RATE ALLOCATION FOR K-USER MAC WITH TURBO EQUALIZATION

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ABSTRACT

A primary objective of this paper is to examine the achievable rate region of frequency selective K-user multiple access channels (MACs) with soft-cancelling frequency domain minimum-mean-squared-error (SC FD-MMSE) turbo equalization. With the aid of the knowledge about the extrinsic information transfer (EXIT) characteristic of the equalizer, we study the problem of determining the optimal rate allocation (RAAL) to the K users, for which the sum rate is maximized, and introduce a heuristic approach based on the dynamic programming principle for solving this optimization problem. We also propose a practical code selection algorithm for the RAAL to the K users given the multidimensional EXIT functions of the equalizer. It is shown through simulations that with SC FD-MMSE turbo equalization, substantial throughput gain can be achieved with the proposed RAAL technique over automatic repeat request with fixed coding rate.

1. INTRODUCTION

Recently, iterative (turbo) techniques [1] have been recognized as practical solutions to the signal detection problem in *frequency selective* channels. In [2], the optimal detector based on the maximum a posteriori probability (MAP) criterion is replaced by simple signal processing that performs soft canceling and minimum mean squared error (SC MMSE) filtering. The turbo equalization technique for single carrier signaling over *frequency-selective* MIMO channels presented in [3] and [4] (referred to as SC FD-MMSE in the following) performs the equivalent processing in the frequency domain, by which the computational complexity is reduced to a logarithmic order of the frame length.

The Extrinsic Information Transfer (EXIT) chart [5] is a powerful tool to analyze the convergence property in terms of mutual information (MI) exchange between the soft input soft output (SfiSfo) components of turbo systems. In [6], the authors showed that the area under the decoder EXIT function is $1 - R$ with R being the rate of the code. This area property leads to the important conclusion that *link* design problems for adaptive transmission with single user turbo equalization, in general, reduce to a *curve-fitting* problem of the two-dimensional (2D) EXIT curves of the equalizer and decoder. It also invokes the idea to change the *link* parameters such as the code and modulation format to increase the information rates of the turbo system, depending on the equalizer EXIT function for a given channel realization. In [7], the authors determine the optimal degree distributions of low density parity check (LDPC) codes for spatially multiplexed MIMO systems to achieve decoding thresholds very close to information-theoretic performance limits. Based on a similar idea, the authors of [8] investigate the optimal code design for bit interleaved coded modulation with iterative detection.

The convergence property analysis for the *frequency selective* K-user MAC with multi-user turbo equalization is possible, in principle, with K-dimensional (KD) EXIT charts, similar to the single user link design described above. There are two detrimental events that dominate performance of such systems, in general, which are: (1) the convergence is stuck before relatively high MI between the transmitted signal and decoder output LLR can be achieved, and (2) an unreasonably low rate channel code is used, resulting in a rate loss. Furthermore, unlike the optimal *single user link* design, the equalizer EXIT function of each user depends on the feedback MI from the multiple decoders. This indicates that the code parameters, including code rate and generator polynomials, used by the multiple users have to be carefully chosen so that the two detrimental outcomes do not happen to the users.

To fully exploit the advantage of the *frequency selective* K-user MAC with SC FD-MMSE turbo equalization, we consider in this paper the problem of determining the rate allocation (RAAL) for the K users that can achieve the maximum sum rate, while properly avoiding the convergence stuck.

This paper is organized as follows: Section 2 introduces the system model. Section 3 examines the achievable rate region of the K-user MAC with turbo equalization. In Section 3.1.1, we formulate the optimal RAAL problem that maxi-

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mizes the total sum rate, and propose a heuristic approach solving this problem using a dynamic programming technique [10]. In Section 3.2 we propose a practical code *selection* algorithm for the RAAL to the K users given the EXIT characteristic of the equalizer. Section 4 presents the results of simulations conducted to evaluate the throughput enhancement with the proposed RAAL technique.

2. SYSTEM MODEL

We consider a symbol-synchronous single carrier CP assisted K-user uplink wireless communication system, where a base station having M receive antennas receives signals from K active users, each equipped with a single transmit antenna. The transmission scheme of each user is bit interleaved coded modulation, where the information bit sequence is independently encoded by a rate-$r_{c,k}$ (with $k = 1, ..., K$ being the user index) binary encoder, randomly bit-interleaved, binary phase-shift keying (BPSK) modulated, and demultiplexed into N transmit blocks, each having a length KQ in BPSK symbols. In this paper, the binary encoder can be a single convolutional code (SCC) or a serially concatenated convolutional code (SCCC) [13]. Note that throughout the paper, we use the index k to denote the k-th user.

The MIMO channel is assumed to be frequency selective, where each of the KM links is comprised of L statistically independent path components $h_{k,m}(l)$, $l = 1, .., L$, $m = 1, .., M$, which are modeled as complex Gaussian random variables with zero mean and unit variance. Further, it is assumed that the channel gains are constant during the transmission of one frame (comprised of N blocks), but varying independently frame-by-frame. We further assume that all channel gains are perfectly known at the receiver.

Employing a cyclic prefix of sufficient length to each transmit block, the received signals can be expressed as[1]

$$\mathbf{r}_n = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{b}_{n,k} + \mathbf{v}_n, n = 1, .., N, \tag{1}$$

where $\mathbf{b}_{n,k} = [b_{n,k}(1), ..., b_{n,k}(Q)]^T$ denotes the transmitted vector of user k. $\mathbf{H}_k$ is the block-circulant channel matrix associated to the k-th user, and $\mathbf{v}_n$ is the AWGN with covariance $\sigma^2\mathbf{I}$. Note that $\mathbf{H}_k$ can be de-composed into a block-diagonal matrix $\mathbf{F}_K^H \mathbf{\Xi}_k \mathbf{F}$ with $\mathbf{F}_K = \mathbf{I}_K \otimes \mathbf{F}$, where $\mathbf{F}$ denotes the Fourier transform matrix of size $Q \times Q$.

At the receiver side, iterative processing for joint equalization and decoding employing an SfiSfo algorithm is performed. It is assumed that the receiver consists of an SC FD-MMSE equalizer and K decoders that perform APP decoding. However, it should be noted that the major outcomes of this paper are also applicable to other types of equalization techniques.

Within the iterative processing, *extrinsic* log-likelihood ratios (LLRs) of the coded bits are exchanged between the SC FD-MMSE equalizer and the K decoders, following the turbo principle [2].

Inputs to the SC FD-MMSE equalizer are the received signals $\mathbf{r}_n$, and the *a priori* LLRs

$$\zeta(n,k)(q) = \ln \frac{P(b_{n,k}(q) = 0)}{P(b_{n,k}(q) = 0)}, \forall n, \forall k, q = 1, .., Q.$$

The equalizer computes the extrinsic LLR for each transmitted bit $b_{n,k}(q)$ as

$$\lambda_{n,k}(q) = \ln \frac{P(z_{n,k}(q)|b_{n,k}(q) = 0)}{P(z_{n,k}(q)|b_{n,k}(q) = 1)}, \tag{2}$$

where $z_{n,k}(q)$ is the FD-MMSE filter output as defined in [4]. Note that during the first iteration of turbo equalization, $\zeta(n,k)(q)$ is zero for all n, k, q, and later on $\zeta(n,k)(q)$ is provided via the k-th interleaver by the extrinsic LLRs of the k-th decoder.

The receiver also selects for each user the code to be used from an available code set according to the criterion derived in the Section 3.2, where the K users are notified of the codes selected via separated feedback links. It is assumed that the feedback links have no delay and are error-free.

3. RATE ALLOCATION WITH TURBO EQUALIZATION

Let the MI between the transmitted bits $b_{n,k}(q)$ and the corresponding LLRs $\lambda_{n,k}(q)$ provided by the k-th equalizer output be denoted as [4]

$$I_{e,k} = \lim_{N\to\infty} \frac{1}{NQ} I(b_{n,k}(1), ..., b_{n,k}(Q); \lambda_{n,k}(1), ..., \lambda_{n,k}(Q)),$$

and let the MI between $b_{n,k}(q)$ and the LLRs $\zeta_{n,k}(q)$ at the k-th equalizer input be denoted as

$$I_{d,k} = \lim_{N\to\infty} \frac{1}{NQ} I(b_{n,k}(1), ..., b_{n,k}(Q); \zeta_{n,k}(1), ..., \zeta_{n,k}(Q)).$$

In the K-user case the convergence characteristic of the equalizer is defined by K EXIT functions, where each function is conditioned on the channel response and the receiver noise variance [4]. These K EXIT functions can be represented by a vector-function, $\mathbf{f}_e : \mathbf{I}_d \to \mathbf{f}_e = \left(f_{e,1}(\mathbf{I}_d), ..., f_{e,K}(\mathbf{I}_d)\right) \in \mathbb{E}^K$, which depends on the feedback MI $\mathbf{I}_d = (I_{d,1}, ..., I_{d,K}) \in \mathbb{E}^K$ from the K decoders. Furthermore, the convergence characteristics of the K decoders are defined by the K EXIT functions $f_{d,k} : I_{d,k} \to$

[1]The transpose operator is denoted by $(\cdot)^T$. The symbol $\otimes$ indicates the Kronecker product. We use the notation $\mathbb{E}$ and $\mathbb{E}^K$ to represent the closed sets $\mathbb{E} = \{x \in [0,1]\}$ and $\mathbb{E}^K = \{\mathbf{x} \in [0,1]^K\}$, respectively. We denote the two vertices given by the K-dimensional row vectors $\mathbf{0}_K$ and $\mathbf{1}_K$ of the region $\mathbb{E}^K$ by $a_0 := \mathbf{0}_K$, and $a_1 := \mathbf{1}_K$, respectively. $\mathcal{D}^1$ denotes the space of continuous and piecewise differentiable functions on the set $\mathbb{E}$. Finally, we use the notation $:=$ to indicate 'by definition'.

$f_{d,k}(I_{d,k}) \in \mathbb{E}$. In the following, we assume that $\mathbf{f}_e$ and $f_{d,k}$, $\forall k$ are monotonically increasing, continuous and piecewise differentiable.

An example of the two equalizer EXIT functions in the $K = 2$-user case, obtained through simulations using LLR histogram measurements [5] and the decoder EXIT function of user 1, is shown in Fig. 1. Also shown is a possible decoding trajectory of the MI exchange, which is plotted as a projection onto the plane region $\mathbb{E}^2$. Note that the decoder EXIT function of user 2 (not shown) is drawn in the $I_{d,2}$-coordinate. For the computation of the decoding trajectory, the codes of the both users were assumed to be identical, and hence the shapes of their EXIT functions are exactly the same.

As observed in Fig. 1, the MI $\mathbf{I}_d$ exchange between the equalizer and the two decoders is limited to a plane region $\mathcal{S}$. This region (referred to as the convergence region in the following) defines the set of all possible $\mathbf{I}_d$-vectors, for which the equalizer EXIT function of each user is above the corresponding decoder EXIT function, i.e.,

$$\mathcal{S} := \left\{\mathbf{I}_d \in \mathbb{E}^2 | f_{e,k}(\mathbf{I}_d) > f_{d,k}(I_{d,k}), \forall k\right\}. \tag{3}$$

Due to the monotonicity of the EXIT functions, the sequence of MI $\mathbf{I}_d$ resulting from the MI exchange between the SfiSfo components over the iterations, converges monotonically to an achievable maximum point $\mathbf{I}_d^* = (I_{d,1}^*, I_{d,2}^*, ..., I_{d,K}^*)$, which is independent of the activation ordering of the equalizer and decoder iterations [11]. The turbo equalizer can possibly converge to a nearly zero bit error rate (BER) if asymptotically $\mathbf{I}_d^* = a_1$ can be attained. Thus, to achieve near zero BER, the following constraints must hold[2]:

$$\mathcal{S} \text{ is pathwise connected and } a_1 \in \mathcal{S}. \tag{4}$$

Let $A_{\mathcal{S}} := \iint_{\mathcal{S}} \mathrm{d}\mathbf{I}_d$ be the K-dimensional volume of $\mathcal{S}$. Assume that each decoder EXIT function $f_{d,k}$ is matched to the corresponding equalizer EXIT function $f_{e,k}$ so that only an infinitesimally low open-tunnel area between all EXIT functions is maintained. Note that such decoder EXIT functions imply (1) an ideally designed channel code for each user of infinite length to achieve infinitesimally small BER and (2) an infinite number of performed turbo iterations. Further note, it does not lead to the conclusion that the whole transmission chain can achieve the capacity region, since the use of sub-optimal (SC FD-MMSE) equalization algorithms already incurs a loss in rate [6]. Under this assumption, the area $A_{\mathcal{S}}$ asymptotically approaches the value zero and $\mathcal{S}$ reduces to a simple curve. This curve is referred in the following to as convergence curve, which is parameterized by a class $\mathcal{D}^1$-vector-function, $\mathbf{u} := \big(u_1(t), u_2(t), ..., u_K(t)\big) : \mathcal{I} \to \mathcal{S}, \mathcal{I} \subset \mathbb{R}$, where each function $u_k(t)$ is monotonically increasing in the parameter t. The monotonicity of $\mathbf{u}$ is a direct result of the monotonically

[2]A set $\mathcal{A}$ is said to be pathwise-connected if every pair of points in $\mathcal{A}$ can be joined by a path in $\mathcal{A}$.

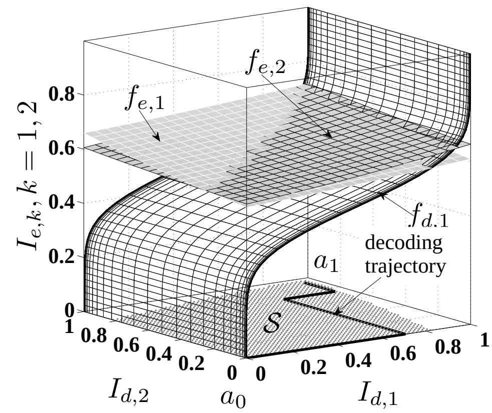

Fig. 1. Equalizer EXIT functions of an 2-user turbo system a single random channel realization and decoder EXIT function of user 1 for a constraint length 5 rate-1/2 SCC. $L = 10$, $Q = 128$, and $E_s/N_0 = 0$ dB.

increasing characteristic of $\mathbf{f}_e$ and $f_{d,k}$, $\forall k$, which ensures that the sequence of MI $\mathbf{I}_d$ does not decrease by the iterations. Let A_k be the area under $f_{d,k}$, $A_k := \int_0^1 f_{d,k}(I_{dk,k})\mathrm{d}I_{d,k}$. In [6] it has been shown that an APP-based decoder for a rate-R_k code satisfies the property $A_k = R_k$[3]. Thus, with the above assumptions on $f_{d,k}$, the rate R_k can be upper bounded by the area under the $(K+1)$-D equalizer EXIT space curve $\mathbf{w}_k(\mathbf{u}(t)) := (u_1(t), u_2(t), ..., u_K(t), f_{e,k}(\mathbf{u}(t)))$,

$$R_k < \int_{\mathcal{S}} f_{e,k}(\mathbf{I}_d)\mathrm{d}I_{d,k} = \int_{\mathcal{I}} f_{e,k}\left(\mathbf{u}(t)\right) u_k'(t)\mathrm{d}t, \tag{5}$$

where $u_k'(t)$ denotes the first derivative of $u_k(t)$. Equation (5) can be equivalently expressed as $R_k < \int_0^1 f_k^{(\mathbf{u})}(I_{d,k})\mathrm{d}I_{d,k}$, where $f_k^{(\mathbf{u})}(I_{d,k})$ denotes the projection of the space curve $\mathbf{w}_k(\mathbf{u}(t))$ onto the $I_{e,k}, I_{d,k}$-plane, where $I_{d,k} = u_k(t)$. An example of the convergence curve for an $K = 2$-user system with the corresponding equalizer EXIT space curves for the two users and the related areas defining the achievable rates R_1 and R_2 for user 1 and user 2, respectively, is shown in Fig. 2.

Let

$$\mathcal{P} := \Big\{\mathbf{p} : \mathcal{I} \to \mathbb{E}^K | \mathbf{p} \text{ of class } \mathcal{D}^1, p_k'(t) \geq 0, \forall t \in \mathcal{I}, \forall k, \\ \mathbf{p}(t_a) = a_0, \mathbf{p}(t_b) = a_1\Big\} \tag{6}$$

be the set of all admissible parametric curves in the region $\mathbb{E}^K$. Each curve $\mathbf{p} \in \mathcal{P}$ is a monotonically increasing func-

[3]The area property has been proved so far only for the case when the a-priori LLRs are assumed to be from a binary erasure channel. Nevertheless, the area property appears also to work well for Gaussian likelihoods, although it does not have a theoretical justification.

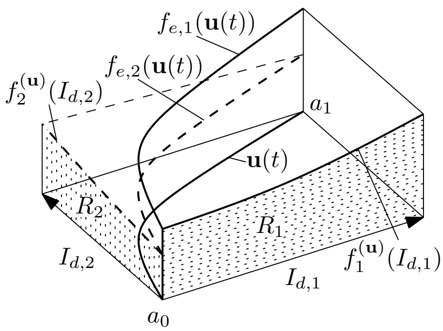

Fig. 2. Example of the parametric convergence curve $\mathbf{u}(t)$ with the two corresponding EXIT space curves and the two areas defining the achievable rates for an $K = 2$-user system.

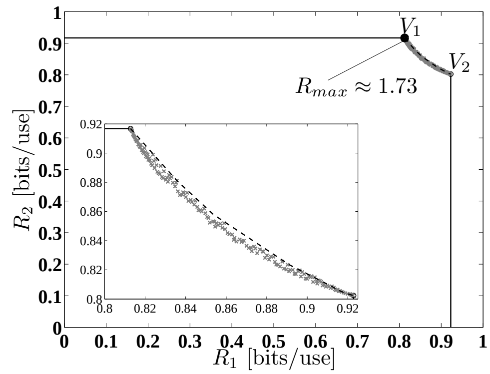

Fig. 3. Approximated rate region of 2-user MAC with SC FD-MMSE turbo equalization for a single random channel realization (a gray dot corresponds to one curve). $L = 10$, $Q = 128$, and $E_s/N_0 = 3$ dB.

tion of $t \in \mathcal{I} = [t_a, t_b]$ that has the prescribed boundary values $\mathbf{p}(t_a) = a_0$ and $\mathbf{p}(t_b) = a_1$. With (5) and (6), the expression of the achievable rate region for the frequency selective K-user MAC with turbo equalization can be expressed as

$$\bigcup_{\mathbf{p}\in\mathcal{P}} \Bigg\{ (R_1, .., R_K) : \sum_{l\in\mathcal{Y}} R_l < \sum_{l\in\mathcal{Y}} \int_{\mathbf{p}} f_{e,l}\left(\mathbf{I}_d\right) \mathrm{d}I_{d,l}, \forall \mathcal{Y} \subseteq \{\forall k\} \Bigg\}. \quad (7)$$

Fig. 3 illustrates an example of the achievable rate region of an 2-user system with SC FD-MMSE turbo equalization for a single random channel realization. In the following, we study the problem of determining the rate tuple and the corresponding convergence curve, for which the sum rate is maximized.

3.1. Sum Rate Maximization

3.1.1. Problem formulation

To identify the rate tuple and the corresponding convergence curve that maximizes the sum rate, we are interested to solve the following optimization problem:

$$R_{max} := \max_{\mathbf{p}\in\mathcal{P}} \sum_{k=1}^{K} \int_{\mathbf{p}} f_{e,k}\left(\mathbf{I}_d\right) \mathrm{d}I_{d,k}. \quad (8)$$

Using (5), the objective can also be written as

$$\max_{\mathbf{p}\in\mathcal{P}} R[\mathbf{p}(\cdot)] := \int_{\mathcal{I}} I(\mathbf{p}(t), \mathbf{p}'(t))\ \mathrm{d}t, \quad (9)$$

where $I(\mathbf{p}(t), \mathbf{p}'(t)) = \sum_{k=1}^{K} f_{e,k}(\mathbf{p}(t))p'_k(t)$. The integral optimization problem (9) falls into the category of the calculus of variations, where the first-order necessary condition of the optimality is given by the Euler-Lagrange differential equations [9]:

$$\frac{\partial I(\mathbf{p}(t), \mathbf{p}'(t))}{\partial p_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial I(\mathbf{p}(t), \mathbf{p}'(t))}{\partial p'_k} \stackrel{!}{=} 0. \quad (10)$$

However, one can easily check that, in general, the candidate curves satisfying the optimality requirement (10) do not satisfy the boundary and monotonicity condition (6). Thus, we cannot use the direct methods of the calculus of variations [9] to solve (9).

3.1.2. An approximate solution using dynamic programming

An approximate solution to (8) can be obtained by transforming the continuous problem into an equivalent discrete dynamic programming problem [10]. In particular, we can discretize the region $\mathbb{E}^K$ into a finite K-dimensional grid of points, which allows us to formulate the variational problem (8) by a path-search problem over a finite weighted graph. Each edge of the graph has its own specific cost and corresponds to a linear curve segment in the region $\mathbb{E}^K$. The admissible curves in this model are assumed to be piecewise linear functions, constrained by the grid points that have to be passed through.

For the ease of notation, we consider in the following only the $K = 2$-user case. But, note that the proposed algorithm can easily be extended to $K > 2$.

Let us denote by $\mathcal{T}$ an uniform 2D grid of $\mathbb{E}^2$,

$$\mathcal{T} := \{ih, jh, 0 \le i \le B, 0 \le j \le B\} \subset \mathbb{E}^2, \quad (11)$$

where the constant $h \in \mathbb{R}^+$ determines the grid spacing in the respective coordinate, and i, j take integer values. We next define a *directed* graph $G = (\mathcal{V}, \mathcal{N})$ for the discrete problem with $\mathcal{V}$ denoting a set of $(B+1)^2$ vertices, $\mathcal{N}$ denoting a finite set of directed edges connecting neighboring pairs of vertices, and $c : \mathcal{N} \rightarrow \mathbb{E}$ being a cost function over the edges. Each vertex $(i, j) \in \mathcal{V}$ of G is an ordered 2-tuple and corresponds to a point (ih, jh) in the cartesian 2D grid $\mathcal{T}$ as defined above. Moreover, we define, for any vertex $(i, j) \in \mathcal{V}$ of G, the set Neighb$[(i, j)]$ of its neighbors by:

$$\text{Neighb}[(i,j)] = \Big\{(i',j') \in \mathcal{V} | (i',j') = (i-1,j), (i,j-1), (i-1,j-1)\Big\}. \tag{12}$$

An edge of this graph is therefore of the form $\big((i',j'),(i,j)\big) \in \mathcal{N}$ with $(i,j) \in \mathcal{V}$, and $(i',j') \in$ Neighb$[(i,j)]$.

Having defined the vertices and edges of G, we can now specify the cost c for each edge. According to (9), a reasonable cost-definition is the sum rate increment that may be approximated for a small step-width τ between neighboring vertices (i',j') and (i,j) as

$$\int_{t'}^{t'+\tau} I(\mathbf{p}(t), \mathbf{p}'(t))\mathrm{d}t \approx \tau f_{e,1}\left(p_1^{(i',i)}, p_2^{(j',j)}\right)\Delta p_1^{(i',i)} + \tau f_{e,2}\left(p_1^{(i',i)}, p_2^{(j',j)}\right)\Delta p_2^{(j',j)} =: c[((i',j'),(i,j))], \tag{13}$$

where the terms $p_k^{(l',l)}$ and $\Delta p_k^{(l',l)}$ are defined as

$$p_k^{(l',l)} := \frac{(l'+l)h}{2}, \; \Delta p_k^{(l',l)} := \frac{(l-l')h}{\tau}. \tag{14}$$

With the above definition of Neighb$[(i,j)]$, any discrete convergence curve through the 2D grid $\mathcal{T}$ from the bottom left corner (vertex $(0,0)$) to the top right corner (vertex (B,B)) is restricted to the direct neighboring grid points, as illustrated in Fig. 4. The neighbors of a point $r \in \mathcal{T}$ are the three nearest grid points from which the point r is accessible in one step. Thus, given two neighboring grid points, their relative path angle $\varphi = \arctan(\Delta p_2^{(j',j)}/\Delta p_1^{(i',i)})$ is restricted to a multiple of $\pi/4$. Furthermore, any admissible path on G satisfies the monotonicity condition (6), since

$$\Delta p_1^{(i',i)} \geq 0 \text{ and } \Delta p_2^{(j',j)} \geq 0, \forall ((i',j'),(i,j)) \in \mathcal{N}. \tag{15}$$

The specified neighborhood structure may be further extended beyond the nearest grid points by extending the definition of the set Neighb$[(i,j)]$. Naturally, this extended definition will increase the path angular resolution, but also reduce the accuracy of approximation (13). However, the optimal choice of the destination grid is out of the scope of this paper.

Based on the above description of the graph G, we next propose a simple recursive dynamic programming algorithm that provides an efficient solution to the problem of finding the path with maximum cost from vertex $(0,0)$ to (B,B) on G. A pseudo-code implementation of the algorithm is shown in Algorithm 1.

For each vertex $(i,j) \in \mathcal{V}$, the algorithm stores the total cost $d[(i,j)]$ of the maximum-cost path found so far between vertex (i,j) and $(0,0)$. Initially, $d[(i,j)] = 0$ for all $(i,j) \in \mathcal{V}$. The computation of $d[(i,j)]$ is performed by evaluating the edge costs of all neighboring vertices $(i',j') \in$ Neighb$[(i,j)]$ to vertex (i,j), and selecting the one yielding the highest edge cost, i.e., $d[(i,j)] = d[(i^*,j^*)] + c[(i^*,j^*),(i,j)]$. Additionally, a predecessor label is stored for each vertex (i,j), which represents the previous vertex (i^*,j^*) in the maximum-cost path to the current vertex (i,j). When the algorithm terminates, the value $d[(B,B)]$ at vertex (B,B) represents the maximum cost corresponding to the optimal path on G. Based on the stored predecessor information, back-tracing is then performed to construct the maximum-cost path to vertex $(0,0)$.

Algorithm 1 Pseudo-code for graph-based path-search

Initialize $d[(i,j)] = 0$, $0 \leq i \leq B$, $0 \leq j \leq B$.
Initialize pre$[(i,j)]$ = none, $0 \leq i \leq B$, $0 \leq j \leq B$.
for i = 0 to B **do**
  for j = 0 to B **do**
    if Neighb$[(i,j)] \neq \emptyset$ **then**
      Find the predecessor to vertex (i,j) with the highest cost value:

$$(i^*,j^*) = \arg \max_{(i',j') \in \text{Neighb}[(i,j)]} \Big\{d[(i',j')] + c[((i',j'),(i,j))]\Big\}$$

      Save the predecessor to vertex (i,j):

$$\text{pre}[(i,j)] = (i^*,j^*)$$

      Update the cost $d[(i,j)]$ of vertex (i,j):

$$d[(i,j)] = d[(i^*,j^*)] + c[((i^*,j^*),(i,j))]$$

    end if
  end for
end for
Initialize the sequence $\bar{\mathbf{p}} = \{(Bh, Bh)\}$.
$(i,j) \leftarrow (B,B)$
while $(i,j) \neq (0,0)$ **do**
  $(i,j) = \text{pre}[(i,j)]$
  Insert the grid point (ih, jh) at the beginning of $\bar{\mathbf{p}}$.
end while
Use linear interpolation between the grid points in $\bar{\mathbf{p}}$ and output the corresponding convergence curve as result.

To estimate the computational complexity of Algorithm 1, let us denote by s the maximal number of neighbors of each vertex $v \in \mathcal{V}$. For the specified neighborhood structure (12),

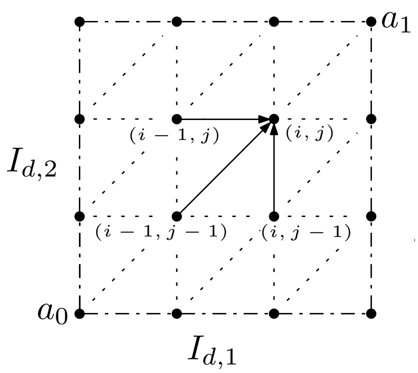

Fig. 4. Discretization of the plane region $\mathbb{E}^2$. Each vertex $(i,j) \in \mathcal{V}$ (corresponding to grid point $(ih, jh) \in \mathcal{T}$) has the three (feasible) neighbors $(i-1,j)$, $(i,j-1)$ and $(i-1,j-1)$.

$s \leq 3$. At each vertex v, the edge cost-values of the s possible neighbors to v have to be computed and compared, which results in $O(s)$ computations. With a total number of $(B+1)^2$ vertices, we can conclude that Algorithm 1 needs $O(sB^2)$ computations to solve the maximum-cost path search problem on G.

3.2. Rate Allocation using EXIT charts

With Algorithm 1, we can calculate for any particular realization of $\mathbf{f}_e$ the convergence curve $\bar{\mathbf{p}} \in \mathcal{P}$ that approximately solves (8). The maximal achievable rate of the k-th user, corresponding to $\bar{\mathbf{p}}$, is then given by the area under the projected equalizer EXIT function $f_k^{(\bar{\mathbf{p}})}$. To closely approach this rate, the decoder EXIT function $f_{d,k}$ of the k-th user should be as close as possible to $f_k^{(\bar{\mathbf{p}})}$. Furthermore, a narrow tunnel between the both functions for all $k = 1, .., K$ should be open so that the conditions for convergence, as stated in (4), are satisfied. In practice, however, optimizing $f_{d,k}$ by adjusting the available code parameters such that the code optimality in a strict sense is always guaranteed is not possible, since the code parameters are presumably limited. Thus, a practical approach for rate allocation is to *select*, for each k, the code $\mathcal{C}_n \in \mathcal{C}$ with the highest possible rate from a finite code set $\mathcal{C} = \{\mathcal{C}_1, ..., \mathcal{C}_m\}$, with m being the number of the codes in the set, with which convergence is achieved, while its decoder EXIT function $f_{d,k}^{(\mathcal{C}_n)}$ best fits to $f_k^{(\bar{\mathbf{p}})}$.

Now, we propose a simple scheme for rate allocation to the K users, which is summarized as follows.

1. Determine $\mathbf{f}_e$ for given ($\mathbf{H}_1$,..,$\mathbf{H}_K$, σ^2).

2. Calculate the convergence curve $\bar{\mathbf{p}} \in \mathcal{P}$ that solves (8) using Algorithm 1.

3. Project the corresponding EXIT space curve $\mathbf{w}_k(\bar{\mathbf{p}}(t))$, for each k, onto the $I_{e,k}$-$I_{d,k}$-plane: $f_k^{(\bar{\mathbf{p}})}(I_{d,k}) = f_{e,k}(\bar{\mathbf{p}}), I_{d,k} = \bar{p}_k(t), \forall t \in \mathcal{I}$.

4. To obtain a high code rate, select the channel code that satisfies:

$$r_{c,k} = \max_{\mathcal{C}_n \in \mathcal{C}} \left\{ r(\mathcal{C}_n) \,\Big|\, f_k^{(\bar{\mathbf{p}})}(I_{d,k}) > f_{d,k}^{(\mathcal{C}_n)}(I_{d,k}) \right., \left. \forall I_{d,k} \in [0,1) \right\}, \quad (16)$$

   where $r(\mathcal{C}_n)$ denotes the code rate of $\mathcal{C}_n$.

5. If $r_{c,k}$ is null, select the code with the lowest possible rate in $\mathcal{C}$, $r_{c,k} = \min_{\mathcal{C}_n \in \mathcal{C}} r(\mathcal{C}_n)$.

6. Output the selected code rates $\{r_{c,1}, ..., r_{c,K}\}$.

The above scheme requires the computation of the equalizer EXIT vector-function $\mathbf{f}_e$ (step 1) *for each channel realization*, which is not feasible, in practice, because of the high computational effort [12]. Therefore, we use in this paper the semi-analytical technique proposed in [4] to compute $\mathbf{f}_e$. For more details, please refer to [4].

4. NUMERICAL RESULTS

In this section, results of the simulations conducted to evaluate the throughput enhancement of the proposed RAAL technique are presented. We consider an 2-user single-carrier block-cyclic transmission system with each transmit block having $Q = 128$ coded BPSK symbols over Rayleigh fading channels with $L = 5$ and $L = 32$ path components and equal average power delay profile.

4.1. Comparison with Capacity

Fig. 5 shows the average (maximal) sum rate of the 2-user turbo system when ideally designed channel codes are allocated to the both users with respect to Algorithm 1. The rate of each user has been calculated by averaging $R_k = \int_0^1 f_k^{(\bar{\mathbf{p}})}(I_{d,k})\mathrm{d}I_{d,k}$ over a large number of channel realizations. For comparison, the ergodic capacity with i.i.d Gaussian inputs and the ergodic constellation constrained capacity with binary-input distribution are also shown. The constrained capacity result has been obtained by Monte Carlo simulations, as described in [14]. Comparing the numerical results in Fig 5, we find that the maximal sum rate achieved with the proposed RAAL technique is only slightly worse than the constrained capacity at moderate to high E_s/N_0-values. In addition, we observe that there is almost no difference between the unconstrained capacity and the constrained capacity as well as the maximal sum rate of the 2-user system in the low E_s/N_0 region.

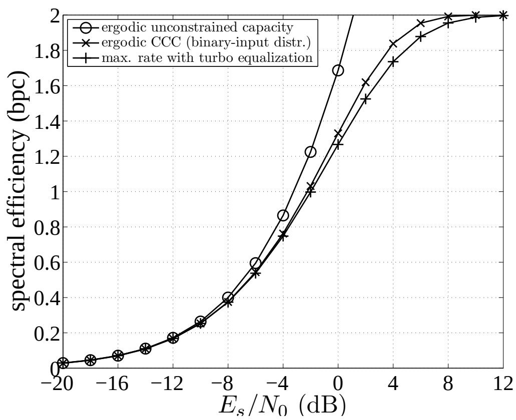

Fig. 5. Comparison of ergodic unconstrained capacity and constrained capacity with binary-input distribution to the maximal achievable rate of an 2-user system with turbo SC FD-MMSE equalization over frequency selective ($L = 5$) Rayleigh channels.

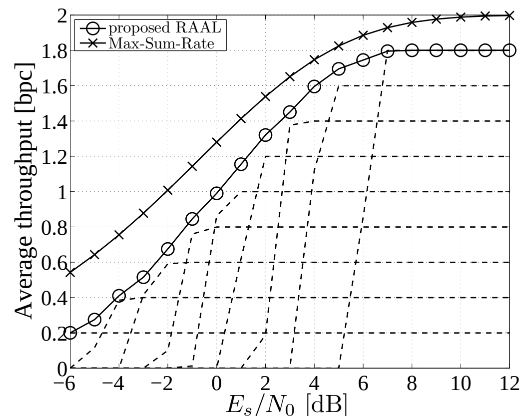

Fig. 6. Average total throughput of the both users versus E_s/N_0 for the proposed RAAL scheme and for ARQ with fixed coding rates $r_{c,1/2} = 0.1(n)$ with $n = 1, .., 9$ (dashed curves, from bottom to top). $L = 32$, $Q = 128$.

4.2. RAAL with practical codes

The rate-compatible punctured SCCCs proposed in [13], consisting of a rate-r_c outer encoder and a recursive rate-1 inner encoder with polynomials $(g_r, g_0) = (3, 2)$ (g_r denotes the feedback polynomial) in octal notation, were assumed for the RAAL to the both users. The outer encoder of the SCCC was selected from a set of $m = 17$ subcodes with rates of $r_c = 0.05(1 + n)$, $n = 1, .., m$. The 17 subcodes are constructed from a systematic, rate-$1/2$ mother code with polynomials $(g_r, g_1) = (23, 35)$, where higher rate codes are obtained through puncturing, and lower rate codes are obtained by adding more generators and using puncturing, as specified in [13]. We generated the EXIT functions of the SCCCs using Monte Carlo simulations, where 30 iterations between the inner and outer decoder were assumed. The block length of the input sequence to the outer encoder was 16384. Thus, the channel gains of the four links were assumed to be static over a frame of $N = 128$ transmitted blocks. The turbo equalizer performed 20 iterations between the equalizer and the both SCCC decoders. The grid spacing parameter in Algorithm 1 is set at $h = 0.02$ ($B = 50$), which is sufficiently small, yielding a fine grid of $(B + 1)^2 = 2601$ points. Note that in general h is a design parameter to be chosen appropriately depending on the particular realization of the equalizer EXIT vector function $\mathbf{f}_e$. Optimizing the grid spacing parameter h, given the particular realization of $\mathbf{f}_e$, is an interesting research topic, but beyond the primary scope of this paper.

When evaluating the throughput efficiency, a selective-repeat ARQ system with infinite buffer size [12] was assumed. Fig. 6 shows the average total throughput (in bpc) of the 2-user system versus the average E_s/N_0 per receive antenna. For comparison, the maximal rate with respect to Algorithm 1 (ideally designed channel codes are allocated to the both users) is shown. Also shown is the average throughput performance for an ARQ scheme with fixed coding rates $r_{c,1/2} = 0.1(n)$, $n = 1, .., 9$ of the both users. As observed in Fig. 6, substantial throughput gain is obtained with the proposed RAAL technique over the fixed rate ARQ. Further, we find that the throughput performance is only 1.4 dB away (in the high E_s/N_0 region) from the maximal information-rate achieved with ideally designed codes for each channel realization. Notably, however, there is still a gap of 2.3 dB in the low E_s/N_0 region between the throughput achieved by the proposed RAAL scheme and the maximal rate. By using more flexible coding techniques, such as irregular SCCCs that allow a code design which is optimal for the corresponding channel realization, the loss in E_s/N_0, or equivalently in throughput, can be further reduced.

5. CONCLUSION

We have examined the achievable rate region of frequency selective K-user MACs with SC FD-MMSE turbo equalization. It has been shown that with the aid of the equalizer EXIT functions the optimal RAAL to the K users can be formulated as an integral optimization problem. Further, we have proposed a heuristic approach based on the dynamic programming principle for solving this problem. In addition, we have also proposed a practical RAAL scheme based on a code *selection* algorithm. Numerical results for the 2-user MAC with SC FD-MMSE turbo equalization show that substantial throughput gain can be observed with the proposed RAAL technique over the fixed rate ARQ. Further, we have found that the spectral efficiency of the proposed practical RAAL technique using rate-compatible punctured SCCCs for each user, is only 1.4 dB away from the maximal rate achieved with ideally designed codes for each channel realization in the high E_s/N_0 region.

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