')$ is defined as follows.

Definition 5.2.1 (i) *For any* $\Gamma \subseteq \mathrm{L}_{\mathrm{F}'}$, $F'(\Gamma)$ *is the smallest set of formulas closed under the rules of* (FMp), (FAf) *and* (Mp), *which contains from* (a1) *to* (a10), *and from* (A1) *to* (A10), (FW1), (FW2) *and* Γ.

(ii) *The element of* $F'(\emptyset)$ *is called the logical theorem of* **F**′.

Then we will get the following basic properties of **F**′, in which (viii) is the result of completeness theorem for **F** relative to Kripke type semantics.

Theorem 5.2.2 *The following are derived rules and logical theorem of* **F**′.

(i) $\dfrac{\alpha \rightharpoonup \beta}{\alpha \to \beta}$ (FIm)

(ii) $\dfrac{\alpha \quad \alpha \rightharpoonup \beta}{\beta}$ (FMp)

(iii) $\dfrac{\alpha \quad \beta}{\alpha \rightleftharpoons \beta}$ (FG)

(iv) $\dfrac{\alpha \leftrightarrow \beta}{\alpha \rightleftharpoons \beta}$ (FQF)

(v) $\dfrac{\alpha \rightleftharpoons \beta}{\alpha \to \beta}$

(vi) $\dfrac{\alpha \rightharpoonup (\beta \rightharpoonup \gamma)}{\alpha \to (\beta \to \gamma)}$

(vii) $((\alpha \rightharpoonup \alpha \wedge \beta) \wedge (\alpha \wedge \beta \rightharpoonup \alpha)) \leftrightarrow (\alpha \rightharpoonup \beta)$

(viii) *For any formula α in $\mathrm{L}_{\mathbf{F}'}$ such that α not contains $\to$ connective at all, $\alpha \in \mathbf{F}'$ if and only if $\alpha \in \mathbf{F}$.*

Proof. (i): this is almost obvious since every axiom and inference rule of **F** is classically valid. (ii), (v) and (vi): these follow from (i) and (Mp). (iii): this follows from (FAf) and (FMp). (iv): suppose $\alpha \leftrightarrow \beta$, then we get $(\alpha \to \beta) \rightleftharpoons (\beta \to \alpha)$ by (A3), (A4) and (FG). So, we get the desired result by (FW1) and (Mp).

(vii) : 1 Put $A = (\alpha \rightharpoonup \alpha \wedge \beta) \wedge (\alpha \wedge \beta \rightharpoonup \alpha)$

2 $A \to (\alpha \rightharpoonup \alpha \wedge \beta)$ (A3)

3 $\alpha \wedge \beta \rightharpoonup \beta$ (a4)

4 $A \to (\alpha \wedge \beta \rightharpoonup \beta)$ (3,A1,Mp)

5 $A \to (\alpha \rightharpoonup \alpha \wedge \beta) \wedge (\alpha \wedge \beta \rightharpoonup \beta)$ (2,4,A5)

6 $(\alpha \rightharpoonup \alpha \wedge \beta) \wedge (\alpha \wedge \beta \rightharpoonup \beta) \rightharpoonup (\alpha \rightharpoonup \beta)$ (a2)

7 $(\alpha \rightharpoonup \alpha \wedge \beta) \wedge (\alpha \wedge \beta \rightharpoonup \beta) \to (\alpha \rightharpoonup \beta)$ (6,FIm)

8 $A \to (\alpha \rightharpoonup \beta)$ (5,7,transitivity of $\to$)

9 $\alpha \rightharpoonup \alpha$ (a1)

10 $(\alpha \rightharpoonup \beta) \to (\alpha \rightharpoonup \alpha)$ (9,A1,Mp)

11 $(\alpha \rightharpoonup \beta) \to (\alpha \rightharpoonup \beta)$

12 $(\alpha \rightharpoonup \beta) \to (\alpha \rightharpoonup \alpha) \wedge (\alpha \rightharpoonup \beta)$ (10,11,A5)

13 $(\alpha \rightharpoonup \alpha) \wedge (\alpha \rightharpoonup \beta) \to (\alpha \rightharpoonup \alpha \wedge \beta)$ (a5,FIm)

14 $(\alpha \rightharpoonup \beta) \to (\alpha \rightharpoonup \alpha \wedge \beta)$ (12,13,trans.of $\to$)

15 $\alpha \wedge \beta \rightharpoonup \alpha$ (a3)

16 $(\alpha \rightharpoonup \beta) \to (\alpha \wedge \beta \rightharpoonup \alpha)$ (15,A1,Mp)

17 $(\alpha \rightharpoonup \beta) \to (\alpha \rightharpoonup \alpha \wedge \beta) \wedge (\alpha \wedge \beta \rightharpoonup \alpha)$ (14,16,A5)

(viii): The if-part is trivial since $\mathbf{F}'$ is an extension of **F** by the above definition. To prove the converse direction we will consider the Kripke model for $\mathbf{F}'$. Given a Kripke model

$\mathcal{M} = (W, R, V)$ for **F**, we get the Kripke model $\mathcal{M}_{\mathrm{F}'}$ for $\mathbf{F}'$ by adding to it one more definition of $\mathbf{F}'$ formula's interpretation as

$$\mathcal{M}_{\mathrm{F}'}, a \models_{\mathrm{F}'} \alpha \to \beta \quad \text{if and only if} \quad \mathcal{M}_{\mathrm{F}'}, a \models_{\mathrm{F}'} \alpha \quad \text{implies} \quad \mathcal{M}_{\mathrm{F}'}, a \models_{\mathrm{F}'} \beta.$$

Then we can easily prove the soundness of $\mathbf{F}'$ with respect to above Kripke model, that is for any formula α in $\mathrm{L}_{\mathrm{F}'}$, $\alpha \in \mathbf{F}'$ implies $\mathcal{F}_{\mathrm{F}'} \models_{\mathrm{F}'} \alpha$ for any frame $\mathcal{F}_{\mathrm{F}'} = (W_{\mathrm{F}'}, R_{\mathrm{F}'})$. Hence if we assume $\alpha \notin \mathbf{F}$ for some formula α in L_{F} then by the completeness result for **F** there exists a world a in the model $\mathcal{M} = (W, R, V)$ for **F** such that $\mathcal{M}, a \not\models_{\mathrm{F}} \alpha$. Then by the above interpretation of $\to$, this model can also be seen as the model $\mathcal{M}_{\mathrm{F}'}$ for $\mathbf{F}'$, so we get $\mathcal{M}_{\mathrm{F}'}, a \not\models_{\mathrm{F}'} \alpha$ for some formula α in $\mathrm{FOR}_{\mathrm{F}}$ and a world a in $\mathcal{M}_{\mathrm{F}'}$. Then by the soundness of $\mathbf{F}'$, we have $\alpha \notin \mathbf{F}'$.

■

Next we will give translations between $\mathbf{F}'$ and $\mathbf{PCI}_{\mathrm{K}}$, and hence prove that they are syntactically equivalent. How to show the syntactically equivalent of two logics follows the previous discipline in Section 3.4. At first we will define two translations t_{F} and t_{P} between $\mathbf{F}'$-language $\mathcal{L}_{\mathrm{F}'}$ and **SCI**-language $\mathcal{L}_{\mathrm{S}}$ in order to show two logics $\mathbf{F}'$ and $\mathbf{PCI}_{\mathrm{K}}$ are syntactically equivalent with respect to these maps.

Definition 5.2.3 *The mapping* $t_{\mathrm{F}} : \mathrm{L}_{\mathrm{F}'} \to \mathrm{L}_{\mathrm{S}}$, *called a* F-translation, *is defined inductively as follows:*

(i) $t_{\mathrm{F}}(p) := p, \quad p \in \mathrm{VAR}$,

(ii) $t_{\mathrm{F}}(\bot) := \bot$,

(iii) $t_{\mathrm{F}}(\alpha \wedge \beta) := (t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta))$,

(iv) $t_{\mathrm{F}}(\alpha \vee \beta) := (t_{\mathrm{F}}(\alpha) \vee t_{\mathrm{F}}(\beta))$,

(v) $t_{\mathrm{F}}(\alpha \to \beta) := (t_{\mathrm{F}}(\alpha) \to t_{\mathrm{F}}(\beta))$,

(vi) $t_{\mathrm{F}}(\alpha \rightharpoonup \beta) := (t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta))$.

Definition 5.2.4 *The mapping* $t_{\mathrm{P}} : \mathrm{L}_{\mathrm{S}} \to \mathrm{L}_{\mathrm{F}'}$, *called a* PCI-translation, *is defined inductively as follows:*

(i) $t_{\mathrm{P}}(p) := p, \quad p \in \mathrm{VAR}$,

(ii) $t_{\mathrm{P}}(\bot) := \bot$,

(iii) $t_{\mathrm{P}}(\neg A) := t_{\mathrm{P}}(A) \to \bot$,

(iv) $t_{\mathrm{P}}(A \wedge B) := (t_{\mathrm{P}}(A) \wedge t_{\mathrm{P}}(B))$,

(v) $t_{\mathrm{P}}(A \vee B) := (t_{\mathrm{P}}(A) \vee t_{\mathrm{P}}(B))$,

(vi) $t_{\mathrm{P}}(A \to B) := (t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(B))$,

(vii) $t_{\mathrm{P}}(A \equiv B) := (t_{\mathrm{P}}(A) \rightleftharpoons t_{\mathrm{P}}(B))$.

For two maps t_{F} and t_{P}, we can prove the following two propositions.

Proposition 5.2.5 *For any formula α in $\mathrm{L}_{\mathrm{F}'}$, $\alpha \in \mathbf{F}'$ implies $t_{\mathrm{F}}(\alpha) \in \mathbf{PCI}_{\mathrm{K}}$.*

Proof. The proof is in the same manner as modal logic in Section 4.3.
Base step: We can easily check all of the following formulas are provable in $\mathbf{PCI}_{\mathrm{K}}$.

(a1) $t_{\mathrm{F}}(\alpha \rightharpoonup \alpha) = t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\alpha)$

(a2) $t_{\mathrm{F}}((\alpha \rightharpoonup \beta) \wedge (\beta \rightharpoonup \gamma) \rightharpoonup (\alpha \rightharpoonup \gamma))$
$= ((t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \wedge (t_{\mathrm{F}}(\beta) \equiv t_{\mathrm{F}}(\beta) \wedge t_{\mathrm{F}}(\gamma)))$
$\equiv ((t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \wedge (t_{\mathrm{F}}(\beta) \equiv t_{\mathrm{F}}(\beta) \wedge t_{\mathrm{F}}(\gamma))) \wedge (t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\gamma))$

(a3) $t_{\mathrm{F}}(\alpha \wedge \beta \rightharpoonup \alpha)$
$= (t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \equiv (t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \wedge t_{\mathrm{F}}(\alpha)$

(a4) $t_{\mathrm{F}}(\alpha \wedge \beta \rightharpoonup \beta)$
$= (t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \equiv (t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \wedge t_{\mathrm{F}}(\beta)$

(a5) $t_{\mathrm{F}}((\alpha \rightharpoonup \beta) \wedge (\alpha \rightharpoonup \gamma) \rightharpoonup (\alpha \rightharpoonup \beta \wedge \gamma))$
$= ((t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \wedge (t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\gamma)))$
$\equiv ((t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \wedge (t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\gamma)))$
$\wedge (t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge (t_{\mathrm{F}}(\beta) \wedge t_{\mathrm{F}}(\gamma)))$

(a6) $t_{\mathrm{F}}(\alpha \rightharpoonup (\alpha \vee \beta)) = t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge (t_{\mathrm{F}}(\alpha) \vee t_{\mathrm{F}}(\beta))$

(a7) $t_{\mathrm{F}}(\beta \rightharpoonup (\alpha \vee \beta)) = t_{\mathrm{F}}(\beta) \equiv t_{\mathrm{F}}(\beta) \wedge (t_{\mathrm{F}}(\alpha) \vee t_{\mathrm{F}}(\beta))$

(a8) $t_{\mathrm{F}}((\alpha \rightharpoonup \beta) \wedge (\gamma \rightharpoonup \beta) \rightharpoonup (\alpha \vee \gamma \rightharpoonup \beta))$
$= ((t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \wedge (t_{\mathrm{F}}(\gamma) \equiv t_{\mathrm{F}}(\gamma) \wedge t_{\mathrm{F}}(\beta)))$
$\equiv ((t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \wedge (t_{\mathrm{F}}(\gamma) \equiv t_{\mathrm{F}}(\gamma) \wedge t_{\mathrm{F}}(\beta)))$
$\wedge ((t_{\mathrm{F}}(\alpha) \vee t_{\mathrm{F}}(\gamma)) \equiv (t_{\mathrm{F}}(\alpha) \vee t_{\mathrm{F}}(\gamma)) \wedge t_{\mathrm{F}}(\beta))$

(a9) $t_{\mathrm{F}}(\alpha \wedge (\beta \vee \gamma) \rightharpoonup (\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
$= (t_{\mathrm{F}}(\alpha) \wedge (t_{\mathrm{F}}(\beta) \vee t_{\mathrm{F}}(\gamma))) \equiv (t_{\mathrm{F}}(\alpha) \wedge (t_{\mathrm{F}}(\beta) \vee t_{\mathrm{F}}(\gamma)))$
$\wedge ((t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\beta)) \vee (t_{\mathrm{F}}(\alpha) \wedge t_{\mathrm{F}}(\gamma)))$

(a10) $t_{\mathrm{F}}(\bot \rightharpoonup \alpha) = \bot \equiv \bot \wedge t_{\mathrm{F}}(\alpha)$

(FW1) $t_{\mathrm{F}}(((\alpha \to \beta) \rightleftharpoons (\beta \to \alpha)) \to (\alpha \rightleftharpoons \beta))$
$= ((t_{\mathrm{F}}(\alpha) \to t_{\mathrm{F}}(\beta)) \equiv (t_{\mathrm{F}}(\beta) \to t_{\mathrm{F}}(\alpha))) \to (t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\beta))$

(FW2) $t_{\mathrm{F}}(((\alpha \to \beta) \rightleftharpoons (\beta \to \alpha)) \to ((\alpha \to \beta) \rightleftharpoons \top) \wedge ((\beta \to \alpha) \rightleftharpoons \top))$
$= ((t_{\mathrm{F}}(\alpha) \to t_{\mathrm{F}}(\beta)) \equiv (t_{\mathrm{F}}(\beta) \to t_{\mathrm{F}}(\alpha)))$
$\to ((t_{\mathrm{F}}(\alpha) \to t_{\mathrm{F}}(\beta)) \equiv \top) \wedge ((t_{\mathrm{F}}(\beta) \to t_{\mathrm{F}}(\alpha)) \equiv \top))$

Induction step: We have to check the admissibility of (FMp), (FAf) and (Mp) in $\mathbf{PCI}_{\mathrm{K}}$ after a t_{F}-translation.
Case1: Assume that $\alpha_1, \ldots, \alpha_n, \alpha_1 \wedge \cdots \wedge \alpha_n \rightharpoonup \alpha$ are provable in $\mathbf{F}'$. Then by I.H. $t_{\mathrm{F}}(\alpha_1), \ldots, t_{\mathrm{F}}(\alpha_n), t_{\mathrm{F}}(\alpha_1 \wedge \cdots \wedge \alpha_n \rightharpoonup \alpha)$ hold in $\mathbf{PCI}_{\mathrm{K}}$. Here $t_{\mathrm{F}}(\alpha_1 \wedge \cdots \wedge \alpha_n \rightharpoonup \alpha) = (t_{\mathrm{F}}(\alpha_1) \wedge \cdots \wedge t_{\mathrm{F}}(\alpha_n) \equiv (t_{\mathrm{F}}(\alpha_1) \wedge \cdots \wedge t_{\mathrm{F}}(\alpha_n)) \wedge t_{\mathrm{F}}(\alpha)$. Hence, it is possible to derive the following proofs in $\mathbf{PCI}_{\mathrm{K}}$, where we only show the case of n=2 for simplicity. At first, from above first two hypothesis we get the following proof in $\mathbf{PCI}_{\mathrm{K}}$:

$$\dfrac{\dfrac{\dfrac{t_{\mathrm{F}}(\alpha_1)}{t_{\mathrm{F}}(\alpha_1) \equiv \top}\,(\mathrm{G}) \qquad \dfrac{t_{\mathrm{F}}(\alpha_2)}{t_{\mathrm{F}}(\alpha_2) \equiv \top}\,(\mathrm{G})}{(t_{\mathrm{F}}(\alpha_1) \equiv \top) \wedge (t_{\mathrm{F}}(\alpha_2) \equiv \top)}\,(\mathrm{A5, Mp})}{(t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \equiv \top}\,(Th.4.2.2\ (\mathrm{xv}), \mathrm{Mp}).$$

Secondly by using above result and third hypothesis we get the following two proofs in $\mathbf{PCI}_{\mathrm{K}}$:

$$\dfrac{\dfrac{\dfrac{(t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \equiv \top}{\top \equiv (t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2))}\,(\mathrm{E2, Mp}) \qquad \begin{matrix} Hypothesis \\ \vdots \\ (t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \equiv (t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \wedge t_{\mathrm{F}}(\alpha) \end{matrix}}{(\top \equiv (t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2))) \wedge ((t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \equiv (t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \wedge t_{\mathrm{F}}(\alpha))}\,(\mathrm{A5, Mp})}{\top \equiv (t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \wedge t_{\mathrm{F}}(\alpha)}\,(\mathrm{E3, Mp}),$$

and

$$\dfrac{\dfrac{(t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \equiv \top \qquad \begin{matrix} (\mathrm{E1}) \\ \vdots \\ t_{\mathrm{F}}(\alpha) \equiv t_{\mathrm{F}}(\alpha) \end{matrix}}{((t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \equiv \top) \wedge (t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2))}\,(\mathrm{A5, Mp})}{(t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \wedge t_{\mathrm{F}}(\alpha) \equiv \top \wedge t_{\mathrm{F}}(\alpha)}\,(\mathrm{C2, Mp}).$$

Hence from both results we can derive the following proof in $\mathbf{PCI}_{\mathrm{K}}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\top \equiv (t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \wedge t_{\mathrm{F}}(\alpha) \quad (t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \wedge t_{\mathrm{F}}(\alpha) \equiv \top \wedge t_{\mathrm{F}}(\alpha)}{(\top \equiv (t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \wedge t_{\mathrm{F}}(\alpha)) \wedge ((t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\alpha_2)) \wedge t_{\mathrm{F}}(\alpha) \equiv \top \wedge t_{\mathrm{F}}(\alpha))}\,(\mathrm{A5, Mp})}{\top \equiv \top \wedge t_{\mathrm{F}}(\alpha)}\,(\mathrm{E3, Mp})}{t_{\mathrm{F}}(\top) \equiv t_{\mathrm{F}}(\top) \wedge t_{\mathrm{F}}(\alpha)}\,(Def.5.2.3\ (\mathrm{ii}))}{t_{\mathrm{F}}(\top \rightharpoonup \alpha)}\,(Def.5.2.3\ (\mathrm{vi}))}{t_{\mathrm{F}}(\alpha)}\,((\top \rightharpoonup \alpha) \rightleftharpoons \alpha)$$

Therefore, we will get $t_{\mathrm{F}}(\alpha) \in \mathbf{PCI}_{\mathrm{K}}$.
Case2: Assume that α_1 is provable in $\mathbf{F}'$, so by I.H. $t_{\mathrm{F}}(\alpha_1) \in \mathbf{PCI}_{\mathrm{K}}$. Also let β_1 be any

formula in $\mathrm{L}_{\mathrm{F}'}$ such that $t_{\mathrm{F}}(\beta_1) \in \mathbf{PCI}_{\mathrm{K}}$. Then we can get the following proof in $\mathbf{PCI}_{\mathrm{K}}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{t_{\mathrm{F}}(\beta_1)\; t_{\mathrm{F}}(\alpha_1)}{t_{\mathrm{F}}(\beta_1) \equiv t_{\mathrm{F}}(\alpha_1)}\,(\mathrm{G}) \qquad \begin{array}{c}(\mathrm{E1})\\ \vdots \\ t_{\mathrm{F}}(\beta_1) \equiv t_{\mathrm{F}}(\beta_1)\end{array}}{(t_{\mathrm{F}}(\beta_1) \wedge t_{\mathrm{F}}(\beta_1)) \equiv (t_{\mathrm{F}}(\beta_1) \wedge t_{\mathrm{F}}(\alpha_1))}\,(\mathrm{C2}, \mathrm{Mp})}{t_{\mathrm{F}}(\beta_1) \equiv (t_{\mathrm{F}}(\beta_1) \wedge t_{\mathrm{F}}(\alpha_1))}\,(\mathrm{E3}, \mathrm{Mp})}{t_{\mathrm{F}}(\beta_1 \rightharpoonup \alpha_1)}\,(Def.5.2.3\ (\mathrm{vi}))$$

Therefore, we will get $t_{\mathrm{F}}(\beta_1 \rightharpoonup \alpha_1) \in \mathbf{PCI}_{\mathrm{K}}$.

Case3: (Mp) rule is almost obvious since $\mathbf{PCI}_{\mathrm{K}}$ also has (Mp) rule.

Thus the t_{F}-translation of any formula provable in $\mathbf{F}'$ is also provable in $\mathbf{PCI}_{\mathrm{K}}$.

■

Proposition 5.2.6 *For any formula A in L_{S}, $A \in \mathbf{PCI}_{\mathrm{K}}$ implies $t_{\mathrm{P}}(A) \in \mathbf{F}'$.*

Proof. This proof is also in the same manner as modal logic in Section 4.3.
Base step: We can easily check all of the following formulas are provable in $\mathbf{F}'$.

(A1) $t_{\mathrm{P}}(A \to (B \to A)) = t_{\mathrm{P}}(A) \to (t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(A))$

(A2) $t_{\mathrm{P}}((A \to (B \to C)) \to ((A \to B) \to (A \to C)))$
$= (t_{\mathrm{P}}(A) \to (t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(C))) \to ((t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(B)) \to (t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(C)))$

(A3) $t_{\mathrm{P}}(A \wedge B \to A) = t_{\mathrm{P}}(A) \wedge t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(A)$

(A4) $t_{\mathrm{P}}(A \wedge B \to B) = t_{\mathrm{P}}(A) \wedge t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(B)$

(A5) $t_{\mathrm{P}}(A \to (B \to (A \wedge B))) = t_{\mathrm{P}}(A) \to (t_{\mathrm{P}}(B) \to (t_{\mathrm{P}}(A) \wedge t_{\mathrm{P}}(B)))$

(A6) $t_{\mathrm{P}}(A \to A \vee B) = t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(A) \vee t_{\mathrm{P}}(B)$

(A7) $t_{\mathrm{P}}(B \to A \vee B) = t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(A) \vee t_{\mathrm{P}}(B)$

(A8) $t_{\mathrm{P}}((A \to C) \to ((B \to C) \to (A \vee B \to C)))$
$= (t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(C)) \to ((t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(C)) \to (t_{\mathrm{P}}(A) \vee t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(C)))$

(A9) $t_{\mathrm{P}}(A \to (\neg A \to B)) = t_{\mathrm{P}}(A) \to (\neg t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(B))$

(A10) $t_{\mathrm{P}}(\neg\neg A \to A) = \neg\neg t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(A)$

(E1) $t_{\mathrm{P}}(A \equiv A) = t_{\mathrm{P}}(A) \rightleftharpoons t_{\mathrm{P}}(B)$

(E2) $t_{\mathrm{P}}((A \equiv B) \to (B \equiv A)) = (t_{\mathrm{P}}(A) \rightleftharpoons t_{\mathrm{P}}(B)) \to (t_{\mathrm{P}}(B) \rightleftharpoons t_{\mathrm{P}}(A))$

(E3) $t_{\mathrm{P}}((A \equiv B) \wedge (B \equiv C) \to (A \equiv C))$
$= (t_{\mathrm{P}}(A) \rightleftharpoons t_{\mathrm{P}}(B)) \wedge (t_{\mathrm{P}}(B) \rightleftharpoons t_{\mathrm{P}}(C)) \to (t_{\mathrm{P}}(A) \rightleftharpoons t_{\mathrm{P}}(C))$

(C1) $t_{\mathrm{P}}((A \equiv B) \to (\neg A \equiv \neg B)) = (t_{\mathrm{P}}(A) \rightleftharpoons t_{\mathrm{P}}(B)) \to (\neg t_{\mathrm{P}}(A) \rightleftharpoons \neg t_{\mathrm{P}}(B))$

(C2) $t_{\mathrm{P}}((A \equiv B) \wedge (C \equiv D) \to ((A \wedge C) \equiv (B \wedge D)))$
$= (t_{\mathrm{P}}(A) \rightleftharpoons t_{\mathrm{P}}(B)) \wedge (t_{\mathrm{P}}(C) \rightleftharpoons t_{\mathrm{P}}(D)) \to ((t_{\mathrm{P}}(A) \wedge t_{\mathrm{P}}(C)) \rightleftharpoons (t_{\mathrm{P}}(B) \wedge t_{\mathrm{P}}(D)))$

(C3) $t_{\mathrm{P}}((A \equiv B) \wedge (C \equiv D) \to ((A \vee C) \equiv (B \vee D)))$
$= (t_{\mathrm{P}}(A) \rightleftharpoons t_{\mathrm{P}}(B)) \wedge (t_{\mathrm{P}}(C) \rightleftharpoons t_{\mathrm{P}}(D)) \to ((t_{\mathrm{P}}(A) \vee t_{\mathrm{P}}(C)) \rightleftharpoons (t_{\mathrm{P}}(B) \vee t_{\mathrm{P}}(D)))$

(C4) $t_{\mathrm{P}}((A \equiv B) \wedge (C \equiv D) \to ((A \to C) \equiv (B \to D)))$
$= (t_{\mathrm{P}}(A) \rightleftharpoons t_{\mathrm{P}}(B)) \wedge (t_{\mathrm{P}}(C) \rightleftharpoons t_{\mathrm{P}}(D)) \to ((t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(C)) \rightleftharpoons (t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(D)))$

(WIA1) $t_{\mathrm{P}}(((A \to B) \equiv (B \to A)) \to (A \equiv B))$
$= ((t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(B)) \rightleftharpoons (t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(A))) \to (t_{\mathrm{P}}(A) \rightleftharpoons t_{\mathrm{P}}(B))$

(WIA2) $t_{\mathrm{P}}(((A \to B) \equiv (B \to A)) \to ((A \to B) \equiv \top) \wedge ((B \to A) \equiv \top))$
$= ((t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(B)) \rightleftharpoons (t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(A))) \to$
$((t_{\mathrm{P}}(A) \to t_{\mathrm{P}}(B)) \rightleftharpoons \top) \wedge ((t_{\mathrm{P}}(B) \to t_{\mathrm{P}}(A)) \rightleftharpoons \top)$

Induction step: Next we must check whether each inference rule is admissible in $\mathbf{F}'$. (Mp) rule is clear so we only consider (G) rule. Assume that A_1, B_1 are provable in $\mathbf{PCI}_{\mathrm{K}}$. Then by I.H. $t_{\mathrm{P}}(A_1), t_{\mathrm{P}}(B_1)$ hold in $\mathbf{F}'$. Here we can derive the following proof in $\mathbf{F}'$:

$$\dfrac{\dfrac{\dfrac{t_{\mathrm{P}}(A_1)}{t_{\mathrm{P}}(B_1) \rightharpoonup t_{\mathrm{P}}(A_1)}\,(\mathrm{FAf}) \quad \dfrac{t_{\mathrm{P}}(B_1)}{t_{\mathrm{P}}(A_1) \rightharpoonup t_{\mathrm{P}}(B_1)}\,(\mathrm{FAf})}{t_{\mathrm{P}}(A_1) \rightleftharpoons t_{\mathrm{P}}(B_1)}\,(\mathrm{A5}, \mathrm{Mp})}{t_{\mathrm{P}}(A_1 \equiv B_1)}\,(\mathit{Def}.5.2.4\ \text{(vii)})$$

Hence, we will get $t_{\mathrm{P}}(A_1 \equiv B_1) \in \mathbf{F}'$. Thus the t_{P}-translation of any formula provable in $\mathbf{PCI}_{\mathrm{K}}$ is also provable in $\mathbf{F}'$.

■

Moreover, we can show the following.

Theorem 5.2.7 (i) *For any formula* α *in* $\mathrm{L}_{\mathrm{F}'}$, $t_{\mathrm{P}}(t_{\mathrm{F}}(\alpha)) \leftrightarrow \alpha \in \mathbf{F}'$.

(ii) *For any formula* A *in* L_{S}, $t_{\mathrm{F}}(t_{\mathrm{P}}(A)) \leftrightarrow A \in \mathbf{PCI}_{\mathrm{K}}$.

Proof. The proof is carried out in the same manner as Theorem 4.3.5. Both case can be proved by induction on the length of formulas. Moreover, it is clear that TF-connectives hold, so we only foucs to $\rightharpoonup$ and $\equiv$ connectives.
(i): Assume that $\alpha = \alpha_1 \rightharpoonup \beta_1$. Then in $\mathbf{F}'$ we have

$t_{\mathrm{P}}(t_{\mathrm{F}}(\alpha_1 \rightharpoonup \beta_1)) \leftrightarrow t_{\mathrm{P}}(t_{\mathrm{F}}(\alpha_1) \equiv t_{\mathrm{F}}(\alpha_1) \wedge t_{\mathrm{F}}(\beta_1))$ (Def.5.2.3(vi))
$\leftrightarrow (t_{\mathrm{P}}(t_{\mathrm{F}}(\alpha_1)) \rightleftharpoons (t_{\mathrm{P}}(t_{\mathrm{F}}(\alpha_1)) \wedge t_{\mathrm{P}}(t_{\mathrm{F}}(\beta_1))))$ (Def.5.2.4(vii))
$\leftrightarrow (\alpha_1 \rightleftharpoons (\alpha_1 \wedge \beta_1))$ (I.H)
$\leftrightarrow (\alpha_1 \rightharpoonup (\alpha_1 \wedge \beta_1)) \wedge ((\alpha_1 \wedge \beta_1) \rightharpoonup \alpha_1)$ (Def.)

$\leftrightarrow (\alpha_1 \rightharpoonup (\alpha_1 \wedge \beta_1))$ (Th.5.2.2(vii))

$\leftrightarrow (\alpha_1 \rightharpoonup \beta_1)$ (a3,a2)

(ii): Assume that $A = A_1 \equiv B_1$. Then in $\mathbf{PCI}_{\mathrm{K}}$ we have

$$
\begin{aligned}
t_{\mathrm{F}}(t_{\mathrm{P}}(A_1 \equiv B_1)) &\leftrightarrow t_{\mathrm{F}}(t_{\mathrm{P}}(A_1) \rightleftharpoons t_{\mathrm{P}}(B_1)) && \text{(Def.5.2.4(vii))}\\
&\leftrightarrow t_{\mathrm{F}}((t_{\mathrm{P}}(A_1) \rightharpoonup t_{\mathrm{P}}(B_1)) \wedge (t_{\mathrm{P}}(B_1) \rightharpoonup t_{\mathrm{P}}(A_1))) && \text{(Def.)}\\
&\leftrightarrow (t_{\mathrm{F}}(t_{\mathrm{P}}(A_1)) \equiv t_{\mathrm{F}}(t_{\mathrm{P}}(A_1)) \wedge t_{\mathrm{F}}(t_{\mathrm{P}}(B_1)))\wedge \\
&\quad\ (t_{\mathrm{F}}(t_{\mathrm{P}}(B_1)) \equiv t_{\mathrm{F}}(t_{\mathrm{P}}(B_1)) \wedge t_{\mathrm{F}}(t_{\mathrm{P}}(A_1))) && \text{(Def.5.2.3(vi))}\\
&\leftrightarrow (A_1 \equiv A_1 \wedge B_1) \wedge (B_1 \equiv B_1 \wedge A_1) && \text{(I.H)}\\
&\leftrightarrow (A_1 \equiv A_1 \wedge B_1) \wedge (A_1 \wedge B_1 \equiv B_1) && \text{(Th.4.2.2(v))}\\
&\leftrightarrow (A_1 \equiv B_1) && \text{(Th.4.2.2(xvii))}
\end{aligned}
$$

■

Theorem 5.2.8 (i) *For any formula α in $\mathrm{L}_{\mathrm{F}'}$, $\alpha \in \mathbf{F}'$ if and only if $t_{\mathrm{F}}(\alpha) \in \mathbf{PCI}_{\mathrm{K}}$.*

(ii) *For any formula A in L_{S}, $A \in \mathbf{PCI}_{\mathrm{K}}$ if and only if $t_{\mathrm{P}}(A) \in \mathbf{F}'$.*

Proof. (i): The only-if-part obtains from Proposition 5.2.5. Also other direction can easily be proved as follows:

$$
\begin{aligned}
t_{\mathrm{F}}(\alpha) \in \mathbf{PCI}_{\mathrm{K}} &\implies t_{\mathrm{P}}(t_{\mathrm{F}}(\alpha)) \in \mathbf{F}' && \text{(Prop.5.2.6)}\\
&\implies \alpha \in \mathbf{F}' && \text{(Th.5.2.7(i))}
\end{aligned}
$$

(ii): The only-if-part obtains from Proposition 5.2.6. Also if-part is as follows:

$$
\begin{aligned}
t_{\mathrm{P}}(A) \in \mathbf{F}' &\implies t_{\mathrm{F}}(t_{\mathrm{P}}(A)) \in \mathbf{PCI}_{\mathrm{K}} && \text{(Prop.5.2.5)}\\
&\implies A \in \mathbf{PCI}_{\mathrm{K}} && \text{(Th.5.2.7(ii))}
\end{aligned}
$$

■

Hence we can conclude that two logics $\mathbf{F}'$ and $\mathbf{PCI}_{\mathrm{K}}$ are syntactically equivalent by Definition 3.4.1, Theorem 5.2.7 and Theorem 5.2.8. Furthermore, from this result and previous Theorem 5.2.2 (viii), we get finally the following corollary.

Corollary 5.2.9 *For any formula α in L_{F}, $\alpha \in \mathbf{F}$ if and only if $t_{\mathrm{F}}(\alpha) \in \mathbf{PCI}_{\mathrm{K}}$.*

Proof. It is clear from Theorem 5.2.2 (viii) and above Theorem 5.2.8 (i).

■

5.3 Classical substructural logics

In this section we will review Girard's classical linear logic **GL** as one category of substructural logic. In general, the linear logic was proposed by J.-Y. Girard, as one of the basic logical systems which would provide a logical framework for investigating the *resource problem* occurring in computer science and its related fields. Here, the **GL** system can be also seen a classical substructural logic lacking the weakening and contraction rules in a classical formulation of Gentzen system (see [31] and [48]). At first we will briefly survey Girard's classical linear logic **GL** and its axiomatic extensions in syntactical point of view. Next we will also briefly explain algebraic semantices of **GL**.

5.3.1 Girard's classical linear logic GL and its axiomatic extensions

Let $\mathcal{L}_{\mathrm{GL}} = \langle \mathrm{L}_{\mathrm{GL}}, \wedge, \vee, *, +, \supset, \perp, 0 \rangle$ be classical Girard's linear language containing of an infinite denumerable set VAR of propositional variables, constants; $\perp$ (false) and 0 (contradict), additive connectives; $\wedge$ (conjunction) and $\vee$ (disjunction), multiplicative connectives; $*$ (conjunction) and $+$ (disjunction), and $\supset$ (linear implication). Formulas L_{GL} of a given **GL**-language $\mathcal{L}_{\mathrm{GL}}$ are defined in the usual way. The propositional constants $\top$ (truth) and 1 (provable), $\sim$ (linear negation) and $\supset\subset$ (linear equivalence) are to be constructed as the abbreviation: $\sim \alpha := \alpha \supset 0$, $\top := \sim \perp := \perp \supset 0$, $1 := \sim 0 := 0 \supset 0$ and $\alpha \supset\subset \beta := (\alpha \supset \beta) \wedge (\beta \supset \alpha)$. Also we will sometime omit parentheses when no confusion will occur, following the assumption that the priority of each connective is weak as $\sim, \wedge, *, \vee, +, \supset, \supset\subset$ in order.

The logical axioms and rules of inference for **GL**-language $\mathcal{L}_{\mathrm{GL}}$ consist of sets of schemata from (a1) to (a18) and, modus ponens (LMp) and adjunction (LAd) as rules of inference below (see e.g.,[72], [4] and [51]):

(a1) $\alpha \supset \alpha$ (Identity)

(a2) $(\alpha \supset \beta) \supset ((\beta \supset \gamma) \supset (\alpha \supset \gamma))$ (Transitivity)

(a3) $(\alpha \supset (\beta \supset \gamma)) \supset (\beta \supset (\alpha \supset \gamma))$ (Exchange)

(a4) $\alpha \wedge \beta \supset \alpha$

(a5) $\alpha \wedge \beta \supset \beta$

(a6) $(\gamma \supset \alpha) \wedge (\gamma \supset \beta) \supset (\gamma \supset \alpha \wedge \beta)$

(a7) $\alpha \supset \alpha \vee \beta$

(a8) $\beta \supset \alpha \vee \beta$

(a9) $(\alpha \supset \gamma) \wedge (\beta \supset \gamma) \supset (\alpha \vee \beta \supset \gamma)$

(a10) $\alpha \supset (\beta \supset \alpha * \beta)$ (Residuation 1)

(a11) $(\alpha \supset (\beta \supset \gamma)) \supset (\alpha * \beta \supset \gamma)$ (Residuation 2)

(a12) $(\alpha + \beta) \supset (\sim \alpha \supset \beta)$

(a13) $(\sim \alpha \supset \beta) \supset (\alpha + \beta)$

(a14) 1

(a15) $1 \supset (\alpha \supset \alpha)$

(a16) $\alpha \supset \top$

(a17) $\bot \supset \alpha$

(a18) $\sim\sim \alpha \supset \alpha$ (Double negation)

(LMp) $\dfrac{\alpha \quad \alpha \supset \beta}{\beta}$

(LAd) $\dfrac{\alpha \quad \beta}{\alpha \wedge \beta}$

Then the axiomatic deductive system $GL(\Gamma)$ for $\mathbf{GL} = (\mathcal{L}_{\mathrm{GL}}, GL)$ is defined as the following way.

Definition 5.3.1 (i) *For any* $\Gamma \subseteq \mathrm{L}_{\mathrm{GL}}$, $GL(\Gamma)$ *is the smallest set of formulas closed under rules of* (LMp) *and* (LAd), *which contains from* (a1) *to* (a18) *and* Γ.

(ii) *The element of* $GL(\emptyset)$ *is called the logical theorem of* **GL**.

Then it is easily verified that GL is a consequence operator. The elementary extension of **GL** with an additional axiom α will be denoted by $\mathbf{GL} \oplus \alpha$. Then the following extensions of **GL** are discussed in [72] and [51]:

(1) $\mathbf{GL}_{\mathrm{c}} = \mathbf{GL} \oplus ((\alpha \supset (\alpha \supset \beta)) \supset (\alpha \supset \beta))$ (Contraction)

(2) $\mathbf{GL}_{\mathrm{w}} = \mathbf{GL} \oplus (\alpha \supset (\beta \supset \alpha))$ (Weaking)

(3) $\mathbf{GL}_{\mathrm{cw}} = \mathbf{GL} \oplus ((\alpha \supset (\alpha \supset \beta)) \supset (\alpha \supset \beta)) \oplus (\alpha \supset (\beta \supset \alpha))$

Next we will introduce an auxiliary system $\mathbf{GL'}$ for investigations in the next section. To restore the balance between both **GL** and **PCI** languages, we need first to extend the **GL**-language by adding a material implication $\rightarrow$. Let this auxiliary language be $\mathcal{L}_{\mathrm{GL'}} = \langle \mathrm{L}_{\mathrm{GL'}}, \wedge, \vee, *, +, \rightarrow, \supset, \bot, 0 \rangle$. Then $\neg$ (classical negation) is the abbreviation of $\neg\alpha := \alpha \rightarrow \bot$. The logical axioms and rules of inference for this language $\mathcal{L}_{\mathrm{GL'}}$ are obtained from a set of schemata from (a1) to (a18) of **GL** and two rules of inference, modus ponens (LMp) and adjunction (LAd), by adding the additional axiom schemata TFA, which are same as from (A1) to (A10) in **SCI** for TF-connectives $(\wedge, \vee, \rightarrow, \bot)$, (LW1), (LW2) and the modus ponens (Mp) rule for $\rightarrow$ below:

(LW1) $((\alpha \rightarrow \beta) \supset\subset (\beta \rightarrow \alpha)) \rightarrow (\alpha \supset\subset \beta)$, (LW2) $((\alpha \to \beta) \supset\subset (\beta \to \alpha)) \to ((\alpha \to \beta) \supset\subset \top) \wedge ((\beta \to \alpha) \supset\subset \top)$,

(Mp) $\dfrac{\alpha \quad \alpha \to \beta}{\beta}$.

Then the axiomatic deductive system $\mathbf{GL}'(\Gamma)$ for $\mathbf{GL}' = (\mathcal{L}_{\mathrm{GL}'}, GL')$ is defined as follows.

Definition 5.3.2 (i) *For any* $\Gamma \subseteq \mathrm{L}_{\mathrm{GL}'}$, $GL'(\Gamma)$ *is the smallest set of formulas closed under the rules of* (LMp), (LAd) *and* (Mp), *which contains from* (a1) *to* (a18), *and from* (A1) *to* (A10), (LW1), (LW2) *and* Γ.

(ii) *The element of* $GL'(\emptyset)$ *is called the logical theorem of* $\mathbf{GL}'$.

Here we can also define auxiliary extensions $\mathbf{GL}'_{\mathrm{c}}$, $\mathbf{GL}'_{\mathrm{w}}$ and $\mathbf{GL}'_{\mathrm{cw}}$ of $\mathbf{GL}'$ as the same way to **GL**. Then we have the following theorem.

Theorem 5.3.3 *The following are derived rules and logical theorem of* $\mathbf{GL}'$.

(i) $\dfrac{\alpha \supset \beta}{\alpha \to \beta}$ (LIm)

(ii) $\dfrac{\alpha \quad \alpha \supset \beta}{\beta}$ (LMp)

(iii) $\dfrac{\alpha \quad \beta}{\alpha \supset\subset \beta}$ (LG)

(iv) $\dfrac{\alpha \leftrightarrow \beta}{\alpha \supset\subset \beta}$ (LQF)

(v) $\dfrac{\alpha \supset\subset \beta}{\alpha \to \beta}$

(vi) $\dfrac{\alpha \supset (\beta \supset \gamma)}{\alpha \to (\beta \to \gamma)}$

(vii) $((\alpha \supset \alpha \wedge \beta) \wedge (\alpha \wedge \beta \supset \alpha)) \leftrightarrow (\alpha \supset \beta)$

Proof. (i): as the same insight in the case of $\mathbf{F}'$, this is clear since every axiom and inference rule of **GL** is classically valid. (ii), (v) and (vi) are straightforward by (LIm) and (Mp). (iii): if α is a theorem of $\mathbf{GL}'$, then so is $\beta \supset \alpha$ by (a16). Similarly, if β is a theorem of $\mathbf{GL}'$, then so is $\alpha \supset \beta$. Hence we have $\alpha \supset\subset \beta$ by (LAd). (iv): suppose $\alpha \leftrightarrow \beta$. Then we get $(\alpha \to \beta) \supset\subset (\beta \to \alpha)$ by (A3), (A4) and (LG). So, we get the desired result by (LW1) and (Mp).

(vii) : 1 Put $A = (\alpha \supset \alpha \wedge \beta) \wedge (\alpha \wedge \beta \supset \alpha)$

2 $A \to (\alpha \supset \alpha \wedge \beta)$ (A3)

3 $\alpha \wedge \beta \supset \beta$ (a5)

4 $(\alpha \supset \alpha \wedge \beta) \supset ((\alpha \wedge \beta \supset \beta) \supset (\alpha \supset \beta))$ (a2)

5 $(\alpha \wedge \beta \supset \beta) \supset ((\alpha \supset \alpha \wedge \beta) \supset (\alpha \supset \beta))$ (4,a3)

6 $(\alpha \supset \alpha \wedge \beta) \supset (\alpha \supset \beta)$ (3,5,LMp)
7 $(\alpha \supset \alpha \wedge \beta) \to (\alpha \supset \beta)$ (6,LIm)
8 $A \to (\alpha \supset \beta)$ (2,7,transitivity of $\to$)
9 $\alpha \supset \alpha$ (a1)
10 $(\alpha \supset \beta) \to (\alpha \supset \alpha)$ (9,A1,Mp)
11 $(\alpha \supset \beta) \to (\alpha \supset \beta)$
12 $(\alpha \supset \beta) \to (\alpha \supset \alpha) \wedge (\alpha \supset \beta)$ (10,11,A5)
13 $(\alpha \supset \alpha) \wedge (\alpha \supset \beta) \to (\alpha \supset \alpha \wedge \beta)$ (a6,LIm)
14 $(\alpha \supset \beta) \to (\alpha \supset \alpha \wedge \beta)$ (12,13,trans.of $\to$)
15 $\alpha \wedge \beta \supset \alpha$ (a4)
16 $(\alpha \supset \beta) \to (\alpha \wedge \beta \supset \alpha)$ (15,A1,Mp)
17 $(\alpha \supset \beta) \to (\alpha \supset \alpha \wedge \beta) \wedge (\alpha \wedge \beta \supset \alpha)$ (14,16,A5)

■

5.3.2 Algebraic semantics of GL

In this subsection we will briefly explain algebraic semantics of **GL**. Here we mainly refer to [72], [4] and [51].

Definition 5.3.4 (i) *$\mathcal{A}_{\mathrm{GL}} = \langle A, \wedge, \vee, *, \supset, \bot, 0, 1\rangle$ is called an* **GL**-algebra *if $\mathcal{A}_{\mathrm{GL}}$ satisfies the following conditions: for every $x, y, z \in A$,*

(1) *$\langle A, \wedge, \vee, \bot\rangle$ is a lattice with bottom $\bot$,*

(2) *$\langle A, *, 1\rangle$ is a commutative monoid with unit 1,*

(3) *$z * (x \vee y) = (z * x) \vee (z * y)$,*

(4) *$x * y \le z$ if and only if $x \le y \supset z$,*

(5) *$x = \sim\sim x$, where $\sim x := x \supset 0$.*

(ii) *Moreover, $\mathcal{A}_{\mathrm{GLc}}$ is called an* **GL**$_{\mathrm{c}}$-algebra *if in addition to* (1)-(5)*, $\mathcal{A}_{\mathrm{GLc}}$ also satisfies*

(6) *$x \le x * x$.*

(iii) *And $\mathcal{A}_{\mathrm{GLw}}$ is called an* **GL**$_{\mathrm{w}}$-algebra *if in addition to* (1)-(5)*, $\mathcal{A}_{\mathrm{GLw}}$ also satisfies*

(7) *$0 = \bot$,*

(8) *$x * y \le x$.*

Here we used the same symbols for both algebraic operations and logical connectives in **GL** for the sake of simplicity. The next lemma is a straightforward by the definition.

Lemma 5.3.5 *For any* **GL***-algebra $\mathcal{A}_{\mathrm{GL}} = \langle A, \wedge, \vee, *, \supset, \bot, 0, 1\rangle$, we have the following equations:*

(i) $x \supset (y \supset z) = x * y \supset z$,

(ii) $x \vee y = \sim (\sim x \wedge \sim y)$,

(iii) $x \supset y = \sim (x * \sim y)$.

A subset F of A is called a **GL**-*filter* of $\mathcal{A}_{\mathrm{GL}}$ if F satisfies the following conditions:

(F1) $1 \in F$,

(F2) $a \in F$ and $a \supset b$ implies $b \in F$,

(F3) $a, b \in F$ implies $a * b \in F$.

For any **GL**-algebras $\mathcal{A}_{\mathrm{GL}}$ and any **GL**-filter F, $\mathfrak{M} = (\mathcal{A}_{\mathrm{GL}}, F)$ is called a **GL**-*model*. For any **GL**-algebras $\mathcal{A}_{\mathrm{GL}}$, a formula α is *valid* in $\mathcal{A}_{\mathrm{GL}}$, in symbols, $\mathcal{A}_{\mathrm{GL}} \models \alpha$, if $h(\alpha) \in F$ for any valuation h of $\mathcal{A}_{\mathrm{GL}}$ and any **GL**-filter F. Moreover, for any valuation h of $\mathcal{A}_{\mathrm{GL}}$, we can define the consequence operator $C_{\mathfrak{M}}$ relative to a **GL**-model $\mathfrak{M}$ as follows.

Definition 5.3.6 *For any $\Gamma \cup \{\alpha\} \subseteq \mathrm{L}_{\mathrm{GL}}$, $\alpha \in C_{\mathfrak{M}}(\Gamma)$ if and only if for every* **GL**-*model $\mathfrak{M} = (\mathcal{A}_{\mathrm{GL}}, F)$ and every valuation h of $\mathcal{L}_{\mathrm{GL}}$ in $\mathfrak{M}$, $h(\alpha) \in F$ whenever $h(\Gamma) \subseteq F$.*

Then the following strong completeness of **GL** can be shown by the results of [72], [4] and [51].

Theorem 5.3.7 **GL** *is strongly complete with respect to a* **GL**-*model, i.e., $GL = C_{\mathfrak{M}}$.*

Moreover, for an auxiliary system **GL**$'$ mentioned in the previous subsection, we get the following Proposition.

Proposition 5.3.8 *For any formula α in $\mathrm{L}_{\mathrm{GL}'}$ such that α not contains $\rightarrow$ connective at all, $\alpha \in$* **GL**$'$ *if and only if $\alpha \in$* **GL**.

Proof. The if-part is trivial since **GL**$'$ is an extension of **GL** by the above definition. To prove the converse direction we will consider the algebraic model for **GL**$'$. Given an algebraic model $\mathcal{M} = (\mathcal{A}_{\mathrm{GL}}, F)$ for **GL**, we can get the algebraic model $\mathcal{M}_{\mathrm{GL}'} = (\mathcal{A}_{\mathrm{GL}'}, F')$ for **GL**$'$ by adding the following definitions:

(A1) $\mathcal{A}_{\mathrm{GL}'} = \langle A, \wedge, \vee, *, \rightarrow, \supset, \perp, 0, 1 \rangle$ is called an **GL**$'$-*algebra* if $\mathcal{A}_{\mathrm{GL}'}$ is a **GL**-algebra, and also satisfies

(9) $x \wedge y \leq' z$ if and only if $x \leq' y \rightarrow z$,

(10) $x = \neg\neg x$, where $\neg x := x \rightarrow \perp$.

(A2) A subset F' of A is called a **GL**$'$-*filter* of $\mathcal{A}_{\mathrm{GL}'}$ if F' is a **GL**-filter, and also satisfies

(F4) $\top \in F'$, where $\top := \perp \rightarrow \perp$, (F5) $a \in F'$ and $a \to b$ implies $b \in F'$,

(F6) $a, b \in F'$ implies $a \wedge b \in F'$.

Then we can easily prove the soundness of $\mathbf{GL}'$ with respect to above algebraic model, that is for any formula α in $\mathrm{L}_{\mathrm{GL}'}$, $\alpha \in \mathbf{GL}'$ implies $\mathcal{M}_{\mathrm{GL}'} \models \alpha$ for any $\mathbf{GL}'$-model $\mathcal{M}_{\mathrm{GL}'} = (\mathcal{A}_{\mathrm{GL}'}, F')$. Hence if we assume $\alpha \notin \mathbf{GL}$ for some formula α in L_{GL} then by the completeness result for **GL** there exists a valuation h in the model $\mathcal{M} = (\mathcal{A}_{\mathrm{GL}}, F)$ for **GL** such that $h(\alpha) \notin F$. Then by the above definition of $\mathbf{GL}'$-model, this valuation falsifies α in the model $\mathcal{M}_{\mathrm{GL}'}$ for $\mathbf{GL}'$, i.e., $h(\alpha) \notin F'$, so we get $\mathcal{M}_{\mathrm{GL}'} \not\models \alpha$. Then by the soundness of $\mathbf{GL}'$, we have $\alpha \notin \mathbf{GL}'$.

■

5.4 $\mathbf{PCI}_{\mathrm{GL}}$ logic with identity as linear implication

In this section we will define $\mathbf{PCI}_{\mathrm{GL}}$ logic as an extension of **PCI** in order to interpret the classical linear implication $\supset$ by identity $\equiv$. Then we need the following conditions in **PCI**:

(R7) $\overrightarrow{\alpha \supset \beta} \longmapsto \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta}$,

(R8) $\overleftarrow{A \equiv B} \longmapsto (\overleftarrow{A} \supset\subset \overleftarrow{B})$,

where $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$, $\overleftarrow{A}$ and $\overleftarrow{B}$ denote the results of translations from **GL** to $\mathbf{PCI}_{\mathrm{GL}}$, and its converse, respectively.

In general, Girard's classical linear logic **GL** can be seen as a classical logic without weakening and contraction rules. Moreover, we notice that Corsi's weak logic **F**, which is a sublogic of intuitionistic logic, can translate into $\mathbf{PCI}_{\mathrm{K}}$ logic as shown in Section 5.2. Hence we will introduce Girard's classical linear logic on **PCI** by adding several multiplicative connective axioms and double negation axiom to $\mathbf{PCI}_{\mathrm{K}}$. So to satisfy the requirement (R7), we need to add the following identity axioms (LT), (LE), (L*1), (L*2) and (LDN), which correspond to axioms (a2), (a3), (a10), (a11) and (a18) in **GL**, respectively, under the system $\mathbf{PCI}_{\mathrm{K}}$ which is also defined by adding identity axioms (WIA1) and (WIA2) , and (G) rule to the original **PCI** logic.

(WIA1) $((A \to B) \equiv (B \to A)) \to (A \equiv B)$

(WIA2) $((A \to B) \equiv (B \to A)) \to ((A \to B) \equiv \top) \wedge ((B \to A) \equiv \top)$

(LT) $(A > B) > ((B > C) > (A > C))$

(LE) $(A > (B > C)) \to (B > (A > C))$

(L*1) $A > (B > A \circ B)$

(L*2) $(A > (B > C)) \to (A \circ B > C)$

(LDN) $\smile\smile A \to A$

(G) $\dfrac{A \quad B}{A \equiv B}$

Here each connectives $>$, $\smile$ and $\circ$ are abbreviations in $\mathbf{PCI}_{\mathrm{GL}}$ as : $A > B := (A \equiv A \wedge B)$, $\smile A := A > \neg(A \equiv A)$ and $A \circ B :=\smile (A >\smile B)$.

Definition 5.4.1 *Let* $\mathbf{PCI} = (\mathcal{L}_{\mathrm{S}}, C)$ *be* **PCI** *logic,* C^{G} *a G-theory of* C *and* $X \subseteq \mathrm{L}_{\mathrm{S}}$*. Then* $\mathbf{PCI}_{\mathrm{GL}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{GL}})$ *is a non-elementary extension of* **PCI***, where* $C^{\mathrm{G}}_{\mathrm{GL}}$ *is a super-consequence of* C *defined by* $C^{\mathrm{G}}_{\mathrm{GL}}(X) = C^{\mathrm{G}}(X; \mathrm{WIA1}, \mathrm{WIA2}, \mathrm{LT}, \mathrm{LE}, \mathrm{L}^*1, \mathrm{L}^*2, \mathrm{LDN})$.

Theorem 5.4.2 *The following are derived rules and logical theorems of* $\mathbf{PCI}_{\mathrm{GL}}$*.*

(i) $\dfrac{A \quad A \equiv B}{B}$ (E)

(ii) $\dfrac{A \quad A > B}{B}$ (LMp′)

(iii) $\dfrac{(A \to B) \equiv (B \to A)}{A \equiv B}$ (I)

(iv) $\dfrac{A \leftrightarrow B}{A \equiv B}$ (QF)

(v) $A \equiv A$

(vi) $(A \equiv B) \equiv (B \equiv A)$

(vii) $((A \to B) \equiv (B \to A)) \equiv (A \equiv B)$

(viii) $A \equiv B \to \neg A \equiv \neg B$

(ix) $A \equiv B \leftrightarrow (A \equiv A \wedge B) \wedge (B \equiv B \wedge A)$

Proof. Since $\mathbf{PCI}_{\mathrm{GL}}$ is an extension of $\mathbf{PCI}_{\mathrm{K}}$, everything is an obvious by Theorem 4.2.2, except for (ii) below.

$$\dfrac{\dfrac{A \qquad \dfrac{\dfrac{A \equiv A \wedge B}{A \leftrightarrow A \wedge B}\,(\mathit{Prop}.3.1.3\ (\mathrm{iii})) \qquad \begin{matrix}(A3)\\ \vdots \\ (A \leftrightarrow A \wedge B) \to (A \to A \wedge B)\end{matrix}}{A \to A \wedge B}\,(\mathrm{Mp})}{A \wedge B}\,(\mathrm{Mp}) \qquad \begin{matrix}(A4)\\ \vdots \\ A \wedge B \to B\end{matrix}}{B}\,(\mathrm{Mp}).$$

■

Next we will introduce elementary extensions of $\mathbf{PCI}_{\mathrm{GL}}$ which are correspond to extensions of classical Girard's linear logic **GL**. So let us first consider the following additional axiom schemata.

(LC) $(A > (A > B)) \to (A > B)$

(LW) $A > (B > A)$

Then we have the following extensions $\mathbf{PCI}_{\mathrm{GLc}}$, $\mathbf{PCI}_{\mathrm{GLw}}$ and $\mathbf{PCI}_{\mathrm{GLcw}}$ of $\mathbf{PCI}_{\mathrm{GL}}$, which can be defined below.

Definition 5.4.3 *Let* $\mathbf{PCI}_{\mathrm{GL}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{GL}})$ *and* $X \subseteq \mathrm{L}_{\mathrm{S}}$. *Then the elementary extensions of* $\mathbf{PCI}_{\mathrm{GL}}$ *are defined as follows:*

(i) $\mathbf{PCI}_{\mathrm{GLc}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{GLc}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{GL}}$, *where* $C^{\mathrm{G}}_{\mathrm{GLc}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{GL}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{GLc}}(X) = C^{\mathrm{G}}_{\mathrm{GL}}(X; \mathrm{LC})$.

(ii) $\mathbf{PCI}_{\mathrm{GLw}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{GLw}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{GL}}$, *where* $C^{\mathrm{G}}_{\mathrm{GLw}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{GL}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{GLw}}(X) = C^{\mathrm{G}}_{\mathrm{GL}}(X; \mathrm{LW})$.

(iii) $\mathbf{PCI}_{\mathrm{GLcw}} = (\mathcal{L}_{\mathrm{S}}, C^{\mathrm{G}}_{\mathrm{GLcw}})$ *is the elementary extension of* $\mathbf{PCI}_{\mathrm{GL}}$, *where* $C^{\mathrm{G}}_{\mathrm{GLcw}}$ *is a superconsequence of* $C^{\mathrm{G}}_{\mathrm{GL}}$ *defined by* $C^{\mathrm{G}}_{\mathrm{GLcw}}(X) = C^{\mathrm{G}}_{\mathrm{GL}}(X; \mathrm{LC}, \mathrm{LW})$.

Theorem 5.4.4 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{GLc}}$.

(i) $A > A \circ A$

(ii) $A \wedge B > A \circ B$

Proof.

(i) : 1 $A > (A > A \circ A)$ (L*1)
2 $A > (A > A \circ A) \to (A > A \circ A)$ (LC)
3 $A > A \circ A$ (1,2,Mp)

(ii) : 1 $A > (B > A \circ B)$ (L*1)
2 $A \wedge B > A$ $(A \wedge B \equiv A \wedge B \wedge A)$
3 $(A \wedge B > A) > ((A > (B > A \circ B)) > (A \wedge B > (B > A \circ B)))$ (LT)
4 $(A > (B > A \circ B)) > (A \wedge B > (B > A \circ B))$ (2,3,LMp)
5 $A \wedge B > (B > A \circ B)$ (1,4,LMp)
6 $A \wedge B > (B > A \circ B) \to B > (A \wedge B > A \circ B)$ (LE)
7 $B > (A \wedge B > A \circ B)$ (5,6,Mp)
8 $A \wedge B > B$ $(A \wedge B \equiv A \wedge B \wedge B)$
9 $(A \wedge B > B) > ((B > (A \wedge B > A \circ B)) > (A \wedge B > (A \wedge B > A \circ B)))$ (LT)
10 $(B > (A \wedge B > A \circ B)) > (A \wedge B > (A \wedge B > A \circ B))$ (8,9,LMp)
11 $A \wedge B > (A \wedge B > A \circ B)$ (7,10,LMp)
12 $(A \wedge B > (A \wedge B > A \circ B)) \to (A \wedge B > A \circ B)$ (LC)
13 $A \wedge B > A \circ B$ (11,12,Mp)

■

Theorem 5.4.5 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{GLw}}$.

(i) $A \circ B > A, \quad A \circ B > B$

(ii) $A \circ B > A \wedge B$

(iii) $\top \equiv 1, \quad \bot \equiv 0$

Proof.

(i) : 1 $A > (B > A)$ (LW)
2 $(A > (B > A)) \to (A \circ B > A)$ (L*2)
3 $A \circ B > A$ (1,2,Mp)
4 $B > (A > B)$ (LW)
5 $(B > (A > B)) \to (A > (B > B))$ (LE)
6 $A > (B > B)$ (4,5,Mp)
7 $A > (B > B) \to (A \circ B > B)$ (L*2)
8 $A \circ B > B$ (6,7,Mp)

(ii) : 1 Put $\alpha = ((A \circ B) \equiv (A \circ B) \wedge A) \wedge ((A \circ B) \equiv (A \circ B) \wedge B)$
2 $\alpha \to (A \circ B) \wedge (A \circ B) \equiv (A \circ B) \wedge (A \circ B) \wedge A \wedge B$ (C4)
3 $(A \circ B) \equiv (A \circ B) \wedge (A \circ B)$ $(A \equiv A \wedge A)$
4 $\alpha \to (A \circ B) \equiv (A \circ B) \wedge (A \circ B)$ (4,A1)
5 $\alpha \to ((A \circ B) \equiv (A \circ B) \wedge (A \circ B)) \wedge$
$((A \circ B) \wedge (A \circ B) \equiv (A \circ B) \wedge (A \circ B) \wedge A \wedge B)$ (2,4,Mp)
6 $\alpha \to (A \circ B) \equiv (A \circ B) \wedge (A \circ B) \wedge A \wedge B$ (E3,Mp)
7 $\alpha \to (A \circ B) \equiv (A \circ B) \wedge (A \wedge B)$ (same way)
8 $(A \circ B > A) \wedge (A \circ B > B) \to (A \circ B > A \wedge B)$ (7,def. of $>$)
9 $A \circ B > A$ (i)
10 $A \circ B > B$ (i)
11 $A \circ B > A \wedge B$ (9,10,8,Mp)

(iii) : It is clear that $\top > 1$. So we will show the converse.
1 $A \circ \top > A \wedge \top$ (ii)
2 $A \wedge \top > A$ ($\top$ is unit of $\wedge$)
3 $(A \circ \top > A \wedge \top) > ((A \wedge \top > A) > (A \circ \top > A))$ (LT)
4 $A \circ \top > A$ (1,2,3,LMp)
Hence we get $\top > 1$.

■

Theorem 5.4.6 *The following are logical theorems of* $\mathbf{PCI}_{\mathrm{GLcw}}$.

(i) $A \wedge B \equiv A \circ B$

Proof. (i): It is clear from Theorem 5.4.4 (ii), 5.4.5 (ii).

■

5.5 Translation of GL into $\mathbf{PCI}_{\mathrm{GL}}$

In this section we will give translations between $\mathbf{GL}'$ and $\mathbf{PCI}_{\mathrm{GL}}$, and hence prove that they are syntactically equivalent. How to show the syntactically equivalent of two logics follows the previous discipline in Section 3.4. At first we will define two translations t_{G} and t_{P} between $\mathbf{GL}'$-language $\mathcal{L}_{\mathrm{GL}'}$ and $\mathbf{PCI}$-language $\mathcal{L}_{\mathrm{P}}$ in order to show two logics $\mathbf{GL}'$ and $\mathbf{PCI}_{\mathrm{GL}}$ are syntactically equivalent with respect to these maps.

Definition 5.5.1 *The mapping* $t_{\mathrm{G}} : \mathrm{L}_{\mathrm{GL}'} \to \mathrm{L}_{\mathrm{S}}$*, called a* G-translation*, is defined inductively as follows:*

(i) $t_{\mathrm{G}}(p) := p, \quad p \in \mathrm{VAR},$

(ii) $t_{\mathrm{G}}(\bot) := \bot,$

(iii) $t_{\mathrm{G}}(0) := \neg(t_{\mathrm{G}}(\alpha) \equiv t_{\mathrm{G}}(\alpha))$*, for any* $\alpha \in \mathrm{L}_{\mathrm{GL}'}$,

(iv) $t_{\mathrm{G}}(\alpha \wedge \beta) := (t_{\mathrm{G}}(\alpha) \wedge t_{\mathrm{G}}(\beta)),$

(v) $t_{\mathrm{G}}(\alpha \vee \beta) := (t_{\mathrm{G}}(\alpha) \vee t_{\mathrm{G}}(\beta)),$

(vi) $t_{\mathrm{G}}(\alpha \to \beta) := (t_{\mathrm{G}}(\alpha) \to t_{\mathrm{G}}(\beta)),$

(vii) $t_{\mathrm{G}}(\alpha \supset \beta) := (t_{\mathrm{G}}(\alpha) \equiv t_{\mathrm{G}}(\alpha) \wedge t_{\mathrm{G}}(\beta)),$

(viii) $t_{\mathrm{G}}(\sim \alpha) := t_{\mathrm{G}}(\alpha \supset 0),$

(ix) $t_{\mathrm{G}}(\alpha * \beta) := t_{\mathrm{G}}(\sim (\alpha \supset \sim \beta)),$

(x) $t_{\mathrm{G}}(\alpha + \beta) := t_{\mathrm{G}}(\sim \alpha \supset \beta).$

Definition 5.5.2 *The mapping* $t_{\mathrm{P}} : \mathrm{L}_{\mathrm{S}} \to \mathrm{L}_{\mathrm{F}'}$*, called a* PCI-translation*, is defined inductively as follows:*

(i) $t_{\mathrm{P}}(p) := p, \quad p \in \mathrm{VAR},$

(ii) $t_{\mathrm{P}}(\bot) := \bot,$

(iii) $t_{\mathrm{P}}(\neg A) := t_{\mathrm{P}}(A) \to \bot,$

(iv) $t_P(A \wedge B) := (t_P(A) \wedge t_P(B))$,

(v) $t_P(A \vee B) := (t_P(A) \vee t_P(B))$,

(vi) $t_P(A \to B) := (t_P(A) \to t_P(B))$,

(vii) $t_P(A \equiv B) := (t_P(A) \supset\subset t_P(B))$.

For two maps t_F and t_P, we can prove the following lemmas and propositions.

Lemma 5.5.3 *All axioms of* $\mathbf{GL}'$ ($\mathbf{GL}'_{\mathrm{c}}$, $\mathbf{GL}'_{\mathrm{w}}$, $\mathbf{GL}'_{\mathrm{cw}}$) *are provable in* $\mathbf{PCI}_{\mathrm{GL}}$ ($\mathbf{PCI}_{\mathrm{GLc}}$, $\mathbf{PCI}_{\mathrm{GLw}}$, $\mathbf{PCI}_{\mathrm{GLcw}}$) *respectively, after* t_G*-translation. Namely, the following formulas are theorems of* $\mathbf{PCI}_{\mathrm{GL}}$ ($\mathbf{PCI}_{\mathrm{GLc}}$, $\mathbf{PCI}_{\mathrm{GLw}}$, $\mathbf{PCI}_{\mathrm{GLcw}}$)*, where* $\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{\gamma}$ *denote the result of* t_G*-translation.*

(a1) $\overrightarrow{\alpha \supset \alpha} = \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\alpha}$

(a2) $\overrightarrow{(\alpha \supset \beta) \supset ((\beta \supset \gamma) \supset (\alpha \supset \gamma))}$
$= (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta}) \equiv (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta})$
$\wedge((\overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge \overrightarrow{\gamma}) \equiv (\overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge \overrightarrow{\gamma}) \wedge (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\gamma}))$

(a3) $\overrightarrow{(\alpha \supset (\beta \supset \gamma)) \supset (\beta \supset (\alpha \supset \gamma))}$
$= (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge (\overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge \overrightarrow{\gamma})) \equiv (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge (\overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge \overrightarrow{\gamma}))$
$\wedge(\overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\gamma}))$

(a4) $\overrightarrow{\alpha \wedge \beta \supset \alpha} = \overrightarrow{\alpha} \wedge \overrightarrow{\beta} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta} \wedge \overrightarrow{\alpha}$

(a5) $\overrightarrow{\alpha \wedge \beta \supset \beta} = \overrightarrow{\alpha} \wedge \overrightarrow{\beta} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta} \wedge \overrightarrow{\beta}$

(a6) $\overrightarrow{(\gamma \supset \alpha) \wedge (\gamma \supset \beta) \supset (\gamma \supset \alpha \wedge \beta)}$
$= (\overrightarrow{\gamma} \equiv \overrightarrow{\gamma} \wedge \overrightarrow{\alpha}) \wedge (\overrightarrow{\gamma} \equiv \overrightarrow{\gamma} \wedge \overrightarrow{\beta}) \equiv (\overrightarrow{\gamma} \equiv \overrightarrow{\gamma} \wedge \overrightarrow{\alpha}) \wedge (\overrightarrow{\gamma} \equiv \overrightarrow{\gamma} \wedge \overrightarrow{\beta})$
$\wedge(\overrightarrow{\gamma} \equiv \overrightarrow{\gamma} \wedge (\overrightarrow{\alpha} \wedge \overrightarrow{\beta}))$

(a7) $\overrightarrow{\alpha \supset \alpha \vee \beta} = \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge (\overrightarrow{\alpha} \vee \overrightarrow{\beta})$

(a8) $\overrightarrow{\beta \supset \alpha \vee \beta} = \overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge (\overrightarrow{\alpha} \vee \overrightarrow{\beta})$

(a9) $\overrightarrow{(\alpha \supset \gamma) \wedge (\beta \supset \gamma) \supset (\alpha \vee \beta \supset \gamma)}$
$= (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\gamma}) \wedge (\overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge \overrightarrow{\gamma}) \equiv (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\gamma}) \wedge (\overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge \overrightarrow{\gamma})$
$\wedge(\overrightarrow{\alpha} \vee \overrightarrow{\beta} \equiv (\overrightarrow{\alpha} \vee \overrightarrow{\beta}) \wedge \overrightarrow{\gamma})$

(a10) $\overrightarrow{\alpha \supset (\beta \supset \alpha * \beta)} = \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge (\overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge (\overrightarrow{\alpha} * \overrightarrow{\beta}))$

(a11) $\overrightarrow{(\alpha \supset (\beta \supset \gamma)) \supset (\alpha * \beta \supset \gamma)}$
$= (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge (\overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge \overrightarrow{\gamma})) \equiv (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge (\overrightarrow{\beta} \equiv \overrightarrow{\beta} \wedge \overrightarrow{\gamma}))$
$\wedge((\overrightarrow{\alpha} * \overrightarrow{\beta}) \equiv (\overrightarrow{\alpha} * \overrightarrow{\beta}) \wedge \overrightarrow{\gamma})$

(a12) $\overrightarrow{(\alpha+\beta)\supset(\sim\alpha\supset\beta)}$
$= (\vec{\alpha} + \vec{\beta}) \equiv (\vec{\alpha} + \vec{\beta}) \wedge ((\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{0}) \equiv (\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{0}) \wedge \vec{\beta})$

(a13) $\overrightarrow{(\sim\alpha\supset\beta)\supset(\alpha+\beta)}$
$= ((\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{0}) \equiv (\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{0}) \wedge \vec{\beta})$
$\equiv ((\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{0}) \equiv (\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{0}) \wedge \vec{\beta}) \wedge (\vec{\alpha} + \vec{\beta})$

(a14) $\vec{1} = \vec{\alpha} \equiv \vec{\alpha}$

(a15) $\overrightarrow{1\supset(\alpha\supset\alpha)} = (\vec{\alpha} \equiv \vec{\alpha}) \equiv (\vec{\alpha} \equiv \vec{\alpha}) \wedge (\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{\alpha})$

(a16) $\overrightarrow{\alpha\supset\top} = \vec{\alpha} \equiv \vec{\alpha} \wedge \top$

(a17) $\overrightarrow{\bot\supset\alpha} = \bot \equiv \bot \wedge \vec{\alpha}$

(a18) $\overrightarrow{\sim\sim\alpha\supset\alpha}$
$= ((\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{0}) \equiv (\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{0}) \wedge \vec{0})$
$\equiv ((\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{0}) \equiv (\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{0}) \wedge \vec{0}) \wedge \vec{0}$

(LW1) $\overrightarrow{(((\alpha\to\beta)\supset\subset(\beta\to\alpha))\to(\alpha\supset\subset\beta))}$
$= ((\vec{\alpha} \to \vec{\beta}) \equiv (\vec{\beta} \to \vec{\alpha})) \to (\vec{\alpha} \equiv \vec{\beta})$

(LW2) $\overrightarrow{(((\alpha\to\beta)\supset\subset(\beta\to\alpha))\to((\alpha\to\beta)\supset\subset\top)\wedge((\beta\to\alpha)\supset\subset\top))}$
$= ((\vec{\alpha} \to \vec{\beta}) \equiv (\vec{\beta} \to \vec{\alpha})) \to ((\vec{\alpha} \to \vec{\beta}) \equiv \top) \wedge ((\vec{\beta} \to \vec{\alpha}) \equiv \top)$

(C) $\overrightarrow{(\alpha\supset(\alpha\supset\beta))\supset(\alpha\supset\beta)}$
$= (\vec{\alpha} \equiv \vec{\alpha} \wedge (\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{\beta})) \equiv (\vec{\alpha} \equiv \vec{\alpha} \wedge (\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{\beta})) \wedge (\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{\beta})$

(W) $\overrightarrow{\alpha\supset(\beta\supset\alpha)} = \vec{\alpha} \equiv \vec{\alpha} \wedge (\vec{\beta} \equiv \vec{\beta} \wedge \vec{\alpha})$

Proof. We will only show the following three cases.

(a9) : 1 $A = (\vec{\alpha} \equiv \vec{\alpha} \wedge \vec{\gamma}) \wedge (\vec{\beta} \equiv \vec{\beta} \wedge \vec{\gamma})$
and $B = \vec{\alpha} \vee \vec{\beta} \equiv (\vec{\alpha} \vee \vec{\beta}) \wedge \vec{\gamma}$ (assume)
2 $A \wedge B \to A$ (A3)
3 $A \to A$
4 $A \to \vec{\alpha} \vee \vec{\beta} \equiv (\vec{\alpha} \wedge \vec{\gamma}) \vee (\vec{\beta} \wedge \vec{\gamma})$ (C3)
5 $(\vec{\alpha} \wedge \vec{\gamma}) \vee (\vec{\beta} \wedge \vec{\gamma}) \leftrightarrow (\vec{\alpha} \vee \vec{\beta}) \wedge \vec{\gamma}$
6 $(\vec{\alpha} \wedge \vec{\gamma}) \vee (\vec{\beta} \wedge \vec{\gamma}) \equiv (\vec{\alpha} \vee \vec{\beta}) \wedge \vec{\gamma}$ (5,QF)
7 $A \to (\vec{\alpha} \wedge \vec{\gamma}) \vee (\vec{\beta} \wedge \vec{\gamma}) \equiv (\vec{\alpha} \vee \vec{\beta}) \wedge \vec{\gamma}$ (6,A1)
8 $A \to \vec{\alpha} \vee \vec{\beta} \equiv (\vec{\alpha} \vee \vec{\beta}) \wedge \vec{\gamma}$ (4,7,E3,Mp)
9 $A \to B$ (def. of B)
10 $A \to A \wedge B$ (3,9,A5,Mp)
11 $A \leftrightarrow A \wedge B$ (2,10,A5,Mp)
12 $A \equiv A \wedge B$ (11,QF)

(C) : 1 $A = \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta})$. (assume)

2 $A \wedge (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta}) \to A$ (A3)

3 $A \to A$

4 $A \to \overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta}$ (LC)

5 $A \to A \wedge (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta})$ (3,4,A5,Mp)

6 $A \leftrightarrow A \wedge (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta})$ (2,5,a5,Mp)

7 $A \equiv A \wedge (\overrightarrow{\alpha} \equiv \overrightarrow{\alpha} \wedge \overrightarrow{\beta})$ (6,QF)

(W) : By (LW) of $\mathbf{PCI}_{\mathrm{GLw}}$.

■

Lemma 5.5.4 *All axioms and inference rules of* $\mathbf{PCI}_{\mathrm{GL}}$ ($\mathbf{PCI}_{\mathrm{GLc}}$, $\mathbf{PCI}_{\mathrm{GLw}}$, $\mathbf{PCI}_{\mathrm{GLcw}}$) *are provable in* $\mathbf{GL}'$ ($\mathbf{GL}'_{\mathrm{c}}$, $\mathbf{GL}'_{\mathrm{w}}$, $\mathbf{GL}'_{\mathrm{cw}}$) *respectively, after* t_{P}*-translation. Namely, the following formulas are theorems of* $\mathbf{GL}'$ ($\mathbf{GL}'_{\mathrm{c}}$, $\mathbf{GL}'_{\mathrm{w}}$, $\mathbf{GL}'_{\mathrm{cw}}$)*, where* $\overleftarrow{A}, \overleftarrow{B}, \overleftarrow{C}, \overleftarrow{D}$ *denote the result of* t_{P}*-translation.*

(E1) $\overleftarrow{A \equiv A} = (\overleftarrow{A} \supset \overleftarrow{A}) \wedge (\overleftarrow{A} \supset \overleftarrow{A})$

(E2) $\overleftarrow{(A \equiv B) \to (B \equiv A)}$
$= (\overleftarrow{A} \supset \overleftarrow{B}) \wedge (\overleftarrow{B} \supset \overleftarrow{A}) \to (\overleftarrow{B} \supset \overleftarrow{A}) \wedge (\overleftarrow{A} \supset \overleftarrow{B})$

(E3) $\overleftarrow{(A \equiv B) \wedge (B \equiv C) \to (A \equiv C)}$
$= (\overleftarrow{A} \supset \overleftarrow{B}) \wedge (\overleftarrow{B} \supset \overleftarrow{A}) \wedge (\overleftarrow{B} \supset \overleftarrow{C}) \wedge (\overleftarrow{C} \supset \overleftarrow{B}) \to (\overleftarrow{A} \supset \overleftarrow{C}) \wedge (\overleftarrow{C} \supset \overleftarrow{A})$

(C1) $\overleftarrow{(A \equiv B) \to (\neg A \equiv \neg B)}$
$= (\overleftarrow{A} \supset \overleftarrow{B}) \wedge (\overleftarrow{B} \supset \overleftarrow{A}) \to (\neg \overleftarrow{A} \supset \neg \overleftarrow{B}) \wedge (\neg \overleftarrow{B} \supset \neg \overleftarrow{A})$

(C2) $\overleftarrow{(A \equiv B) \wedge (C \equiv D) \to (A \wedge C) \equiv (B \wedge D)}$
$= (\overleftarrow{A} \supset \overleftarrow{B}) \wedge (\overleftarrow{B} \supset \overleftarrow{A}) \wedge (\overleftarrow{C} \supset \overleftarrow{D}) \wedge (\overleftarrow{D} \supset \overleftarrow{C})$
$\to (\overleftarrow{A} \wedge \overleftarrow{C} \supset \overleftarrow{B} \wedge \overleftarrow{D}) \wedge (\overleftarrow{B} \wedge \overleftarrow{D} \supset \overleftarrow{A} \wedge \overleftarrow{C})$

(C3) $\overleftarrow{(A \equiv B) \wedge (C \equiv D) \to (A \vee C) \equiv (B \vee D)}$
$= (\overleftarrow{A} \supset \overleftarrow{B}) \wedge (\overleftarrow{B} \supset \overleftarrow{A}) \wedge (\overleftarrow{C} \supset \overleftarrow{D}) \wedge (\overleftarrow{D} \supset \overleftarrow{C})$
$\to (\overleftarrow{A} \vee \overleftarrow{C} \supset \overleftarrow{B} \vee \overleftarrow{D}) \wedge (\overleftarrow{B} \vee \overleftarrow{D} \supset \overleftarrow{A} \vee \overleftarrow{C})$

(C4) $\overleftarrow{(A \equiv B) \wedge (C \equiv D) \to (A \to C) \equiv (B \to D)}$
$= (\overleftarrow{A} \supset \overleftarrow{B}) \wedge (\overleftarrow{B} \supset \overleftarrow{A}) \wedge (\overleftarrow{C} \supset \overleftarrow{D}) \wedge (\overleftarrow{D} \supset \overleftarrow{C})$
$\to ((\overleftarrow{A} \to \overleftarrow{C}) \supset (\overleftarrow{B} \to \overleftarrow{D})) \wedge ((\overleftarrow{B} \to \overleftarrow{D}) \supset (\overleftarrow{A} \to \overleftarrow{C}))$

(WIA1) $\overleftarrow{((A \to B) \equiv (B \to A)) \to (A \equiv B)}$
$= ((\overleftarrow{A} \to \overleftarrow{B}) \supset (\overleftarrow{B} \to \overleftarrow{A})) \wedge ((\overleftarrow{B} \to \overleftarrow{A}) \supset (\overleftarrow{A} \to \overleftarrow{B})) \to (\overleftarrow{A} \supset \overleftarrow{B}) \wedge (\overleftarrow{B} \supset \overleftarrow{A})$

(WIA2) $\overleftarrow{((A \to B) \equiv (B \to A)) \to ((A \to B) \equiv \top) \wedge ((B \to A) \equiv \top)}$
$= ((\overleftarrow{A} \to \overleftarrow{B}) \supset (\overleftarrow{B} \to \overleftarrow{A})) \wedge ((\overleftarrow{B} \to \overleftarrow{A}) \supset (\overleftarrow{A} \to \overleftarrow{B})) \to$
$((\overleftarrow{A} \to \overleftarrow{B}) \supset \top) \wedge (\top \supset (\overleftarrow{A} \to \overleftarrow{B})) \wedge ((\overleftarrow{B} \to \overleftarrow{A}) \supset \top) \wedge (\top \supset (\overleftarrow{B} \to \overleftarrow{A}))$

(LT) $\overleftarrow{(A \supset B) \supset ((B \supset C) \supset (A \supset C))}$
$= (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}) \wedge (\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A})$
$\supset ((\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{C}) \wedge (\overleftarrow{B} \wedge \overleftarrow{C} \supset \overleftarrow{B}) \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{C}) \wedge (\overleftarrow{A} \wedge \overleftarrow{C} \supset \overleftarrow{A}))$

(LE) $\overleftarrow{(A \supset (B \supset C)) \to (B \supset (A \supset C))}$
$= (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{C}) \wedge (\overleftarrow{B} \wedge \overleftarrow{C} \supset \overleftarrow{B})) \wedge (\overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{C})$
$\wedge(\overleftarrow{B} \wedge \overleftarrow{C} \supset \overleftarrow{B}) \supset \overleftarrow{A}) \to (\overleftarrow{B} \supset \overleftarrow{B} \wedge (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{C}) \wedge (\overleftarrow{A} \wedge \overleftarrow{C} \supset \overleftarrow{A}))$
$\wedge(\overleftarrow{B} \wedge (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{C}) \wedge (\overleftarrow{A} \wedge \overleftarrow{C} \supset \overleftarrow{A}) \supset \overleftarrow{B})$

(L $*$ 1) $\overleftarrow{A \supset (B \supset A * B)}$
$= \overleftarrow{A} \supset (\overleftarrow{B} \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{0}) \wedge (\overleftarrow{B} \wedge \overleftarrow{0} \supset \overleftarrow{B}))$
$\wedge(\overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{0}) \wedge (\overleftarrow{B} \wedge \overleftarrow{0} \supset \overleftarrow{B}) \supset \overleftarrow{A}) \supset \overleftarrow{0}))$

(L $*$ 2) $\overleftarrow{(A \supset (B \supset C)) \to (A * B \supset C)}$
$= (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{C}) \wedge (\overleftarrow{B} \wedge \overleftarrow{C} \supset \overleftarrow{B})) \wedge (\overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{C})$
$\wedge(\overleftarrow{B} \wedge \overleftarrow{C} \supset \overleftarrow{B}) \supset \overleftarrow{A}) \to ((\overleftarrow{A} * \overleftarrow{B}) \supset (\overleftarrow{A} * \overleftarrow{B}) \wedge \overleftarrow{C}) \wedge ((\overleftarrow{A} * \overleftarrow{B}) \wedge \overleftarrow{C} \supset (\overleftarrow{A} * \overleftarrow{B}))$

(LDN) $\overleftarrow{\sim\sim A \to A}$
$= ((\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{0}) \wedge (\overleftarrow{A} \wedge \overleftarrow{0} \supset \overleftarrow{A}) \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{0}) \wedge (\overleftarrow{A} \wedge \overleftarrow{0} \supset \overleftarrow{A}) \wedge \overleftarrow{0}) \wedge$
$((\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{0}) \wedge (\overleftarrow{A} \wedge \overleftarrow{0} \supset \overleftarrow{A}) \wedge \overleftarrow{0} \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{0}) \wedge (\overleftarrow{A} \wedge \overleftarrow{0} \supset \overleftarrow{A})) \to \overleftarrow{A}$

(LC) $\overleftarrow{(A \supset (A \supset B)) \to (A \supset B)}$
$= (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}) \wedge (\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A})) \wedge (\overleftarrow{A} \wedge (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B})$
$\wedge(\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A}) \supset \overleftarrow{A}) \to (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}) \wedge (\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A})$

(LW) $\overleftarrow{A \supset (B \supset A)}$
$= (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{A}) \wedge (\overleftarrow{B} \wedge \overleftarrow{A} \supset \overleftarrow{B})) \wedge (\overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{A})$
$\wedge(\overleftarrow{B} \wedge \overleftarrow{A} \supset \overleftarrow{B}) \supset \overleftarrow{A})$

(G) $(\overleftarrow{A},\ \overleftarrow{B}\ \Rightarrow\ \overleftarrow{A \equiv B}) \Longrightarrow (\overleftarrow{A},\ \overleftarrow{B}\ \Rightarrow\ (\overleftarrow{A} \supset \overleftarrow{B}) \wedge (\overleftarrow{B} \supset \overleftarrow{A}))$

Proof. We will only show the following five cases.

(C4) : 1 $\alpha = (\overleftarrow{A} \supset \overleftarrow{B}) \wedge (\overleftarrow{B} \supset \overleftarrow{A})$ and $\beta = (\overleftarrow{C} \supset \overleftarrow{D}) \wedge (\overleftarrow{D} \supset \overleftarrow{C})$ (assume)
2 $(\overleftarrow{B} \supset \overleftarrow{A}) \supset ((\overleftarrow{A} \supset \overleftarrow{C}) \supset (\overleftarrow{B} \supset \overleftarrow{C}))$ (a2)
3 $(\overleftarrow{A} \supset \overleftarrow{C}) \supset ((\overleftarrow{B} \supset \overleftarrow{A}) \supset (\overleftarrow{B} \supset \overleftarrow{C}))$ (2,a3,LMp)
4 $(\overleftarrow{B} \supset \overleftarrow{C}) \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset (\overleftarrow{B} \supset \overleftarrow{D}))$ (a2)
5 $((\overleftarrow{B} \supset \overleftarrow{A}) \supset (\overleftarrow{B} \supset \overleftarrow{C})) \supset (((\overleftarrow{B} \supset \overleftarrow{C}) \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset (\overleftarrow{B} \supset \overleftarrow{D})))$
$\supset ((\overleftarrow{B} \supset \overleftarrow{A}) \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset (\overleftarrow{B} \supset \overleftarrow{D}))))$ (a2)
6 $(\overleftarrow{A} \supset \overleftarrow{C}) \supset (((\overleftarrow{B} \supset \overleftarrow{C}) \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset (\overleftarrow{B} \supset \overleftarrow{D})))$
$\supset ((\overleftarrow{B} \supset \overleftarrow{A}) \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset (\overleftarrow{B} \supset \overleftarrow{D}))))$ (3,5,a2,LMp)
7 $((\overleftarrow{B} \supset \overleftarrow{C}) \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset (\overleftarrow{B} \supset \overleftarrow{D}))) \supset ((\overleftarrow{A} \supset \overleftarrow{C})$
$\supset ((\overleftarrow{B} \supset \overleftarrow{A}) \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset (\overleftarrow{B} \supset \overleftarrow{D}))))$ (5,a3,LMp)
8 $(\overleftarrow{A} \supset \overleftarrow{C}) \supset ((\overleftarrow{B} \supset \overleftarrow{A}) \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset (\overleftarrow{B} \supset \overleftarrow{D})))$ (4,7,LMp)
9 $(\overleftarrow{B} \supset \overleftarrow{A}) \supset ((\overleftarrow{A} \supset \overleftarrow{C}) \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset (\overleftarrow{B} \supset \overleftarrow{D})))$ (8,a3,LMp)
10 $(\overleftarrow{B} \supset \overleftarrow{A}) \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset ((\overleftarrow{A} \supset \overleftarrow{C}) \supset (\overleftarrow{B} \supset \overleftarrow{D})))$ (9,a3,LMp)
11 $\alpha \supset (\overleftarrow{B} \supset \overleftarrow{A})$ (a5)
12 $\beta \supset (\overleftarrow{C} \supset \overleftarrow{D})$ (a4)
13 $\alpha \supset ((\overleftarrow{C} \supset \overleftarrow{D}) \supset ((\overleftarrow{A} \supset \overleftarrow{C}) \supset (\overleftarrow{B} \supset \overleftarrow{D})))$ (11,10,a2,LMp)
14 $(\overleftarrow{C} \supset \overleftarrow{D}) \supset (\alpha \supset ((\overleftarrow{A} \supset \overleftarrow{C}) \supset (\overleftarrow{B} \supset \overleftarrow{D})))$ (13,a3,LMp)
15 $\beta \supset (\alpha \supset ((\overleftarrow{A} \supset \overleftarrow{C}) \supset (\overleftarrow{B} \supset \overleftarrow{D})))$ (12,13,a2,LMp)
16 $\alpha \supset (\beta \supset ((\overleftarrow{A} \supset \overleftarrow{C}) \supset (\overleftarrow{B} \supset \overleftarrow{D})))$ (15,a3,LMp)

17 $\alpha \supset (\beta \supset ((\overleftarrow{B} \supset \overleftarrow{D}) \supset (\overleftarrow{A} \supset \overleftarrow{C})))$ (same way)

18 $\alpha \supset (\beta \supset ((\overleftarrow{A} \supset \overleftarrow{C}) \supset (\overleftarrow{B} \supset \overleftarrow{D})) \wedge ((\overleftarrow{B} \supset \overleftarrow{D}) \supset (\overleftarrow{A} \supset \overleftarrow{C})))$ (16,17,a6,LMp)

19 $\alpha \to (\beta \to ((\overleftarrow{A} \to \overleftarrow{C}) \supset (\overleftarrow{B} \to \overleftarrow{D})) \wedge ((\overleftarrow{B} \to \overleftarrow{D}) \supset (\overleftarrow{A} \to \overleftarrow{C})))$
(18,Th.5.3.3 (vi),LIm,LMp)

(L $*$ 1) : 1 $\alpha = (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{0}) \wedge (\overleftarrow{B} \wedge \overleftarrow{0} \supset \overleftarrow{B}))$
$\wedge(\overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{0}) \wedge (\overleftarrow{B} \wedge \overleftarrow{0} \supset \overleftarrow{B}) \supset \overleftarrow{A})$ (assume)

2 $\alpha \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{0}) \wedge (\overleftarrow{B} \wedge \overleftarrow{0} \supset \overleftarrow{B}))$ (a4)

3 $\overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{0}) \wedge (\overleftarrow{B} \wedge \overleftarrow{0} \supset \overleftarrow{B}) \supset (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{0})$ (a4,a5)

4 $\overleftarrow{B} \wedge \overleftarrow{0} \supset \overleftarrow{0}$ (a5)

5 $\alpha \supset (\overleftarrow{A} \supset (\overleftarrow{B} \supset \overleftarrow{0}))$ (2,3,4,a2,LMp)

6 $\overleftarrow{A} \supset (\alpha \supset (\overleftarrow{B} \supset \overleftarrow{0}))$ (5,a3,LMp)

7 $\overleftarrow{A} \supset (\overleftarrow{B} \supset (\alpha \supset \overleftarrow{0}))$ (6,a3,LMp)

(L $*$ 2) : 1 $\alpha = (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{C}) \wedge (\overleftarrow{B} \wedge \overleftarrow{C} \supset \overleftarrow{B}))$
$\wedge(\overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{C}) \wedge (\overleftarrow{B} \wedge \overleftarrow{C} \supset \overleftarrow{B}) \supset \overleftarrow{A})$ (assume)

2 $\alpha \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{C}) \wedge (\overleftarrow{B} \wedge \overleftarrow{C} \supset \overleftarrow{B}))$ (a4)

3 $\overleftarrow{A} \wedge (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{C}) \wedge (\overleftarrow{B} \wedge \overleftarrow{C} \supset \overleftarrow{B}) \supset (\overleftarrow{B} \supset \overleftarrow{B} \wedge \overleftarrow{C})$ (a4,a5)

4 $\overleftarrow{B} \wedge \overleftarrow{C} \supset \overleftarrow{C}$ (a5)

5 $\alpha \supset (\overleftarrow{A} \supset (\overleftarrow{B} \supset \overleftarrow{C}))$ (2,3,4,a2,LMp)

6 $(\overleftarrow{A} \supset (\overleftarrow{B} \supset \overleftarrow{C})) \supset (\overleftarrow{A} * \overleftarrow{B} \supset \overleftarrow{C})$ (a11)

7 $\alpha \supset (\overleftarrow{A} * \overleftarrow{B} \supset \overleftarrow{C})$ (5,6,a2,LMp)

8 $\overleftarrow{A} * \overleftarrow{B} \supset \overleftarrow{A} * \overleftarrow{B}$

9 $\alpha \supset (\overleftarrow{A} * \overleftarrow{B} \supset \overleftarrow{A} * \overleftarrow{B})$ (8,a16)

10 $\alpha \supset (\overleftarrow{A} * \overleftarrow{B} \supset (\overleftarrow{A} * \overleftarrow{B}) \wedge \overleftarrow{C})$ (7,9,a6,LMp)

11 $(\overleftarrow{A} * \overleftarrow{B}) \wedge \overleftarrow{C} \supset \overleftarrow{A} * \overleftarrow{B}$ (a4)

12 $\alpha \supset ((\overleftarrow{A} * \overleftarrow{B}) \wedge \overleftarrow{C} \supset \overleftarrow{A} * \overleftarrow{B})$ (11,a16)

13 $\alpha \supset (\overleftarrow{A} * \overleftarrow{B} \supset (\overleftarrow{A} * \overleftarrow{B}) \wedge \overleftarrow{C}) \wedge ((\overleftarrow{A} * \overleftarrow{B}) \wedge \overleftarrow{C} \supset \overleftarrow{A} * \overleftarrow{B})$ (10,12,a6,LMp)

14 $\alpha \to (\overleftarrow{A} * \overleftarrow{B} \supset (\overleftarrow{A} * \overleftarrow{B}) \wedge \overleftarrow{C}) \wedge ((\overleftarrow{A} * \overleftarrow{B}) \wedge \overleftarrow{C} \supset \overleftarrow{A} * \overleftarrow{B})$ (13,LIm,Mp)

(LC) : 1 $\alpha = (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}) \wedge (\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A}))$
$\wedge(\overleftarrow{A} \wedge (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}) \wedge (\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A}) \supset \overleftarrow{A})$ (assume)

2 $\alpha \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}) \wedge (\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A}))$ (a4)

3 $\overleftarrow{A} \wedge (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}) \wedge (\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A}) \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B})$ (a4,a5)

4 $\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{B}$ (a5)

5 $\alpha \supset (\overleftarrow{A} \supset (\overleftarrow{A} \supset \overleftarrow{B}))$ (2,3,4,a2,LMp)

6 $(\overleftarrow{A} \supset (\overleftarrow{A} \supset \overleftarrow{B})) \supset (\overleftarrow{A} \supset (\overleftarrow{A} \supset \overleftarrow{B}))$ (a3)

7 $\alpha \supset (\overleftarrow{A} \supset (\overleftarrow{A} \supset \overleftarrow{B}))$ (5,6,a2,LMp)

8 $\overleftarrow{A} \supset \overleftarrow{A}$

9 $\alpha \supset (\overleftarrow{A} \supset (\overleftarrow{A} \supset \overleftarrow{A}))$ (8,a16)

10 $\alpha \supset (\overleftarrow{A} \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}))$ (7,9,a6,LMp)

11 $(\overleftarrow{A} \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B})) \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B})$ (C)

$$
\begin{array}{lll}
 & 12\ \alpha \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}) & (10,11,\text{a2},\text{LMp}) \\
 & 13\ \overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A} & (\text{a4}) \\
 & 14\ \alpha \supset (\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A}) & (13,\text{a16}) \\
 & 15\ \alpha \supset (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}) \wedge (\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A}) & (12,15,\text{a6},\text{LMp}) \\
 & 16\ \alpha \to (\overleftarrow{A} \supset \overleftarrow{A} \wedge \overleftarrow{B}) \wedge (\overleftarrow{A} \wedge \overleftarrow{B} \supset \overleftarrow{A}) & (15,\text{LIm},\text{LMp}) \\
(\text{G}): & 1\ \overleftarrow{A} & (\text{assume}) \\
 & 2\ \overleftarrow{B} \supset \overleftarrow{A} & (1,\text{a16}) \\
 & 3\ \overleftarrow{B} & (\text{assume}) \\
 & 4\ \overleftarrow{A} \supset \overleftarrow{B} & (3,\text{a16}) \\
 & 5\ (\overleftarrow{A} \supset \overleftarrow{B}) \wedge (\overleftarrow{B} \supset \overleftarrow{A}) & (2,4,\text{LAd})
\end{array}
$$

■

Proposition 5.5.5 *For any formula α in $\mathrm{L}_{\mathrm{GL}'}$, $\alpha \in \mathbf{GL}'$ implies $t_{\mathrm{G}}(\alpha) \in \mathbf{PCI}_{\mathrm{GL}}$.*

Proof. By induction on the length of derivation in $\mathbf{GL}'$.

(1) Base step: We have to check the provability of each axioms of $\mathbf{GL}'$ in $\mathbf{PCI}_{\mathrm{GL}}$ after a t_{G}-translation. The case of TFA is trivial since every t_{G}-translation preserves the structure of TF-connectives and also $\mathbf{PCI}_{\mathrm{GL}}$ has TFA axioms. Also by Lemma 5.5.3 all linear axioms (a1)-(a18), (LW1) and (LW2) are provable in $\mathbf{PCI}_{\mathrm{GL}}$ after a t_{G}-translation.

(2) Induction step: Assume that we have established the theorem for some step, and consider a new derivation from these by applying inference rules (LMp), (LAd) and (Mp) of $\mathbf{GL}'$. Here (LMp) and (LAd) are easily conclusions from Theorem 5.4.2 (ii) and (A5), respectively. Also (Mp) is trivial since every t_{G}-translation preserves the structure of TF-connectives and $\mathbf{PCI}_{\mathrm{GL}}$ also has (Mp) rule.

Thus the t_{G}-translation of any formula provable in $\mathbf{GL}'$ is also provable in $\mathbf{PCI}_{\mathrm{GL}}$.

■

Proposition 5.5.6 *For any formula A in L_{P}, $A \in \mathbf{PCI}_{\mathrm{GL}}$ implies $t_{\mathrm{P}}(A) \in \mathbf{GL}'$.*

Proof. By induction on the length of derivation in $\mathbf{PCI}_{\mathrm{GL}}$.

(1) Base step: We have to check the provability of each axioms of $\mathbf{PCI}_{\mathrm{GL}}$ in $\mathbf{GL}'$ after a t_{P}-translation. The case of TFA is trivial because of the similar reason in the above Proposition 5.5.5. Also by Lemma 5.5.4 all IDA are provable in $\mathbf{GL}'$ after a t_{P}-translation.

(2) Induction step: Assume that we have established the theorem for some step, and consider a new derivation from these by applying inference rules of $\mathbf{PCI}_{\mathrm{GL}}$.
(Mp): This case is trivial because of the similar reason in the above Proposition 5.5.5. (G): Assume that both $t_{\mathrm{P}}(A_1)$ and $t_{\mathrm{P}}(B_1)$ are theorem of $\mathbf{GL'}$ by I.H. Then, it is possible to derive the following proof in $\mathbf{GL'}$:

$$\dfrac{\dfrac{t_{\mathrm{P}}(A_1)}{t_{\mathrm{P}}(B_1) \supset t_{\mathrm{P}}(A_1)} \quad \dfrac{t_{\mathrm{P}}(B_1)}{t_{\mathrm{P}}(A_1) \supset t_{\mathrm{P}}(B_1)}\ \text{(a16)}}{(t_{\mathrm{P}}(A_1) \supset t_{\mathrm{P}}(B_1)) \wedge (t_{\mathrm{P}}(B_1) \supset t_{\mathrm{P}}(A_1))}\ \text{(LAd)}$$

Hence, by the definition we get $(t_{\mathrm{P}}(A_1) \supset t_{\mathrm{P}}(B_1)) \wedge (t_{\mathrm{P}}(B_1) \supset t_{\mathrm{P}}(A_1)) \in \mathbf{GL'}$. But $t_{\mathrm{P}}(A) = t_{\mathrm{P}}(A_1 \equiv B_1) = (t_{\mathrm{P}}(A_1) \supset t_{\mathrm{P}}(B_1)) \wedge (t_{\mathrm{P}}(B_1) \supset t_{\mathrm{P}}(A_1))$, so $t_{\mathrm{P}}(A) \in \mathbf{GL'}$.

Thus the t_{P}-translation of any formula provable in $\mathbf{PCI}_{\mathrm{GL}}$ is also provable in $\mathbf{GL'}$.

■

Moreover, we can show the following.

Theorem 5.5.7 (i) *For any formula α in $\mathrm{L}_{\mathrm{GL'}}$, $t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha)) \leftrightarrow \alpha \in \mathbf{GL'}$.*

(ii) *For any formula A in L_{P}, $t_{\mathrm{G}}(t_{\mathrm{P}}(A)) \leftrightarrow A \in \mathbf{PCI}_{\mathrm{GL}}$.*

Proof. (i): By induction on the length of the formula α.
Base step: It is clear for $\alpha = p(\in \mathrm{VAR})$ or $\bot$. For $\alpha = 0$ we in $\mathbf{GL'}$ have

$$\begin{aligned}
& t_{\mathrm{P}}(t_{\mathrm{G}}(0)) \leftrightarrow t_{\mathrm{P}}(0) && \text{(Def.5.5.1(iii))}\\
& \leftrightarrow t_{\mathrm{P}}(\neg(t_{\mathrm{G}}(\alpha_1) \equiv t_{\mathrm{G}}(\alpha_1))) && \text{(Def. of 0)}\\
& \leftrightarrow t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1) \equiv t_{\mathrm{G}}(\alpha_1) \to \bot) && \text{(Def. of } \neg)\\
& \leftrightarrow t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1) \equiv t_{\mathrm{G}}(\alpha_1)) \to t_{\mathrm{P}}(\bot) && \text{(Def.5.5.2 (vi))}\\
& \leftrightarrow (t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1)) \supset t_{\mathrm{P}}(tg(\alpha_1))) \wedge (t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1)) \supset t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1))) \to \bot && \text{(Def.5.5.2 (ii),(vii))}\\
& \leftrightarrow (\alpha_1 \supset \alpha_1) \wedge (\alpha_1 \supset \alpha_1) \to \bot && (t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1)) \leftrightarrow \alpha_1\text{:H.I.})\\
& \leftrightarrow \neg 1 && ((\alpha_1 \supset \alpha_1) \wedge (\alpha_1 \supset \alpha_1) = (\alpha_1 \equiv \alpha_1) = 1)\\
& \leftrightarrow 0 && (0 = \neg 1)
\end{aligned}$$

Induction step: It is clear that TF connectives $(\neg, \wedge, \vee, \to)$ hold. So we have only to check the $\supset$ connective. Assume that for any $\alpha_1, \beta_1 \in \mathrm{L}_{\mathrm{GL'}}$, $t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1)) \leftrightarrow \alpha_1 \in \mathbf{GL'}$ and $t_{\mathrm{P}}(t_{\mathrm{G}}(\beta_1)) \leftrightarrow \beta_1 \in \mathbf{GL'}$. Then we in $\mathbf{GL'}$ have

$$\begin{aligned}
& t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1 \supset \beta_1)) \leftrightarrow t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1) \equiv t_{\mathrm{G}}(\alpha_1) \wedge t_{\mathrm{G}}(\beta_1)) && \text{(Def.5.5.1 (vii))}\\
& \leftrightarrow (t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1)) \supset t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1) \wedge t_{\mathrm{G}}(\beta_1)))\\
& \quad \wedge (t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1) \wedge t_{\mathrm{G}}(\beta_1)) \supset t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1))) && \text{(Def.5.5.2 (vii))}\\
& \leftrightarrow (t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1)) \supset t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1)) \wedge t_{\mathrm{P}}(t_{\mathrm{G}}(\beta_1)))\\
& \quad \wedge (t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1)) \wedge t_{\mathrm{P}}(t_{\mathrm{G}}(\beta_1)) \supset t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1))) && \text{(Def.5.5.2 (iv))}\\
& \leftrightarrow t_{\mathrm{P}}(t_{\mathrm{G}}(\alpha_1)) \supset t_{\mathrm{P}}(t_{\mathrm{G}}(\beta_1)) && \text{(Th.5.3.3 (vii))}\\
& \leftrightarrow \alpha_1 \supset \beta_1 && \text{(I.H)}
\end{aligned}$$

(ii): By induction on the length of the formula A. For the same reasons of (i) we will only attention to identity connective. Assume that for any $A_1, B_1 \in \mathrm{L_P}$, $t_\mathrm{G}(t_\mathrm{P}(A_1)) \leftrightarrow A_1 \in \mathbf{PCI}_\mathrm{GL}$ and $t_\mathrm{G}(t_\mathrm{P}(B_1)) \leftrightarrow B_1 \in \mathbf{PCI}_\mathrm{GL}$. Then we in $\mathbf{PCI}_\mathrm{GL}$ have

$$
\begin{aligned}
& t_\mathrm{G}(t_\mathrm{P}(A_1 \equiv B_1)) \\
& \quad \leftrightarrow t_\mathrm{G}((t_\mathrm{P}(A_1) \supset t_\mathrm{P}(B_1)) \wedge (t_\mathrm{P}(B_1) \supset t_\mathrm{P}(A_1))) && \text{(Def.5.5.2 (vii))} \\
& \quad \leftrightarrow t_\mathrm{G}(t_\mathrm{P}(A_1) \supset t_\mathrm{P}(B_1)) \wedge t_\mathrm{G}(t_\mathrm{P}(B_1) \supset t_\mathrm{P}(A_1)) && \text{(Def.5.5.2 (iv))} \\
& \quad \leftrightarrow (t_\mathrm{G}(t_\mathrm{P}(A_1)) \equiv t_\mathrm{G}(t_\mathrm{P}(A_1)) \wedge t_\mathrm{G}(t_\mathrm{P}(B_1))) \\
& \qquad \wedge (t_\mathrm{G}(t_\mathrm{P}(B_1)) \equiv t_\mathrm{G}(t_\mathrm{P}(B_1)) \wedge t_\mathrm{G}(t_\mathrm{P}(A_1))) && \text{(Def.5.5.1 (vii))} \\
& \quad \leftrightarrow t_\mathrm{G}(t_\mathrm{P}(A_1)) \equiv t_\mathrm{G}(t_\mathrm{P}(B_1)) && \text{(Th.5.4.2 (ix))} \\
& \quad \leftrightarrow A_1 \equiv B_1 && \text{(I.H)}
\end{aligned}
$$

■

Theorem 5.5.8 (i) *For any formula α in $\mathrm{L_{GL'}}$, $\alpha \in \mathbf{GL'}$ if and only if $t_\mathrm{G}(\alpha) \in \mathbf{PCI}_\mathrm{GL}$.*

(ii) *For any formula A in $\mathrm{L_P}$, $A \in \mathbf{PCI}_\mathrm{GL}$ if and only if $t_\mathrm{P}(A) \in \mathbf{GL'}$.*

Proof. (i): The only if part obtains from Proposition 5.5.5. Also other direction can easily be proved as follows:

$$
\begin{aligned}
t_\mathrm{G}(\alpha) \in \mathbf{PCI}_\mathrm{GL} & \Longrightarrow t_\mathrm{P}(t_\mathrm{G}(\alpha)) \in \mathbf{GL'} && \text{(Prop.5.5.6)} \\
& \Longrightarrow \alpha \in \mathbf{GL'} && \text{(Th.5.5.7 (i))}
\end{aligned}
$$

(ii): The only if part obtains from from Proposition 5.5.6. Also other direction is as follows:

$$
\begin{aligned}
t_\mathrm{P}(A) \in \mathbf{GL'} & \Longrightarrow t_\mathrm{G}(t_\mathrm{P}(A)) \in \mathbf{PCI}_\mathrm{GL} && \text{(Prop.5.5.5)} \\
& \Longrightarrow A \in \mathbf{PCI}_\mathrm{GL} && \text{(Th.5.5.7 (ii))}
\end{aligned}
$$

■

Hence we may conclude that two logics $\mathbf{GL'}$ and $\mathbf{PCI}_\mathrm{GL}$ are syntactically equivalent by Definition 3.4.1, Theorem 5.5.7 and Theorem 5.5.8. Furthermore, from this result and previous Proposition 5.3.8, we get finally the following corollary.

Corollary 5.5.9 *For any formula α in $\mathrm{L_{GL}}$, $\alpha \in \mathbf{GL}$ if and only if $t_\mathrm{G}(\alpha) \in \mathbf{PCI}_\mathrm{GL}$.*

5.6 Notes

In this chapter we discussed how two types of weak logics, e.g., **F** with strict implication and **GL** with linear implication, are simulated on **PCI** logic. As another example of such a weak logic, we can consider basic propositional logic **BPL** in [70]. **BPL** was firstly introduced by A. Visser as an embeddable system into **K4**, and extensively studied by Y. Suzuki from a point of transitive frames. Although we have not checked precisely yet, it is conjectured that **BPL** can be simulated on $\mathbf{PCI}_\mathrm{K4}$ extension.

Chapter 6

Algebraic properties of PCI logics

In this chapter we will investigate algebraic properties of **PCI** logics. In Section 1, we will first survey broad informations of various methods for the algebraization of deductive systems. The most famous method to algebraize a logic is to construct a Lindenbaum-Tarski algebra by factoring the algebras of formulas by the congruence relative to theories of the logic. Furthermore, we will explain equivalential algebras and congruence operators, which also contribute to algebraize a logic. At the end of this section, we will consider the case of **PCI** logics introduced so far. In Section 2, we will show that the class of **PCI**-algebras, defined by the above algebraization, forms a variety. In fact, we only consider a class of $\mathbf{PCI}_{\mathrm{K}}$-algebas whether this class forms a variety or not. In Section 3, we will check a variety of $\mathbf{PCI}_{\mathrm{K}}$-algebras to have EDPC property, and show a necessary and sufficient condition to have EDPC property. Finally, in Section 4, we will also give further information on related results shown in this chapter.

6.1 Algebraization of deductive systems

In this section we will explain various methods for the algebraization of deductive systems. If a deductive system $\mathfrak{L} = (\mathcal{L}, C)$ has the equivalence connective $\leftrightarrow$, then this connective expresses in the material sense the fact that two formulas have the same logical value, while it also expresses in the strict sense the fact that two formulas are interderivable on the basis of the deductive system $\mathfrak{L}$. The process of identification of equivalent formulas relative to theories of C defines a class of abstract algebras, in which each member is called *Lindenbaum-Tarski* algebra. The above abstraction from a deductive system to its Lindenbaum-Tarski algebra enable to investigate the deductive system by using various powerful methods of contemporary algebra in metalogic. From this direction the concept of an *algebraizable* deductive system is clarified by Blok and Pigozzi (see [8]). Roughly speaking, a deductive system is called algebraizable if a certain class of algebas can be associated with this deductive system and moreover, the properties of this deductive system are fully reducible to the algebraic properties of the associated class of algebras. But the Lindenbaum-Tarski algebra itself is in general not sufficient for covering of all deductive systems. Indeed, there exist numerous deductive systems to which the Lindenbaum-Tarski algebra cannot be directly applied since there will not exist a connective $\leftrightarrow$ in the language of the deductive system which defines a congruence on the language. In order to overcome this problem, Prucnal and Wroński (see [56]) have proposed a generalization of the Lindenbaum-Tarski algebra by replacing the equivalence connective with a set of sentential formulas which has many properties of the equivalence connective. Any deductive system having such a set is called *equivalential.* Roughly speaking, a deductive system is equivalential if and only if the greatest matrix congruences in its matrics (models) are determined by polynomials. In [8], Blok and Pigozzi have proposed the approach based on the concept of the *Leibniz operator* Ω for characterizing their concept of an algebraizable deductive system. The Leibniz operator Ω is a function which assigns to each theory $T \subseteq L$ a congruence on L, denoted by ΩT. This definition is independent from various kinds of deductive systems admitting in the language $\mathcal{L}$. If we restrict the domain of the Leibniz operator Ω to the family of all theories of a given deductive system C, then it assigns the congruence ΩT to each closed theory $T \in \mathrm{Th}(C)$. So the Leibniz operator give us the possibility of building a certain natural hierarchy of deductive systems based on properties of the operator Ω (see [20] and [33]). A variety of algebras has *equationally definable principal congruences* (EDPC for short) if the principal congruence relation $c \equiv d \pmod{\Theta(a,b)}$ is definable in each member of the variety by the conjunction of a fixed, finite set of polynomial equations $p_i(a,b,c,d) = q_i(a,b,c,d)$. Since for the varieties arising in algebraic deductive system the EDPC is closely connected with the deduction theorem, it seems to be their most characteristic property (see [39] and [5]).

In this section we assume that $\mathcal{L}$ is some fixed but arbitrary sentential language and L is the set of all $\mathcal{L}$ formulas. We write $\gamma[\alpha/p]$ as the result of simultaneously replacing the variable p in γ by the formula α.

6.1.1 Lindenbaum-Tarski algebra and its equational theory

The Lindenbaum-Tarski algebra is a powerful method to study various kind of an algebraizable deductive system. If $\mathfrak{L} = (\mathcal{L}, C)$ is the classical deductive system, then it is well-known that the relation $\equiv_T$:

$(*)$ $\alpha \equiv_T \beta$ if and only if $\alpha \leftrightarrow \beta \in C(T)$

defines a congruence of the algebra of formulas L. Then quotient algebras obtained by factoring the algebra of formulas by the congruence $(*)$ form the class of Lindenbaum-Tarski algebras, and coincide with the class of Boolean algebras. Indeed, many important logics, e.g., the intuitionistic logics of Heyting, the many-valued logics of Post and Łukasiewicz, and the modal logics **S4**, **S5** of Lewis were algebraized in this way. In the result, the process of algebraization of deductive systems can be reduced to the study of the equational theory of the Lindenbaum-Tarski algebra, and more precisely, to the quasi-equational theory of a certain quasivariety to which it belongs (see [8] and [20]).

By an $\mathcal{L}$-*equation*, or simply an *equation*, we mean a formal expression $\alpha \approx \beta$ for any $\alpha, \beta \in L$. We denote the set of all $\mathcal{L}$-equations by $\mathrm{Eq}(L)$. For any class K of $\mathcal{L}$-algebras, any subset $X \cup \{\alpha \approx \beta\}$ of $\mathrm{Eq}(L)$, we will define the operator E_{K} between a set X of equations and a single equation $\alpha \approx \beta$, in symbols $\alpha \approx \beta \in E_{\mathrm{K}}(X)$, by for every $\mathcal{A} \in \mathrm{K}$ and every homomorphism $h : \mathrm{Eq}(L) \to \mathcal{A}$, $h(\alpha) = h(\beta)$ whenever $h(\delta) = h(\epsilon)$ for any $\delta \approx \epsilon \in X$. Then E_{K} is called the *semantic equational consequence operator* determined by K. We say that E_{K} is *finitary* if $\alpha \approx \beta \in E_{\mathrm{K}}(X)$ implies $\alpha \approx \beta \in E_{\mathrm{K}}(X')$ for some finite $X' \subseteq X$, and moreover, if $X = \{\delta_0 \approx \epsilon_0, \ldots, \delta_{n-1} \approx \epsilon_{n-1}\}$, then $\alpha \approx \beta \in E_{\mathrm{K}}(X)$ if and only if K satisfies the quasi-identities: $\delta_0 \approx \epsilon_0 \wedge \cdots \wedge \delta_{n-1} \approx \epsilon_{n-1} \to \alpha \approx \beta$. Thus if E_{K} is finitary, then $E_{\mathrm{K}} = E_{\mathrm{K}^{\mathrm{Q}}}$ where K^{Q} is the quasivariety generated by K. Conversely, if K is a quasivariety, then it is easy to show that E_{K} is finitary. Therefore for any class K of $\mathcal{L}$-algebras, E_{K} is finitary if and only if $E_{\mathrm{K}} = E_{\mathrm{K}^{\mathrm{Q}}}$.

Definition 6.1.1 *Let $\mathfrak{L} = (\mathcal{L}, C)$ be a deductive system and* K *a class of algebras.*

(i) *Then* K *is called an* algebraic semantics *for $\mathfrak{L}$ if C can be interpreted in E_{K} in the following way: there exists a finite system $\{\delta_i(p) \approx \epsilon_i(p); i < n\} \subseteq \mathrm{Eq}(L)$ of equations with a single variable p such that for all $X \cup \{\alpha\} \subseteq L$ and each $j < n$,*
$\alpha \in C(X)$ if and only if $\delta_j[\alpha/p] \approx \epsilon_j[\alpha/p] \in E_{\mathrm{K}}(\{\delta_i[\beta/p] \approx \epsilon_i[\beta/p] : i < n, \beta \in X\})$.

(ii) *Moreover, the system $\{\delta_i(p) \approx \epsilon_i(p); i < n\}$ is called* defining equations *for $\mathfrak{L}$ and* K.

In order to simplify notation we will use $\delta(p) \approx \epsilon(p)$, $\delta \approx \epsilon \in X$ and $\delta(\alpha) \approx \epsilon(\alpha) \in E_{\mathrm{K}}(X)$ as abbreviations for $\delta_i(p) \approx \epsilon_i(p)$, $i < n$, $\{\delta_i \approx \epsilon_i : i < n\} \subseteq X$ and $\delta_i[\alpha/p] \approx \epsilon_i[\alpha/p] \in E_{\mathrm{K}}(X)$ for all $i < n$, respectively. Since C is always assumed to be finitary, we can also assume that the set X in definition 6.1.1 (i) is always finite. As previously observed, the operator E_{K} on the right hand side in definition 6.1.1 (i) holds if and only if K satisfies the quasi-identities: $\bigwedge_{\beta \in X} \delta(\beta) \approx \epsilon(\beta) \to \delta(\alpha) \approx \epsilon(\alpha)$. Hence if K is an algebraic sematics for a deductive system $\mathfrak{L}$, then so is the quasivariety K^{Q}.

Let K be an algebraic semantics for $\mathfrak{L}$ with defining equations $\delta(p) \approx \epsilon(p)$. For any $\mathcal{A} \in \mathrm{K}$ and any $h : \mathrm{Eq}(L) \to \mathcal{A}$, let $F_{\mathcal{A}}^{\delta \approx \epsilon} = \{a \in A : h(\delta(a)) = h(\epsilon(a))\}$. Then it is easy to see that $(\mathcal{A}, F_{\mathcal{A}}^{\delta \approx \epsilon})$ is the logical matrix for $\mathcal{L}$ as follows.

Theorem 6.1.2 *Let $\mathfrak{L} = (\mathcal{L}, C)$ be a deductive system,* K *a quasivariety, and $\delta(p) \approx \epsilon(p)$ a system of single variable equations. Then the following are equivalent:*

(i) K *is an algebraic semantics of $\mathfrak{L}$ with defining equations $\delta(p) \approx \epsilon(p)$.*

(ii) *The class $\mathrm{M} = \{(\mathcal{A}, F_{\mathcal{A}}^{\delta \approx \epsilon}) : \mathcal{A} \in \mathrm{K}\}$ is a matrix semantics for $\mathfrak{L}$.*

Definition 6.1.3 *Let $\mathfrak{L} = (\mathcal{L}, C)$ be a deductive system and* K *an algebraic semantics for $\mathfrak{L}$ with defining equations $\delta_i \approx \epsilon_i$, for $i < n$.*

(i) *Then* K *is said to be* equivalent *to* $\mathfrak{L}$ *if there exists a finite system* $\Delta_j(p,q)$*, for* $j < m$*, of formulas with two distinct variables such that for every* $\alpha \approx \beta \in \mathrm{Eq}(L)$*,*
$$E_{\mathrm{K}}(\alpha \approx \beta) = E_{\mathrm{K}}(\{\delta_i(\Delta_j(\alpha,\beta)) \approx \epsilon_i(\Delta_j(\alpha,\beta)); i < n, j < m\}).$$

(ii) *Moreover, the system* Δ_j*,* $j < m$*, satisfying the above condition is called a system of* equivalence formulas for $\mathfrak{L}$ *and* K.

Corollary 6.1.4 *Let* K *be an algebraic semantics for a deductive system* $\mathfrak{L}$*. Then* K *is equivalent to* $\mathfrak{L}$ *if and only if so is the quasivariety* K^{Q}*.*

Corollary 6.1.5 *Let* $\mathfrak{L} = (\mathcal{L}, C)$ *be a deductive system and* K *an algebraic semantics for* $\mathfrak{L}$ *with defining equations* $\delta \approx \epsilon$*. If* K *is equivalent to* $\mathfrak{L}$ *with equivalence formulas* Δ*, then we have:*

(i) *for all* $X \cup \{\alpha \approx \beta\} \subseteq \mathrm{Eq}(L)$*,*
$$\alpha \approx \beta \in E_{\mathrm{K}}(X) \text{ if and only if } \Delta(\alpha,\beta) \in C(\{\Delta(\xi,\mu) : \xi \approx \mu \in X\}),$$

(ii) $C(\vartheta) = C(\Delta(\delta(\vartheta), \epsilon(\vartheta)))$.

Conversely, if there exists a system of formulas Δ *satisfying conditions* (i) *and* (ii)*, then* K *is equivalent to* $\mathfrak{L}$ *with equivalence formulas* Δ*.*

Thus if K is an equivalent algebraic semantics for $\mathfrak{L}$, then above definition 6.1.3 and corollary 6.1.4 guarantee that C and E_{K} are mutually interpretable.

Definition 6.1.6 *A deductive system* $\mathfrak{L}$ *is said to be* algebraizable *if it has an equivalent algebraic semantics.*

The class of algebraizable logic contain many of traditionally considered logics, e.g., the classical and intutionistic propositional logics, the intermediate logics, the normal modal logics, many valued logics and quantum logics. But there exist also important logics that fail to be algebraizable, e.g., the non-normal modal logics (**S1**, **S2**, **S3**). Nevertheless, most of these systems follow the methods of universal algebra when applied to the matrix models of the system. This class is clarified as *protoalgebraic* logics by Blok and Pigozzi (see [7]). Let $\mathfrak{L} = (\mathcal{L}, C)$ be a deductive system and $X \cup \{\alpha, \beta\} \subseteq L$. Two formulas α and β are said to be *X-equivalent relative* to $\mathfrak{L}$ if for every $\gamma \in L$, and every variable p occurring in γ, $\gamma[\alpha/p] \in C(X)$ if and only if $\gamma[\beta/p] \in C(X)$. Moreover, α and β are *X-interderivable relative* to $\mathfrak{L}$ if $\beta \in C(X;\alpha)$ if and only if $\alpha \in C(X;\beta)$.

Definition 6.1.7 *A deductive system* $\mathfrak{L} = (\mathcal{L}, C)$ *is called* protoalgebraic *if for every* $X \subseteq L$*, any two formulas that are* X*-equivalent relative to* $\mathfrak{L}$ *are* X*-interderivable relative to* $\mathfrak{L}$*.*

First of all noticed that it follows immediately from the definition that α and β are X-equivalent relative to $\mathfrak{L}$ if and only if $\alpha \equiv \beta \pmod{\Theta_T}$ where T is the C-theory generated by X and Θ_T is the greatest congruence on L compatible with T. And they are X-interderivable if and only if, whenever one of them is contained in a theory including X, so is the other. Thus $\mathfrak{L}$ is protoalgebraic if and only if, for every pair of theories T and S, $T \subseteq S$ implies that Θ_T is compatible with S, i.e., $\Theta_T \subseteq \Theta_S$.

6.1.2 Equivalential algebra

Let $\mathfrak{L} = (\mathcal{L}, C)$ be a deductive system. If $\Delta(p,q) \subseteq L$ and α, β are formulas of $\mathcal{L}$, then we write $\Delta(\alpha,\beta)$ to denote the set of formulas which result by the simultaneous substitution of α for p and β for q in all formulas from $\Delta(p,q)$ (see [18], [8] and [32]).

Definition 6.1.8 *A deductive system $\mathfrak{L} = (\mathcal{L}, C)$ is said to be* equivalential *if there exists a set $\Delta(p,q)$ of formulas with two distinct variables such that for any $\alpha, \beta, \gamma \in L$ the following conditions are satisfied:*

(i) $\Delta(\alpha,\alpha) \subseteq C(\emptyset)$,

(ii) $\Delta(\beta,\alpha) \subseteq C(\Delta(\alpha,\beta))$,

(iii) $\Delta(\alpha,\gamma) \subseteq C(\Delta(\alpha,\beta) \cup \Delta(\beta,\gamma))$,

(iv) *for every natural $n \geq 0$, every n-ary connective § of $\mathcal{L}$ and any formulas α_i, β_i, $1 \leq i \leq n$, $\Delta(\S(\alpha_1,\dots,\alpha_n), \S(\beta_1,\dots,\beta_n)) \subseteq C(\Delta(\alpha_1,\beta_1) \cup \cdots \cup \Delta(\alpha_n,\beta_n))$,*

(v) $\beta \in C(\Delta(\alpha,\beta);\alpha)$.

If $\mathfrak{L} = (\mathcal{L}, C)$ is equivalential with respect to a set $\Delta(p,q)$, then $\Delta(p,q)$ can be seen as a C-equivalence. Therefore for each theory $T \in \mathrm{Th}(C)$, the relation $\equiv_T$:

$(**)$ $\alpha \equiv_T \beta$ if and only if $\Delta(\alpha,\beta) \subseteq C(T)$

defines a congruence on the language L compatible with T. Given a matrix $\mathfrak{M} = (\mathcal{A}, D)$ for $\mathcal{L}$ we will define the *polynomial relation* $\Delta_{\mathfrak{M}}$ in $\mathfrak{M}$ as $a\Delta_{\mathfrak{M}}b$ if and only if $\gamma_{\mathfrak{M}}(a,b) \in D$ for every $\gamma \in \Delta$, where $\gamma_{\mathfrak{M}}$ is the polynomial over $\mathcal{A}$ corresponding to a formula γ. Then we have the following proposition.

Proposition 6.1.9 *Let $\mathfrak{M} = (\mathcal{A}, D)$ be a matrix for $\mathcal{L}$ and $\Delta_{\mathfrak{M}}$ the polynomial relation in $\mathfrak{M}$. Then we have:*

(i) *if $\Delta_{\mathfrak{M}}$ is reflexive then $\Theta_{\mathfrak{M}} \subseteq \Delta_{\mathfrak{M}}$,*

(ii) *if $\Delta_{\mathfrak{M}}$ is a matrix congruence of $\mathfrak{M}$, then it is the greatest matrix congruence of $\mathfrak{M}$, i.e., $\Delta_{\mathfrak{M}} = \Theta_{\mathfrak{M}}$.*

Theorem 6.1.10 *A deductive system $\mathfrak{L} = (\mathcal{L}, C)$ is equivalential relative to a set $\Delta(p, q)$ of formulas if and only if for any matrix $\mathfrak{M} \in$ Matr(C), $\Delta_{\mathfrak{M}} = \Theta_{\mathfrak{M}}$.*

A deductive system $\mathfrak{L} = (\mathcal{L}, C)$ is called *1-equivalential* if it is equivalential and so it has a set $\Delta(p, q)$ of formulas, and $p, q/\Delta(p, q)$ (called G-rule) is a set of rules of C for any $\Delta(p, q)$. In other word, 1-equivalential systems are equivalential systems in which the members of an arbitrary theory T are all identified under the congruence relation generated by T.

6.1.3 Congruence operators

The congruence ΩT being assigned by the Leibniz operator Ω for any theory $T \subseteq L$, is the synonymy relation on L relative to T. Thus

$$\alpha \equiv \beta \quad (\text{mod } \Omega T) \text{ if and only if } \bigwedge_{\gamma \in L} \bigwedge_{p \in Var(\gamma)} (\gamma[\alpha/p] \in T \Leftrightarrow \gamma[\beta/p] \in T),$$

where $Var(\gamma)$ is the set of variables occurring in γ. Here ΩT is the greatest congruence on L compatible with T. The definition of ΩT is related to the well-known method of defining the equality relation in second order logic that goes back to Leibniz. For this reason ΩT is called the *Leibniz congruence* associated with T, and the operator Ω, assigning the congruence ΩT to each theory T in L, is called the *Leibniz operator*. In metalogic, the format of the operator Ω is restricted by admitting that the domain of Ω is the family of all theories of a given system C, and this restricted Leibniz operator thus assigns the congruence ΩT to each closed theory $T \in \text{Th}(C)$. The hierarchy of deductive systems outlined below directly refers to the following list of properties of the Leibniz operator Ω. Here C is assumed to be a fixed sentential system and $T, T_1, T_2, T_i (i \in I)$ range over arbitrary theories of C (see [20] and [33]).

(1) $T_1 \subseteq T_2$ implies $\Omega T_1 \subseteq \Omega T_2$ (monotonicity)

(2) $\Omega T_1 = \Omega T_2$ implies $T_1 = T_2$ (injectivity)

(3) For all directed system $T_i (i \in I)$ such that the union $\bigcup\{T_i : i \in I\}$ is a theory of C, $\Omega \bigcup\{T_i : i \in I\} = \bigcup\{\Omega T_i : i \in I\}$ (continuity)

(4) $\Omega \bigcap\{T_i : i \in I\} = \bigcap\{\Omega T_i : i \in I\}$ (meet-continuity)

(5) For every substitution e, $\Omega e^{-1} T = e^{-1} \Omega T$ (commutativity with inverse substitutions)

Then we have the following characteristic theorems for protoalgebraic, equivalential and algebraizable logics which are mentioned so far (see [20] and [33]). The class hierarchy of degrees of algebraization is also shown in Fig 6.1. A system is located above another one if it is stronger than the other. In this figure *implicative* deductive systems were firstly introduced by Rasiowa in [57]. Every implicative system can be seen equivalential relative to $\Delta(\alpha, \beta) = \{a \to \beta, \beta \to \alpha\}$, where $\to$ is a material implication, but the converse does not hold in general.

Theorem 6.1.11 *For any system $\mathfrak{L} = (\mathcal{L}, C)$ the following conditions are equivalent:*

(i) *C is protoalgebraic,*

(ii) *for all $T \in \mathrm{Th}(C)$, $\alpha, \beta \in L$, $\alpha \equiv \beta \pmod{\Omega C(T)}$ implies $C(T, \alpha) = C(T, \beta)$,*

(iii) *the Leibniz operator Ω is meet-continuous on $\mathrm{Th}(C)$,*

(iv) *there exists a set $\Delta(p, q)$ of formulas with two distinct variables such that for any $\alpha, \beta \in L$, $\Delta(\alpha, \alpha) \subseteq C(\emptyset)$ and $\beta \in C(\Delta(\alpha, \beta); \alpha)$.*

Theorem 6.1.12 *For any system $\mathfrak{L} = (\mathcal{L}, C)$ the following conditions are equivalent:*

(i) *C is equivalential,*

(ii) *the Leibniz operator Ω is monotonic and commutes with inverse substitutions on $\mathrm{Th}(C)$, i.e., $e^{-1}\Omega T \subseteq \Omega e^{-1}T$ for any substitution e in L and any $T \in \mathrm{Th}(C)$,*

(iii) *Ω is monotonic and $e\Omega T \subseteq \Omega C(eT)$ for all substitutions e and all $T \in \mathrm{Th}(C)$.*

Theorem 6.1.13 *For any system $\mathfrak{L} = (\mathcal{L}, C)$ the following conditions are equivalent:*

(i) *C is finitely equivalential,*

(ii) *the Leibniz operator Ω is continuous on $\mathrm{Th}(C)$.*

Theorem 6.1.14 *For any system $\mathfrak{L} = (\mathcal{L}, C)$ the following conditions are equivalent:*

(i) *C is algebraizable,*

(ii) *the Leibniz operator Ω is injective, monotonic and commutes with inverse substitutions on $\mathrm{Th}(C)$,*

(iii) *C is equivalential and Ω is injective on $\mathrm{Th}(C)$.*

Theorem 6.1.15 *For any system $\mathfrak{L} = (\mathcal{L}, C)$ the following conditions are equivalent:*

(i) *C is finitely algebraizable,*

(ii) *the Leibniz operator Ω is injective and continuity.*

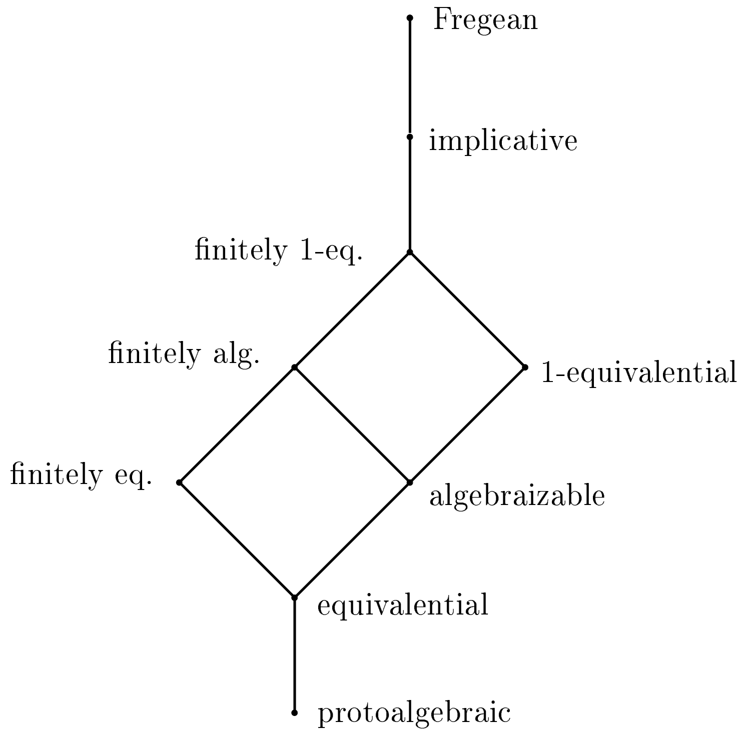

Figure 6.1: The class hierarchy of degrees of algebraization

The Fregean axiom (FA) being mentioned in Section 2.2, leads to distinguishing the class of Fregean deductive system. Formally, a protoalgebraic system C is called *Fregean* if C is not almost inconsistent, i.e., $C(\emptyset) \neq \emptyset$ and $C(X) \neq L$ for any nonempty $X \subseteq L$, and the Leibniz operator Ω satisfies the following condition: for any $T \cup \{\alpha, \beta\} \subseteq L$, $\alpha \equiv \beta \pmod{\Omega C(T)}$ if and only if $C(T; \alpha) = C(T; \beta)$. For example, classical and intuitionistic logics are Fregean since the above condition reduces to the well-known Tarski's condition: for any $T \cup \{\alpha, \beta\} \subseteq L$, $\alpha \leftrightarrow \beta \in C(T)$ if and only if $C(T; \alpha) = C(T; \beta)$ (see [55] and [20]).

The class of protoalgebraic system is too restrictive because there exists at least a variety which can not be characterized by the class of protoalgebraic, e.g., the conjunction-disjunction fragment of classical logic. As the alternation of Leibniz operator which overcome this restriction, we can consider the general notion of an operator from [20], in particular, the Suszko operator \$ which maps a theory of C to the greatest congruence on L that has a certain interderivability property. For every theory $T \subseteq L$ we define the binary relation $\$T$ on L by means of the condition:

$$\alpha \equiv \beta \pmod{\$T} \text{ if and only if } \bigwedge_{\gamma \in L} \bigwedge_{p \in Var(\gamma)} C(T; \gamma[\alpha/p]) = C(T; \gamma[\beta/p]),$$

where $C(T; \gamma[\alpha/p]) = C(T; \gamma[\beta/p])$ if and only if $\gamma[\beta/p] \in C(T; \gamma[\alpha/p])$ and $\gamma[\alpha/p] \in C(T; \gamma[\beta/p])$. So as previous mentioned, $\gamma[\alpha/p]$ and $\gamma[\beta/p]$ are T-interderivable relative to a system C. A congruence Θ on L is said to have the *T-interderivability property* relative to a system C if $\alpha \equiv \beta \pmod{\Theta}$ implies $C(T;\alpha) = C(T;\beta)$ for any $\alpha, \beta \in L$.

The definition of $\$T$ is strictly relativised to the logic C and, unlike the definition of the Leibniz congruence, it does not have the absolute character. $\$T$ can be shown to be a congruence on L compatible with T. Therefore $\$T \subseteq \Omega T$ for all $T \in \mathrm{Th}(C)$ and this inclusion may be proper unless C is protoalgebraic. The congruence $\$T$ is called the *Suszko congruence* corresponding the theory T. The operator $\$$ which to each theory $T \in \mathrm{Th}(C)$ assigns the congruence $\$T$ is called the *Suszko operator*. The condition on the right hand side of the above definition was used by Suszko to define the identity connective in **SCI**. It follows from the definition of $\$$ that, for any deductive system, not necessarily protoalgebraic, the operator $\$$ is monotonic on $\mathrm{Th}(C)$, i.e., $\$T_1 \subseteq \$T_2$ whenever $T_1 \subseteq T_2$.

Theorem 6.1.16 *For any system $\mathfrak{L} = (\mathcal{L}, C)$ the following conditions are equivalent:*

(i) *C is protoalgebraic,*

(ii) *For all $T \in \mathrm{Th}(C)$, $\$T = \Omega T$.*

A deductive system C is said to have the *strong congruence property* if for all $\alpha, \beta \in L$, all $T \in \mathrm{Th}(C)$ the T-interderivable relation $C(T;\alpha) = C(T;\beta)$ is not only an equivalence relation but also a congruence relation, namely $\alpha \equiv \beta \pmod{\$T}$ if and only if $C(T;\alpha) = C(T;\beta)$.

Theorem 6.1.17 *For any system $\mathfrak{L} = (\mathcal{L}, C)$ the following conditions are equivalent:*

(i) *C is Fregean,*

(ii) *C is protoalgebraic and has the strong congruence property.*

6.1.4 The case of PCI logics

In this subsection we will consider the algebraization of **PCI** logics. The deductive system **SCI**, which was introduced in Section 2.2, is equivalential relative to a set $\Delta(p,q) := \{p \equiv q\}$ since identity axioms IDA of **SCI** satisfy the conditions from (i) to (v) in Definition 6.1.8. This fact is also confirmed by Theorem 6.1.12. But **SCI** is not algebraizable in the sense of Definition 6.1.6 since the Leibniz operator is not injective on all theories of **SCI**. Next we will consider the algebraization of **PCI** and $\mathbf{PCI}_{\mathrm{K}}$. At first we have the following conjecture, which make a contrast with the fact that **SCI** is protoalgebraic.

Conjecture 6.1.18 *The deductive system* $\mathbf{PCI} = (\mathrm{L_P}, C)$ *is not protoalgebraic.*

Theorem 6.1.19 *The deductive system* $\mathbf{PCI}_{\mathrm{K}} = (\mathrm{L_P}, C^{\mathrm{G}}_{\mathrm{K}})$ *is 1-equivalential.*

Proof. Let $\Delta(p,q) := \{p \leftrightarrow q\}$. Then at first we will show that $\mathbf{PCI}_{\mathrm{K}}$ is equivalential relative to $\Delta(p,q)$. In Definition 6.1.8, conditions (i)-(iii) and (v) are obviously satisfied in $\mathbf{PCI}_{\mathrm{K}}$ relative to $\Delta(p,q)$. For the condition (iv) in Definition 6.1.8, we have only to show that $\leftrightarrow$ is also congruence relation. As $\mathbf{PCI}_{\mathrm{K}}$ is a conservative extension of the classical logic $\mathbf{CL}$, this is almost obvious except that $(A \leftrightarrow B) \wedge (C \leftrightarrow D) \in \mathbf{PCI}_{\mathrm{K}}$ implies $(A \equiv C) \leftrightarrow (B \equiv D) \in \mathbf{PCI}_{\mathrm{K}}$. But we have the following derivations:

$$
\begin{aligned}
&(A \leftrightarrow B) \wedge (C \leftrightarrow D) \in \mathbf{PCI}_{\mathrm{K}} \\
&\qquad \Longrightarrow (A \leftrightarrow B) \leftrightarrow (C \leftrightarrow D) \in \mathbf{PCI}_{\mathrm{K}}, \\
&\qquad \Longrightarrow (A \leftrightarrow C) \leftrightarrow (B \leftrightarrow D) \in \mathbf{PCI}_{\mathrm{K}}, \\
&\qquad \Longrightarrow ((A \leftrightarrow C) \leftrightarrow (B \leftrightarrow D)) \equiv \top \in \mathbf{PCI}_{\mathrm{K}}, && \text{(G rule in Section 4.2)} \\
&\qquad \Longrightarrow ((A \leftrightarrow C) \equiv \top) \leftrightarrow ((B \leftrightarrow D) \equiv \top) \in \mathbf{PCI}_{\mathrm{K}}, && \text{(Lemma 4.2.2 (xiv))} \\
&\qquad \Longrightarrow (A \equiv C) \leftrightarrow (B \equiv D) \in \mathbf{PCI}_{\mathrm{K}}. && \text{(Lemma 4.2.2 (xvi))}
\end{aligned}
$$

Thus $\leftrightarrow$ is a congruence relation on $\mathbf{PCI}_{\mathrm{K}}$. So the condition (iv) in definition 6.1.8 also holds in $\mathbf{PCI}_{\mathrm{K}}$, and we conclude that $\mathbf{PCI}_{\mathrm{K}}$ is equivalential. Here we have also that $A \wedge B \in \mathbf{PCI}_{\mathrm{K}}$ implies $A \leftrightarrow B \in \mathbf{PCI}_{\mathrm{K}}$. Hence $\mathbf{PCI}_{\mathrm{K}}$ is 1-equivalential.

■

6.2 Varieties of PCI algebras

In this section we will show that the class of $\mathbf{PCI}_{\mathrm{K}}$-algebras forms a variety. In general, the class of all algebras of the same similarity type is called a *variety of algebras* of this type if all identities in a given set X are valid in this class, and denoted by $\mathrm{Va}(X)$. Here if X is a set of $\mathbf{PCI}_{\mathrm{K}}$ formulas then $\mathrm{Va}(X)$ means the variety of $\mathbf{PCI}_{\mathrm{K}}$-algebras generated by the identities $A = \mathrm{t}$ such that for all $A \in X$.

Theorem 6.2.1 *The deductive system* $\mathbf{PCI}_{\mathrm{K}} = (\mathrm{L_P}, C^{\mathrm{G}}_{\mathrm{K}})$ *forms a variety.*

Proof. At first we recall that $\mathcal{A}_{\mathrm{K}} = \langle A, -, \cap, \cup, \supset, \Delta, \mathrm{f}, \mathrm{t}\rangle$ is a $\mathbf{PCI}_{\mathrm{K}}$-algebra introduced in 4.5. Let $\mathcal{V}_{\mathrm{K}}$ be the class of all $\mathbf{PCI}_{\mathrm{K}}$-algebras. Then since the Representation Theorem 4.5.8 of $\mathbf{PCI}_{\mathrm{K}}$-algebras and Theorem 4.5.10, there exists a 1-1 correspondence between $\mathbf{PCI}_{\mathrm{K}}$ Kripke model $\mathcal{M} = (W, R, V)$ and $\mathbf{PCI}_{\mathrm{K}}$ matrix model $\mathcal{M}_{\mathrm{K}} = (\mathcal{A}_{\mathrm{K}}, \{\mathrm{t}\})$. Furthermore, as we know in Theorem 4.4.3 that $\mathbf{PCI}_{\mathrm{K}}$ logic is complete with respect to $\mathbf{PCI}_{\mathrm{K}}$ Kripke model, so by composing the above results we conclude that for any $X \cup \{B\} \subseteq \mathrm{L_P}$ and every valuation $v : \mathrm{L_P} \to A$, $B \in C^{\mathrm{G}}_{\mathrm{K}}(X)$ if and only if for any $\mathcal{A}_{\mathrm{K}} \in \mathcal{V}_{\mathrm{K}}$ such that $\mathcal{A}_{\mathrm{K}} \models \{v(D) = \mathrm{t}; \forall D \in X\}$ implies $\mathcal{A}_{\mathrm{K}} \models v(B) = \mathrm{t}$. Here we define the class $X^* = \{\mathcal{A}_{\mathrm{K}} \in \mathcal{V}_{\mathrm{K}}; \mathcal{A}_{\mathrm{K}} \models v(D) = \mathrm{t} \text{ for all } D \in X\}$. Then this class forms clearly a variety, and characterizes the logic $C^{\mathrm{G}}_{\mathrm{K}}(X)$. Moreover, the function $\mathrm{L_P} \mapsto \mathrm{L_P}^*$, which assigns each extension of $\mathbf{PCI}_{\mathrm{K}}$ to a variety of $\mathbf{PCI}_{\mathrm{K}}$-algebras, is 1-1. On the other hand, if $\mathcal{K} \subseteq \mathrm{Va}(\mathcal{V}_{\mathrm{K}})$, then $\mathcal{K}^+ = \{B \in \mathrm{L_P}; \forall \mathcal{A}_{\mathrm{K}} \in \mathcal{K}, \mathcal{A}_{\mathrm{K}} \models v(B) = \mathrm{t}\}$ can be seen an extension of $\mathbf{PCI}_{\mathrm{K}}$ logic, in which satisfies $\mathcal{K}^{+*} = \mathcal{K}$ because every equation $v(B) = v(D)$ in $\mathcal{K}^+$ can be written $(-v(B) \cup v(D)) \cap (v(B) \cup -v(D)) = \mathrm{t}$. Thus there exists a 1-1 correspondence between the family of extensions of $\mathbf{PCI}_{\mathrm{K}}$ logic and that of subvarieties of $\mathrm{Va}(\mathcal{V}_{\mathrm{K}})$.

■

6.3 Equationally definable principal congruences

6.3.1 General theory of EDPC

The algebraization of a deductive system is usually accomplished by transforming each well-formed formulas into terms of an appropriate functional language. Then logical connectives and atomic formulas in the deductive system are replaced by operator symbols and individual constants in the corresponding algebra, respectively. Furthermore, the transfornations determine which pairs of terms are to be identified in the corresponding algebras. Namely, by means of their equivalence connective the axioms and rules of inference of the deductive system take the form of equations. In this way various metalogical properties of the deductive system translate into algebraic and metamathematical properties of its associated variety. The algebraic analogue of the deduction process is congruence generation, and the algebraic analogue for a variety of the deduction theorem is the ability to represent congruence generation by means of equations (see [39], [5] and [6]).

Definition 6.3.1 *A variety $\mathcal{V}$ is said to have* EDPC *if there exist 4-ary terms $p_i(x, y, z, w)$, $q_i(x, y, z, w)$, $i = 0, \ldots, n-1$ for some natural number n, such that for every algebra $\mathcal{A} \in \mathcal{V}$, and all $a, b, c, d \in A$,*

$$c \equiv d \pmod{\Theta(a, b)} \text{ if and only if } \mathcal{A} \models p_i(a, b, c, d) = q_i(a, b, c, d) \quad 0 \leq i < n,$$

where $\theta(a, b)$ is the principal congruence generated by the pair a and b, and also $c \equiv d$ (mod $\theta(a, b)$) means that c and d are congruent under the relation $\theta(a, b)$.

A *Brouwerian semilattice* is an algebra $\langle A, \cdot, \rightarrow, 1\rangle$ such that $\langle A, \cdot, 1\rangle$ is a meet semilattice with greatest element 1, and $a \rightarrow b$ is a *pseudo-complement of a relative to b*, i.e., for all $c \in A$, $c \leq a \rightarrow b$ if and only if $a \cdot c \leq b$. A semilattice $\langle A, \cdot, 1\rangle$ is called a *semilattice with relative pseudo-complementation* if $a \rightarrow b$ exists for all $a, b \in A$. By a *dual Brouwerian semilattice* we mean an algebra $\langle A, +, *, I\rangle$ such that $\langle A, +, I\rangle$ is a join semilattice with least element I and $a * b$ is a *dual relative pseudo-complement*, i.e., for all $c \in A$, $a * b \leq c$ if and only if $b \leq a + c$. If we write $\mathrm{Cp}(\mathcal{A})$ as the set of all finitely generated congruences (called *compact congruences*) of an algebra $\mathcal{A}$, then it is well-known the following results (see [5] and [55]).

Theorem 6.3.2 *A variety $\mathcal{V}$ has* EDPC *if and only if for every $\mathcal{A} \in \mathcal{V}$, $\langle \mathrm{Cp}(\mathcal{A}), +, \mathrm{I}\rangle$ is dual Brouwerian semilattice.*

Let $\mathcal{V}$ be any class of algebras with distinguished constant 1, and $\mathcal{A} \in \mathcal{V}$. Then a subset F of A is called a *1-filter* of $\mathcal{A}$ if $F = 1/\Theta$ $(:= \{a; a \equiv 1 \pmod{\Theta}\})$ for some $\Theta \in \mathrm{Co}(\mathcal{A})$. The 1-filter generated by a subset X of A is the intersection of all 1-filters including X, denoted by Fi(X). Moreover, if the set X is finite, then it is called a *compact 1-filter* and denoted by Fp(X). Then for any class $\mathcal{V}$ of algebras with distinguished constant 1, we can introduce the class of matrices $\mathrm{M}_{\mathcal{V}} = \{(\mathcal{A}, F); \mathcal{A} \in \mathcal{V}, F$ is a 1-filter of A$\}$ which constructs the deductive system $\mathfrak{A} = (\mathcal{A}, C_{\mathrm{M}_{\mathcal{V}}})$ by the following way: for any $X \cup \{a\} \subseteq A$, $a \in C_{\mathrm{M}_{\mathcal{V}}}(X)$ if and only if for any matrix $\mathfrak{M} = (\mathcal{A}, F) \in \mathrm{M}_{\mathcal{V}}$, $a \in F$ whenever $X \subseteq F$.

Definition 6.3.3 *Let $\mathcal{V}$ be any variety with distinguished constant* 1*, $\mathcal{A} \in \mathcal{V}$ and $\mathfrak{A} = (\mathcal{A}, C_{\mathrm{M}_{\mathcal{V}}})$ its deductive system. Then a binary term $\rightarrow$ in $\mathcal{A}$ is called* conditional *for $\mathfrak{A} = (\mathcal{A}, C_{\mathrm{M}_{\mathcal{V}}})$ if the following conditions hold:*

(i) *for all $x, y \in A$, $y \in C_{\mathrm{M}_{\mathcal{V}}}(x, x \rightarrow y)$,* (*Modus ponens*)

(ii) *for all $X \cup \{x, y\} \subseteq A$, $y \in C_{\mathrm{M}_{\mathcal{V}}}(X; x)$ implies $x \rightarrow y \in C_{\mathrm{M}_{\mathcal{V}}}(X)$.* (*Deduction theorem*)

Then we have the following relationship between a conditional and EDPC in the deductive system $\mathfrak{A}$ (see [55]).

Theorem 6.3.4 *Assume that $\mathcal{V}$ is a variety in which each compact 1-filter is principal. Then for any $\mathcal{A} \in \mathcal{V}$, $\mathfrak{A} = (\mathcal{A}, C_{\mathrm{M}_{\mathcal{V}}})$ has a conditional if and only if $\mathcal{V}$ has* EDPC.

We will show some typical examples of varieties occurring in the literature that are known to have EDPC.

Example 6.3.5 *Hilbert algebras $\langle A, \rightarrow, 1\rangle$ are defined by the following equations:*

(i) $a \rightarrow (b \rightarrow a) = 1,$

(ii) $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1,$

(iii) *if $a \rightarrow b = 1$ and $b \rightarrow a = 1$ then $a = b$,*

(iv) $a \rightarrow 1 = 1.$

This algebra is also called positive implication *algebras. Let $p(x, y, z) = (x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow z)$. Then for any Hilbert algebra $\mathcal{A}$, $a, b, c, d \in A$, we have $c \equiv d \pmod{\Theta(a, b)}$ if and only if $p(a, b, c) = p(a, b, d)$. Thus the variety of Hilbert algebras has* EDPC.

Example 6.3.6 *Heyting algebras $\langle A, +, \cdot, \rightarrow, 0, 1\rangle$ where $\langle A, +, \cdot, 0, 1\rangle$ is a bounded distributive lattice and $\rightarrow$ is relative pseudo-complementation. This algebra is also called* pseudo-Boolean *algebras. We can get the same result as the previous example, i.e., for $p(x, y, z)$ within example 6.3.5, $a, b, c, d \in A$, we have $c \equiv d \pmod{\Theta(a, b)}$ if and only if $p(a, b, c) = p(a, b, d)$. Thus the variety of Heyting algebras has* EDPC.

Example 6.3.7 *Modal algebras* $\langle A, +, \cdot, -, \Box, 0, 1\rangle$ *where* $\langle A, +, \cdot, -, 0, 1\rangle$ *is a Boolean algebras and* $\Box$ *a modal operator satisfying*

(i) $\Box(x \cdot y) = \Box x \cdot \Box y$,

(ii) $\Box 1 = 1$.

Let $\mathcal{V}_n$ *be the variety of modal algebras satisfying the equation* $\underbrace{\Box \cdots \Box}_{n} x \le \Box(\underbrace{\Box \cdots \Box}_{n})x$, *where* $\underbrace{\Box \cdots \Box}_{0} x = x$, $\underbrace{\Box \cdots \Box}_{n} x = x \cdot \Box(\underbrace{\Box \cdots \Box}_{n-1} x)$, $n = 1, 2, \ldots$. *Then for any* $\mathcal{A} \in \mathcal{V}_n$, $a, b, c, d \in A$, *we have* $c \equiv d \pmod{\Theta(a, b)}$ iff $\underbrace{\Box \cdots \Box}_{n}(a\Delta b) \cdot c = \underbrace{\Box \cdots \Box}_{n}(a\Delta b) \cdot d$. *Here* $a\Delta b$ *is the dual difference* $(-a + b) \cdot (-b + a)$. *Thus* $\mathcal{V}_n$ *has* EDPC.

6.3.2 EDPC property of PCI varieties

In this subsection we will formulate a necessary and sufficient condition for a variety of $\mathbf{PCI}_{\mathrm{K}}$-algebras to have EDPC property (see [36]). To show this property we will first introduce a unary operator r on a $\mathbf{PCI}_{\mathrm{K}}$-algebra defined by

$$r(x) = (x\Delta \mathrm{t}) \cap x.$$

Moreover, let :

$$r^0(x) = x,$$
$$r^{n+1}(x) = r(r^n(x)).$$

Definition 6.3.8 *A nonempty subset* F *of a* $\mathbf{PCI}_{\mathrm{K}}$*-algebra* $\mathcal{A}_{\mathrm{K}}$ *is* a congruence filter *if*

(1) *F is a Boolean filter and,*

(2) *F is closed under r, i.e., $x \in F$ implies $r(x) \in F$.*

By the above definition, $x \in F$ implies $x\Delta \mathrm{t} \in F$ for any congruence filter F of a $\mathbf{PCI}_{\mathrm{K}}$-algebra. Moreover, we can show that F is closed under Δ, i.e., $x, y \in F$ implies $x\Delta y \in F$, by using the symmetry and the transitivity of Δ. We notice that the above congruence filter of a $\mathbf{PCI}_{\mathrm{K}}$-algebra is clearly identical to a $\mathbf{PCI}_{\mathrm{K}}$-filter defined in Section 4.5.

Proposition 6.3.9 *Let* $\mathcal{A}_{\mathrm{K}}$ *be a* $\mathbf{PCI}_{\mathrm{K}}$*-algebra. Then, there exists an isomorphism between the lattice of congruence filters of* $\mathcal{A}_{\mathrm{K}}$ *and the lattice of congruences on* $\mathcal{A}_{\mathrm{K}}$*. Namely, let* θ *be a congruence on* $\mathcal{A}_{\mathrm{K}}$*. Then*

$$\mathbf{F}(\theta) = \{a \supset\subset b; \langle a, b\rangle \in \theta\}$$

is a congruence filter of $\mathcal{A}_{\mathrm{K}}$*, where* $a \supset\subset b$ *means* $(a \supset b) \cap (b \supset a)$*. Conversely, if* F *is a congruence filter of* $\mathcal{A}_{\mathrm{K}}$*, then*

$$\Theta(F) = \{\langle a, b\rangle; a \supset\subset b \in F\}$$

is a congruence on $\mathcal{A}_{\mathrm{K}}$*. Moreover,* $\mathbf{F}(\Theta(F)) = F$ *and* $\Theta(\mathbf{F}(\theta)) = \theta$*.*

Proof. Let θ be a congruence on an $\mathbf{PCI}_{\mathrm{K}}$-algebra $\mathcal{A}_{\mathrm{K}}$. Then, we have to check whether $\mathbf{F}(\theta) = \{a \supset\subset b; \langle a, b\rangle \in \theta\}$ is a congruence filter of $\mathcal{A}_{\mathrm{K}}$. So, there are four things to check as follows:

(1) $\mathrm{t} \in \mathbf{F}(\theta)$,

(2) $x, y \in \mathbf{F}(\theta)$ implies $x \cap y \in \mathbf{F}(\theta)$,

(3) $x \in \mathbf{F}(\theta), x \leq y$ implies $y \in \mathbf{F}(\theta)$,

(4) $x \in \mathbf{F}(\theta)$ implies $x\Delta\mathrm{t} \in \mathbf{F}(\theta)$.

(1) : Notice that $a \supset\subset a = (a \supset a) \cap (a \supset a) = \mathrm{t} \cap \mathrm{t} = \mathrm{t}$. So, by reflexivity of θ we get $a \supset\subset a \in \mathbf{F}(\theta)$, i.e., $\mathrm{t} \in \mathbf{F}(\theta)$.

(2) : Assume $x, y \in \mathbf{F}(\theta)$, then $x = a \supset\subset b$ and $y = c \supset\subset d$ for some $\langle a, b\rangle, \langle c, d\rangle \in \theta$. We have to show $x \cap y = e \supset\subset f$ for some $\langle e, f\rangle \in \theta$. First notice that (i) $\langle a, b\rangle \in \theta$ iff $\langle a \cap b, a \cup b\rangle \in \theta$, and moreover (ii) $a \supset\subset b = a \cup b \supset a \cap b$. To prove the former, if $\langle a, b\rangle \in \theta$ then from $\langle a, a\rangle \in \theta$ and the definition of congruence relation, $\langle a \cup a, a \cup b\rangle \in \theta$ and $\langle a \cap a, a \cap b\rangle \in \theta$. By $a \cup a = a = a \cap a$ and using the aboves, we get $\langle a, a \cup b\rangle \in \theta$ and $\langle a, a \cap b\rangle \in \theta$. So

$$\begin{aligned}
&\langle a, a \cup b\rangle, \langle a, a \cap b\rangle \in \theta \\
&\quad \Longrightarrow \langle a \cap b, a\rangle, \langle a, a \cup b\rangle \in \theta && \text{(symmetry)} \\
&\quad \Longrightarrow \langle a \cap b, a \cup b\rangle \in \theta. && \text{(transitivity)}
\end{aligned}$$

Conversely, assume $\langle a \cap b, a \cup b\rangle \in \theta$. Then, we have $a \cap b \leq a \leq a \cup b$, so $\langle a \cap b, a\rangle \in \theta$ and similarly $\langle a \cap b, b\rangle \in \theta$. Thus

$$\begin{aligned}
&\langle a \cap b, a\rangle, \langle a \cap b, b\rangle \in \theta \\
&\quad \Longrightarrow \langle a, a \cap b\rangle, \langle a \cap b, b\rangle \in \theta && \text{(symmetry)} \\
&\quad \Longrightarrow \langle a, b\rangle \in \theta. && \text{(transitivity)}
\end{aligned}$$

Next we will show the latter:

$$\begin{aligned}
a \supset\subset b &= (a \supset b) \cap (b \supset a) = (-a \cup b) \cap (-b \cup a) \\
&= ((-a \cup b) \cap -b) \cup ((-a \cup b) \cap a) \\
&= ((-a \cap -b) \cup (b \cap -b)) \cup ((-a \cap a) \cup (a \cap b)) \\
&= (-a \cap -b) \cup (a \cap b) = -(a \cup b) \cup (a \cap b) \\
&= a \cup b \supset a \cap b.
\end{aligned}$$

To continue the proof of case (2), take the element e as

$$\begin{aligned}
e &= (a \cup b) \cap x \cap (c \cup d) \cap y \\
&= (a \cup b) \cap (a \cup b \supset a \cap b) \cap (c \cup d) \cap (c \cup d \supset c \cap d).
\end{aligned}$$

Then, we have $(a \cap b)\theta(a \cup b)$ and $(a \cap b)\theta(a \cap b)$. Thus $(a \cap b \supset a \cap b)\theta(a \cup b \supset a \cap b)$, i.e., $\mathrm{t}\theta(a \cup b \supset a \cap b)$. Similarly, we get $\mathrm{t}\theta(c \cup d \supset c \cap d)$. From $\mathrm{t}\theta(a \cup b \supset a \cap b)$ and $(a \cup b)\theta\,(a \cup b)$, we get $((a \cup b) \cap \mathrm{t})\theta((a \cup b) \cap (a \cup b \supset a \cap b))$, i.e., $(a \cup b)\theta((a \cup b) \cap(a \cup b \supset a \cap b))$. And similarly $(c \cup d)\theta((c \cup d) \cap (c \cup d \supset c \cap d))$. Thus $((a \cup b)\cap$

$(c\cup d))\theta((a\cup b)\cap(a\cup b\supset a\cap b)\cap(c\cup d)\cap(c\cup d\supset c\cap d))$, i.e., $((a\cup b)\cap(c\cup d))\theta e$. Moreover, we have $(a\cap b)\theta(a\cup b)$ and $(c\cap d)\theta(c\cup d)$. Thus $((a\cap b)\cap(c\cap d))\theta((a\cup b)$ $\cap(c\cup d))$. From the aboves we have $((a\cap b)\cap(c\cap d))\theta((a\cup b)\cap(c\cup d))\theta e$. Also we have $(a\cup b)\cap(c\cup b)\cap(x\cap y)=e$ iff $x\cap y=(a\cup b)\cap(c\cup d)\supset e$. Hence $x\cap y=(a\cup b)\cap(c\cup d)\supset e$ and we are done.

(3) : Assume $x\in\mathbf{F}(\theta)$ and $x\leq y$. Then $x=a\supset\subset b$ for some $\langle a,b\rangle\in\theta$. We have to show that $y=c\supset\subset d$ for some $\langle c,d\rangle\in\theta$. First notice that $x=a\supset\subset b=a\cup b\supset a\cap b$. Also $x\geq a\cap b$ because $x=a\cup b\supset a\cap b=-(a\cup b)\cup(a\cap b)\geq(a\cap b)$. And $x\cup(a\cup b)=\mathrm{t}$ because $x\cup(a\cup b)=(a\cup b\supset a\cap b)\cup(a\cup b)=-(a\cup b)\cup(a\cap b)\cup$ $(a\cup b)=\mathrm{t}$. Now, because y is being between x and t, take the element $e=(a\cup b)\cap y$. Then clearly $e\leq a\cup b$ because $e=(a\cup b)\cap y\leq(a\cup b)$ by the assumption $x\leq y$. Also from $a\cap b\leq a\leq a\cup b$ and $a\cap b\leq x\leq y$ we get $a\cap b\leq(a\cup b)\cap y=e$. Thus $a\cap b\leq e\leq a\cup b$. Also we have $(a\cup b)\cap y=e$ iff $y=a\cap b\supset e$. Hence $y=a\cap b\supset\subset e$ and we are done.

(4) : Assume $x\in\mathbf{F}(\theta)$ then $x=a\supset\subset b=a\cup b\supset a\cap b$ for some $\langle a,b\rangle\in\theta$. By the definition of $r(x)$, $r(x)=(x\Delta\mathrm{t})\cap x=((a\cup b\supset a\cap b)\Delta\mathrm{t})\cap(a\cup b\supset a\cap b)$. Then, we have to show $r(x)=c\supset\subset d$ for some $\langle c,d\rangle\in\theta$. Notice that $r(x)\leq x$ and define $e=a\cap b\cap r(x)$. We will show $e\theta(a\cap b)$. First, we know $a\theta b$ iff $(a\cap b)\theta(a\cup b)$. So, $e=a\cap b\cap r(x)=(a\cap b)\cap((a\cup b\supset a\cap b)\Delta\mathrm{t})\cap(a\cup b\supset a\cap b)$. Notice that we have easily:

$$\begin{aligned}(a\cap b)\cap(a\cup b\supset a\cap b)&=(a\cap b)\cap(-(a\cup b)\cup(a\cap b))\\&=(a\cap b)\cap((-a\cap -b)\cup(a\cap b))\\&=((a\cap b)\cap(-a\cap -b))\cup(a\cap b)=(a\cap b),\end{aligned}$$

so $e=(a\cap b)\cap((a\cup b\supset a\cap b)\Delta\mathrm{t})$. Now, because we have $(a\cap b)\theta(a\cup b)$ and $(a\cap b)$ $\theta(a\cap b)$, we get $(a\cap b\supset a\cap b)\theta(a\cup b\supset a\cap b)$, i.e., $\mathrm{t}\theta(a\cup b\supset a\cap b)$. And obviously $\mathrm{t}\theta\mathrm{t}$, so also $\mathrm{t}\Delta\mathrm{t}\theta(a\cup b\supset a\cap b)\Delta\mathrm{t}$, i.e., $\mathrm{t}\theta(a\cup b\supset a\cap b)\Delta\mathrm{t}$. From $(a\cap b)\theta(a\cap b)$ and the above we get $((a\cap b)\cap\mathrm{t})\theta((a\cap b)\cap((a\cup b\supset a\cap b)\Delta\mathrm{t}))$, i.e., $(a\cap b)\theta((a\cap b)$ $\cap((a\cup b\supset a\cap b)\Delta\mathrm{t}))$. So $(a\cap b)\theta e$. Also we have $(a\cap b)\cap r(x)=e$ iff $r(x)=a\cap b\supset$ e. Hence $r(x)=a\cap b\supset\subset e$ and we are done.

Conversely, let F be a congruence filter of $\mathcal{A}_{\mathrm{K}}$. Then we have to check whether $\Theta(F)=\{\langle a,b\rangle;a\supset\subset b\in F\}$ is a congruence on $\mathcal{A}_{\mathrm{K}}$. As $\mathcal{A}_{\mathrm{K}}$ is a Boolean algebra, this is almost obvious, except that $\langle a,b\rangle,\langle c,d\rangle\in\Theta(F)$ implies $\langle a\Delta c,b\Delta d\rangle\in\Theta(F)$. Assume $\langle a,b\rangle,\langle c,d\rangle\in\Theta(F)$. Then we have,

$\langle a,b\rangle,\langle c,d\rangle\in\Theta(F)\Longrightarrow a\supset\subset b,c\supset\subset d\in F$ (definition of $\Theta(F)$)

$\Longrightarrow(a\supset\subset b)\supset\subset(c\supset\subset d)\in F$

$\Longrightarrow(a\supset\subset c)\supset\subset(b\supset\subset d)\in F$

$\Longrightarrow((a\supset\subset c)\supset\subset(b\supset\subset d))\Delta\mathrm{t}\in F$ (F is closed under r)

$\Longrightarrow((a\supset\subset c)\Delta\mathrm{t})\supset\subset((b\supset\subset d)\Delta\mathrm{t})\in F$ (Lemma 4.5.1 (i))

$\Longrightarrow(a\Delta c)\supset\subset(b\Delta d)\in F.$ (Lemma 4.5.1 (iii))

Thus $\Theta(F)$ is a congruence relation on $\mathcal{A}_{\mathrm{K}}$. Moreover,

$$\begin{aligned} a \in \mathbf{F}(\Theta(F)) &\text{ iff } a \supset\subset \mathrm{t} \in \mathbf{F}(\Theta(F)) \\ &\text{ iff } \langle a, \mathrm{t} \rangle \in \Theta(F) \\ &\text{ iff } a \supset\subset \mathrm{t} \in F \\ &\text{ iff } a \in F. \end{aligned}$$

Hence $\mathbf{F}(\Theta(F)) = F$. Also,

$$\begin{aligned} \langle a, b \rangle \in \Theta(\mathbf{F}(\theta)) &\text{ iff } a \supset\subset b \in \mathbf{F}(\theta) \\ &\text{ iff } \langle a, b \rangle \in \theta. \end{aligned}$$

So $\Theta(\mathbf{F}(\theta)) = \theta$. It is also straightforward to check that the above map preserves joins and meets of filters and congruences.

■

As a result of Proposition 6.3.9, we can discuss EDPC property of $\mathbf{PCI}_{\mathrm{K}}$-algebras by using not a congruence relation but a congruence filter.

Definition 6.3.10 (i) *A congruence filter F of a $\mathbf{PCI}_{\mathrm{K}}$-algebra $\mathcal{A}_{\mathrm{K}}$ is* principal *if it is generated by a single element $a \in A$. The principal filter generated by a is denoted by $F(\{a\})$ or $F(a)$.*

(ii) *A principal congruence filter is* equationally definable *if there is a finite set $E(x, y)$ of equations in two distinct variables such that for every $a, b \in A$ it satisfies*

$$a \in F(b) \textit{ iff } \mathcal{A}_{\mathrm{K}} \models E(a, b).$$

Proposition 6.3.11 *A variety $\mathcal{V}_{\mathrm{K}}$ of $\mathbf{PCI}_{\mathrm{K}}$-algebras has* EDPC *if and only if principal congruence filters of $\mathbf{PCI}_{\mathrm{K}}$-algebras in $\mathcal{V}_{\mathrm{K}}$ are equationally definable.*

Proof. To show the only-if-part, assume $\mathcal{V}_{\mathrm{K}}$ has EDPC. Let $\pi(x, y, z, w)$ be a principal congruence formula of a given type. Then we can show that $E(x, y) = \{\pi(x, \mathrm{t}, y, \mathrm{t})\}$ defines a principal congruence filter as the following way:

$$\begin{aligned} a \in F(b) &\Longleftrightarrow \theta(a, \mathrm{t}) \subseteq \theta(b, \mathrm{t}) && \text{(Prop. 6.3.9)} \\ &\Longleftrightarrow a \equiv \mathrm{t}(mod\ \theta(b, \mathrm{t})) \\ &\Longleftrightarrow \mathcal{A}_{\mathrm{K}} \models \pi(a, \mathrm{t}, b, \mathrm{t}) && (\mathcal{V}_{\mathrm{K}} \text{ has EDPC by hypothesis}) \\ &\Longleftrightarrow \mathcal{A}_{\mathrm{K}} \models E(a, b). \end{aligned}$$

Conversely, assume that $\forall \mathcal{A}_{\mathrm{K}} \in \mathcal{V}_{\mathrm{K}}, a, b \in A$ $a \in F(b)$ iff $\mathcal{A}_{\mathrm{K}} \models E(a, b)$. Then we can show that the principal congruence formula $\pi(x, y, z, w) \in E(x \supset\subset y, z \supset\subset w)$ defines EDPC :

$$\begin{aligned} c \equiv d(mod\ \theta(a, b)) &\Longleftrightarrow \theta(c, d) \subseteq \theta(a, b) \\ &\Longleftrightarrow \theta(c \supset\subset d, \mathrm{t}) \subseteq \theta(a \supset\subset b, \mathrm{t}) \\ &\Longleftrightarrow c \supset\subset d \in F(a \supset\subset b) && \text{(Prop. 6.3.9)} \\ &\Longleftrightarrow \mathcal{A}_{\mathrm{K}} \models E(a \supset\subset b, c \supset\subset d) && \text{(hypothesis)} \\ &\Longleftrightarrow \mathcal{A}_{\mathrm{K}} \models \pi(a, b, c, d). \end{aligned}$$

■

Proposition 6.3.12 *For every* $\mathbf{PCI}_{\mathrm{K}}$*-algebra* $\mathcal{A}_{\mathrm{K}}$ *and all* $a, b \in A$ $a \in F(b)$ *holds if and only if there exists some* $m \in \mathbf{N}$ *such that* $\mathcal{A}_{\mathrm{K}} \models r^m(b) \leq a$.

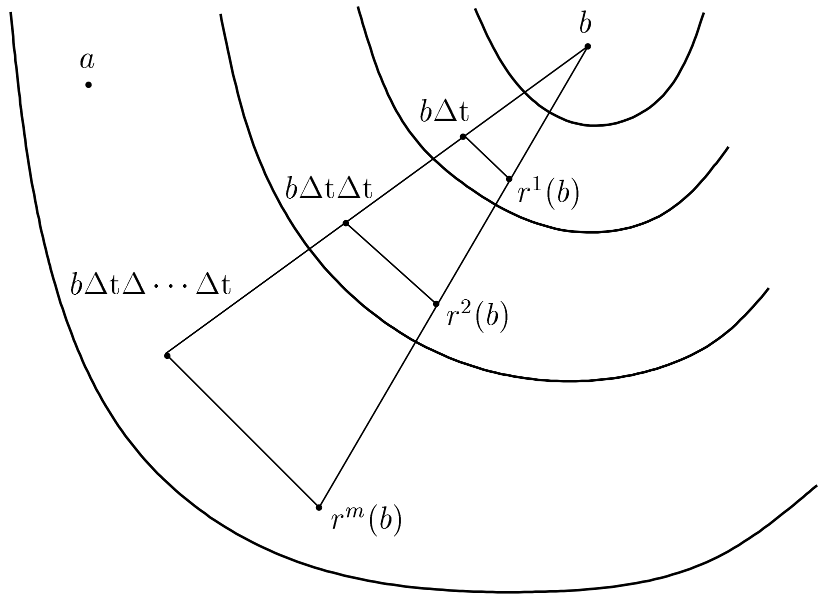

Figure 6.2: The situation of $\mathcal{A}_{\mathrm{K}} \models r^m(b) \leq a$ in Proposition 6.3.12

Proof. Recall first that $F(b)$ is the smallest congruence filter containing b. Then we claim that

$$F(b) = \bigcup_{n \in \mathbf{N}} [r^n(b)),$$

where $[z) = \{x; z \leq x\}$ is the Boolean filter generated by z. Now we define F_∞ as $\bigcup_{n \in \mathbf{N}} [r^n(b))$ and we will show $F(b) = F_\infty$. By the definition of F_∞, it is clear that $F(b) \subseteq F_\infty$ and that F_∞ is a congruence filter.

Conversely, let F be any congruence filter of $\mathcal{A}_{\mathrm{K}}$ such that $b \in F$. Assume that $x \in F_\infty$, i.e., $r^m(b) \leq x$ for some $m \in \mathbf{N}$. Then, since $b \in F$ and F is closed under r, we have $r^m(b) \in F$. So $x \in F$ by the hypothesis $r^m(b) \leq x$. Thus $F_\infty \subseteq F$. Thus F_∞ is the smallest congruence filter containing b. This means $F(b) = F_\infty = \bigcup_{n \in \mathbf{N}} [r^n(b))$. Hence the following equivalences hold:

$a \in F(b)$ iff $a \in \bigcup_{n \in \mathbf{N}} [r^n(b))$ (above claim)
iff $a \in [r^m(b))$ for some $m \in \mathbf{N}$
iff $\mathcal{A}_{\mathrm{K}} \models r^m(b) \leq a$ for some $m \in \mathbf{N}$.

■

Fact 6.3.13 *There exists an* $\mathbf{PCI}_{\mathrm{K}}$*-algebra* $\mathcal{A}_{\mathrm{K}}$ *such that for any positive integer* $m \in \mathbf{N}$ $\mathcal{A}_{\mathrm{K}} \models r^{m+1}(x) \neq r^m(x)$ *holds.*

Proof. We will show this fact by actually constructing an $\mathbf{PCI}_{\mathrm{K}}$-algebra. Let N_{fc} be a set of all finite or co-finite subset of $\mathbf{N}$, i.e., $N_{fc} = \{A \subseteq \mathbf{N}; \text{either } A \text{ or } \mathbf{N} \setminus A \text{ is finite}\}$. Then it is well known that $\mathcal{B} = \langle N_{fc}, +, \cdot, \to, -, 0, 1\rangle$ such that for any $a, b \in N_{fc}$ (i) $a + b = a \cup b$, (ii) $a \cdot b = a \cap b$, (iii) $a \leq b = a \subseteq b$, (iv) $-a = \mathbf{N} \setminus a$ and (v) $1 = \mathbf{N}, 0 = \emptyset$ is a set theoretical Boolean algebra on $\mathbf{N}$. Now let us consider a Boolean algebra $\mathcal{A} = \langle N_{fc}, +, \cdot, \to, \Delta, -, 0, 1\rangle$ with additional connective Δ that satisfies the following conditions :

(1) for any co-atom $c_n \in N_{fc}$ $(n \in \mathbf{N})$, that is $c_n = \mathbf{N} \setminus \{n\}$,

$$
\begin{aligned}
&c_0 \Delta 1 = c_0 \cdot c_1 \\
&(c_0 \cdot c_1)\Delta 1 = c_0 \cdot c_1 \cdot c_2 \\
&\quad\vdots \\
&(c_0 \cdot c_1 \cdot \cdots \cdot c_n)\Delta 1 = c_0 \cdot c_1 \cdot \cdots \cdot c_n \cdot c_{n+1} \\
&\quad\vdots \\
&(c_0 \cdot c_1 \cdot c_2 \cdot \cdots)\Delta 1 = c_0 \cdot c_1 \cdot c_2 \cdot \cdots
\end{aligned}
$$

(2) for any atom $a_n \in N_{fc}$ $(n \in \mathbf{N})$, that is $a_n = \{n\}$,

$$
\begin{aligned}
&a_0 \Delta 1 = 0 \\
&a_1 \Delta 1 = 0 \\
&\quad\vdots \\
&a_n \Delta 1 = 0 \\
&\quad\vdots
\end{aligned}
$$

Then we claim that $\mathcal{A}$ is an $\mathbf{PCI}_{\mathrm{K}}$-algebra such that for any positive integer $m \in \mathbf{N}$ $\mathcal{A} \models r^{m+1}(x) \neq r^m(x)$ holds. So, at first we have to chech this algebra $\mathcal{A}$ satisfies conditions (1)-(7) in Section 4.5. By Lemma 4.5.1, for every $a, b, c, d \in A$ we have the following equalities :

$$
\begin{aligned}
(1):\ a\Delta a &= (a \leftrightarrow a)\Delta 1 && \text{(Lemma 4.5.1 (iii))} \\
&= 1\Delta 1 = 1 \\
(2):\ a\Delta b &= (a \leftrightarrow b)\Delta 1 && \text{(Lemma 4.5.1 (iii))} \\
&= ((a \to b)\Delta 1) \cdot ((b \to a)\Delta 1) && \text{(Lemma 4.5.1 (ii))} \\
&\leq ((b \to a)\Delta 1) \cdot ((a \to b)\Delta 1) \\
&= (b \leftrightarrow a)\Delta 1 && \text{(Lemma 4.5.1 (ii))} \\
&= b\Delta a && \text{(Lemma 4.5.1 (iii))} \\
(3):\ (a\Delta b) \cdot (b\Delta c) &= ((a \leftrightarrow b)\Delta 1) \cdot ((b \leftrightarrow c)\Delta 1) && \text{(Lemma 4.5.1 (iii))} \\
&= ((a \leftrightarrow b) \cdot (b \leftrightarrow c))\Delta 1 && \text{(Lemma 4.5.1 (ii))} \\
&\leq (a \leftrightarrow c)\Delta 1 \\
&= a\Delta c && \text{(Lemma 4.5.1 (iii))} \\
(4):\ a\Delta b &= (a \leftrightarrow b)\Delta 1 && \text{(Lemma 4.5.1 (iii))}
\end{aligned}
$$

$$
\begin{aligned}
&\leq (-a \leftrightarrow -b)\Delta 1 \\
&= -a\Delta - b && \text{(Lemma 4.5.1 (iii))}
\end{aligned}
$$

$$
\begin{aligned}
(5) : (a\Delta b) \cdot (c\Delta d) &= ((a \leftrightarrow b)\Delta 1) \cdot ((c \leftrightarrow d)\Delta 1) && \text{(Lemma 4.5.1 (iii))} \\
&= ((a \leftrightarrow b) \cdot (c \leftrightarrow d))\Delta 1 && \text{(Lemma 4.5.1 (ii))} \\
&\leq ((a * c) \leftrightarrow (b * d))\Delta 1 \\
&= (a * c)\Delta(b * d) && \text{(Lemma 4.5.1 (iii))}
\end{aligned}
$$

$$
\begin{aligned}
(6) : (a \to b)\Delta(b \to a) &= ((a \to b) \leftrightarrow (b \to a))\Delta 1 && \text{(Lemma 4.5.1 (iii))} \\
&\leq ((a \to b) \cdot (b \to a))\Delta 1 \\
&= ((a \to b)\Delta 1) \cdot ((b \to a)\Delta 1) && \text{(Lemma 4.5.1 (ii))}
\end{aligned}
$$

Thus $\mathcal{A}$ is an $\mathbf{PCI}_{\mathrm{K}}$-algebra.

Next we will show this algebra $\mathcal{A}$ satisfies $\forall m \in \mathbf{N}, \forall x \in N_{fc}\ \mathcal{A} \models r^{m+1}(x) \neq r^m(x)$. Notice that we should only consider co-atoms as elements of algebra by the above construction of an algebra $\mathcal{A}$. From the condition (1) of Δ, it is easy to see that $c_n\Delta 1 = c_{n+1}$ $(n \in \mathbf{N})$. So for every $c_i \in N_{fc}$ $(i \in \mathbf{N})$, we get the following sequences :

$$
\begin{aligned}
r^0(c_i) &= c_i \\
r^1(c_i) &= (c_i\Delta 1) \cdot c_i \\
&= c_{i+1} \cdot c_i \\
&= c_{i+1} \cdot r^0(c_i) \neq r^0(c_i) \\
r^2(c_i) &= ((c_{i+1} \cdot c_i)\Delta 1) \cdot c_{i+1} \cdot c_i \\
&= (c_{i+1}\Delta 1) \cdot (c_i\Delta 1) \cdot c_{i+1} \cdot c_i \\
&= c_{i+2} \cdot c_{i+1} \cdot c_{i+1} \cdot c_i \\
&= c_{i+2} \cdot c_{i+1} \cdot c_i \\
&= c_{i+2} \cdot r^1(c_i) \neq r^1(c_i) \\
&\vdots \\
r^{m+1}(c_i) &= (r^m(c_i)\Delta 1) \cdot r^m(c_i) \\
&= c_{i+m+1} \cdot r^m(c_i) \neq r^m(c_i) \\
&\vdots
\end{aligned}
$$

Since the set of all co-atoms is a infinite set, there exists a infinite descending chain of $r^m(c_i)$ not including 0 by the above fact. Hence we have $r^{m+1}(c_i) \neq r^m(c_i)$ for any positive integer $i, m \in \mathbf{N}$.

■

From the above fact, we can show that the whole variety of $\mathbf{PCI}_{\mathrm{K}}$-algebras does not have EDPC by Propositions 6.3.11 and 6.3.12. However, we have the following.

Theorem 6.3.14 *A subvariety $\mathcal{V}_{\mathrm{K}}$ of $\mathbf{PCI}_{\mathrm{K}}$-algebras has* EDPC *if and only if there exists some $m \in \mathbf{N}$ such that for every $\mathcal{A}_{\mathrm{K}} \in \mathcal{V}_{\mathrm{K}}$ $\mathcal{A}_{\mathrm{K}} \models r^{m+1}(x) = r^m(x)$.*

Proof. To show the only-if-part, assume $\mathcal{V}_{\mathrm{K}}$ is a variety of $\mathbf{PCI}_{\mathrm{K}}$-algebra such that there exists some $m \in \mathbf{N}$ with $\forall \mathcal{A}_{\mathrm{K}} \in \mathcal{V}_{\mathrm{K}}$, $\mathcal{A}_{\mathrm{K}} \models r^{m+1}(x) = r^m(x)$. For any $b \in A$, let $F(b)$ be a principal congruence filter generated by b, and take any $a \in F(b)$.

Then, we have $\mathcal{A}_{\mathrm{K}} \models r^{m+1}(b) = r^m(b)$. Also, by the hypothesis $a \in F(b)$ and applying Proposition 6.3.12 to this, we get that there exists some $k \in \mathbf{N}$ such that $\mathcal{A}_{\mathrm{K}} \models r^k(b) \leq a$. Since the equality $r^{m+1}(b) = r^m(b)$ holds in $\mathcal{A}_{\mathrm{K}}$, we can take m for k. Thus by Proposition 6.3.12, we have: $x \in F(y)$ iff $\mathcal{A}_{\mathrm{K}} \models r^m(y) \leq x$ iff $\mathcal{A}_{\mathrm{K}} \models -r^m(y) \cup x = \mathrm{t}$. Hence, let us take $E(x, y) = \{-r^m(y) \cup x = \mathrm{t}\}$, then we conclude that $\mathcal{V}_{\mathrm{K}}$ has EDPC by Proposition 6.3.11.

Conversely, assume that $\mathcal{V}_{\mathrm{K}}$ has EDPC, but for any $m \in \mathbf{N}$, there exists an algebra $\mathcal{B}_{\mathrm{K}} \in \mathcal{V}_{\mathrm{K}}$ such that $\mathcal{B}_{\mathrm{K}} \not\models r^{m+1}(x) = r^m(x)$. Then by Proposition 6.3.11, there exists a finite set $E(x, y)$ of equations in two variables, which defines principal congruence filters . Let us take a sequence of algebras $\mathcal{A}_n$ $(n \in \mathbf{N})$ such that $\mathcal{A}_n \not\models r^{n+1}(x) = r^n(x)$, i.e., there exists an element $a_n \in A_n$ with $\mathcal{A}_n \models r^{n+1}(a_n) < r^n(a_n)$. Define $\mathcal{A}_{\mathrm{K}} = \prod_{n \in \mathbf{N}} \mathcal{A}_n$ and $a = \langle a_n; n \in \mathbf{N}\rangle \in A$, and consider a principal congruence filter $F(a)$ generated by a. Since $\mathcal{V}_{\mathrm{K}}$ is a variety, we have $\mathcal{A}_{\mathrm{K}} \in \mathcal{V}_{\mathrm{K}}$. Moreover, it is easily noticed that $F(a)$ is nontrivial and proper, since we have $r^{n+1}(\mathrm{t}) = r^n(\mathrm{t})$ and $r^{n+1}(\mathrm{f}) = r^n(\mathrm{f})$ in $\mathcal{A}_n$, respectively.

Now take $c = \langle r^n(a_n); n \in \mathbf{N}\rangle \in A$. Then by the hypothesis of $\mathcal{A}_n$, we have $\mathcal{A}_n \models r^{n+1}(a_n) < r^n(a_n) = c_n$ for any $n \in \mathbf{N}$. So by applying Proposition 6.3.12 to this, we get $c_n = r^n(a_n) \in F(a_n)$, i.e., the n-th projection of c belongs to the filter $F(a_n)$ on the n-th coordinate. Hence, by the hypothesis of $\mathcal{V}_{\mathrm{K}}$ having EDPC and Proposition 6.3.11, we have $\mathcal{A}_n \models E(c_n, a_n)$ $(n \in \mathbf{N})$. Thus, in the product $\mathcal{A}_{\mathrm{K}} = \prod_{n \in \mathbf{N}} \mathcal{A}_n$, we get $\mathcal{A}_{\mathrm{K}} \models E(c, a)$, where $a = \langle a_n; n \in \mathbf{N}\rangle$, $c = \langle r^n(a_n); n \in \mathbf{N}\rangle \in A$. Again, by the hypothesis of $\mathcal{V}_{\mathrm{K}}$ having EDPC and Proposition 6.3.11, this means $c \in F(a)$, and by applying Proposition 6.3.12 to this, we get that there exists some $m \in \mathbf{N}$ such that $\mathcal{A}_{\mathrm{K}} \models r^m(a) \leq c$.

However, since $c = \langle r^n(a_n); n \in \mathbf{N}\rangle$ by our definition, we have $r^k(a_n) > r^n(a_n) = c_n$ on every coordinate $n > k$. Thus, we conclude that there exists no fixed $k \in \mathbf{N}$ such that $\mathcal{A}_{\mathrm{K}} \models r^k(a) \leq c$, where $r^k(a) = \langle r^k(a_n); n \in \mathbf{N}\rangle$. This contradicts $\mathcal{A}_{\mathrm{K}} \models r^m(a) \leq c$.

■

Example 6.3.15 (The case of $\mathbf{PCI}_{\mathrm{K4}}$) *This logic is defined by*

$$\mathbf{PCI}_{\mathrm{K4}} = \mathbf{PCI}_{\mathrm{K}} \oplus ((A \equiv B) \wedge (C \equiv D) \to (A \equiv C) \equiv (B \equiv D)).$$

So, in $\mathbf{PCI}_{\mathrm{K4}}$-algebras $\mathcal{A}_{\mathrm{K4}} = \langle \mathcal{A}_0, \Delta\rangle$, Δ have to satisfy the following condition besides from (1) to (7) in Section 4.5:

(8) $(x\Delta y) \cap (r\Delta z) \supset (x\Delta r)\Delta(y\Delta z)$.

Then we have the following calculations:

$$\begin{aligned} r^0(x) &= x, \\ r^1(x) &= x \cap (x\Delta \mathrm{t}), \\ r^2(x) &= x \cap (x\Delta \mathrm{t}) \cap (x \cap (x\Delta \mathrm{t}))\Delta \mathrm{t} \\ &= x \cap (x\Delta \mathrm{t}) \cap (x\Delta \mathrm{t}) \cap (x\Delta \mathrm{t})\Delta \mathrm{t} \\ &= x \cap (x\Delta \mathrm{t}) \qquad (\star) \end{aligned}$$

$= r^1(x)$.

Here the above $(\star)$ *can be calculate because that both* $(x\Delta \mathrm{t}) \cap (\mathrm{t}\Delta \mathrm{t}) \supset (x\Delta \mathrm{t})\Delta(\mathrm{t}\Delta \mathrm{t})$ *by* (8)*, and* $(\mathrm{t}\Delta \mathrm{t})\Delta \mathrm{t}$ *imply* $(x\Delta \mathrm{t}) \supset (x\Delta \mathrm{t})\Delta \mathrm{t}$. *Hence a subvariety of* $\mathbf{PCI}_{\mathrm{K4}}$*-algebras have* EDPC *property.*

6.4 Notes

In this chatper, we explained Lindenbaum-Tarski algebra, equivalential algebra and congruence operators as various methods for the algebraization of deductive system. Besides these methods, we can give one more method of such an aim, namely *abstract logics*. The theory of abstract logics was initiated in 1970's under the inspiration of R. Suszko and the elaboration of his collaborators, S. L. Bloom and D. J. Brown (see [12] and [13]). After that, this approach was continued mainly by the Barcelona Group in Algebraic Logic (see [28]). In general, abstract logics are couples $\mathfrak{L} = (\mathcal{A}, C)$ where $\mathcal{A}$ is an algebra and C is a closure operator defined on the pwoer set of its underlying set. Then by the duality, we can also express abstract logics by $\mathfrak{L} = (\mathcal{A}, \mathcal{C})$ where $\mathcal{C}$ is the closure system associated with the closure operator C. Here abstract logics can be viewed as a generalization of both concepts of formal deductive system (i.e., syntactical formalism) and logical matrix (i.e., semantic matrices). If we view $\mathcal{A}$ in $\mathfrak{L}$ as an algebra of sentential formulas, then abstract logics $\mathfrak{L} = (\mathcal{A}, C)$ just correspond to the deductive system mentioned in Section 2.1. Moreover, if we view them from the semantical standpoint, then abstract logics correspond to a family of logical matrices. This approach is useful in the study of deductive systems that are not protoalgebraic, e.g., the conjunction-disjunction fragment of classical logic mentioned in Section 6.1 (see also [26]). As other examples of this approach, we can find modal logics (see [24] and [37]), and relevance logics (see [25] and [27]).

Chapter 7

Conclusions

In this chapter, we will summerize our achievements in this thesis (Section 1), and also discuss some remaining problems and several further subjects in our research field (Section 2). Our main interest in this thesis is making a contribution to understanding a logic as a unified form. So as to do, we first observed that some kinds of nonclassical logic can be reconstructed, commonly based on identity connective, as theories of *non-Fregean logic*, i.e., **SCI**, created by R. Suszko, when we gave an appropriate interpretation to *identity* connective in **SCI**. We called them the *simulation* property of **SCI**. If we assume that two notions of *identity* and *distinction* (that is a dual notion of the former) are the fundamentals of the knowledge acquisition and/or the construction of logic, then we can view Suszko' formalization as that based on the former notion (i.e., identity). Then, our main achievements in this thesis are concerned with a generalization of Suszko's **SCI**, and hence these are also viewed as a formalization of logic by the former notion. Furthermore, we also touch upon the latter notion (i.e., distinction) briefly as one of the further subjects in Section 2.

7.1 Achievements

In this section we will summerize our achievements in this thesis. We have formalized a logical system **PCI**, based on the notion of identity, as a generalization of Suszko's **SCI**. Roughly speaking, our results can be classified two subjects, namely *syntactical translations* between various kinds of nonclassical logics and **PCI** logics, and *algebraic characterizations* of a specific $\mathbf{PCI}_{\mathrm{K}}$ extension. We will summerize each result in the following subsections.

7.1.1 Syntactical translations

In general, we can classify nonclassical logics, according to its construction, into two types, namely (i): classical logics with additional operators and (ii): weak logics with various kinds of weak implication, e.g., strict/relevance/linear implication. Since **SCI** logic is a bit strong to simulate a weak modal logic like **K** on it, so we have first introduced more weak system, which is obtained from **SCI** by deleting two identity axioms of reflexivity and transitivity, and we called it **PCI** logic. Then, by defining the simulation property precisely as the syntactical equivalence of two logics, we gave the following simulation properties of **PCI** on each types of nonclassical logics. Here two logics are called *syntacticall equivalent* if there exist two translations between two langauges such that both translations preserve the logical relationships of two logics.

(i) Classical logics with additional operators
In fact, we have considered as this case the classical modal logics with necessary operator $\Box$. At first, we defined two translations between **K** and **PCI** languages by imposing the equality $A \equiv B$ iff $\Box(A \leftrightarrow B)$ to hold. As a result we have introduced the $\mathbf{PCI}_{\mathrm{K}}$ extension, which is obtained from **PCI** by adding two identity axioms (WIA1) and (WIA2), and one inference rule (G). Then, for above two translations and $\mathbf{PCI}_{\mathrm{K}}$ extension, we showed that **K** and $\mathbf{PCI}_{\mathrm{K}}$ are syntactically equivalent in a sense of Definition 3.4.1, and hence we also showed that modal logic **K** is a *simulationable* on $\mathbf{PCI}_{\mathrm{K}}$ logic. Furthermore, we showed that the same situation holds for various extensions of **K**. Here of cource, if we consider modal extensions **KT4** and **KT5** (which are also called **S4** and **S5**, respectively), then our system $\mathbf{PCI}_{\mathrm{S4}}$ and $\mathbf{PCI}_{\mathrm{S5}}$ are identical to the original extensions $\mathbf{W}_{\mathrm{T}}$ and $\mathbf{W}_{\mathrm{H}}$ of **SCI**, respectively, which are first introduced by R. Suszko in [65] and [67]. In this sense, our results of **PCI** can be seen as a generalization of Suszko's **SCI**, while **PCI** logic is no longer *non-Fregean logic* in a sense of Suszko's intention. Finally, as a semantical investigation of $\mathbf{PCI}_{\mathrm{K}}$ logic, we have introduced Kripke type semantics for $\mathbf{PCI}_{\mathrm{K}}$ logic by exchanging the validity of modal formulas in modal Kripke type semantics with new validity of identity formulas. Then, we showed that $\mathbf{PCI}_{\mathrm{K}}$ and **K** are also semantically equivalent relative to the same Kripke frame. So by invoking the completeness of modal logic, we gave a completeness theorem of $\mathbf{PCI}_{\mathrm{K}}$ relative to Kripke type semantics.

(ii) Weak logics with various kinds of weak implication
As this case, we have considered three types of weak logic, namely (1): weak logic with relevance implication $\rightsquigarrow$ (concretely, Angell's analytic containment logic **AC**), (2): weak logic with strict implication $\rightharpoonup$ (concretely, Corsi's weak logic **F**) and (3): weak logic with linear implication $\supset$ (concretely, Girard's classical linear logic **GL**). At first, since **AC** has a synonymity connective $\approx$ defined by $\alpha \approx \beta$ iff $(\alpha \rightsquigarrow \beta) \wedge (\beta \rightsquigarrow \alpha)$, we defined two translations between **AC** and **PCI** languages by imposing the equality $A \equiv B$ iff $A \approx B$ to hold. Then, the $\mathbf{PCI}_{\mathrm{W}}$ extension obtained from **PCI** by adding two identity axioms of reflexivity and transitivity, is known as nothing but non-Fregean logic **SCI**. Furthermore, we showed that **AC** and $\mathbf{PCI}_{\mathrm{W}}$ are syntactically equivalent in a sense of Definition 3.4.1, under the restriction of no nestings of identity. Secondly, since **F** has a strict implication, if

we define two translations between **F** and **PCI** languages by imposing the equality $A \equiv B$ iff $(A \rightleftharpoons B)$ to hold, then we got the same extension $\mathbf{PCI}_{\mathrm{K}}$ in the sight of both Kripke models between **K** and **F**. Then, we showed that every formulas in **F**-language can be translated into $\mathbf{PCI}_{\mathrm{K}}$ with keeping logical validity by introducing an auxiliary language with an extra material implication $\rightarrow$ to restore the balance of both **F** and **PCI** languages. Finally, **GL** can be seen as a logic, in which each of conjunction, disjunction and constant is splitted into additive and multiplicative parts, namely $(\wedge, \vee, \perp)$ and $(*, +, 0)$, respectively, and moreover, linear implication $\supset$ depends not on additive part but on multiplicative part. So if we define two translations between **GL** and **PCI** languages by imposing the equality $A \equiv B$ iff $(A \supset B) \wedge (B \supset A)$ to hold, then we got the $\mathbf{PCI}_{\mathrm{GL}}$ extension from **PCI** by adding the identity axioms (LT), (LE), (L*1), (L*2) and (LDN), which corresponded to the axioms of multiplicative part in **GL**, under the weak system $\mathbf{PCI}_{\mathrm{K}}$ in the above case of (i). Then, for above two translations and $\mathbf{PCI}_{\mathrm{GL}}$ extension, we showed that every formulas in **GL**-language can be translated into $\mathbf{PCI}_{\mathrm{GL}}$ with keeping logical validity by applying the similar discussion as the case of **F**. In $\mathbf{PCI}_{\mathrm{GL}}$, there exist the extra connectives $>$, $\smile$ and $\circ$, which correspond to multiplicative part in **GL**, abbreviated in $\mathbf{PCI}_{\mathrm{GL}}$ as: $A > B := (A \equiv A \wedge B)$, $\smile A := A > \neg(A \equiv A)$ and $A \circ B := \smile (A > \smile B)$.

7.1.2 Algebraic characterizations

We have investigated the algebraic characterizations of a specific $\mathbf{PCI}_{\mathrm{K}}$ extension mentioned in the above (i). At first, as a semantical counterpart of $\mathbf{PCI}_{\mathrm{K}}$ logic, we have introduced a $\mathbf{PCI}_{\mathrm{K}}$-algebras $\mathcal{A}_{\mathrm{K}} = \langle \mathcal{A}_0, \Delta \rangle$, which is a Boolean algebra $\mathcal{A}_0$ with an additional binary operation Δ. Then, we gave the representation theorem of this algebras in the similar way to the case of modal algebras by using the definitions of duality of frame and algebra. Moreover, we gave an alternative completeness result of $\mathbf{PCI}_{\mathrm{K}}$ logic by using the above representation theorem. Next, since it is easily shown that $\mathbf{PCI}_{\mathrm{K}}$-algebras form a variety, we have investigated a necessary and sufficient condition for a subvariety of $\mathbf{PCI}_{\mathrm{K}}$-algebras to have *equationally definable principal congruences* (EDPC for short) property, which is closely connected with the deduction theorem of a logic. At first, by invoking an isomorphism between the lattice of filters of $\mathbf{PCI}_{\mathrm{K}}$-algebras and the lattice of congruences of $\mathbf{PCI}_{\mathrm{K}}$-algebras, we can restate equivalently the EDPC property as that principal filters of $\mathbf{PCI}_{\mathrm{K}}$-algebras are equationally definable. As a result, by introducing an extra unary operator r on $\mathbf{PCI}_{\mathrm{K}}$-algebra such that $r(x) = (x \Delta \mathrm{t}) \cap x$, $r^0(x) = x$ and $r^{n+1}(x) = r(r^n(x))$, we gave a desired condition that a subvariety of $\mathbf{PCI}_{\mathrm{K}}$-algebras to have EDPC is equivalent to the equation $r^{m+1}(x) = r^m(x)$ for some m in $\mathbf{N}$.

7.2 Further researches

In this section we will discuss some remaining problems and several further subjects. At this point in time, these are listed as the following subsections.

7.2.1 Develop the semantics of PCI logic

In this thesis, we have introduced a system **PCI**, that is weaker than the original **SCI**, because of lacking the reflexivity (SI) and transitivity (C5) axioms for identity below:

(C5) $(A \equiv B) \wedge (C \equiv D) \rightarrow (A \equiv C) \equiv (B \equiv D)$,

(SI) $(A \equiv B) \rightarrow (A \rightarrow B)$.

At first, we would like to construct a logical matrix model of **PCI** logics, as the similar manner to the case of **SCI** matrix model. In **SCI** matrix model, the identity connective $\equiv$ is typically interpreted as the arithmetic equality $=$ of Boolean algebras. In **PCI** matrix model, however, we need more weak interpretation of identity because that the reflexivity and transitivity of identity are generally not assumed in **PCI** logic. To define such a interpretation of identity, now we are investigating the q-matrix model, proposed by G. Malinowski to fit his many-valued matrix semantics, which has a form of $\mathfrak{M} = (\mathcal{A}, D, \overline{D})$, where D and $\overline{D}$ denote to accepted and rejected designated elements of $\mathfrak{M}$, respectively. Secondly, we want to consider the Kripke type semantics of both **SCI** and **PCI** logics. Since both are a kind of situation theory, so we think worth developing the relational semantics of these. Lastly, we also would like to develop the semantics of **PCI** by using the discussion of the abstract logic, because that we have conjectured **PCI** as being not protoalgebraic.

7.2.2 Expand the target of simulations by PCI logic

All nonclassical logics investigated in this thesis are classical base. There exist, however, a considerable number of known nonclassical logic based on the intutionistic logic, for instance, intuitionistic modal logic, intuitionistic linear logic and so on. Hence our next target of simulations by **PCI** logic are those of intuitionistic base. Then, since P. Łukowski studied the intuitionistic version of **SCI** (he called **ISCI** for short) in the semantical point of view, we can refer to his results when we will construct the above simulations. The next candidate of simulations by **PCI** logic is a predicate logic. At the beginning of the development of non-Fregean logic, Suszko constructed his situation theory on the two sorted languages, namely *sentential* and *nominal* languages which are devoted to express the ontology of *situations* and *objects*, respectively. After that, his main interest, however, moved to **SCI** system for the sake of simplicity. So if we want to investigate totally the Suszko's or Wittgenstein's situation theory, then we also need to consider the case of predicate logics. Furthermore, we would like to investigate the Gentzen type formalization of **PCI** system based on the natural deduction. In **SCI**, Gentzen type formalization was already introduced in proceeding of the Cut-elemination. Hence as the similar manner to **SCI**, we would like to construct the Gentzen type formalization of **PCI**, and moreover, to consider the connection with computer science.

7.2.3 Consider PCI logic as a uniform framework

At present, there exist many kinds of logic which were born from their individual background or object, for instance, intuitionistic logic has the epistemic motivation, while classical logic has the ontological basis. So if we want to understand them uniformly, then we need some kinds of metalogic. As such a kind of metalogic, we can consider the **FL** proposed by H. Ono in [51]. In **FL**, various kinds of logic are lined up as some extensions of **FL** in view of substructural rules, i.e., weakening/contraction/exchange rules. Why it is possible is that **FL** was constructed to express the common property among logics, that is a structural rule in Gentzen formalization. Similarly, our system **PCI** has been constructed to express the sameness situation of individual logic by identity connective. In this thesis, we have demonstrated the typical cases by the simulation property of **PCI** logic. However, in order to make up our system **PCI** to suit the real cases, we have to refine it more and more.

7.2.4 Expect another logical framework based on distinction

Finally in this subsection, we will discuss another direction based on the notion of *distinction*, which is a dual notion of identity. In fact, we can find the similar notion in some literatures. For example, D. Van Dalen proposed the theory of *apartness* for the purpose of proceeding the intuitionistic mathematics (see [21]). In this theory, it was employed the positive inequality relation, which was first introduced by L. E. J. Brouwer and axiomatized by A. Heyting, below.

Definition 7.2.1 *A binary relation # is called an* apartness relation *if for any x, y and z, it satisfies the following conditions:*

(i) $x = y \leftrightarrow \neg(x \# y)$,

(ii) $x \# y \leftrightarrow y \# x$,

(iii) $x \# y \rightarrow (x \# z) \vee (y \# z)$.

Secondly, P. Łukowski used the new connective of *nonidentity* $\not\equiv$ to define the intuitionistic possibility in [42], which has the following properties:

(B1) $\neg(\alpha \not\equiv \alpha)$,

(B2) $(\sim \alpha \not\equiv \sim \beta) \rightarrow (\alpha \not\equiv \beta)$, (B3) $((\alpha \star \gamma) \not\equiv (\beta \star \delta)) \to ((\alpha \not\equiv \beta) \vee (\gamma \not\equiv \delta))$, where $\star \in \{\wedge, \vee, \leftharpoondown, \rightleftharpoons, \not\equiv\}$,

(B4) $(\alpha \leftharpoondown \beta) \to (\alpha \not\equiv \beta)$.

Here, $\sim$ and $\leftharpoondown$ are weak negation and coimplication, respectively (for detail, refer to [42]). Thirdly, G. Spencer-Brown published his book *Laws of Form* [62] in 1969, which was another formalization of Wittgenstein's *Tractatus*. In this book, he constructed the primary arithmetic by *cross* operation based on the primary action of *distinction*. There were a few followers, e.g., F. Varela and L. Kauffman to refine his theory, but unfortunately, there has been almost nothing of influences to the logical field until now. Lastly, from the philosophical site, G. Deleuze emphasized in [23] that both *difference* and *repetition* were the most primary acts for everything, through studying of Leibniz's infinitesimal analysis. From the above several pieces of approach, we are fully convinced that it is worth developing the another theory based on the dual notion of identity, i.e., distinction.

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