Notes on "Reducing Algorithm Complexity for Computing an Aggregate Uncertainty Measure"

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***Abstract*—In a recent paper, Liu *et al.* have proposed the so-called $\mathcal{F}$-algorithm which conditionally reduces the computational complexity of the Meyerowitz–Richman–Walker algorithm for the computation of the aggregate-uncertainty measure in the Dempster–Shafer theory of evidence, along with an illustration of its application in a practical scenario of target identification. In this correspondence, we will point out several technical mistakes, which some of them lead to some inexact or incomplete statements in the paper of Liu *et al.* The corrections of these mistakes will be made, and some further improvement and results will be derived.**

***Index Terms*—Aggregate uncertainty (AU), computational complexity, Dempster–Shafer (D-S) theory, uncertainty measures.**

I. Introduction

Since its inception, the Dempster–Shafer (D-S) theory of evidence [3], [7] has appeared as one of the most popular theories for modeling and reasoning with uncertainty. In the D-S theory, two types of uncertainty referred to in the literature as nonspecificity and discord coexist and are formally modeled by the notion of a belief function or equivalently its basic probability assignment (BPA). Since the early 1990s, several attempts have been made to define justifiable measures of uncertainty which would capture both nonspecificity and discord and would play a similar role in generalized information theory to that of the Shannon measure of entropy in the classical information theory. In particular, Maeda *et al.* [9] and Harmanec and Klir [10] proposed a well-justified measure of aggregate uncertainty (shortly, AU) that aggregates both nonspecificity and discord in its formulation and perfectly satisfies all axiomatic requirements that an AU measure should meet [4]. However, this AU measure suffers from some significant shortcomings, one of which is its computational complexity. Meyerowitz *et al.* [8] developed an algorithm for computing the AU measure of a belief function Bel, denoted by AU(Bel), which also faces the same problem of computational complexity due to the required computation of belief values over the power set of the frame of discernment. Therefore, any improvement or new algorithm for efficiently computing AU(Bel) is desirable. In fact, Harmanec *et al.* [11] provided some important suggestions for improving the performance of the algorithm of Meyerowitz *et al.*

Recently, Liu *et al.* [1] have proposed the so-called $\mathcal{F}$-algorithm for computing AU(Bel), which conditionally reduces the computational complexity of the algorithm of Meyerowitz *et al.* In addition, an application of the $\mathcal{F}$-algorithm in a practical scenario of target identification has been also illustrated. Essentially, Liu *et al.* first provided justifications for suggestions, given by Harmanec *et al.* in [11], of possible simplifications of the algorithm of Meyerowitz *et al.* and then developed the $\mathcal{F}$-algorithm based on these justifications. The authors also provided interesting properties of the proposed $\mathcal{F}$-algorithm. However, there are several technical mistakes which make some statements in the paper of Liu *et al.* inexact or incomplete. In the following, we will point out these mistakes and discuss the corrections for them, which yield improvement on the work of Liu *et al.*

The rest of this correspondence is organized as follows. In Section II, we briefly review a few concepts in the D-S theory that are necessary for the discussion, as well as the algorithm of Meyerowitz *et al.* and the remarks given by Harmanec *et al.* on it. Section III is then devoted to the $\mathcal{F}$-algorithm of Liu *et al.* and their main results, as well as our discussion and improvement on their work. Finally, Section IV presents some conclusions.

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II. AU Measure In D-S Theory of Evidence

For convenience, we keep the notation unchanged from Liu *et al.* [1]. Let X be a finite set called the frame of discernment. A belief function on X is defined as a mapping Bel from the power set of X, 2^X, to the unit interval [0, 1] and satisfies the following conditions:

1) $\text{Bel}(\emptyset) = 0$;
2) $\text{Bel}(X) = 1$;
3) For any finite family $\{A_i\}_{i=1}^n$ in 2^X

$$\text{Bel}\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1}\text{Bel}\left(\bigcap_{i\in I} A_i\right).$$

In the D-S theory, a belief function is often derived from its corresponding BPA m, which is also defined as a mapping from 2^X to [0, 1] satisfying $m(\emptyset) = 0$ and

$$\sum_{A\in 2^X} m(A) = 1. \tag{1}$$

Then, Bel, for any $A \in 2^X$

$$\text{Bel}(A) = \sum_{B\subseteq A} m(B) \tag{2}$$

is a belief function.

Definition 1: Let Bel be a belief function over X and m be its corresponding BPA. A subset $A \in 2^X$ with $m(A) > 0$ is called a *focal element* of Bel. Let us denote

$$\mathcal{F} = \left\{A \in 2^X | m(A) > 0\right\}$$

the set of all focal elements, and

$$\mathcal{C} = \bigcup_{A\in\mathcal{F}} A$$

the core of Bel, i.e., the union of all its focal elements.

The AU measure proposed by Maeda *et al.* [9] and Harmanec and Klir [10] then aims at quantifying both aspects of uncertainty, namely, nonspecificity and discord, modeled by a belief function "in an aggregate fashion" [4]. It is defined as follows.

Definition 2: Let Bel be a belief function over the frame of discernment X. The AU measure associated with Bel, denoted by AU(Bel), is defined by

$$\mathrm{AU(Bel)} = \max_{\mathcal{P}_{\mathrm{Bel}}} \left[- \sum_{x \in X} p_x \log_2 p_x \right] \tag{3}$$

where the maximum is taken over $\mathcal{P}_{\mathrm{Bel}}$—the set of all probability distributions $\langle p_x | x \in X \rangle$ that are consistent with Bel. Namely, distributions $\langle p_x | x \in X \rangle$ from $\mathcal{P}_{\mathrm{Bel}}$ must satisfy the following constraints:

$$p_x \in [0,1] \qquad \forall x \in X; \sum_{x \in X} p_x = 1 \tag{4}$$

$$\mathrm{Bel}(A) \leq \sum_{x \in A} p_x \qquad \forall A \in 2^X. \tag{5}$$

More details of this measure and its justification regarding axiomatic requirements for measures of uncertainty in generalized information theory could be found in [4]. A recent attempt was made in [5] to overcome the computational complexity of AU by introducing a new functional as an alternative measure of AU. However, it was pointed out in [6] that this new functional does not qualify as a measure of AU since it violates the essential requirement of subadditivity.

Meyerowitz *et al.* [8] developed an algorithm (called MRW algorithm, for short) for computing AU (Bel) associated with a belief function Bel as formulated in Algorithm 1.

Algorithm 1 The MRW algorithm

In: A frame of discernment X, a belief function Bel on X.

Out: AU(Bel), $\langle p_x | x \in X \rangle \in \mathcal{P}_{\mathrm{Bel}}$ such that $\mathrm{AU(Bel)} = -\sum_{x \in X} p_x \log_2 p_x$.

1: Find a nonempty set $A \in 2^X$ that maximizes $\mathrm{Bel}(A)/|A|$. If there is more than one such set A, the one with maximal cardinality should be selected.
2: For $x \in A$, put $p_x = \mathrm{Bel}(A)/|A|$.
3: For each $B \subseteq X \setminus A$, put $\mathrm{Bel}(B) = \mathrm{Bel}(B \cup A) - \mathrm{Bel}(A)$.
4: Put $X := X \setminus A$.
5: If $X \neq \emptyset$ and $\mathrm{Bel}(X) > 0$, then go to 1.
6: If $\mathrm{Bel}(X) = 0$ and $X \neq \emptyset$, then put $p_x = 0$ for all $x \in X$.
7: Calculate $\mathrm{AU(Bel)} = -\sum_{x \in X} p_x \log_2 p_x$.

Due to the computation of all belief values on the power set of X required in step 1, the MRW algorithm suffers from a high computational complexity when the cardinality of X becomes large. This makes the algorithm potentially computationally intractable [1]. Interestingly, Harmanec *et al.* [11] provided some important suggestions for improving the computational efficiency of the MRW algorithm, which are summarized as follows [1].

H-1. The elements of X outside the core of Bel could be excluded for the computation of AU(Bel).

H-2. It is enough to consider only

$$\left\{ A \subseteq X | \exists \{F_1, \ldots, F_l\} \subseteq 2^X, \right.$$
$$\left. \text{such that } m(F_i) > 0 \text{ and } A = \bigcup_{i=1}^{l} F_i \right\}. \tag{6}$$

The first suggestion H-1) for restricting the computation of AU(Bel) to the core $\mathcal{C}$ of Bel instead of using the whole X is justified by the following theorem.

Theorem 1 ([1, Th. 2]): Let Bel be a belief function (generalized or not), and let $\mathcal{C}$ be its core. Then, the set A which maximizes the ratio $(\mathrm{Bel}(A)/|A|)$ is included in $\mathcal{C}$.

III. Complexity-Reducing Algorithm for AU Measure

A. $\mathcal{F}$-Algorithm

To reduce the computational complexity for calculating AU(Bel), Liu *et al.* [1] have recently developed the $\mathcal{F}$-algorithm, which was named according to and based on the following idea as quoted from their paper.

> "The idea behind this algorithm is based on the observation that if one restricts a subset to the set of the focal elements included in it, its belief stays unchanged (from the definition of a belief), while its size may decrease. Consequently, one may consider only unions of focal elements rather than all the subsets. For this reason, this algorithm has been called the $\mathcal{F}$-algorithm, since $\mathcal{F}$ represents the set of all focal elements of a belief function. This supports the remarks of Harmanec *et al.* in [20]"[1] (see previous section).

The $\mathcal{F}$-algorithm is formulated specifically as described in Algorithm 2. Here, again, for convenience of discussion, we keep all formulations of $\mathcal{F}$-algorithm and its properties the same as shown in [1]. The main properties of $\mathcal{F}$-algorithm are stated in the following theorems.

Theorem 2 ([1, Th. 3]): If $\mathcal{F}$ is the set of the focal elements of a belief function Bel, then the set A maximizing the ratio $\mathrm{Bel}(A)/|A|$ is an element of the power set of $\mathcal{F}$. [*sic*]

Proposition 1 ([1, Proposition 1]): Both the MRW algorithm and the $\mathcal{F}$-algorithm produce the same results for AU(Bel).

Proposition 2 ([1, Proposition 2]): If $|\mathcal{F}| < |X|$, then the $\mathcal{F}$-algorithm reduces the computational complexity of AU compared to the MRW algorithm.

By observing that the computational complexity of the $\mathcal{F}$-algorithm is directly linked to the size of $2^{\mathcal{F}}$ and that the size of $\mathcal{F}$ changes in successive steps of the algorithm, Liu *et al.* [1] also suggested the comparison, at each loop, of the size of the current support of Bel (its core) with the number of current focal elements. If $|\mathcal{F}| < |\mathcal{C}|$, then the $\mathcal{F}$-algorithm should be used, and whenever $|\mathcal{C}| < |\mathcal{F}|$, the MRW algorithm with the core restriction should be chosen.

Algorithm 2 The $\mathcal{F}$-algorithm (From [1], with errors)

In: The set of focal elements $\mathcal{F}$ of a belief function Bel and their corresponding BPA.

Out: AU(Bel), $\langle p_x | x \in \mathcal{C} \rangle$ such that $\mathrm{AU(Bel)} = -\sum_{x \in X} p_x \log_2 p_x$, $p_x \geq 0$, $\sum_{x \in \mathcal{C}} p_x = 1$, and $\mathrm{Bel}(A) \leq \sum_{x \in A} p_x \ \forall \emptyset \neq A \subseteq \mathcal{C}$.

1: Initialize $\mathrm{AU(Bel)} = 0$.
2: Compute the belief measures for all elements of $\mathcal{F}$ and their unions. Suppose $|\mathcal{F}| = M$

$$\mathrm{Bel}(A_i) = m(A_i) \qquad \forall A_i \in \mathcal{F}[sic]$$
$$\mathrm{Bel}(A_i \cup A_j) = m(A_i) + m(A_j) \qquad \forall A_i,\ A_j \in \mathcal{F}[sic]$$
$$\vdots$$
$$\mathrm{Bel}(A_1 \cup A_2, \ldots, \cup A_M) = \sum_{i=1}^{M} m(A_i).\ [sic]$$

3: Find a set $A \in 2^{\mathcal{F}}$, such that $\mathrm{Bel}(A)/|A|$ is maximal. If there is more than one such set A, the one with the highest cardinality should be selected.

[1]The reference for Harmanec *et al.* is given in [11] in this correspondence.

4: For $x \in A$, put $p_x = \text{Bel}(A)/|A|$; calculate $\text{AU(Bel)} := \text{AU(Bel)} - \text{Bel}(A) \times \log_2 p_x$.
5: For each focal element $B_i \in \mathcal{F}$, put $S_i = B_i \setminus A \cap B_i$.
1) If $S_i = \emptyset$, the $\mathcal{F} = \mathcal{F} \setminus \{S_i\}$.
2) Otherwise, put $m(S_i) = m(B_i)$.
For each new focal element S, set $m(S) = \sum_{S_i = S} m(S_i)$ and prune all S_i so that $S = S_i$.
6: If $|\mathcal{F}| > 1$, return to Step 2.
7: If $|\mathcal{F}| = 1$, put $p_x = m(A)/|A|$ and $\text{AU(Bel)} := \text{AU(Bel)} - m(A) \times \log_2 p_x$.

B. Discussion and Further Improvement

Before discussing the $\mathcal{F}$-algorithm formulated previously, let us denote

$$U(\mathcal{F}) = \left\{ \bigcup_{A_i \in F | F \in 2^{\mathcal{F}}} A_i \right\} \tag{7}$$

i.e., the set of the subsets of X which are generated by the unions of the focal elements from $\mathcal{F}$. It is worth emphasizing that $U(\mathcal{F})$ represents exactly the set described by (6) of the second remark H-2) due to Harmanec *et al.* [11]. Note that

$$2^{\mathcal{F}} \neq U(\mathcal{F}) \subseteq 2^X \tag{8}$$

and that, mathematically, any element of $2^{\mathcal{F}}$ is never a member of 2^X, i.e., a subset of X, but a member of 2^{2^X}.

Now, we can see that the most important mistake by Liu *et al.* in the formulation of the $\mathcal{F}$-algorithm earlier (as well as in the proof of Theorem 2 given in [1]) is the confusion of using $2^{\mathcal{F}}$ and $U(\mathcal{F})$ identically. Consequently, it leads to several improper formulations as pointed out next.

1) The formulas for calculating belief values $\text{Bel}(\cup_i A_i)$ in step 2 of the $\mathcal{F}$-algorithm are incorrect in general. For example, consider $X = \{a, b, c, d\}$ and assume that the focal set of Bel is $\mathcal{F} = \{\{a, b, c\}, \{c\}, \{b, c, d\}\}$. Then, we have

$$\text{Bel}\left(\{a, b, c\} \cup \{b, c, d\}\right) > m\left(\{a, b, c\}\right) + m\left(\{b, c, d\}\right)$$

irrespective of whatever m. Note that $2^{\mathcal{F}}$ consists of elements that are sets of sets, such as $\{\{a, b, c\}, \{c\}\}$.
2) The statement "Find a set $A \in 2^{\mathcal{F}}$, such that…" in step 3 should read "Find a set $A \in 2^X$ such that A is a union of focal elements from $\mathcal{F}$ and …". In addition, the statement "$\mathcal{F} = \mathcal{F} \setminus \{S_i\}$" in step 5-1) should be "$\mathcal{F} = \mathcal{F} \setminus \{B_i\}$."
3) The statement "the set A maximizing the ratio $\text{Bel}(A)/|A|$ is an element of the power set of $\mathcal{F}$" in Theorem 2 is inexact. Such a set A is a union of focal elements from $\mathcal{F}$, i.e., $A \in U(\mathcal{F})$.

In addition, in the proof of Theorem 2 [1], given $\mathcal{F} = \{A_1, \ldots, A_M\}$, the authors stated "let $2^{\mathcal{F}}$ be its power set

$$2^{\mathcal{F}} = \{\{A_1\}, \ldots, \{A_M\}, \{A_1 \cup A_2\}, \ldots, \mathcal{C}\}$$

where $\mathcal{C} = \{A_1 \cup, \ldots, \cup A_M\}$ is the core of Bel" which, clearly, is a confusion throughout the proof between sets of sets and unions of sets. The confusion encourages one to think of using $U(\mathcal{F})$ rather than $2^{\mathcal{F}}$ in the formulation of the $\mathcal{F}$-algorithm.

However, as stated "This supports the remarks of Harmanec et al" in the aforementioned quotation and the result formulated by Proposition 2, it seems that Liu *et al.* might actually intend to use $2^{\mathcal{F}}$ in their development of the $\mathcal{F}$-algorithm. This would be a good and solid idea, but the following revisions should be made.

First, one can define $\Gamma : 2^{\mathcal{F}} \to [0, 1]$ by

$$\Gamma(F) = \sum_{A_i \in F} m(A_i) \qquad \forall F \in 2^{\mathcal{F}} \tag{9}$$

and the statement in step 2 of Algorithm 2 is replaced with "Compute values $\Gamma(F)$ for all $F \in 2^{\mathcal{F}}$." The statement in step 3 is then "Find a set $F \in 2^{\mathcal{F}}$, such that

$$\Gamma(F)/|\underset{A_i \in F}{\cup} A_i|$$

is maximal and put $A = \cup_{A_i \in F} A_i \ldots$". Note that, with such a maximal set $F \in 2^{\mathcal{F}}$, we always have

$$\Gamma(F) = \text{Bel}\left(\underset{A_i \in F}{\cup} A_i\right) = \text{Bel}(A).$$

Indeed, if $\Gamma(F) < \text{Bel}(A)$, by definition, there must be some $\emptyset \neq F' \subseteq (\mathcal{F} \setminus F)$ such that $\forall A_j \in F'$, $A_j \subseteq A$ and

$$\text{Bel}(A) = \Gamma(F \cup F')$$

which violates the maximum of $\Gamma(F)/|A|$ as then $\Gamma(F)/|A| < \Gamma(F \cup F')/|A|$. Then, the remaining steps of Algorithm 2 could be kept unchanged except the minor errors mentioned previously that should be corrected.

To see clearly the relation between the two algorithms, we prove the following.

Lemma 1: Let m' denote the new BPA defined in step 5 of the $\mathcal{F}$-algorithm, and let Bel' be the new belief function defined in step 3 of the MRW algorithm. Then, m' is the corresponding generalized BPA of the generalized belief function Bel'. Namely, we have

$$\forall B \subseteq X \setminus A,\ \text{Bel}'(B) = \sum_{S \in \mathcal{F}'; S \subseteq B} m'(S) \tag{10}$$

where A is the set with the largest cardinality which maximizes the ratio $\text{Bel}(A)/|A|$ and $\mathcal{F}' = \{A_i \setminus A | A_i \in \mathcal{F}\} \setminus \{\emptyset\}$.

Proof: By definition, in step 5 of the $\mathcal{F}$-algorithm, we have

$$\forall S \in \mathcal{F}',\ m'(S) = \sum_{A_i \in \mathcal{F}, A_i \setminus A = S} m(A_i) \tag{11}$$

while step 3 of the MRW algorithm defines

$$\text{Bel}'(B) = \text{Bel}(A \cup B) - \text{Bel}(A) \qquad \forall B \subseteq X \setminus A.$$

Let us consider $F = \{A_i \in \mathcal{F} | A_i \subseteq A \cup B\}$. It is easily seen that $F = F_1 \cup F_2$, where

$$F_1 = \{A_i \in \mathcal{F} | A_i \subseteq A\}$$
$$F_2 = \{A_i \in \mathcal{F} | A_i \setminus A \neq \emptyset, A_i \subseteq A \cup B\}.$$

Moreover, for any $S \in \mathcal{F}'$ and $S = A_i \setminus A$ for some $A_i \in \mathcal{F}$, if $A_i \subseteq B \cup A$, then we have $S \subseteq B$. Thus, the set F_2 can be represented by

$$F_2 = \bigcup_{S \in \mathcal{F}', S \subseteq B} \{A_i \in \mathcal{F} | A_i \setminus A = S\}. \tag{12}$$

As

$$\text{Bel}(A \cup B) = \sum_{A_i \in F} m(A_i) \quad \text{Bel}(A) = \sum_{A_i \in F_1} m(A_i)$$

it follows that

$$\begin{aligned}\text{Bel}'(B) &= \sum_{A_i \in F_2} m(A_i) \\ &= \sum_{S \in \mathcal{F}', S \subseteq B} \sum_{A_i \in \mathcal{F}, A_i \setminus A = S} m(A_i) \qquad \text{[by (12)]} \\ &= \sum_{S \in \mathcal{F}', S \subseteq B} m'(S) \qquad \text{[by (11)]}\end{aligned}$$

which justifies (10) as desired. ■

Consequently, Lemma 1 provides a justification for the correctness of the $\mathcal{F}$-algorithm formulated by Proposition 1, the proof of which, in [1], is mathematically unclear. At this moment, we can see the similarity between the MRW algorithm and the $\mathcal{F}$-algorithm. The only difference is that, while the MRW algorithm initially computes all belief values over the power set of X and later calculates the values for the generalized belief function[2] over the reduced frame $X \setminus A$ (step 3 of Algorithm 1), instead, the (revised) $\mathcal{F}$-algorithm initially computes all values for the function Γ over the power set of $\mathcal{F}$ and later calculates the values for the generalized BPA (step 5 of Algorithm 2). As for the value of AU (Bel), instead of computing it once at the last step, as in the MRW algorithm, the $\mathcal{F}$-algorithm computes it in a cumulative fashion.

Let us now further show that an even better improvement could be obtained if Liu *et al.* [1] perfectly took into account the aforementioned second remark H-2) by Harmanec *et al.* [11] in their development of the $\mathcal{F}$-algorithm. First, it is of interest to present the following.

Lemma 2: Let Bel be a belief function on X and $\mathcal{F}$ be the focal set of Bel. For any $B \in 2^X$, if Bel$(B) > 0$, then there exists $A \in U(\mathcal{F})$ [cf. (7)] such that

$$A \subseteq B \quad \text{Bel}(B)/|B| \leq \text{Bel}(A)/|A|.$$

Proof: We have

$$\text{Bel}(B) = \sum_{C \in \mathcal{F}, C \subseteq B} m(C).$$

Therefore, by taking $A = \bigcup_{C \in \mathcal{F}, C \subseteq B} C$, it follows that

$$A \in U(\mathcal{F}) \quad A \subseteq B \quad \text{Bel}(B)/|B| \leq \text{Bel}(A)/|A|$$

as Bel$(B) = $ Bel(A) and $|A| \leq |B|$. ■

As a direct consequence of Lemma 2, we obtain the following theorem which is an exact restatement of Theorem 2.

Theorem 3: If $\mathcal{F}$ is the set of the focal elements of a belief function Bel, then the set A maximizing the ratio Bel$(A)/|A|$ is an element of the set $U(\mathcal{F})$.

Furthermore, noting that any member of the set $U(\mathcal{F})$ is a subset of the core C, it then obviously follows Theorem 1, of which a long proof was given in [1].

With Lemma 1 and Theorem 3, we are ready to formulate an improvement of the $\mathcal{F}$-algorithm as follows.

Algorithm 3 The improved $\mathcal{F}$-algorithm

In: The set of focal elements $\mathcal{F}$ of a belief function Bel and their corresponding BPA.

Out: AU(Bel), $\langle p_x | x \in X \rangle \in \mathcal{P}_{\text{Bel}}$ such that AU(Bel) $= -\sum_{x \in X} p_x \log_2 p_x$.

1: Initialize AU(Bel) $= 0$.

2: Compute the belief measures for all elements of $U(\mathcal{F})$.

3: Find a set $A \in U(\mathcal{F})$, such that Bel$(A)/|A|$ is maximal. If there is more than one such set A, the one with the largest cardinality should be selected.

4: For $x \in A$, put $p_x = $ Bel$(A)/|A|$; calculate AU(Bel) := AU(Bel) − Bel$(A) \times \log_2 p_x$.

5: Set $\mathcal{F}' = \{A_i \setminus A | A_i \in \mathcal{F}\} \setminus \{\emptyset\}$.

1) If $\mathcal{F}' = \emptyset$, stop.

2) Otherwise, for each $S \in \mathcal{F}'$, put

$$m(S) = \sum_{A_i \in \mathcal{F}, A_i \setminus A = S} m(A_i)$$

and set $\mathcal{F} = \mathcal{F}'$.

6: If $|\mathcal{F}| > 1$, return to Step 2.

7: If $|\mathcal{F}| = 1$ and $\mathcal{F} = \{S\}$, put $p_x = m(S)/|S|$ and AU(Bel) := AU(Bel) $- m(S) \times \log_2 p_x$.

As discussed previously, it is worth noting here that, in the formulation of Algorithm 3, if we replace the statement of step 2 by "Compute values $\Gamma(F)$ [refer to (9)] for all $F \in 2^{\mathcal{F}}$"and the first statement of step 3 by "Find a set $F \in 2^{\mathcal{F}}$, such that

$$\Gamma(F)/|\underset{A_i \in F}{\cup} A_i|$$

is maximal and put $A = \cup_{A_i \in F} A_i$," we then obtain an accurate formulation of the original $\mathcal{F}$-algorithm.

Now, it follows by definition from (7) that

$$\left|U(\mathcal{F})\right| \leq \left|2^{\mathcal{F}}\right|.$$

Furthermore, as $U(\mathcal{F}) \subseteq 2^X$, we then have

$$|U(\mathcal{F})| \leq \min\left(|2^{\mathcal{F}}|, |2^X|\right).$$

Thus, we obtain the following result regarding the efficiency of the improved $\mathcal{F}$-algorithm in comparison to the MRW algorithm, which is stronger than the result of Liu *et al.* formulated by Proposition 2.

Proposition 3: The improved $\mathcal{F}$-algorithm generally reduces the computational complexity of AU compared to both the MRW algorithm and the (corrected) original $\mathcal{F}$-algorithm.

IV. Conclusion

In this correspondence, we have pointed out and corrected several mistakes in the formulation of the $\mathcal{F}$-algorithm developed in [1]. The simplification of some results obtained in [1] and further improvement of the $\mathcal{F}$-algorithm have been also made. Hopefully, this will support a better understanding of the $\mathcal{F}$-algorithm, as well as improve its computational efficiency in calculating the AU measure of uncertainty in potential practical applications.

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