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Algebras and Frames for Modal Logics

by

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Abstract

In this thesis, we investigate modal logics semantically by using both algebraic semantics and general Kripke type semantics. We will discuss several topics on modal logic. Though the topic varies, there is a unique underlying motif through the whole thesis, i.e. *the duality between algebras and frames*.

Kripke type semantics for modal logics has made a great success in these years. This is mainly due to the fact that Kripke type semantics offers us intuitively comprehensible and easily manageable, mathematical models for modal logics. On the other hand, while algebraic structures lack these properties, they have one quite important merit which Kripke type semantics lacks. That is, every modal logic is complete with respect to algebraic semantics.

To supplement this defect, Kripke type semantics based on *general Kripke frames* was introduced. This semantics bridges between original Kripke type semantics and algebraic semantics. In fact, by the Stone duality, we have a nice correspondence between general Kripke frames and algebras. Through this duality, it becomes possible to get important results on general Kripke frames from results on the corresponding class of algebras, which are obtained by using the fruits of universal algebra.

The first topic of our thesis is pseudo-Euclidean logics. For fixed non-negative integers m and n, let E_k be the logic which is obtained from the smallest normal (classical) modal logic $\mathbf{K}$ by adding the axiom $\Diamond^k\phi \to \Box^m\Diamond^n\phi$, where $k \geq 0$. We will give a complete answer to the question when $E_k \supseteq E_{k'}$ holds.

Second, we discuss intuitionistic modal logics. For Kripke type semantics, we discuss *finite model property* of intuitionistic modal logics by *filtration method*. For algebraic semantics, we have succeeded to give a description of subdirectly irreducible algebras for various kinds of modal Heyting algebras. By using the duality theory, this result can be translated into a result on a description of irreducible (finite) Kripke frames.

Finally, we introduce a new type of products of modal logics, called *normal products*. Normal products resemble products familiar to researcher of measure theory and topology, and are defined as a generalization of *products of algebras of sets*. Our products of modal logics can be defined either by means of *normal products of general frames*, or by means of *normal products of modal algebras*. Since our notion of products is based highly on the duality theory, it has such a nice property as follows; the product of two general frames is isomorphic to the dual of the product of the corresponding dual algebras. This brought us a desired effect that the definition of the normal product of modal logics L_1 and L_2 is not affected by the choice of classes of general frames (or, modal algebras) which determine L_1 and L_2. Note that this is not the case for usual products of modal logics.

The notion of normal products is quite natural from the view point of duality theory. Therefore this enables us to extend the notion of products to other logics like intuitionistic modal logics.

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Contents

Chapter 1

Introduction

Our aim in this thesis is to develop a semantical study of modal logic, using both algebraic semantics and general Kripke type semantics. We will discuss several topics on modal logic, including pseudo-Euclidean logics (Chapters 3), intuitionistic modal logics (Chapters 4 and 5) and a new type of products of modal logics, called normal products (Chapter 6 and 7). Though the topic varies in every chapter, there is a unique underlying motif through the whole thesis, i.e. *the duality between algebras and frames*.

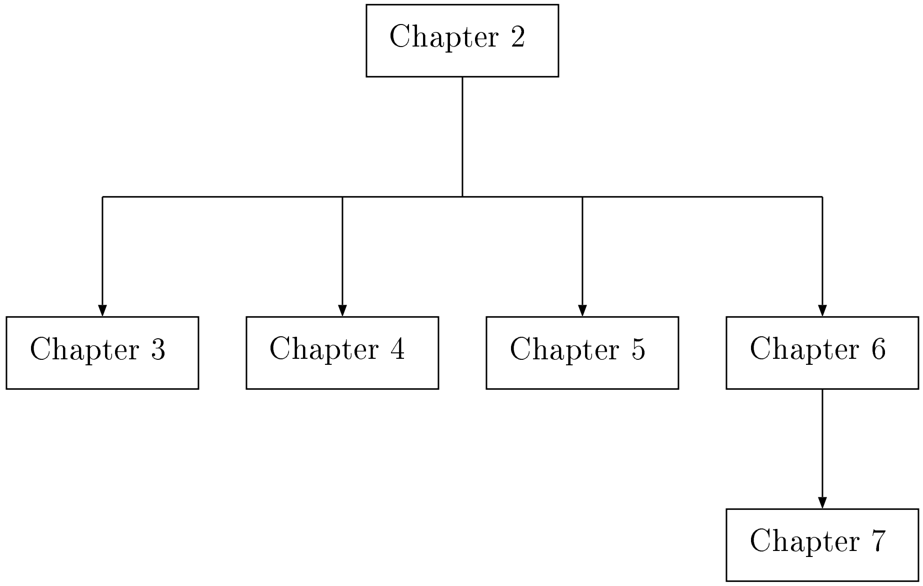
Kripke type semantics for modal logics has made a great success in these years. This is mainly due to the fact that Kripke type semantics offers us intuitively comprehensible and easily manageable, mathematical models for modal logics. On the other hand, while algebraic structures lack these properties, they have one quite important merit which Kripke type semantics lacks. That is, every modal logic is complete with respect to algebraic semantics.

To supplement this defect, Kripke type semantics based on *general Kripke frames* was introduced. This semantics bridges between original Kripke type semantics and algebraic semantics. In fact, by the Stone duality, we have a nice correspondence between general Kripke frames and algebras. Through this duality, it becomes possible to get important results on general Kripke frames from results on the corresponding class of algebras, which are obtained by using the fruits of universal algebra.

For instance, we have succeeded to give a description of subdirectly irreducible algebras for various kinds of modal Heyting algebras, in Chapter 5. By using the duality theory, this result can be translated into a result on a description of irreducible (finite) Kripke frames.

Also, we introduce a new type of products of modal logics in Chapter 6. Since our notion of products, i.e. normal products, is based highly on the duality theory, it has such a nice property as follows; the product of two general frames is isomorphic to the dual of the product of the corresponding dual algebras. In other words, the notion of normal products is quite natural from the view point of duality theory. This enables us to extend the notion of products to other logics, like superintuitionistic logics and intuitionistic modal logics, and obtain similar results about them (see Chapter 7).

The dependencies among chapters are given by the following diagram:



The organization of the thesis is as follows:

In Chapter 2, we will introduce basic idea of algebraic and Kripke-type semantics for intuitionistic modal logics, and general frame for intuitionistic modal logics.

In Chapter 3, we will consider *pseudo-Euclidean logics* Let E_k be the logic which is obtained from the smallest normal (classical) modal logic **K** by adding the axiom $\Diamond^k\phi \to \Box^m\Diamond^n\phi$, where $k \geq 0$ for fixed non-negative integers m and n. Since each axiom $\Diamond^k\phi \to \Box^m\Diamond^n\phi$ is a Sahlqvist formula, we can show that the logic E_k is Kripke complete for each k. A binary relation R on a set W is k*-pseudo-Euclidean* if for any $x, y, z \in W$, xR^ky and xR^mz imply zR^ny. Note that when $m = n = 1$, 1-pseudo-Euclidean relations are equal to usual Euclidean relations. Let $\mathcal{PE}_k$ be the class of all Kripke frames of the form (W, R), where R is a k-pseudo-Euclidean relation on W. Then, it is easy to see that E_k is Kripke complete with respect to $\mathcal{PE}_k$ and that $E_k \supseteq E_{k'}$ if and only if $\mathcal{PE}_k \subseteq \mathcal{PE}_{k'}$. Here, we identify the axiom system E_k with the set of all formulas provable in E_k. We will give a complete answer to the question when $E_k \supseteq E_{k'}$ holds.

Modal logics based on classical logic **Cl** have been thoroughly investigated. Classical logic, however, is sometimes considered to be too strong from a view point of computer science or constructive mathematics. This is a motivation for studying *intuitionistic modal logics*, i.e., modal logics based on intuitionistic logic **Int**. In intuitionistic modal logics the necessity operator $\Box$ and the possibility operator $\Diamond$ are not necessarily considered to be dual. In other words, we do not always assume that $\Box p \leftrightarrow \neg\Diamond\neg p$ and $\Diamond p \leftrightarrow \neg\Box\neg p$ hold. This provides more possibilities for introducing various kinds of intuitionistic modal logics (see e.g. [34]).

Indeed, several intuitionistic modal analogues of the classical normal modal logic **K** have been considered. One example is the logic $\mathbf{IntK}_\Box$ with one modal operator $\Box$ which is axiomatized by adding to **Int** the axioms

$$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q) \quad \text{and} \quad \Box\top,$$

and the congruence rule for $\Box$ (i.e., $\vdash_L \psi \leftrightarrow \varphi \;/\; \vdash_L \Box\psi \leftrightarrow \Box\varphi$). Another is the logic $\mathbf{IntK}_\Diamond$ with one modal operator $\Diamond$ which is axiomatized by adding to **Int** the axioms

$$\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q) \text{ and } \neg\Diamond\bot,$$

and the congruence rule for $\Diamond$ (i.e., $\vdash_L \psi \leftrightarrow \varphi \;/\; \vdash_L \Diamond\psi \leftrightarrow \Diamond\varphi$). Also, the logic $\mathbf{IntK}_{\Box\Diamond}$, the smallest logic containing both $\mathbf{IntK}_\Box$ and $\mathbf{IntK}_\Diamond$, having independent modalities $\Box$ and $\Diamond$ and congruence rules for both of them have been considered ([26]). Yet another important example is the logic **FS**, introduced by Fischer Servi ([7],[8]), which is an extension of $\mathbf{IntK}_{\Box\Diamond}$ obtained from $\mathbf{IntK}_{\Box\Diamond}$ by adding

$$\Diamond(p \to q) \to (\Box p \to \Diamond q) \text{ and } (\Diamond p \to \Box q) \to \Box(p \to q),$$

as an axiom. This expresses a weak connection between $\Box$ and $\Diamond$.

In Chapter 4, we will consider two method for completeness on Kripke type semantics. One of them is the *method of canonical models*, and the other is *filtration method*. In particular, we will show some logic enjoys the finite model property by using filtration method. Although the filtration method for classical modal logics has been studied comprehensively, the method is not applied to intuitionistic modal logics yet.

In Chapter 5, we will give a uniform description of subdirectly irreducible algebras for various classes of (multi-) modal Heyting algebras (Theorem 5.3.2 and Corollary 5.3.3), which answers the problem posed in [30] (Proposition 5.3.4).

In recent years, products of modal logics have been studied intensively mainly by D. Gabbay and V. Shehtman. These products, first introduced by Shehtman in [24], have been defined through products of Kripke frames. Shehtman's product of modal logics L_1 and L_2 is the modal logic determined by the class of Kripke frames $\mathcal{F} \times \mathcal{G}$ such that $\mathcal{F}$ and $\mathcal{G}$ validate L_1 and L_2, respectively.

In Chapter 6, we will propose a new notion of products of modal logics, which we will call a *normal product*. Normal products resemble products familiar to researcher of measure theory and topology, and are defined as a generalization of *products of algebras of sets* (see § 6.2). Our products of modal logics can be defined either by means of *normal products of general frames*, or by means of *normal products of modal algebras*. It enables us to develop a duality theory between these two, as shown in § 6.3. This brought us a desired effect that the definition of the normal product of modal logics L_1 and L_2 is not affected by the choice of classes of general frames (or, modal algebras) which determine L_1 and L_2. Note that this is not the case for usual products of modal logics, as pointed out in [21]. We also show some important transfer results, including the transfer of the finite model property.

In Chapter 7, we will apply normal products to infinitely many products and products of intuitionistic modal logics. Although in infinitely many products we assume that logics are extensions of **D** and in shifted products of intuitionistic modal logics we doesn't shift implications, our argument in this chapter goes in parallel with that in Chapter 6.

Chapter 2

Preliminaries

Classical modal logics are the classical logic with some axioms and rules for modal operators. When a modal logic has a single modal operator $\Box$ (or $\Diamond$), it is called *mono modal* logic. Usually, $\Diamond$ is considered to be equal to $\neg\Box\neg$ in any classical modal logic, where $\neg$ denotes the negation. In this thesis, in addition to classical modal logics we will study not only intuitionistic modal logics, i.e. modal logics based on the intuitionistic logic (and its extensions), but also *multi modal* logics (in Chapter 6).

For the brevity's sake, we will be mainly concerned with intuitionistic modal logics and semantics for them. For general information on classical modal logics, see e.g. [11], [13], [19], [34], [5].

2.1 Modal logics

Let $\mathcal{L}_{\Box\Diamond}$ be the language of propositional modal logics with countably many propositional variables, $p, q, r, \ldots$ and the connectives $\wedge, \vee, \rightarrow, \bot, \Box, \Diamond$. Let $\mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ be the set of all formulas of $\mathcal{L}_{\Box\Diamond}$. Formulas $\neg\alpha$ and $\top$ are defined as the abbreviation of $\alpha \rightarrow \bot$ and $\bot \rightarrow \bot$, respectively.

The basic classical modal logic is **K**. In this thesis, we will sometimes identify a logical system L with the set of all theorems of L. Thus, **K** can be defined as follows.

Definition 2.1.1 *The* modal propositional logic **K** *is the least set of formulas of* $\mathcal{L}_{\Box\Diamond}$ *which contains axioms from* (1) *to* (3)*, and is closed under the rules of inference from* (a) *to* (c)*.*

(1) *all theorems of the classical logic* **Cl***, i.e. the logic which is axiomatized by the following axioms;*

(i) $p \rightarrow (q \rightarrow p)$,

(ii) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$,

(iii) $(p \wedge q) \rightarrow p$,

(iv) $(p \wedge q) \rightarrow q$,

(v) $(r \rightarrow p) \rightarrow ((r \rightarrow q) \rightarrow (r \rightarrow (p \wedge q)))$,

(vi) $p \rightarrow (p \vee q)$,

(vii) $q \rightarrow (p \vee q)$,

(viii) $(p \to r) \to ((q \to r) \to (p \lor q) \to r)$,

(ix) $\bot \to p$,

(x) $p \lor \neg p$,

(2) $\Box(p \to q) \to (\Box p \to \Box q)$,

(3) $\neg\Diamond\neg p \leftrightarrow \Box p$,

(*a*) *modus ponens*

$$\frac{\vdash \alpha \to \beta \quad \vdash \alpha}{\vdash \ \beta} \ (\mathbf{MP}),$$

(*b*) *substitution* (**Sub**),

(*c*) *rule of necessitation*

$$\frac{\vdash \ \alpha}{\vdash \Box\alpha} \ (\mathbf{RN}).$$

On the other hand, *the intuitionistic version of* **K**, there will be some possibilities of defining since it is not necessary to define $\Diamond$ as the dual of $\Box$. The following logic $\mathbf{IntK}_{\Box\Diamond}$ is introduced by Wolter and Zakharyaschev.in [33] (also Sotirov [26]). It is easy to see that if we add the law of excluded middle $p \lor \neg p$ and $\neg\Diamond\neg p \leftrightarrow \Box p$ to $\mathbf{IntK}_{\Box\Diamond}$, then it becomes equal to **K**.

Definition 2.1.2 *The* intuitionistic bi-modal propositional logic $\mathbf{IntK}_{\Box\Diamond}$ *is the least set of formulas of* $\mathcal{L}_{\Box\Diamond}$ *which contains axioms from* (1) *to* ($3_\Diamond$)*, and is closed under the rules of inference from* (*a*) *to* (*c*).

(1) *all theorems of the intuitionistic logic* **Int**, *i.e. the logic which is axiomatized by the following axioms;*

(i) $p \to (q \to p)$,

(ii) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$,

(iii) $(p \land q) \to p$,

(iv) $(p \land q) \to q$,

(v) $(r \to p) \to ((r \to q) \to (r \to (p \land q)))$,

(vi) $p \to (p \lor q)$,

(vii) $q \to (p \lor q)$,

(viii) $(p \to r) \to ((q \to r) \to (p \lor q) \to r)$,

(ix) $\bot \to p$,

($2_\Box$) $(\Box p \land \Box q) \to \Box(p \land q)$ *and* ($2_\Diamond$) $\Diamond(p \lor q) \to (\Diamond p \lor \Diamond q)$,

($3_\Box$) $\Box\top$ *and* ($3_\Diamond$) $\neg\Diamond\bot$,

(*a*) *modus ponens*

$$\frac{\vdash \alpha \to \beta \quad \vdash \alpha}{\vdash \ \beta} \ (\mathbf{MP}),$$

(*b*) *substitution* (**Sub**),

(*c*) *rules of regularity*

$$\frac{\vdash\ \alpha \to \beta}{\vdash \Box\alpha \to \Box\beta}\ (\mathbf{RR}_\Box) \quad \textit{and} \quad \frac{\vdash\ \alpha \to \beta}{\vdash \Diamond\alpha \to \Diamond\beta}\ (\mathbf{RR}_\Diamond).$$

It is easily seen that the converses $\Box(p \wedge q) \to (\Box p \wedge \Box q)$ and $(\Diamond p \vee \Diamond q) \to \Diamond(p \vee q)$ of ($2\Box$) and ($2\Diamond$) are derivable in $\mathbf{IntK}_{\Box\Diamond}$.

We can take alternative definitions. For example, the axiom $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ is replaced by $\Box(p \to q) \to (\Box p \to \Box q)$. The rule of inference ($\mathbf{RR}_\Box$) is equivalent to the rule of inference (**RN**) under the formula $\Box(p \to q) \to (\Box p \to \Box q)$.

But the corresponding replacement doesn't work for $\Diamond$-operator. The axiom $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$ is not equivalent to the formula $\Diamond(p \to q) \to (\Diamond p \to \Diamond q)$. The rule of inference ($\mathbf{RR}_\Diamond$) is not equivalent to the rule of inference $\vdash \alpha\ /\ \vdash \Diamond\alpha$ even if we take the formula $\Diamond(p \to q) \to (\Diamond p \to \Diamond q)$ as an axiom. As another example, the axiom ($2\Box$) with the rule of inference $\mathbf{RR}_\Box$ is replaced by the axiom $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$ with the rule of inference $\vdash \alpha \leftrightarrow \beta\ /\ \vdash \Box\alpha \leftrightarrow \Box\beta$, while the axiom ($2\Diamond$) with the rule of inference $\mathbf{RR}_\Diamond$ is replaced by the axiom $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$ with the rule of inference $\vdash \alpha \leftrightarrow \beta\ /\ \vdash \Diamond\alpha \leftrightarrow \Diamond\beta$.

Let $\mathcal{L}_\Box$ ($\mathcal{L}_\Diamond$) be the language of propositional modal logics with countably many propositional variables, $p, q, r, \ldots$ and the connectives $\wedge, \vee, \to, \bot, \Box$ ($\Diamond$), respectively. Let $\mathcal{F}orm(\mathcal{L}_\Box)$ ($\mathcal{F}orm(\mathcal{L}_\Diamond)$) be the set of all formulas of $\mathcal{L}_\Box$ ($\mathcal{L}_\Diamond$), respectively. The *intuitionistic mono-modal propositional logic* $\mathbf{IntK}_\Box$ is the least set of formulas of $\mathcal{L}_\Box$ which contains axioms (1), ($2_\Box$) and ($3_\Box$), and is closed under the rules of inference (*a*), (*b*) and $\mathbf{RR}_\Box$. The *intuitionistic mono-modal propositional logic* $\mathbf{IntK}_\Diamond$ is the least set of formulas of $\mathcal{L}_\Diamond$ which contains axioms (1), ($2_\Diamond$) and ($3_\Diamond$), and is closed under the rules of inference (*a*), (*b*) and $\mathbf{RR}_\Diamond$.

A set L of formulas of $\mathcal{L}_{\Box\Diamond}$ is said an *intuitionistic modal logic* if L contains $\mathbf{IntK}_{\Box\Diamond}$ and is closed under all of rules of inference from (*a*) to (*c*). Our logics are called *normal* because of containing ($2_\Box$), ($2_\Diamond$), ($3_\Box$) and ($3_\Diamond$). We denote by $\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$ the set of all normal intuitionistic modal logics. In general, for an intuitionistic modal logic L, $\mathrm{NExt}L$ denotes the set of all normal intuitionistic modal logics containing L.

Let L_i be any logics or sets of formulas. Then $\bigoplus_{i \in I} L_i$ denotes the smallest logic which contains all of L_i's.

Theorem 2.1.3 $(\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}, \cap, \oplus)$ *is a complete lattice.*

Proof. It is easy to see that $(\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}, \cap, \oplus)$ is closed with respect to infinite intersections and that by the definition $\bigoplus_{i \in I} L_i$ is the smallest logic which contains all of L_i's. ∎

2.2 Algebraic semantics

First we will introduce algebraic semantics for intuitionistic modal logics. By translating the language of logic into that of algebra, we have algebraic semantics which will be an adequate semantics for intuitionistic modal logics.

Recall that an algebra $\boldsymbol{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is called a *Heyting algebra* if the following conditions hold in $\boldsymbol{A}$ for every $a, b \in A$:

(1) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$,

(2) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$,

(3) $(a \vee b) \vee b = b$, $a \wedge (a \vee b) = a$,

(4) $c \wedge a \leq b$ iff $c \leq a \rightarrow b$,

(5) $0 \leq a$,

(6) $1 = 0 \rightarrow 0$,

where for every $a, b \in A$,

$$a \leq b \text{ iff } a \wedge b = a.$$

As usual, $\neg a$ is defined as the abbreviation of $a \rightarrow 0$.

Definition 2.2.1 *An algebra* $\boldsymbol{A} = (\boldsymbol{A'}, \Box, \Diamond)$ *is called a* $\Box\Diamond$-modal Heyting algebra *if the following conditions are satisfied;*

(1) $\boldsymbol{A'} = (A, \wedge, \vee, \rightarrow, 0, 1)$ *is a Heyting algebra,*

$(2_\Box)$ $\Box(a \wedge b) = \Box a \wedge \Box b$ *and* $(2_\Diamond)$ $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$,

$(3_\Box)$ $\Box 1 = 1$ *and* $(3_\Diamond)$ $\Diamond 0 = 0$.

By $\boldsymbol{m_{\Box\Diamond}HA}$, we denote the class of all $\Box\Diamond$-modal Heyting algebras.

A $\Box\Diamond$-modal Heyting algebra $\boldsymbol{A} = (\boldsymbol{A'}, \Box, \Diamond)$ with a Boolean algebra $\boldsymbol{A'}$ satisfying $\Diamond a = \neg\Box\neg a$ is usually called *modal algebra*.

An algebra $\boldsymbol{A} = (\boldsymbol{A'}, \Box)$ is called a $\Box$-*modal Heyting algebra* if $\boldsymbol{A'}$ is a Heyting algebra and both $(2_\Box)$ and $(3_\Box)$ hold. Also an algebra $\boldsymbol{A} = (\boldsymbol{A'}, \Diamond)$ is called a $\Diamond$-*modal Heyting algebra* if $\boldsymbol{A'}$ is a Heyting algebra and both $(2_\Diamond)$ and $(3_\Diamond)$ hold.

Notice that from $(2_\Box)$ and $(2_\Diamond)$ both $\Box$ and $\Diamond$ are monotone operators, i.e.

(1) $a \leq b \Rightarrow \Box a \leq \Box b$,

(2) $a \leq b \Rightarrow \Diamond a \leq \Diamond b$.

Definition 2.2.2 *(1) A* valuation v *on a modal Heyting algebra* $\boldsymbol{A}$ *is a function from* $\mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ *to* A *which satisfies the following conditions;*

(i) $v(\bot) = 0$,

(ii) $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$,

(iii) $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$,

(iv) $v(\alpha \to \beta) = v(\alpha) \to v(\beta)$,

(v) $v(\Box\alpha) = \Box v(\alpha)$,

(vi) $v(\Diamond\alpha) = \Diamond v(\alpha)$.

(2) *For any* $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$, *any* $\boldsymbol{A} \in \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$ *and any valuation* v *on* $\boldsymbol{A}$, α *is* true *in* $\boldsymbol{A}$ *under* v *(in symbol,* $(\boldsymbol{A}, v) \models \alpha$*)* *if* $v(\alpha) = 1$.

(3) *For any* $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ *and any* $\boldsymbol{A} \in \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$, α *is* valid *in* $\boldsymbol{A}$ *(in symbol,* $\boldsymbol{A} \models \alpha$ *)* *if* $v(\alpha) = 1$ *for any valuation* v *on* $\boldsymbol{A}$.

(4) *For any* $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ *and any class* $\mathcal{K} \subseteq \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$, α *is* valid *in* $\mathcal{K}$ *(in symbol,* $\mathcal{K} \models \alpha$*)* *if* $\boldsymbol{A} \models \alpha$ *for any* $\boldsymbol{A}$ *in* $\mathcal{K}$.

Note that the value of a given valuation v is uniquely determined only by its value for each propositional variable.

Proposition 2.2.3

(1) *Let* $\boldsymbol{A}$ *be a* $\Box\Diamond$*-modal Heyting algebra. The set of formulas which are valid in* $\boldsymbol{A}$ *is an intuitionistic modal logic.*

(2) *Let* $\mathcal{K}$ *be a class of* $\Box\Diamond$*-modal Heyting algebras. The set of formulas which are valid in all algebras in* $\mathcal{K}$ *is an intuitionistic modal logic.*

They are called the logic characterized by $\boldsymbol{A}$ and are the logic characterized by $\mathcal{K}$ and denoted by $\boldsymbol{L}(\boldsymbol{A})$ and $\boldsymbol{L}(\mathcal{K})$, respectively.

Proof. Since $\boldsymbol{L}(\mathcal{K}) = \bigcap_{A \in \mathcal{K}} \boldsymbol{L}(\boldsymbol{A})$, it is enough to show (1). To show this, it suffices to show that each axiom of $\mathbf{IntK}_{\Box\Diamond}$, is valid in $\boldsymbol{A}$ and that each rule preserves the validity. Let v be any valuation on $\boldsymbol{A}$. $v((\Box p \wedge \Box q) \to \Box(p \wedge q)) = (\Box v(p) \wedge \Box v(q)) \to \Box(v(p) \wedge v(q)) = 1$. Next, suppose $v(\alpha \to \beta) = 1$. Since $v(\alpha) \leq v(\beta)$, by using monotonicity of $\Box$ $\Box v(\alpha) \leq \Box v(\beta)$. Therefore $v(\Box\alpha \to \Box\beta) = 1$. The validity of other axioms and rules can be shown in the same way. ∎

Let $\boldsymbol{A} = (\boldsymbol{A'}, \Box, \Diamond)$ be a modal Heyting algebra.

(1) A set $F \subseteq A$ is called a *filter* in $\boldsymbol{A}$ if

(i) $F \neq \emptyset$,

(ii) $a \in F$ and $a \leq b \Rightarrow b \in F$,

(iii) both $a \in F$ and $b \in F \Rightarrow a \wedge b \in F$,

(2) A filter F is called to be *proper* if $F \neq A$,

(3) A proper filter F is called to be *prime* if $a \vee b \in F$ implies either $a \in F$ or $b \in F$.

We denote by $PF(\boldsymbol{A})$ the set of all prime filters in $\boldsymbol{A}$.

An equivalence relation θ in a modal Heyting algebra $\boldsymbol{A} = (\boldsymbol{A'}, \Box, \Diamond)$ is said to be a *congruence* if the following conditions hold: for every $a, b, c, d \in A$,

(i) $a\,\theta\, b$ and $c\,\theta\, d$ imply $(a \wedge c)\,\theta\,(b \wedge d)$, $(a \vee c)\,\theta\,(b \vee d)$ and $(a \to c)\,\theta\,(b \to d)$,

(ii) $a\,\theta\, b$ implies $(\Box a)\,\theta\,(\Box b)$ and $(\Diamond a)\,\theta\,(\Diamond b)$.

Suppose $\boldsymbol{A} = (\boldsymbol{A'}, \Box, \Diamond)$ and $\boldsymbol{B} = (\boldsymbol{B'}, \Box, \Diamond)$ are modal Heyting algebras. A map f from A into B is called a *homomorphism* of $\boldsymbol{A}$ in $\boldsymbol{B}$ if f preserves the operators in the following sense: for every $a, a_1, a_2 \in A$,

(i) $f(a_1 \wedge a_2) = f(a_1) \wedge f(a_2)$,

(ii) $f(a_1 \vee a_2) = f(a_1) \vee f(a_2)$,

(iii) $f(a_1 \to a_2) = f(a_1) \to f(a_2)$,

(iv) $f(0) = 0$,

(v) $f(1) = 1$,

(vi) $f(\Box a) = \Box f(a)$,

(vii) $f(\Diamond a) = \Diamond f(a)$.

A homomorphism f of $\boldsymbol{A}$ in $\boldsymbol{B}$ is an *isomorphism* if f is an *injection* and also a *surjection*.

A modal Heyting algebra $\boldsymbol{B}$ is said to be a *subalgebra* of a modal Heyting algebra $\boldsymbol{A}$ if $B \subseteq A$ and $\boldsymbol{B}$'s operators are the restrictions of $\boldsymbol{A}$'s operators to B.

If f is a homomorphism of $\boldsymbol{A}$ in $\boldsymbol{B}$ then the set $f(A)$ is clearly closed under operators in $\boldsymbol{B}$ and so $(f(A), \wedge, \vee, \to, 0, 1, \Box, \Diamond)$ is a subalgebra of $\boldsymbol{B}$. We call it the *homomorphic image* of $\boldsymbol{A}$ (under the homomorphism f).

Given a family $\{\boldsymbol{A_\lambda} = (\boldsymbol{A'_\lambda}, \Box, \Diamond) \mid \lambda \in \Lambda\}$ of modal Heyting algebras, the *direct product* of $\{\boldsymbol{A_\lambda} \mid \lambda \in \Lambda\}$ is the modal Heyting algebra

$$\prod_{\lambda \in \Lambda} \boldsymbol{A_\lambda} = (\prod_{\lambda \in \Lambda} A_\lambda, \wedge, \vee, \to, 0, 1, \Box, \Diamond)$$

in which $\prod_{\lambda \in \Lambda} A_\lambda$ is the set of all function a from Λ into $\bigcup_{\lambda \in \Lambda} A_\lambda$ such that $a(\lambda) \in A_\lambda$ and each operator is defined by as follows:

(i) $(a \wedge b)(\lambda) = a(\lambda) \wedge b(\lambda)$,

(ii) $(a \vee b)(\lambda) = a(\lambda) \vee b(\lambda)$,

(iii) $(a \to b)(\lambda) = a(\lambda) \to b(\lambda)$,

(iv) $0(\lambda) = 0$,

(v) $1(\lambda) = 1$,

(vi) $(\Box a)(\lambda) = \Box a(\lambda)$,

(vii) $(\Diamond a)(\lambda) = \Diamond a(\lambda)$, for every $a, b \in \prod_{\lambda\in\Lambda} A_\lambda$ and $\lambda \in \Lambda$. We often write a_λ instead of $a(\lambda)$.

The *projection* π_λ is the function from $\prod_{\lambda\in\Lambda} A_\lambda$ onto A_λ defined by

$$\pi_\lambda(a) = a_\lambda,$$

for every $a \in \prod_{\lambda\in\Lambda} A_\lambda$. Also, for $\Lambda' \subseteq \Lambda$ the *projection* $\pi_{\Lambda'}$ is the function from $\prod_{\lambda\in\Lambda} A_\lambda$ onto $\prod_{\lambda\in\Lambda'} A_\lambda$ such that $\pi_{\Lambda'}(a)$ is a restriction of a to Λ', for every $a \in \prod_{\lambda\in\Lambda} A_\lambda$.

Let $\mathcal{K}$ be a class of $\Box\Diamond$-modal Heyting algebra. Then, $\boldsymbol{H}(\mathcal{K})$, $\boldsymbol{S}(\mathcal{K})$ and $\boldsymbol{P}(\mathcal{K})$ denote the class of all homomorphic images of algebras from $\mathcal{K}$, the class of all subalgebras of algebras from $\mathcal{K}$ and the class of all direct products of algebras from $\mathcal{K}$, respectively.

When $\boldsymbol{H}(\mathcal{K}) = \boldsymbol{S}(\mathcal{K}) = \boldsymbol{P}(\mathcal{K}) = \mathcal{K}$ holds, $\mathcal{K}$ is said to be a *variety* . It is well-known that $\mathcal{K}$ is a variety iff $\boldsymbol{HSP}(\mathcal{K}) = \mathcal{K}$, because $\boldsymbol{SH}(\mathcal{K}) \subseteq \boldsymbol{HS}(\mathcal{K}), \boldsymbol{PH}(\mathcal{K}) \subseteq \boldsymbol{HP}(\mathcal{K}), \boldsymbol{PS}(\mathcal{K}) \subseteq \boldsymbol{SP}(\mathcal{K})$(see e.g. [28]). $\mathcal{K}$ is a variety if and only if $\mathcal{K}$ is a equivalent class.

We can easily show that the class of all $\Box\Diamond$-modal Heyting algebras forms variety. In the following, $\Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA})$ denotes the set of all subvarieties of $\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$.

Proposition 2.2.4 *The set* $(\Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}), \wedge, \vee)$ *forms a complete lattice, where* $\mathcal{K}_1 \wedge \mathcal{K}_2$ *is* $\mathcal{K}_1 \cap \mathcal{K}_2$ *and* $\mathcal{K}_1 \vee \mathcal{K}_2$ *is* $\boldsymbol{HSP}(\mathcal{K}_1 \cup \mathcal{K}_2)$.

Proof. It is easy to see that $(\Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}), \wedge, \vee)$ is closed with respect to infinite intersections and that $\boldsymbol{HSP}(\bigcup_{i\in I} \mathcal{K}_i)$ is the least variety containing all $\mathcal{K}_i$'s. ∎

The following proposition holds, similarly to (classical) modal algebras.

Proposition 2.2.5 *Suppose that* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are* $\Box\Diamond$*-modal Heyting algebras.*

(1) If $\boldsymbol{B}$ *is a homomorphic image of* $\boldsymbol{A}$*, then* $\boldsymbol{L}(\boldsymbol{A}) \subseteq \boldsymbol{L}(\boldsymbol{B})$.

(2) If $\boldsymbol{B}$ *is a subalgebra of* $\boldsymbol{A}$*, then* $\boldsymbol{L}(\boldsymbol{A}) \subseteq \boldsymbol{L}(\boldsymbol{B})$.

(3) If $\boldsymbol{A}$ *is the direct product of* $\{\boldsymbol{A}_i\}_{i\in I}$ *, then* $\boldsymbol{L}(\boldsymbol{A}) = \bigcap_{i\in I} \boldsymbol{L}(\boldsymbol{A}_i)$.

Proof. (1). Let f be the homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$. There exists a map g of $\boldsymbol{B}$ to $\boldsymbol{A}$ such that $f \circ g = id_{\boldsymbol{B}}$. For each valuation v on $\boldsymbol{B}$, define the valuation $v_{\boldsymbol{A}}$ on $\boldsymbol{A}$ by $v_{\boldsymbol{A}} = (g \circ v)(p)$ for each propositional variable p. Then we have $f(v_{\boldsymbol{A}}(\alpha)) = v(\alpha)$ for every formula α. So, if $v_{\boldsymbol{A}}(\alpha) = 1$ in $\boldsymbol{A}$ then $v(\alpha) = 1$ in $\boldsymbol{B}$. Thus, we have (1).

(2). If $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$, any valuation on $\boldsymbol{B}$ is also a valuation on $\boldsymbol{A}$.

(3). Let v be a valuation on $\boldsymbol{A}$. Since for each i the projection π_i is a homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}_i$, $\pi_i \circ v$ is a valuation on $\boldsymbol{A}_i$. Moreover, the valuation v can be represented $(\pi_i \circ v)_{i\in I}$. Hence, $(\pi_i \circ v)(\alpha) = 1$ in $\boldsymbol{A}_i$ for each i iff $\pi_i(\alpha) = 1$ in $\boldsymbol{A}$. Thus, we have $\boldsymbol{L}(\boldsymbol{A}) \supseteq \bigcap_{i\in I} \boldsymbol{L}(\boldsymbol{A}_i)$. The converse direction follows from (1). ∎

Let $\boldsymbol{V}(L)$ denote $\{\boldsymbol{A} \in \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA} \mid \boldsymbol{A} \models L\}$. Then, since $\boldsymbol{V}(L)$ is closed under $\boldsymbol{H}$, $\boldsymbol{S}$ and $\boldsymbol{P}$ by Proposition 2.2.5, we have the following corollary.

Corollary 2.2.6 *For a given* $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$, $\boldsymbol{V}(L)$ *is a variety.*

Our algebraic semantics is *adequate*, since the following completeness theorem holds.

Theorem 2.2.7 *For any* $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$, $L \vdash \alpha$ *iff* $\boldsymbol{V}(L) \models \alpha$.

The theorem can be shown in the standard way. It is clear that $L \vdash \alpha$ implies $\boldsymbol{V}(L) \models \alpha$. To prove the converse, define the algebra $\boldsymbol{A}_L$ called the Lindenbaum algebra of a logic $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$ as follows. First, for a given logic L, define a congruence relation $\sim_L$ on the set of formulas by taking

$$\alpha \sim_L \beta \text{ iff } L \vdash (\alpha \to \beta) \wedge (\alpha \leftarrow \beta).$$

Let $|\alpha|_L$ be the equivalent class to which a formula α belongs. Now the Lindenbaum algebra $\boldsymbol{A}_L = (\mathcal{F}orm(\mathcal{L}_{\Box\Diamond})/_{\sim_L}, \wedge, \vee, \to, 0, 1, \Box, \Diamond)$ is constructed by taking

$$\begin{aligned}
\mathcal{F}orm(\mathcal{L}_{\Box\Diamond})/_{\sim_L} &:= \{|\alpha|_L \mid \alpha \text{ a formula }\},\\
|\alpha|_L \wedge |\beta|_L &:= |\alpha \wedge \beta|_L,\\
|\alpha|_L \vee |\beta|_L &:= |\alpha \vee \beta|_L,\\
|\alpha|_L \to |\beta|_L &:= |\alpha \to \beta|_L,\\
0 &:= |\bot|_L,\\
1 &:= |\top|_L,\\
\Box|\alpha|_L &:= |\Box\alpha|_L,\\
\Diamond|\alpha|_L &:= |\Diamond\alpha|_L.
\end{aligned}$$

The fact that $\boldsymbol{A}_L$ is indeed a modal Heyting algebra is easily shown by using the axioms and rules of $\mathbf{IntK}_{\Box\Diamond}$. Also, 1 of $\boldsymbol{A}_L$ consists exactly of the set of all provable formulas. Now, define a function v_L by

$$v_L(\alpha) = |\alpha|_L, \quad \text{for each formula } \alpha.$$

It is obvious that v_L is a valuation. Then we have

$$v_L(\alpha) = 1 \text{ iff } \alpha \in L.$$

Furthermore, for each valuation v on $\boldsymbol{A}_L$, $v(\alpha)$ is $|\beta|_L$ for some substitution instance β of α. So, in particular if $\alpha \in L$ then $v_L(\alpha) = |\beta|_L = 1$, since L is closed under substitution. Thus, $\boldsymbol{A}_L$ validates L. Now suppose that $L \not\vdash \alpha$. Then $v_L(\alpha) \neq 1$ for $\boldsymbol{A}_L \in \boldsymbol{V}(L)$. Thus, $\boldsymbol{V}(L) \not\models \alpha$ Hence, we have our theorem.

Proposition 2.2.8

(1) For any $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$, $\boldsymbol{L}(\boldsymbol{V}(L)) = L$.

(2) For any $\mathcal{K} \in \Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA})$, $\boldsymbol{V}(\boldsymbol{L}(\mathcal{K})) = \mathcal{K}$.

(3) For any $L_1, L_2 \in \Lambda(\mathbf{IntK}_{\Box\Diamond})$, $L_1 \subseteq L_2$ *iff* $\boldsymbol{V}(L_2) \subseteq \boldsymbol{V}(L_1)$

Proof. (1). This is the completeness theorem itself.

(2). By Birkhoff's theorem [2], any variety $\mathcal{K}$ is of the form $\boldsymbol{V}(L)$ for some logic L, so that $\boldsymbol{V}(\boldsymbol{L}(\mathcal{K})) = \boldsymbol{V}(\boldsymbol{L}(\boldsymbol{V}(L))) = \boldsymbol{V}(L) = \mathcal{K}$.

(3). By the definition, both $\boldsymbol{V}(\cdot)$ and $\boldsymbol{L}(\cdot)$ are monotone decreasing. ∎

Since any (dually) order isomorphism between lattices is also a (dually) lattice isomorphism, we have following corollary.

Corollary 2.2.9 $\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$ *is dually isomorphic to* $\Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA})$.

2.3 Kripke type semantics

In this section we will consider Kripke-type semantics for intuitionistic modal logics.

2.3.1 Kripke frames

Definition 2.3.1 *(1) A structure* $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond)$ *is called an* intuitionistic modal Kripke frame *if the following conditions are satisfied.*

(i) $W \neq \emptyset$,

(ii) $\lhd$ *is a partial order on* W,

(iii) both $R_\Box$ *and* $R_\Diamond$ *are binary relations on* W,

(iv) $\lhd \circ R_\Box \circ \lhd = R_\Box$, *where* $R_1 \circ R_2$ *is the* relational composition *of* R_1, R_2 *defined by* $x(R_1 \circ R_2)y$ *iff there is a* z *such that* xR_1z *and* zR_2y,

(v) $\lhd^{-1} \circ R_\Diamond \circ \lhd^{-1} = R_\Diamond$, *where* $\lhd^{-1}$ *is the reverse of* $\lhd$.

(2) UpW *is the set of all upward closed sets of* W *with respect to* $\lhd$, *i.e.* $UpW = \{V \subseteq W \mid (x \in V \text{ and } x \lhd y) \Rightarrow y \in V\}$

(3) A valuation v *on* $\mathcal{F}$ *is a function* $: \mathcal{F}orm(\mathcal{L}_{\Box\Diamond}) \to UpW$ *which satisfies the following conditions*

(i) $v(\bot) = \emptyset$,

(ii) $v(\alpha \wedge \beta) = v(\alpha) \cap v(\beta)$,

(iii) $v(\alpha \vee \beta) = v(\alpha) \cup v(\beta)$,

(iv) $v(\alpha \to \beta) = v(\alpha) \to v(\beta)$,

(v) $v(\Box\alpha) = \Box v(\alpha)$,

(vi) $v(\Diamond\alpha) = \Diamond v(\alpha)$,

where operators $\to$, $\Box$ *and* $\Diamond$ *are defined respectively as follows.*

$$\begin{aligned} X \to Y &:= \{w \in W \mid \text{ for any } v,\ w \lhd v \text{ and } v \in X \text{ implies } v \in Y\}, \\ \Box X &:= \{w \in W \mid \text{ for any } v,\ wR_\Box v \text{ implies } v \in X\}, \\ \Diamond X &:= \{w \in W \mid wR_\Diamond v \text{ for some } v \in X\}. \end{aligned}$$

(4) A pair $\mathcal{M} = (\mathcal{F}, v)$ *of an intuitionistic modal Kripke frame* $\mathcal{F}$ *and a valuation* v *on* $\mathcal{F}$ *is called a* model. *In this case,* $\mathcal{F}$ *is called the* base *of a model* $\mathcal{M}$.

(5) For any $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$, *any model* $\mathcal{M}$ *and any* $x \in W, \alpha$ *is* true *at* x *in* $\mathcal{M}$ *(in symbol,* $(\mathcal{M}, x) \models \alpha$ *or simply* $x \models \alpha$ *if* $\mathcal{M}$ *is understood) if* $x \in v(\alpha)$.

(6) For any $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ *and any model* $\mathcal{M}, \alpha$ *is* true *in* $\mathcal{M}$ *(in symbol,* $\mathcal{M} \models \alpha$*) if* $W = v(\alpha)$. *If it is not true in* $\mathcal{M}$ *then it is* refuted *in* $\mathcal{M}$.

(7) For any $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ *and any intuitionistic modal Kripke frame* $\mathcal{F}$, α *is* valid *in* $\mathcal{F}$ *(in symbol,* $\mathcal{F} \models \alpha$*) if* $W = v(\alpha)$ *for any valuation* v *on* $\mathcal{F}$.

Note that for modal logics we can take Δ as $\lhd$, and consider $R_\Diamond$ equal to $R_\Box$, where $x\Delta y \Leftrightarrow x = y$. Then we write $(W, R_\Box)$ instead of $(W, \Delta, R_\Box, R_\Box)$. Then the above conditions from (1) are obviously satisfied and $UpW = \mathcal{P}(W)$ holds.

Note that the value of a given valuation v is uniquely determined only by its value for each propositional variable.

We denote by ***IMF*** the set of all intuitionistic modal Kripke frames.

Proposition 2.3.2 *(1) Let $\mathcal{F}$ be an intuitionistic modal Kripke frame. The set of formulas which are valid in $\mathcal{F}$ is an intuitionistic modal logic.*

(2) Let $\mathcal{C}$ be a class of intuitionistic modal Kripke frames. The set of formulas which are valid in all frames in $\mathcal{C}$ is an intuitionistic modal logic.

They are called the logic characterized by $\mathcal{F}$ and the logic characterized by $\mathcal{C}$ and are denoted by $\boldsymbol{L}(\mathcal{F})$ and $\boldsymbol{L}(\mathcal{C})$, respectively.

Proof. Since $\boldsymbol{L}(\mathcal{C}) = \bigcap_{\mathcal{F}\in\mathcal{C}} \boldsymbol{L}(\mathcal{F})$, it is enough to show (1). Let v be any valuation on $\mathcal{F}$. If xRy, $y \models \Box p$ and $y \models \Box q$ then $z \models p$ and $z \models q$ for any z such that $yR_\Box z$. Hence $x \models (\Box p \wedge \Box q) \to \Box(p \wedge q)$ at any x.
Next, suppose that $u \models p \to q$ for all u and $y \models \Box p$ for $x \lhd y$. Then if $yR_\Box z$, $z \models p$. Since $z \models q$, $y \models \Box q$. Therefore $x \models \Box p \to \Box q$. The other axioms and rules can be treated in the same way. ∎

We denote by $\Box^n\alpha$ and $\Diamond^n\alpha$ the formulas $\underbrace{\Box\cdots\Box}_{n}\alpha$ and $\underbrace{\Diamond\cdots\Diamond}_{n}\alpha$, respectively. For brevity's sake, both $\Box^0\alpha$ and $\Diamond^0\alpha$ denote α. We denote also by $\Box^{(n)}\alpha$ and $\Diamond^{(n)}\alpha$ the formulas $\Box^0 \wedge \cdots \wedge \Box^n\alpha$ and $\Diamond^0 \vee \cdots \vee \Diamond^n\alpha$, respectively. In particular, we denote $\Box^{(1)}\alpha$ and $\Diamond^{(1)}\alpha$ by $\Box^+\alpha$ and $\Diamond^+\alpha$, respectively.

Also, for $n > 0$ we denote $\underbrace{R_\Box \circ \cdots \circ R_\Box}_{n}$ and $\underbrace{R_\Diamond \circ \cdots \circ R_\Diamond}_{n}$ by $R_\Box^n$ and $R_\Diamond^n$, respectively. We understand $R_\Box^0$ and $R_\Diamond^0$ as $\lhd$ and $\lhd^{-1}$, respectively. We denote also by $R_\Box^{(n)}$ and $R_\Diamond^{(n)}$ the binary relations $R_\Box^0 \cup \cdots \cup R_\Box^n$ and $R_\Diamond^0 \cup \cdots \cup R_\Diamond^n$, respectively. In particular,we denote $R_\Box^{(1)}$ and $R_\Diamond^{(1)}$ by $R_\Box^+$ and $R_\Diamond^+$, respectively. For each binary relation S, S^∞ denotes the *transitive closure*, i.e. $\bigcup_{n>0} S^n$, of a given binary relation S.

Similarly to classical modal logic, we can develop the correspondence theory. Here are some examples.

Proposition 2.3.3 *For any intuitionistic modal frame $\mathcal{F}$, $\mathcal{F}$ validates each formula in the following list iff $\mathcal{F}$ satisfies the corresponding condition in the list .*

$\Box(p \to q) \to (\Diamond p \to \Diamond q)$	$yR_\Diamond x \Rightarrow \exists z(x \lhd z \ \&\ yR_\Diamond z \ \&\ yR_\Box z)$	(2.1)
$\Diamond(p \to q) \to (\Box p \to \Diamond q)$	$yR_\Diamond x \Rightarrow \exists z(x \lhd z \ \&\ yR_\Diamond z \ \&\ yR_\Box z)$	(2.2)
$(\Diamond p \to \Box q) \to \Box(p \to q)$	$xR_\Box y \Rightarrow \exists z(x \lhd z \ \&\ zR_\Diamond y \ \&\ zR_\Box y)$	(2.3)
$\Box^+(p \to q) \to (\Diamond p \to \Diamond q)$	$yR_\Diamond x \Rightarrow \exists z(x \lhd z \ \&\ yR_\Diamond z \ \&\ yR_\Box^+ z)$	(2.4)

$\Box^{(m)}(p \to q) \to (\Diamond p \to \Diamond q)$	$yR_\Diamond x \Rightarrow \exists z(x \lhd z \;\&\; yR_\Diamond z \;\&\; yR_\Box^{(m)} z)$	(2.5)
$\neg\Box\bot$	$R_\Box : serial, i.e., \forall x \exists y\; xR_\Box y$	(2.6)
$\Diamond\top$	$R_\Diamond : serial, i.e., \forall x \exists y\; xR_\Diamond y$	(2.7)
$\Box p \to p$	$R_\Box : reflexive$	(2.8)
$p \to \Diamond p$	$R_\Diamond : reflexive$	(2.9)
$\Box p \to \Box\Box p$	$R_\Box : transitive$	(2.10)
$\Diamond\Diamond p \to \Diamond p$	$R_\Diamond : transitive$	(2.11)
$\Diamond^k\Box^l p \to \Box^m\Diamond^n p$	$(xR_\Box^m y \;\&\; xR_\Diamond^k z) \Rightarrow \exists u(yR_\Diamond^n u \;\&\; zR_\Box^l u)$	(2.12)
$\Box p \vee \Box\neg\Box p$	$(xR_\Box y \;\&\; xR_\Box z) \Rightarrow yR_\Box z$	(2.13)
$\Box(\Box p \vee q) \to (\Box p \vee \Box q)$	$(xR_\Box y \;\&\; xR_\Box z) \Rightarrow \exists u(xR_\Box u \;\&\; u \lhd z \;\&\; uR_\Box y)$	(2.14)
$\Box(\Box p \to q) \vee \Box(\Box q \to p)$	$(xR_\Box y \;\&\; xR_\Box z) \Rightarrow (yR_\Box z \text{ or } zR_\Box y)$	(2.15)

Proof. We will take up several of them. The rest can be checked similarly.

(2.5). Suppose that $\mathcal{M} = (\mathcal{F}, v)$ is a counter-model for it. Then $y \models \Box^{(m)}(p \to q)$ and $y \models \Diamond p$ and $y \not\models \Diamond q$, for some y in $\mathcal{F}$. Since $x \models p$ for some x such that $yR_\Diamond x$, if there exist z and number n such that $x \lhd z$, $yR_\Diamond z$, $yR_\Box^n z$ and $0 \le n \le m$, then we have $z \models p$ and $z \models p \to q$. Hence $z \models q$. This is a contradiction. Conversely, suppose that there are x, y such that $yR_\Diamond x$ and there is no point z for which $x \lhd z$, $yR_\Diamond z$ and $yR_\Box^{(m)} z$. Define a valuation v in $\mathcal{F}$ by taking $v(p) = \{w \mid x \lhd w\}, v(q) = \{w \mid x \lhd w \text{ and } yR_\Box^{(m)} w\}$. Then for any n such that $0 \le n \le m$ if $xR_\Box^n w$ and $w \models p$ then $w \models q$. Hence $y \models \Box(p \to q)$. we can also show $y \models \Diamond p$ and $y \not\models \Diamond q$, since $x \models p$ and there is no point z for which $yR_\Diamond z$, $x \lhd z$ and $yR_\Box^{(m)} z$. Thus, $y \not\models \Box(p \to q) \to (\Diamond p \to \Diamond q)$.

(2.12). Suppose that $\mathcal{M} = (\mathcal{F}, v)$ is a counter-model for it. Then $x \models \Diamond^k\Box^l p$ and $x \not\models \Box^m\Diamond^n p$, for some x in $\mathcal{F}$. Hence there are y, z such that $xR_\Box^m y$, $y \not\models \Diamond^n p$ and $xR_\Diamond^k z$, $z \models \Box^l p$. If there is u such that $yR_\Diamond^n u$ and $zR_\Box^l u$ then this is a contradiction. Conversely, suppose that there are x, y, z such that $xR_\Box^m y$, $xR_\Diamond^k z$ and there is no point u for which $yR_\Diamond^n u$ and $zR_\Box^l u$. Define a valuation v in $\mathcal{F}$ by taking $v(p) = \{w \mid zR_\Box^l w\}$. Then we can show $z \models \Box^l p$ and $y \not\models \Diamond^n p$, whence $x \models \Diamond^k\Box^l p$ and $x \not\models \Box^m\Diamond^n p$. Thus, $x \not\models \Diamond^k\Box^l p \to \Box^m\Diamond^n p$.

(2.14). Suppose that $\mathcal{M} = (\mathcal{F}, v)$ is a counter-model for it. Then $x \models \Box(\Box p \vee q)$ and $x \not\models \Box p \vee \Box q$, for some x in $\mathcal{F}$. Hence there are y, z such that $xR_\Box y, y \not\models p$ and $xR_\Box z, z \not\models q$. If there is u such that $xR_\Box u$, $u \lhd z$ and $uR_\Box y$ then $u \not\models \Box p$ and $u \not\models q$. Hence $u \not\models \Box p \vee q$. This is a contradiction. Conversely, suppose that there are x, y, z such that $xR_\Box y$, $xR_\Box z$ and there is no point u, for which $xR_\Box u$, $u \lhd z$ and $uR_\Box y$. Define a valuation v in $\mathcal{F}$ by taking $v(p) = \{w \mid w \not\lhd y\}, v(q) = \{w \mid w \not\lhd z\}$. Then we can show $y \not\models p$, $z \not\models q$ and $u \models \Box p \vee q$. Indeed, there is no point u such that $xR_\Box u$, $u \models \Box p$ and $u \models q$. Hence $x \models \Box(\Box p \vee q)$ and $x \not\models \Box p \vee \Box q$. Thus, $x \not\models \Box(\Box p \vee q) \to (\Box p \vee \Box q)$. ∎

The following is a list of some intuitionistic modal logics, which are discussed often in the literature.

$\mathbf{IntK}_{\Box\Diamond}^{+} = \mathbf{IntK}_{\Box\Diamond} \oplus \Box^{+}(p \to q) \to (\Diamond p \to \Diamond q)$,
$\mathbf{IntK}_{\Box\Diamond}^{*} = \mathbf{IntK}_{\Box\Diamond} \oplus \Box(p \to q) \to (\Diamond p \to \Diamond q)$,
$\mathbf{FS} = \mathbf{IntK}_{\Box\Diamond} \oplus \{\Diamond(p \to q) \to (\Box p \to \Diamond q),\ (\Diamond p \to \Box q) \to \Box(p \to q)\}$,
$\mathbf{IntD}_{\Box\Diamond} = \mathbf{IntK}_{\Box\Diamond} \oplus \{\neg\Box\bot,\ \Diamond\top\}$,
$\mathbf{IntT}_{\Box\Diamond} = \mathbf{IntK}_{\Box\Diamond} \oplus \{\Box p \to p,\ p \to \Diamond p\}$,

$\mathbf{IntK4}_{\Box\Diamond} = \mathbf{IntK}_{\Box\Diamond} \oplus \{\Box p \to \Box\Box p,\ \Diamond\Diamond p \to \Diamond p\}$,
$\mathbf{IntS4}_{\Box\Diamond} = \mathbf{IntK4}_{\Box\Diamond} \oplus \{\Box p \to p,\ p \to \Diamond p\}$,
$\mathbf{IntS4.3}_{\Box\Diamond} = \mathbf{IntS4}_{\Box\Diamond} \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$,
$\mathbf{IntK5}_{\Box\Diamond} = \mathbf{IntK}_{\Box\Diamond} \oplus \{\Diamond\Box p \to \Box p,\ \Diamond p \to \Box\Diamond p\}$,
$\mathbf{IntS5}_{\Box\Diamond} = \mathbf{IntK5}_{\Box\Diamond} \oplus \{\Box p \to p,\ p \to \Diamond p\}$,
$\mathbf{MIPC} = \mathbf{IntS5}_{\Box\Diamond} \oplus \Box(p \to q) \to (\Diamond p \to \Diamond q)$.

Then, the conditions of frames validating these logics are given as follows.

$\mathbf{IntK}^{+}_{\Box\Diamond}$	$yR_\Diamond x \Rightarrow \exists z(x \lhd z \ \&\ yR_\Diamond z \ \&\ yR^{+}_{\Box} z)$
$\mathbf{IntK}^{*}_{\Box\Diamond}$	$yR_\Diamond x \Rightarrow \exists z(x \lhd z \ \&\ yR_\Diamond z \ \&\ yR_\Box z)$
FS	$yR_\Diamond x \Rightarrow \exists z(x \lhd z \ \&\ yR_\Diamond z \ \&\ yR_\Box z)$
	$xR_\Box y \Rightarrow \exists z(x \lhd z \ \&\ zR_\Diamond y \ \&\ zR_\Box y)$
$\mathbf{IntD}_{\Box\Diamond}$	$R_\Box, R_\Diamond$: serial
$\mathbf{IntT}_{\Box\Diamond}$	$R_\Box, R_\Diamond$: reflexive
$\mathbf{IntK4}_{\Box\Diamond}$	$R_\Box, R_\Diamond$: transitive
$\mathbf{IntS4}_{\Box\Diamond}$	$R_\Box, R_\Diamond$: reflexive and transitive
$\mathbf{IntS4.3}_{\Box\Diamond}$	$R_\Box, R_\Diamond$: reflexive and transitive
	$(xR_\Box y \ \&\ xR_\Box z) \Rightarrow (yR_\Box z \text{ or } zR_\Box y)$
$\mathbf{IntK5}_{\Box\Diamond}$	$(xR_\Box y \ \&\ xR_\Diamond z) \Rightarrow (zR_\Box y \ \&\ yR_\Diamond z)$
$\mathbf{IntS5}_{\Box\Diamond}$	$R_\Diamond = R_\Box^{-1}$ and $R_\Box$:reflexive and transitive
MIPC	$R_\Diamond = R_\Box^{-1}$ and $R_\Box$:reflexive and transitive
	$xR_\Box y \Rightarrow \exists z(x \lhd z \ \&\ yR_\Box z \ \&\ zR_\Box y)$

2.3.2 Truth-preserving operations

In this subsection we will introduce three important operations on intuitionistic modal frames which preserve the validity.

Definition 2.3.4 *(1) A Kripke frame* $\mathcal{F}_1 = (W_1, \lhd_1, R_{\Box_1}, R_{\Diamond_1})$ *is called a* generated subframe *of a Kripke frame* $\mathcal{F}_2 = (W_2, \lhd_2, R_{\Box_2}, R_{\Diamond_2})$ *if the following conditions are satisfied.*

(i) $W_1 \subseteq W_2$,

(ii) $\lhd_1$ *(*$R_{\Box_1}$ *and* $R_{\Diamond_1}$*) is the restriction of* $\lhd_2$ *(*$R_{\Box_2}$ *and* $R_{\Diamond_2}$*, respectively) to* W_1.

(iii) $x \in W_1 \ \&\ x \lhd_2 y \Rightarrow y \in W_1$,

(iv) $x \in W_1 \ \&\ xR_{\Box_2} y \Rightarrow y \in W_1$,

(v) $x \in W_1 \ \&\ xR_{\Diamond_2} y \Rightarrow \exists z \in W_1 \ :\ xR_{\Diamond_1} z \ \&\ y \lhd_2 z$.

(2) A map $f : W_1 \to W_2$ *is said to be a* p-morphism *from a Kripke frame* $\mathcal{F}_1$ *to a Kripke frame* $\mathcal{F}_2$ *if for all* $x \in W_1, y \in W_2$,

(i) $f(x) \lhd_2 y \Leftrightarrow \exists z \in W_1 \ :\ x \lhd_1 z \ \&\ f(z) = y$,

(ii) $f(x) R_{\Box_2} y \Leftrightarrow \exists z \in W_1 \ :\ xR_{\Box_1} z \ \&\ f(z) = y$,

(iii) $xR_{\Diamond_1} y \Rightarrow f(x) R_{\Diamond_2} f(y)$,

(iv) $f(x) R_{\Diamond_2} y \Rightarrow \exists z \in W_1 \ :\ xR_{\Diamond_1} z \ \&\ y \lhd_2 f(z)$.

(3) A Kripke frame $\mathcal{F}_1$ is said to be reducible *to a Kripke frame $\mathcal{F}_2$ if there exists a onto p-morphism (called a* reduction*) $f : W_1 \to W_2$.*

(4) The Kripke frame $\sum_{i \in I} \mathcal{F}_i = (\sum_{i \in I} W_i, \bigcup_{i \in I} \lhd_i, \bigcup_{i \in I} R_{\Box_i}, \bigcup_{i \in I} R_{\Diamond_i})$ is called the disjoint union *of a disjoint family $\{\mathcal{F}_i \mid i \in I\}$.*

Note that each $\mathcal{F}_i$ is a generated subframe of the disjoint union of $\{\mathcal{F}_i \mid i \in I\}$.

Theorem 2.3.5 *Suppose that $\mathcal{F}_1$ is a generated subframe of $\mathcal{F}_2$, and that $\mathcal{M}_1$ (and $\mathcal{M}_2$) is a model with the base $\mathcal{F}_1$ (and with the base $\mathcal{F}_2$, respectively). If for every propositional variable p and every x in $\mathcal{F}_1$*

$$(\mathcal{M}_1, x) \models p \quad \textit{iff} \quad (\mathcal{M}_2, x) \models p$$

then for every formula α and every x in $\mathcal{F}_1$

$$(\mathcal{M}_1, x) \models \alpha \quad \textit{iff} \quad (\mathcal{M}_2, x) \models \alpha.$$

Proof. We prove by the induction on the construction of α. The basis of induction is obvious. Let $\alpha = \Diamond\beta$. If $(\mathcal{M}_1, x) \models \Diamond\beta$ then there is a point $y \in W_1$ such that $xR_{\Diamond_1}y$ and $(\mathcal{M}_1, y) \models \beta$. By the induction hypothesis, $(\mathcal{M}_2, y) \models \beta$, and by (1)(ii) of Definition 2.3.4, $xR_{\Diamond_2}y$. Therefore $(\mathcal{M}_2, x) \models \Diamond\beta$. Conversely, suppose that $(\mathcal{M}_2, x) \models \Diamond\beta$. Then there is a point $y \in W_2$ such that $xR_{\Diamond_2}y$ and $(\mathcal{M}_2, y) \models \beta$. By (1)(v) of Definition 2.3.4, There is $z \in W_1$ such that $xR_{\Diamond_1}z$ and $y \lhd_2 z$. Since $(\mathcal{M}_2, z) \models \beta$, $(\mathcal{M}_1, z) \models \beta$ by the induction hypothesis, whence $(\mathcal{M}_1, x) \models \Diamond\beta$.

The cases $\alpha = \beta \to \gamma$ and $\alpha = \Box\beta$ can be treated in the same way, and the cases $\alpha = \beta \wedge \gamma$ and also $\alpha = \beta \vee \gamma$ are trivial. ∎

Corollary 2.3.6 *If $\mathcal{F}_1$ is a generated subframe of $\mathcal{F}_2$, then $\boldsymbol{L}(\mathcal{F}_2) \subseteq \boldsymbol{L}(\mathcal{F}_1)$.*

Proof. Suppose $\mathcal{F}_1 \not\models \alpha$. Then $((\mathcal{F}_1, v_1), x) \not\models \alpha$ for some v_1 on $\mathcal{F}_1$ and some $x \in \mathcal{F}_1$. Define a valuation v_2 on $\mathcal{F}_2$ by taking

$$v_2(p) := v_1(p) \quad \text{for all propositional variables } p.$$

By Theorem 2.3.5, $((\mathcal{F}_2, v_2), x) \not\models \alpha$. Therefore, $\mathcal{F}_2 \not\models \alpha$. ∎

Theorem 2.3.7 *Suppose that f is a reduction of $\mathcal{F}_1$ to $\mathcal{F}_2$, , and that $\mathcal{M}_1$ (and $\mathcal{M}_2$) is a model with the base $\mathcal{F}_1$ (and with the base $\mathcal{F}_2$, respectively). If for every propositional variable p and every x in $\mathcal{F}_1$*

$$(\mathcal{M}_1, x) \models p \quad \textit{iff} \quad (\mathcal{M}_2, f(x)) \models p$$

then for every formula α and every x in $\mathcal{F}_1$

$$(\mathcal{M}_1, x) \models \alpha \quad \textit{iff} \quad (\mathcal{M}_2, f(x)) \models \alpha.$$

Proof. We prove by the induction on the construction of α. The basis of induction is obvious. Let $\alpha = \Diamond\beta$. If $(\mathcal{M}_1, x) \models \Diamond\beta$ then there is a point $y \in W_1$ such that $xR_{\Diamond_1}y$ and $(\mathcal{M}_1, y) \models \beta$. By the induction hypothesis, $(\mathcal{M}_2, f(y)) \models \beta$, and by (2)(iii) of Definition 2.3.4, $f(x)R_{\Diamond_2}f(y)$. Therefore $(\mathcal{M}_2, f(x)) \models \Diamond\beta$. Conversely, suppose that $(\mathcal{M}_2, f(x)) \models \Diamond\beta$. Then there is a point $y \in W_2$ such that $f(x)R_{\Diamond_2}y$ and $(\mathcal{M}_2, y) \models \beta$. By (2)(iv) of Definition 2.3.4, There is $z \in W_1$ such that $xR_{\Diamond_1}z$ and $y \lhd_2 f(z)$. Since $(\mathcal{M}_2, f(z)) \models \beta$, $(\mathcal{M}_1, z) \models \beta$ by the induction hypothesis, whence $(\mathcal{M}_1, x) \models \Diamond\beta$.

The cases $\alpha = \beta \to \gamma$ and $\alpha = \Box\beta$ can be treated in the same way, and the cases $\alpha = \beta \wedge \gamma$ and also $\alpha = \beta \vee \gamma$ are trivial. ∎

Corollary 2.3.8 *If $\mathcal{F}_1$ is reducible to $\mathcal{F}_2$, then $\boldsymbol{L}(\mathcal{F}_1) \subseteq \boldsymbol{L}(\mathcal{F}_2)$.*

Proof. Let f be a reduction of $\mathcal{F}_1$ to $\mathcal{F}_2$. Suppose $\mathcal{F}_2 \not\models \alpha$. Then $((\mathcal{F}_2, v_2), f(x)) \not\models \alpha$ for some v_2 on $\mathcal{F}_2$ and some $x \in \mathcal{F}_1$. (Note that f is onto.) Define a valuation v_1 on $\mathcal{F}_1$ by taking

$$v_1(p) := f^{-1}(v_2(p)) \ \text{ for all propositional variables } p.$$

By Theorem 2.3.7, $((\mathcal{F}_1, v_1), x) \not\models \alpha$. Therefore, $\mathcal{F}_1 \not\models \alpha$. ∎

Again, as a corollary of Theorem 2.3.5 we have the following.

Corollary 2.3.9 *If $\mathcal{F}$ is the disjoint union of a family $\{\mathcal{F}_i \mid i \in I\}$, then $\boldsymbol{L}(\mathcal{F}) = \bigcap_{i \in I} \boldsymbol{L}(\mathcal{F}_i)$.*

2.4 Correspondence between algebraic semantics and Kripke type semantics

In the following, we will show some relations between modal Heyting algebras and intuitionistic modal Kripke frames.

Definition 2.4.1 *(1) The map $(\ \cdot\ )^\dagger : \boldsymbol{IMF} \to \boldsymbol{m_{\Box\Diamond}HA}$ is defined as follows :*
For any $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond)$, $\mathcal{F}^\dagger = (\boldsymbol{UpW}, \Box, \Diamond)$, where $\boldsymbol{UpW} = (UpW, \cap, \cup, \to, \emptyset, W)$. Then $\mathcal{F}^\dagger$ is called the dual of $\mathcal{F}$.

(2) The map $(\ \cdot\ )_\dagger : \boldsymbol{m_{\Box\Diamond}HA} \to \boldsymbol{IMF}$ is defined as follows :
For any $\boldsymbol{A} = (\boldsymbol{A'}, \Box, \Diamond)$, $\boldsymbol{A}_\dagger = (W_{\boldsymbol{A}}, \lhd_{\boldsymbol{A}}, R_{\Box_{\boldsymbol{A}}}, R_{\Diamond_{\boldsymbol{A}}})$, where

(i) $W_{\boldsymbol{A}}$ is the set $PF(\boldsymbol{A})$ of all prime filters in $\boldsymbol{A}$,

(ii) $x \lhd_{\boldsymbol{A}} y \overset{def}{\Leftrightarrow} x \subseteq y$,

(iii) $xR_{\Box_{\boldsymbol{A}}}y \overset{def}{\Leftrightarrow} \forall a \in A(\Box a \in x \Rightarrow a \in y)(\Leftrightarrow x_\Box \subseteq y$, where $x_\Box := \{a \mid \Box a \in x\})$,

(iv) $xR_{\Diamond_{\boldsymbol{A}}}y \overset{def}{\Leftrightarrow} \forall a \in A(a \in y \Rightarrow \Diamond a \in x)(\Leftrightarrow y \subseteq x_\Diamond$, where $x_\Diamond := \{a \mid \Diamond a \in x\})$.

Then $\boldsymbol{A}_\dagger$ is called the dual of $\boldsymbol{A}$.

Proposition 2.4.2

(1) For every intuitionistic modal Kripke frame $\mathcal{F}$, its dual $\mathcal{F}^\dagger$ is a modal Heyting algebra.

(2) For every modal Heyting algebra $\boldsymbol{A}$, its dual $\boldsymbol{A}_\dagger$ is an intuitionistic modal Kripke frame.

Proof. It is routine to check our proposition. Compare (1) with Proposition 2.3.2. ∎

In relation to prime filters, the following well-known theorem can be obtained by using Zorn's Lemma.

Theorem 2.4.3 *Let G and J be a filter and an ideal of a given distributive lattice A such that $G \cap J = \emptyset$. Then there exists a prime filter F such that $G \subseteq F$ and $F \cap J = \emptyset$.*

The least filter $[X)$ containing a given non-empty set X in a lattice A is called the *filter generated by* X which can be represented by

$$[X) = \{y \in A \mid x_1 \wedge \cdots \wedge x_n \leq y \text{ for some } x_1, \ldots, x_n \in X\}.$$

The least ideal $(X]$ containing a given non-empty set X in a lattice A is called the *ideal generated by* X which can be represented by

$$(X] = \{y \in A \mid y \leq x_1 \vee \cdots \vee x_n \text{ for some } x_1, \ldots, x_n \in X\}.$$

Proposition 2.4.4 *(1) For every intuitionistic modal frame $\mathcal{F}$, $\mathcal{F}$ is embedded into $(\mathcal{F}^\dagger)_\dagger$.*

(2) For every modal Heyting algebra $\boldsymbol{A}$, $\boldsymbol{A}$ is embedded into $(\boldsymbol{A}_\dagger)^\dagger$.

Proof. (1). Let $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond)$ and $(\mathcal{F}^\dagger)_\dagger = (W_{\mathcal{F}^\dagger}, \lhd_{\mathcal{F}^\dagger}, R_{\Box_{\mathcal{F}^\dagger}}, R_{\Diamond_{\mathcal{F}^\dagger}})$. Define a map $f_\mathcal{F}$ from W into $W_{\mathcal{F}^\dagger}$ as follows; for all $x \in W$

$$f_\mathcal{F}(x) := \{a \in UpW \mid x \in a\} \in W_{\mathcal{F}^\dagger} (= PF(\boldsymbol{UpW})).$$

Suppose $x \lhd y$. If $a \in f_\mathcal{F}(x)$, then $y \in a$ since $x \in a$ and $a \in UpW$. Therefore $a \in f_\mathcal{F}(y)$, whence $f_\mathcal{F}(x) \lhd_{\mathcal{F}^\dagger} f_\mathcal{F}(y)$. Conversely, suppose $f_\mathcal{F}(x) \subseteq f_\mathcal{F}(y)$. Since $\{z \mid x \lhd z\} \in f_\mathcal{F}(x)$, $\{z \mid x \lhd z\} \in f_\mathcal{F}(y)$. Therefore $y \in \{z \mid x \lhd z\}$, whence $x \lhd y$.

Suppose $xR_\Box y$. If $\Box a \in f_\mathcal{F}(x)$, then $y \in a$ since $x \in \Box a$. Therefore $a \in f_\mathcal{F}(y)$, whence $f_\mathcal{F}(x) R_{\Box_{\mathcal{F}^\dagger}} f_\mathcal{F}(y)$. Conversely, suppose $(f_\mathcal{F}(x))_\Box \subseteq f_\mathcal{F}(y)$. Since $\Box\{z \mid xR_\Box z\} \in f_\mathcal{F}(x)$, $\{z \mid xR_\Box z\} \in f_\mathcal{F}(y)$. Therefore $y \in \{z \mid xR_\Box z\}$, whence $xR_\Box y$.

Suppose $xR_\Diamond y$. If $a \in f_\mathcal{F}(y)$, then $x \in \Diamond a$ since $y \in a$. Therefore $\Diamond a \in f_\mathcal{F}(x)$, whence $f_\mathcal{F}(x) R_{\Diamond_{\mathcal{F}^\dagger}} f_\mathcal{F}(y)$. Conversely, suppose $f_\mathcal{F}(y) \subseteq (f_\mathcal{F}(x))_\Diamond$. Since $\{z \mid y \lhd z\} \in f_\mathcal{F}(y)$, $\Diamond\{z \mid y \lhd z\} \in f_\mathcal{F}(x)$. Therefore since $x \in \Diamond\{z \mid y \lhd z\}$, there is z such that $xR_\Diamond z$ and $y \lhd z$, whence $xR_\Diamond y$.

(2). Let $\boldsymbol{A} = (\boldsymbol{A'}, \Box, \Diamond)$ and $(\boldsymbol{A}_\dagger)^\dagger = (UpW_{\boldsymbol{A}}, \Box, \Diamond)$. Define a map $f_{\boldsymbol{A}}$ from A into $UpW_{\boldsymbol{A}}$ as follows; for all $a \in A$

$$f_{\boldsymbol{A}}(a) := \{x \in PF(\boldsymbol{A}) \mid a \in x\} \in UpW_{\boldsymbol{A}} (= UpPF(\boldsymbol{A})).$$

Let $a \not\leq b$. Since there is a prime filter z in A such that $a \in z$ and $b \notin z$ by Theorem 2.4.3, $f_{\boldsymbol{A}}(a) \not\subseteq f_{\boldsymbol{A}}(b)$. Therefore $f_{\boldsymbol{A}}$ is a injection.

Let's check that $f_{\boldsymbol{A}}$ preserves the operations. It is easy to show $f_{\boldsymbol{A}}$ preserves $\wedge, \vee, \perp$. Suppose $x \in f_{\boldsymbol{A}}(a \to b)$. If $x \subseteq y$ and $a \in y$, then $b \in y$ since $a \to b \in x$. Therefore $x \in f_{\boldsymbol{A}}(a) \to f_{\boldsymbol{A}}(b)$. Conversely, suppose $x \notin f_{\boldsymbol{A}}(a \to b)$. If $a \wedge c \leq b$ for some $c \in x$ then $c \leq a \to b$. But this leads to a contradiction. Therefore $[x \cup \{a\})$ and $(\{b\}]$ are disjoint. Then there is a prime filter y such that $x \subseteq y$, $a \in y$ and $b \notin y$ by Theorem 2.4.3. This means $x \notin f_{\boldsymbol{A}}(a) \to f_{\boldsymbol{A}}(b)$. Next, suppose $x \in f_{\boldsymbol{A}}(\Box a)$. If $x_\Box \subseteq y$, then $a \in y$. Therefore $x \in \Box f_{\boldsymbol{A}}(a)$. Conversely, suppose $x \notin f_{\boldsymbol{A}}(\Box a)$. Since the filter $x_\Box$ does not contain a, there is a prime filter y such that $x_\Box \subseteq y$ and $a \notin y$ by Theorem 2.4.3. This means $x \notin \Box f_{\boldsymbol{A}}(a)$.

Suppose $x \in f_{\boldsymbol{A}}(\Diamond a)$. Since the ideal $-(x_\Diamond)$ does not contain a, there is a prime filter y such that $y \subseteq x_\Diamond$ and $a \in y$ by Theorem 2.4.3. This implies that $x \in \Diamond f_{\boldsymbol{A}}(a)$. Conversely, suppose $x \in \Diamond f_{\boldsymbol{A}}(a)$. There is a prime filter y such that $y \subseteq x_\Diamond$ and $f_{\boldsymbol{A}}(y) \in a$. Then $\Diamond a \in x$, since $a \in y$. Therefore $x \in f_{\boldsymbol{A}}(\Diamond a)$. ∎

Proposition 2.4.5

(1) For every intuitionistic modal frame $\mathcal{F}$, $\boldsymbol{L}(\mathcal{F}^\dagger) = \boldsymbol{L}(\mathcal{F})$.

(2) For every modal Heyting algebra $\boldsymbol{A}$, $\boldsymbol{L}(\boldsymbol{A}_\dagger) \subseteq \boldsymbol{L}(\boldsymbol{A})$.

Proof. (1). By the definition, each valuation on $\mathcal{F}$ is at the same time regarded as a valuation on $\mathcal{F}^\dagger$, and vice versa.

(2). Since $\boldsymbol{A}$ is isomorphic to a subalgebra of $(\boldsymbol{A}_\dagger)^\dagger$ by Proposition 2.4.4 (2), $\boldsymbol{L}((\boldsymbol{A}_\dagger)^\dagger) \subseteq \boldsymbol{L}(\boldsymbol{A})$ by corollary 2.3.6. Also by (1), $\boldsymbol{L}((\boldsymbol{A}_\dagger)^\dagger) = \boldsymbol{L}(\boldsymbol{A}_\dagger)$. Hence $\boldsymbol{L}(\boldsymbol{A}_\dagger) \subseteq \boldsymbol{L}(\boldsymbol{A})$. ∎

2.5 General frame semantics

In this section we will consider general frame semantics for intuitionistic modal logics.

2.5.1 General frames

Definition 2.5.1 *(1) A structure $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond, \mathcal{P})$ is called an* intuitionistic modal general frame *if the following conditions are satisfied.*

(i) $(W, \lhd, R_\Box, R_\Diamond)$ is an intuitionistic modal Kripke frame,

(ii) A subset $\mathcal{P}$ of UpW is called a modal Heyting algebra *on a Kripke frame $(W, \lhd, R_\Box, R_\Diamond)$ satisfies as follows;*

- $\emptyset \in \mathcal{P}$,
- $X, Y \in \mathcal{P} \Rightarrow X \cap Y, X \cup Y \in \mathcal{P}$,
- $X, Y \in \mathcal{P} \Rightarrow X \to Y \in \mathcal{P}$,
- $X \in \mathcal{P} \Rightarrow \Box X, \Diamond X \in \mathcal{P}$.

(iii) If $\mathcal{P} = UpW$, then $\mathcal{F}$ is called a full (or Kripke) frame and is sometimes written $(W, \lhd, R_\Box, R_\Diamond)$ instead of $(W, \lhd, R_\Box, R_\Diamond, UpW)$. The underlying full frame of $\mathcal{F}$ is denoted by $\kappa\mathcal{F}$.

(2) *A* valuation *v on $\mathcal{F}$ is a function* : $\mathcal{F}orm(\mathcal{L}_{\Box\Diamond}) \to \mathcal{P}$ *which satisfies the following conditions*

(i) $v(\bot) = \emptyset$,

(ii) $v(\alpha \wedge \beta) = v(\alpha) \cap v(\beta)$,

(iii) $v(\alpha \vee \beta) = v(\alpha) \cup v(\beta)$,

(iv) $v(\alpha \to \beta) = v(\alpha) \to v(\beta)$,

(v) $v(\Box\alpha) = \Box v(\alpha)$,

(vi) $v(\Diamond\alpha) = \Diamond v(\alpha)$,

(3) *A pair $\mathcal{M} = (\mathcal{F}, v)$ of an intuitionistic modal general frame $\mathcal{F}$ and a valuation v on $\mathcal{F}$ is called a* model. *In this case, $\mathcal{F}$ is called the* base *of a model $\mathcal{M}$.*

(4) *For any $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$, any model $\mathcal{M}$ and any $x \in W, \alpha$ is* true *at x in $\mathcal{M}$ (in symbol, $(\mathcal{M}, x) \models \alpha$ or simply $x \models \alpha$ if $\mathcal{M}$ is understood) if $x \in v(\alpha)$.*

(5) *For any $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ and any model $\mathcal{M}, \alpha$ is* true *in $\mathcal{M}$ (in symbol, $\mathcal{M} \models \alpha$) if $W = v(\alpha)$. If it is not true in $\mathcal{M}$ then it is* refuted *in $\mathcal{M}$.*

(6) *For any $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ and any intuitionistic modal general frame $\mathcal{F}$, α is* valid *in $\mathcal{F}$ (in symbol, $\mathcal{F} \models \alpha$) if $W = v(\alpha)$ for any valuation v on $\mathcal{F}$.*

Note that the value of a given valuation v is uniquely determined only by its value for each propositional variable.

Proposition 2.5.2 *(1) Let $\mathcal{F}$ be an intuitionistic modal general frame. The set of formulas which are valid in $\mathcal{F}$ is an intuitionistic modal logic.*

(2) Let $\mathcal{C}$ be a class of intuitionistic modal general frames. The set of formulas which are valid in all frames in $\mathcal{C}$ is an intuitionistic modal logic.

They are called the logic characterized by $\mathcal{F}$ and the logic characterized by $\mathcal{C}$ and are denoted by $\boldsymbol{L}(\mathcal{F})$ and $\boldsymbol{L}(\mathcal{C})$, respectively.

2.5.2 Truth-preserving operations

In this subsection we will introduce three important operations on intuitionistic modal general frames which preserve the validity.

Definition 2.5.3 *(1) A general frame $\mathcal{F}_1 = (W_1, \lhd_1, R_{\Box_1}, R_{\Diamond_1}, \mathcal{P}_1)$ is called a* generated subframe *of a general frame $\mathcal{F}_2 = (W_2, \lhd_2, R_{\Box_2}, R_{\Diamond_2}, \mathcal{P}_2)$ if the following conditions are satisfied.*

(i) $W_1 \subseteq W_2$,

(ii) $\lhd_1$ *($R_{\Box_1}$ and $R_{\Diamond_1}$) is the restriction of $\lhd_2$ ($R_{\Box_2}$ and $R_{\Diamond_2}$, respectively) to W_1.*

(iii) $x \in W_1 \;\&\; x \lhd_2 y \Rightarrow y \in W_1$, *(iv)* $x \in W_1$ & $xR_{\Box_2}y \Rightarrow y \in W_1$,

(v) $x \in W_1$ & $xR_{\Diamond_2}y \Rightarrow \exists z \in W_1 \ : \ xR_{\Diamond_1}z$ & $y \lhd_2 z$.

(vi) $\mathcal{P}_1 = \{Y \cap W_1 \mid Y \in \mathcal{P}_2\}$.

(2) A map $f : W_1 \to W_2$ is said to be a p-morphism *from a general frame $\mathcal{F}_1$ to a general frame $\mathcal{F}_2$ if for all $x \in W_1, y \in W_2$,*

(i) $f(x) \lhd_2 y \Leftrightarrow \exists z \in W_1 \ : \ x \lhd_1 z$ & $f(z) = y$,

(ii) $f(x)R_{\Box_2}y \Leftrightarrow \exists z \in W_1 \ : \ xR_{\Box_1}z$ & $f(z) = y$,

(iii) $xR_{\Diamond_1}y \Rightarrow f(x)R_{\Diamond_2}f(y)$,

(iv) $f(x)R_{\Diamond_2}y \Rightarrow \exists z \in W_1 \ : \ xR_{\Diamond_1}z$ & $y \lhd_2 f(z)$.

(v) $f^{-1}(Y) \in \mathcal{P}_1$ *for* $Y \in \mathcal{P}_2$.

(3) A general frame $\mathcal{F}_1$ is said to be reducible *to a general frame $\mathcal{F}_2$ if there exists a onto p-morphism (say* reduction*) $f : W_1 \to W_2$.*

(4) The general frame $\sum_{i\in I} \mathcal{F}_i = (\sum_{i\in I} W_i, \bigcup_{i\in I} \lhd_i, \bigcup_{i\in I} R_{\Box_i}, \bigcup_{i\in I} R_{\Diamond_i}, \{\bigcup_{i\in I} X_i \mid X_i \in \mathcal{P}_i, i \in I\})$ *is called the* disjoint union *of a disjoint family* $\{\mathcal{F}_i \mid i \in I\}$.

Note that each $\mathcal{F}_i$ is a generated subframe of the disjoint union of $\{\mathcal{F}_i \mid i \in I\}$.
We can show the following results similarly to former results.

Theorem 2.5.4 *Suppose that $\mathcal{F}_1$ is a generated subframe of $\mathcal{F}_2$, and that $\mathcal{M}_1$ (and $\mathcal{M}_2$) is a model with the base $\mathcal{F}_1$ (and with the base $\mathcal{F}_2$, respectively). If for every propositional variable p and every x in $\mathcal{F}_1$*

$$(\mathcal{M}_1, x) \models p \ \text{ iff } \ (\mathcal{M}_2, x) \models p$$

then for every formula α and every x in $\mathcal{F}_1$

$$(\mathcal{M}_1, x) \models \alpha \ \text{ iff } \ (\mathcal{M}_2, x) \models \alpha.$$

Corollary 2.5.5 *If $\mathcal{F}_1$ is a generated subframe of $\mathcal{F}_2$, then $\boldsymbol{L}(\mathcal{F}_2) \subseteq \boldsymbol{L}(\mathcal{F}_1)$.*

Theorem 2.5.6 *Suppose that f is a reduction of $\mathcal{F}_1$ to $\mathcal{F}_2$, , and that $\mathcal{M}_1$ (and $\mathcal{M}_2$) is a model with the base $\mathcal{F}_1$ (and with the base $\mathcal{F}_2$, respectively). If for every propositional variable p and every x in $\mathcal{F}_1$*

$$(\mathcal{M}_1, x) \models p \ \text{ iff } \ (\mathcal{M}_2, f(x)) \models p$$

then for every formula α and every x in $\mathcal{F}_1$

$$(\mathcal{M}_1, x) \models \alpha \ \text{ iff } \ (\mathcal{M}_2, f(x)) \models \alpha.$$

Corollary 2.5.7 *If $\mathcal{F}_1$ is reducible to $\mathcal{F}_2$, then $\boldsymbol{L}(\mathcal{F}_1) \subseteq \boldsymbol{L}(\mathcal{F}_2)$.*

Again, as a corollary of Theorem 2.5.4 we have the following.

Corollary 2.5.8 *If $\mathcal{F}$ is the disjoint union of a family $\{\mathcal{F}_i \mid i \in I\}$, then $\boldsymbol{L}(\mathcal{F}) = \bigcap_{i\in I} \boldsymbol{L}(\mathcal{F}_i)$.*

2.6 Correspondence between algebraic semantics and general frame semantics

In the following, we will show some relations between modal Heyting algebras and intuitionistic modal general frames.

Definition 2.6.1 *(1) For any intuitionistic modal general frame $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond, \mathcal{P})$, define a modal Heyting algebra $\mathcal{F}^+ = (\boldsymbol{\mathcal{P}}, \Box, \Diamond)$, where $\boldsymbol{\mathcal{P}} = (\mathcal{P}, \cap, \cup, \rightarrow, \emptyset, W)$. Then $\mathcal{F}^+$ is called the dual of $\mathcal{F}$. Also define the map $f_\mathcal{F}$ from W into $PF(\boldsymbol{\mathcal{P}})$ as follows; for all $x \in W$*

$$f_\mathcal{F}(x) := \{X \in \mathcal{P} \mid x \in X\} \in PF(\boldsymbol{\mathcal{P}}).$$

(2) For any modal Heyting algebra $\boldsymbol{A} = (\boldsymbol{A'}, \Box, \Diamond)$, we define an intuitionistic modal general frame $\boldsymbol{A}_+ = (W_{\boldsymbol{A}}, \lhd_{\boldsymbol{A}}, R_{\Box_{\boldsymbol{A}}}, R_{\Diamond_{\boldsymbol{A}}}, \mathcal{P}_{\boldsymbol{A}})$, where

(i) $W_{\boldsymbol{A}}$ is the set of all prime filters of $\boldsymbol{A}$

(ii) $x \lhd_{\boldsymbol{A}} y \overset{def}{\Leftrightarrow} x \subseteq y$,

(iii) $x R_{\Box_{\boldsymbol{A}}} y \overset{def}{\Leftrightarrow} \forall a \in A(\Box a \in x \Rightarrow a \in y)$,

(iv) $x R_{\Diamond_{\boldsymbol{A}}} y \overset{def}{\Leftrightarrow} \forall a \in A(a \in y \Rightarrow \Diamond a \in x)$,

(v) The map $f_{\boldsymbol{A}}$ from A into $UpW_{\boldsymbol{A}}$ is defined as follows; for all $a \in A$

$$f_{\boldsymbol{A}}(a) := \{x \in PF(\boldsymbol{A}) \mid a \in x\} \in UpW_{\boldsymbol{A}}(= UpPF(\boldsymbol{A})).$$

(vi) $\mathcal{P}_{\boldsymbol{A}} := \{f_{\boldsymbol{A}}(a) \mid a \in A\}$

Then $\boldsymbol{A}_+$ is called the dual of $\boldsymbol{A}$.

Proposition 2.6.2

(1) For every intuitionistic modal general frame $\mathcal{F}$, its dual $\mathcal{F}^+$ is a modal Heyting algebra.

(2) For every modal Heyting algebra $\boldsymbol{A}$, its dual $\boldsymbol{A}_+$ is an intuitionistic modal general frame.

Proposition 2.6.3 *For every modal Heyting algebra $\boldsymbol{A}$, $\boldsymbol{A}$ is isomorphic to $(\boldsymbol{A}_+)^+$ with being $f_{\boldsymbol{A}}$ an isomorphism.*

Proof. By Proposition 2.4.4 (2), $f_{\boldsymbol{A}}$ is a one to one homomorphism. Since $\mathcal{P}_{\boldsymbol{A}}$ is the image of A by $f_{\boldsymbol{A}}$, $f_{\boldsymbol{A}}$ is a surjection. ∎

Proposition 2.6.4

(1) For every intuitionistic modal general frame $\mathcal{F}$, $\boldsymbol{L}(\mathcal{F}^+) = \boldsymbol{L}(\mathcal{F})$.

(2) For every modal Heyting algebra $\boldsymbol{A}$, $\boldsymbol{L}(\boldsymbol{A}_+) = \boldsymbol{L}(\boldsymbol{A})$.

By Proposition 2.6.3 every modal Heyting algebra $\boldsymbol{A}$ is isomorphic to its *bidual* $(\boldsymbol{A}_+)^+$. On the other hand, there are intuitionistic modal general frames $\mathcal{F}$ which are not isomorphic to its *bidual* $(\mathcal{F}^+)_+$. For example, for intuitionistic modal general frames of the form $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond, \{\emptyset, W\})$, their bidual $(\mathcal{F}^+)_+$ are always singletons. So the relation $\mathcal{F} \cong (\mathcal{F}^+)_+$ does not generally hold.

Definition 2.6.5 *An intuitionistic modal general frame $\mathcal{F}$ is said to be* descriptive *if $\mathcal{F} \cong (\mathcal{F}^+)_+$.*

Definition 2.6.6 *For every intuitionistic modal general frame $\mathcal{F}$,*

1. *$\mathcal{F}$ is* differentiated *if $\forall x, y \in W (x \neq y \Rightarrow \exists X \in \mathcal{P} : x \in X, y \notin X)$,*
2. *$\mathcal{F}$ is* i-tight *if $\forall x, y \in W (x \not\lhd y \Rightarrow \exists X \in \mathcal{P} : x \in X, y \notin X)$,*
3. *$\mathcal{F}$ is* $\Box$-tight *if $\forall x, y \in W (x \not{R}_\Box y \Rightarrow \exists X \in \mathcal{P} : x \in \Box X, y \notin X)$,*
4. *$\mathcal{F}$ is* $\Diamond$-tight *if $\forall x, y \in W (x \not{R}_\Diamond y \Rightarrow \exists X \in \mathcal{P} : y \in X, x \notin \Diamond X)$,*
5. *$\mathcal{A} \subseteq \mathcal{P}(W)$ has* finite intersection property *if $\bigcap \mathcal{A}_0 \neq \emptyset$ for finite $\mathcal{A}_0 \subseteq \mathcal{A}$.*
6. *$\mathcal{F}$ is* compact *if $\forall \mathcal{A} \subseteq \mathcal{P} \cup \overline{\mathcal{P}}(\mathcal{A}$ has finite intersection property $\Rightarrow \bigcap \mathcal{A} \neq \emptyset)$, where $\overline{\mathcal{P}} = \{-X \mid X \in \mathcal{P}\}$.*

Note that $\mathcal{F}$ is i-tight only if it is differentiated, and that $\overline{\mathcal{P}} = \mathcal{P}$ if $\mathcal{P}$ is a modal algebra.

Proposition 2.6.7 *An intuitionistic modal general frame $\mathcal{F}$ is descriptive iff $\mathcal{F}$ is differentiated, i-tight, $\Box$-tight, $\Diamond$-tight and compact.*

Proof. It is enough to show that the map $f_\mathcal{F}$ is an isomorphism from $\mathcal{F}$ to $(\mathcal{F}^+)_+$ iff $\mathcal{F}$ is differentiated, i-tight, $\Box$-tight, $\Diamond$-tight and compact. We can show that $f_\mathcal{F}$ is an injection iff $\mathcal{F}$ is differentiated, that $f_\mathcal{F}$ is a surjection iff $\mathcal{F}$ is compact, that $(x \lhd y \Leftrightarrow f_\mathcal{F}(x) \lhd_{\mathcal{F}^+} f_\mathcal{F}(y))$ iff $\mathcal{F}$ is i-tight, that $(x_\Box y \Leftrightarrow f_\mathcal{F}(x) R_{\Box_{\mathcal{F}^+}} f_\mathcal{F}(y))$ iff $\mathcal{F}$ is $\Box$-tight and that $(x R_\Diamond y \Leftrightarrow f_\mathcal{F}(x) R_{\Diamond_{\mathcal{F}^+}} f_\mathcal{F}(y))$ iff $\mathcal{F}$ is $\Diamond$-tight. ∎

Proposition 2.6.8

(1) If h is an isomorphism of an intuitionistic modal general frame $\mathcal{F}_2 = (W_2, \lhd_2, R_{\Box_2}, R_{\Diamond_2}, \mathcal{P}_2)$ onto a generated subframe of an intuitionistic modal general frame $\mathcal{F}_1 = (W_1, \lhd_1, R_{\Box_1}, R_{\Diamond_1}, \mathcal{P}_1)$ then the map h^+ defined by

$$h^+(X) = h^{-1}(X), \text{ for every } X \in \mathcal{P}_1,$$

is a homomorphism of $\mathcal{F}_1^+$ onto $\mathcal{F}_2^+$.

(2) If h is a homomorphism of a modal Heyting algebra $\boldsymbol{A_1}$ onto a modal Heyting algebra $\boldsymbol{A_2}$ then the map h_+ defined by

$$h_+(F) = h^{-1}(F), \text{ for every prime filter } F \text{ in } \boldsymbol{A_2},$$

is an isomorphism of $(\boldsymbol{A_2})_+$ onto a generated subframe of $(\boldsymbol{A_1})_+$.

Proposition 2.6.9

(1) If f is a reduction of an intuitionistic modal general frame $\mathcal{F}_1 = (W_1, \lhd_1, R_{\Box_1}, R_{\Diamond_1}, \mathcal{P}_1)$ to an intuitionistic modal general frame $\mathcal{F}_2 = (W_2, \lhd_2, R_{\Box_2}, R_{\Diamond_2}, \mathcal{P}_2)$ then the map f^+ defined by

$$f^+(X) = f^{-1}(X), \text{ for every } X \in \mathcal{P}_2,$$

is an isomorphism of $\mathcal{F}_2^+$ onto a subalgebra of $\mathcal{F}_1^+$.

(2) If f is an isomorphism of a modal Heyting algebra $\boldsymbol{A_2}$ onto a subalgebra of a modal Heyting algebra $\boldsymbol{A_1}$ then the map f_+ defined by

$$f_+(F) = f^{-1}(F), \text{ for every prime filter } F \text{ in } \boldsymbol{A_1},$$

is a reduction of $(\boldsymbol{A_1})_+$ to $(\boldsymbol{A_2})_+$.

Proposition 2.6.10 *Suppose $\{\mathcal{F}_i = (W_i, \lhd_i, R_{\Box_i}, R_{\Diamond_i}, \mathcal{P}_i) \mid i \in I\}$ is a family of descriptive frames. Then $\sum_{i \in I} \mathcal{F}_i$ is descriptive iff I is finite.*

Proposition 2.6.11

(1) Let $\{\mathcal{F}_i = (W_i, \lhd_i, R_{\Box_i}, R_{\Diamond_i}, \mathcal{P}_i) \mid i \in I\}$ is a family of intuitionistic modal general frames and $\sum_{i \in I} \mathcal{F}_i = (W, \lhd, R_\Box, R_\Diamond, \mathcal{P})$ their disjoint union. Then the map f defined by

$$f(X)(i) = X \cap W_i, \text{ for every } X \in \mathcal{P} \text{ and } i \in I,$$

is an isomorphism of $(\sum_{i \in I} \mathcal{F}_i)^+$ onto $\prod_{i \in I} \mathcal{F}_i^+$.

(2) Suppose that both $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are modal Heyting algebras. Then the map f defined by

$$f(F_1) = \{(a_1, a_2) \in \boldsymbol{A_1} \times \boldsymbol{A_2} \mid a_1 \in F_1, a_2 \in A_2\}, \text{ for every prime filter } F_1 \text{ in } \boldsymbol{A_1},$$

and

$$f(F_2) = \{(a_1, a_2) \in \boldsymbol{A_1} \times \boldsymbol{A_2} \mid a_1 \in A_1, a_2 \in F_2\}, \text{ for every prime filter } F_2 \text{ in } \boldsymbol{A_2},$$

is an isomorphism of $(\boldsymbol{A_1})_+ + (\boldsymbol{A_2})_+$ onto $(\boldsymbol{A_1} \times \boldsymbol{A_2})_+$.

2.7 Note

Many ways of defining intuitionistic analogues of classical normal modal logics have been considered. First, one can take the family of logics extending $\mathbf{IntK}_\Box$. A model theory for a logic extending $\mathbf{IntK}_\Box$ was developed by H.Ono [17], M. Bošić and K. Došen [3], V. H. Sotirov [26] and F. Wolter and M. Zakharyaschev [33]. A possibility operator $\Diamond$ in those logics can be defined in the classical way by taking $\Diamond\varphi$ as $\neg\Box\neg\varphi$. Note, however, that in general this $\Diamond$ doesn't distribute over disjunction and that the connection via negation between $\Box$ and $\Diamond$ is too strong from intuitionistic point of view.

Another family of normal logics is logics extending $\mathbf{IntK}_{\Diamond}$ These logics were studied by M. Bošić and K. Došen [3], V. H. Sotirov [26] and F. Wolter [29].

Fischer Servi [7] constructed **FC**, the logic $\mathbf{IntK}_{\Box\Diamond}$ with a weak condition between the necessity operator $\Box$ and possibility operator $\Diamond$. The standard translation of modal formulas into first order formulas not only embeds **K** into classical predicate logic but also **FC** into intuitionistic predicate logic. Various extensions of **FC** were studied by R.A. Bull [4], H. Ono [17], G. Fischer Servi [6][7], F. Wolter and M. Zakharyaschev [33], F. Wolter [29] and C. Grefe [12]. A well-known extension of **FC** is the logic **MIPC** introduced A. Prior [20]. R.A. Bull noticed that **MIPC** is embedded into the monadic fragment of intuitionistic predicate logic. H. Ono [17], H. Ono and N.-Y. Suzuki [18] and G. Bezhanishvili [1] investigated the relation between logics extending **MIPC** and superintuitionistic predicate logics, and their models.

V. H. Sotirov studied in [26] weaker logics than $\mathbf{IntK}_{\Box\Diamond}$, logics which are not necessarily normal, and their models. In [26] he studied as a part of those logics $\mathbf{IntK}_{\Box\Diamond}$ (he called $\mathbf{IK}(\Box\Diamond)$) and its extensions.

Our based logic $\mathbf{IntK}_{\Box\Diamond}$ is induced by F. Wolter and M. Zakharyaschev. The algebraic semantics and the dual relation semantics, i.e. Kripke type semantics and General frame semantics presented in this chapter have been originally defined in [33]. The theorems which is presented in this chapter are straightforward generalizations of analogous results in superintuitionistic and classical modal logic (see e.g.[5]).

Chapter 3

Pseudo-Euclidean logics

3.1 Introduction

Throughout this chapter, m and n are fixed non-negative integers. Let E_k be the logic which is obtained from the smallest normal modal logic **K** by adding the axiom $\Diamond^k\phi \to \Box^m\Diamond^n\phi$, where $k \geq 0$. Since each axiom $\Diamond^k\phi \to \Box^m\Diamond^n\phi$ is a Sahlqvist formula, we can show that the logic E_k is Kripke complete for each k. A binary relation R on a set W is *k-pseudo-Euclidean* if for any $x, y, z \in W$, xR^ky and xR^mz imply zR^ny. Note that when $m = n = 1$, 1-pseudo-Euclidean relations are equal to Euclidean relation. Let $\mathcal{PE}_k$ be the class of all Kripke frames of the form (W, R), where R is a k-pseudo-Euclidean relation on W. Then, it is easy to see that E_k is Kripke complete with respect to $\mathcal{PE}_k$ and that $E_k \supseteq E_{k'}$ if and only if $\mathcal{PE}_k \subseteq \mathcal{PE}_{k'}$. Here, we identify the axiom system E_k with the set of all formulas provable in E_k. Our main goal of this chapter is to show when $E_k \supseteq E_{k'}$ holds. The answer is given as follows.

Theorem 3.1.1 *1. If $m > n \geq 0$ then*

$$E_k \supseteq E_{k'} \quad \text{iff} \quad \text{either}$$
$$1)\, k' = k \text{ if } m + n > k > m, \text{ or}$$
$$2)\, k' \geq k \text{ and } (k - m - n) \,\Big|\, (k' - m - n) \text{ if either } k \geq m + n \text{ or } m \geq k \geq 0.$$

2. If $m = n > 0$

$$E_k \supseteq E_{k'} \quad \text{iff} \quad \text{either}$$
$$1)\, k' = k \text{ if either } 2m > k > m \text{ or } k = 0, \text{ or}$$
$$2)\, k' \geq k \text{ and } (k - 2m) \,\Big|\, (k' - 2m) \text{ if either } k \geq 2m \text{ or } m \geq k > 0.$$

3. If $n > m > 0$

$$E_k \supseteq E_{k'} \quad \text{iff} \quad \text{either}$$
$$1)\, k' = k \text{ if } m + n > k \geq 0, \text{ or}$$
$$2)\, k' \geq k \text{ and } (k - m - n) \,\Big|\, (k' - m - n) \text{ if } k \geq m + n.$$

4. If $m = 0$

$$E_k \supseteq E_{k'} \quad \textit{iff} \quad \textit{either}$$
$$1) k' = k \textit{ or } k' = n \textit{ if } n > k \geq 0, \textit{ or}$$
$$2) k' \geq n \textit{ and } (k-n) \,\Big|\, (k'-n) \textit{ if } k \geq n.$$

For example, when $m = n = 1$ and $k, k' > 0$

$$E_k \supseteq E_{k'} \text{ iff } k' \geq k \text{ and } (k-2) \,\Big|\, (k'-2)$$

$$E_k \supseteq E_{k'} \Leftrightarrow (k, k') \text{ in the graphs}$$

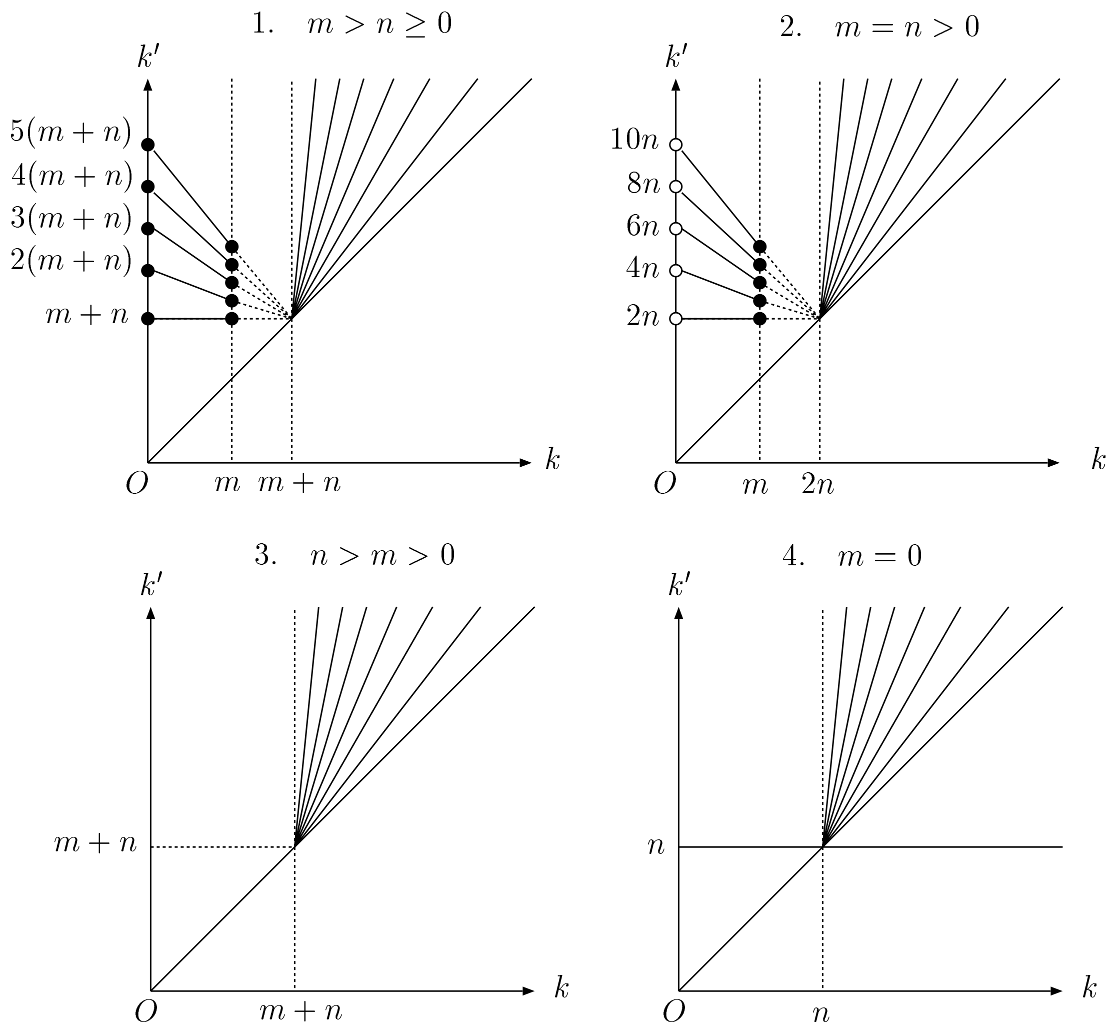


Figure 3.1:

We will give a proof of this theorem in the following section.

3.2 Proof of the theorem

When $k' = k$, it is clear that $E_k = E_{k'}$. We will show the rests in the following.

Lemma 3.2.1 *If $m = 0$ and $k' = n$ then $E_k \supseteq E_{k'}$.*

Proof. It is clear that $E_k \supseteq E_{k'}$ since $E_{k'}$ coincides with **K**.

∎

Lemma 3.2.2 *If $k > k'$ and either $m > 0$ or $k' \neq n$ then $E_k \not\supseteq E_{k'}$.*

Proof. When $k > m$, define a frame $\mathcal{F} = (W, R)$ as follows;

$$
\begin{aligned}
W &= \{w_i \mid 0 \leq i \leq k' + m\}, \\
w_i R w_j &\Leftrightarrow \text{either} \\
&\quad 1) j = i + 1 \text{ if } m \leq i \leq k' + m - 1 \text{ or} \\
&\quad 2) j = i - 1 \text{ if } 1 \leq i \leq m.
\end{aligned}
$$

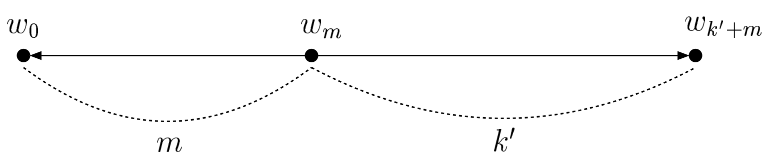


Figure 3.2:

Then, we can show that both $w_m R^{k'} w_{k'+m}$ and $w_m R^m w_0$ hold, while $w_0 R^n w_{k'+m}$ doesn't, since either $m > 0$ or $k' \neq n$. Thus, $\mathcal{F} \notin \mathcal{PE}_{k'}$. On the other hand, for each $x \in W$, there is no $y \in W$ such that $xR^k y$ since $k > k'$ and $k > m$. Therefore $\mathcal{F} \in \mathcal{PE}_k$ since R is k-pseudo-Euclidean.

When $k \leq m$, take a frame $\mathcal{F} = (W, R)$ as follows;

$$
\begin{aligned}
W &= \{w_i \mid 0 \leq i \leq k' + 1\}, \\
w_i R w_j &\Leftrightarrow \text{either} \\
&\quad 1) j = i + 1 \text{ if } 1 \leq i \leq k' \text{ or} \\
&\quad 2) i = 1 \text{ and } j = 0 \text{ or} \\
&\quad 3) i = 0 \text{ and } j = 0.
\end{aligned}
$$

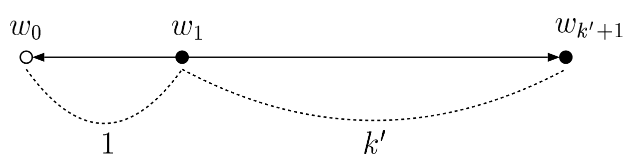


Figure 3.3:

Then, we can show that both $w_1 R^{k'} w_{k'+1}$ and $w_1 R^m w_0$ hold, while $w_0 R^n w_{k'+1}$ doesn't. Thus $\mathcal{F} \notin \mathcal{PE}_{k'}$. Next, suppose that both $xR^k y$ and $xR^m z$ hold, for a given $x \in W$. Since

$k > k'$ and $m > k'$, both y and z are equal to w_0. Hence $zR^n y$, i.e. $w_0 R^n w_0$, holds since w_0 is a reflexive point. Therefore $\mathcal{F} \in \mathcal{PE}_k$. ∎

Lemma 3.2.3 *If $k' \geq k \geq m+n$ and $(k-m-n) \bigm| (k'-m-n)$ then $E_k \supseteq E_{k'}$.*

Proof. By the assumption, $k'-m-n = h(k-m-n)$, that is $k' = k+(h-1)(k-m-n)$, for a certain number $h \in \boldsymbol{Z}$. Since $k' \geq k$ and $k-m-n \geq 0$, we can assume that $k' = k+(h-1)(k-m-n)$ for a certain number $h \geq 1$. To show that $E_k \supseteq E_{k'} = E_{k+(h-1)(k-m-n)}$, it is enough to show that every $(W,R) \in \mathcal{PE}_k$ belongs also to $\mathcal{PE}_{k+(h-1)(k-m-n)}$ for any $h \geq 1$. This can be shown by the induction on h.

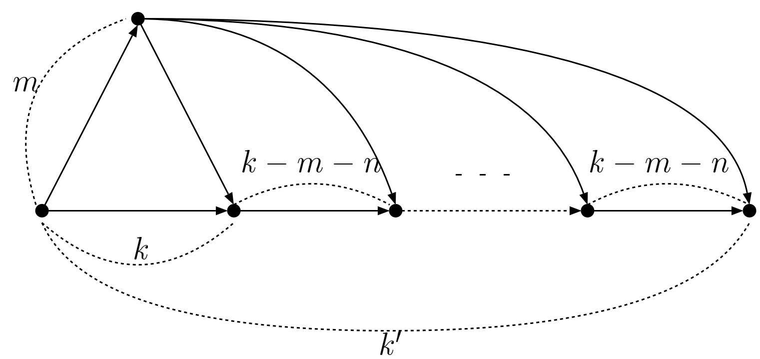


Figure 3.4:

If $h=1$, this is trivial since $k+(h-1)(k-m-n) = k$. So, we assume that this holds for h. To show that (W,R) belongs to $\mathcal{PE}_{k+h(k-m-n)}$, we assume that $xR^{k+h(k-m-n)}y$ and $xR^m z$. Then, for some $w \in W$, $xR^{k+(h-1)(k-m-n)}w$ and $wR^{k-m-n}y$, since $k+(h-1)(k-m-n) \geq 0$ and $k-m-n \geq 0$. Since (W,R) belongs to $\mathcal{PE}_{k+(h-1)(k-m-n)}$ by the hypothesis of induction, $xR^{k+(h-1)(k-m-n)}w$ and $xR^m z$ imply $zR^n w$. Since $xR^m z$, $zR^n w$ and $wR^{k-m-n}y$ hold, $xR^k y$. But since (W,R) is in $\mathcal{PE}_k$, $zR^n y$. Thus, we have shown that (W,R) belongs to $\mathcal{PE}_{k+h(k-m-n)}$. ∎

Lemma 3.2.4 *If $E_k \supseteq E_{k'}$ then $(k-m-n) \bigm| (k'-m-n)$.*

Proof. Suppose that $E_k \supseteq E_{k'}$ but $(k-m-n) \bigm| (k'-m-n)$ doesn't hold. For $a = k-m-n$, we define a frame $\mathcal{F} = (W,R)$ as follows;

$$\begin{aligned} W &= \{w_i \mid i \in \boldsymbol{Z}/a\boldsymbol{Z}\}, \\ w_i R w_j &\Leftrightarrow j \equiv i+1 \pmod{a}. \end{aligned}$$

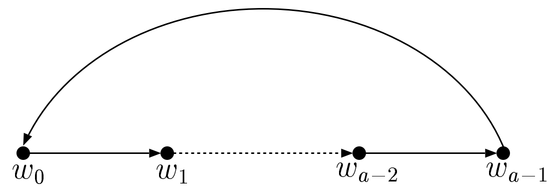


Figure 3.5:

By the assumption, since $k' - m \neq n + h(k - m - n)$ for any $h \in \boldsymbol{Z}$, i.e. $k' - m \not\equiv n \pmod a$, $w_m R^n w_{k'}$ doesn't hold. On the other hand, both $w_0 R^{k'} w_{k'}$ and $w_0 R^m w_m$ hold. Thus $\mathcal{F} \notin \mathcal{PE}_{k'}$. Next, suppose that $w_i R^k w_j$ and $w_i R^m w_s$. Then, $j - i \equiv k \pmod a$ and $s - i \equiv m \pmod a$. Hence $j - s \equiv k - m \pmod a$. But $k - m \equiv n \pmod a$ since $a = k - m - n$. Thus $j - s \equiv n \pmod a$, i.e. $w_s R^n w_j$. Hence $\mathcal{F} \in \mathcal{PE}_k$. This contradicts that $E_k \supseteq E_{k'}$. ∎

Lemma 3.2.5 *If $k' > k$ and $m + n \geq k > m > 0$ then $E_k \not\supseteq E_{k'}$.*

Proof. Define a frame $\mathcal{F} = (W, R)$ as follows;

$$
\begin{aligned}
W &= \{w_i \mid 0 \leq i \leq m+n+1\}, \\
w_i R w_j &\Leftrightarrow \text{either} \\
& \quad 1) j = i+1 \text{ if } 0 \leq i \leq m+n \text{ or} \\
& \quad 2) j = i-1 \text{ if } m+2 \leq i \leq m+n+1 \text{ or} \\
& \quad 3) j = i \text{ if } m+1 \leq i \leq m+n+1 \text{ or} \\
& \quad 4) i = 0 \text{ and } j = m+n+1-k.
\end{aligned}
$$

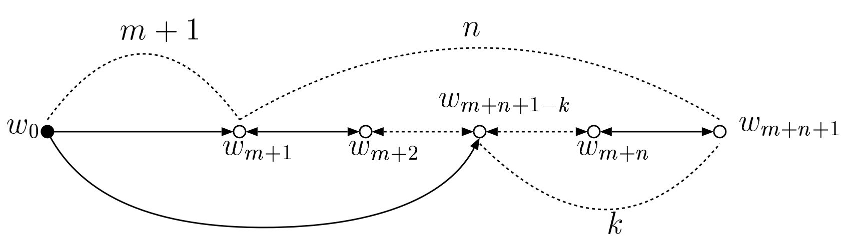


Figure 3.6:

First, we will show that $\mathcal{F} \in \mathcal{PE}_k$. If $i \geq 1$, $w_i R^k w_j$ and $w_i R^m w_{j'}$ then both w_j and $w_{j'}$ are between w_{m+1} and w_{m+n+1} since $i + k \geq m + 1$ and $i + k \geq m + 1$. Thus $w_{j'} R^n w_j$. If $w_0 R^k w_j$ and $w_0 R^m w_{j'}$ then $w_{j'} R^n w_j$ since $m + 1 \leq j \leq m + n$ and $m \leq j' < m + n$. Hence $\mathcal{F} \in \mathcal{PE}_k$. On the other hand, $w_m R^n w_{m+n+1}$ doesn't hold since $m \neq 0$, while both $w_0 R^{k'} w_{m+n+1}$ and $w_0 R^m w_m$ hold. (Note here that $w_0 R^{k+1} w_{m+n+1}$ and $k+1 \leq k'$.) Hence $\mathcal{F} \notin \mathcal{PE}_{k'}$. ∎

Lemma 3.2.6 *If $k' > k$, $m \geq k \geq 0$ and $n > m > 0$ then $E_k \not\supseteq E_{k'}$.*

Proof. If $k' < m + n$ then $m + n - k > m + n - k' > 0$, so $(k - m - n) \mid (k' - m - n)$ doesn't hold. Thus, we can derive our conclusion by using Lemma 3.2.4. It is therefore sufficient to consider the case where $k' \geq m + n$. We will divide the case into two.

For $n \geq k + m$, we define a frame $\mathcal{F} = (W, R)$ as follows;

$$
\begin{aligned}
W &= \{w_i \mid 0 \leq i \leq m + n\}, \\
w_i R w_j &\Leftrightarrow |i - j| \leq 1.
\end{aligned}
$$

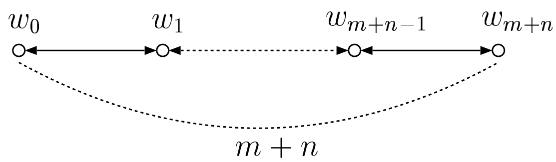


Figure 3.7:

Since $m + n > n$ by $m > 0$, $w_0 R^n w_{m+n}$ doesn't hold while both $w_0 R^m w_0$ and $w_0 R^{k'} w_{m+n}$ hold for $k' \geq m + n$. Therefore $\mathcal{F} \notin \mathcal{PE}_{k'}$. We will next show that $\mathcal{F} \in \mathcal{PE}_k$. We first note that $w_i R^t w_j$ holds if and only if $|i - j| \leq t$. Now, suppose that $w_i R^k w_j$ and $w_i R^m w_s$. Then, $|i - j| \leq k$ and $|i - s| \leq m$. Therefore, $|s - j| \leq |s - i| + |i - j| \leq m + k \leq n$. Hence, $w_s R^n w_j$

For $n < k + m$, define a frame $\mathcal{G} = (V, S)$ as follows;

$$
\begin{aligned}
V &= \{v_i \mid 0 \leq i \leq m + n + 1\}, \\
v_i S v_j &\Leftrightarrow \text{either} \\
&\quad 1) |i - j| \leq 1 \text{ if } 0 \leq i, j \leq m + n + 1 \text{ or} \\
&\quad 2) j = k + m - n + 2 \text{ if } 1 \leq i < k + m - n + 2 \text{ or} \\
&\quad 3) j = n - 1 \text{ if } n - 1 < j \leq k + m.
\end{aligned}
$$

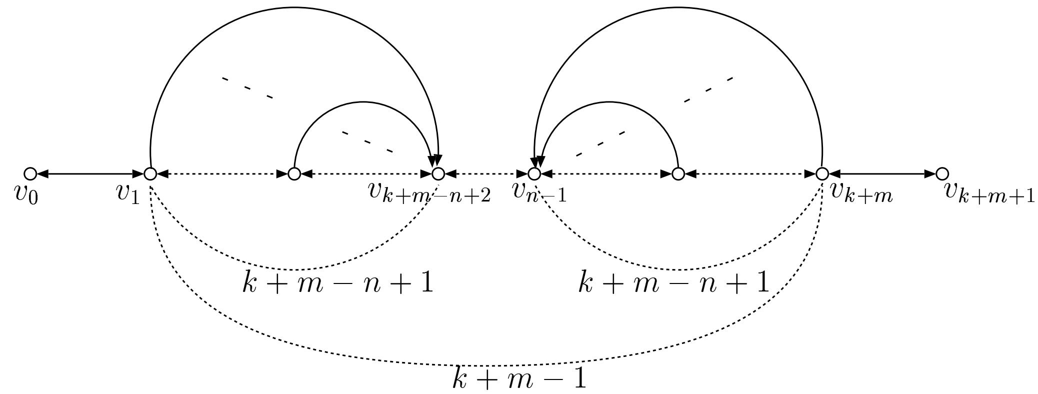


Figure 3.8:

Note that the frame takes at least $n+1$ steps from v_0 to v_{k+m+1} by the relation S. Thus $v_0 S^n v_{k+m+1}$ doesn't hold. But both $v_m S^{k'} v_{k+m+1}$ and $v_m S^m v_0$ hold because of $k+m+1 \leq k'+m$. Thus $\mathcal{G} \notin \mathcal{PE}_{k'}$.

Assume that $xS^k y$ and $xS^m z$ for any $x, y, z \in V$. Then both y and z must be either between v_0 and v_{k+m}, or between v_1 and v_{k+m+1}, depending on x. For each case, y is accessible from z by n steps, i.e. $zS^n y$. Therefore $\mathcal{G} \in \mathcal{PE}_k$. ∎

Lemma 3.2.7 *If $k' > k$, $m = 0$, $n > k$ and $k' \neq n$ then $E_k \not\supseteq E_{k'}$.*

Proof. Similarly to Lemma 3.2.6, we can show our lemma easily when $k' \leq n$. So, suppose that $k' > n$. If $k' < 2n-k$ then $n-k > k'-n > 0$, so $(k-n) \bigm| (k'-n)$ doesn't hold. This case has been discussed already in Lemma 3.2.4. It is therefore sufficient to consider the case $k' \geq 2n-k$. Then we define a frame $\mathcal{F} = (W, R)$ as follows;

$$
\begin{aligned}
W &= \{w_i \mid 0 \leq i \leq 2n-k\}, \\
w_i R w_j &\Leftrightarrow |i-j| \leq 1.
\end{aligned}
$$

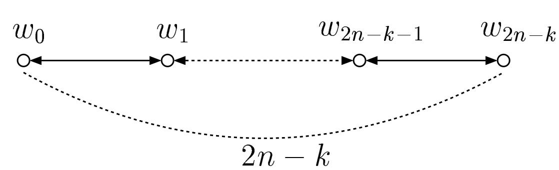


Figure 3.9:

Since $2n-k > n$ by $n-k > 0$, $w_0 R^n w_{2n-k}$ doesn't hold while $w_0 R^{k'} w_{2n-k}$ hold, therefore $\mathcal{F} \notin \mathcal{PE}_{k'}$.

On the other hand, if $xR^k y$ then $xR^n y$ for any $x, y \in W$, since $n > k$. Thus $\mathcal{F} \in \mathcal{PE}_k$. ∎

Lemma 3.2.8 *If $k' > k$, $m = n$ and $k = 0$ then $E_0 \not\supseteq E_{k'}$.*

Proof. We define a frame $\mathcal{F} = (W, R)$ as follows;

$$\begin{aligned} W &= \{w_i \mid 0 \le i \le m+1\}, \\ w_i R w_j &\Leftrightarrow |i-j| \le 1. \end{aligned}$$

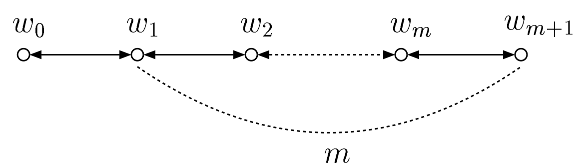


Figure 3.10:

Then $w_0 R^n w_{m+1}$ doesn't hold while both $w_1 R^{k'} w_0$ and $w_1 R^m w_{m+1}$ hold. Hence $\mathcal{F} \notin \mathcal{PE}_{k'}$. On the other hand, $xR^m y$ implies $yR^n x$ since the frame R is symmetric. Thus $\mathcal{F} \in \mathcal{PE}_0$. ∎

Lemma 3.2.9 *If $k' \ge k$, $m \ge n \ge 0$, $m \ge k \ge 0$, either $m - n > 0$ or $k > 0$, and $(k - m - n) \bigm| (k' - m - n)$ then $E_k \supseteq E_{k'}$.*

Proof. By the assumption, $k' - m - n = h(m + n - k)$, that is $k' = k + (h+1)(m+n-k)$, for a certain number $h \in \boldsymbol{Z}$. Since $k' \ge k$ and $m + n - k \ge 0$, we can assume that $k' = k + (h+1)(m+n-k)$ with $h \ge -1$. To show that $E_k \supseteq E_{k'} = E_{k+(h+1)(m+n-k)}$, it is enough to show that every $(W, R) \in \mathcal{PE}_k$ belongs also to $\mathcal{PE}_{k+(h+1)(m+n-k)}$ for any $h \ge -1$. This can be shown by the induction on h.

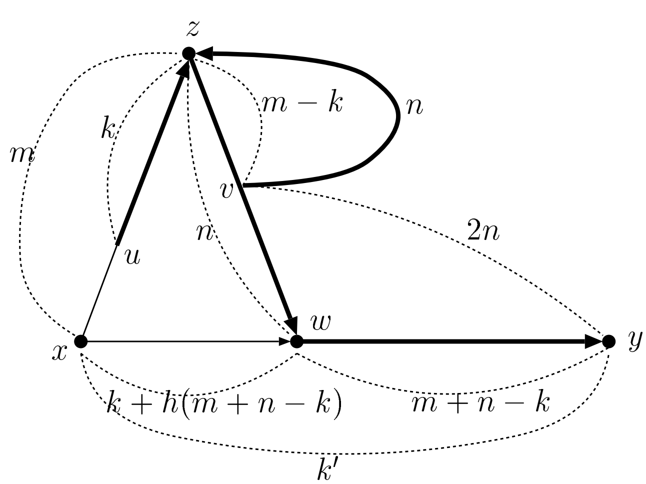


Figure 3.11:

If $h = -1$, this is trivial since $k + (h+1)(m+n-k) = k$. So, we assume that this holds for h. To show that (W, R) belongs to $\mathcal{PE}_{k+(h+2)(m+n-k)}$, we assume that $xR^{k+(h+2)(m+n-k)}y$ and $xR^m z$. Then, for some $w \in W$, $xR^{k+(h+1)(m+n-k)}w$ and $wR^{m+n-k}y$, since $k + (h+1)(m+n-k) \geq 0$ and $m + n - k \geq 0$. Since (W, R) belongs to $\mathcal{PE}_{k+(h+1)(m+n-k)}$ by the hypothesis of induction, $xR^{k+(h+1)(m+n-k)}w$ and $xR^m z$ imply $zR^n w$. Thus, $zR^{m-k+2n}y$.

Then, for some $u, v \in W$, $xR^{m-k}u$, $uR^k z$, $zR^{m-k}v$ and $vR^{2n}y$, since $m - k \geq 0$ and $k \geq 0$. Since $uR^k z$ and $zR^{m-k}v$ hold, $uR^m v$. But since (W, R) is in $\mathcal{PE}_k$, $vR^n z$.

But by using the next lemma, $zR^n y$ by taking $l = n$. Thus, we have shown that (W, R) belongs to $\mathcal{PE}_{k+(h+2)(m+n-k)}$. ∎

Lemma 3.2.10 *Let (W, R) be in $\mathcal{PE}_k$. Suppose that $m \geq n$, $m \geq k$ and either $m - n > 0$ or $k > 0$. Also, suppose that $M \geq \max(m-n-1, k-1) \geq 0$. Then, for every non-negative integer l, if $xR^{n+l}y$, $xR^l z$ and $x'R^M x$ then $zR^n y$.*

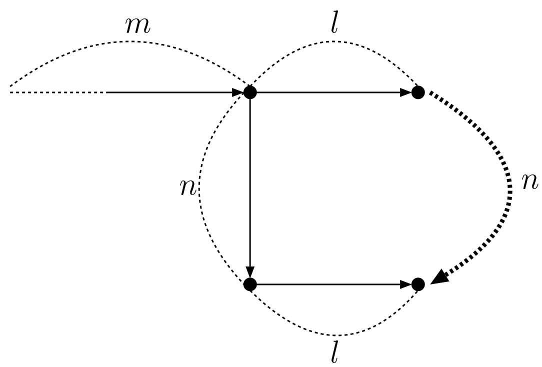


Figure 3.12:

Proof. We will show by the induction on l. If $l = 0$, this is trivial. When $l = 1$, we will divide the case into two.

First, suppose that $k \geq m - n$. Then, for some $w, u \in W$, $x'R^{M-(k-1)}w$, $wR^{k-1}x$, $xR^{m-k+1}u$ and $uR^{k+n-m}y$, since $M \geq k - 1 \geq 0$, $m - k + 1 > 0$ and $k + n - m \geq 0$. Since $wR^{k-1}x$ and xRz hold, $wR^k z$. Also, since $wR^{k-1}x$ and $xR^{m-k+1}u$ hold, $wR^m u$. Since (W, R) is in $\mathcal{PE}_k$, $uR^n z$. Then, for some $v \in W$, $x'R^{M+m-k+1-(m-n)}v$ and $vR^{m-n}u$, since $M + m - k + 1 - (m - n) > 0$ and $m - n \geq 0$. Since $vR^{m-n}u$ and $uR^{k+n-m}y$ hold, $vR^k y$. Also, since $vR^{m-n}u$ and $uR^n z$ hold, $vR^m z$. Therefore $zR^n y$ since (W, R) is in $\mathcal{PE}_k$.

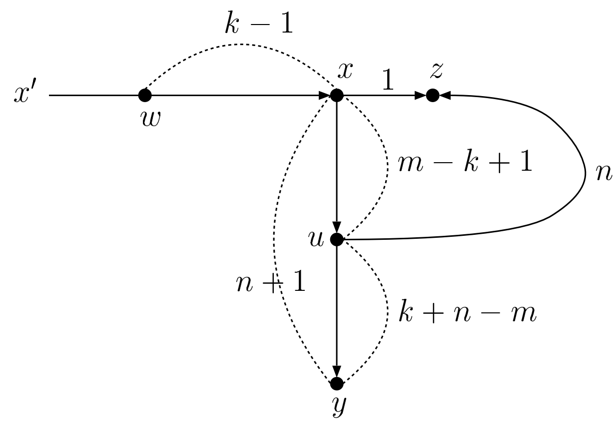


Figure 3.13:

When $k < m-n$, for some $w, u \in W$, $x'R^{M-k-(m-n-k-1)}w$, $wR^k u$ and $uR^{m-n-k-1}x$, since $M \geq k+(m-n-k-1)$, $k \geq 0$ and $m-n-k-1 \geq 0$. Since $wR^k u$, $uR^{m-n-k-1}x$ and $xR^{n+1}y$ hold, $wR^m y$. Since (W, R) is in $\mathcal{PE}_k$, $yR^n u$. Then, for some $v \in W$, $wR^{m-k}v$ and $vR^k y$, since $m-k \geq 0$ and $k \geq 0$. Since $vR^k y$, $yR^n u$, $uR^{m-n-k-1}x$ and xRz hold, $vR^m z$. Since (W, R) is in $\mathcal{PE}_k$, $zR^n y$.

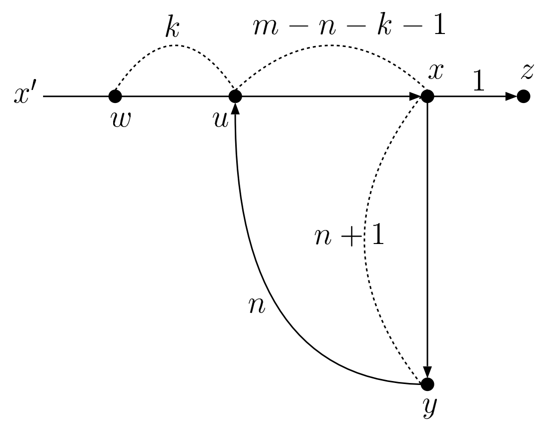


Figure 3.14:

Therefore, we have shown, when $l = 1$.

Now, we assume that this holds for l. To show for $l+1$, we assume that $xR^{n+l+1}y$, $xR^{l+1}z$ and $x'R^M x$. Then, for some $y', z' \in W$, $xR^{n+1}y'$, $y'R^l y$, xRz' and $z'R^l z$. Hence $z'R^n y'$ by the result when $l = 1$. Since $z'R^n y'$ and $y'R^l y$ hold, $z'R^{n+l}y$. Since $x'R^M x$ and xRz' hold, $x'R^{M+1}z'$. Since $z'R^{n+l}y$, $z'R^l z$, $x'R^{M+1}z'$ and $M+1 \geq M \geq \max(m-n-1, k-1)$, $zR^n y$ by the hypothesis of induction. ∎

3.3 Note

About inclusion relations on family of logics, for example, a class of logics extending **K45** is shown in [23].

As generalization of our results, it is interested in what happen if we allow both m and n to change.

Chapter 4

Intuitionistic modal logics I —— finite model property

4.1 Introduction

In this chapter we will consider two methods for completeness on Kripke type semantics for intuitionistic modal logics. One is the *method of canonical models*, and the other is *filtration method*.

Definition 4.1.1 *An intuitionistic modal logic L is called* Kripke complete *if there is a class $\mathcal{C}$ of intuitionistic modal Kripke frames such that*

$$L \vdash \alpha \quad \textit{iff} \quad \mathcal{C} \models \alpha.$$

To show Kripke completeness of a given intuitionistic modal logic L, it is necessary to construct a model in which each formula which doesn't belong L is refuted, but whose frame validates L.

Definition 4.1.2

*(1) Let L be an intuitionistic modal logic. A set T of formulas is said to be a L-*theory *if*

(i) $L \subseteq T$,

(ii) $\alpha, \alpha \to \beta \in T \Rightarrow \beta \in T$ for each α, β.

(2) A L-theory T is consistent *iff $\perp \notin T$.*

(3) A L-theory T is prime *iff*

(i) T is consistent,

(ii) $\alpha \vee \beta \in T \Rightarrow \alpha \in T$ or $\beta \in T$ for each α, β.

Definition 4.1.3

(1) Let L be an intuitionistic modal logic. The canonical frame *$\mathcal{F}_L = (W_L, \lhd_L, R_{\Box_L}, R_{\Diamond_L})$ is defined as follows.*

(i) W_L *is the set of all prime L-theories,*

(ii) $\lhd_L$ *is the set-inclusion relation* $\subseteq$, *i.e.,* $T_1 \lhd_L T_2 \Leftrightarrow T_1 \subseteq T_2$,

(iii) $T_1 R_{\Box_L} T_2 \overset{def}{\Leftrightarrow} \forall\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})(\Box a \in T_1 \Rightarrow a \in T_2)$, *in other words,* $(T_1)_\Box \subseteq T_2$, *where* $(T_1)_\Box := \{\alpha \mid \Box\alpha \in T_1\}$,

(iv) $T_1 R_{\Diamond_L} T_2 \overset{def}{\Leftrightarrow} \forall\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})(a \in T_2 \Rightarrow \Diamond a \in T_1)$, *in other words,* $T_2 \subseteq (T_1)_\Diamond$, *where* $(T_1)_\Diamond := \{\alpha \mid \Diamond\alpha \in T_1\}$.

(2) The canonical model $\mathcal{M}_L = (\mathcal{F}_L, v_L)$ *is the pair of the canonical frame and the valuation* v_L *defined by,*

$$v_L(p) := \{x \in W_L \mid p \in x\},$$

for every propositional variable p.

We can regard a (prime) L-theory as a (prime) filter of the Lindenbaum algebra $\boldsymbol{A}_L$. Compare Definition 4.1.3 with Definition 2.4.1(2), and we can also consider the canonical frame $\mathcal{F}_L$ as the dual $(\boldsymbol{A}_L)_\dagger$ of the Lindenbaum algebra $\boldsymbol{A}_L$. Actually a map : $T \mapsto T_L := \{|\alpha|_L \mid \alpha \in T\}$ is an isomorphism.

Definition 4.1.4 *Let L be an intuitionistic modal logic. For a given non-empty set X of formulas, we define* $[X)_L$ *and* $(X]_L$ *by*

$$[X)_L = \{\beta \mid L \vdash \alpha_1 \wedge \cdots \wedge \alpha_n \to \beta \text{ for some } \alpha_1, \ldots, \alpha_n \in X\},$$

$$(X]_L = \{\beta \mid L \vdash \beta \to \alpha_1 \vee \cdots \vee \alpha_n \text{ for some } \alpha_1, \ldots, \alpha_n \in X\}.$$

Then, similarly to Theorem 2.4.3, we can show the following theorem.

Theorem 4.1.5 *Let L be an intuitionistic modal logic. Given non-empty sets X and Y of formulas such that* $[X)_L \cap (Y]_L = \emptyset$, *there exists a prime L-theory T such that* $X \subseteq T$ *and* $T \cap Y = \emptyset$.

Theorem 4.1.6 *Let* $\mathcal{M}_L = (\mathcal{F}_L, v_L)$ *be the canonical model. Then for every formula* α,

$$v_L(\alpha) = \{x \in W_L \mid \alpha \in x\}.$$

Proof. This can be shown in the same way as the proof of Proposition 2.4.4(2), by taking v_L instead of h. ∎

Theorem 4.1.7 *For any* $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$, $L \vdash \alpha$ *iff* $\mathcal{M}_L \models \alpha$.

Proof. For any $x \in W_L$, $L \subseteq x$. So $\alpha \in L$ implies $x \models \alpha$. Conversely, suppose $L \nvdash \alpha$. Since L and $(\{\alpha\}]_L$ are disjoint, by Theorem 4.1.5 there is a prime L-theory x such that $\alpha \notin x$. Thus $\mathcal{M}_L \not\models \alpha$. ∎

In order to show that a given logic L is Kripke complete, it is sufficient that the canonical frame validates L. Such a logic L is called a *canonical logic*.

Proposition 4.1.8

(1) If a logic L_1 is an extension of a logic L_2, then the canonical frame $\mathcal{F}_{L_1}$ is a generated subframe of $\mathcal{F}_{L_2}$.

(2) If L_i is a canonical logic for each $i \in I$, then $\bigoplus_{i \in I} L_i$ is also canonical.

Proof. (1). It suffices to check the conditions of generated subframes (see Definition 2.3.4.) Clearly W_{L_2} contains W_{L_1}. Suppose that $x \in W_{L_1}$ and $y \in W_{L_2}$. If $x \subseteq y$ then $L_1 \subseteq y$, and if $x_\Box \subseteq y$ then $L_1 \subseteq y$, too, since $L_1 \subseteq (L_1)_\Box$. Thus, both conditions (iii) and (iv) of Definition 2.3.4(1) hold. Suppose $y \subseteq x_\Diamond$. For any formula $\alpha \in y$ and any formula β, if $\alpha \to \beta \in L_1$ then $\Diamond\beta \in x$ since $\Diamond\alpha \to \Diamond\beta \in L_1 \subseteq x$. Therefore since $[L_1 \cup y)_{L_1}$ and $(-(x_\Diamond)]_{L_1} = -(x_\Diamond)$ are disjoint, by Theorem 4.1.5 there is $z \in W_{L_1}$ such that $y \subseteq z$ and $z \cap -(x_\Diamond) = \emptyset$. Hence $z \subseteq x_\Diamond$. Thus, both condition (v) of Definition 2.3.4(1) holds. (2). By (1), $\mathcal{F}_{\oplus L_i}$ is a generated subframe of $\mathcal{F}_{L_i}$ for every $i \in I$. Since $\mathcal{F}_{L_i} \models L_i$, $\mathcal{F}_{\oplus L_i} \models L_i$ by Corollary 2.3.6. Thus, $\mathcal{F}_{\oplus L_i} \models \bigoplus_{i \in I} L_i$ ∎

Theorem 4.1.9 $\mathbf{IntK}_{\Box\Diamond} \oplus \mathbf{\Gamma}$ *is a canonical logic, if* $\mathbf{\Gamma}$ *is any combination of formulas in the following list.*

$$\Box(p \to q) \to (\Diamond p \to \Diamond q), \tag{4.1}$$
$$\Diamond(p \to q) \to (\Box p \to \Diamond q), \tag{4.2}$$
$$(\Diamond p \to \Box q) \to \Box(p \to q), \tag{4.3}$$
$$\Box^+(p \to q) \to (\Diamond p \to \Diamond q), \tag{4.4}$$
$$\Box^{(m)}(p \to q) \to (\Diamond p \to \Diamond q), \tag{4.5}$$
$$\neg\Box\bot, \tag{4.6}$$
$$\Diamond\top, \tag{4.7}$$
$$\Box p \to p, \tag{4.8}$$
$$p \to \Diamond p, \tag{4.9}$$
$$\Box p \to \Box\Box p, \tag{4.10}$$
$$\Diamond\Diamond p \to \Diamond p, \tag{4.11}$$
$$\Diamond^k \Box^l p \to \Box^m \Diamond^n p, \tag{4.12}$$
$$\Box p \vee \Box\neg\Box p, \tag{4.13}$$
$$\Box(\Box p \vee q) \to (\Box p \vee \Box q) \tag{4.14}$$
$$\Box(\Box p \to q) \vee \Box(\Box q \to p) \tag{4.15}$$

As a consequence, each $\mathbf{IntK}_{\Box\Diamond} \oplus \mathbf{\Gamma}$ in the above theorem is Kripke complete.

Proof. We will check only formulas which are shown in Proposition 2.3.3. The rest can be checked similarly.

(4.5). Let us suppose that $\Box^{(m)}(p \to q) \to (\Diamond p \to \Diamond q) \in L$. We will show that the canonical frame $\mathcal{F}_L$ satisfies the condition of (2.5), i.e.

$$x \subseteq y_\Diamond \Rightarrow \exists z \exists n (x \subseteq z \ \& \ z \subseteq y_\Diamond \ \& \ y_{\Box^n} \subseteq z \ \& \ 0 \leq n \leq m).$$

Suppose $x \subseteq y_\diamond$. Then we will show $[x \cup y_{\Box^n})_L \subseteq y_\diamond$ for some n. Suppose otherwise. Then for each n ($0 \leq n \leq m$) there are formulas $\alpha_n, \beta_n, \gamma_n$ such that $\alpha_n \in x$, $\Box^n \beta_n \in y$, $\diamond\gamma_n \notin y$ and $\alpha_n \wedge \beta_n \to \gamma_n \in L$. Since

$$(\beta_0 \vee \cdots \vee \beta_m) \to ((\alpha_0 \wedge \cdots \wedge \alpha_m) \to (\gamma_0 \vee \cdots \vee \gamma_m)) \in L,$$

for each n

$$\Box^n(\beta_0 \vee \cdots \vee \beta_m) \to \Box^n((\alpha_0 \wedge \cdots \wedge \alpha_m) \to (\gamma_0 \vee \cdots \vee \gamma_m)) \in L.$$

Since $\Box^n \beta_n \in y$ implies $\Box^n(\beta_0 \vee \cdots \vee \beta_m) \in y$, for each n

$$\Box^n((\alpha_0 \wedge \cdots \wedge \alpha_m) \to (\gamma_0 \vee \cdots \vee \gamma_m)) \in y.$$

Since also

$$\Box^{(m)}((\alpha_0 \wedge \cdots \wedge \alpha_m) \to (\gamma_0 \vee \cdots \vee \gamma_m)) \in y,$$

by using the axiom (4.5),

$$\diamond(\alpha_0 \wedge \cdots \wedge \alpha_m) \to \diamond(\gamma_0 \vee \cdots \vee \gamma_m) \in y.$$

Since $\alpha_0 \wedge \cdots \wedge \alpha_m \in x$ and $x \subseteq y_\diamond$, we have $\diamond(\alpha_0 \wedge \cdots \wedge \alpha_m) \in y$. Therefore

$$\diamond\gamma_0 \vee \cdots \vee \diamond\gamma_m \in y.$$

This is a contradiction. Thus, we have $[x \cup y_{\Box^n})_L \subseteq y_\diamond$ for some n. Therefore since $[x \cup y_{\Box^n})_L$ and $-(y_\diamond)$ are disjoint, by Theorem 4.1.5 there is an L-theory z such that $[x \cup y_{\Box^n})_L \subseteq z$ and $z \cap -(y_\diamond) = \emptyset$. Thus, $x \subseteq z$, $z \subseteq y_\diamond$ and $y_{\Box^n} \subseteq z$ for some n.

(4.12). Suppose that $\diamond^k \Box^l p \to \Box^m \diamond^n p \in L$ for some $k, l, m, n \geq 0$. Then we will show that the canonical frame $\mathcal{F}_L$ satisfies the condition of (2.12), i.e.

$$(x_{\Box^m} \subseteq y \ \&\ z \subseteq x_{\diamond^k}) \Rightarrow \exists u(u \subseteq y_{\diamond^n} \ \&\ z_{\Box^l} \subseteq u).$$

Suppose $x_{\Box^m} \subseteq y$ and $z \subseteq x_{\diamond^k}$. Hence, $x_{\Box^m \diamond^n} \subseteq y_{\diamond^n}$ and $z_{\Box^l} \subseteq x_{\diamond^k \Box^l}$. On the other hand $x_{\diamond^k \Box^l} \subseteq x_{\Box^m \diamond^n}$ since $\diamond^k \Box^l \alpha \to \Box^m \diamond^n \alpha \in x$. Thus $z_{\Box^l} \subseteq y_{\diamond^n}$. Therefore Theorem 4.1.5 guarantees the existence of L-theory u such that $u \subseteq y_{\diamond^n}$ and $z_{\Box^l} \subseteq u$.

(4.14). Let $\Box(\Box p \vee q) \to (\Box p \vee \Box q)$ belongs to L. We will show that the canonical frame $\mathcal{F}_L$ satisfies the condition of (2.14), i.e.

$$(x_\Box \subseteq y \ \&\ x_\Box \subseteq z) \Rightarrow \exists u(x_\Box \subseteq u \ \&\ u \subseteq z \ \&\ u_\Box \subseteq y).$$

Suppose $x_\Box \subseteq y$ and $x_\Box \subseteq z$. Then we will show $x_\Box \cap (\{\Box\beta \mid \beta \notin y\} \cup -z]_L = \emptyset$. Suppose otherwise. Then there are formulas $\alpha, \beta_1, \ldots, \beta_n, \gamma$ such that $\Box\alpha \in x$, $\beta_1, \ldots, \beta_n \notin y$ $\gamma \notin z$, and

$$\alpha \to \Box\beta_1 \vee \ldots \vee \Box\beta_n \vee \gamma \in L.$$

Hence

$$\Box\alpha \to \Box(\Box\beta_1 \vee \ldots \vee \Box\beta_n \vee \gamma) \in L.$$

Since

$$\Box(\Box\beta_1 \vee \ldots \vee \Box\beta_n \vee \gamma) \in x,$$

by using the axiom (4.14)

$$\Box\beta_1 \vee \Box(\Box\beta_2 \vee \ldots \vee \Box\beta_n \vee \gamma) \in x.$$

By iterating this,

$$\Box\beta_1 \vee \Box\beta_2 \vee \ldots \vee \Box\beta_n \vee \Box\gamma \in x.$$

Therefore, either $\Box\beta_i \in x$ for some i or $\Box\gamma \in x$. If $\Box\beta_i \in x$ for some i, $\beta_i \in x_\Box \subseteq y$. This contradicts $\beta_i \notin y$. If $\Box\gamma \in x$, $\gamma \in x_\Box \subseteq z$. This contradicts $\gamma \notin z$. Thus, we have $x_\Box \cap (\{\Box\beta \mid \beta \notin y\} \cup -z]_L = \emptyset$. Therefore by Theorem 4.1.5 there is an L-theory u such that $x_\Box \subseteq u$, $u \subseteq z$ and $u \cap \{\Box\beta \mid \beta \notin y\} = \emptyset$. Since $\emptyset = (u \cap \{\Box\beta \mid \beta \notin y\})_\Box = u_\Box \cap \{\Box\beta \mid \beta \notin y\}_\Box = u_\Box \cap -y$, we have $u_\Box \subseteq y$. ∎

Corollary 4.1.10 *Logics* $\mathbf{IntK}_{\Box\Diamond}$, $\mathbf{IntK}^+_{\Box\Diamond}$, $\mathbf{IntK}^*_{\Box\Diamond}$, **FS**, $\mathbf{IntD}_{\Box\Diamond}$, $\mathbf{IntT}_{\Box\Diamond}$, $\mathbf{IntK4}_{\Box\Diamond}$, $\mathbf{IntS4}_{\Box\Diamond}$, $\mathbf{IntS4.3}_{\Box\Diamond}$, $\mathbf{IntK5}_{\Box\Diamond}$, $\mathbf{IntS5}_{\Box\Diamond}$ *and* **MIPC** *are Kripke complete.*

4.2 Filtration method

The canonical model of a given L refutes any formula which does not belong to L. The frame will contain continuum many points. But it will be nice if we can find a *finite frame* refuting each formula α which does not belong to L. For, if L is moreover *finitely axiomatizable*, then L is *decidable* in this case. Here, we say that L is *finitely axiomatizable* if $L = \mathbf{IntK}_{\Box\Diamond} \oplus \mathbf{\Gamma}$ for some finite set $\mathbf{\Gamma}$ of formulas.

Definition 4.2.1 *A logic L has the* finite model property *if for every non-theorem φ of L, there exists a finite frame $\mathcal{F}$ such that $\mathcal{F} \models L$ and $\mathcal{F} \not\models \varphi$.*

In the following, by using filtration method we will show that many of basic intuitionistic modal logics have the finite model property.

Definition 4.2.2 *(1) Let $\mathcal{M}$ be a model and $\mathbf{\Sigma}$ be a set of formulas closed under subformulas, i.e., $\mathbf{Sub}\varphi \subseteq \mathbf{\Sigma}$ whenever $\varphi \in \mathbf{\Sigma}$, where $\mathbf{Sub}\varphi$ is the set of all subformulas of φ. Define an equivalence relation $\sim_{\mathbf{\Sigma}}$ on W, by taking*

$$x \sim_{\mathbf{\Sigma}} y \overset{def}{\Leftrightarrow} (\mathcal{M}, x) \models \varphi \ \ \textit{iff} \ \ \textit{for every } \varphi \in \mathbf{\Sigma} (\mathcal{M}, y) \models \varphi,$$

and say that x, y are $\mathbf{\Sigma}$-equivalent *in $\mathcal{M}$. Denote by $[x]_{\mathbf{\Sigma}}$ the equivalence class generated by x. If understood, $[x]_{\mathbf{\Sigma}}$ is written simply $[x]$.*

(2) A model $\mathcal{M}_{\mathbf{\Sigma}} = (W_{\mathbf{\Sigma}}, \lhd_{\mathbf{\Sigma}}, R_{\Box\mathbf{\Sigma}}, R_{\Diamond\mathbf{\Sigma}}, v_{\mathbf{\Sigma}})$ is called a filtration of $\mathcal{M}$ through $\mathbf{\Sigma}$ if the following conditions are satisfied.

(i) $W_{\mathbf{\Sigma}} = \{[x] \mid x \in W\}$

(ii) $v_{\mathbf{\Sigma}}(p) = \{[x] \mid x \in v(p)\}$, *for every propositional variable* $p \in \mathbf{\Sigma}$,

(iii) for all $x, y \in W$ $x \lhd y$ *implies* $[x] \lhd_{\mathbf{\Sigma}} [y]$,

(iv) for all $x, y \in W$ $xR_\Box y$ *implies* $[x]R_{\Box\mathbf{\Sigma}}[y]$,

(v) for all $x, y \in W$ $xR_\Diamond y$ *implies* $[x]R_{\Diamond\mathbf{\Sigma}}[y]$,

(vi) *for* $x, y \in W$ *and* $\varphi \in \mathbf{\Sigma}$ *if* $[x] \lhd_{\mathbf{\Sigma}} [y]$ *then* $y \models \varphi$ *whenever* $x \models \varphi$,

(vii) *for* $x, y \in W$ *and* $\Box\varphi \in \mathbf{\Sigma}$ *if* $[x] R_{\Box\mathbf{\Sigma}} [y]$ *then* $y \models \varphi$ *whenever* $x \models \Box\varphi$,

(viii) *for* $x, y \in W$ *and* $\Diamond\varphi \in \mathbf{\Sigma}$ *if* $[x] R_{\Diamond\mathbf{\Sigma}} [y]$ *then* $x \models \Diamond\varphi$ *whenever* $y \models \varphi$.

Theorem 4.2.3 *Let* $\mathcal{M}_{\mathbf{\Sigma}}$ *be a filtration of a model* $\mathcal{M}$ *through a set* $\mathbf{\Sigma}$ *of formulas. Then for every* x *in* $\mathcal{M}$ *and every formula* $\varphi \in \mathbf{\Sigma}$,

$$(\mathcal{M}, x) \models \varphi \Leftrightarrow (\mathcal{M}_{\mathbf{\Sigma}}, [x]) \models \varphi.$$

Proof. we prove our theorem by induction on the construction of φ. The basis of induction follows from (ii). Now let $\varphi = \psi \to \chi \in \mathbf{\Sigma}$. Suppose that $x \models \psi \to \chi$, $[x] \lhd_{\mathbf{\Sigma}} [y]$ and $[y] \models \psi$. Then, by (vi), $y \models \psi \to \chi$ and by the induction hypothesis, $y \models \psi$. Hence, $y \models \chi$. Again by the induction hypothesis, $[y] \models \chi$. Thus, $[x] \models \psi \to \chi$. Conversely, suppose that $[x] \models \psi \to \chi$, $x \lhd y$ and $y \models \psi$. Then, by (iii), $[x] \lhd_{\mathbf{\Sigma}} [y]$ and by the induction hypothesis, $[y] \models \psi$. Hence, $[y] \models \chi$. Again by the induction hypothesis, $y \models \chi$. Thus, $x \models \psi \to \chi$.

Next let $\varphi = \Box\psi \in \mathbf{\Sigma}$. Suppose that $x \models \Box\psi$ and $[x] R_{\Box\mathbf{\Sigma}} [y]$. Then, by (vii), $y \models \psi$ and by the induction hypothesis, $[y] \models \psi$. Thus, $[x] \models \Box\psi$. Conversely, suppose that $[x] \models \Box\psi$ and xRy. Then, by (iv), $[x] R_{\Box\mathbf{\Sigma}} [y]$ and so $[y] \models \psi$. Hence, by the induction hypothesis, $y \models \psi$. Thus, $x \models \Box\psi$.

Let $\varphi = \Diamond\psi \in \mathbf{\Sigma}$. Suppose that $x \models \Diamond\psi$. Then, there is y such that $xR_{\Diamond}y$ and $y \models \psi$. Hence, by (v), $[x] R_{\Diamond\mathbf{\Sigma}} [y]$ and by the induction hypothesis, $[y] \models \psi$. Thus, $[x] \models \Diamond\psi$. Conversely, suppose that $[x] \models \Diamond\psi$. Then, there is $[y]$ such that $[x] R_{\Diamond\mathbf{\Sigma}} [y]$ and $[y] \models \psi$. Hence, by the induction hypothesis, $y \models \psi$ and by (viii), $x \models \Diamond\psi$. ∎

In general, the conditions from (iii) to (viii) do not determine the binary relations $\lhd_{\mathbf{\Sigma}}, R_{\Box\mathbf{\Sigma}}, R_{\Diamond\mathbf{\Sigma}}$ uniquely. Actually, they allow us to choose any relations $\lhd_{\mathbf{\Sigma}}, R_{\Box\mathbf{\Sigma}}, R_{\Diamond\mathbf{\Sigma}}$ such that $\underline{\lhd}_{\mathbf{\Sigma}} \subseteq \lhd_{\mathbf{\Sigma}} \subseteq \overline{\lhd}_{\mathbf{\Sigma}}$, $\underline{R}_{\Box\mathbf{\Sigma}} \subseteq R_{\Box\mathbf{\Sigma}} \subseteq \overline{R}_{\Box\mathbf{\Sigma}}$, $\underline{R}_{\Diamond\mathbf{\Sigma}} \subseteq R_{\Diamond\mathbf{\Sigma}} \subseteq \overline{R}_{\Diamond\mathbf{\Sigma}}$, where

$$\begin{aligned}
\underline{\lhd}_{\mathbf{\Sigma}} &= \{([x],[y]) \mid \exists x', y'(x \sim_{\mathbf{\Sigma}} x' \;\&\; y \sim_{\mathbf{\Sigma}} y' \;\&\; x'Ry')\},\\
\underline{R}_{\Box\mathbf{\Sigma}} &= \{([x],[y]) \mid \exists x', y'(x \sim_{\mathbf{\Sigma}} x' \;\&\; y \sim_{\mathbf{\Sigma}} y' \;\&\; x'R_{\Box}y')\},\\
\underline{R}_{\Diamond\mathbf{\Sigma}} &= \{([x],[y]) \mid \exists x', y'(x \sim_{\mathbf{\Sigma}} x' \;\&\; y \sim_{\mathbf{\Sigma}} y' \;\&\; x'R_{\Diamond}y')\},\\
\overline{\lhd}_{\mathbf{\Sigma}} &= \{([x],[y]) \mid \forall\varphi \in \mathbf{\Sigma}(x \models \varphi \Rightarrow y \models \varphi)\},\\
\overline{R}_{\Box\mathbf{\Sigma}} &= \{([x],[y]) \mid \forall\Box\varphi \in \mathbf{\Sigma}(x \models \Box\varphi \Rightarrow y \models \varphi)\},\\
\overline{R}_{\Diamond\mathbf{\Sigma}} &= \{([x],[y]) \mid \forall\Diamond\varphi \in \mathbf{\Sigma}(y \models \varphi \Rightarrow x \models \Diamond\varphi)\}.
\end{aligned}$$

Indeed, if $[x] \lhd_{\mathbf{\Sigma}} [y]$, $[x] R_{\Box\mathbf{\Sigma}} [y]$ and $[x] R_{\Diamond\mathbf{\Sigma}} [y]$ hold then, by (vi), (vii) and (viii), $[x] \overline{\lhd}_{\mathbf{\Sigma}} [y]$, $[x] \overline{R}_{\Box\mathbf{\Sigma}} [y]$ and $[x] \overline{R}_{\Diamond\mathbf{\Sigma}} [y]$, respectively. And if $[x] \underline{\lhd}_{\mathbf{\Sigma}} [y]$, $[x] \underline{R}_{\Box\mathbf{\Sigma}} [y]$ and $[x] \underline{R}_{\Diamond\mathbf{\Sigma}} [y]$ then $x' \lhd y'$, $x'R_{\Box}y'$ and $x'R_{\Diamond}y'$ for some $x' \in [x], y' \in [y]$. Hence, by (iii), (iv) and (v), $[x] \lhd_{\mathbf{\Sigma}} [y]$, $[x] R_{\Box\mathbf{\Sigma}} [y]$ and $[x] R_{\Diamond\mathbf{\Sigma}} [y]$, respectively. Note here that the fact that $[x] \underline{\lhd}_{\mathbf{\Sigma}} [y]$, $[x] \underline{R}_{\Box\mathbf{\Sigma}} [y]$ and $[x] \underline{R}_{\Diamond\mathbf{\Sigma}} [y]$ satisfy (vi), (vii) and (viii), respectively, and $[x] \overline{\lhd}_{\mathbf{\Sigma}} [y]$, $[x] \overline{R}_{\Box\mathbf{\Sigma}} [y]$ and $[x] \overline{R}_{\Diamond\mathbf{\Sigma}} [y]$, satisfy (iii), (iv) and (v), respectively, follows directly from the definition of the valuation.

Note that $\overline{\lhd}_{\mathbf{\Sigma}}$ is a partial order, $\overline{\lhd}_{\mathbf{\Sigma}} \circ \overline{R}_{\Box\mathbf{\Sigma}} \circ \overline{\lhd}_{\mathbf{\Sigma}} = \overline{R}_{\Box\mathbf{\Sigma}}$, and $\overline{\lhd}_{\mathbf{\Sigma}}^{-1} \circ \overline{R}_{\Diamond\mathbf{\Sigma}} \circ \overline{\lhd}_{\mathbf{\Sigma}}^{-1} = \overline{R}_{\Diamond\mathbf{\Sigma}}$. The reflexivity of $\overline{\lhd}_{\mathbf{\Sigma}}$ follows from (iii). The anti-symmetry of $\overline{\lhd}_{\mathbf{\Sigma}}$ follows from (vi) and the definition of $\mathbf{\Sigma}$-equivalence. Suppose that $[x] \overline{\lhd}_{\mathbf{\Sigma}} [y]$ and $[y] \overline{\lhd}_{\mathbf{\Sigma}} [z]$ and $\psi \in \mathbf{\Sigma}$. If $x \models \psi$, then $y \models \psi$. Hence $z \models \psi$. Thus, $[x] \overline{R}_{\Box\mathbf{\Sigma}} [z]$. Next, suppose that $[x] \overline{\lhd}_{\mathbf{\Sigma}} [y]$,

$[y]\overline{R}_{\Box\mathbf{\Sigma}}[z]$, $[z]\overline{\lhd}_{\mathbf{\Sigma}}[w]$, and $\Box\psi \in \mathbf{\Sigma}$. If $x \models \Box\psi$, then $y \models \Box\psi$. Hence $z \models \psi$. So, $w \models \psi$. Thus, $[x]\overline{R}_{\Box\mathbf{\Sigma}}[w]$. Finally, suppose that $[x]\overline{\lhd}_{\mathbf{\Sigma}}^{-1}[y]$, $[y]\overline{R}_{\Diamond\mathbf{\Sigma}}[z]$, $\overline{\lhd}_{\mathbf{\Sigma}}^{-1}[w]$, and $\Diamond\psi \in \mathbf{\Sigma}$. If $w \models \psi$, then $z \models \psi$. Hence $y \models \Diamond\psi$. So, $x \models \Diamond\psi$. Thus, $[x]\overline{R}_{\Diamond\mathbf{\Sigma}}[w]$. Therefore, $(W_{\mathbf{\Sigma}}, \overline{\lhd}_{\mathbf{\Sigma}}, \overline{R}_{\Box\mathbf{\Sigma}}, \overline{R}_{\Diamond\mathbf{\Sigma}})$ is an intuitionistic modal frame.

But not all of $\lhd_{\mathbf{\Sigma}}, R_{\Box\mathbf{\Sigma}}, R_{\Diamond\mathbf{\Sigma}}$ in these intervals give rise to filtrations of intuitionistic modal frames. More preciously, the reflexivity and the anti-symmetry of $\lhd_{\mathbf{\Sigma}}$ hold always, while $\lhd_{\mathbf{\Sigma}}$ may be non-transitive and neither $\lhd_{\mathbf{\Sigma}} \circ R_{\Box\mathbf{\Sigma}} \circ \lhd_{\mathbf{\Sigma}} = R_{\Box\mathbf{\Sigma}}$ nor $\lhd_{\mathbf{\Sigma}}^{-1} \circ R_{\Diamond\mathbf{\Sigma}} \circ \lhd_{\mathbf{\Sigma}}^{-1} = R_{\Diamond\mathbf{\Sigma}}$ may hold.

To get a transitive relation it is enough to take the transitive closure $\lhd_{\mathbf{\Sigma}}^{\infty}$ of $\lhd_{\mathbf{\Sigma}}$. Clearly, $\lhd_{\mathbf{\Sigma}}^{\infty}$ satisfies (iii). By the transitivity of $\overline{\lhd}_{\mathbf{\Sigma}}$, $\lhd_{\mathbf{\Sigma}}^{\infty}$ satisfies (vi). We also defines $R^{\star}_{\Box\mathbf{\Sigma}}$ and $R^{\star}_{\Diamond\mathbf{\Sigma}}$ by

$$R^{\star}_{\Box\mathbf{\Sigma}} = \lhd_{\mathbf{\Sigma}}^{\infty} \circ R_{\Box\mathbf{\Sigma}} \circ \lhd_{\mathbf{\Sigma}}^{\infty}$$

and

$$R^{\star}_{\Diamond\mathbf{\Sigma}} = \lhd_{\mathbf{\Sigma}}^{-1\infty} \circ R_{\Diamond\mathbf{\Sigma}} \circ \lhd_{\mathbf{\Sigma}}^{-1\infty}.$$

Then, it is easily shown that they satisfy (iv), (v), (vii) and (viii). Thus, $(W_{\mathbf{\Sigma}}, \lhd_{\mathbf{\Sigma}}^{\infty}, R^{\star}_{\Box\mathbf{\Sigma}}, R^{\star}_{\Diamond\mathbf{\Sigma}})$ is also an intuitionistic modal frame. Clearly, if $(W_{\mathbf{\Sigma}}, \lhd_{\mathbf{\Sigma}}, R_{\Box\mathbf{\Sigma}}, R_{\Diamond\mathbf{\Sigma}})$ is an intuitionistic modal frame, $\lhd_{\mathbf{\Sigma}}^{\infty} = \lhd_{\mathbf{\Sigma}}$, $R^{\star}_{\Box\mathbf{\Sigma}} = R_{\Box\mathbf{\Sigma}}$ and $R^{\star}_{\Diamond\mathbf{\Sigma}} = R_{\Diamond\mathbf{\Sigma}}$ hold. Therefore, for any filtration $(W_{\mathbf{\Sigma}}, \lhd_{\mathbf{\Sigma}}, R_{\Box\mathbf{\Sigma}}, R_{\Diamond\mathbf{\Sigma}}, v_{\mathbf{\Sigma}})$, we have $\underline{\lhd}_{\mathbf{\Sigma}}^{\infty} \subseteq \lhd_{\mathbf{\Sigma}} \subseteq \overline{\lhd}_{\mathbf{\Sigma}}$, $\underline{R}^{\star}_{\Box\mathbf{\Sigma}} \subseteq R_{\Box\mathbf{\Sigma}} \subseteq \overline{R}_{\Box\mathbf{\Sigma}}$ and $\underline{R}^{\star}_{\Diamond\mathbf{\Sigma}} \subseteq R_{\Diamond\mathbf{\Sigma}} \subseteq \overline{R}_{\Diamond\mathbf{\Sigma}}$.

Definition 4.2.4 *The filtration on the frame* $\underline{\mathcal{F}}_{\mathbf{\Sigma}} = (W_{\mathbf{\Sigma}}, \underline{\lhd}_{\mathbf{\Sigma}}^{\infty}, \underline{R}^{\star}_{\Box\mathbf{\Sigma}}, \underline{R}^{\star}_{\Diamond\mathbf{\Sigma}})$ *is called the* finest *filtration of* $\mathcal{M}$ *through* $\mathbf{\Sigma}$, *while the filtration on the frame* $\overline{\mathcal{F}}_{\mathbf{\Sigma}} = (W_{\mathbf{\Sigma}}, \overline{\lhd}_{\mathbf{\Sigma}}, \overline{R}_{\Box\mathbf{\Sigma}}, \overline{R}_{\Diamond\mathbf{\Sigma}})$ *is called the* coarsest *filtration of* $\mathcal{M}$ *through* $\mathbf{\Sigma}$.

If $\mathbf{\Sigma}$ is finite then $W_{\mathbf{\Sigma}}$ is finite (in fact, it contains at most $2^{|\mathbf{\Sigma}|}$ elements.). Therefore, to prove the finite model property of a logic L, it suffices to show that for every non-theorem φ of L and a model $\mathcal{M}$ of L such that $\mathcal{M} \not\models \varphi$, if there exists a filtration of $\mathcal{M}$ through a finite set $\mathbf{\Sigma}$ containing φ such that $\mathcal{F}_{\mathbf{\Sigma}} \models L$. If this is really the case then we say that L *admits filtration*.

Suppose that P is a property of a frame. Suppose moreover that a given logic L is sound with respect to the class $\mathcal{C}$ of frames satisfying a property P, and that the canonical frames of L satisfies P. In such a case, to prove that L has finite model property it suffices to show that for each non-theorem φ of L, there exists a finite set $\mathbf{\Sigma}$ containing φ such that a filtration $\mathcal{F}_{\mathbf{\Sigma}}$ of any $\mathcal{M}$ in $\mathcal{C}$ through $\mathbf{\Sigma}$ satisfies P.

The following theorem will show how filtration method works well also for intuitionistic modal logics.

Theorem 4.2.5 *Any of* $\mathbf{IntK}_{\Box\Diamond}$, $\mathbf{IntD}_{\Box\Diamond}$, $\mathbf{IntT}_{\Box\Diamond}$, $\mathbf{IntK4}_{\Box\Diamond}$, $\mathbf{IntS4}_{\Box\Diamond}$ *and* $\mathbf{IntS5}_{\Box\Diamond}$ *admits filtration and hence has the finite model property.*

Proof. For $\mathbf{IntK}_{\Box\Diamond}$: Our basic logic $\mathbf{IntK}_{\Box\Diamond}$ is characterized by the class of all intuitionistic modal frames, it trivially admits filtration.

For $\mathbf{IntD}_{\Box\Diamond}$: When $R_{\Box}$ and $R_{\Diamond}$ are serial, by (iv) and (v) $R_{\Box\mathbf{\Sigma}}$ and $R_{\Diamond\mathbf{\Sigma}}$ in any filtration are also serial, respectively.

For $\mathbf{IntT}_{\Box\Diamond}$: When $R_{\Box}$ and $R_{\Diamond}$ are reflexive, by (iv) and (v) $R_{\Box\mathbf{\Sigma}}$ and $R_{\Diamond\mathbf{\Sigma}}$ in any filtration are also reflexive, respectively.

For $\mathbf{IntK4}_{\Box\Diamond}$: Let $\mathcal{M}$ be a model with transitive relations $R_\Box$ and $R_\Diamond$.

First, we will consider the finest filtration. Let $\mathbf{\Sigma}$ be a set of formulas closed under subformulas. We take the transitive closures $(\underline{R}^\star_{\Box\mathbf{\Sigma}})^\infty$ and $(\underline{R}^\star_{\Diamond\mathbf{\Sigma}})^\infty$ of $\underline{R}^\star_{\Box\mathbf{\Sigma}}$ and $\underline{R}^\star_{\Diamond\mathbf{\Sigma}}$. It is easily shown that they satisfy (iv) and (v).

We show that $(\underline{R}^\star_{\Box\mathbf{\Sigma}})^\infty$ satisfies (vii). Suppose that $[x]\underline{R}_{\Box\mathbf{\Sigma}}[y]$, $\Box\psi \in \mathbf{\Sigma}$. Then, there exist x', y' such that $x' \in [x], y' \in [y]$ and $x' R_\Box y'$. If $x \models \Box\psi$, $x' \models \Box\psi$. Since $R_\Box$ is transitive, $y' \models \Box^+\psi$. Hence $y \models \Box^+\psi$. By iterating this argument, for any $\Box\psi \in \mathbf{\Sigma}$, if $(\underline{R}^\star_{\Box\mathbf{\Sigma}})^\infty$ and $x \models \Box\psi$, then $y \models \Box^+\psi$. Thus, $(\underline{R}^\star_{\Box\mathbf{\Sigma}})^\infty$ satisfies (vii). Similarly, we can show that $(\underline{R}^\star_{\Diamond\mathbf{\Sigma}})^\infty$ satisfies (viii). Thus, the frame $(W_{\mathbf{\Sigma}}, \underline{\trianglelefteq}^\infty_{\mathbf{\Sigma}}, (\underline{R}^\star_{\Box\mathbf{\Sigma}})^\infty, (\underline{R}^\star_{\Diamond\mathbf{\Sigma}})^\infty)$ is also a frame which we want to get.

Next, we will consider the coarsest filtration. Suppose that

$$\mathbf{\Sigma}_0 = \mathbf{Sub}\varphi \cup \{\Box\Box\psi \mid \Box\psi \in \mathbf{Sub}\varphi\} \cup \{\Diamond\Diamond\psi \mid \Diamond\psi \in \mathbf{Sub}\varphi\}.$$

We will check that in the coarsest filtration both $\overline{R}_{\Box\mathbf{\Sigma}_0}$ and $\overline{R}_{\Diamond\mathbf{\Sigma}_0}$ are transitive.

Suppose that $[x]\overline{R}_{\Box\mathbf{\Sigma}_0}[y]$, $[y]\overline{R}_{\Box\mathbf{\Sigma}_0}[z]$ and $\Box\psi \in \mathbf{\Sigma}_0$. When $\Box\psi \in \mathbf{Sub}\varphi$, if $x \models \Box\psi$, then $x \models \Box\Box\psi$ by the transitivity. Since $\Box\Box\psi \in \mathbf{\Sigma}_0$, $y \models \Box\psi$. Hence, $z \models \psi$. Thus, $[x]\overline{R}_{\Box\mathbf{\Sigma}_0}[z]$. Otherwise, $\Box\psi = \Box\Box\chi \in \{\Box\Box\zeta \mid \Box\zeta \in \mathbf{Sub}\varphi\}$ for some χ. If $x \models \Box\Box\chi$, then $y \models \Box\chi$, since $\Box\Box\chi \in \mathbf{\Sigma}_0$. By the transitivity, $y \models \Box\Box\chi$. Hence, $z \models \Box\chi$. Thus, $[x]\overline{R}_{\Box\mathbf{\Sigma}_0}[z]$.

Suppose that $[x]\overline{R}_{\Diamond\mathbf{\Sigma}_0}[y]$, $[y]\overline{R}_{\Diamond\mathbf{\Sigma}_0}[z]$ and $\Diamond\psi \in \mathbf{\Sigma}_0$. When $\Diamond\psi \in \mathbf{Sub}\varphi$, if $z \models \psi$, then $y \models \Diamond\psi$. Since $\Diamond\Diamond\psi \in \mathbf{\Sigma}_0$, $x \models \Diamond\Diamond\psi$. Hence, $x \models \Diamond\psi$, by the transitivity. Thus, $[x]\overline{R}_{\Diamond\mathbf{\Sigma}_0}[z]$. Otherwise,$\Diamond\psi = \Diamond\Diamond\chi \in \{\Diamond\Diamond\zeta \mid \Diamond\zeta \in \mathbf{Sub}\varphi\}$ for some χ. If $z \models \Diamond\chi$, then $y \models \Diamond\Diamond\chi$. By the transitivity, $y \models \Diamond\chi$. Since $\Diamond\Diamond\chi \in \mathbf{\Sigma}_0$, $x \models \Diamond\Diamond\chi$. Therefore, $[x]\overline{R}_{\Diamond\mathbf{\Sigma}_0}[z]$. Thus, the frame $(W_{\mathbf{\Sigma}_0}, \overline{R}_{\mathbf{\Sigma}_0}, \overline{R}_{\Box\mathbf{\Sigma}_0}, \overline{R}_{\Diamond\mathbf{\Sigma}_0})$ is also a frame for $\mathbf{IntK4}_{\Box\Diamond}$.

Therefore, for any filtration $(W_{\mathbf{\Sigma}}, \lhd_{\mathbf{\Sigma}_0}, R_{\Box\mathbf{\Sigma}_0}, R_{\Diamond\mathbf{\Sigma}_0}, v_{\mathbf{\Sigma}_0})$, we have $\underline{\trianglelefteq}^\infty_{\mathbf{\Sigma}_0} \subseteq \lhd_{\mathbf{\Sigma}_0} \subseteq \overline{\lhd}_{\mathbf{\Sigma}_0}$, $\underline{R}^\star_{\Box\mathbf{\Sigma}_0} \subseteq (\underline{R}^\star_{\Box\mathbf{\Sigma}_0})^\infty \subseteq (R_{\Box\mathbf{\Sigma}_0})^\infty \subseteq \overline{R}_{\Box\mathbf{\Sigma}_0}$ and $\underline{R}^\star_{\Diamond\mathbf{\Sigma}_0} \subseteq (\underline{R}^\star_{\Diamond\mathbf{\Sigma}_0})^\infty \subseteq (R_{\Diamond\mathbf{\Sigma}_0})^\infty \subseteq \overline{R}_{\Diamond\mathbf{\Sigma}_0}$. Thus, the frame $(W_{\mathbf{\Sigma}_0}, \lhd_{\mathbf{\Sigma}_0}, (R_{\Box\mathbf{\Sigma}_0})^\infty, (R_{\Diamond\mathbf{\Sigma}_0})^\infty)$ is also a frame which we want to get.

For $\mathbf{IntS4}_{\Box\Diamond}$: When $R_\Box$ and $R_\Diamond$ are reflexive, by (iv) and (v) $R_{\Box\mathbf{\Sigma}}$ and $R_{\Diamond\mathbf{\Sigma}}$ in any filtration are reflexive, respectively. The filtrations for $\mathbf{IntK4}_{\Box\Diamond}$ work well also for $\mathbf{IntS4}_{\Box\Diamond}$. Moreover, $(\underline{R}^\star_{\Box\mathbf{\Sigma}})^\infty = \underline{R}^\infty_{\Box\mathbf{\Sigma}}$ and $(\underline{R}^\star_{\Diamond\mathbf{\Sigma}})^\infty = \underline{R}^\infty_{\Diamond\mathbf{\Sigma}}$ by the reflexivities of $\underline{R}_{\Box\mathbf{\Sigma}}$ and $\underline{R}_{\Diamond\mathbf{\Sigma}}$.

For $\mathbf{IntS5}_{\Box\Diamond}$: Let $\mathcal{M}$ be a model such that both $R_\Box$ and $R_\Diamond$ are reflexive and transitive, and that $R_\Box = R_\Diamond^{-1}$.

First, we will consider the finest filtration. Let $\mathbf{\Sigma}$ be a set of formulas closed under subformulas. By the assumption we can easily show that $\underline{R}_{\Diamond\mathbf{\Sigma}} = \underline{R}^{-1}_{\Diamond\mathbf{\Sigma}}$. Thus both $\underline{R}^\infty_{\Box\mathbf{\Sigma}}$ and $\underline{R}^\infty_{\Diamond\mathbf{\Sigma}}$ are reflexive and transitive, and $\underline{R}^\infty_{\Box\mathbf{\Sigma}} = \underline{R}^{\infty-1}_{\Diamond\mathbf{\Sigma}}$. Therefore, the frame $(W_{\mathbf{\Sigma}}, \underline{\trianglelefteq}^\infty_{\mathbf{\Sigma}}, \underline{R}^\infty_{\Box\mathbf{\Sigma}}, \underline{R}^\infty_{\Diamond\mathbf{\Sigma}})$ is a frame which we want to get.

Next, we will consider the coarsest filtration. Put

$$\mathbf{\Sigma}_1 = \mathbf{Sub}\varphi \cup \{\Box\Box\psi, \Diamond\Box\psi \mid \Box\psi \in \mathbf{Sub}\varphi\} \cup \{\Diamond\Diamond\psi, \Box\Diamond\psi \mid \Diamond\psi \in \mathbf{Sub}\varphi\}.$$

We will check that the coarsest filtration satisfies $\overline{R}_{\Box\mathbf{\Sigma}_1} = \overline{R}^{-1}_{\Diamond\mathbf{\Sigma}_1}$. Notice that by properties of $\mathbf{IntS5}_{\Box\Diamond}$, if ψ is either $\Box\chi$ or $\Diamond\chi$ for some χ then

$$x \models \psi \text{ iff } x \models \Box\psi \text{ iff } x \models \Diamond\psi.$$

Suppose $[x]\overline{R}_{\Box\mathbf{\Sigma}_1}[y]$ and $\Diamond\psi \in \mathbf{\Sigma}_1$. When $\Diamond\psi \in \mathbf{Sub}\varphi$, if $x \models \psi$ then $x \models \Diamond\psi$ by the reflexivity of $R_\Diamond$. Hence, $x \models \Box\Diamond\psi$. Then, $y \models \Diamond\psi$ since $\Box\Diamond\psi \in \mathbf{\Sigma}_1$. Thus, $[y]\overline{R}_{\Diamond\mathbf{\Sigma}_1}[x]$.

When $\Diamond\psi \in \{\Box\Box\chi, \Diamond\Box\chi \mid \Box\chi \in \mathbf{Sub}\varphi\} \cup \{\Diamond\Diamond\chi, \Box\Diamond\chi \mid \Diamond\chi \in \mathbf{Sub}\varphi\}$, $x \models \psi$ implies $x \models \Box\psi$. Since $\Box\psi \in \mathbf{\Sigma}_1$, $y \models \psi$. Hence, $y \models \Diamond\psi$. Thus, $[y]\overline{R}_{\Diamond\mathbf{\Sigma}_1}[x]$. Conversely, suppose $[y]\overline{R}_{\Diamond\mathbf{\Sigma}_1}[x]$ and $\Box\psi \in \mathbf{\Sigma}_1$. When $\Box\psi \in \mathbf{Sub}\varphi$, if $x \models \Box\psi$ then $y \models \Diamond\Box\psi$, since $\Diamond\Box\psi \in \mathbf{\Sigma}_1$. Hence, $y \models \Box\psi$. By the reflexivity of $R_\Box$, $y \models \psi$. Thus, $[x]\overline{R}_{\Box\mathbf{\Sigma}_1}[y]$. When $\Diamond\psi \in \{\Box\Box\chi, \Diamond\Box\chi \mid \Box\chi \in \mathbf{Sub}\varphi\} \cup \{\Diamond\Diamond\chi, \Box\Diamond\chi \mid \Diamond\chi \in \mathbf{Sub}\varphi\}$, if $x \models \Box\psi$ then $x \models \psi$. Since $\Diamond\psi \in \mathbf{\Sigma}_1$, $y \models \Diamond\psi$. Hence, $y \models \psi$. Therefore, $[x]\overline{R}_{\Box\mathbf{\Sigma}_1}[y]$. Thus, the frame $(W_{\mathbf{\Sigma}_1}, \overline{R}_{\mathbf{\Sigma}_1}, \overline{R}_{\Box\mathbf{\Sigma}_1}, \overline{R}_{\Diamond\mathbf{\Sigma}_1})$ is also a frame for $\mathbf{IntS5}_{\Box\Diamond}$.

Therefore, for any filtration $(W_{\mathbf{\Sigma}}, \lhd_{\mathbf{\Sigma}_1}, R_{\Box\mathbf{\Sigma}_1}, R_{\Diamond\mathbf{\Sigma}_1}, v_{\mathbf{\Sigma}_1})$, we have $\underline{\unlhd}^{\infty}_{\mathbf{\Sigma}_1} \subseteq \lhd_{\mathbf{\Sigma}_1} \subseteq \overline{\lhd}_{\mathbf{\Sigma}_1}$, $\underline{R}^\star_{\Box\mathbf{\Sigma}_1} \subseteq (\underline{R}^\star_{\Box\mathbf{\Sigma}_1} \circ {\underline{R}^\star_{\Diamond\mathbf{\Sigma}_1}}^{-1})^\infty \subseteq (R_{\Box\mathbf{\Sigma}_1} \circ R^{-1}_{\Diamond\mathbf{\Sigma}_1})^\infty \subseteq \overline{R}_{\Box\mathbf{\Sigma}_1}$ and $\underline{R}^\star_{\Diamond\mathbf{\Sigma}_1} \subseteq (\underline{R}^\star_{\Diamond\mathbf{\Sigma}_1} \circ {\underline{R}^\star_{\Box\mathbf{\Sigma}_1}}^{-1})^\infty \subseteq (R_{\Diamond\mathbf{\Sigma}_1} \circ R^{-1}_{\Box\mathbf{\Sigma}_1})^\infty \subseteq \overline{R}_{\Diamond\mathbf{\Sigma}_1}$. Thus, the frame $(W_{\mathbf{\Sigma}_1}, \lhd_{\mathbf{\Sigma}_1}, (R_{\Box\mathbf{\Sigma}_1} \circ R^{-1}_{\Diamond\mathbf{\Sigma}_1})^\infty, (R_{\Diamond\mathbf{\Sigma}_1} \circ R^{-1}_{\Box\mathbf{\Sigma}_1})^\infty)$ is also a frame which we want to get. ∎

4.3 Note

V. H. Sotirov proved in [26] the finite model property for $\mathbf{IntK}_{\Box\Diamond}$, $\mathbf{IntT}_{\Box\Diamond}$ $\mathbf{IntS4}_{\Box\Diamond}$ and $\mathbf{IntS5}_{\Box\Diamond}$ by using coarsest filtrations.

In this thesis, we gave alternative proofs of the finite model property for these logics by using more general filtration method.

As another result of the finite model property for intuitionistic modal logics, C. Grefe showed that **FC** has the finite model property [12].

In general, the finite model property for bimodal logics is much more difficult than that for mono-modal logics. Many problems remain on the finite model property for intuitionistic modal logics yet.

So far, we have treated the logics of $\mathcal{L}_{\Box\Diamond}$. But, if we restrict the modal operators only to $\Box$ operator, Some logics can admit filtration. For example, we fail to prove that $\mathbf{IntS4.3}_{\Box\Diamond}$ admits filtration. But $\mathbf{IntS4.3}_{\Box}$ on which the modal operator is restricted to $\Box$ operator admits filtration, because we can take $\Box$-rooted counter-model.

Chapter 5

Intuitionistic modal logics II —— subdirectly irreducible algebras

5.1 Introduction

In this chapter, we will discuss modal Heyting algebras. From the algebraic point of view, to each of these intuitionistic modal logics there corresponds a *variety* of modal Heyting algebras ($\Box$-modal Heyting algebras, $\Diamond$-modal Heyting algebras, $\Box\Diamond$-modal Heyting algebras and **FS**-algebras, respectively) with one or two operators (see [33]). Since every algebra is isomorphic to a subdirect product of *subdirectly irreducible* algebras by Birkhoff's subdirect representation theorem, it is important to find a nice description of subdirectly irreducible modal Heyting algebras. In the case of $\Box$-modal Heyting algebras and **FS**-algebras, the following result was proved in [30].

The logic **FS** defined in page 14 corresponds **FS**-algebras, i.e. $\Box\Diamond$-modal Heyting algebras with $\Diamond(a \to b) \leq \Box a \to \Diamond b$ and $\Diamond a \to \Box b \leq \Box(a \to b)$.

Proposition 5.1.1 *Let $\boldsymbol{A}$ be either a $\Box$-modal Heyting algebra or a $\boldsymbol{FS}$-algebra.*

1. *A* nontrivial *algebra $\boldsymbol{A}$ is subdirectly irreducible iff there exists an element $a \in \boldsymbol{A}$ with $a \neq 1$ such that for all $b \in \boldsymbol{A}$ with $b \neq 1$ there exists a number n such that $b \wedge \Box b \wedge \Box^2 b \wedge \cdots \wedge \Box^n b \leq a$.*

2. *A finite algebra $\boldsymbol{A}$ is subdirectly irreducible iff the* dual frame *of $\boldsymbol{A}$ is* rooted.

In [30] the following problem is posed; "Is there a nice description of subdirectly irreducible finite $\Diamond$-modal Heyting algebras as for a $\Box$-modal Heyting algebra in Proposition 5.1.1?" In this chapter, we will give a uniform description of subdirectly irreducible algebras for various classes of (multi-) modal Heyting algebras (see Theorem 5.3.2 and Corollary 5.3.3), from which an answer to the above problem immediately follows (see Proposition 5.3.4).

5.2 Normalizing operators

In this section, we will discuss Heyting algebras with operators. In the rest of the chapter, we assume that $\boldsymbol{A} = (\boldsymbol{A'}, M)$ is a Heyting algebra $\boldsymbol{A'} = (A, \wedge, \vee, \to, 0, 1)$ with a finite set M of *unary* operators $m_1, \ldots, m_k$. Let $a \leftrightarrow b$ be the abbreviation of $(a \to b) \wedge (b \to a)$.

A filter F of $\boldsymbol{A}$ is called an *M-filter* if for each $i \in \{1, \ldots, k\}$,

$$a \leftrightarrow b \in F \text{ implies } m_i a \leftrightarrow m_i b \in F \qquad \text{for all } a, b \in A. \tag{5.1}$$

Let $F_M(\boldsymbol{A})$ be the set of all M-filters of $\boldsymbol{A}$. We can show that $\boldsymbol{F}_M(\boldsymbol{A}) = (F_M(\boldsymbol{A}), \cap, \vee, \{1\}, A)$ forms a complete lattice, where for all $G, H \in F_M(\boldsymbol{A}), G \vee H$ denotes the M-filter generated by the set $G \cup H$. Let $\Theta(\boldsymbol{A})$ be the set of all congruence relations on $\boldsymbol{A}$. A binary relation θ on A will be identified with its graph $\{(a, b) \in A^2 \mid a\,\theta\,b\}$. We can show easily that $\boldsymbol{\Theta}(\boldsymbol{A}) = (\Theta(\boldsymbol{A}), \cap, \vee, \Delta, A^2)$ forms a complete lattice. Here Δ denotes the *diagonal* $\{(a, a) \in A^2 \mid a \in A\}$ and $\theta_1 \vee \theta_2$ denotes the congruence relation generated by the set $\theta_1 \cup \theta_2$ for all $\theta_1, \theta_2 \in \Theta(\boldsymbol{A})$.

It is well-known that for any Heyting algebra $\boldsymbol{A'}$, the complete lattice of all filters of $\boldsymbol{A'}$ is isomorphic to the complete lattice of all congruence relations on $\boldsymbol{A'}$. This result can be easily extended as follows.

Proposition 5.2.1 *The map*

$$f : F \mapsto \theta_F = \{(a, b) \mid a \leftrightarrow b \in F\}$$

is an isomorphism from the complete lattice $\boldsymbol{F}_M(\boldsymbol{A})$ onto the complete lattice $\boldsymbol{\Theta}(\boldsymbol{A})$. The inverse map is given by

$$g : \theta \mapsto F_\theta = \{a \in A \mid a\,\theta\,1\}.$$

Proof. We note first that θ_F is a congruence relation by definitions if F is an M-filter. Next, F_θ is an M-filter whenever θ is a congruence relation, since $(a \leftrightarrow b)\,\theta\,1$ iff $a\,\theta\,b$. It is easy to see that $F_{\theta_F} = F$ and $\theta_{F_\theta} = \theta$ i.e., $(g \circ f)(F) = F$ and $(f \circ g)(\theta) = \theta$. So, f is an isomorphism from $\boldsymbol{F}_M(\boldsymbol{A})$ onto $\boldsymbol{\Theta}(\boldsymbol{A})$. ∎

A unary operator m on A is *monotone* if

$$a \leq b \text{ implies } m\,a \leq m\,b$$

for all $a, b \in A$. A unary operator m is *normal* if both

$$m\,1 = 1 \text{ and } m(a \wedge b) = m\,a \wedge m\,b$$

hold for all $a, b \in A$. It is clear that every normal operator is monotone.

Proposition 5.2.2 *The condition (5.1) can be replaced by the following condition*

$$a \rightarrow b \in F \textit{ implies } m_i a \rightarrow m_i b \in F \qquad \textit{for all } a, b \in A \tag{5.2}$$

when m_i is monotone; and by the condition

$$a \in F \textit{ implies } m_i a \in F \qquad \textit{for all } a \in A \tag{5.3}$$

when m_i is normal.

Proof. First, suppose m_i is monotone. We will show that the condition (5.1) implies the condition (5.2). Suppose $a \to b \in F$. Then $a \leftrightarrow (a \wedge b) \in F$, and, by the condition (5.1) $m_i a \leftrightarrow m_i(a \wedge b) \in F$. On the other hand, $m_i(a \wedge b) \leq m_i b$ holds by using monotonicity. Therefore $m_i a \to m_i b \in F$. Conversely, it is clear that the condition (5.2) implies the condition (5.1). Next, suppose m_i is normal. Then, the condition (5.2) implies the condition (5.3) by using $1 \to x = x$ for any x and $m_i 1 = 1$. Conversely, the condition (5.3) implies the condition (5.2) by using $m_i(a \to b) \leq m_i a \to m_i b$, which is a consequence of the normality. ∎

Next, we will introduce an operator which characterizes M-filters. We will define (partial) operators $[m_i]$ and $[M]$ as follows: for each $a \in A$,

$$[m_i]a := \bigwedge_{a \leq b \leftrightarrow c} (m_i b \leftrightarrow m_i c),$$

$$[M]a := \bigwedge_{\substack{a \leq b \leftrightarrow c \\ 1 \leq i \leq k}} (m_i b \leftrightarrow m_i c),$$

if the infima exist, and they are undefined, otherwise. It is easily seen that if all $[m_i]$ exist then $[M]$ exists and $[M] = \bigwedge_{1 \leq i \leq k} [m_i]$.

Theorem 5.2.3 *Suppose that both $[m_i]a$ and $[M]a$ exist for all $a \in A$. Then $[m_i]$ and $[M]$ are normal.*

Proof. First $[m_i]1 = \bigwedge_{b=c}(m_i b \leftrightarrow m_i c) = \bigwedge 1 = 1$. Next $[m_i](a \wedge b) \leq [m_i]a \wedge [m_i]b$ holds, since $\{(c,d) \mid a \leq c \leftrightarrow d\} \cup \{(c,d) \mid b \leq c \leftrightarrow d\} \subseteq \{(c,d) \mid a \wedge b \leq c \leftrightarrow d\}$. Conversely suppose that $a \wedge b \leq c \leftrightarrow d$. Then $a \leq (b \wedge c) \leftrightarrow (b \wedge d)$, $b \leq c \leftrightarrow (b \wedge c)$ and $b \leq d \leftrightarrow (b \wedge d)$. Hence $[m_i]a \wedge [m_i]b \leq (m_i(b \wedge c) \leftrightarrow m_i(b \wedge d)) \wedge (m_i c \leftrightarrow m_i(b \wedge c)) \wedge (m_i d \leftrightarrow m_i(b \wedge d)) \leq m_i c \leftrightarrow m_i d$. Therefore $[m_i]a \wedge [m_i]b \leq [m_i](a \wedge b)$ holds. Similarly, we can show that $[M]$ is normal. ∎

Proposition 5.2.4 *Suppose that each m_i is monotone. Then for all $a \in A$, the following equations hold.*

1.

$$[m_i]a = \bigwedge_{b \in A} (m_i b \to m_i(a \wedge b)),$$

2.

$$[M]a = \bigwedge_{\substack{b \in A \\ 1 \leq i \leq k}} (m_i b \to m_i(a \wedge b)).$$

More precisely, whenever the infimum on one side exists in the above equations, the infimum on the other side exists and they are equal.

Proof. Since $a \leq b \leftrightarrow (a \wedge b)$ and by the monotonicity $m_i b \geq m_i(a \wedge b)$ for all $a, b \in A$, $\{m_i b \to m_i(a \wedge b) \mid b \in A\} \subseteq \{m_i b \leftrightarrow m_i c \mid a \leq b \leftrightarrow c\}$. Suppose $a \leq b \leftrightarrow c$. Then $a \wedge b \leq c$ and $a \wedge c \leq b$. Hence $(m_i b \to m_i(a \wedge b)) \wedge (m_i c \to m_i(a \wedge c)) \leq m_i b \leftrightarrow m_i c$ by the monotonicity of m_i. The second equality is proved, similarly. ∎

Proposition 5.2.5 *Suppose that m_i is normal for a given i. Then $[m_i]a$ always exists and $[m_i]a = m_i a$, for all $a \in A$.*

Proof. By Proposition 5.2.4, it is enough to show that $m_i a$ is the minimum of $\{m_i b \to m_i(a \wedge b) \mid b \in A\}$. Since m_i is normal, $m_i a = m_i 1 \to m_i(a \wedge 1)$ and $m_i a \leq m_i b \to m_i(a \wedge b)$, for all $b \in A$. ∎

Corollary 5.2.6 *Suppose that both $[m_i]a$ and $[M]a$ exist for all $a \in A$. Then $\big[[m_i]\big] = [m_i]$ and $\big[[M]\big] = [M]$.*

Proof. By Theorem 5.2.3 and Proposition 5.2.5. ∎

5.3 A description of subdirectly irreducible algebras

Recall that a nontrivial algebra $\boldsymbol{A}$, i.e. $0 \neq 1$ holds in $\boldsymbol{A}$, is *subdirectly irreducible* (s.i., for short) iff it has the second smallest congruence relation. In particular, $\boldsymbol{A}$ is s.i. iff $\bigcap(\Theta(\boldsymbol{A}) - \{\Delta\}) \supsetneqq \Delta$. For a subset $B \subseteq A$, we denote by $[B)$ the smallest filter containing B, and by $[B)_M$ the smallest M-filter containing B. Sometimes, $[\{a\})$ is denoted by $[a)$ and $[\{a\})_M$ is denoted by $[a)_M$. For $a \in A$ we define $[M]^n a$ by induction on n as follows;

$$[M]^0 a = a, \ [M]^{n+1} a = [M]([M]^n a).$$

Moreover, we put $[M]^{(n)} a = [M]^0 a \wedge \cdots \wedge [M]^n a$. In particular, $[M]^{(1)}$ is denoted by $[M]^+$. When $[m_i]b$ exists for all $b \in A$ and all $i \in \{1, 2, \ldots, k\}$, we can show that

$$[M]^{(n)} a = \bigwedge [m_{i_1}] \cdots [m_{i_d}] a$$

where $(i_1, \ldots, i_d)$ ranges over all sequences consisting of elements in $\{1, 2, \ldots, k\}$ with the length d.

Lemma 5.3.1 *Suppose that $[M]a$ exists for all $a \in A$. Then*

1. *for any filter F, if F is a $[M]$-filter then F is also a M-filter,*
2. *for any nonempty subset B of A, $[B)_M$ is a subset of $[B)_{[M]}$, which in turn is equal to $\{a \in A \mid [M]^{(n)}(b_1 \wedge \cdots \wedge b_j) \leq a$ for some $b_1, \ldots, b_j \in B$ and some $n, j \in \boldsymbol{N}\}$,*
3. *$\Theta(\boldsymbol{A'}, [M])$ is a subset of $\Theta(\boldsymbol{A'}, M)$,*
4. *if $(\boldsymbol{A'}, M)$ is s.i., then $(\boldsymbol{A'}, [M])$ is also s.i., i.e. there exists such an element $a \in A$ with $a \neq 1$ that for all $b \in A$ with $b \neq 1$ there exists a number n such that $[M]^{(n)} b \leq a$.*

Proof. Before proving our lemma, we note here that while $(\boldsymbol{A}', M)$ with $M = \{m_1, \ldots, m_k\}$ is a Heyting algebra with k operators, $(\boldsymbol{A}', [M])$ is a Heyting algebra with a single unary operator $[M]$. For (1), suppose that $F \ni a \leftrightarrow b$ for a $[M]$-filter F. Since $[m_i](a \leftrightarrow b) \leq m_i a \leftrightarrow m_i b$, $F \ni m_i a \leftrightarrow m_i b$. Hence F is an M-filter. Next, we will show that (1),(2) and (3) are equivalent. (1) implies (2) since $[B)_M$ is the smallest M-filter containing B. (2) implies (1) since $F \subseteq [F)_M \subseteq [F)_{[M]} = F$ for any $[M]$-filter F. Clearly, (1) and (3) are equivalent by Proposition 5.2.1. (4) holds because $\bigcap(\Theta(\boldsymbol{A}', [M]) - \{\Delta\}) \supseteq \bigcap(\Theta(\boldsymbol{A}', M) - \{\Delta\}) \supsetneqq \Delta$ by (3) and the condition that $(\boldsymbol{A}', M)$ is s.i.. ∎

Theorem 5.3.2 *The converse of each of (1)–(3) in Lemma 5.3.1 also holds iff $[M]a$ belongs to $[a)_M$ for every $a \in A$. Moreover, when $[M]a$ exists in $[a)_M$ for every $a \in A$, a nontrivial algebra $(\boldsymbol{A}', M)$ is s.i. iff $(\boldsymbol{A}', [M])$ is s.i.*

Proof. Suppose that any M-filter is a $[M]$-filter. Since $[a)_M \supseteq [a)_{[M]}$ and $[a)_{[M]} \ni [M]a$, $[M]a$ belongs to $[a)_M$. Conversely, suppose that $[M]a$ belongs to $[a)_M$. If F is an M-filter and $F \ni a$, then $F \ni [M]a$ since $F \supseteq [a)_M$ and $[a)_M \ni [M]a$. Since $[M]$ is normal by Theorem 5.2.3, F is a $[M]$-filter. Thus, the set of $[M]$-filters is the same as that of M-filters. The rest follows immediately from this. ∎

The following corollary is an immediate consequence of Theorem 5.3.2.

Corollary 5.3.3 *When $[M]a$ exists in $[a)_M$ for every $a \in A$, a nontrivial algebra $\boldsymbol{A} = (\boldsymbol{A}', M)$ is s.i. iff there exists such an element $a \in A$ with $a \neq 1$ that for all $b \in A$ with $b \neq 1$ there exists a number n such that $[M]^{(n)}b \leq a$.*

Proposition 5.3.4 *Suppose that $\boldsymbol{A} = (\boldsymbol{A}', M)$ is finite. Then $[M]a$ always exists in $[a)_M$. Hence the consequence of Theorem 5.3.2 holds.*

Proof. For all $b, c \in A$ and all $i \in \{1, \ldots, k\}$, $[a)_M \ni m_i b \leftrightarrow m_i c$, if $[a)_M \ni b \leftrightarrow c$. Hence $[a)_M \ni [M]a$, since the infimum in the definition of $[M]a$ consists of finitely many elements. ∎

From Theorem 5.3.2 and Proposition 5.3.4 we can derive the result corresponding to Proposition 5.1.1(2) for modal Heyting algebras, in general. As special case, we will discuss the case of $\Box\Diamond$-modal Heyting algebras in the next section (Theorem 5.4.9).

Fix-points of $[M]^+$ correspond to M-filters as the following theorems show.

Theorem 5.3.5 *Suppose that $[M]a$ exists for all $a \in A$. Then $(\{a \in A \mid a = [M]^+a\}, \wedge, \vee, 0, 1)$ is a sublattice of $\boldsymbol{A}'$, and the map*

$$h : a \mapsto [a) = \{b \in A \mid a \leq b\}$$

dually embeds the lattice $(\{a \in A \mid a = [M]^+a\}, \wedge, \vee, 0, 1)$ into the lattice $\boldsymbol{F}_M(\boldsymbol{A})$.

Proof. First we show that $(\{a \in A \mid a = [M]^+a\}, \wedge, \vee, 0, 1)$ is a sublattice of $\boldsymbol{A}'$. Suppose that $a_1 = [M]^+a_1$ and $a_2 = [M]^+a_2$. Then $a_1 \wedge a_2 \leq [M]a_1 \wedge [M]a_2 = [M](a_1 \wedge a_2)$ by the normality of $[M]$, and $a_1 \vee a_2 \leq [M]a_1 \vee [M]a_2 \leq [M](a_1 \vee a_2)$ by the monotonicity of $[M]$.

Also, $0 \leq [M]0$ and $1 = [M]1$. Next, we show that $[a)$ is an M-filter if $a = [M]^+a$ holds. Suppose $a \leq b \leftrightarrow c$. Then $a \leq m_i b \leftrightarrow m_i c$, since $[M]a \leq m_i b \leftrightarrow m_i c$ and $a \leq [M]a$. Finally, it is easily seen that $[a_1) \vee [a_2) = \{b \in A \mid a_1 \wedge a_2 \leq b\}$, $[a_1) \cap [a_2) = \{b \in A \mid a_1 \vee a_2 \leq b\}$, $[0) = A$ and $[1) = \{1\}$ hold. So, since h is clearly injective, h is a dual embedding from $(\{a \in A \mid a = [M]^+a\}, \wedge, \vee, 0, 1)$ into $\boldsymbol{F}_M(\boldsymbol{A})$. ∎

When an algebra $\boldsymbol{A}$ is complete, we have the following theorem.

Theorem 5.3.6 *Suppose that $\boldsymbol{A} = (\boldsymbol{A'}, M)$ is a complete lattice. Then $\{a \in A \mid a = [M]^+a\}$ is closed under arbitrary joins, and the map h from Theorem 5.3.5 translates arbitrary joins to arbitrary intersections in $\boldsymbol{F}_M(\boldsymbol{A})$. Hence if a nontrivial algebra $\boldsymbol{A}$ is moreover s.i. then $\{a \in A \mid a = [M]^+a\}$ has the second greatest element.*

Proof. We note first that $[M]a$ always exists for all $a \in A$. Suppose that B is a subset of $\{a \in A \mid a = [M]^+a\}$. Then $\bigvee B \leq \bigvee\{[M]b \mid b \in B\} \leq [M](\bigvee B)$ by the monotonicity of $[M]$, and $\bigcap h(B) = \bigcap\{[b) \mid b \in B\} = \{a \in A \mid \bigvee B \leq a\} = h(\bigvee B)$. Hence, $h(\bigvee(\{a \in A \mid a = [M]^+a\} - \{1\})) = \bigcap h(\{a \in A \mid a = [M]^+a\} - \{1\}) \supseteq \bigcap(F_M(\boldsymbol{A}) - \{\{1\}\}) \supsetneq \{1\}$ when $\boldsymbol{A}$ is s.i.. Therefore $\bigvee(\{a \in A \mid a = [M]^+a\} - \{1\})$ is the second greatest element since $\bigvee(\{a \in A \mid a = [M]^+a\} - \{1\}) < 1$ holds. ∎

When an algebra $\boldsymbol{A}$ is finite, we have following results.

Theorem 5.3.7 *Suppose $\boldsymbol{A} = (\boldsymbol{A'}, M)$ is finite. Then the map h in Theorem 5.3.5 becomes a dual isomorphism. Hence a nontrivial algebra $\boldsymbol{A}$ is s.i. iff $\{a \in A \mid a = [M]^+a\}$ has the second greatest element.*

Proof. Since algebra $\boldsymbol{A}$ is finite, any M-filter is principal. For any M-filter F there exists an element $a \in A$ such that $F = [a)$. (Take a as $\bigwedge F$.) Then $[a) = [a)_M$ holds. Since $[M]a \in [a)_M$ holds by Proposition 5.3.4, $a = [M]^+a$ holds. Thus, $F = h(a)$. The second part follows immediately from this and Proposition 5.2.1. ∎

5.4 Applications to □◇-modal Heyting algebras

As an application of results in the previous section, we will give a characterization theorem of finite *irreducible* intuitionistic modal Kripke frames (Theorem 5.4.9), which gives an answer to a question put by Wolter in [30].

Any □◇-modal Heyting algebra $\boldsymbol{A} = (\boldsymbol{A'}, \Box, \Diamond)$ can be regarded as a Heyting algebra with the set $M = \{\Box, \Diamond\}$. In the following we will write $[\Box\Diamond]$ instead of $[M]$. Note that $\Box$ is normal and $\Diamond$ is monotone. As an immediate consequence of Corollary 5.3.3, we have the following which implies the result in [30] for $\boldsymbol{FS}$-algebras.

Corollary 5.4.1 *Let $\boldsymbol{A}$ be a □◇-modal Heyting algebra with $\Box(a \to b) \leq \Diamond a \to \Diamond b$ for all $a, b \in A$. A nontrivial algebra $\boldsymbol{A}$ is s.i. iff there exists an $a \in A$ with $a \neq 1$ such that for all $b \in A$ with $b \neq 1$ there exists a number n such that $\Box^{(n)}b \leq a$.*

Proof. It is easy to see that $\Box a \leq \Diamond b \to \Diamond(a \wedge b)$ holds for all $b \in A$ by our assumption. Then, by using Proposition 5.2.4 and 5.2.5, we have $[\Box\Diamond]a = \Box a$. Thus $[\Box\Diamond]a = \Box a \in [a)_{\Box\Diamond}$. Hence, we have our corollary by using Corollary 5.3.3. ∎

As another consequence of Theorem 5.3.2, we have the following, which implies the well-known result for Boolean algebras.

Corollary 5.4.2 *Let $\boldsymbol{A} = (\boldsymbol{A'}, \Box, \Diamond)$ be a $\Box\Diamond$-modal Heyting algebra in which $\boldsymbol{A'}$ is a Boolean algebra. Then*

$$(\boldsymbol{A'}, \Box, \Diamond) \text{ is s.i.} \quad \text{iff} \quad (\boldsymbol{A'}, \Box, \neg\Diamond\neg) \text{ is s.i..}$$

Proof. First note that $[\Diamond]a \leq \Diamond\neg a \to \Diamond(a \wedge \neg a) = \neg\Diamond\neg a$, for any $a \in A$. Conversely, for any $a, b \in A$, $\Diamond b \leq \Diamond\neg a \vee \Diamond(a \wedge b)$, since $b \leq \neg a \vee (b \wedge a)$. Hence $\neg\Diamond\neg a \leq \Diamond b \to \Diamond(a \wedge b)$. Therefore $[\Diamond] = \neg\Diamond\neg$. Next, $\neg\Diamond\neg a \in [a)_\Diamond$ follows from $\neg\neg a \in [a)_\Diamond$. We have the corollary by Theorem 5.3.2. ∎

An intuitionistic modal Kripke frame $\mathcal{F}$ is called *irreducible* if the dual $\mathcal{F}^\dagger$ is s.i..

Let $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond)$ be an intuitionistic modal Kripke frame and V a nonempty subset of W satisfying the following conditions:

$$\text{for any } x \in V, y \in W,\ x \lhd y \text{ implies } \ y \in V, \tag{5.4}$$

$$\text{for any } x \in V, y \in W,\ xR_\Box y \text{ implies } \ y \in V, \tag{5.5}$$

$$\text{for any } x \in V, y \in W,\ xR_\Diamond y \text{ implies that there exists } z \in V \text{ such that } xR_\Diamond z \text{ and } yRz. \tag{5.6}$$

Then we can show that $\mathcal{G} = (V, R \cap V^2, R_\Box \cap V^2, R_\Diamond \cap V^2)$ is also an intuitionistic modal Kripke frame. Following [30], we say that $\mathcal{G}$ is the *generated subframe of $\mathcal{F}$ induced by V*. For an intuitionistic modal Kripke frame $\mathcal{F}$ and $r \in W$, $W(r)$ denotes the smallest set which induces a generated subframe and contains r, if such a set exists.

Proposition 5.4.3 *Suppose that $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond)$ is an intuitionistic modal Kripke frame. If both V_1 and V_2 induce generated subframes and their intersection is nonempty, then $V_1 \cap V_2$ induces a generated subframe. If each V_i ($i \in I$) induces a generated subframe, then $\bigcup_{i \in I} V_i$ induces a generated subframe.*

Proof. It is clear that the family of sets satisfying conditions (5.4) and (5.5) is closed under intersection. For condition (5.6), suppose that both V_1 and V_2 induce generated subframes. Let $x \in V_1 \cap V_2$ and $y \in W$ be elements satisfying $xR_\Diamond y$. Then there exists $z \in V_1$ such that $xR_\Diamond z$ and $y \lhd z$. Hence since $xR_\Diamond z$, there exists $v \in V_2$ such that $xR_\Diamond v$ and $z \lhd v$. Therefore $v \in V_1 \cap V_2, xR_\Diamond v$ and $y \lhd v$ hold, since V_1 is upward closed with respect to $\lhd$ and $\lhd$ is transitive. Thus the family of sets satisfying condition (5.6) is closed under intersection. It is straightforward that the family of sets satisfying conditions (5.4), (5.5) and (5.6) is closed under union. ∎

Corollary 5.4.4 *Suppose that $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond)$ is a finite intuitionistic modal Kripke frame. For all $r \in W$, $W(r)$ exists.*

Proof. We note that W induces a generated subframe which contains r. Let U be the set of subsets of W, each of which induces a generated subframe and contains r. Then U is closed under finite intersection, by Proposition 5.4.3. The intersection of all sets in U gives $W(r)$. ∎

For any $w \in W$ we denote by $\lhd(w)$ the set $\{v \in W \mid w \lhd v\}$.

Theorem 5.4.5 *Suppose that $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond)$ is an intuitionistic modal Kripke frame. Then for any nonempty $X \in UpW$,*

$$X = [\Box\Diamond]^+ X \quad \textit{iff} \quad X \textit{ induces a generated subframe.}$$

Proof. We note first that

$$X \in UpW \quad \text{iff} \quad X \text{ satisfies the condition (5.4).}$$

Obviously,

$$X \subseteq \Box X \quad \text{iff} \quad X \text{ satisfies the condition (5.5).}$$

Since $\Diamond$ is monotone, $[\Diamond]X = \bigcap_{Y \in UpW}(\Diamond Y \to \Diamond(X \cap Y))$. Hence,
$r \in [\Diamond]X$
iff $\forall Y \in UpW\ r \in \Diamond Y \to \Diamond(X \cap Y)$,
iff $\forall Y \in UpW\ \forall w \in W\ ((r \lhd w, w \in \Diamond Y) \Rightarrow w \in \Diamond(X \cap Y))$,
iff $\forall Y \in UpW\ \forall w \in W\ ((r \lhd w, \exists v \in Y(wR_\Diamond v)) \Rightarrow w \in \Diamond(X \cap Y))$,
iff $\forall Y \in UpW\ \forall w, v \in W\ ((r \lhd w, v \in Y, wR_\Diamond v) \Rightarrow w \in \Diamond(X \cap Y))$,
iff $\forall w, v \in W\ ((r \lhd w, wR_\Diamond v) \Rightarrow \forall Y \in UpW\ ((v \in Y) \Rightarrow w \in \Diamond(X \cap Y)))$,
iff $\forall w, v \in W\ ((r \lhd w, wR_\Diamond v) \Rightarrow w \in \Diamond(X \cap \lhd(v)))$,
iff $\forall w, v \in W\ ((r \lhd w, wR_\Diamond v) \Rightarrow \exists u \in X(wR_\Diamond u, v \lhd u))$.
Therefore for $X \in UpW$,

$$X \subseteq [\Diamond]X \quad \text{iff} \quad X \text{ satisfies the condition (5.6).}$$

∎

Note that we can also show Proposition 5.4.3 by Theorem 5.3.5, 5.3.6 and 5.4.5, in an alternative way.

Theorem 5.4.6 *If an intuitionistic modal Kripke frame $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond)$ is irreducible then there exists $r \in W$ such that the only set which induces a generated subframe and contains r is W.*

Proof. By Theorem 5.3.6 and 5.4.5, $\bigcup\{X \subsetneq W \mid X$ induces a generated subframe$\}$ is a proper subset of W. Therefore there exists $r \in W$ such that $r \notin \bigcup\{X \subsetneq W \mid X$ induces a generated subframe$\}$. That is to say, there exists $r \in W$ such that the only set which induces a generated subframe and contains r is W. ∎

Proposition 5.4.7 *A finite intuitionistic modal Kripke frame $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond)$ is irreducible iff there exists $r \in W$ such that $W(r) = W$.*

Proof. By Theorem 5.3.7 and 5.4.5, similarly to Theorem 5.4.6, we have that a finite intuitionistic modal Kripke frame $\mathcal{F}$ is irreducible iff there exists $r \in W$ such that the only set which induces a generated subframe and contains r is W. Since $W(r)$ exists for such r by finiteness, W must be equal to $W(r)$. ∎

We will define a binary relation $R_{[\diamond]}$ on W as follows.

$$wR_{[\diamond]}v \overset{def}{\Leftrightarrow} \forall X \in UpW(w \in [\diamond]X \Rightarrow v \in X).$$

We say that an intuitionistic modal Kripke frame $\mathcal{F} = (W, \lhd, R_\Box, R_\diamond)$ is $\Box[\diamond]$-*rooted* if there exists such a *root* r that $W = \{w \in W \mid r \lhd w \text{ or } r\,(R_\Box \cup R_{[\diamond]})^\infty\, w\}$.

Proposition 5.4.8 *Suppose that $\mathcal{F} = (W, \lhd, R_\Box, R_\diamond)$ is an intuitionistic modal Kripke frame. If V induces a generated subframe containing $r \in W$, then $V \supseteq \{w \in W \mid r \lhd w \text{ or } r\,(R_\Box \cup R_{[\diamond]})^\infty\, w\}$. Moreover, if $[\diamond]$ distributes over infinite intersections (i.e., $[\diamond](\bigcap_{\lambda\in\Lambda} X_\lambda) = \bigcap_{\lambda\in\Lambda}[\diamond]X_\lambda$), then $W(r) = \{w \in W \mid r \lhd w \text{ or } r\,(R_\Box \cup R_{[\diamond]})^\infty\, w\}$.*

Proof. Since $[\diamond]X \subseteq \{w \in W \mid \text{ for any } v,\ wR_{[\diamond]}v \text{ implies } v \in X\}$ for any $X \in UpW$ by definition, $V \supseteq \{w \in W \mid r \lhd w \text{ or } r\,(R_\Box \cup R_{[\diamond]})^\infty\, w\}$. Suppose that $[\diamond]$ distributes infinitely many intersection. Note that $wR_{[\diamond]}v \Leftrightarrow v \in \bigcap_{w\in[\diamond]X} X$. Then $[\diamond]V \supseteq [\diamond](\bigcap_{w\in[\diamond]X} X) = \bigcap_{w\in[\diamond]X}[\diamond]X \ni w$, when $V \supseteq \bigcap_{w\in[\diamond]X} X$. Hence $[\diamond]X = \{w \in W \mid \text{ for any } v,\ wR_{[\diamond]}v \text{ implies } v \in X\}$ for any $X \in UpW$. Therefore $\{w \in W \mid r \lhd w \text{ or } r\,(R_\Box \cup R_{[\diamond]})^\infty\, w\}$ is the smallest set which induces a generated subframe and contains r. ∎

Recall here that a finite algebra (frame) is isomorphic to its bidual. The following theorem answers to a question put by Wolter in [30].

Theorem 5.4.9 *A finite intuitionistic modal Kripke frame $\mathcal{F} = (W, \lhd, R_\Box, R_\diamond)$ is irreducible iff it is $\Box[\diamond]$-rooted.*

Proof. By Propositions 5.4.7 and 5.4.8. ∎

5.5 Some remarks

So far, we have dealt with Heyting algebras with unary operators. But these arguments can be extended to Heyting algebras with (a finite number of) operators having arbitrary arities. Suppose that $\boldsymbol{A} = (\boldsymbol{A'}, M)$ is a Heyting algebra with the set M of operators $m_1, \ldots, m_k$ whose arities are $n_1, \ldots, n_k$. For $a \in A$, define (partial) operators

$$[m_i]a := \bigwedge_{\substack{a \le b_j \leftrightarrow c_j \\ 1 \le j \le n_i}} (m_i(b_1, \ldots, b_{n_i}) \leftrightarrow m_i(c_1, \ldots, c_{n_i})),$$

$$[M]a := \bigwedge_{\substack{a \le b_j \leftrightarrow c_j \\ 1 \le j \le n_i \\ 1 \le i \le k}} (m_i(b_1, \ldots, b_{n_i}) \leftrightarrow m_i(c_1, \ldots, c_{n_i})),$$

if the infima exist. Again note that $(\boldsymbol{A'}, [M])$ is a Heyting algebra with a single unary operator $[M]$. Then most of results on Heyting algebras with unary operators hold also for algebras in this section. The only alteration we are required to make in the proofs to change Proposition 5.2.5 into Proposition 5.5.1, because a n_i-ary operator m is *normal* if both $m(a,\ldots,a,1,a,\ldots,a)=1$ and $m(a,\ldots,a,b\wedge c,a,\ldots,a)=m(a,\ldots,a,b,a,\ldots,a)\wedge m(a,\ldots,a,c,a,\ldots,a)$ hold for all $a,b,c\in A$. Denote by $a^j=(0,\ldots,0,a,0,\ldots,0)$ the element of A^{n_i} whose j-th coordinate equals a and all other coordinates equal 0.

Proposition 5.5.1 *Suppose that m_i is normal n_i-ary operator for a given i. Then $[m_i]a$ exists and $[m_i]a = \bigwedge_{1\le j\le n_i} m_i a^j$ for any $a\in A$.*

As another application of the relationship between M-filters and $[M]$-filters in Lemma 5.3.1, we can show the deduction theorem for intuitionistic modal logics. By $\mathcal{L}_M$ we denote the language of propositional intuitionistic logic with connectives $\wedge,\vee,\to,\top,\bot$ and operators $M=\{m_1,\ldots,m_k\}$. Let L be a logic, i.e. a subset L of $\mathcal{L}_M$ containing **Int**, and closed under modus ponens, substitution, the *congruence rule* (i.e., $\vdash_L \psi\leftrightarrow\varphi \,/\, \vdash_L m_i\,\psi\leftrightarrow m_i\,\varphi$, for all i, $1\le i\le k$). Recall that a *derivation* of φ from assumptions $\mathbf{\Gamma}$ is a sequence $\varphi_1,\ldots,\varphi_n$ of formulas such that $\varphi_n=\varphi$ and for every i, $1\le i\le n$, φ_i is either an axiom, an assumption or obtained from some of the preceding formulas in the sequence by one of the inference rules, with substitution being applied only to axioms. We say that a formula φ_k *depends on a formula* φ_i in the derivation if either $k=i$ or φ_k is obtained by modus ponens or a congruence rule from formulas, at least one of which depends on φ_k. If there is a derivation of φ from assumptions $\mathbf{\Gamma}$, we write $\mathbf{\Gamma}\vdash_L\varphi$. For simplicity, we will write $\mathbf{\Gamma},\psi_1,\ldots,\psi_n\vdash_L\varphi$ instead of $\mathbf{\Gamma}\cup\{\psi_1,\ldots,\psi_n\}\vdash_L\varphi$.

Theorem 5.5.2 (deduction theorem for intuitionistic modal logic L) *Suppose $\mathbf{\Gamma},\psi\vdash_L\varphi$ and there exists a derivation of φ from the assumptions $\mathbf{\Gamma}\cup\{\psi\}$ in which congruence rules are applied to formulas depending on ψ $n(\ge 0)$ times. Also suppose that for any formula α there exists such a formula $[\![M]\!]\alpha$ that for any β,γ,δ, $\vdash_L\alpha\to(\beta\leftrightarrow\gamma)$ implies that $\vdash_L\delta\to[\![M]\!]\alpha \Leftrightarrow$ for each i $\vdash_L\delta\to(m_i\;\beta\leftrightarrow m_i\;\gamma)$. Then*

$$\mathbf{\Gamma}\vdash_L \psi\wedge[\![M]\!]\psi\wedge[\![M]\!]([\![M]\!]\psi)\wedge\cdots\wedge[\![M]\!]^n\psi\to\varphi.$$

Proof. The proof of this theorem can be directly shown by induction on the length of derivation. ∎

In the case of the logic $\boldsymbol{FS}$, for example, we can take $\Box\alpha$ for $[\![M]\!]\alpha$. Let $\boldsymbol{A}_L$ be the Lindenbaum algebra of L. Let $\|\varepsilon\|$ be the element of $\boldsymbol{A}_L$, to which a formula ε belongs. Then, our assumption means that $\|[\![M]\!]\alpha\|=[M]\|\alpha\|$.

5.6 Note

A description of subdirectly irreducible Heyting algebras is wellknown. A description of subdirectly irreducible $\Box$-modal Heyting algebra or a $\boldsymbol{FS}$-algebra is shown in [30]. In [30] this description is applied to splittings. In order to be applied to splittings for $\Box$-modal logics it is needed to a description of subdirectly irreducible $\Diamond$-modal Heyting algebra.

Chapter 6

Products of modal logics

6.1 Introduction

In this chapter, we will propose a new concept of products of modal logics, which we will call a *normal product*. Normal products will resemble the products familiar from measure theory and topology, and will be defined as a generalization of *products of algebras of sets* (see § 6.2). Our products of modal logics can be defined either by means of *normal products of general frames* or by means of *normal products of modal algebras*. It enables us to develop a duality theory between these two, as shown in § 6.3. This brings about a desired effect that the definition of the normal product of modal logics L_1 and L_2 is not affected by the choice of classes of general frames (or, modal algebras) which determine L_1 and L_2. Note, that this is not the case for usual products of modal logics, as pointed out in [21]. We also show some transfer results, including transfer of the finite model property, in § 6.4.

One conceptual difference between normal products and usual products is, that in the usual case, when L_1 and L_2 are m-modal and n-modal logics, then their product $L_1 \times L_2$ is a $(m+n)$-modal logic. On the other hand, the normal product can be defined only when both L_1 and L_2 are m-modal logics and their normal product is also a m-modal logic. This difference, however, is not essential. In fact, when a m-modal logic L_1 and a n-modal logic L_2 are given, we first consider two $(m+n)$-modal logics ${L_1}^\circ$ and ${}^\circ L_2$ and then take their normal product. Here, ${L_1}^\circ$ and ${}^\circ L_2$ are essentially the same as L_1 and L_2, respectively, but in addition they have n and m *dummy* modal operators, respectively (see § 6.5 for the detailed definition). We will denote it as $L_1 \otimes L_2$ and call it the *shifted product* of L_1 and L_2. Relations between shifted and usual products will be discussed in the last section.

As the definition of our normal products is quite general, by a slight modification of definitions, we can also introduce normal products of two superintuitionistic logics, two intuitionistic modal logics, and even of *infinitely many* of them, and obtain results similar to these from the present paper.

We will assume a certain familiarity with [9], and basically follow the terminology in [5].

6.2 Normal products of general frames

In this section, we will define normal products of general frames, following the standard method used in measure theory and topology (see e.g. [16]). For simplicity, we will consider only general frames for mono-modal logics, but it is easily seen that every definition and result can be naturally extended to m-modal case for any given $m > 1$.

Definition 6.2.1 *A set $\mathcal{P}$ of subsets of W is called a* modal algebra *on a Kripke frame (W, R) if*

(i) $\emptyset \in \mathcal{P}$,

(ii) $X, Y \in \mathcal{P} \Rightarrow X \cap Y \in \mathcal{P}$,

(iii) $X \in \mathcal{P} \Rightarrow -X \in \mathcal{P}$,

(iv) $X \in \mathcal{P} \Rightarrow \Diamond X \in \mathcal{P}$.

A modal algebra on (W, R) contains W and is closed under $\cup$, $\Box$. Clearly, $\mathcal{P}(W)$ is an example of a modal algebra on (W, R).

Definition 6.2.2 *A set $\mathcal{S}$ of subsets of W is called a* modal semi-algebra *on a Kripke frame (W, R) if*

(i) $\emptyset \in \mathcal{S}$,

(ii) $X, Y \in \mathcal{S} \Rightarrow X \cap Y \in \mathcal{S}$,

(iii) $X \in \mathcal{S} \Rightarrow -X$ *is a union of finitely many members of* $\mathcal{S}$,

(iv) $X \in \mathcal{S} \Rightarrow \Diamond X \in \mathcal{S}$.

Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are modal algebras on (W, R) and (V, S), respectively. We consider the product $(W \times V, R \times S)$ of Kripke frames (W, R) and (V, S), where $W \times V$ is the direct product of W and V and $R \times S$ is the product relation of R and S, i.e., a binary relation on $W \times V$ defined by

$$(w_1, v_1) R{\times}S (w_2, v_2) \text{ if and only if } w_1 R w_2 \text{ and } v_1 S v_2,$$

for $w_1, w_2 \in W$ and $v_1, v_2 \in V$.

Now, consider the set $\{X \times Y \mid X \in \mathcal{P} \text{ and } Y \in \mathcal{Q}\}$ of *rectangle sets*, which is a set of subsets of $W \times V$. This set is not always a modal algebra on $(W \times V, R \times S)$, but it is always a modal semi-algebra on it. For a given Kripke frame (W, R) and a set $\mathcal{S}$ of subsets of W, the smallest modal algebra on (W, R) containing $\mathcal{S}$ is called the *modal algebra generated by* $\mathcal{S}$. Then, the following lemma holds.

Lemma 6.2.3 *Suppose that $\mathcal{S}$ is a modal semi-algebra on a Kripke frame (W, R). Then the modal algebra generated by $\mathcal{S}$ is* $\{\bigcup_{i=1}^{n} X_i \mid X_i \in \mathcal{S} \text{ and some } n < \omega\}$.

Proof. Since $\{\bigcup_{i=1}^{n} X_i \mid X_i \in \mathcal{S}$ and some $n < \omega\}$ contains $\mathcal{S}$, we will show that $\{\bigcup_{i=1}^{n} X_i \mid X_i \in \mathcal{S}$ and some $n < \omega\}$ satisfies conditions (i)–(iv) of modal algebras. Since $\mathcal{S}$ contains $\emptyset$, (i) holds. Since $\bigcup_{i=1}^{n} X_i \cap \bigcup_{j=1}^{m} Y_j = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m} X_i \cap Y_j$, (ii) holds. Since $-\bigcup_{i=1}^{n} X_i = \bigcap_{i=1}^{n} -X_i$ and $-X$ is a union of finitely many members of $\mathcal{S}$, (ii) implies (iii). Since $\Diamond\bigcup_{i=1}^{n} X_i = \bigcup_{i=1}^{n} \Diamond X_i$, (iv) holds. ∎

For modal algebras $\mathcal{P}$ on (W, R) and $\mathcal{Q}$ on (V, S), define $\mathcal{P} \divideontimes \mathcal{Q}$ to be the modal algebra on $(W \times V, R \times S)$ generated by $\{X \times Y \mid X \in \mathcal{P}$ and $Y \in \mathcal{Q}\}$.

Corollary 6.2.4

$$\mathcal{P} \divideontimes \mathcal{Q} = \{\bigcup_{i\in I}(X_i \times Y_i) \mid X_i \in \mathcal{P} \textit{ and } Y_i \in \mathcal{Q} \textit{ for a finite } I\ \}.$$

Now, we come to the definition of normal products of general frames. In the following, $\mathcal{F}_\lambda$ and $\mathcal{G}_\mu$ (with or without indices) denote general frames of the form $(W_\lambda, R_\lambda, \mathcal{P}_\lambda)$ and $(V_\mu, S_\mu, \mathcal{Q}_\mu)$, respectively.

For given general frames $\mathcal{F} = (W, R, \mathcal{P})$ and $\mathcal{G} = (V, S, \mathcal{Q})$, the *normal product* $\mathcal{F} \divideontimes \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ is a general frame $(W \times V, R \times S, \mathcal{P} \divideontimes \mathcal{Q})$. Note that the normal product thus obtained is also a general frame for mono-modal logics.

In the following, we will characterize the operators in $\mathcal{F} \divideontimes \mathcal{G}$. Suppose that both $\bigcup_{i\in I}(X_i \times Y_i)$ and $\bigcup_{j\in J}(T_j \times U_j)$ are elements of $\mathcal{P} \divideontimes \mathcal{Q}$.

$$\emptyset = \emptyset \times \emptyset (= \emptyset \times V = W \times \emptyset). \tag{6.1}$$

$$\bigcup_{i\in I}(X_i \times Y_i) \cap \bigcup_{j\in J}(T_j \times U_j) = \bigcup_{(i,j)\in I\times J} (X_i \cap T_j) \times (Y_i \cap U_j). \tag{6.2}$$

$$\bigcup_{i\in I}(X_i \times Y_i) \cup \bigcup_{j\in J}(T_j \times U_j) = \bigcup_{k\in I\sqcup J} A_k \times B_k, \tag{6.3}$$

where $I \sqcup J = \{(i, 0) \mid i \in I\} \cup \{(j, 1) \mid j \in J\}, A_{(i,0)} = X_i, A_{(j,1)} = T_j, B_{(i,0)} = Y_j, B_{(j,1)} = U_j$.

$$-\bigcup_{i\in I}(X_i \times Y_i) = \bigcup_{K\in\mathcal{P}(I)} (-\bigcup_{i\in K} X_i) \times (-\bigcup_{i\in -K} Y_i). \tag{6.4}$$

$$\Diamond \bigcup_{i\in I}(X_i \times Y_i) = \bigcup_{i\in I}(\Diamond X_i \times \Diamond Y_i). \tag{6.5}$$

$$\Box \bigcup_{i\in I}(X_i \times Y_i) = \bigcup_{\mathcal{C}\subseteq\mathcal{P}(I)} (\Box \bigcap_{K\in\mathcal{C}} \bigcup_{i\in K} X_i) \times (\Box \bigcup_{K\in\mathcal{C}} \bigcap_{i\in K} Y_i). \tag{6.6}$$

Moreover, the following holds in $\mathcal{P} \divideontimes \mathcal{Q}$.

$$\bigcup_{i\in I}(X_i \times Y_i) = \bigcup_{j\in J}(T_j \times U_j) \tag{6.7}$$

$$\text{iff} \quad \forall i \in I, \exists \mathcal{C} \subseteq \mathcal{P}(J) \text{ such that } X_i \subseteq \bigcap_{K\in\mathcal{C}} \bigcup_{j\in K} T_j, Y_i \subseteq \bigcup_{K\in\mathcal{C}} \bigcap_{j\in K} U_j,$$

$$\text{and} \quad \forall j \in J, \exists \mathcal{C} \subseteq \mathcal{P}(I) \text{ such that } T_j \subseteq \bigcap_{K\in\mathcal{C}} \bigcup_{i\in K} X_i, U_j \subseteq \bigcup_{K\in\mathcal{C}} \bigcap_{i\in K} Y_i.$$

Note that for Kripke frames $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \divideontimes \mathcal{G}$ is a general frame $(W \times V, R \times S, \mathcal{P}(W) \divideontimes \mathcal{P}(V))$ while $\mathcal{F} \times \mathcal{G}$ is a Kripke frame $(W \times V, R \times S)$.

Theorem 6.2.5 *Let $\mathcal{F}$ and $\mathcal{G}$ be Kripke frames. Then, $\mathcal{F} \divideontimes \mathcal{G}$ is a Kripke frame if and only if either $\mathcal{F}$ or $\mathcal{G}$ is finite. In other words, $\mathcal{P}(W) \divideontimes \mathcal{P}(V) = \mathcal{P}(W \times V)$ if and only if either W or V is finite.*

Proof. Suppose that W is finite. For $Z \in \mathcal{P}(W \times V)$, we can show $Z = \bigcup_{w \in W} (\{w\} \times Z_w)$, where $Z_w = \{v \in V \mid (w, v) \in Z\}$. Conversely, suppose that both W and V are infinite. Then we can take distinct sequences $\{w_i\}_{i<\omega} \subseteq W$ and $\{v_i\}_{i<\omega} \subseteq V$. Hence we can show that $\{(w_i, v_i) \mid i < \omega\} \notin \mathcal{P}(W) \divideontimes \mathcal{P}(V)$. ∎

It is well-known that each projection from a product of topological spaces to any of its component spaces is continuous. As in the realm of modal logic reductions are natural counterparts of continuous maps in topology, we can ask whether each projection from a normal product of general frames to any of its component frames is a reduction. The following theorem gives a necessary and sufficient condition. Here, $S(v)$ and $R(w)$ denote sets $\{v' \mid vSv'\}$ and $\{w' \mid wRw'\}$, respectively.

Theorem 6.2.6 *A projection $\pi : W \times V \to W$ is a reduction of $\mathcal{F} \divideontimes \mathcal{G}$ to $\mathcal{F}$ iff either $S(v) \neq \emptyset$ for each $v \in V$ or $R(w) = \emptyset$ for each $w \in W$.*

Proof. Since it is easily seen that a projection π always satisfies the conditions (i),(iii) of reduction, we will consider condition (ii).
A projection π satisfies the condition (ii),
iff for all $w, w' \in W$ and $v, v' \in V$ $wRw' \Rightarrow \exists u \in V : (w, v)R \times S(w', u)$,
iff for all $w, w' \in W$ and $v, v' \in V$ either $w \not R w'$ or $\exists u \in V : vSu$,
iff either $S(v) \neq \emptyset$ for each $v \in V$ or $R(w) = \emptyset$ for each $w \in W$. ∎

The next theorem gathers together several essential properties of normal products.

Theorem 6.2.7

1. *If f_i is a reduction of $\mathcal{F}_i$ to $\mathcal{G}_i$ for $i = 1, 2$, then the* product map *$f_1 \divideontimes f_2$ is a reduction of $\mathcal{F}_1 \divideontimes \mathcal{F}_2$ to $\mathcal{G}_1 \divideontimes \mathcal{G}_2$, where $f_1 \divideontimes f_2$ is defined by $(f_1 \divideontimes f_2)(w_1, w_2) = (f_1(w_1), f_2(w_2))$ for each $w_1 \in W_1$ and $w_2 \in W_2$.*

2. *If $\mathcal{G}_i$ is a generated subframe of $\mathcal{F}_i$ for $i = 1, 2$, then $\mathcal{G}_1 \divideontimes \mathcal{G}_2$ is also a generated subframe of $\mathcal{F}_1 \divideontimes \mathcal{F}_2$.*

3. *The identity map $id : \sum_{\lambda \in \Lambda\, \mu \in M} (W_\lambda \times V_\mu) \to (\sum_{\lambda \in \Lambda} W_\lambda) \times (\sum_{\mu \in M} V_\mu)$ is a reduction of $\sum_{\lambda \in \Lambda\, \mu \in M} (\mathcal{F}_\lambda \divideontimes \mathcal{G}_\mu)$ to $(\sum_{\lambda \in \Lambda} \mathcal{F}_\lambda) \divideontimes (\sum_{\mu \in M} \mathcal{G}_\mu)$, where $\sum$ denotes disjoint unions.*

Proof. Here, we will only show the part of algebras of sets. The rest is routine.
For 1, $(f_1 \divideontimes f_2)^{-1}(\bigcup_{i \in I}(X_i \times Y_i)) = \bigcup_{i \in I}(f_1^{-1}(X_i) \times f_2^{-1}(Y_i)) \in \mathcal{P}_1 \divideontimes \mathcal{P}_2$.
For 2, $(\bigcup_{i \in I}(X_i \times Y_i)) \cap (V_1 \times V_2) = \bigcup_{i \in I}((X_i \cap V_1) \times (Y_i \cap V_2)) \in \mathcal{Q}_1 \divideontimes \mathcal{Q}_2$.
For 3, $\bigcup_{i \in I}(\sum_{\lambda \in \Lambda} X^i_\lambda) \times (\sum_{\mu \in M} Y^i_\mu) = \sum_{\lambda \in \Lambda\, \mu \in M} \bigcup_{i \in I}(X^i_\lambda \times Y^i_\mu) \in \sum_{\lambda \in \Lambda\, \mu \in M}(\mathcal{P}_\lambda \divideontimes \mathcal{Q}_\mu)$. ∎

6.3 Normal products of modal algebras and duality theory

In the previous section, we have defined normal products of general frames employing products $\mathcal{P} \divideontimes \mathcal{Q}$ of modal algebras $\mathcal{P}$ and $\mathcal{Q}$ on Kripke frames. In this section, we will define *normal products of arbitrary modal algebras*. A structure $\boldsymbol{A} = (A, \wedge, \vee, \neg, 0, 1, \Box)$ is a *modal algebra* if $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra and $\Box$ is unary operator on A satisfying $\Box 1 = 1$ and $\Box(x \wedge y) = \Box x \wedge \Box y$ for $x, y \in A$. Any modal algebra $\mathcal{P}$ on a Kripke frame (W, R) can be regarded as a modal algebra in the present sense, if we identify $\mathcal{P}$ with $\boldsymbol{\mathcal{P}} = (\mathcal{P}, \cap, \cup, -, \emptyset, W, \Box)$, where $\Box$ is the operator defined in § 6.2. Below, we will take this identification for granted.

For a given modal algebra $\boldsymbol{A}$, let $\boldsymbol{A}_+$ stand, as usual, for the *dual* of $\boldsymbol{A}$. Also, for a given general frame $\mathcal{F}$, let $\mathcal{F}^+$ stand for the *dual* of $\mathcal{F}$. For modal algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we define their *normal product* $\boldsymbol{A} \divideontimes \boldsymbol{B}$ as $(\boldsymbol{A}_+ \divideontimes \boldsymbol{B}_+)^+$. We can show that for modal algebras $\mathcal{P}$ and $\mathcal{Q}$ on some Kripke frames, normal product $\mathcal{P} \divideontimes \mathcal{Q}$ in the present sense is isomorphic to $\mathcal{P} \divideontimes \mathcal{Q}$ in the sense of the previous section. Though our definition of the normal product $\boldsymbol{A} \divideontimes \boldsymbol{B}$ is stated by means of the normal product of general frames $\boldsymbol{A}_+$ and $\boldsymbol{B}_+$, we can also give a direct definition by representing equations (6.1)–(6.6) from the previous section in algebraic terms, and introducing an equivalence relation $\sim$ following (6.7), over the set $\mathcal{P}_f(A \times B)$ of all finite sets of $A \times B$. More preciously, for modal algebras $\boldsymbol{A} = (A, \wedge, \vee, \neg, 0, 1, \Box)$ and $\boldsymbol{B} = (B, \wedge, \vee, \neg, 0, 1, \Box)$, $\boldsymbol{A} \divideontimes \boldsymbol{B}$ is isomorphic to a modal algebras

$$\Big(\mathcal{P}_f(A \times B)/_\sim, \wedge, \vee, \neg, 0, 1, \Box\Big),$$

where $\mathcal{P}_f(A \times B)$ is the set $\{\{(a_i, b_i) \mid i \in I\} \mid a_i \in A \text{ and } b_i \in B \text{ for a finite } I\}$ of all finite sets of $A \times B$, and $\mathcal{P}_f(A \times B)/_\sim$ is the quotient set of $\mathcal{P}_f(A \times B)$ under the equivalence relation $\sim$ defined by, for $\{(a_i, b_i) \mid i \in I\}$, $\{(c_j, d_j) \mid j \in J\} \in \mathcal{P}_f(A \times B)$,

$$\begin{aligned} &\{(a_i, b_i) \mid i \in I\} \sim \{(c_j, d_j) \mid j \in J\} \\ &\quad\text{iff}\quad \forall i \in I, \exists \mathcal{C} \subseteq \mathcal{P}(J) \text{ such that } a_i \leq \bigwedge_{K \in \mathcal{C}} \bigvee_{j \in K} c_j, b_i \leq \bigvee_{K \in \mathcal{C}} \bigwedge_{j \in K} d_j, \\ &\quad\text{and}\quad \forall j \in J, \exists \mathcal{C} \subseteq \mathcal{P}(I) \text{ such that } c_j \leq \bigwedge_{K \in \mathcal{C}} \bigvee_{i \in K} a_i, d_j \leq \bigvee_{K \in \mathcal{C}} \bigwedge_{i \in K} b_i. \end{aligned} \tag{6.8}$$

Let $[(a_i, b_i) \mid i \in I]$ be the equivalent class to which $\{(a_i, b_i) \mid i \in I\}$ belongs. The operators are defined as follows; for $[(a_i, b_i) \mid i \in I]$, $[(c_j, d_j) \mid j \in J] \in \mathcal{P}_f(A \times B)/_\sim$,

$$[(a_i, b_i) \mid i \in I] \wedge [(c_j, d_j) \mid j \in J] = [(a_i \wedge c_j, b_i \wedge d_j) \mid (i, j) \in I \times J], \tag{6.9}$$

$$[(a_i, b_i) \mid i \in I] \vee [(c_j, d_j) \mid j \in J] = [(x_k, y_k) \mid k \in I \sqcup J], \tag{6.10}$$

where $I \sqcup J = \{(i, 0) \mid i \in I\} \cup \{(j, 1) \mid j \in J\}, x_{(i,0)} = a_i, x_{(j,1)} = c_j, y_{(i,0)} = b_j, y_{(j,1)} = d_j$,

$$\neg[(a_i, b_i) \mid i \in I] = [(\neg \bigvee_{i \in K} a_i, \neg \bigvee_{i \in -K} b_i) \mid K \in \mathcal{P}(I)], \tag{6.11}$$

$$0 = [(0, 0)], \tag{6.12}$$

$$1 = [(1, 1)], \tag{6.13}$$

$$\Diamond[(a_i, b_i) \mid i \in I] = [(\Diamond a_i, \Diamond b_i) \mid i \in I], \tag{6.14}$$

$$\Box[(a_i, b_i) \mid i \in I] = [(\Box \bigwedge_{K\in\mathcal{C}} \bigvee_{i\in K} a_i, \Box \bigvee_{K\in\mathcal{C}} \bigwedge_{i\in K} b_i) \mid \mathcal{C} \subseteq \mathcal{P}(I)]. \tag{6.15}$$

Corollary 6.3.1 *For general frames $\mathcal{F}$ and $\mathcal{G}$,*

$$(\mathcal{F} \divideontimes \mathcal{G})^+ \cong \mathcal{F}^+ \divideontimes \mathcal{G}^+. \tag{6.16}$$

Theorem 6.3.2 *If both general frames $\mathcal{F}, \mathcal{G}$ are descriptive, then their normal product $\mathcal{F} \divideontimes \mathcal{G}$ is also descriptive.*

Proof. It is easily shown that it is differentiated and tight. We can show compactness applying the method of Tikhonov's theorem. For the case of finitely many products of modal frames we can prove without Zorn's Lemma. First, note that we can express an element of $\mathcal{P}$ as the form of $\bigcap_{i\in I}(X_i \times V \cup W \times Y_i)$. Therefore it is enough to show that for any $\mathcal{A} = \{X_\lambda \times V \cup W \times Y_\lambda \mid X_\lambda \in \mathcal{P}, Y_\lambda \in \mathcal{Q}, \lambda \in \Lambda\}$, ($\mathcal{A}$ has finite intersection property $\Rightarrow$ $\bigcap \mathcal{A} \neq \emptyset$). Define $\mathcal{A}_w = \{X_\lambda \times V \cup W \times Y_\lambda \in \mathcal{A} \mid X_\lambda \times V \cup W \times Y_\lambda \not\supseteq \{w\} \times V\}$ and $\mathcal{B}_w = \{Y_\lambda \mid X_\lambda \times V \cup W \times Y_\lambda \in \mathcal{A}_w\}$ for each $w \in W$. Suppose that $\mathcal{B}_w$ doesn't have finite intersection property for any $w \in W$. Hence for each $w \in W$ there exist $X_1^w, \ldots, X_{n_w}^w, Y_1^w, \ldots, Y_{n_w}^w$ such that $Y_1^w \cap \cdots \cap Y_{n_w}^w = \emptyset$ and $X_{i_w}^w \times V \cup W \times Y_{i_w}^w \in \mathcal{A}_w$. Therefore, $\bigcap_{i_w=1}^{n_w} X_{i_w}^w \times V \cup W \times Y_{i_w}^w \supseteq (\bigcup_{i_w=1}^{n_w} X_{i_w}^w) \times V$. Since by finite intersection property of $\mathcal{A}$ $\{(\bigcup_{i_w=1}^{n_w} X_{i_w}^w) \times V \mid w \in W\}$ also has finite intersection property, $\bigcap_{w\in W} (\bigcup_{i_w=1}^{n_w} X_{i_w}^w) \times V \neq \emptyset$ by compactness of $\mathcal{F}$. But since $(\bigcup_{i_w=1}^{n_w} X_{i_w}^w) \times V \supseteq -(\{w\} \times V)$, we have $\bigcap_{w\in W} (\bigcup_{i_w=1}^{n_w} X_{i_w}^w) \times V = \emptyset$. It is contradiction. Therefore, $\mathcal{B}_w$ has finite intersection property for some w. Since $\bigcap \mathcal{A} \supseteq \{w\} \times (\bigcap \mathcal{A}_w) \neq \emptyset$, $\mathcal{F} \divideontimes \mathcal{G}$ is also compact by compactness of $\mathcal{G}$. ∎

Corollary 6.3.3 *For modal algebras $\boldsymbol{A}$ and $\boldsymbol{B}$,*

$$(\boldsymbol{A} \divideontimes \boldsymbol{B})_+ \cong \boldsymbol{A}_+ \divideontimes \boldsymbol{B}_+. \tag{6.17}$$

We can show the duality of Theorem 6.2.7 as follows.

Theorem 6.3.4

1. *If $\boldsymbol{A}_2$ is a homomorphic image of $\boldsymbol{A}_1$ and $\boldsymbol{B}_2$ is a homomorphic image of $\boldsymbol{B}_1$, then $\boldsymbol{A}_2 \divideontimes \boldsymbol{B}_2$ is also a homomorphic image of $\boldsymbol{A}_1 \divideontimes \boldsymbol{B}_1$.*

2. *If $\boldsymbol{A}_1$ is a subalgebra of $\boldsymbol{A}_2$ and $\boldsymbol{B}_1$ is a subalgebra of $\boldsymbol{B}_2$, then $\boldsymbol{A}_1 \divideontimes \boldsymbol{B}_1$ is isomorphic to a subalgebra of $\boldsymbol{A}_2 \divideontimes \boldsymbol{B}_2$.*

3. *$\prod_{\lambda\in\Lambda} \boldsymbol{A}_\lambda \divideontimes \prod_{\mu\in M} \boldsymbol{B}_\mu$ is isomorphic to a subalgebra of $\prod_{(\lambda,\mu)\in\Lambda\times M} \boldsymbol{A}_\lambda \divideontimes \boldsymbol{B}_\mu$, where $\prod$ denotes direct products.*

Let $\mathcal{K}$ be a class of modal algebras. As usual, $\boldsymbol{V}(\mathcal{K})$, $\boldsymbol{H}(\mathcal{K})$, $\boldsymbol{I}(\mathcal{K})$, $\boldsymbol{S}(\mathcal{K})$ and $\boldsymbol{P}(\mathcal{K})$ denote, respectively: the variety generated by $\mathcal{K}$, the class of homomorphic images of algebras in $\mathcal{K}$, the class of isomorphic copies of algebras in $\mathcal{K}$, the class of subalgebras of algebras in $\mathcal{K}$ and the class of direct products of nonempty families of algebras in $\mathcal{K}$, respectively. For classes $\mathcal{K}_1$ and $\mathcal{K}_2$ of modal algebras, $\mathcal{K}_1 \divideontimes \mathcal{K}_2$ is the class $\{\boldsymbol{A}_1 \divideontimes \boldsymbol{A}_2 \mid \boldsymbol{A}_1 \in \mathcal{K}_1, \boldsymbol{A}_2 \in \mathcal{K}_2\}$. By Theorem 6.3.4, we have the following.

Theorem 6.3.5 *For classes $\mathcal{K}_1, \mathcal{K}_2$ of modal algebras,*

1. $\boldsymbol{H}(\mathcal{K}_1) \divideontimes \boldsymbol{H}(\mathcal{K}_2) \subseteq \boldsymbol{H}(\mathcal{K}_1 \divideontimes \mathcal{K}_2)$,

2. $\boldsymbol{S}(\mathcal{K}_1) \divideontimes \boldsymbol{S}(\mathcal{K}_2) \subseteq \boldsymbol{IS}(\mathcal{K}_1 \divideontimes \mathcal{K}_2)$,

3. $\boldsymbol{P}(\mathcal{K}_1) \divideontimes \boldsymbol{P}(\mathcal{K}_2) \subseteq \boldsymbol{ISP}(\mathcal{K}_1 \divideontimes \mathcal{K}_2)$.

Hence,

4. $\boldsymbol{V}(\boldsymbol{V}(\mathcal{K}_1) \divideontimes \boldsymbol{V}(\mathcal{K}_2)) = \boldsymbol{V}(\mathcal{K}_1 \divideontimes \mathcal{K}_2)$.

6.4 Normal products of modal logics

For a given modal logic L, $\boldsymbol{F}(L)$ is the class of all general frames which validate L, and $\boldsymbol{V}(L)$ is the class of all modal algebras which validate L. For a class $\mathcal{K}$ of modal algebras (a class $\mathcal{C}$ of general frames), $\boldsymbol{L}(\mathcal{K})$ ($\boldsymbol{L}(\mathcal{C})$) denotes the set of formulas valid in every modal algebra in $\mathcal{K}$ (in every general frame in $\mathcal{C}$). For classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of general frames, $\mathcal{C}_1 \divideontimes \mathcal{C}_2$ is the class $\{\mathcal{F}_1 \divideontimes \mathcal{F}_2 \mid \mathcal{F}_1 \in \mathcal{C}_1 \text{ and } \mathcal{F}_2 \in \mathcal{C}_2\}$.

Now, for modal logics L_1, L_2, define the *normal product* $L_1 \divideontimes L_2$ to be $\boldsymbol{L}(\boldsymbol{F}(L_1) \divideontimes \boldsymbol{F}(L_2))$. By (6.16) and (6.17), $L_1 \divideontimes L_2 = \boldsymbol{L}(\boldsymbol{V}(L_1) \divideontimes \boldsymbol{V}(L_2))$ holds. Thus, we may also take the latter as the definition of $L_1 \divideontimes L_2$. By Theorem 6.3.5(4), we have the following.

Theorem 6.4.1 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be classes of modal algebras, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of general frames. Then*

1. $\boldsymbol{L}(\mathcal{K}_1) \divideontimes \boldsymbol{L}(\mathcal{K}_2) = \boldsymbol{L}(\mathcal{K}_1 \divideontimes \mathcal{K}_2)$,

2. $\boldsymbol{L}(\mathcal{C}_1) \divideontimes \boldsymbol{L}(\mathcal{C}_2) = \boldsymbol{L}(\mathcal{C}_1 \divideontimes \mathcal{C}_2)$.

This theorem says that the logic $\boldsymbol{L}(\mathcal{C}_1 \divideontimes \mathcal{C}_2)$ is uniquely determined by the logics $\boldsymbol{L}(\mathcal{C}_1)$ and $\boldsymbol{L}(\mathcal{C}_2)$. More precisely, if $\boldsymbol{L}(\mathcal{C}_1) = \boldsymbol{L}(\mathcal{C}_1')$ and $\boldsymbol{L}(\mathcal{C}_2) = \boldsymbol{L}(\mathcal{C}_2')$, then $\boldsymbol{L}(\mathcal{C}_1 \divideontimes \mathcal{C}_2) = \boldsymbol{L}(\mathcal{C}_1' \divideontimes \mathcal{C}_2')$. Since the normal product of general frames is associative, the normal product of modal logics is also associative, by the above theorem.

Following [9], we say that a logic L has the *product f.m.p.* if L is a logic of some class of finite products. It is obvious that if L has the product f.m.p., then L has the f.m.p.. By Theorem 6.2.5 and Theorem 6.4.1, we have the following.

Theorem 6.4.2

1. *If both L_1 and L_2 have the f.m.p., then $L_1 \divideontimes L_2$ has the product f.m.p.*

2. *If L_1 has the f.m.p. and L_2 is Kripke complete, then $L_1 \divideontimes L_2$ is also Kripke complete.*

We say that L is *product-persistent* if, for every Kripke frames $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \divideontimes \mathcal{G} \models L$ implies $\mathcal{F} \times \mathcal{G} \models L$.

As usual we define a Sahlqvist formula in the following way:

A formula φ is called a *Sahlqvist formula* if φ is a formula which is equivalent to a conjunction of formulas of the form $\Box^m(\psi \to \chi)$, where $m \geq 0$, χ is constructed by $\wedge, \vee, \Box, \Diamond, \top, \bot$ from propositional variables and ψ is obtained from propositional variables and their negations applying $\wedge, \vee, \Box, \Diamond, \top, \bot$ in such a way that no positive occurrence of a variable is in a subformula of the form either $\psi_1 \vee \psi_2$ or $\Diamond\psi_1$ with the scope of any $\Box$.

Theorem 6.4.3 *Suppose that L is a product-persistent logic and Γ is any set of Sahlqvist formulas. Then the logic $L \oplus \Gamma$ is also product-persistent.*

Proof. The point of proof is that a $(R \times S)$-*expression* $(R \times S)^{k_1}(w_1, v_1) \cup \cdots \cup (R \times S)^{k_n}(w_n, v_n) = \bigcup_{i=1}^{n} R^{k_i}(w_i) \times S^{k_i}(v_i)$ belongs to $\mathcal{P}(W) \divideontimes \mathcal{P}(V)$. ∎

Recall that, given a formula $\varphi(p_1, \ldots, p_n)$ (whose variables are listed among $p_1, \ldots, p_n$), a general frame $\mathcal{F} = (W, R, \mathcal{P})$ and sets $X_1, \ldots, X_n$ in $\mathcal{P}$, we denote by $\varphi(X_1, \ldots, X_n)$ the set of points in $\mathcal{F}$ at which φ is true under the valuation v defined by $v(p_i) = X_i$, for $i = 1, \ldots, n$, i.e., $\varphi(X_1, \ldots, X_n) = v(\varphi)$.

Theorem 6.4.4 *Suppose that L_1 is a modal logic and L_2 is an extension of the logic* **D**. *Then $L_1 \supseteq L_1 \divideontimes L_2$.*
Moreover, if both L_1 and L_2 are extensions of the logic **D**,

$$L_1 \cap L_2 \supseteq L_1 \divideontimes L_2.$$

Proof. By induction on construction of formulas, we can show that for any formula φ, $\varphi(X_1 \times V, \ldots, X_n \times V) = \varphi(X_1, \ldots, X_n) \times V$. Therefore this implies that $L_1 \divideontimes L_2$ is included in L_1. ∎

A formula φ is *preserved* under normal products if, whenever φ belongs to $L_1 \cap L_2$, then φ belongs also to $L_1 \divideontimes L_2$. The next theorem shows that restricted Sahlqvist formulas are preserved.

Theorem 6.4.5 *Let φ be a formula which is equivalent to a conjunction of formulas of the form $\Box^m(\psi \to \chi)$, where $m \geq 0$, χ is constructed by $\wedge, \Box, \Diamond, \top, \bot$ from propositional variables and ψ is obtained from propositional variables applying $\wedge, \vee, \Box, \Diamond, \top, \bot$ in such a way that no subformula of the form either $\psi_1 \vee \psi_2$ or $\Diamond\psi_1$ occurs in the scope of any $\Box$ and that ψ_1 and ψ_2 have no common propositional variable in a subformula of the form $\psi_1 \wedge \psi_2$. Then φ is preserved under normal products.*

Proof. By induction on construction of formulas, we can show
$\chi(\bigcup_{i_1 \in I_1} (X^1_{i_1} \times Y^1_{i_1}), \ldots, \bigcup_{i_n \in I_n} (X^n_{i_n} \times Y^n_{i_n})) \supseteq \bigcup_{i_1 \in I_1, \ldots, i_n \in I_n} \chi(X^1_{i_1}, \ldots, X^n_{i_n}) \times \chi(Y^1_{i_1}, \ldots, Y^n_{i_n})$ and
$\psi(\bigcup_{i_1 \in I_1} (X^1_{i_1} \times Y^1_{i_1}), \ldots, \bigcup_{i_n \in I_n} (X^n_{i_n} \times Y^n_{i_n})) \subseteq \bigcup_{i_1 \in I_1, \ldots, i_n \in I_n} \psi(X^1_{i_1}, \ldots, X^n_{i_n}) \times \psi(Y^1_{i_1}, \ldots, Y^n_{i_n})$. Hence, since we can show

$\varphi(\bigcup_{i_1\in I_1}(X^1_{i_1}\times Y^1_{i_1}),\ldots,\bigcup_{i_n\in I_n}(X^n_{i_n}\times Y^n_{i_n}))\supseteq\bigcap_{i_1\in I_1,\ldots,i_n\in I_n}\varphi(X^1_{i_1},\ldots,X^n_{i_n})\times\varphi(Y^1_{i_1},\ldots,Y^n_{i_n})$, we have our theorem. ∎

6.5 Shifted products

As we have noticed in the Introduction, the normal product of two m-modal logics is a m-modal logic. On the other hand, products introduced in [9] have a property that if L_1 and L_2 are m-modal and n-modal logics, the product $L_1 \times L_2$ is a $(m+n)$-modal logic. To compare this type of products with our normal products, we will introduce yet another kind of products, called *shifted products*. Though we will discuss only mono-modal cases for brevity, but we can extend the results to (n,m)-modal cases easily.

Suppose that $\boldsymbol{A} = (A,\wedge,\vee,\neg,0,1,\Box_1)$ and $\boldsymbol{B} = (B,\wedge,\vee,\neg,0,1,\Box_2)$ are mono-modal algebras. Define bi-modal algebras $\boldsymbol{A}^\circ$ and ${}^\circ\boldsymbol{B}$ as $\boldsymbol{A}^\circ = (A,\wedge,\vee,\neg,0,1,\Box_1,\Box_2)$ with $\Box_2 a = a$ for all $a \in A$ and ${}^\circ\boldsymbol{B} = (B,\wedge,\vee,\neg,0,1,\Box_1,\Box_2)$ with $\Box_1 b = b$ for all $b \in B$. For general frames $\mathcal{F}$ and $\mathcal{G}$, define general frames $\mathcal{F}^\circ$ and ${}^\circ\mathcal{G}$, with two binary relations, as $\mathcal{F}^\circ = (W,R,\Delta,\mathcal{P})$ and ${}^\circ\mathcal{G} = (V,\Delta,S,\mathcal{Q})$, where Δ denotes the diagonal relation. For a class $\mathcal{K}$ of mono-modal algebras and a class $\mathcal{C}$ of general frames, $\mathcal{K}^\circ$, ${}^\circ\mathcal{K}$, $\mathcal{C}^\circ$ and ${}^\circ\mathcal{C}$ are defined in a natural way. Now, define the *shifted product* $\boldsymbol{A}\otimes\boldsymbol{B}$ of $\boldsymbol{A}$ and $\boldsymbol{B}$ (the shifted product $\mathcal{F}\otimes\mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$) by $\boldsymbol{A}\otimes\boldsymbol{B} = \boldsymbol{A}^\circ \divideontimes {}^\circ\boldsymbol{B}$ ($\mathcal{F}\otimes\mathcal{G} = \mathcal{F}^\circ \divideontimes {}^\circ\mathcal{G}$). We can define *shifted product* of classes of modal algebras and those of classes of general frames in an obvious way. Then, we define the shifted product $L_1 \otimes L_2$ of modal logics L_1 and L_2 by $L_1 \otimes L_2 = \boldsymbol{L}(\boldsymbol{F}(L_1)\otimes\boldsymbol{F}(L_2))$.

Theorem 6.5.1 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be classes of modal algebras, and $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of general frames. Then*

1. $\boldsymbol{L}(\mathcal{K}_1)\otimes\boldsymbol{L}(\mathcal{K}_2) = \boldsymbol{L}(\mathcal{K}_1\otimes\mathcal{K}_2)$,
2. $\boldsymbol{L}(\mathcal{C}_1)\otimes\boldsymbol{L}(\mathcal{C}_2) = \boldsymbol{L}(\mathcal{C}_1\otimes\mathcal{C}_2)$.

Proof. Since $\boldsymbol{V}(\mathcal{K}^\circ) = \boldsymbol{V}(\mathcal{K})^\circ$ and $\boldsymbol{V}({}^\circ\mathcal{K}) = {}^\circ\boldsymbol{V}(\mathcal{K})$ hold, Theorem 6.4.1 implies this theorem. ∎

Suppose that L_1 and L_2 are modal logics. Define bi-modal logics ${L_1}^\circ$ and ${}^\circ L_2$ as ${L_1}^\circ = L_1 * \mathbf{Triv}$ and ${}^\circ L_2 = \mathbf{Triv} * L_2$, where $\mathbf{Triv} = \mathbf{K} \oplus \Box p \leftrightarrow p$ and $*$ denotes the *fusion* of logics.

Theorem 6.5.2 *For modal logics L_1 and L_2,*

$$L_1 \otimes L_2 = {L_1}^\circ \divideontimes {}^\circ L_2.$$

Proof. Since $\boldsymbol{V}(L^\circ) = \boldsymbol{V}(L)^\circ$ and $\boldsymbol{V}({}^\circ L) = {}^\circ\boldsymbol{V}(L)$ hold, by definition we have the result. ∎

Define a translation $\mathbf{T}$ from the set of formulas in the language of $L_1 \divideontimes L_2$ (with one modal operator $\Box$) to the set of formulas in the language of $L_1 \otimes L_2$ (with two modal operators $\Box_1$ and $\Box_2$) inductively as follows. (i) $\mathbf{T}(p) = p$ for all propositional variables, (ii) $\mathbf{T}(\varphi\wedge\psi) = \mathbf{T}(\varphi)\wedge\mathbf{T}(\psi)$, (iii) $\mathbf{T}(\varphi\vee\psi) = \mathbf{T}(\varphi)\vee\mathbf{T}(\psi)$, (iv) $\mathbf{T}(\neg\varphi) = \neg\mathbf{T}(\varphi)$, (v) $\mathbf{T}(\bot) = \bot$, (vi) $\mathbf{T}(\top) = \top$, (vii) $\mathbf{T}(\Box\varphi) = \Box_1\Box_2\mathbf{T}(\varphi)$. Then, we have the following.

Theorem 6.5.3 *A formula φ is a theorem of $L_1 \divideontimes L_2$ iff $\mathbf{T}(\varphi)$ is a theorem of $L_1 \otimes L_2$.*

Proof. Since $\Diamond_1\Diamond_2\bigcup_{i\in I}(X_i \times Y_i) = \Diamond_1\bigcup_{i\in I}(X_i \times \Diamond_2 Y_i) = \bigcup_{i\in I}(\Diamond_1 X_i \times \Diamond_2 Y_i) = \Diamond\bigcup_{i\in I}(X_i \times Y_i)$ holds, we can show this result by induction on construction of formulas. ∎

After [9], we define the bi-modal logic $[L_1, L_2]$ to be $(L_1 * L_2) \oplus \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p \oplus \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$. The following theorem lists some properties of $\otimes$ and enables us to compare them with the corresponding properties of $\times$.

Theorem 6.5.4 *Let L_1, L_2 be modal logics. Then*

1. *If L_1 and L_2 are consistent logics, then $L_1 \otimes L_2$ is a conservative extension of L_1 and L_2,*
2. *$L_1 \otimes L_2 \supseteq [L_1, L_2]$,*
3. *if both L_1 and L_2 are extensions of the logic $\mathbf{D}$, $L_1^\circ \cap {}^\circ L_2 \supseteq L_1 \otimes L_2$,*
4. *If L_1 and L_2 have the f.m.p., then $L_1 \otimes L_2$ has the product f.m.p.,*
5. *If L_1 has the f.m.p. and L_2 is Kripke complete, then $L_1 \otimes L_2$ is also Kripke complete.*

Proof. For 1, by induction on construction of formulas, we can show that for any formula φ in the language of L_1,
$\varphi(\sum_{i_1\in I_1}(X^1_{i_1} \times Y^1_{i_1}), \ldots, \sum_{i_n\in I_n}(X^n_{i_n} \times Y^n_{i_n})) = \sum_{i_1\in I_1,\ldots,i_n\in I_n}(\varphi(X^1_{i_1}, \ldots, X^n_{i_n}) \times \bigcap_{j=1}^{n} Y^j_{i_j})$, where for any $j = 1, \ldots, n$, $V = \sum_{i_j\in I_j} Y^j_{i_j}$ holds. Therefore this implies that $L_1 \otimes L_2$ is a conservative extension of L_1.
For 2, since both L_1° and ${}^\circ L_2$ contain $\{\Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p, \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p\}$, by Theorem 6.4.5 $L_1 \otimes L_2 \supseteq [L_1, L_2]$ holds.
3 immediately follows Theorems 6.4.4 and 6.5.2.
4 and 5 are shown similarly to Theorem 6.4.2. ∎

6.6 Comparison of our $\otimes$ and the usual product $\times$

Here we will compare properties of our $\otimes$ and the usual product $\times$ (cf. Table 1 in [9]). For a modal logic L, $\boldsymbol{F}^K(L)$ is the class of all Kripke frames which validate L. For modal logics L_1 and L_2, the usual product $L_1 \times L_2$ is defined by $\boldsymbol{L}(\boldsymbol{F}^K(L_1)^\circ \times {}^\circ\boldsymbol{F}^K(L_2))$.

By $\otimes$-*transfer* ($\times$-*transfer*) of a property P, we mean that P carries over from modal logics L_1 and L_2 to the product of $L_1 \otimes L_2$ ($L_1 \times L_2$).

Clearly, $\times$-transfer of Kripke completeness holds, whereas $\otimes$-transfer of Kripke completeness remains an open question. We have, however, shown a partial result in Theorem 6.5.4(4). On the other hand, while $\times$-transfer of the f.m.p. does not hold, its $\otimes$-transfer holds as shown in Theorem 6.5.4(3). It is easy to see that both $\otimes$-transfer and $\times$-transfer of tabularity hold.

We say that the modal logics L_1 and L_2 are called $\otimes$-*product-matching* ($\times$-*product-matching*) if $L_1 \otimes L_2 = [L_1, L_2]$ ($L_1 \times L_2 = [L_1, L_2]$). A *PTC-logic* is a modal logic axiomatized by a set of *PTC-formulas*, i.e., either *pseudo-transitive* formulas $\Diamond^m\Box p \to \Box^n p$ for $m, n \geq 0$ or closed formulas. It is shown in [9] that every pair of PTC-logic is $\times$-product-matching. For a given modal logic L, $\boldsymbol{F}_f(L)$ is the class of all finite general frames which validate L, and $\boldsymbol{F}_f^K(L)$ is the class of all finite Kripke frames which validate L. For modal logics L_1 and L_2, $L_1 \otimes_f L_2$ ($L_1 \times_f L_2$) denotes $\boldsymbol{L}(\boldsymbol{F}_f(L_1) \otimes \boldsymbol{F}_f(L_2))$ ($\boldsymbol{L}(\boldsymbol{F}_f^K(L_1) \times \boldsymbol{F}_f^K(L_2))$). It is easy to see that $L_1 \otimes_f L_2 \supseteq L_1 \otimes L_2 \supseteq [L_1, L_2]$ and $L_1 \times_f L_2 \supseteq L_1 \times L_2 \supseteq [L_1, L_2]$ hold. The next theorem follows from Theorem 6.2.5.

Theorem 6.6.1 *Let L_1 and L_2 be modal logics. Then*

$$L_1 \otimes_f L_2 = L_1 \times_f L_2.$$

As a corollary, we obtain:

Corollary 6.6.2 *Let L_1 and L_2 be PTC-logics which have the f.m.p. (e.g.* **K, D, S4, S5** *and so on). Then L_1 and L_2 are $\otimes$-product-matching if and only if $L_1 \times L_2$ has the product f.m.p. if and only if $L_1 \otimes L_2 = L_1 \times L_2$.*

Since both **K** $\times$ **K** and **S5** $\times$ **S5** have the product f.m.p. (cf. [10]) but **K4** $\times$ **K4** doesn't, **K** $\otimes$ **K** = **K** $\times$ **K**, **S5** $\otimes$ **S5** = **S5** $\times$ **S5** and **K4** $\otimes$ **K4** $\neq$ **K4** $\times$ **K4**.

Theorem 6.6.3 *Let L_1 and L_2 be Kripke complete. Then $L_1 \otimes L_2 \supseteq L_1 \times L_2$.*

Proof. Since L_1 and L_2 be Kripke complete, $L_1 \otimes L_2 = \boldsymbol{L}(\boldsymbol{F}^K(L_1) \otimes \boldsymbol{F}^K(L_2))$. Hence since $\boldsymbol{L}(\boldsymbol{F}^K(L_1) \otimes \boldsymbol{F}^K(L_2)) \supseteq \boldsymbol{L}(\boldsymbol{F}^K(L_1)^\circ \times {}^\circ\boldsymbol{F}^K(L_2))$ holds, $L_1 \otimes L_2 \supseteq L_1 \times L_2$ holds. ∎

Theorem 6.6.4 *Let φ be a Sahlqvist formula. Then if φ belongs to $L_1 \otimes L_2$, φ belongs also to $L_1 \times L_2$.*

Proof. Since φ belongs to $\boldsymbol{L}(\boldsymbol{F}(L_1)^\circ \divideontimes {}^\circ\boldsymbol{F}(L_2))$, φ belongs also to $\boldsymbol{L}(\boldsymbol{F}^K(L_1)^\circ \divideontimes {}^\circ\boldsymbol{F}^K(L_2))$. Therefore by Theorem 6.4.3, φ belongs also to $\boldsymbol{L}(\boldsymbol{F}^K(L_1)^\circ \times {}^\circ\boldsymbol{F}^K(L_2))$. ∎

6.7 Note

Usual products of modal logics induced by V. Shehtman [24] have been studied by Gabbay and Shehtman [9][10], Marx and Venema [15], Reynolds [21] and Wolter [31], in recent years. Numerous interesting problems on normal products remain unexplored yet.

Chapter 7

Infinitely many products and products of intuitionistic modal logics

7.1 Introduction

The definition of our products introduced in the previous chapter is based on the standard way of introducing "products" in measure theory and topology. In this chapter, we will show that the same idea is also applicable to intuitionistic modal logics and infinitely many products.

In general, our argument in this chapter goes in parallel with that in Chapter 6. So, we will give an outline of it.

7.2 Products of infinitely many modal logics

For simplicity, we will consider only modal general frames for mono-modal logics. We assume the logics in this section are extensions of **D**.

Now, consider the set $\{\pi_\lambda^{-1}(X_\lambda) \mid X_\lambda \in \mathcal{P}_\lambda \text{ and } \lambda \in \Lambda\}$, which is a set of subsets of $\prod_{\lambda\in\Lambda} W_\lambda$.

For modal algebras $\mathcal{P}_\lambda$ on (W_λ, R_λ), define $\mathop{\divideontimes}\limits_{\lambda\in\Lambda} \mathcal{P}_\lambda$ to be the modal algebra on $(\prod\limits_{\lambda\in\Lambda} W_\lambda, \prod\limits_{\lambda\in\Lambda} R_\lambda)$ generated by $\{\pi_\lambda^{-1}(X_\lambda) \mid X_\lambda \in \mathcal{P}_\lambda \text{ and } \lambda \in \Lambda\}$.

We say $\pi^{-1}_{\{\lambda_1,\ldots,\lambda_n\}}(X_{\lambda_1} \times \cdots \times X_{\lambda_n})$ to be *cylindric set* of $\{\mathcal{P}_\lambda \mid \lambda \in \Lambda\}$ for a finite subset $\{\lambda_1, \ldots, \lambda_n\}$ of Λ and $X_{\lambda_i} \in \mathcal{P}_{\lambda_i}$.

Note that $\pi^{-1}_{\{\lambda_1,\ldots,\lambda_n\}}(X_{\lambda_1} \times \cdots \times X_{\lambda_n}) = \pi_{\lambda_1}^{-1}(X_{\lambda_1}) \cap \cdots \cap \pi_{\lambda_1}^{-1}(X_{\lambda_n})$ and that C is a cylindric set of $\{\mathcal{P}_\lambda \mid \lambda \in \Lambda\}$ iff $C = \prod\limits_{\lambda\in\Lambda} X_\lambda$, where $X_\lambda \in \mathcal{P}_\lambda$ for every $\lambda \in \Lambda$ and $X_\lambda = W_\lambda$ finitely except $\lambda \in \Lambda$.

The set of all cylindric sets is not always a modal algebra on $(\prod\limits_{\lambda\in\Lambda} W_\lambda, \prod\limits_{\lambda\in\Lambda} R_\lambda)$ but it is always a modal semi-algebra on it.

Corollary 7.2.1

$$\mathop{\divideontimes}\limits_{\lambda\in\Lambda} \mathcal{P}_\lambda = \{\bigcup_{i\in I} C_i \mid C_i \textit{ is a cylindric set of } \{\mathcal{P}_\lambda \mid \lambda \in \Lambda\} \textit{ for a finite } I\ \}.$$

Normal products of infinitely many modal general frames can be defined in the same way as normal products introduced in Chapter 6.

For given modal general frames $\mathcal{F}_\lambda = (W_\lambda, R_\lambda)$ $(\lambda \in \Lambda)$, the *normal product* $\divideontimes_{\lambda\in\Lambda} \mathcal{F}_\lambda$ of $\mathcal{F}_\lambda$'s is a modal general frame $(\prod_{\lambda\in\Lambda} W_\lambda, \prod_{\lambda\in\Lambda} R_\lambda, \divideontimes_{\lambda\in\Lambda} \mathcal{P}_\lambda)$.

Since any frame in this section is serial by our assumption. the following theorem holds similarly to Theorem 6.2.6

Theorem 7.2.2 *A projection* $\pi_\lambda : \prod_{\lambda\in\Lambda} W_\lambda \to W_\lambda$ *is always a reduction of* $\divideontimes_{\lambda\in\Lambda} \mathcal{F}_\lambda$ *to* $\mathcal{F}_\lambda$.

The next theorem gathers together several essential properties of normal products.

Theorem 7.2.3

1. *If* f_λ *is a reduction of* $\mathcal{F}_\lambda$ *to* $\mathcal{G}_\lambda$ *for* $\lambda \in \Lambda$, *then the* product map $\divideontimes_{\lambda\in\Lambda} f_\lambda$ *is a reduction of* $\divideontimes_{\lambda\in\Lambda} \mathcal{F}_\lambda$ *to* $\divideontimes_{\lambda\in\Lambda} \mathcal{G}_\lambda$, *where* $\divideontimes_{\lambda\in\Lambda} f_\lambda$ *is defined by* $((\divideontimes_{\lambda\in\Lambda} f_\lambda)(w))(\lambda) = f_\lambda(w_\lambda)$ *for* $w \in \prod_{\lambda\in\Lambda} W_\lambda$.

2. *If* $\mathcal{G}_\lambda$ *is a generated subframe of* $\mathcal{F}_\lambda$ *for* $\lambda \in \Lambda$, *then* $\divideontimes_{\lambda\in\Lambda} \mathcal{G}_\lambda$ *is also a generated subframe of* $\divideontimes_{\lambda\in\Lambda} \mathcal{F}_\lambda$.

3. *The identity map id* : $\sum_{\lambda\in\prod_{\mu\in M}\Lambda_\mu} \prod_{\lambda\in\Lambda} W^\mu_{\lambda_\mu} \to \prod_{\mu\in M} \sum_{\lambda_\mu\in\Lambda_\mu} W^\mu_{\lambda_\mu}$ *is a reduction of* $\sum_{\lambda\in\prod_{\mu\in M}\Lambda_\mu} \divideontimes_{\lambda\in\Lambda} \mathcal{F}^\mu_{\lambda_\mu}$ *to* $\divideontimes_{\mu\in M} \sum_{\lambda_\mu\in\Lambda_\mu} \mathcal{F}^\mu_{\lambda_\mu}$.

Similarly to Chapter 6 for modal algebras $\boldsymbol{A_\lambda}$'s, we define their *normal product* $\divideontimes_{\lambda\in\Lambda} \boldsymbol{A_\lambda}$ as $(\divideontimes_{\lambda\in\Lambda} (\boldsymbol{A_\lambda})_+)^+$.

Corollary 7.2.4 *For modal general frames* $\mathcal{F}_\lambda$ *for* $\lambda \in \Lambda$,

$$(\divideontimes_{\lambda\in\Lambda} \mathcal{F}_\lambda)^+ \cong \divideontimes_{\lambda\in\Lambda} (\mathcal{F}_\lambda)^+. \tag{7.1}$$

Theorem 7.2.5 *If modal general frame* $\mathcal{F}_\lambda$ *is descriptive for* $\lambda \in \Lambda$, *then their normal product* $\divideontimes_{\lambda\in\Lambda} \mathcal{F}_\lambda$ *is also descriptive.*

Proof. We will show compactness, i.e., $\forall \mathcal{A} \subseteq \divideontimes_{\lambda\in\Lambda} \mathcal{P}_\lambda$($\mathcal{A}$ has finite intersection property $\Rightarrow$ $\bigcap \mathcal{A} \neq \emptyset$). Since $\mathcal{A}$ has finite intersection property, $\emptyset$ doesn't belongs to the filter $[\mathcal{A})$ generated by $\mathcal{A}$ in $\divideontimes_{\lambda\in\Lambda} \mathcal{P}_\lambda$. Since $[\mathcal{A}) \cap (\emptyset] = \emptyset$, there exists a prime filter $\mathcal{B}$ such that $[\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{B} \cap (\emptyset] = \emptyset$ by Theorem 2.4.3. Define $\mathcal{B}_{cy} = \{C \in \mathcal{B} \mid C$is a cylindric set of$\{\mathcal{P}_\lambda \mid \lambda \in \Lambda\}\}$ Since $\mathcal{B}$ is a prime filter, $\bigcup_{i\in I} C_i \in \mathcal{B}$ implies $C_i \in \mathcal{B}_{cy}$ for some $i \in I$. Hence $\bigcap \mathcal{B} \supset \bigcap \mathcal{B}_{cy}$. Since $\mathcal{B} \cap (\emptyset] = \emptyset$ implies $\emptyset \notin \mathcal{B}$, $\mathcal{B}$ has finite intersection property and so $\mathcal{B}_{cy}$ does. Therefore, since each $\mathcal{P}_\lambda$ has finite intersection property, $\bigcap \mathcal{B}_{cy} \neq \emptyset$ by the compactness of each $\mathcal{F}_\lambda$. Since $\bigcap \mathcal{A} \bigcap \mathcal{B} \supset \bigcap \mathcal{B}_{cy}$, $\bigcap \mathcal{A} \neq \emptyset$. ∎

Corollary 7.2.6 *For modal algebras $\boldsymbol{A_\lambda}$'s,*

$$(\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{A_\lambda})_+ \cong \mathop{\divideontimes}_{\lambda\in\Lambda} (\boldsymbol{A_\lambda})_+. \tag{7.2}$$

Similarly to Theorem 6.3.4, we can show the duality of Theorem 7.2.3 as follows.

Theorem 7.2.7

1. *If $\boldsymbol{B_\lambda}$ is a homomorphic image of $\boldsymbol{A_\lambda}$, then $\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{B_\lambda}$ is also a homomorphic image of $\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{A_\lambda}$.*

2. *If $\boldsymbol{B_\lambda}$ is a subalgebra of $\boldsymbol{A_\lambda}$, then $\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{B_\lambda}$ is isomorphic to a subalgebra of $\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{A_\lambda}$.*

3. *$\mathop{\divideontimes}_{\mu\in M} \sum_{\lambda_\mu\in\Lambda_\mu} \boldsymbol{A}^\mu_{\lambda_\mu}$ is isomorphic to a subalgebra of $\sum_{\lambda\in\prod_{\mu\in M}\Lambda_\mu} \mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{A}^\mu_{\lambda_\mu}$.*

By Theorem 7.2.7, we have the following.

Theorem 7.2.8 *For classes $\mathcal{K}_\lambda$ of modal algebras for $\lambda \in \Lambda$,*

1. $\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{H}(\mathcal{K}_\lambda) \subseteq \boldsymbol{H}(\mathop{\divideontimes}_{\lambda\in\Lambda} \mathcal{K}_\lambda)$,

2. $\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{S}(\mathcal{K}_\lambda) \subseteq \boldsymbol{IS}(\mathop{\divideontimes}_{\lambda\in\Lambda} \mathcal{K}_\lambda)$,

3. $\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{P}(\mathcal{K}_\lambda) \subseteq \boldsymbol{ISP}(\mathop{\divideontimes}_{\lambda\in\Lambda} \mathcal{K}_\lambda)$.

Hence,

4. $\boldsymbol{V}(\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{V}(\mathcal{K}_\lambda)) = \boldsymbol{V}(\mathop{\divideontimes}_{\lambda\in\Lambda} \mathcal{K}_\lambda)$.

Now, similarly to 6.4 we will define normal products of infinitely many modal logics.

For modal logics L_λ's, define the *normal product* $\mathop{\divideontimes}_{\lambda\in\Lambda} L_\lambda$ to be $\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{L}(\boldsymbol{F}(L_\lambda))$. By (7.1) and (7.2), $\mathop{\divideontimes}_{\lambda\in\Lambda} L_\lambda = \boldsymbol{L}(\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{V}(L_\lambda))$ holds. Thus, we may also take the latter as the definition of $\mathop{\divideontimes}_{\lambda\in\Lambda} L_\lambda$. By Theorem 7.2.8(4), we have the following.

Theorem 7.2.9 *Let $\mathcal{K}_\lambda$ be classes of modal algebras, and let $\mathcal{C}_\lambda$ be classes of modal general frames for $\lambda \in \Lambda$. Then*

1. $\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{L}(\mathcal{K}_\lambda) = \boldsymbol{L}(\mathop{\divideontimes}_{\lambda\in\Lambda} \mathcal{K}_\lambda)$,

2. $\mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{L}(\mathcal{C}_\lambda) = \boldsymbol{L}(\mathop{\divideontimes}_{\lambda\in\Lambda} \mathcal{C}_\lambda)$.

Since any logic in this section is extension of **D**, the following theorem holds similarly to Theorem 6.2.6

Theorem 7.2.10 *Suppose that L_λ is modal logic for $\lambda \in \Lambda$. Then*

$$\bigcap_{\lambda\in\Lambda} L_\lambda \supseteq \mathop{\divideontimes}_{\lambda\in\Lambda} L_\lambda.$$

Theorem 7.2.11 *Let φ be a formula which is equivalent to a conjunction of formulas of the form $\Box^m(\psi \to \chi)$, where $m \geq 0$, χ is constructed by $\wedge, \Box, \Diamond, \top, \bot$ from propositional variables and ψ is obtained from propositional variables applying $\wedge, \vee, \Box, \Diamond, \top, \bot$ in such a way that no subformula of the form either $\psi_1 \vee \psi_2$ or $\Diamond\psi_1$ occurs in the scope of any $\Box$ and that ψ_1 and ψ_2 have no common propositional variable in a subformula of the form $\psi_1 \wedge \psi_2$. Then φ is preserved under normal products.*

Now, similarly to 6.5 we will define shifted products of infinitely many modal logics.

Suppose that $\boldsymbol{A_\lambda} = (A_\lambda, \wedge, \vee, \to, 0, 1, \Box_\lambda)$ is mono-modal algebra, for $\lambda \in \Lambda$. Define multi-modal algebra $\boldsymbol{A_\lambda}^\circ$ as $\boldsymbol{A_\lambda}^\circ = (A_\lambda, \wedge, \vee, \to, 0, 1, \{\Box_\mu \mid \mu \in \Lambda\})$ with $\Box_\mu a = a$ for all $\mu \neq \lambda$ and all $a \in A$. For modal general frame $\mathcal{F}_\lambda$ for $\lambda \in \Lambda$, define modal general frame $\mathcal{F}_\lambda^\circ$, with binary relations, as $\mathcal{F}_\lambda^\circ = (W_\lambda, \{R_\mu \mid \mu \in \Lambda\}, \mathcal{P})$, where $R_\mu = \Delta$ for all $\mu \neq \lambda$. For a class $\mathcal{K}$ of mono-modal algebras and a class $\mathcal{C}$ of modal general frames, $\mathcal{K}^\circ$ and $\mathcal{C}^\circ$ are defined in a natural way. Now, define the *shifted product* $\bigotimes_{\lambda\in\Lambda} \boldsymbol{A_\lambda}$ of $\boldsymbol{A_\lambda}$'s, (the shifted product $\bigotimes_{\lambda\in\Lambda} \mathcal{F}_\lambda$ of $\mathcal{F}_\lambda$'s) by $\bigotimes_{\lambda\in\Lambda} \boldsymbol{A_\lambda} = \mathop{\divideontimes}_{\lambda\in\Lambda} \boldsymbol{A_\lambda}^\circ$ ($\bigotimes_{\lambda\in\Lambda} \mathcal{F}_\lambda = \mathop{\divideontimes}_{\lambda\in\Lambda} \mathcal{F}_\lambda^\circ$). We can define *shifted product* of classes of modal algebras and those of classes of modal general frames in an obvious way. Then, we define the shifted product $\bigotimes_{\lambda\in\Lambda} L_\lambda$ of modal logics L_λ's by $\bigotimes_{\lambda\in\Lambda} L_\lambda = \boldsymbol{L}(\bigotimes_{\lambda\in\Lambda} \boldsymbol{F}(L_\lambda))$.

Theorem 7.2.12 *Let $\mathcal{K}_\lambda$ be classes of modal algebras, and let $\mathcal{C}_\lambda$ be classes of modal general frames for $\lambda \in \Lambda$. Then*

1. $\bigotimes_{\lambda\in\Lambda} \boldsymbol{L}(\mathcal{K}_\lambda) = \boldsymbol{L}(\bigotimes_{\lambda\in\Lambda} \mathcal{K}_\lambda)$,

2. $\bigotimes_{\lambda\in\Lambda} \boldsymbol{L}(\mathcal{C}_\lambda) = \boldsymbol{L}(\bigotimes_{\lambda\in\Lambda} \mathcal{C}_\lambda)$.

Suppose that L_λ is a modal logic. Define multi-modal logics L_λ° as $L_\lambda^\circ = \mathop{*}_{\mu\in\Lambda} L_\mu$, where $L_\mu = \textbf{Triv}$ for $\mu \neq \lambda$ and $*$ denotes the *fusion* of logics.

Theorem 7.2.13 *For modal logics L_λ's,*

$$\bigotimes_{\lambda\in\Lambda} L_\lambda = \mathop{\divideontimes}_{\lambda\in\Lambda} L_\lambda^\circ.$$

Let $\mathbf{T}$ be the translation in Chapter 6. Then, we have the following.

Theorem 7.2.14 *A formula φ is a theorem of $\mathop{\divideontimes}_{\lambda\in\Lambda} L_\lambda$ iff $\mathbf{T}(\varphi)$ is a theorem of $\bigotimes_{\lambda\in\Lambda} L_\lambda$.*

We define the multi-modal logic $[L_\lambda \mid \lambda \in \Lambda]$ to be $\mathop{*}_{\lambda\in\Lambda} L_\lambda \oplus \Box_{\lambda_i}\Box_{\lambda_j} p \leftrightarrow \Box_{\lambda_j}\Box_{\lambda_i} p \oplus \Diamond_{\lambda_i}\Box_{\lambda_j} p \to \Box_{\lambda_j}\Diamond_{\lambda_i} p$, for $i \neq j$. The following theorem lists some properties of $\otimes$.

Theorem 7.2.15 *Let L_λ be modal logic for $\lambda \in \Lambda$. Then*

1. *If L_λ's are consistent logics, then $\bigotimes_{\lambda\in\Lambda} L_\lambda$ is a conservative extension of L_λ,*

2. $\bigotimes_{\lambda\in\Lambda} L_\lambda \supseteq [L_\lambda \mid \lambda \in \Lambda]$.

7.3 Products of intuitionistic modal logics

For simplicity, we will consider only intuitionistic modal general frames for intuitionistic $\Box\Diamond$-modal logics, although every definition and result can be naturally extended to multi-modal case.

Definition 7.3.1 *A subset $\mathcal{P}$ of UpW is called a* modal Heyting algebra *on a Kripke frame $(W, \lhd, R_\Box, R_\Diamond)$ if*

(i) $\emptyset \in \mathcal{P}$,

(ii) $X, Y \in \mathcal{P} \Rightarrow X \cap Y, X \cup Y \in \mathcal{P}$,

(iii) $X, Y \in \mathcal{P} \Rightarrow X \to Y \in \mathcal{P}$,

(iv) $X \in \mathcal{P} \Rightarrow \Box X, \Diamond X \in \mathcal{P}$.

Clearly, UpW is an example of a modal Heyting algebra on $(W, \lhd, R_\Box, R_\Diamond)$.

Definition 7.3.2 *A subset $\mathcal{S}$ of UpW is called a* modal Heyting semi-algebra *on a Kripke frame $(W, \lhd, R_\Box, R_\Diamond)$ if*

(i) $\emptyset \in \mathcal{S}$,

(ii) $X, Y \in \mathcal{S} \Rightarrow X \cap Y \in \mathcal{S}$,

(iii) $X, Y_i \in \mathcal{S} \Rightarrow X \to \bigcup_{i=1}^{n} Y_i$ *is a union of finitely many members of* $\mathcal{S}$,

(iv) $X_i \in \mathcal{S} \Rightarrow \Box \bigcup_{i=1}^{n} X_i$ *is a union of finitely many members of* $\mathcal{S}$,

(v) $X \in \mathcal{S} \Rightarrow \Diamond X \in \mathcal{S}$.

Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are modal Heyting algebras on $(W, \lhd, R_\Box, R_\Diamond)$ and $(V, \lhd', S_\Box, S_\Diamond)$, respectively. We consider the product $(W \times V, \lhd \times \lhd', R_\Box \times S_\Box, R_\Diamond \times S_\Diamond)$ of Kripke frames $(W, \lhd, R_\Box, R_\Diamond)$ and $(V, \lhd', S_\Box, S_\Diamond)$.

Now, consider the set $\{X \times Y \mid X \in \mathcal{P} \text{ and } Y \in \mathcal{Q}\}$, which is a subset of $Up(W \times V)$. This set is not always a modal Heyting algebra on $(W \times V, \lhd \times \lhd', R_\Box \times S_\Box, R_\Diamond \times S_\Diamond)$, but it is always a modal Heyting semi-algebra on it. For a given Kripke frame $(W, \lhd, R_\Box, R_\Diamond)$ and a subset $\mathcal{S}$ of UpW, the smallest modal Heyting algebra on $(W, \lhd, R_\Box, R_\Diamond)$ containing $\mathcal{S}$ is called the *modal Heyting algebra generated by* $\mathcal{S}$. Then, the following lemma holds.

Lemma 7.3.3 *Suppose that $\mathcal{S}$ is a modal Heyting semi-algebra on a Kripke frame (W, R). Then the modal Heyting algebra generated by $\mathcal{S}$ is $\{\bigcup_{i=1}^{n} X_i \mid X_i \in \mathcal{S}$ and some $n < \omega\}$.*

For modal Heyting algebras $\mathcal{P}$ on $(W, \lhd, R_\Box, R_\Diamond)$ and $\mathcal{Q}$ on $(V, \lhd', S_\Box, S_\Diamond)$, define $\mathcal{P} \divideontimes \mathcal{Q}$ to be the modal Heyting algebra on $(W \times V, \lhd \times \lhd', R_\Box \times S_\Box, R_\Diamond \times S_\Diamond)$ generated by $\{X \times Y \mid X \in \mathcal{P} \text{ and } Y \in \mathcal{Q}\}$.

Corollary 7.3.4

$$\mathcal{P} \divideontimes \mathcal{Q} = \{\bigcup_{i\in I}(X_i \times Y_i) \mid X_i \in \mathcal{P} \textit{ and } Y_i \in \mathcal{Q} \textit{ for a finite } I\ \}.$$

Normal products of intuitionistic modal general frames can be defined in the same way as normal products introduced in Chapter 6. In the following, $\mathcal{F}_\lambda$ and $\mathcal{G}_\mu$ (with or without indices) denote intuitionistic modal general frames of the form $(W_\lambda, \lhd_\lambda, R_{\Box_\lambda}, R_{\Diamond_\lambda}, \mathcal{P}_\lambda)$ and $(V_\mu, \lhd'_\mu, S_{\Box_\mu}, S_{\Diamond_\mu}, \mathcal{Q}_\mu)$, respectively.

For given intuitionistic modal general frames $\mathcal{F} = (W, \lhd, R_\Box, R_\Diamond, \mathcal{P})$ and $\mathcal{G} = (V, \lhd', S_\Box, S_\Diamond, \mathcal{Q})$, the *normal product* $\mathcal{F} \divideontimes \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ is an intuitionistic modal general frame $(W \times V, \lhd \times \lhd', R_\Box \times S_\Box, R_\Diamond \times S_\Diamond, \mathcal{P} \divideontimes \mathcal{Q})$. The normal product thus obtained is also an intuitionistic modal general frame for mono-modal logics.

In the following, we will characterize the operators in $\mathcal{F} \divideontimes \mathcal{G}$. Take notice of (7.6). Suppose that both $\bigcup_{i\in I}(X_i \times Y_i)$ and $\bigcup_{j\in J}(T_j \times U_j)$ are elements of $\mathcal{P} \divideontimes \mathcal{Q}$.

$$\emptyset = \emptyset \times \emptyset (= \emptyset \times V = W \times \emptyset). \tag{7.3}$$

$$\bigcup_{i\in I}(X_i \times Y_i) \cap \bigcup_{j\in J}(T_j \times U_j) = \bigcup_{(i,j)\in I\times J}(X_i \cap T_j) \times (Y_i \cap U_j). \tag{7.4}$$

$$\bigcup_{i\in I}(X_i \times Y_i) \cup \bigcup_{j\in J}(T_j \times U_j) = \bigcup_{k\in I\sqcup J} A_k \times B_k, \tag{7.5}$$

where $I \sqcup J = \{(i,0) \mid i \in I\} \cup \{(j,1) \mid j \in J\}$, $A_{(i,0)} = X_i$, $A_{(j,1)} = T_j$, $B_{(i,0)} = Y_j$, $B_{(j,1)} = U_j$.

$$\bigcup_{j\in J}(T_j \times U_j) \to \bigcup_{i\in I}(X_i \times Y_i) = \bigcup_{f\subseteq J\times\mathcal{P}(I)} (\bigcap_{j\in J}(T_j \to \bigcup_{I'\in f_j}\bigcap_{i\in I'} X_i)) \times (\bigcap_{j\in J}(U_j \to \bigcup_{I'\in f_j}\bigcap_{i\in I'} Y_i)). \tag{7.6}$$

$$\Diamond \bigcup_{i\in I}(X_i \times Y_i) = \bigcup_{i\in I}(\Diamond X_i \times \Diamond Y_i). \tag{7.7}$$

$$\Box \bigcup_{i\in I}(X_i \times Y_i) = \bigcup_{\mathcal{C}\subseteq\mathcal{P}(I)} (\Box \bigcap_{J\in\mathcal{C}}\bigcup_{i\in J} X_i) \times (\Box \bigcup_{J\in\mathcal{C}}\bigcap_{i\in J} Y_i). \tag{7.8}$$

Moreover, the following holds in $\mathcal{P} \divideontimes \mathcal{Q}$.

$$\bigcup_{i\in I}(X_i \times Y_i) = \bigcup_{j\in J}(T_j \times U_j) \tag{7.9}$$

$$\text{iff} \quad \forall i \in I, \exists \mathcal{C} \subseteq \mathcal{P}(J) \text{ such that } X_i \subseteq \bigcap_{K\in\mathcal{C}}\bigcup_{j\in K} T_j, Y_i \subseteq \bigcup_{K\in\mathcal{C}}\bigcap_{j\in K} U_j,$$

$$\text{and} \quad \forall j \in J, \exists \mathcal{C} \subseteq \mathcal{P}(I) \text{ such that } T_j \subseteq \bigcap_{K\in\mathcal{C}}\bigcup_{i\in K} X_i, U_j \subseteq \bigcup_{K\in\mathcal{C}}\bigcap_{i\in K} Y_i.$$

Theorem 7.3.5 *Let $\mathcal{F}$ and $\mathcal{G}$ be intuitionistic modal Kripke frames. Then, $\mathcal{F} \divideontimes \mathcal{G}$ is a Kripke frame if and only if either $\mathcal{F}$ or $\mathcal{G}$ is finite. In other words, $UpW \divideontimes UpV = Up(W \times V)$ if and only if either W or V is finite.*

The following theorem holds similarly to Theorem 6.2.6

Theorem 7.3.6 *A projection $\pi : W \times V \to W$ is a reduction of $\mathcal{F} \divideontimes \mathcal{G}$ to $\mathcal{F}$ iff either $S_\Box(v) \neq \emptyset$ for each $v \in V$ or $R_\Box(w) = \emptyset$ for each $w \in W$, and either $S_\Diamond(v) \neq \emptyset$ for each $v \in V$ or $R_\Diamond(w) = \emptyset$ for each $w \in W$.*

The next theorem gathers together several essential properties of normal products.

Theorem 7.3.7

1. *If f_i is a reduction of $\mathcal{F}_i$ to $\mathcal{G}_i$ for $i = 1, 2$, then the* product map *$f_1 \divideontimes f_2$ is a reduction of $\mathcal{F}_1 \divideontimes \mathcal{F}_2$ to $\mathcal{G}_1 \divideontimes \mathcal{G}_2$, where $f_1 \divideontimes f_2$ is defined by $(f_1 \divideontimes f_2)(w_1, w_2) = (f_1(w_1), f_2(w_2))$ for each $w_1 \in W_1$ and $w_2 \in W_2$.*

2. *If $\mathcal{G}_i$ is a generated subframe of $\mathcal{F}_i$ for $i = 1, 2$, then $\mathcal{G}_1 \divideontimes \mathcal{G}_2$ is also a generated subframe of $\mathcal{F}_1 \divideontimes \mathcal{F}_2$.*

3. *The identity map id : $\sum\limits_{\lambda \in \Lambda\, \mu \in M} (W_\lambda \times V_\mu) \to (\sum\limits_{\lambda \in \Lambda} W_\lambda) \times (\sum\limits_{\mu \in M} V_\mu)$ is a reduction of $\sum\limits_{\lambda \in \Lambda\, \mu \in M} (\mathcal{F}_\lambda \divideontimes \mathcal{G}_\mu)$ to $(\sum\limits_{\lambda \in \Lambda} \mathcal{F}_\lambda) \divideontimes (\sum\limits_{\mu \in M} \mathcal{G}_\mu)$, where $\sum$ denotes disjoint unions.*

Similarly to Chapter 6 for modal Heyting algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we define their *normal product* $\boldsymbol{A} \divideontimes \boldsymbol{B}$ as $(\boldsymbol{A}_+ \divideontimes \boldsymbol{B}_+)^+$. Also, $\boldsymbol{A} \divideontimes \boldsymbol{B}$ is isomorphic to a modal algebras

$$\left(\mathcal{P}_f(A \times B)/_\sim, \wedge, \vee, \to, 0, 1, \Box, \Diamond\right),$$

where the operator $\to$ is defined as follows; for $[(a_i, b_i) \mid i \in I]$, $[(c_j, d_j) \mid j \in J] \in \mathcal{P}_f(A \times B)/_\sim$,

$$\begin{aligned} &[(a_i, b_i) \mid i \in I] \to [(c_j, d_j) \mid j \in J] \\ &= \Big[\Big(\bigwedge_{j \in J} (T_j \to \bigvee_{I' \in f_j} \bigwedge_{i \in I'} X_i), \bigwedge_{j \in J} (U_j \to \bigvee_{I' \in f_j} \bigwedge_{i \in I'} Y_i)\Big) \mid f \subseteq J \times \mathcal{P}(I)\Big]. \end{aligned} \tag{7.10}$$

Corollary 7.3.8 *For intuitionistic modal general frames $\mathcal{F}$ and $\mathcal{G}$,*

$$(\mathcal{F} \divideontimes \mathcal{G})^+ \cong \mathcal{F}^+ \divideontimes \mathcal{G}^+. \tag{7.11}$$

Theorem 7.3.9 *If both intuitionistic modal general frames $\mathcal{F}, \mathcal{G}$ are descriptive, then their normal product $\mathcal{F} \divideontimes \mathcal{G}$ is also descriptive.*

Proof. We will show compactness, i.e., $\forall \mathcal{A} \subseteq \mathcal{P} \divideontimes \mathcal{Q} \cup \overline{\mathcal{P} \divideontimes \mathcal{Q}}$($\mathcal{A}$ has finite intersection property $\Rightarrow \bigcap \mathcal{A} \neq \emptyset$). Since $\mathcal{A}$ has finite intersection property, $\emptyset$ doesn't belongs to the filter $[\mathcal{A})$ generated by $\mathcal{A}$ in $\{\bigcup\limits_{i \in I}((X_i \cap X'_i) \times (Y_i \cap Y'_i)) \mid X_i, X'_i \in \mathcal{P}$ and $Y_i, Y'_i \in \mathcal{Q}$ for a finite I $\}$ Since $[\mathcal{A}) \cap (\emptyset] = \emptyset$, there exists a prime filter $\mathcal{B}$ such that $[\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{B} \cap (\emptyset] = \emptyset$ by Theorem 2.4.3. Define $\mathcal{B}_{rec} = \{C \in \mathcal{B} \mid C$is a rectangle set$\}$ Since $\mathcal{B}$ is a prime filter, $\bigcup\limits_{i \in I} C_i \in \mathcal{B}$ implies $C_i \in \mathcal{B}_{rec}$ for some $i \in I$. Hence $\bigcap \mathcal{B} \supset \bigcap \mathcal{B}_{rec}$. Since $\mathcal{B} \cap (\emptyset] = \emptyset$ implies $\emptyset \notin \mathcal{B}$, $\mathcal{B}$ has finite intersection property and so $\mathcal{B}_{rec}$ does. Therefore, since both $\mathcal{P} \cup \overline{\mathcal{P}}$ and $\mathcal{Q} \cup \overline{\mathcal{Q}}$ have finite intersection property, $\bigcap \mathcal{B}_{rec} \neq \emptyset$ by the compactness of both $\mathcal{F}$ and $\mathcal{G}$. Since $\bigcap \mathcal{A} \bigcap \mathcal{B} \supset \bigcap \mathcal{B}_{rec}$, $\bigcap \mathcal{A} \neq \emptyset$. ∎

Corollary 7.3.10 *For modal Heyting algebras* $\boldsymbol{A}$ *and* $\boldsymbol{B}$,

$$(\boldsymbol{A} \divideontimes \boldsymbol{B})_+ \cong \boldsymbol{A}_+ \divideontimes \boldsymbol{B}_+. \tag{7.12}$$

Similarly to Theorem 6.3.4, we can show the duality of Theorem 7.3.7 as follows.

Theorem 7.3.11

1. *If* $\boldsymbol{A}_2$ *is a homomorphic image of* $\boldsymbol{A}_1$ *and* $\boldsymbol{B}_2$ *is a homomorphic image of* $\boldsymbol{B}_1$, *then* $\boldsymbol{A}_2 \divideontimes \boldsymbol{B}_2$ *is also a homomorphic image of* $\boldsymbol{A}_1 \divideontimes \boldsymbol{B}_1$.
2. *If* $\boldsymbol{A}_1$ *is a subalgebra of* $\boldsymbol{A}_2$ *and* $\boldsymbol{B}_1$ *is a subalgebra of* $\boldsymbol{B}_2$, *then* $\boldsymbol{A}_1 \divideontimes \boldsymbol{B}_1$ *is isomorphic to a subalgebra of* $\boldsymbol{A}_2 \divideontimes \boldsymbol{B}_2$.
3. $\prod_{\lambda\in\Lambda} \boldsymbol{A}_\lambda \divideontimes \prod_{\mu\in M} \boldsymbol{B}_\mu$ *is isomorphic to a subalgebra of* $\prod_{(\lambda,\mu)\in\Lambda\times M} \boldsymbol{A}_\lambda \divideontimes \boldsymbol{B}_\mu$, *where* $\prod$ *denotes direct products.*

By Theorem 7.3.11, we have the following.

Theorem 7.3.12 *For classes* $\mathcal{K}_1, \mathcal{K}_2$ *of modal Heyting algebras,*

1. $\boldsymbol{H}(\mathcal{K}_1) \divideontimes \boldsymbol{H}(\mathcal{K}_2) \subseteq \boldsymbol{H}(\mathcal{K}_1 \divideontimes \mathcal{K}_2)$,
2. $\boldsymbol{S}(\mathcal{K}_1) \divideontimes \boldsymbol{S}(\mathcal{K}_2) \subseteq \boldsymbol{IS}(\mathcal{K}_1 \divideontimes \mathcal{K}_2)$,
3. $\boldsymbol{P}(\mathcal{K}_1) \divideontimes \boldsymbol{P}(\mathcal{K}_2) \subseteq \boldsymbol{ISP}(\mathcal{K}_1 \divideontimes \mathcal{K}_2)$.

Hence,

4. $\boldsymbol{V}(\boldsymbol{V}(\mathcal{K}_1) \divideontimes \boldsymbol{V}(\mathcal{K}_2)) = \boldsymbol{V}(\mathcal{K}_1 \divideontimes \mathcal{K}_2)$.

Now, similarly to 6.4 we will define normal products of intuitionistic modal logics. We also use $\boldsymbol{F}(L)$ and $\boldsymbol{L}(\mathcal{K})$ for intuitionistic modal general frames, and also use $\boldsymbol{V}(L)$ and $\boldsymbol{L}(\mathcal{C})$ for modal Heyting algebras.

For intuitionistic modal logics L_1 and L_2, define the *normal product* $L_1 \divideontimes L_2$ to be $\boldsymbol{L}(\boldsymbol{F}(L_1) \divideontimes \boldsymbol{F}(L_2))$. By (7.11) and (7.12), $L_1 \divideontimes L_2 = \boldsymbol{L}(\boldsymbol{V}(L_1) \divideontimes \boldsymbol{V}(L_2))$ holds. Thus, we may also take the latter as the definition of $L_1 \divideontimes L_2$. By Theorem 7.3.12(4), we have the following.

Theorem 7.3.13 *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be classes of modal Heyting algebras, and let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be classes of intuitionistic modal general frames. Then*

1. $\boldsymbol{L}(\mathcal{K}_1) \divideontimes \boldsymbol{L}(\mathcal{K}_2) = \boldsymbol{L}(\mathcal{K}_1 \divideontimes \mathcal{K}_2)$,
2. $\boldsymbol{L}(\mathcal{C}_1) \divideontimes \boldsymbol{L}(\mathcal{C}_2) = \boldsymbol{L}(\mathcal{C}_1 \divideontimes \mathcal{C}_2)$.

Similarly to Theorem 6.4.2, we have the following by Theorem 7.3.5 and Theorem 7.3.13.

Theorem 7.3.14

1. *If both* L_1 *and* L_2 *have the f.m.p., then* $L_1 \divideontimes L_2$ *has the product f.m.p.*
2. *If* L_1 *has the f.m.p. and* L_2 *is Kripke complete, then* $L_1 \divideontimes L_2$ *is also Kripke complete.*

Theorem 7.3.15 *Suppose that L is a product-persistent logic and Γ is any set of Sahlqvist formulas. Then the logic $L \oplus \Gamma$ is also product-persistent.*

Theorem 7.3.16 *Suppose that L_1 is an intuitionistic modal logic and L_2 is an extension of the logic* $\mathbf{IntD}_{\Box\Diamond}$. *Then $L_1 \supseteq L_1 \divideontimes L_2$.*
Moreover, if both L_1 and L_2 are extensions of the logic $\mathbf{IntD}_{\Box\Diamond}$,

$$L_1 \cap L_2 \supseteq L_1 \divideontimes L_2.$$

Theorem 7.3.17 *Let φ be a formula which is equivalent to a conjunction of formulas of the form $\Box^m(\psi \to \chi)$, where $m \geq 0$, χ is constructed by $\wedge, \Box, \Diamond, \top, \bot$ from propositional variables and ψ is obtained from propositional variables applying $\wedge, \vee, \Box, \Diamond, \top, \bot$ in such a way that no subformula of the form either $\psi_1 \vee \psi_2$ or $\Diamond\psi_1$ occurs in the scope of any $\Box$ and that ψ_1 and ψ_2 have no common propositional variable in a subformula of the form $\psi_1 \wedge \psi_2$. Then φ is preserved under normal products.*

Now, similarly to 6.5 we will define shifted products of intuitionistic modal logics. Note that we don't shift implication because we need the only one implication in shifted products, while we shift modalities.

Suppose that $\boldsymbol{A} = (A, \wedge, \vee, \to, 0, 1, \Box_1)$ and $\boldsymbol{B} = (B, \wedge, \vee, \to, 0, 1, \Box_2)$ are mono-modal Heyting algebras. Define bi-modal Heyting algebras $\boldsymbol{A}^\circ$ and ${}^\circ\boldsymbol{B}$ as $\boldsymbol{A}^\circ = (A, \wedge, \vee, \to, 0, 1, \Box_1, \Box_2)$ with $\Box_2 a = a$ for all $a \in A$ and ${}^\circ\boldsymbol{B} = (B, \wedge, \vee, \to, 0, 1, \Box_1, \Box_2)$ with $\Box_1 b = b$ for all $b \in B$. For intuitionistic modal general frames $\mathcal{F}$ and $\mathcal{G}$, define intuitionistic modal general frames $\mathcal{F}^\circ$ and ${}^\circ\mathcal{G}$, with two binary relations, as $\mathcal{F}^\circ = (W, \lhd, R_\Box, \lhd, R_\Diamond, \lhd^{-1}, \mathcal{P})$ and ${}^\circ\mathcal{G} = (V, \lhd', \lhd', S_\Box, \lhd'^{-1}, S_\Diamond, \mathcal{Q})$. For a class $\mathcal{K}$ of mono-modal Heyting algebras and a class $\mathcal{C}$ of intuitionistic modal general frames, $\mathcal{K}^\circ$, ${}^\circ\mathcal{K}$, $\mathcal{C}^\circ$ and ${}^\circ\mathcal{C}$ are defined in a natural way. Now, define the *shifted product* $\boldsymbol{A} \otimes \boldsymbol{B}$ of $\boldsymbol{A}$ and $\boldsymbol{B}$ (the shifted product $\mathcal{F} \otimes \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$) by $\boldsymbol{A} \otimes \boldsymbol{B} = \boldsymbol{A}^\circ \divideontimes {}^\circ\boldsymbol{B}$ ($\mathcal{F} \otimes \mathcal{G} = \mathcal{F}^\circ \divideontimes {}^\circ\mathcal{G}$). We can define *shifted product* of classes of modal Heyting algebras and those of classes of intuitionistic modal general frames in an obvious way. Then, we define the shifted product $L_1 \otimes L_2$ of intuitionistic modal logics L_1 and L_2 by $L_1 \otimes L_2 = \boldsymbol{L}(\boldsymbol{F}(L_1) \otimes \boldsymbol{F}(L_2))$.

Theorem 7.3.18 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be classes of modal Heyting algebras, and $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of intuitionistic modal general frames. Then*

1. $\boldsymbol{L}(\mathcal{K}_1) \otimes \boldsymbol{L}(\mathcal{K}_2) = \boldsymbol{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$,
2. $\boldsymbol{L}(\mathcal{C}_1) \otimes \boldsymbol{L}(\mathcal{C}_2) = \boldsymbol{L}(\mathcal{C}_1 \otimes \mathcal{C}_2)$.

Suppose that L_1 and L_2 are intuitionistic modal logics. Define bi-modal logics ${L_1}^\circ$ and ${}^\circ L_2$ as ${L_1}^\circ = L_1 * \mathbf{Triv}$ and ${}^\circ L_2 = \mathbf{Triv} * L_2$, where $\mathbf{Triv} = \mathbf{IntK}_{\Box\Diamond} \oplus \Box p \leftrightarrow p$ and $*$ denotes the *fusion* of logics.

Theorem 7.3.19 *For intuitionistic modal logics L_1 and L_2,*

$$L_1 \otimes L_2 = {L_1}^\circ \divideontimes {}^\circ L_2.$$

Let $\mathbf{T}$ be the translation in Chapter 6. Then, we have the following.

Theorem 7.3.20 *A formula φ is a theorem of $L_1 \divideontimes L_2$ iff $\mathbf{T}(\varphi)$ is a theorem of $L_1 \otimes L_2$.*

The following theorem lists some properties of $\otimes$.

Theorem 7.3.21 *Let L_1 and L_2 be intuitionistic modal logics. Then*

1. *For any theorem φ of L_1, φ belongs to $L_1 \otimes L_2$,*

2. $L_1 \otimes L_2 \ni \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p, \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p, \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p, \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p,$

3. *if both L_1 and L_2 are extensions of the logic* $\mathbf{IntD}_{\Box\Diamond}$, $L_1^\circ \cap {}^\circ L_2 \supseteq L_1 \otimes L_2$,

4. *If L_1 and L_2 have the f.m.p., then $L_1 \otimes L_2$ has the product f.m.p.,*

5. *If L_1 has the f.m.p. and L_2 is Kripke complete, then $L_1 \otimes L_2$ is also Kripke complete.*

Chapter 8

Conclusions and further studies

In this chapter, we mention brief survey and open problems

1. Inclusion Relationship between Pseudo-Euclidean Logics
   In this thesis, for *fixed* non-negative integers m and n, we have shown when $E_k \supseteq E_{k'}$ holds. Now, what will happen if we allow both m and n to change? More preciously, let $E_k^{m,n}$ be the logic which is obtained from the smallest normal modal logic **K** by adding the axiom $\Diamond^k\phi \to \Box^m\Diamond^n\phi$, where $k, m, n \geq 0$. Then it will be interesting to see when $E_k^{m,n} \supseteq E_{k'}^{m',n'}$ holds.

2. Finite model property for intuitionistic modal logics
   For classical modal logics, We have already a lot of general results on the finite model property (see e.g. [5]). In fact, the finite model property of classical modal logics can be obtained not only by filtration method but also by various methods, including algebraic methods and the method of selecting points. On the other hand, because of $\Diamond$-operator, the situation is much more complicated for intuitionistic modal logics. Thus, it is quite interesting and important to develop methods for obtaining the finite model property for intuitionistic modal logics. A certain attempt is made in e.g. [12].

3. Subdirectly irreducible modal Heyting algebras

   In this thesis, by normalizing we have shown a description of subdirectly irreducible modal Heyting algebras under some weak conditions. But most general case remains. Therefore, we have to investigate $\mathbf{IntK}_{\Box\Diamond}$ in the various points of view in order to see how $\Diamond$-operator behave. An interesting problem is, for example, *splitting* which is discussed in [30].

4. Products of modal logics
   There are many open problems in this area:

   (a) Which property carries over from modal logics L_1 and L_2 to the product of $L_1 \otimes L_2$ ($L_1 \times L_2$) ?

   (b) Decidability of logics in the following list:

   - $\mathbf{K4} \times \mathbf{K4}$, $\mathbf{S4} \times \mathbf{S4}$, $\mathbf{K4} \times \mathbf{K4.3}$, $\mathbf{S4} \times \mathbf{S4.3}$
   - $\mathbf{K} \otimes \mathbf{K4}$, $\mathbf{S5} \otimes \mathbf{K4}$, $\mathbf{K4} \otimes \mathbf{K4}$, $\mathbf{S4} \otimes \mathbf{S4}$

- $\mathbf{K} \divideontimes \mathbf{K}$, $\mathbf{S5} \divideontimes \mathbf{S5}$, $\mathbf{K4} \divideontimes \mathbf{K4}$, $\mathbf{S4} \divideontimes \mathbf{S4}$

(c) Finite axiomatizability of logics in the following list:

- $\mathbf{K4} \otimes \mathbf{K4}$, $\mathbf{K} \otimes \mathbf{K4}$, $\mathbf{S5} \otimes \mathbf{K4}$
- $\mathbf{K} \divideontimes \mathbf{K}$, $\mathbf{S5} \divideontimes \mathbf{S5}$, $\mathbf{K4} \divideontimes \mathbf{K4}$, $\mathbf{S4} \divideontimes \mathbf{S4}$

(d) $\mathbf{K}^3 \times \mathbf{K} = \mathbf{K}^4$?

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[2] Y. Hasimoto, *Normal products of modal logics*, to appear in *Advances in Modal Logic, Volume 3*, CSLI Publications.

[3] Y. Hasimoto and A. Maruyama, *Inclusion relationship between pseudo-Euclidean logics*, in preparation.

Presentations at Conferences

- International Conferences (refereed)

  1. Y. Hasimoto, *A description of subdirectly irreducible Heyting algebras*, Twelfth European Summer School in Logic, Language and Information, ESSLLI-2000, Workshop on Many-dimensional logical systems, Birmingham, Great Britain, 2000.8.6–18.
  2. Y. Hasimoto, *Normal products of modal logics*, Advances in Modal Logic – International Conference on Temporal Logic 2000, University of Leipzig, Germany, 2000.10.4–7.

- Domestic Workshops

  1. Y. Hasimoto, *Completeness theory for intuitionistic modal logics*, the 31st MLG meeting, Miho, Shimizu, Japan, 1997.11.24–26.
  2. Y. Hasimoto, *Algebraic semantics for intuitionistic modal logics*, the 32nd MLG meeting, Sizuoka, Japan, 1998.11.26–28.
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  4. Y. Hasimoto, *Products of modal logics by general frames*, the 33rd MLG meeting, Echigo-Yuzawa, Japan, 2000.1.10–13.