Title	フィッサーの論理,古典論理および一階算術に関する証明論的研究
Author(s)	石井，克正
Citation	
Issue Date	2002-03
Type	Thesis or Dissertation
Text version	author
URL	http://hdl.handle.net/10119/915
Rights	
Description	Supervisor:石原 哉，情報科学研究科，博士

Proof theoretical investigations for Visser's logics, classical logic and the first-order arithmetic

by

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submitted to
Japan Advanced Institute of Science and Technology
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

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March 2002

Abstract

In this thesis, we present proof theoretical investigations for Visser's logics, classical logic and the first-order arithmetic. We discuss the following three topics.

First, we introduce sequent calculi for Basic Propositional Logic (BPL) and Formal Propositional Logic (FPL) which are introduced by Visser, and prove the cut-elimination theorems for these by syntactical method. As is well-known, modal logic **S4** corresponds to the intuitionistic propositional logic by the Gödel translation. Visser introduced a logic to which modal logic GL corresponds by the Gödel translation. This logic is FPL. Visser also introduced BPL as a preliminary one for the development of FPL. A cut-free sequent calculus for BPL can be found in Ardeshir's Ph.D.thesis, but it is not satisfactory since it does not satisfies the subformula property. Later, Sasaki introduced another sequent calculus for BPL and prove the cut-elimination theorem. However his system involves an ad hoc expression as $(A \supset B)^+$ which departs from ordinary formulations of sequent calculus and subformula property in this system becomes a weak form. In this thesis, we introduce another sequent calculi for BPL and FPL, both of which satisfy subformula property. Furthermore, we prove the cut-elimination theorem for these by syntactical method.

Next, we introduce another reduction procedure for the first-order classical natural deduction **NK** and prove the strong normalization theorem and the Church-Rosser property. For the first-order classical natural deduction, Prawitz proved the strong normalization theorem for **NK** which is restricted in the sense that $\vee$ and $\exists$ are not treated as primitive logical symbols. For **NK** with full logical symbols, Stålmarck introduced a reduction procedure and proved the strong normalization theorem. However this result is not satisfactory since Stålmarck's reduction does not satisfy the Church-Rosser property. Then we introduce another reduction procedure for **NK** and prove the strong normalization theorem and the Church-Rosser property. This yields the strong normalization theorem with respect to Andou's reduction introduced in 1995 since Andou's reduction steps are expressed by several steps of ours.

Finally, we discuss proof theoretical study for the first-order arithmetic. We treat here is provable well-founded relation of $I\Sigma_k$, where $I\Sigma_k$ is a subsystem of **PA** which is obtained by restricting induction formulae to Σ_k-formulae. Let $\prec$ be a recursive well-ordering of the natural numbers and let $TI(\prec)$ be $\forall x(\forall y(y \prec x \supset \varepsilon(y)) \supset \varepsilon(x)) \to \varepsilon(a)$. Gentzen proved that if $TI(\prec)$ is provable in **PA**, then the ordertype of $\prec$ is less than ε_0. Later, Takeuti refined this, i.e., he constructed recursive function f such that if $TI(\prec)$ is provable in **PA**, then $a \prec b \Leftrightarrow f(a) <^* f(b)$ holds where $<^*$ denotes the standard ordering of type ε_0, and there exists an ordinal $\mu < \varepsilon_0$ such that for every a, $f(a) <^* \mu$. Furthermore, Arai weakened the assumption "$\prec$ is well-ordering" to "$\prec$ is an irreflexive and transitive well-founded binary relation". In this thesis, we consider this problem for $I\Sigma_k$ and obtain the similar result to the one for **PA**, in which we can replace ε_0 to ω_{k+2}.

Acknowledgments

The author wishes to express gratitude Associated Professor Hajime Ishihara for his constant encouragement and kind guidance during this studies. The author would like to thank Dr. Masahiro Hamano and Mr. Kentaro Kikuchi for their helpful discussions and suggestions. The author would also like to thank Professor Hiroakira Ono, Professor Atsushi Ohori, Professor Tetsuo Asano and Dr. Ryo Kashima for their valuable suggestions.

Contents

Chapter 1

Introduction

In this thesis, we will investigate subsystems of the first-order arithmetic and some logics by proof theoretical methods. In particular, we will discuss three topics:

1. cut-elimination theorem for Visser's Basic Propositional Logic and Formal Propositional Logic,
2. strong normalization theorem for the first-order classical natural deduction and
3. provable well-founded relations of subsystems of the first-order arithmetic.

In this introductory chapter, we first survey historical background of proof theory, in which one can find what proof theory is. Next, we explain motivations of our studies and overlook the organization of this thesis.

1.1 Historical background of proof theory

1.1.1 The origin of proof theory

Proof theory is one of the fields of study of the foundations of mathematics or mathematical logic in which we consider proofs as objects of our investigation. However, one may ask what "proofs as objects" means. Then first of all, we explain the origin of proof theory in which one can find the reason why we come to treat proofs as objects.

As is well-known, from the end of the nineteenth century to the beginning of the twentieth century, various paradoxes of set theory were discovered and many mathematicians have obliged to reconsider the foundations of mathematics. For this purpose, three principles arisen: the logicism by B.Russel, the intuitionism by L.E.J.Brouwer and the formalism by D.Hilbert. Since proof theory was occurred from Hilbert's formalism, here we only explain the formalism and omit other two principles.

So far, we have been considered that mathematical theory developed from some premises which are called axioms. This is the axiomatism initiated by Hilbert. For example, the system of axioms for Euclidean geometry due to Hilbert is well-known. Hilbert also proved the consistency of this system of axioms by constructing a model for this system of axioms using real numbers (hence strictly speaking, what Hilbert proved is that if the theory of real numbers is consistent, then Hilbert's system is also consistent). The theory of real numbers is based on the theory of natural number (namely, the arithmetic) and set theory. These notions had been considered as the most fundamental ones. However, the discovery of paradoxes in the set theory made ones reconsider Hilbert's plan.

Then Hilbert thought that we must axiomatize the arithmetic and set theory, and then we must proof the consistency of these. However, how do we prove the consistency of these? As mentioned above, the consistency of Hilbert's system for Euclidean geometry was proved by constructing a model. However, the "natural numbers" and the "sets" are the most fundamental notions and hence when we try to prove the consistency of these theories, we can not carry out the consistency proof by constructing "models" for these. The unique conceivable method is that we must examine the mathematical proofs deduced from the system of axioms for set theory or the arithmetic, and try to prove *directly* that these systems can not derive a contradiction. For this aim, we must formalize mathematical proofs in order to make mathematical proofs an object of study. This is the origin of the *proof theory*.

Remark. In order to analyze formalized proofs and prove consistency, what method can we use? For this, Hilbert proposed a *finitist standpoint*. Today, Hilbert's plan which is proving consistency of formalized mathematical theories by finitist standpoints is called *Hilbert's program*. However, we do not explain this in detail here.

1.1.2 Completeness theorem

It had been well-known that mathematical proofs is expressed by using (classical) predicate logic. Therefore, in order to formalize proofs, we must formalize predicate logic. At the present day, formalization of (classical) predicate logic is carried out by several ways. In this thesis, we will use mainly Gentzen's two formalizations which are called *natural deduction system* and *sequent calculus* respectively. However, the original accomplishment of formalization of classical predicate logic was due to Whitehead and Russell [36], and Hilbert and Ackermann [15]. Today this is called *Russell-Hilbert style*.

For a system which formalize a logic, the following two conditions are required:

1. this system can not derive any "false" formulae, that is to say, formulae which are derived by this system are only "true" ones;

2. this system can derive all "true" formulae.

Condition 1 is called *soundness* and proving soundness for classical predicate logic is not so difficult. On the other hand, condition 2, which is called *completeness*, is generally difficult to prove. In fact it was open problem in [15] whether the system due to Hilbert and Ackermann is complete or not. This was solved affirmatively by K.Gödel in [12]:

Theorem 1.1.1 (Gödel) *Any valid formula is provable* (*in the system due to Hilbert and Ackermann*).

Therefore, this theorem guarantees that formalization of classical predicate logic is completed and investigation of formalized proofs is appropriate to analyzing mathematical proofs.

1.1.3 Incompleteness theorems

By Gödel's completeness theorem, formalization of mathematical proofs is accomplished. Moreover when we develop mathematical theories in this formalized system, we must axiomatize these theories, and obtain axiom systems for these theories. These axiom systems are desired that either A or $\neg A$ (negation of A) is provable from these axioms for each sentence A. This property is called *completeness* of an axiom system (or theory). However, in [13], Gödel proved the following result which answer the above negatively with respect to the system PM defined in [36] and its extensions:

Theorem 1.1.2 (Gödel) *Let K be a set of formulae which is primitive recursive and is ω-consistent. Then there exists a sentence A such that neither A nor $\neg A$ is provable in* PM+K.

That is to say, the above theorem asserts that not only PM is not complete but also any extension of PM, which is as long as primitive recursive extension and ω-consistent, is not complete. Later Rosser weakened the assumption of the ω-consistency to the consistency in [25].

Furthermore by formalizing the above theorem in PM, Gödel proved the following marvelous result with respect to the consistency problem:

Theorem 1.1.3 (Gödel) *Let K be a set of formulae which is primitive recursive and is consistent. Then a sentence which asserts consistency of K is not provable in* PM+K.

By this theorem, proving consistency of mathematical theories by (strictly) Hilbert's finitist standpoint became impossible.

1.1.4 Gentzen's works

Hilbert's program was struck by Gödel's incompleteness theorems, but it is not necessarily impossible to carry out consistency proofs. Gödel himself said that there may exist consistency proofs for a mathematical theory (which is formalized in a certain system **S**) by finitist standpoint but can not be formalized in **S**. In fact, there exists consistency proof for the first order arithmetic due to Gentzen.

As mentioned above, Gentzen introduced two formalizations for classical predicate logic other than Russell-Hilbert style, which are called *natürlicher klassischer Kalkül* (or **NK**) and *logistischer klassischer Kalkül* (or **LK**). Today, we call the system like **NK** *natural deduction system* and the system like **LK** *sequent calculus.* In addition to this, Gentzen also introduced two formalizations for intuitionistic predicate logic, namely, **NJ** due to natural deduction system and **LJ** due to sequent culculus. Since these systems will play very important roles in this thesis, in the following, we present the definitions of **NK** and **LK** as well as **NJ** and **LJ**.

First, a first-order language $\mathcal{L}$ must be given. $\mathcal{L}$ consists of the following symbols:

- Individual constants, function constants and predicate constants.
- Free and bound variables.
- Logical symbols.

Then *terms* and *formulae* are defined in a usual way (see [33]).

In the following, we introduce both natural deduction systems and sequent calculi.

Natural deduction systems

We first present inference rules of **NK**, where A, A_1, A_2, B, etc. denote formulae. Furthermore, $\neg A$ is considered as an abbreviation for $A \supset \bot$ and hence the rules for $\neg$ are omitted.

$$\frac{A_1 \quad A_2}{A_1 \wedge A_2}\ (\wedge I) \qquad\qquad \frac{A_1 \wedge A_2}{A_i}\ (\wedge E) \quad (i = 1, 2)$$

$$\frac{A_i}{A_1 \vee A_2}\ (\vee I) \quad (i = 1, 2) \qquad\qquad \frac{A_1 \vee A_2 \quad \begin{matrix}[A_1] \\ \vdots \\ B\end{matrix} \quad \begin{matrix}[A_2] \\ \vdots \\ B\end{matrix}}{B}\ (\vee E)$$

$$\frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \supset B}\ (\supset I) \qquad\qquad \frac{A \supset B \quad A}{B}\ (\supset E)$$

$$\frac{A(a)}{\forall x A(x)}\ (\forall I) \qquad\qquad \frac{\forall x A(x)}{A(t)}\ (\forall E)$$

$$\frac{A(t)}{\exists x A(x)}\ (\exists I) \qquad\qquad \frac{\exists x A(x) \quad \begin{matrix}[A(a)] \\ \vdots \\ B\end{matrix}}{B}\ (\exists E)$$

$$\frac{\begin{matrix}[\neg A] \\ \vdots \\ \bot\end{matrix}}{A}\ (\mathrm{RAA})$$

In $(\forall E)$ and $(\exists I)$, t is an arbitrary term. Remark that $(\forall I)$ and $(\exists E)$ are subject to the restriction of eigenvariable described below (see clause 2 of definition of proofs). In $(\supset I)$, $(\vee E)$, $(\exists E)$ and (RAA), formulae with bracket [ ] (for example, $[A]$ of $(\supset I)$, etc.) are the assumptions *discharged* by this rule.

We now define simultaneously *proofs* of **NK** and its *assumptions* and *end-formula* inductively as follows.

1. If A is a formula, then A itself is a proof with the assumption A and with the end-formula A;

2. Suppose that $P_1, \dots, P_n$ be proofs where $n = 1, 2$ or 3, and their end-formulae are $A_1, \dots, A_n$ respectively. If
$$\frac{A_1 \quad \dots \quad A_n}{A}$$
is an inference rule, say R, then
$$\frac{P_1 \quad \dots \quad P_n}{A}$$
is a proof and its end-formula is A, provided that

   - if R is $(\forall I)$ of the form
   $$\frac{A(a)}{\forall x A(x)}\ (\forall I)$$
   then a does not occur in the assumptions of P_1;
   - if R is $(\exists E)$ of the form
   $$\frac{\exists x A(x) \qquad \begin{matrix}[A(a)] \\ \vdots \\ B\end{matrix}}{B}\ (\exists E)$$
   then a does not occur in $\exists x A(x)$, B and the assumptions of P_2 except $A(a)$.

   If R is $(\supset I)$, $(\vee E)$, $(\exists E)$ or (RAA), then the assumptions of the above proof are those of $P_1, \dots, P_n$ except the ones discharged by R. Otherwise the assumptions of the above proof are those of $P_1, \dots, P_n$.

A formula A is said to be *provable* in **NK** or **NK**-*provable* if there exists a proof of **NK** with no assumption whose end-formula is A.

NJ is obtained from **NK** by replacing the rule (RAA) to
$$\frac{\bot}{A}\ (\bot)$$

Sequent calculi

For sequent calculi, we use *sequents* instead of formulae. Let $A_1, A_2, \dots, A_m$ and $B_1, B_2, \dots, B_n$ be formulae. Then
$$A_1, A_2, \dots, A_m \to B_1, B_2, \dots, B_n$$
is called a *sequent*. Note that m and n may be 0.

Now **LK** is defined as follows. We first introduce inference rules. In the following, Greek capital letters Γ, Δ, etc. are denote finite (possibly empty) sequences of formulae separated by commas.

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta} \text{ (weakening : left)} \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A} \text{ (weakening : right)}$$

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta} \text{ (contraction : left)} \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A} \text{ (contraction : right)}$$

$$\frac{\Gamma, A, B, \Pi \to \Delta}{\Gamma, B, A, \Pi \to \Delta} \text{ (exchange : left)} \qquad \frac{\Gamma \to \Delta, A, B, \Lambda}{\Gamma \to \Delta, B, A, \Lambda} \text{ (exchange : right)}$$

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda} \text{ (cut)}$$

$$\frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta} \text{ } (\neg : \text{left}) \qquad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A} \text{ } (\neg : \text{right})$$

$$\frac{A_i, \Gamma \to \Delta}{A_1 \wedge A_2, \Gamma \to \Delta} \text{ } (\wedge : \text{left}) \quad (i = 1, 2) \qquad \frac{\Gamma \to \Delta, A_1 \quad \Gamma \to \Delta, A_2}{\Gamma \to \Delta, A_1 \wedge A_2} \text{ } (\wedge : \text{right})$$

$$\frac{A_1, \Gamma \to \Delta \quad A_2, \Gamma \to \Delta}{A_1 \vee A_2, \Gamma \to \Delta} \text{ } (\vee : \text{left}) \qquad \frac{\Gamma \to \Delta, A_i}{\Gamma \to \Delta, A_1 \vee A_2} \text{ } (\vee : \text{right}) \quad (i = 1, 2)$$

$$\frac{\Gamma \to \Delta, A \quad B, \Pi \to \Lambda}{A \supset B, \Gamma, \Pi \to \Delta, \Lambda} \text{ } (\supset: \text{left}) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B} \text{ } (\supset: \text{right})$$

$$\frac{A(t), \Gamma \to \Delta}{\forall x A(x), \Gamma \to \Delta} \text{ } (\forall : \text{left}) \qquad \frac{\Gamma \to \Delta, A(a)}{\Gamma \to \Delta, \forall x A(x)} \text{ } (\forall : \text{right})$$

$$\frac{A(a), \Gamma \to \Delta}{\exists x A(x), \Gamma \to \Delta} \text{ } (\exists : \text{left}) \qquad \frac{\Gamma \to \Delta, A(t)}{\Gamma \to \Delta, \exists x A(x)} \text{ } (\exists : \text{right})$$

In ($\forall$: left) and ($\exists$: right), t is an arbitrary term and in ($\forall$: right) and ($\exists$: left), a does not occur in the lower sequent. a is called the *eigenvariable*.

We now define simultaneously *proofs* of **LK** and its *end-sequent* inductively as follows.

1. $A \to A$ is a proof and its end-sequent is $A \to A$. This is called the *initial-sequent*.

2. Suppose that P_1 and P_2 are proofs and their end-sequents are S_1 and S_2 respectively. If

$$\frac{S_1}{S}$$

or

$$\frac{S_1 \quad S_2}{S}$$

is an inference rule, then

$$\frac{P_1}{S}$$

or

$$\frac{P_1 \quad P_2}{S}$$

is a proof and its end-sequent is S

A sequent S is said to be *provable* in **LK** or **LK**-*provable* if there exists a proof of **LK** whose end-sequent is S. A formula A is said to be *provable* in **LK** or **LK**-*provable* if a sequent $\rightarrow A$ is *provable* in **LK**.

LJ is obtained from **LK** by modifying it as follows:

1. The succedent (i.e., right side) of each sequent in **LJ** consists of at most one formula;

2. Inference rules of **LJ** are obtained from those of **LK** by imposing the restriction that the succedent of each upper and lower sequent consists of at most one formula.

"S is *provable* in **LJ** (**LJ**-*provable*)" is defined as same as the one of **LK**.

Gentzen first introduced the natural deduction systems in order to have a logical calculus close to actual reasoning. However, he thought the natural deduction systems are not convenient for investigations of logics. Then he invented sequent calculi. In [7], he proved the equivalency between **NK** and **LK**. More precisely, he proved that if A is provable in **NK** if and only if A is provable in **LK**. He also proved that equivalency between Russell-Hilbert style and his systems.

Furthermore, Gentzen proved the very important property for **LK** and **LJ** which is known as Gentzen's *Hauptsatz* or *cut-elimination theorem*, i.e.,

Theorem 1.1.4 (Gentzen) *If a sequent is provable in* **LK** (*resp.***LJ**)*, then it is provable in* **LK** (*resp.***LJ**) *without using cut rule.*

There exist numerous applications of this theorem. In fact, cut-free proofs have good properties. The typical one among them is the subformula property. This property yields several logical properties, for example, consistency, decidability, interpolation property, etc. As applications of cut-elimination theorem, Gentzen himself proved that the consistency of **LK** and **LJ**, decidability of the propositional classical logic and the consistency of the first-order arithmetic without the mathematical induction. Recently many kinds of non-classical logics are investigated vigorously. In such studies, formalizations by use of sequent calculi and cut-elimination theorems are useful methods.

We now turn back to Gentzen's works. In 1936, Gentzen proved consistency of the first order arithmetic **PA** which was formalized in natural deduction system ([8]). Furthermore, in [9], he proved consistency of the first order arithmetic which was formalized in sequent calculus in 1938. Here we must note that we need a method for consistency proof for **PA** which is not derivable in **PA** by Gödel's incompleteness theorem. As this kind of methods, Gentzen used transfinite induction up to first ε-number (ε_0). We also note that Gentzen introduced the theory of ordinal numbers up to ε_0 by finitist standpoint and prove transfinite induction up to ε_0 by finitary argument (but beyond Hilbert's finitist standpoint). Hence Gentzen's work can be considered as the extension of Hilbert's program.

Now we can see that transfinite induction up to ε_0 does not provable in **PA** indirectly by virtue of Gödel's incompleteness theorem and Gentzen's consistency proof. In [10], Gentzen gave a direct proof for unprovability of transfinite induction up to ε_0 in **PA**. Namely, he proved the following:

Theorem 1.1.5 *If transfinite induction up to an ordinal number α can be proved in* **PA**, *then α is less than ε_0.*

Then conversely, can transfinite induction up to an ordinal number α be proved in **PA** whenever α is less than ε_0? For each $k \in \mathbf{N}$, we define an ordinal ω_k inductively as follows: $\omega_0 := 1$; $\omega_{k+1} := \omega^{\omega_k}$. Note that $\omega_k < \varepsilon_0$ for all $k \in \mathbf{N}$. Then Gentzen affirmatively answered to the above question:

Theorem 1.1.6 *For each natural number n, transfinite induction up to ω_n can be proved in* **PA**.

Hence ε_0 is the least ordinal number α such that transfinite induction up to α can not be proved in **PA**. In general, the least ordinal number α such that transfinite induction up to α can not be proved in a certain system is called the *proof-theoretical ordinal* of this system. Gentzen showed that the proof-theoretical ordinal of **PA** is ε_0.

Beyond Gentzen's works for consistency problems, several results for consistency problems for stronger systems than the first-order arithmetic are obtaiened by G.Takeuti, K.Shütte, W.Pohlers, W.Buchholz, M.Rathjen and T.Arai. But we do not argue these issues further.

1.2 Motivations of the studies in this thesis

In the last section, we explain what proof theory is and the author expects that one can understand the meaning of "proofs as objects". By the observation of the last section, in proof theory (or proof theoretical study), the followings are considered as the main problem:

- cut-elimination theorems for sequent calculi;
- consistency for mathematics (arithmetic).

In addition to these, there are normalization theorems for natural deduction systems (which is explained later) can be considered. The studies in this thesis are related with the above three matters. In the following, we explain motivations of our studies.

First, we are interested in cut-elimination theorems for sequent calculi. When one intends to investigate logics, formalizing logics by use of sequent calculi and proving cut-elimination theorem are very useful. We here consider Visser's propositional logics BPL and FPL.

Basic Propositional Logic (BPL) and Formal Propositional Logic (FPL) were first introduced by Visser in [35]. In fact, Visser introduced FPL by interpreting implication as formal provability and BPL was introduced as a preliminary one for the development of FPL. We first explain his motivation briefly.

IPL denotes intuitionistic propositional logic. As is well-known, IPL can be embedded to a modal logic S4 by Gödel translation. Namely, if T denotes Gödel translation, then

$$\mathrm{IPL} \vdash A \Leftrightarrow \mathrm{S4} \vdash T(A)$$

holds where IPL $\vdash$ and S4 $\vdash$ denote the provability in IPL and S4 respectively. Visser raised the question: if we take GL (Gödel-Löb's provablity logic) instead of S4, then what logic is considered instead of IPL? His answer is that this is FPL. FPL is characterized by finite *transitive* and *irreflexive* Kripke-style models. Then he first investigated a logic which characterized by transitive models, which is BPL.

He described BPL and FPL in the form of natural deduction systems and proved their completeness with respect to transitive models and finite transitive and irreflexive models respectively. In 1991, Ruitenburg reintroduced BPL with a philosophical motivation [26] and extended BPL to the first order logic, BQC (Ruitenburg[27]). Ardeshir and Ruitenburg [5] and Suzuki, Wolter and Zakharyaschev [31] explored the structure of propositional logics over BPL by model theoretic and algebraic methods.

Now, how do we formalize BPL and FPL by use of sequent calculi? For BPL, a cut-free sequent calculus already can be found in Ardeshir [4], but it is not satisfactory since it does not satisfies the subformula property. Later, Sasaki [28] introduced another sequent calculus $\mathbf{GVPL}^+$ for BPL and prove the cut-elimination theorem. However his system involves an ad hoc expression as $(A \supset B)^+$ which departs from ordinary formulations of sequent calculus and subformula property in this system becomes a weak form. Therefore we need another system without an expression like $(A \supset B)^+$.

Recently Kikuchi introduced another sequent calculus for BPL, which is called **LBP** ([18]). This system satisfies the subformula property and is shown to be complete with respect to transitive models. Furthermore a sequent calculus for FPL, which is called **LFP**, is introduced by extending the system for BPL. Roughly speaking, **LBP** is obtained from (propositional part of) **LK** by relpacing rules for $\supset$ to

$$\frac{\Delta_1, \Sigma, A \to B, \Gamma_1 \quad \Delta_2, \Sigma, A \to B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \to B, \Gamma_{2^n}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B}\ (\supset)$$

and **LFP** is obtained from **LK** by relpacing rules for $\supset$

$$\frac{\Delta_1, \Sigma, A \supset B, A \to B, \Gamma_1 \quad \Delta_2, \Sigma, A \supset B, A \to B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \supset B, A \to B, \Gamma_{2^n}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B}\ (\supset)$$

where $n \geq 0$, $\Gamma_i = \{C_j \mid j \in \gamma(i)\}$, $\Delta_i = \{D_j \mid j \in \delta(i)\}$, and the sets $\gamma(i)$ and $\delta(i)$ of natural numbers are defined as follows: $\delta(i)$ runs through the subsets of $\{1, ..., n\}$ ordered according to size and $\gamma(i) = \{1, ..., n\} \setminus \delta(i)$. Kikuchi proved that the cut rule is admissible in both **LBP** and **LFP** by virtue of the completeness theorems. In other words, cut-elimination theorems are proved by semantical methods. Then the question arises: can the cut-elimination theorems for them be proved by syntactical methods? The rules for $(\supset)$ which are introduced to define **LBP** and **LFP** are quite complicated.

Therefore it seems that Gentzen's method can not apply to **LBP** and **LFP**. However we can overcome this difficulty by analyzing Gentzen's proof more circumstantially and succeed to prove cut-elimination theorems for **LBP** and **LFP**. In Chapter 2, we will deal with these problems and will present the syntactical proofs of cut-elimination theorems for **LBP** and **LFP**.

We mentioned in 1.1.4 that Gentzen introduced two formalizations for classical and intuitionistic predicate logic. For sequent calculi, we explained that the cut-elimination theorems are important for these. Then it is natural to ask that what the theorems which are corresponding to the cut-elimination theorems are for natural deduction systems. The answer for this is "normalization theorem". This theorem asserts that all proofs of **NK** or **NJ** can be transformed into normal forms, where normal form means proof without redundancy[1]. (This transformations of proofs are also called *reductions*, and hence the normalization theorem asserts that for each proof $\mathcal{D}$, there exists a finite reduction sequence starting from $\mathcal{D}$.) As the cut-elimination theorem for **LK**, Gentzen tried to establish the normalization theorem for **NK**. However his original **NK** was not fitted to formulate the normalization theorem since the law of excluded middle (tertium non datur) $A \vee \neg A$ is adopted as an axiom schema. In order to overcome this, Prawitz reformulated the natural deduction system for the classical logic by adding the rule (RAA) to **NJ** and omitted the law of excluded middle in [23]. The system **NK** which was introduced in 1.1.4 is actually the Prawitz's system[2]. In [23] and [24], the normalization theorems for **NK** and **NJ** were proved.

As is well-known, there is a close relationship between natural deduction systems and λ-calculi by the Curry-Howard isomorphism and reductions for proofs correspond to reductions for λ-terms. In λ-calculi, a reduction sequence represents a process of calculation and a normal form represents a result of calculation. Then it is desirable that every reduction sequence terminates. In the same manner, we strength the normalization theorems as follows: for each proof $\mathcal{D}$, every reduction sequences starting from $\mathcal{D}$ are finite. This statement is called *strong normalization theorem*. In [24], the strong normalization theorem for **NJ** was proved. For **NK**, however, situations are more difficult. In [24], Prawitz proved the strong normalization theorem for **NK** which is restricted in the sense that $\vee$ and $\exists$ are not treated as primitive logical symbols. For **NK** with full logical symbols, Stålmarck introduced a reduction procedure and proved the strong normalization theorem([30]). However this result is not satisfactory. In the following, we explain this dissatisfaction.

Needless to say, the difficulty in establishing the strong normalization theorem for **NK** is treating the rule (RAA). Consider the following (RAA)

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots \\ \bot\end{array}}{A}$$

[1]For precise definition of normal forms, see Chapter 3

[2]But we also call this system **NK**.

and suppose that A is the major premise of an elimination rule. For example, we consider the case where A is of the form $A_1 \supset A_2$ as follows:

$$\dfrac{\dfrac{\begin{matrix}[\neg(A_1 \supset A_2)]\\ \vdots\ \mathcal{D}_1\\ \bot\end{matrix}}{A_1 \supset A_2}\,R \qquad \begin{matrix}\vdots\ \mathcal{D}_2\\ A_1\end{matrix}}{A_2}$$

where R is (RAA). Then we would like to define a reduction at $A_1 \supset A_2$ which is the conclusion of R. This is defined as follows:

$$\dfrac{\dfrac{\begin{matrix}[\neg(A_1 \supset A_2)]\\ \vdots\ \mathcal{D}_1\\ \bot\end{matrix}}{A_1 \supset A_2}\,R \qquad \begin{matrix}\vdots\ \mathcal{D}_2\\ A_1\end{matrix}}{A_2} \qquad \Rightarrow \qquad \begin{matrix}\dfrac{\dfrac{[\neg A_2]^2 \qquad \dfrac{[A_1 \supset A_2]^1 \qquad \begin{matrix}\vdots\ \mathcal{D}_2\\ A_1\end{matrix}}{A_2}}{\bot}}{\neg(A_1 \supset A_2)}\,(\supset)^1\\ \vdots\ \mathcal{D}_1\\ \dfrac{\bot}{A_2}\,(\mathrm{RAA})^2\end{matrix}$$

The cases where A is of the form $A_1 \wedge A_2$ or $\forall x A_1(x)$ are defined similarly. If we try to treat the cases where A is of the form $A_1 \vee A_2$ or $\exists x A_1(x)$ in the same way, then, for the case where A is of the form $A_1 \vee A_2$, for example, a reduction is defined as follows (we call this *Red-1*):

$$\dfrac{\dfrac{\begin{matrix}[\neg(A_1 \vee A_2)]\\ \vdots\ \mathcal{D}_1\\ \bot\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}[A_1]\\ \vdots\ \mathcal{D}_2\\ B\end{matrix} \qquad \begin{matrix}[A_2]\\ \vdots\ \mathcal{D}_3\\ B\end{matrix}}{B} \qquad \Rightarrow \qquad \begin{matrix}\dfrac{\dfrac{[\neg B]^2 \qquad \dfrac{[A_1 \vee A_2]^1 \qquad \begin{matrix}[A_1]\\ \vdots\ \mathcal{D}_2\\ B\end{matrix} \qquad \begin{matrix}[A_2]\\ \vdots\ \mathcal{D}_3\\ B\end{matrix}}{B}}{\bot}}{\neg(A_1 \vee A_2)}\,(\supset)^1\\ \vdots\ \mathcal{D}_1\\ \dfrac{\bot}{B}\,(\mathrm{RAA})^2\end{matrix}$$

However, Stålmarck's definitions are slightly different (we call this *Red-2*):

$$\dfrac{\dfrac{\begin{matrix}[\neg(A_1 \vee A_2)]\\ \vdots\ \mathcal{D}_1\\ \bot\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}[A_1]\\ \vdots\ \mathcal{D}_2\\ B\end{matrix} \qquad \begin{matrix}[A_2]\\ \vdots\ \mathcal{D}_3\\ B\end{matrix}}{B} \qquad \Rightarrow \qquad \begin{matrix}\dfrac{\dfrac{[A_1 \vee A_2]^1 \qquad \dfrac{[\neg B]^2 \qquad \begin{matrix}[A_1]\\ \vdots\ \mathcal{D}_2\\ B\end{matrix}}{\bot} \qquad \dfrac{[\neg B]^2 \qquad \begin{matrix}[A_2]\\ \vdots\ \mathcal{D}_3\\ B\end{matrix}}{\bot}}{\bot}}{\neg(A_1 \vee A_2)}\,(\supset)^1\\ \vdots\ \mathcal{D}_1\\ \dfrac{\bot}{B}\,(\mathrm{RAA})^2\end{matrix}$$

In the author's opinion, Red-2 seems to be unnatural. Furthermore, the existence of this reduction causes that failure of the Church-Rosser property[3]. Therefore, we want to adopt Red-1. However Stålmarck's proof for the strong normalization theorem heavily depends on the definition of Red-2 and hence his proof does not work if we adopt Red-1.

In 1995, Andou introduced a reduction procedure for **NK** with full logical symbols which is distinct from Stålmarck's one[4] and proved normalization theorem([1]). Furthermore Andou proved that his reduction procedure satisfies the Church-Rosser property([2]). However it was not known whether strong normalization theorem holds.

In the above observation, our aim is to define a reduction procedure for **NK** with full logical symbols satisfying the Church-Rosser property and to prove the strong normalization theorem with respect to this reduction procedure. In Chapter 4, we will discuss this problem.

Our interests now reach the consistency problem. In 1.1.4, we explain that the proof-theoretical ordinal of the first-order arithmetic is ε_0. This fact is expressed more generally as follows (see [33]). $\mathbf{PA}(\varepsilon)$ denotes Peano arithmetic with an additional unary predicate ε. Let $\prec$ be a recursive well-ordering. $\prec$ is called a *provable well-ordering* of **PA** if

$$\forall x(\forall y(y \prec x \supset \varepsilon(y)) \supset \varepsilon(x)) \to \varepsilon(a)$$

is provable in $\mathbf{PA}(\varepsilon)$. Then Gentzen's result can be described as follows:

$$\prec \text{ is provable well-ordering of } \mathbf{PA} \Leftrightarrow |\prec| < \varepsilon_0$$

where $|\prec|$ denotes the order type of $\prec$. Takeuti refined this result ([33]). He proved that if $\prec$ is a provable well-ordering of **PA**, then there exists a recursive function f such that $a \prec b \Leftrightarrow f(a) <^* f(b)$, where $<^*$ denotes the standard ordering of type ε_0, and there exists an ordinal $\mu < \varepsilon_0$ such that for every a, $f(a) <^* \ulcorner\mu\urcorner$, where $\ulcorner\mu\urcorner$ denotes the Gödel number of μ. Recently Arai extended the above result to the case where $\prec$ is well-founded ([3, Section 1]).

Now, as mentioned 1.1.4, the proof-theoretical ordinals are obtained for several systems which are stronger than **PA**. However, not only for stronger systems, but also for systems which is weaker than **PA**, i.e., for subsystems of **PA**, the proof-theoretical ordinals are calculated. Now, for each $k \in \mathbf{N}$, let $I\Sigma_k$ be a subsystem of **PA** which is obtained by restricting induction formulae to Σ_k-formulae[5]. Then the following holds ([19]).

Theorem 1.2.1 (Mints) *For each $k \in \mathbf{N}$, the proof-theoretical ordinal of $I\Sigma_k$ is ω_{k+1}.*

Then it is natural to ask whether Takeuti and Arai's refinements are also carried out for $I\Sigma_k$ or not. In fact, this question is treated as an open problem in [3].

[3] In Chapter 3, a counter-example to Church-Rosser property will be given.
[4] In Andou's reduction procedure, the treatment for (RAA) is quitely different. See Chapter 4.
[5] For precise definition of Σ_k-formulae, see Chapter 5.

Theorem 1.2.1 is proved in a similar way to Gentzen's proof for **PA**. In fact, by Gentzen's method, if $\prec$ is provable well-ordering of **PA**, then there exists an ordinal $\alpha < \varepsilon_0$ such that $|\prec| < \alpha$. This estimate, however, is a little rough. That is to say, while $\omega_n \leq \alpha < \omega_{n+1}$ holds for some n in Gentzen's proof, we can take $\alpha < \omega_n$ in Mints' proof. Takeuti's refinement is deeply depend on Gentzen's proof and hence this is the first difficulty for treating $I\Sigma_k$.

Secondly, in Takeuti and Arai's refinements, the above μ may be greater than $|\prec|$. More precisely, from the construction of f, though $|\prec| < \omega_n$ for some n, μ may be greater than ω_n. Therefore Takeuti and Arai's methods can not apply directly when we consider the case where the base system is $I\Sigma_k$.

In Chapter 5, we will discuss this problem. We can overcome only the first difficulty and hence our result is a partial one. For the second difficulty, it still remains a open problem.

1.3 Overview of the results in this thesis

In this section, we list the main results of this thesis.

1. We prove cut-elimination theorems for **LBP** and **LFP** where **LBP** and **LFP** are sequent calculi for Visser's BPL and FPL respectively (Theorem 2.4.1, Theorem 2.4.2).

For proving cut-elimination theorem for **LBP**, Gentzen's proof for **LK** is almost work. However since the rule ($\supset$) in **LBP** is a little complicated, the ordinary definition of the rank does not work. Then we have to refining the definition of the rank. Furthermore, in the rule ($\supset$), the succedent of the lower sequent restricts to one formula. This causes the difficulty in proving by means of the usual technique of cut-elimination. To overcome this difficulty, we have to consider a special case which is proved in the same way as the cut-elimination theorem for **LK** in advance (Lemma 2.4.3).

For proving cut-elimination theorem for **LFP**, the existence of the diagonal formulae in the rule ($\supset$) (the diagonal formulae are the formulae $A \supset B$ in the upper sequents) causes the difficulty in proving by means of the usual technique of cut-elimination. To overcome this difficulty, we introduce third measure for proofs called the *width* and prove cut-elimination theorem by triple induction. This technique is an analogue of that used in [34].

2. We introduce another reduction procedure satisfying the Church-Rosser property and prove the strong normalization theorem with respect to this reduction procedure (Theorem 4.3.1). This reduction procedure is an improvement of the one due to Stålmarck.

For this aim, as Stålmarck's is used in [30], we introduce the notion of "validity" for proofs and rules[6]. Then the proof of the strong normalization theorem is carried out in three steps as follows.

1. Valid proofs are SN.
2. Proofs built of valid rules are valid.
3. All rules are valid.

In the above, proof of 3. is very hard. One of the most difficulty is as follows. In order to prove the validity of $(\vee E)$, we have to show the validity of

$$\dfrac{\begin{matrix} \\ \vdots\ \mathcal{D}_1 \\ A_1 \vee A_2 \end{matrix} \quad \begin{matrix} [A_1] \\ \vdots\ \mathcal{D}_2 \\ B \end{matrix} \quad \begin{matrix} [A_2] \\ \vdots\ \mathcal{D}_3 \\ B \end{matrix}}{B}$$

where $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ is valid. Especially, a problematic case is the one where the last inference of $\mathcal{D}_1$ is (RAA) as follows:

$$\dfrac{\dfrac{\begin{matrix} [\neg(A_1 \vee A_2)] \\ \vdots \\ \bot \end{matrix}}{A_1 \vee A_2} \quad \begin{matrix} [A_1] \\ \vdots\ \mathcal{D}_2 \\ B \end{matrix} \quad \begin{matrix} [A_2] \\ \vdots\ \mathcal{D}_3 \\ B \end{matrix}}{B}$$

In Stålmarck's proof, this is not so difficult. However, for our reduction procedure, Stålmarck's proof can not be applied. In order to overcome this difficulty, we need more delicate and complicated syntactical arguments for reductions.

3. If $\prec$ is a provable well-founded relation of $I\Sigma_k$, then there exists primitive recursive function f such that $a \prec b \Leftrightarrow f(a) <^* f(b)$, where $<^*$ denotes the standard ordering of type ω_{k+2}, and there exists an ordinal $\mu < \omega_{k+2}$ such that for every a, $f(a) <^* \ulcorner \mu \urcorner$ (Theorem 5.2.2).

This is obtained by combining Arai's proof in [3, Section 1] and Mints' proof in [19] with a slight modification to overcome the first difficulty mentioned in 1.2.

1.4 Organization of this thesis

In this section, we summarize the contents of the chapters.

In Chapter 2, we consider the cut-elimination theorems for **LBP** and **LFP**.
First, we introduce the systems **LBP** and **LFP**, and Kripke-style models for BPL and FPL. Then we summarize that both soundness and completeness are hold for them

[6]This notion is first adopted by Prawitz in [24] to prove the strong normalization theorem for **NJ**.

without proofs. Next, we present proofs for cut-elimination theorems for **LBP** and **LFP** by syntactical methods.

In Chapter 3, as preparation for discussing the strong normalization theorem for **NK**, we survey Stålmarck's result. First, we present Stålmarck's reduction procedure and summarize the outline of proof due to Stålmarck. We also present a dissatisfaction with this result. Actually, we give a counter-example for the Church-Rosser property with respect to Stålmarck's reduction procedure.

In Chapter 4, we prove the strong normalization theorem for **NK** with respect to another reduction procedure. First, we introduce another reduction procedure and remark the relation between this new reduction procedure and the one due to Andou [1]. Actually, one can easily verify that strong normalization theorem of our reduction yields the one of Andou's reduction. Next, we prove the storg normalizaition theorem. For this aim, we introduce the notion of validity of proofs, which is a main tool for proof of strong normalization theorem. There, we show that valid proofs are strongly normalizable. Hence our purpose change to prove that all proofs are valid. For this aim, we introduce the validity of inference rules and proof that all rules are valid, which implies that all proofs are valid. Finally, we remark the Church-Rosser property for this reduction.

In Chapter 5, we discuss provable well-founded relation of $I\Sigma_k$. First, we briefly recall the system **PA** and its subsystems $I\Sigma_k$, and present the definition of provable well-founded relations in $I\Sigma_k$. Next, we introduce **TJ**-proofs and reduction steps for **TJ**-proofs in order to prove the theorem. Finally, we prove the theorem by using the notion of **TJ**-proofs.

Finally, in Chapter 6, we summarize conclusions of this thesis and survey briefly further studies.

Chapter 2

Cut-elimination for Visser's propositional logics

In this chapter, we shall investigate sequent calculi for Visser's propositional logics. we introduce sequent calculi **LBP** for Basic Propositional Logic (BPL) and **LFP** for Formal Propositional Logic (FPL), and prove the cut-elimination theorems for these by syntactical method. We note that the propositional part of **LK** and **LJ** are denoted by also **LK** and **LJ** to the end of this chapter since we consider only propositional logics in this chapter.

The contents of this chapter are based on [18].

2.1 Introduction

In this section, we first recall modal logics since Visser's motivation for BPL and FPL is related to these.

Modal logics are obtained by adding *modal operator* $\Box$ to classical logic such that $\Box A$ is intended for "A is necessarily true". There are many variations of modal logics. Among them, we first introduce **K**. **K** is obtained from **LK** by adding the following rule:

$$\frac{\Gamma \to A}{\Box\Gamma \to \Box A}\ (\Box)$$

where $\Box\Gamma$ denotes the sequence of formulae $\Box A_1, \dots, \Box A_n$ when Γ is $A_1, \dots, A_n$.

Next we introduce the system **S4**. **S4** is obtained from **K** by adding axiom schemes

$$\Box A \supset A$$

and

$$\Box A \supset \Box\Box A$$

i.e., by adding sequents of the forms

$$\to \Box A \supset A$$

and

$$\rightarrow \Box A \supset \Box\Box A$$

as initial sequents.

Modal logics themselves are important objects of study and have been studied (and now are studying) by many researchers. Furthermore, an interpretation of the provability of formulae in intuitionistic propositional logic were studied by using a modal operator. In [14], Gödel describes an interpretation of intuitionistic propositional logic in a system which is equivalent to **S4**. In the following, we summarize Gödel's work.

Gödel translation T which is a mapping from formulae of a propositional logic without modal operators to formulae of one with modal operator $\Box$ is defined inductively as follows:

1. $T(p) = \Box p$ for each propositional variable p;

2. $T(A \wedge B) = T(A) \wedge T(B)$;

3. $T(A \vee B) = T(A) \vee T(B)$;

4. $T(A \supset B) = \Box(T(A) \supset T(B))$;

5. $T(\neg A) = \Box\neg T(A)$.

As is well-known, the following holds.

Theorem 2.1.1 (Gödel, McKinsey, Tarski) *A is* **LJ***-provable if and only if $T(A)$ is* **S4***-provable.*

Remark. "Only if" part of this theorem is due to Gödel and "if" part is due to McKinsey and Tarski.

In the above observation, Gödel interpreted a modal operator $\Box$ as "provable by any correct means" and not as "provable in a given formal system". Then we next consider the interpretation of the provability in a given formal system. As this formal system, we here take Peano arithmetic **PA**. An *interpretation* from formulae of a propositional logic with modal operator $\Box$ to sentences of **PA** is defined by assigning a sentence of **PA** to each propositional variable and $\Box A$ to $\mathrm{Bew}(\bar{a})$ where $\mathrm{Bew}(x)$ is the provability predicate[1] and a is the Gödel number of A. Then as is well-known, there exists a modal logic **GL** such that A is **GL**-provable if and only if A^* is **PA**-provable for each interpretation * (Solovay [29]). **GL**, which is called Gödel-Löb's provablity logic, is obtained from **K** by adding axiom schemes

$$\Box A \supset \Box\Box A$$

and

$$\Box(\Box A \supset A) \supset \Box A.$$

[1]For details for provability predicate, see [13].

Now question arises: what logic L is considered such that A is provable in L if and only if $T(A)$ is **GL**-provable? As this L, Visser introduced a logic called Formal Propositional Logic (abbreviated by FPL). FPL is characterized by finite *transitive* and *irreflexive* Kripke-style models. Then he first investigated a logic which characterized by transitive models, which is called Basic Propositional Logic (abbreviated by BPL).

As mentioned in 1.1.4, a cut-free sequent calculus for BPL can be found in Ardeshir [4], but it is not satisfactory since it does not satisfies the subformula property. Later, Sasaki [28] introduced another sequent calculus $\mathbf{GVPL}^+$ for BPL and prove the cut-elimination theorem. However his system involves an ad hoc expression as $(A \supset B)^+$ which departs from ordinary formulations of sequent calculus and subformula property in this system becomes a weak form. Therefore we need another system without an expression like $(A \supset B)^+$.

In the following, we introduce sequent calculi **LBP** for BPL and **LFP** for FPL in the ordinary formulation (without ad hoc expressions), both of which satisfy the subformula property. Roughly speaking, **LBP** is obtained from **LK** by relpacing rules for $\supset$ to

$$\frac{\Delta_1, \Sigma, A \to B, \Gamma_1 \quad \Delta_2, \Sigma, A \to B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \to B, \Gamma_{2^n}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B} \ (\supset)$$

and **LFP** is obtained from **LK** by relpacing rules for $\supset$

$$\frac{\Delta_1, \Sigma, A \supset B, A \to B, \Gamma_1 \quad \Delta_2, \Sigma, A \supset B, A \to B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \supset B, A \to B, \Gamma_{2^n}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B} \ (\supset)$$

where $n \geq 0$, $\Gamma_i = \{C_j \mid j \in \gamma(i)\}$, $\Delta_i = \{D_j \mid j \in \delta(i)\}$, and the sets $\gamma(i)$ and $\delta(i)$ of natural numbers are defined as follows: $\delta(i)$ runs through the subsets of $\{1, ..., n\}$ ordered according to size and $\gamma(i) = \{1, ..., n\} \setminus \delta(i)$. In the following, we prove the cut-elimination theorem for these by syntactical method as main result of this chapter.

The existence of these rules make difficult to prove cut-elimination theorem by syntactical methods because the principal formulae are not single in these rules and the ordinary definition of the rank does not work by these rules. We can overcome this difficulty by refining the definition of the rank.

2.2 A sequent calculus for BPL

In this section, we introduce a sequent calculus for BPL, which is called **LBP**.

Our propositional language has a denumerably infinite set of propositional variables, the propositional constant $\bot$ and the binary connectives $\wedge$, $\vee$ and $\supset$. Formulae are constructed from these in the usual way. Propositional variables are denoted by $p, q, \ldots$, and formulae are denoted by $A, B, \ldots$, possibly with subscripts or superscripts. Capital Greek letters $\Gamma, \Delta, \ldots$ are used for finite sequences of formulae. A *sequent* is an expression of the form $\Gamma \to \Delta$. Γ and Δ are called the *antecedent* and the *succedent* of a sequent $\Gamma \to \Delta$ respectively.

Before the definition of our sequent calculus, we will first explain semantics for BPL, which is similar to Kripke semantics for intuitionistic propositional logic except that the accessibility relation is not necessarily reflexive.

A *transitive model* is a triple $\langle W, R, V\rangle$ where W is a non-empty set, R is a transitive relation on W, and V is a mapping from the set of propositional variables to the power set of W such that

$$x \in V(p) \text{ and } xRy \text{ imply } y \in V(p).$$

We say simply a *model* instead of a transitive model. If W is a finite set, we say that a model $\langle W, R, V\rangle$ is *finite*.

Given a model $M = \langle W, R, V\rangle$, the truth-relation $\Vdash$ is defined inductively as follows:

$$\begin{array}{lll} (M,x) \Vdash p & \text{iff} & x \in V(p) \quad \text{for each propositional variable } p, \\ (M,x) \nVdash \bot, & & \\ (M,x) \Vdash A \wedge B & \text{iff} & (M,x) \Vdash A \text{ and } (M,x) \Vdash B, \\ (M,x) \Vdash A \vee B & \text{iff} & (M,x) \Vdash A \text{ or } (M,x) \Vdash B, \\ (M,x) \Vdash A \supset B & \text{iff} & \forall y \in W[xRy \text{ and } (M,y) \Vdash A \text{ imply } (M,y) \Vdash B]. \end{array}$$

If M is understood, we write simply $x \Vdash A$ instead of $(M,x) \Vdash A$. We say that a formula A is *true* in a model $\langle W, R, V\rangle$ if $x \Vdash A$ for every $x \in W$.

Formulae which are theorems of intuitionistic propositional logic but not true in every model defined above are, for example, $(p \wedge (p \supset q)) \supset q$ and $(p \supset (p \supset q)) \supset (p \supset q)$.

For a given model M, the truth-relation for sequents is defined as follows:

$$(M,x) \Vdash \Gamma \to \Delta \quad \text{iff} \quad \forall A \in \Gamma[(M,x) \Vdash A] \text{ implies } \exists A \in \Delta[(M,x) \Vdash A].$$

We write simply $x \Vdash \Gamma \to \Delta$ for $(M,x) \Vdash \Gamma \to \Delta$, if M is understood. We say that a sequent $\Gamma \to \Delta$ is *true* in a model $\langle W, R, V\rangle$ if $x \Vdash \Gamma \to \Delta$ for every $x \in W$.

Now we introduce a sequent calculus, which is called **LBP**. Initial sequents of **LBP** are of the following forms:

$$A \to A,$$
$$\bot \to \ .$$

Rules of inference of **LBP** consist of the following:

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta} \text{ (weakening : left)} \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A} \text{ (weakening : right)}$$

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta} \text{ (contraction : left)} \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A} \text{ (contraction : right)}$$

$$\frac{\Gamma, A, B, \Pi \to \Delta}{\Gamma, B, A, \Pi \to \Delta} \text{ (exchange : left)} \qquad \frac{\Gamma \to \Delta, A, B, \Lambda}{\Gamma \to \Delta, B, A, \Lambda} \text{ (exchange : right)}$$

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda} \text{ (cut)}$$

$$\frac{A_i, \Gamma \to \Delta}{A_1 \wedge A_2, \Gamma \to \Delta} \ (\wedge : \text{left}) \quad (i = 1, 2) \qquad \frac{\Gamma \to \Delta, A_1 \quad \Gamma \to \Delta, A_2}{\Gamma \to \Delta, A_1 \wedge A_2} \ (\wedge : \text{right})$$

$$\frac{A_1, \Gamma \to \Delta \quad A_2, \Gamma \to \Delta}{A_1 \vee A_2, \Gamma \to \Delta} \ (\vee : \text{left}) \qquad \frac{\Gamma \to \Delta, A_i}{\Gamma \to \Delta, A_1 \vee A_2} \ (\vee : \text{right}) \quad (i = 1, 2)$$

$$\frac{\Delta_1, \Sigma, A \to B, \Gamma_1 \quad \Delta_2, \Sigma, A \to B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \to B, \Gamma_{2^n}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B} \ (\supset)$$

where $n \geq 0$, $\Gamma_i = \{C_j \mid j \in \gamma(i)\}$, $\Delta_i = \{D_j \mid j \in \delta(i)\}$, and the sets $\gamma(i)$ and $\delta(i)$ of natural numbers are defined as follows: $\delta(i)$ runs through the subsets of $\{1, ..., n\}$ ordered according to size and $\gamma(i) = \{1, ..., n\} \setminus \delta(i)$.

For example, when $n = 0, 1, 2$, the rule $(\supset)$ is of the forms

$$\frac{\Sigma, A \to B}{\Sigma \to A \supset B} \ (\supset),$$

$$\frac{\Sigma, A \to B, C_1 \quad D_1, \Sigma, A \to B}{\Sigma, C_1 \supset D_1 \to A \supset B} \ (\supset),$$

and

$$\frac{\Sigma, A \to B, C_1, C_2 \quad D_1, \Sigma, A \to B, C_2 \quad D_2, \Sigma, A \to B, C_1 \quad D_1, D_2, \Sigma, A \to B}{\Sigma, C_1 \supset D_1, C_2 \supset D_2 \to A \supset B} \ (\supset),$$

respectively.

The formulae $C_1 \supset D_1, \ldots, C_n \supset D_n$ and $A \supset B$ in the rule $(\supset)$ are called the *principal formulae* of this rule. As for the other rules, the notion is defined in the usual way. Note that it allows more than one formulae in the succedent of a sequent, although BPL is a logic weaker than intuitionistic propositional logic for which the Gentzen **LJ** does not allow them.

Let $\mathbf{LBP}^-$ be a system obtained from **LBP** by deleting (cut).

Now, we present the followings without proofs. For proofs, see [18].

Theorem 2.2.1 (Soundness and Completeness) *For every sequent $\Gamma \to \Delta$, we have that $\Gamma \to \Delta$ is provable in $\mathbf{LBP}^-$ if and only if $\Gamma \to \Delta$ is true in any model.*

Corollary 2.2.2 (cut) *is admissible in $\mathbf{LBP}^-$.*

2.3 A sequent calculus for FPL

Next we introduce a sequent calculus for FPL, which is called **LFP**.

The sequent calculus **LFP** is obtained from **LBP** by providing the antecedent of each upper sequent in the rule $(\supset)$ with the formula $A \supset B$. Then the rule $(\supset)$ of **LFP** is of the form

$$\frac{\Delta_1, \Sigma, A \supset B, A \to B, \Gamma_1 \quad \Delta_2, \Sigma, A \supset B, A \to B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \supset B, A \to B, \Gamma_{2^n}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B} \ (\supset)$$

where $n \geq 0$, and Γ_i and Δ_i are as in the rule ($\supset$) of **LBP**. This rule extends the rule ($\supset$) of **LBP** which is derivable from this rule and (weakening:left).

Let $\mathbf{LFP}^-$ be a system obtained from **LFP** by deleting (cut).

An *irreflexive model* is a model $\langle W, R, V \rangle$ in which R is irreflexive, i.e., there is no $x \in W$ such that xRx.

Then, as same as $\mathbf{LBP}^-$, the followings hold.

Theorem 2.3.1 (Soundness and Completeness) *For every sequent* $\Gamma \to \Delta$*, we have that* $\Gamma \to \Delta$ *is provable in* $\mathbf{LFP}^-$ *if and only if* $\Gamma \to \Delta$ *is true in any finite irreflexive model.*

Corollary 2.3.2 (cut) *is admissible in* $\mathbf{LFP}^-$*.*

Remark. By Corollaries 2.2.2 and 2.3.2, both **LBP** and **LFP** enjoy the cut-elimination theorems. However the proofs are semantical. Then it is natural to ask whether these can be proved by syntactical method or not. The answer for this is actually "yes". In the next section, we present syntactical proofs of the cut-eliminations of **LBP** and **LFP**.

2.4 Cut-elimination theorems

In this section, we prove the cut-elimination theorems for **LBP** and **LFP** following Gentzen's method (see, e.g., Takeuti [33]).

2.4.1 Preliminaries

Our aim is to prove the following theorems:

Theorem 2.4.1 *If a sequent is provable in* **LBP***, then it is provable in* **LBP** *without using the cut rule.*

Theorem 2.4.2 *If a sequent is provable in* **LFP***, then it is provable in* **LFP** *without using the cut rule.*

In order to prove these, we introduce as usual the mix rule as follows:

$$\frac{\Gamma \to \Delta \quad \Pi \to \Lambda}{\Gamma, \Pi^A \to \Delta^A, \Lambda}$$

where both Δ and Π contain the formula A, and Δ^A and Π^A are obtained from Δ and Π respectively by deleting all the occurrences of A in them. A is called the *mix formula* of this inference. The systems $\mathbf{LBP}^*$ and $\mathbf{LFP}^*$ are obtained from **LBP** and **LFP** respectively by replacing the cut rule by the mix rule.

Then in order to prove Theorems 2.4.1 and 2.4.2, it suffices to show the following lemmata:

Lemma 2.4.1 *If P is a proof of a sequent S in* $\mathbf{LBP}^*$ *which contains only one mix occurring as the last inference, then we can transform P into a proof of S in* $\mathbf{LBP}^*$ *in which no mix occurs.*

Lemma 2.4.2 *If P is a proof of a sequent S in* $\mathbf{LFP}^*$ *which contains only one mix occurring as the last inference, then we can transform P into a proof of S in* $\mathbf{LFP}^*$ *in which no mix occurs.*

A proof which contains no mix is called a *mix-free* proof.

Before proving the above lemmata, we need some auxiliary definitions.
Let P be a proof in $\mathbf{LBP}^*$ or $\mathbf{LFP}^*$ and let E be an occurrence of a formula in P. The *direct ancestors* of E are defined inductively as follows:

1. E is a direct ancestor of itself.

2. If a direct ancestor E_1 of E is in the lower sequent of a contraction rule in P as the principal formula, e.g.
$$\frac{E_1, E_1, \Gamma \to \Delta}{E_1, \Gamma \to \Delta}$$
then the two E_1's in the upper sequent are direct ancestors of E.

3. If a direct ancestor E_1 of E is in the lower sequent of an exchange rule in P as one of the principal formulas, e.g.
$$\frac{\Gamma, E_1, E_2, \Sigma \to \Delta}{\Gamma, E_2, E_1, \Sigma \to \Delta}$$
then the E_1 in the upper sequent is a direct ancestor of E.

4. If a direct ancestor E_1 of E is in the lower sequent of an inference in P not as one of the principal formulas, e.g., if E_1 is the k-th occurrence of Σ in the lower sequent of a rule ($\supset$)
$$\frac{\Delta_1, \Sigma, A \supset B, A \to B, \Gamma_1 \quad \Delta_2, \Sigma, A \supset B, A \to B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \supset B, A \to B, \Gamma_{2^n}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B}$$
then the k-th occurrence of Σ in each upper sequent is a direct ancestor of E.

The *grade* of a formula A, denoted by $g(A)$, is defined inductively as follows:

1. $g(p) = 0$ for each propositional variable p,

2. $g(\perp) = 0$,

3. $g(A \supset B) = g(A \wedge B) = g(A \vee B) = g(A) + g(B) + 1$.

The grade of a mix is the grade of the mix formula. When a proof P has a mix only as the last inference, we define the grade of P, denoted by $g(P)$, to be the grade of this mix.

Let P be a proof in $\mathbf{LBP}^*$ or $\mathbf{LFP}^*$ which has a mix only as the last inference and P_1 and P_2 be the subproofs of P whose end-sequents are the left upper sequent and the right upper sequent of the mix respectively. We define the *rank* of a sequent S contained in P, denoted by $r(S)$, as follows:

1. S is contained in P_1.
   (a) If the succedent of S contains no direct ancestor of the occurrences of the mix formula, then $r(S) = 0$.
   (b) Otherwise, if S is an initial sequent, then $r(S) = 1$, if S is the lower sequent of an inference whose upper sequents are $S_1, ..., S_n$, then $r(S) = \max\{r(S_1), ..., r(S_n)\} + 1$.
2. S is contained in P_2.
   (a) If the antecedent of S contains no direct ancestor of the occurrences of the mix formula, then $r(S) = 0$.
   (b) Otherwise, similar to 1.(b) above.

Let P be a proof in $\mathbf{LBP}^*$ or $\mathbf{LFP}^*$ which has a mix only as the last inference and S_1 (S_2) be the left (the right) upper sequent of this mix. We define $r_l(P) = r(S_1)$ and $r_r(P) = r(S_2)$. The *rank* of P, denoted by $r(P)$, is defined as $r_l(P) + r_r(P)$.

2.4.2 Proof for LBP

In this subsection, we prove Lemma 2.4.1. One of the difficulties in proving this lemma by means of the usual technique of cut-elimination is caused by the rule ($\supset$), which only restricts the succedent of the lower sequent to one formula. The same difficulty arises in proof of cut-elimination even for a version of sequent calculus for intuitionistic logic with similar restriction. To overcome this difficulty, we first consider a special case which is proved in the same way as the cut-elimination theorem for **LK** (and so we omit the proof).

Lemma 2.4.3 *Let P be a proof of a sequent S in $\mathbf{LBP}^*$ which contains only one mix occurring as the last inference and let P_1 be the subproof of P whose end-sequent is the left upper sequent of this mix. If P_1 contains no ($\supset$) and if the succedent of each sequent occurring in P_1 consists of at most one formula, then S is provable in $\mathbf{LBP}^*$ with no mix.*

This lemma is used in the subcase (2-2-1)(c) of the following proof.

Proof of Lemma 2.4.1. We prove the lemma by transfinite induction on $\omega \cdot g(P) + r(P)$.

Case1: $r(P) = 2$.
We treat the case where each upper sequent of the mix is the lower sequent of $(\supset)$. For simplicity, we write

$$\dfrac{\begin{matrix} \vdots\ Q_k \\ [\Delta_k, \Sigma, A \to B, \Gamma_k]_{1 \le k \le 2^n} \end{matrix}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B}$$

instead of

$$\dfrac{\begin{matrix} \vdots\ Q_1 \\ \Delta_1, \Sigma, A \to B, \Gamma_1 \end{matrix} \quad \begin{matrix} \vdots\ Q_2 \\ \Delta_2, \Sigma, A \to B, \Gamma_2 \end{matrix} \quad \cdots \quad \begin{matrix} \vdots\ Q_{2^n} \\ \Delta_{2^n}, \Sigma, A \to B, \Gamma_{2^n} \end{matrix}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B}$$

We also remark that if $C_i \supset D_i$ is identical with $C_j \supset D_j$ for $i \neq j$ then the lower sequent (without $C_j \supset D_j$) is derivable only from the upper sequents $\Delta_k, \Sigma, A \to B, \Gamma_k$ such that either $C_i, C_j \in \Gamma_k$ or $D_i, D_j \in \Delta_k$. Modified so that the mix formula may appear just once in the antecedent of the right upper sequent of the mix, the last part of P is as follows:

$$\dfrac{\dfrac{\begin{matrix} \vdots\ Q'_l \\ [\Delta'_l, \Sigma', C_i \to D_i, \Gamma'_l]_{1 \le l \le 2^m} \end{matrix}}{\Sigma', E_1 \supset F_1, ..., E_m \supset F_m \to C_i \supset D_i} \quad \dfrac{\begin{matrix} \vdots\ Q_k \\ [\Delta_k, \Sigma, A \to B, \Gamma_k]_{1 \le k \le 2^n} \end{matrix}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B}}{\Sigma', E_1 \supset F_1, ..., E_m \supset F_m, \Sigma, C_1 \supset D_1, ..., C_{i-1} \supset D_{i-1}, C_{i+1} \supset D_{i+1}, ..., C_n \supset D_n \to A \supset B}$$

Now take any l and k such that $1 \le l \le 2^m$, $1 \le k \le 2^n$ and C_i is contained in Γ_k. We consider the following proof P_1:

$$\dfrac{\begin{matrix} \vdots\ Q_k \\ \Delta_k, \Sigma, A \to B, \Gamma_k \end{matrix} \quad \begin{matrix} \vdots\ Q'_l \\ \Delta'_l, \Sigma', C_i \to D_i, \Gamma'_l \end{matrix}}{\Delta_k, \Sigma, A, (\Delta'_l)^{C_i}, (\Sigma')^{C_i} \to B^{C_i}, (\Gamma_k)^{C_i}, D_i, \Gamma'_l} \text{ (mix)}$$

Since $g(P_1) < g(P)$, we can obtain a mix-free proof P_2 of $\Delta_k, \Sigma, A, (\Delta'_l)^{C_i}, (\Sigma')^{C_i} \to B^{C_i}, (\Gamma_k)^{C_i}, D_i, \Gamma'_l$ by the induction hypothesis.

Next, let $\Gamma_{k'}$ be the sequence obtained from Γ_k by deleting C_i. Then D_i is contained in $\Delta_{k'}$, and we consider the following proof P_3:

$$\dfrac{\begin{matrix} \vdots\ P_2 \\ \Delta_k, \Sigma, A, (\Delta'_l)^{C_i}, (\Sigma')^{C_i} \to B^{C_i}, (\Gamma_k)^{C_i}, D_i, \Gamma'_l \end{matrix} \quad \begin{matrix} \vdots\ Q_{k'} \\ \Delta_{k'}, \Sigma, A \to B, \Gamma_{k'} \end{matrix}}{\Delta_k, \Sigma, A, (\Delta'_l)^{C_i}, (\Sigma')^{C_i}, (\Delta_{k'})^{D_i}, \Sigma^{D_i}, A^{D_i} \to (B^{C_i})^{D_i}, ((\Gamma_k)^{C_i})^{D_i}, (\Gamma'_l)^{D_i}, B, \Gamma_{k'}} \text{ (mix)}$$

Since $g(P_3) < g(P)$, we can eliminate the above mix by the induction hypothesis. Noticing that $(\Gamma_k)^{C_i}$ is identical with or a subsequence of $\Gamma_{k'}$ and that $(\Delta_{k'})^{D_i}$ is identical with or a subsequence of Δ_k, we obtain a mix-free proof ending with $\Delta_k, \Sigma, A, (\Delta'_l)^{C_i}, (\Sigma')^{C_i}, \Delta_k, \Sigma^{D_i}, A^{D_i} \to (B^{C_i})^{D_i}, (\Gamma_{k'})^{D_i}, (\Gamma'_l)^{D_i}, B, \Gamma_{k'}$ and hence obtain a mix-free proof of $\Delta_k, \Delta'_l, \Sigma, \Sigma', A \to B, \Gamma_{k'}, \Gamma'_l$. This holds for any l and k such that $1 \le l \le 2^m$, $1 \le k \le 2^n$ and C_i is contained in Γ_k.

Now we set $C_{n+h} = E_h$ and $D_{n+h} = F_h$ for all h such that $1 \leq h \leq m$. Let $\langle[q_1, ..., q_t], [q_{t+1}, ..., q_{n+m-1}]\rangle$ be the q-th division of $[1, ..., i-1, i+1, ..., n, n+1, ..., n+m]$ (cf. the definition of **LBP** in Section 2.2). Then there exists a (k, l) considered above such that $\Delta_k, \Delta'_l = D_{q_1}, ..., D_{q_t} (= \Delta^*_q)$ and $\Gamma_{k'}, \Gamma'_l = C_{q_{t+1}}, ..., C_{q_{n+m-1}} (= \Gamma^*_q)$. Therefore we can construct a required proof as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots & \vdots & & \vdots \\ \Delta^*_1, \Sigma, \Sigma', A \to B, \Gamma^*_1 & \Delta^*_2, \Sigma, \Sigma', A \to B, \Gamma^*_2 & \cdots & \Delta^*_{2^{n+m-1}}, \Sigma, \Sigma', A \to B, \Gamma^*_{2^{n+m-1}}\end{matrix}}{\Sigma, \Sigma', C_1 \supset D_1, ..., C_{i-1} \supset D_{i-1}, C_{i+1} \supset D_{i+1}, ..., C_n \supset D_n, E_1 \supset F_1, ..., E_m \supset F_m \to A \supset B}\,(\supset)}{\Sigma', E_1 \supset F_1, ..., E_m \supset F_m, \Sigma, C_1 \supset D_1, ..., C_{i-1} \supset D_{i-1}, C_{i+1} \supset D_{i+1}, ..., C_n \supset D_n \to A \supset B}\,\text{(some exchanges)}$$

Case2: $r(P) > 2$.
(2-1) $r_l(P) > 1$.
The last part of P is as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Phi \to \Psi\end{matrix}}{\Gamma \to \Delta}\,I \qquad \begin{matrix}\vdots \\ \Pi \to \Lambda\end{matrix}}{\Gamma, \Pi^A \to \Delta^A, \Lambda}$$

Since $r_l(P) > 1$, the inference I can not be $(\supset)$. Then the proof is carried out in the same way as that for **LK**.
(2-2) $r_l(P) = 1$ and $r_r(P) > 1$.
(2-2-1) The right upper sequent of the mix is the lower sequent of either a logical inference whose principal formulae contain no A or a structural inference. The last part of P is as follows:

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \to \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\vdots \\ \Phi \to \Psi\end{matrix}}{\Pi \to \Lambda}\,I}{\Gamma, \Pi^A \to \Delta^A, \Lambda}$$

We treat the case where I is $(\supset)$.
(a) $\Gamma \to \Delta$ is an initial sequent or the lower sequent of a weakening rule.
In this case, the claim is easy to see and we omit the detail.
(b) $\Gamma \to \Delta$ is the lower sequent of $(\supset)$.
Since the succedent Δ consists of one formula A, the last part of P is as follows:

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Gamma \to A\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\, R_1 & \vdots\, R_2 & & \vdots\, R_{2^n} \\ \Delta_1, \Sigma, B \to C, \Gamma_1 & \Delta_2, \Sigma, B \to C, \Gamma_2 & \cdots & \Delta_{2^n}, \Sigma, B \to C, \Gamma_{2^n}\end{matrix}}{\Pi \to B \supset C}\,I}{\Gamma, \Pi^A \to B \supset C}$$

Consider the following proof P_k:

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Gamma \to A\end{matrix} \qquad \begin{matrix}\vdots\, R_k \\ \Delta_k, \Sigma, B \to C, \Gamma_k\end{matrix}}{\Gamma, (\Delta_k)^A, \Sigma^A, B^A \to C, \Gamma_k}\,\text{(mix)}$$

Since $r(P_k) < r(P)$, we can eliminate the above mix by the induction hypothesis. Therefore for all k such that $1 \leq k \leq 2^n$, we can obtain mix-free proofs P'_k ending with $\Delta_k, \Gamma, \Sigma^A, B \to C, \Gamma_k$. Noticing that no A is in the principal formulae of I, we can construct a mix-free proof of $\Gamma, \Pi^A \to B \supset C$ as follows:

$$\dfrac{\begin{array}{c}\vdots\, P'_1 \\ \Delta_1, \Gamma, \Sigma^A, B \to C, \Gamma_1\end{array} \quad \begin{array}{c}\vdots\, P'_2 \\ \Delta_2, \Gamma, \Sigma^A, B \to C, \Gamma_2\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, P'_{2^n} \\ \Delta_{2^n}, \Gamma, \Sigma^A, B \to C, \Gamma_{2^n}\end{array}}{\Gamma, \Pi^A \to B \supset C}\ (\supset)$$

(c) $\Gamma \to \Delta$ is the lower sequent of ($\wedge$: right) or ($\vee$: right).
We treat the case of ($\wedge$: right). The last part of P is as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, Q_1 \\ \Gamma \to \Delta', B\end{array} \quad \begin{array}{c}\vdots\, Q_2 \\ \Gamma \to \Delta', C\end{array}}{\Gamma \to \Delta', B \wedge C} \quad \begin{array}{c}\vdots\, R \\ \Pi \to \Lambda\end{array}}{\Gamma, \Pi^{B\wedge C} \to \Delta', \Lambda}$$

Consider the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{C, B \to B}}{B, C \to B} \quad \dfrac{C \to C}{B, C \to B}}{B, C \to B \wedge C} \quad \begin{array}{c}\vdots\, R \\ \Pi \to \Lambda\end{array}}{B, C, \Pi^{B\wedge C} \to \Lambda}$$

By applying Lemma 2.4.3 to the above proof, we obtain a mix-free proof P_1 ending with $B, C, \Pi^{B\wedge C} \to \Lambda$. Then consider the following proof P_2:

$$\dfrac{\begin{array}{c}\vdots\, Q_1 \\ \Gamma \to \Delta', B\end{array} \quad \begin{array}{c}\vdots\, P_1 \\ B, C, \Pi^{B\wedge C} \to \Lambda\end{array}}{\Gamma, C^B, (\Pi^{B\wedge C})^B \to (\Delta')^B, \Lambda}\ (\text{mix})$$

Since $g(P_2) < g(P)$, we can eliminate the above mix by the induction hypothesis, and obtain a mix-free proof P_3 ending with $\Gamma, C, \Pi^{B\wedge C} \to \Delta', \Lambda$.

Next, consider the following proof P_4:

$$\dfrac{\begin{array}{c}\vdots\, Q_2 \\ \Gamma \to \Delta', C\end{array} \quad \begin{array}{c}\vdots\, P_3 \\ \Gamma, C, \Pi^{B\wedge C} \to \Delta', \Lambda\end{array}}{\Gamma, \Gamma^C, (\Pi^{B\wedge C})^C \to (\Delta')^C, \Delta', \Lambda}\ (\text{mix})$$

Since $g(P_4) < g(P)$, we can obtain a mix-free proof ending with $\Gamma, \Gamma^C, (\Pi^{B\wedge C})^C \to (\Delta')^C, \Delta', \Lambda$ by the induction hypothesis, and hence obtain a mix-free proof of $\Gamma, \Pi^{B\wedge C} \to \Delta', \Lambda$.
(2-2-2) The right upper sequent of the mix is the lower sequent of a logical inference whose principal formulae contain the mix formula A. This case is treated similarly to that of the cut-elimination theorem for **LK**.

This completes the proof of Lemma 2.4.1, and hence of the cut-elimination theorem of **LBP** (Theorem 2.4.1). □

2.4.3 Proof for LFP

In this subsection, we prove Lemma 2.4.2. We assume here that the initial sequents of the form $A \to A$ are restricted to the form such that A is a propositional variable. It is easy to see that the resulting system is equivalent to the original one. Hence we call this also **LFP***.

Let P be a proof in **LFP*** which has a rule $(\supset)$

$$\frac{\Delta_1, \Sigma, A \supset B, A \to B, \Gamma_1 \quad \Delta_2, \Sigma, A \supset B, A \to B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \supset B, A \to B, \Gamma_{2^n}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B}$$

The $(A \supset B)$'s in the upper sequents are called the *diagonal formulae* of this inference. We define the *width* of this inference I to be the number of the inferences I' whose principal formulae contain a direct ancestor of the diagonal formulae of I.

Let P be a proof in **LFP*** which has a mix only as the last inference. The *width* of P, denoted by $w(P)$, is defined as follows:

1. The case where the mix formula is of the form $A \supset B$.
   Let P' be the subproof of P whose end-sequent is the left upper sequent of the mix. Then $w(P)$ is the sum of the width of all lowermost $(\supset)$'s in P'.

2. Otherwise, $w(P) = 0$.

Now we prove Lemma 2.4.2. The following technique is an analogue of that used in [34].

Proof of Lemma 2.4.2. We prove the lemma by transfinite induction on $\omega^2 \cdot g(P) + \omega \cdot w(P) + r(P)$. Here we treat only the case where $r(P) = 2$ and each upper sequent of the mix is the lower sequent of $(\supset)$. With the modification remarked in the proof of Lemma 2.4.1, the last part of P is as follows:

$$\frac{\dfrac{\begin{matrix} \vdots\ Q'_l \\ [\Delta'_l, \Sigma', C_i \supset D_i, C_i \to D_i, \Gamma'_l]_{1 \le l \le 2^m} \end{matrix}}{\Sigma', E_1 \supset F_1, ..., E_m \supset F_m \to C_i \supset D_i}\ I \quad \dfrac{\begin{matrix} \vdots\ Q_k \\ [\Delta_k, \Sigma, A \supset B, A \to B, \Gamma_k]_{1 \le k \le 2^n} \end{matrix}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \to A \supset B}}{\Sigma', E_1 \supset F_1, ..., E_m \supset F_m, \Sigma, C_1 \supset D_1, ..., C_{i-1} \supset D_{i-1}, C_{i+1} \supset D_{i+1}, ..., C_n, \supset D_n \to A \supset B}$$

Case1: $w(P) = 0$.
In this case, each topmost direct ancestor of the diagonal formulae $C_i \supset D_i$ of I is the principal formula of a weakening rule (note that all the initial sequents of the form $A \to A$ of our system are of the form $p \to p$ for propositional variable p). Hence by deleting the diagonal formulae $C_i \supset D_i$ as well as all direct ancestors of them from P and some trivial modifications, we obtain proofs ending with $[\Delta'_l, \Sigma', C_i \to D_i, \Gamma'_l]_{1 \le l \le 2^m}$. Then we proceed in the same way as Case1 in the proof of Lemma 2.4.1.
Case2: $w(P) > 0$.
Let P_1 be the subproof of P whose end-sequent is the left upper sequent of the mix. In

this case, there exists an inference I' whose principal formulae contain a direct ancestor of the diagonal formulae $C_i \supset D_i$ of I. Then P_1 looks like this:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ R_j \\ [\Delta_j^*, \Sigma^*, G \supset H, G \rightarrow H, \Gamma_j^*]_{1 \leq j \leq 2^r}\end{array}}{\begin{array}{c}\Sigma^*, J_1 \supset K_1, ...(C_i \supset D_i)..., J_r \supset K_r \rightarrow G \supset H \\ \vdots\ Q'_{l'} \\ \cdots \quad \Delta'_{l'}, \Sigma', C_i \supset D_i, C_i \rightarrow D_i, \Gamma'_{l'} \quad \cdots\end{array}}\ I'}{\Sigma', E_1 \supset F_1, ..., E_m \supset F_m \rightarrow C_i \supset D_i}\ I$$

where $C_i \supset D_i$ appears in $J_1 \supset K_1, ..., J_r \supset K_r$, which are all distinct with the modification remarked in the proof of Lemma 2.4.1. We transform P_1 as follows:

1. Delete the part over $\Sigma^*, J_1 \supset K_1, ..., J_r \supset K_r \rightarrow G \supset H$.
2. Transform each remaining sequent $\Pi \rightarrow \Lambda$ to $G \supset H, \Pi \rightarrow \Lambda$.

In the figure obtained in this way, each inference is correct, and each topmost sequent is provable using structural inferences only. Therefore from this figure, we can obtain a mix-free proof P_2 ending with $G \supset H, \Sigma', E_1 \supset F_1, ..., E_m \supset F_m \rightarrow C_i \supset D_i$.

Now take any l such that $1 \leq l \leq 2^m$, and consider the following proof P'_l:

$$\dfrac{\begin{array}{c}\vdots\ P_2 \\ G \supset H, \Sigma', E_1 \supset F_1, ..., E_m \supset F_m \rightarrow C_i \supset D_i\end{array} \quad \begin{array}{c}\vdots\ Q'_l \\ \Delta'_l, \Sigma', C_i \supset D_i, C_i \rightarrow D_i, \Gamma'_l\end{array}}{G \supset H, \Sigma', E_1 \supset F_1, ..., E_m \supset F_m, (\Delta'_l)^{C_i \supset D_i}, (\Sigma')^{C_i \supset D_i}, C_i, \rightarrow D_i, \Gamma'_l}\ (\text{mix})$$

Since $w(P'_l) < w(P)$, we can obtain a mix-free proof R'_l of this end-sequent. Furthermore take any j with $1 \leq j \leq 2^r$ such that C_i is contained in Γ_j^*, and by the same argument as Case1 in the proof of Lemma 2.4.1 with R_j and R'_l instead of Q_k and Q'_l there, we finally obtain a mix-free proof R ending with

$$\Sigma', E_1 \supset F_1, ..., E_m \supset F_m, \Sigma^*, (J_1 \supset K_1, ..., J_r \supset K_r)^{C_i \supset D_i} \rightarrow G \supset H.$$

Next, we again transform P_1 as follows:

1. Delete the part over $\Sigma^*, J_1 \supset K_1, ..., J_r \supset K_r \rightarrow G \supset H$.
2. Transform each remaining sequent $\Pi \rightarrow \Lambda$ to $\Sigma', E_1 \supset F_1, ..., E_m \supset F_m, \Pi \rightarrow \Lambda$.
3. Put on the deleted part the mix-free proof R followed by a weakening rule (and some exchange rules) whose principal formula is $C_i \supset D_i$.

Since each topmost sequent not in R is provable by structural inferences, we can obtain a mix-free proof P_3 ending with $\Sigma', E_1 \supset F_1, ..., E_m \supset F_m \rightarrow C_i \supset D_i$.

Finally, consider the following proof P_4:

$$\dfrac{\begin{array}{c}\vdots\ P_3 \\ \Sigma', E_1 \supset F_1, ..., E_m \supset F_m \rightarrow C_i \supset D_i\end{array} \quad \dfrac{\begin{array}{c}\vdots\ Q_k \\ [\Delta_k, \Sigma, A \supset B, A \rightarrow B, \Gamma_k]_{1 \leq k \leq 2^n}\end{array}}{\Sigma, C_1 \supset D_1, ..., C_n \supset D_n \rightarrow A \supset B}}{\Sigma', E_1 \supset F_1, ..., E_m \supset F_m, \Sigma, C_1 \supset D_1, ..., C_{i-1} \supset D_{i-1}, C_{i+1} \supset D_{i+1}, ..., C_n, \supset D_n \rightarrow A \supset B}$$

It is easy to see that $w(P_4) < w(P)$. Therefore we can eliminate the above mix by the induction hypothesis, and obtain a required mix-free proof.

This completes the proof of Lemma 2.4.2, and hence of the cut-elimination theorem of **LFP** (Theorem 2.4.2). □

Chapter 3

Strong normalization theorem for NK (I) – Survey of Stålmarck's result

In this and next chapters, we shall discuss the strong normalization theorem for the natural deduction system for the first-order classical logic.

For the first-order classical natural deduction with full logical symbols, Stålmarck already proved the strong normalization theorem. However this result is not satisfactory. Actually, his reduction procedure does not satisfy the Church-Rosser property. In this chapter, we survey of Stålmarck's result and explain the failure of Church-Rosser property with respect to his reduction procedure.

3.1 System NK

The natural deduction system for which we prove the strong normalization theorem is the one defined in Chapter 1. For reader's convenience, we present again the inference rules of this system. In the following, A, A_1, A_2, B, etc. denote formulae and $\neg A$ is considered as an abbreviation for $A \supset \bot$ and hence the rules for $\neg$ are omitted.

$$\frac{A_1 \quad A_2}{A_1 \wedge A_2}\ (\wedge I) \qquad\qquad \frac{A_1 \wedge A_2}{A_i}\ (\wedge E) \quad (i = 1, 2)$$

$$\frac{A_i}{A_1 \vee A_2}\ (\vee I) \quad (i = 1, 2) \qquad\qquad \frac{A_1 \vee A_2 \quad \begin{array}{c}[A_1]\\ \vdots \\ B\end{array} \quad \begin{array}{c}[A_2]\\ \vdots \\ B\end{array}}{B}\ (\vee E)$$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \supset B}\ (\supset I) \qquad\qquad \frac{A \supset B \quad A}{B}\ (\supset E)$$

$$\frac{A(a)}{\forall x A(x)}\ (\forall I) \qquad\qquad \frac{\forall x A(x)}{A(t)}\ (\forall E)$$

$$\frac{A(t)}{\exists x A(x)}\ (\exists I) \qquad\qquad \frac{\exists x A(x) \quad \begin{array}{c}[A(a)]\\ \vdots \\ B\end{array}}{B}\ (\exists E)$$

$$\frac{\begin{array}{c}[\neg A]\\ \vdots \\ \bot\end{array}}{A}\ (\mathrm{RAA})$$

In $(\forall E)$ and $(\exists I)$, t is an arbitrary term. Furthermore $(\forall I)$ and $(\exists E)$ are subject to the restriction of eigenvariable.

Remark. As usual, we can assume that the eigenvariables must be separated in a proof.

Before defining the reduction procedure, we need a notion of maximum formulae.

Let A be a formula-occurrence in a proof $\mathcal{D}$. A is called a *maximum formula* in $\mathcal{D}$ if A satisfies the following conditions:

1. A is a conclusion of an introduction rule, $(\vee E)$, $(\exists E)$ or (RAA);

2. A is the major premise of an elimination rule.

A proof is said to be *normal* if it contains no maximum formula.

3.2 Stålmarck's reduction procedure

In this section, we present the reduction procedure due to Stålmarck [30].

Let C be a maximum formula in a proof which is the conclusion of a rule R. The reduction at C is defined as follows.

1. R is $(\wedge I)$ and C is $A_1 \wedge A_2$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ A_1\end{matrix}\qquad\begin{matrix}\vdots\,\mathcal{D}_2\\ A_2\end{matrix}}{A_1\wedge A_2}\,R}{A_i}\qquad\Rightarrow\qquad\begin{matrix}\vdots\,\mathcal{D}_i\\ A_i\end{matrix}$$

2. R is $(\vee I)$ and C is $A_1 \vee A_2$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ A_i\end{matrix}}{A_1\vee A_2}\,R\qquad\begin{matrix}[A_1]\\ \vdots\,\mathcal{E}_1\\ B\end{matrix}\qquad\begin{matrix}[A_2]\\ \vdots\,\mathcal{E}_2\\ B\end{matrix}}{B}\qquad\Rightarrow\qquad\begin{matrix}\vdots\,\mathcal{D}\\ A_i\\ \vdots\,\mathcal{E}_i\\ B\end{matrix}$$

3. R is $(\supset I)$ and C is $A \supset B$:

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\,\mathcal{D}\\ B\end{matrix}}{A\supset B}\,R\qquad\begin{matrix}\vdots\,\mathcal{E}\\ A\end{matrix}}{B}\qquad\Rightarrow\qquad\begin{matrix}\vdots\,\mathcal{E}\\ A\\ \vdots\,\mathcal{D}\\ B\end{matrix}$$

4. R is $(\forall I)$ and C is $\forall x A(x)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}(a)\\ A(a)\end{matrix}}{\forall x A(x)}\,R}{A(t)}\qquad\Rightarrow\qquad\begin{matrix}\vdots\,\mathcal{D}(t)\\ A(t)\end{matrix}$$

5. R is $(\exists I)$ and C is $\exists x A(x)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ A(t)\end{matrix}}{\exists x A(x)}\,R\qquad\begin{matrix}[A(a)]\\ \vdots\,\mathcal{E}(a)\\ B\end{matrix}}{B}\qquad\Rightarrow\qquad\begin{matrix}\vdots\,\mathcal{D}\\ A(t)\\ \vdots\,\mathcal{E}(t)\\ B\end{matrix}$$

6. R is $(\vee E)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ A_1\vee A_2\end{matrix}\quad\begin{matrix}[A_1]\\ \vdots\,\mathcal{D}_2\\ C\end{matrix}\quad\begin{matrix}[A_2]\\ \vdots\,\mathcal{D}_3\\ C\end{matrix}}{C}\,R\qquad\mathcal{E}_1\quad\mathcal{E}_2}{B}\,R'\qquad\Rightarrow\qquad\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ A_1\vee A_2\end{matrix}\quad\dfrac{\begin{matrix}[A_1]\\ \vdots\,\mathcal{D}_2\\ C\end{matrix}\quad\mathcal{E}_1\quad\mathcal{E}_2}{B}\quad\dfrac{\begin{matrix}[A_2]\\ \vdots\,\mathcal{D}_3\\ C\end{matrix}\quad\mathcal{E}_1\quad\mathcal{E}_2}{B}}{B}$$

where the subproof

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ A_1 \vee A_2\end{array}\qquad \begin{array}{c}[A_1]\\ \vdots\,\mathcal{D}_2\\ C\end{array}\qquad \begin{array}{c}[A_2]\\ \vdots\,\mathcal{D}_3\\ C\end{array}}{C}\,R$$

has to be normal.

7. R is $(\exists E)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \exists x A(x)\end{array}\qquad \begin{array}{c}[A(a)]\\ \vdots\,\mathcal{D}_2\\ C\end{array}}{C}\,R \qquad \mathcal{E}_1 \quad \mathcal{E}_2}{B}\,R' \quad \Rightarrow \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \exists x A(x)\end{array}\qquad \dfrac{\begin{array}{c}[A(a)]\\ \vdots\,\mathcal{D}_2\\ C\end{array}\quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}}{B}$$

where the subproof

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \exists x A(x)\end{array}\qquad \begin{array}{c}[A(a)]\\ \vdots\,\mathcal{D}_2\\ C\end{array}}{C}\,R$$

has to be normal.

8. R is (RAA):

(a) C is $A_1 \wedge A_2$:

$$\dfrac{\dfrac{\begin{array}{c}[\neg(A_1 \wedge A_2)]\\ \vdots\,\mathcal{D}\\ \bot\end{array}}{A_1 \wedge A_2}\,R}{A_i} \quad \Rightarrow \quad \dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg A_i]^2 \qquad \dfrac{[A_1 \wedge A_2]^1}{A_i}}{\bot}}{\neg(A_1 \wedge A_2)}\,(\supset I)^1\\ \vdots\,\mathcal{D}\\ \bot\end{array}}{A_i}\,(\mathrm{RAA})^2$$

(b) C is $A \supset B$:

$$\dfrac{\dfrac{\begin{array}{c}[\neg(A \supset B)]\\ \vdots\,\mathcal{D}\\ \bot\end{array}}{A \supset B}\,R \qquad \begin{array}{c}\vdots\,\mathcal{E}\\ A\end{array}}{B} \quad \Rightarrow \quad \dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg B]^2 \qquad \dfrac{[A \supset B]^1 \quad \begin{array}{c}\vdots\,\mathcal{E}\\ A\end{array}}{B}}{\bot}}{\neg(A \supset B)}\,(\supset I)^1\\ \vdots\,\mathcal{D}\\ \bot\end{array}}{B}\,(\mathrm{RAA})^2$$

(c) C is $\forall x A(x)$:

$$
\dfrac{\dfrac{\begin{matrix}[\neg\forall x A(x)]\\ \vdots\ \mathcal{D}\\ \bot\end{matrix}}{\forall x A(x)}\ R}{A(t)}
\quad\Rightarrow\quad
\dfrac{\begin{matrix}\dfrac{\dfrac{[\neg A(t)]^2 \qquad \dfrac{[\forall x A(x)]^1}{A(t)}}{\bot}}{\neg\forall x A(x)}\ (\supset I)^1 \\ \vdots\ \mathcal{D} \\ \bot\end{matrix}}{A(t)}\ (\mathrm{RAA})^2
$$

(d) C is $A_1 \vee A_2$:

$$
\dfrac{\dfrac{\begin{matrix}[\neg(A_1\vee A_2)]\\ \vdots\ \mathcal{D}\\ \bot\end{matrix}}{A_1\vee A_2} \qquad \begin{matrix}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{matrix} \qquad \begin{matrix}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{matrix}}{B}
\quad\Rightarrow\quad
\dfrac{\begin{matrix}\dfrac{\dfrac{[A_1\vee A_2]^1 \qquad \dfrac{[\neg B]^2 \quad \begin{matrix}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{matrix}}{\bot} \qquad \dfrac{[\neg B]^2 \quad \begin{matrix}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{matrix}}{\bot}}{\bot}}{\neg(A_1\vee A_2)}\ (\supset)^1 \\ \vdots\ \mathcal{D} \\ \bot\end{matrix}}{B}\ (\mathrm{RAA})^2
$$

(e) C is $\exists x A(x)$:

$$
\dfrac{\dfrac{\begin{matrix}[\neg\exists x A(x)]\\ \vdots\ \mathcal{D}\\ \bot\end{matrix}}{\exists x A(x)} \qquad \begin{matrix}[A(a)]\\ \vdots\ \mathcal{E}\\ B\end{matrix}}{B}
\quad\Rightarrow\quad
\dfrac{\begin{matrix}\dfrac{\dfrac{[\exists x A(x)]^1 \qquad \dfrac{[\neg B]^2 \quad \begin{matrix}[A(a)]\\ \vdots\ \mathcal{E}\\ B\end{matrix}}{\bot}}{\bot}}{\neg\exists x A(x)}\ (\supset)^1 \\ \vdots\ \mathcal{D} \\ \bot\end{matrix}}{B}\ (\mathrm{RAA})^2
$$

9. In addition to the above, we also introduce reductions for “redundant application”:

$$
\dfrac{\neg\bot \qquad \begin{matrix}\vdots\ \mathcal{D}\\ \bot\end{matrix}}{\bot}
\quad\Rightarrow\quad
\begin{matrix}\vdots\ \mathcal{D}\\ \bot\end{matrix}
$$

and

$$
\dfrac{\begin{matrix}\vdots\ \mathcal{D}\\ \bot\end{matrix}}{\bot}\ (\mathrm{RAA})
\quad\Rightarrow\quad
\begin{matrix}\vdots\ \mathcal{D}\\ \bot\end{matrix}
$$

where no assumption is discharged by this (RAA).

We note that in clauses 6 and 7, both $\mathcal{E}_1$ and $\mathcal{E}_2$ are the proofs of the minor premises of R', if they exist. In the following, we shall not mention this kind of remarks each time.

If $\mathcal{D}'$ is obtained from $\mathcal{D}$ by the reduction at C, we also say "$\mathcal{D}'$ is obtained from $\mathcal{D}$ by reducing at C" and denoted by $\mathcal{D} \Rightarrow \mathcal{D}'$. As usual, $\overset{+}{\Rightarrow}$ denotes the transitive closure of $\Rightarrow$ and $\overset{*}{\Rightarrow}$ denotes the reflexive transitive closure of $\Rightarrow$. If $\mathcal{D} \overset{+}{\Rightarrow} \mathcal{D}'$, $\mathcal{D}'$ is called a *reduct* of $\mathcal{D}$. Especially, if $\mathcal{D} \Rightarrow \mathcal{D}'$, $\mathcal{D}'$ is called a *direct reduct* of $\mathcal{D}$.

Remark. In the author's opinion, the above reduction procedure is not satisfactory in the following sense.

First, in clauses 6 and 7, the normality of the subproof is required. However these restrictions are not desirable in establishing the strong normalization theorem. Furthermore, clauses 8 (d) and (e) seem to be unnatural. Actually, these cause the failure of the Church-Rosser property (see 3.4). However clauses 8 (d) and (e) are necessary in order to prove the strong normalization theorem. In the next section, we summarize this point.

3.3 Strong normalization theorem

In order to grasp the strong normalization theorem, we need the following definitions. A *reduction sequence* is a sequence $\mathcal{D}_0, \mathcal{D}_1, \ldots$ of proofs such that $\mathcal{D}_i \Rightarrow \mathcal{D}_{i+1}$ for all $i \geq 0$ and the last term in the sequence, if the sequence is finite, is normal.

A proof $\mathcal{D}$ is said to be *strongly normalizable* (abbreviated to SN) if each reduction sequence starting from $\mathcal{D}$ is terminates.

In [30], Stålmarck proved the following.

Theorem 3.3.1 *All proofs are strongly normalizable* (*with respect to the above reduction procedure*).

In order to prove this, we introduce the notion of the validity of proofs and rules.

A proof of the form

$$\frac{\mathcal{D}_1 \cdots \mathcal{D}_n}{A}\ R$$

is said to be in *I-form* if A is not an atomic formula and R is an introduction rule or (RAA).

First, we present the definition of the validity of proofs.

A proof $\mathcal{D}$ is said to be *valid* if one of the following conditions is satisfied:

1. $\mathcal{D}$ is in I-form of the form

$$\frac{\mathcal{D}_1 \cdots \mathcal{D}_n}{A}\ R$$

   and each $\mathcal{D}_i$ is valid and each proof of the form

$$\frac{\mathcal{D} \quad \mathcal{E}_1 \quad \mathcal{E}_2}{A}\ R'$$

   is valid where R' is an elimination rule, $\mathcal{E}_1$ and $\mathcal{E}_2$ are valid (if they exist) and A is $\perp$ if R' is $(\vee E)$ or $(\exists E)$.

2. $\mathcal{D}$ is not in I-form and each direct reduct of $\mathcal{D}$ is valid and if $\mathcal{D}$ is of the form

$$\dfrac{\begin{matrix} \\ \vdots \\ A_1 \vee A_2 \end{matrix} \quad \begin{matrix} [A_1] \\ \vdots\, \mathcal{E}_1 \\ B \end{matrix} \quad \begin{matrix} [A_2] \\ \vdots\, \mathcal{E}_2 \\ B \end{matrix}}{B}$$

or

$$\dfrac{\begin{matrix} \\ \vdots \\ \exists x A(x) \end{matrix} \quad \begin{matrix} [A(a)] \\ \vdots\, \mathcal{E}_1 \\ B \end{matrix}}{B}$$

then $\mathcal{E}_1$ and $\mathcal{E}_2$ are valid.

Then the following lemma is easily verified.

Lemma 3.3.1 *Let $\mathcal{D}$ be a valid proof. If $\mathcal{D} \Rightarrow \mathcal{D}'$, then $\mathcal{D}'$ is also valid.*

Next we define the validity of rules.

Let $\mathcal{D}$ be a proof and let $\mathcal{D}^*$ be a proof which is obtained from $\mathcal{D}^{**}$ by substituting an arbitrary valid proofs for open assumptions is valid where $\mathcal{D}^{**}$ is obtained from $\mathcal{D}$ by substituting an arbitrary terms for free variables which are not used as eigenvariables. For simplicity, we say this " $\mathcal{D}^*$ is obtained from $\mathcal{D}$ by substitution". A proof $\mathcal{D}$ is said to be *valid under substitution* if each proof $\mathcal{D}^*$ which is obtained from $\mathcal{D}$ by substitution is valid.

An inference rule R is said to be *valid* if each proof of the form

$$\frac{\mathcal{D}_1 \cdots \mathcal{D}_n}{A}\ R$$

is valid under substitution where each $\mathcal{D}_i$ is valid under substitution.

Remark. The notion of validity defined above is an analogue to the notion of *strong computability* in typed λ-calculus due to Tait [32]. In fact, this notion is used in order to prove the strong normalization theorem for typed λ-calculus. In the following, we summarize the outline of this proof.

λ-terms are either x, MN and $\lambda x.M$ where x is a variable, and M and N are λ-terms. *Types* are constructed from atomic types and a binary connection $\to$ in a usual way. *Typed λ-terms* are defined inductively as follows:

1. each x^α is a typed λ-term where x is a variable and α is a type;
2. if $M^{\alpha\to\beta}$ and N^α is typed λ-terms, then $(M^{\alpha\to\beta}N^\alpha)^\beta$ is a typed λ-term;
3. if M^β is a typed λ-term, then $(\lambda x^\alpha.M^\beta)^{\alpha\to\beta}$ is a typed λ-term.

β-reduction $\to_\beta$ is defined by

$$(\lambda x.M)N \to_\beta M[N/x]$$

where $M[N/x]$ is obtained from M by replacing all x in M by N. Then the following theorem holds.

Theorem 3.3.2 *All typed λ-terms are strongly normalizable with respect to β-reduction.*

For the proof of this, we introduce the notion of strong computablity of terms. *Strongly computable terms* (abbreviated SC terms) are defined inductively as follows:

1. a term of atomic type is SC if it is SN;
2. a term $M^{\alpha \to \beta}$ is SC if the term $(MN)^{\beta}$ is SC for each SC term N^{α}.

Then the proof of this theorem is carried out in two steps as follows.

1. Each SC term is SN.
2. Every term is SC.

By the help of the Curry-Howard isomorphism, one can easily understand the similarity between validity (for the implicational fragment) and strong computability.

Now we turn back to Stålmarck's result. The proof of the strong normalization theorem is carried out in three steps as follows.

1. Valid proofs are SN.
2. Proofs built of valid rules are valid.
3. All rules are valid.

Among the above, 1. and 2. are easy while the proof of 3. is difficult. In the following, we see that where one uses clauses 8(d) of the definition of the reduction procedure in the proof of 3.

In order to prove the validity of rules, we first show the validity of introduction rules and next show the validity of (RAA), and finally show the validity of elimination rules. In this proof, clause 8(d) is needed when we show that the validity of $(\vee E)$. In the following, we see the outline of this proof. For the details, see [30].

When we show that the validity of $(\vee E)$, the following situation arises. Consider the following proof $\mathcal{D}$ where $\mathcal{D}_1$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are valid.

$$\dfrac{\dfrac{\begin{array}{c}[\neg(A_1 \vee A_2)] \\ \vdots\ \mathcal{D}_1 \\ \bot\end{array}}{A_1 \vee A_2} \qquad \begin{array}{c}[A_1] \\ \vdots\ \mathcal{E}_1 \\ B\end{array} \qquad \begin{array}{c}[A_2] \\ \vdots\ \mathcal{E}_2 \\ B\end{array}}{B}\ R$$

In order to prove that R is valid, we have to show that $\mathcal{D}$ is valid. For this aim, we have to show that the validity of the following proof $\mathcal{D}'$, which is obtained from $\mathcal{D}$ by the reduction at $A_1 \vee A_2$:

$$\begin{array}{c}\dfrac{\dfrac{[A_1 \vee A_2] \quad \dfrac{[\neg B] \quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{array}}{\bot} \quad \dfrac{[\neg B] \quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{array}}{\bot}}{\bot}}{\neg(A_1 \vee A_2)}\\ \vdots\ \mathcal{D}_1\\ \dfrac{\bot}{B}\end{array}$$

Now, consider the following valid proof (Note that the end-formula is $\bot$):

$$\dfrac{\dfrac{\begin{array}{c}[\neg(A_1 \vee A_2)]\\ \vdots\ \mathcal{D}_1\\ \bot\end{array}}{A_1 \vee A_2} \quad \dfrac{\neg B \quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{array}}{\bot} \quad \dfrac{\neg B \quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{array}}{\bot}}{\bot}$$

This is reduced to

$$\begin{array}{c}\dfrac{\dfrac{[A_1 \vee A_2] \quad \dfrac{\neg\bot \quad \dfrac{\neg B \quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{array}}{\bot}}{\bot} \quad \dfrac{\neg\bot \quad \dfrac{\neg B \quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{array}}{\bot}}{\bot}}{\bot}}{\neg(A_1 \vee A_2)}\\ \vdots\ \mathcal{D}_1\\ \bot\end{array}$$

and using "redundant application", this is reduced to

$$\begin{array}{c}\dfrac{\dfrac{[A_1 \vee A_2] \quad \dfrac{\neg B \quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{array}}{\bot} \quad \dfrac{\neg B \quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{array}}{\bot}}{\bot}}{\neg(A_1 \vee A_2)}\\ \vdots\ \mathcal{D}_1\\ \bot\end{array}$$

which is valid using Lemma 3.3.1. Finally, using the validity of (RAA), we can show that $\mathcal{D}'$ is valid.

3.4 Failure of Church-Rosser property

As mentioned above, clauses 8 (d) and (e) cause that failure of the Church-Rosser property. In this section we present a counter-example. Consider the following proof $\mathcal{D}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A_1 \vee A_2]^1 \quad \begin{array}{c}[A_1]\\ \vdots\,\mathcal{D}_1\\ B_1 \vee B_2\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\,\mathcal{D}_2\\ B_1 \vee B_2\end{array}}{B_1 \vee B_2} \quad \begin{array}{c}[B_1]\\ \vdots\,\mathcal{D}_3\\ C\end{array} \quad \begin{array}{c}[B_2]\\ \vdots\,\mathcal{D}_4\\ C\end{array}}{C}}{(A_1 \vee A_2) \supset C}\,(\supset)^1 \qquad \dfrac{\begin{array}{c}[\neg(A_1 \vee A_2)]\\ \vdots\,\mathcal{D}_5\\ \bot\end{array}}{A_1 \vee A_2}}{C}$$

where each $\mathcal{D}_i$, $1 \leq i \leq 5$, is normal form and each assumption $\neg(A_1 \vee A_2)$ in $\mathcal{D}_5$ (discharged by (RAA)) is not the major premise of an elimination rule. Then it is easy to see that $\mathcal{D}$ is reduced to two normal forms $\mathcal{E}_1$ and $\mathcal{E}_2$ where $\mathcal{E}_1$ is

$$\begin{array}{c}\dfrac{\dfrac{[A_1 \vee A_2] \quad \dfrac{[\neg C] \quad \dfrac{\begin{array}{c}[A_1]\\ \vdots\,\mathcal{D}_1\\ B_1 \vee B_2\end{array} \quad \begin{array}{c}[B_1]\\ \vdots\,\mathcal{D}_3\\ C\end{array} \quad \begin{array}{c}[B_2]\\ \vdots\,\mathcal{D}_4\\ C\end{array}}{C}}{\bot} \quad \dfrac{[\neg C] \quad \dfrac{\begin{array}{c}[A_2]\\ \vdots\,\mathcal{D}_2\\ B_1 \vee B_2\end{array} \quad \begin{array}{c}[B_1]\\ \vdots\,\mathcal{D}_3\\ C\end{array} \quad \begin{array}{c}[B_2]\\ \vdots\,\mathcal{D}_4\\ C\end{array}}{C}}{\bot}}{\bot}}{\neg(A_1 \vee A_2)}\\ \vdots\,\mathcal{D}_5\\ \dfrac{\bot}{C}\end{array}$$

and $\mathcal{E}_2$ is

$$\begin{array}{c}\dfrac{\dfrac{[A_1 \vee A_2] \quad \dfrac{\begin{array}{c}[A_1]\\ \vdots\,\mathcal{D}_1\\ B_1 \vee B_2\end{array} \quad \dfrac{[\neg C] \quad \begin{array}{c}[B_1]\\ \vdots\,\mathcal{D}_3\\ C\end{array}}{\bot} \quad \dfrac{[\neg C] \quad \begin{array}{c}[B_2]\\ \vdots\,\mathcal{D}_3\\ C\end{array}}{\bot}}{\bot} \quad \dfrac{\begin{array}{c}[A_2]\\ \vdots\,\mathcal{D}_2\\ B_1 \vee B_2\end{array} \quad \dfrac{[\neg C] \quad \begin{array}{c}[B_1]\\ \vdots\,\mathcal{D}_3\\ C\end{array}}{\bot} \quad \dfrac{[\neg C] \quad \begin{array}{c}[B_2]\\ \vdots\,\mathcal{D}_3\\ C\end{array}}{\bot}}{\bot}}{\bot}}{\neg(A_1 \vee A_2)}\\ \vdots\,\mathcal{D}_5\\ \dfrac{\bot}{C}\end{array}$$

Note that if we adopt the reduction step ("Red-1" in Chapter 1)

$$\dfrac{\dfrac{\begin{matrix}[\neg(A_1\vee A_2)]\\ \vdots\ \mathcal{D}_1\\ \bot\end{matrix}}{A_1\vee A_2} \quad \begin{matrix}[A_1]\\ \vdots\ \mathcal{D}_2\\ B\end{matrix} \quad \begin{matrix}[A_2]\\ \vdots\ \mathcal{D}_3\\ B\end{matrix}}{B} \quad\Rightarrow\quad \dfrac{\begin{matrix}\dfrac{\dfrac{[\neg B]^2 \quad \dfrac{[A_1\vee A_2]^1 \quad \begin{matrix}[A_1]\\ \vdots\ \mathcal{D}_2\\ B\end{matrix} \quad \begin{matrix}[A_2]\\ \vdots\ \mathcal{D}_3\\ B\end{matrix}}{B}}{\bot}}{\neg(A_1\vee A_2)}\,(\supset)^1\\ \vdots\ \mathcal{D}_1\\ \bot\end{matrix}}{B}\,(\mathrm{RAA})^2$$

instead of clause 8(d) of Stålmarck's definition, then one can easily seen that $\mathcal{D}$ has the unique normal form.

As we have seen above, Stålmarck's result is not satisfactory. We would like to adopt the reduction procedure for (RAA) like "Red-1" in Chapter 1 as follows:

$$\dfrac{\dfrac{\begin{matrix}[\neg C]\\ \vdots\ \mathcal{D}\\ \bot\end{matrix}}{C}\,R \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B} \quad\Rightarrow\quad \dfrac{\begin{matrix}\dfrac{\dfrac{[\neg B]^2 \quad \dfrac{[C]^1 \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}}{\bot}}{\neg C}\,(\supset I)^1\\ \vdots\ \mathcal{D}\\ \bot\end{matrix}}{B}\,(\mathrm{RAA})^2$$

However, when we adopt the above reduction, Stålmarck's proof can not work. In the next chapter, we present how this difficulty is overcome.

Chapter 4

Strong normalization theorem for NK (II) – New reduction procedure

This chapter is a sequel to the previous chapter. In this chapter, we introduce another reduction procedure for NK which is an improvement of Stålmarck's one satisfying the Church-Rosser property, and prove the strong normalization theorem with respect to this reduction procedure. This result yields the strong normalization theorem with respect to Andou's reduction procedure introduced in [1]. The contents of this chapter are based on [17].

4.1 Introduction

As we have seen in Chapter 3, Stålmarck's reduction procedure is not satisfactory. If we would like to recover the Church-Rosser property, we have to adopt the reduction procedure for (RAA) as follows:

$$\dfrac{\dfrac{\begin{matrix}[\neg C]\\ \vdots\ \mathcal{D}\\ \bot\end{matrix}}{C}\,R \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B} \quad\Rightarrow\quad \dfrac{\begin{matrix}\dfrac{\dfrac{[\neg B]^2 \quad \dfrac{[C]^1 \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}}{\bot}}{\neg C}\,(\supset I)^1\\ \vdots\ \mathcal{D}\\ \bot\end{matrix}}{B}\,(\mathrm{RAA})^2$$

However this makes a proof for the strong normalization theorem very hard. In this introductory part, we explain this difficulty and the idea for how this difficulty can be overcome.

In order to prove the strong normalization theorem using the notion of the validity, we have to show that all rules are valid. When we show the validity of elimination rules, the following situation occurs.

Let $\mathcal{D}$ be the following proof:

$$\dfrac{\begin{matrix} \\ \vdots\, \mathcal{D}_1 \\ A_1 \vee A_2 \end{matrix} \qquad \begin{matrix} [A_1] \\ \vdots\, \mathcal{D}_2 \\ B \end{matrix} \qquad \begin{matrix} [A_2] \\ \vdots\, \mathcal{D}_3 \\ B \end{matrix}}{B}$$

where $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ is valid. Then in order to prove the validity of $(\vee E)$, we have to show that $\mathcal{D}$ is valid. Especially, a problematic case is the one where the last inference of $\mathcal{D}_1$ is (RAA) as follows:

$$\dfrac{\dfrac{\begin{matrix} [\neg(A_1 \vee A_2)] \\ \vdots \\ \bot \end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix} [A_1] \\ \vdots\, \mathcal{D}_2 \\ B \end{matrix} \qquad \begin{matrix} [A_2] \\ \vdots\, \mathcal{D}_3 \\ B \end{matrix}}{B}$$

Genarally, in order to prove the validity of a proof whose last inference is an elimination rule whose major premise is the conclusion of (RAA) like

$$\dfrac{\dfrac{\begin{matrix} [\neg A] \\ \vdots\, \mathcal{D} \\ \bot \end{matrix}}{A} \qquad \mathcal{E}_1 \quad \mathcal{E}_2}{B}$$

we need the validity of

$$\dfrac{\begin{matrix} \dfrac{\dfrac{[\neg B]^2 \qquad \dfrac{[A]^1 \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}}{\bot}}{\neg A}\,(\supset I)^1 \\ \vdots\, \mathcal{D} \\ \bot \end{matrix}}{B}\,(\mathrm{RAA})^2$$

This proof is ending with (RAA). Then (if B is not atomic) in order to prove the validity of this, we have to show the validity of

$$\dfrac{\dfrac{\begin{matrix} \dfrac{\dfrac{[\neg B]^2 \qquad \dfrac{[A]^1 \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}}{\bot}}{\neg A}\,(\supset I)^1 \\ \vdots\, \mathcal{D} \\ \bot \end{matrix}}{B}\,(\mathrm{RAA})^2 \qquad \mathcal{C}_1 \quad \mathcal{C}_2}{C}\, R_0$$

where R_0 is an elimination rule. This is a proof whose last inference is an elimination rule whose major premise is the conclusion of (RAA), i.e., this is the same form of

$$\dfrac{\dfrac{\begin{matrix}[\neg A]\\ \vdots\ \mathcal{D}\\ \bot\end{matrix}}{A}\qquad \mathcal{E}_1\quad \mathcal{E}_2}{B}$$

Hence in general, we have to repeat the above argument in several time. This is the main difficulty of the proof. Then how do we overcome this? We next explain the idea to overcome this difficulty.

By repeating the above argument, we need generally the validity of a proof of the following form:

$$\begin{matrix}\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[\neg(A_1 \vee A_2)]\\ \vdots\ \mathcal{D}_1\\ \bot\end{matrix}}{A_1 \vee A_2}\qquad \begin{matrix}[A_1]\\ \vdots\ \mathcal{E}_1\\ B_0\end{matrix}\qquad \begin{matrix}[A_2]\\ \vdots\ \mathcal{E}_2\\ B_0\end{matrix}}{B_0}\,R_0\qquad \mathcal{E}_1^1\quad \mathcal{E}_2^1}{B_1}\,R_1\qquad \mathcal{E}_1^2\quad \mathcal{E}_2^2}{B_2}\,R_2}{\vdots} \\ \dfrac{B_{k-1}\qquad\qquad \mathcal{E}_1^k\quad \mathcal{E}_2^k}{B_k}\,R_k\end{matrix}$$

where $R_0, \dots, R_k$ are elimination rules. By recalling the definition of validity, there exists k such that B_k becomes atomic. On the other hand, strongly normalizability of a proof of the above form can be proved (this is very hard and complicated) and if the end-formula is atomic, it is easy to see that SN implies valid. Therefore we can prove that the above proof is valid.

Remark. Needless to say, the above explanation is only the idea of the proof. As a matter of fact, we have to modify the definition of validity and for this modification, the case in which B_k for each k in the above is not necessarily atomic arises. Therefore we have to treat this case in a different way.

In the rest of this chapter, we introduce the new reduction procedure and prove the strong normalization theorem as well as the Church-Rosser property with respect to this reduction procedure.

4.2 New reduction procedure

In this section, we introduce a new reduction procedure for which we prove the strong normalization theorem and the Church-Rosser property. This reduction procedure is an improvement of Stålmarck's one presented in Chapter 3.

Let C be a maximum formula in a proof which is the conclusion of a rule R. The reduction at C is defined as follows.

1. R is $(\wedge I)$ and C is $A_1 \wedge A_2$:

$$\frac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ A_1\end{array}\qquad\begin{array}{c}\vdots\ \mathcal{D}_2\\ A_2\end{array}}{A_1\wedge A_2}\,R}{A_i}\qquad\Rightarrow\qquad\begin{array}{c}\vdots\ \mathcal{D}_i\\ A_i\end{array}$$

2. R is $(\vee I)$ and C is $A_1 \vee A_2$:

$$\frac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ A_i\end{array}}{A_1\vee A_2}\,R\qquad\begin{array}{c}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{array}\qquad\begin{array}{c}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{array}}{B}\qquad\Rightarrow\qquad\begin{array}{c}\vdots\ \mathcal{D}\\ A_i\\ \vdots\ \mathcal{E}_i\\ B\end{array}$$

3. R is $(\supset I)$ and C is $A \supset B$:

$$\frac{\dfrac{\begin{array}{c}[A]\\ \vdots\ \mathcal{D}\\ B\end{array}}{A\supset B}\,R\qquad\begin{array}{c}\vdots\ \mathcal{E}\\ A\end{array}}{B}\qquad\Rightarrow\qquad\begin{array}{c}\vdots\ \mathcal{E}\\ A\\ \vdots\ \mathcal{D}\\ B\end{array}$$

4. R is $(\forall I)$ and C is $\forall x A(x)$:

$$\frac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}(a)\\ A(a)\end{array}}{\forall x A(x)}\,R}{A(t)}\qquad\Rightarrow\qquad\begin{array}{c}\vdots\ \mathcal{D}(t)\\ A(t)\end{array}$$

5. R is $(\exists I)$ and C is $\exists x A(x)$:

$$\frac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ A(t)\end{array}}{\exists x A(x)}\,R\qquad\begin{array}{c}[A(a)]\\ \vdots\ \mathcal{E}(a)\\ B\end{array}}{B}\qquad\Rightarrow\qquad\begin{array}{c}\vdots\ \mathcal{D}\\ A(t)\\ \vdots\ \mathcal{E}(t)\\ B\end{array}$$

6. R is $(\vee E)$:

$$\frac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ A_1\vee A_2\end{array}\quad\begin{array}{c}[A_1]\\ \vdots\ \mathcal{D}_2\\ C\end{array}\quad\begin{array}{c}[A_2]\\ \vdots\ \mathcal{D}_3\\ C\end{array}}{C}\,R\quad\mathcal{E}_1\quad\mathcal{E}_2}{B}\quad\Rightarrow\quad\frac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ A_1\vee A_2\end{array}\quad\dfrac{\begin{array}{c}[A_1]\\ \vdots\ \mathcal{D}_2\\ C\end{array}\quad\mathcal{E}_1\quad\mathcal{E}_2}{B}\quad\dfrac{\begin{array}{c}[A_2]\\ \vdots\ \mathcal{D}_3\\ C\end{array}\quad\mathcal{E}_1\quad\mathcal{E}_2}{B}}{B}$$

7. R is $(\exists E)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \exists x A(x)\end{matrix} \quad \begin{matrix}[A(a)]\\ \vdots\,\mathcal{D}_2\\ C\end{matrix}}{C}\,R \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B} \quad \Rightarrow \quad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \exists x A(x)\end{matrix}\quad \dfrac{\begin{matrix}[A(a)]\\ \vdots\,\mathcal{D}_2\\ C\end{matrix}\quad \mathcal{E}_1\quad\mathcal{E}_2}{B}}{B}$$

8. R is (RAA):

$$\dfrac{\dfrac{\begin{matrix}[\neg C]\\ \vdots\,\mathcal{D}\\ \bot\end{matrix}}{C}\,R \quad \mathcal{E}_1\quad\mathcal{E}_2}{B} \quad \Rightarrow \quad \dfrac{\begin{matrix}\dfrac{\dfrac{[\neg B]^2 \quad \dfrac{[C]^1\quad \mathcal{E}_1\quad\mathcal{E}_2}{B}}{\bot}}{\neg C}\,(\supset I)^1\\ \vdots\,\mathcal{D}\\ \bot\end{matrix}}{B}\,(\text{RAA})^2$$

Remark. The above definition is almost the same as Stålmarck's one except the followings.

- In clauses 6 and 7, restrictions about the normality of the subproof are deleted;
- Clause 8 is changed;
- Clause 9 is deleted.

The following remarks are the same as the one for Stålmarck's reduction procedure. For reader's convenience, we repeat these again. If $\mathcal{D}'$ is obtained from $\mathcal{D}$ by the reduction at C, we also say "$\mathcal{D}'$ is obtained from $\mathcal{D}$ by reducing at C" and denoted by $\mathcal{D} \Rightarrow \mathcal{D}'$. $\overset{+}{\Rightarrow}$ denotes the transitive closure of $\Rightarrow$ and $\overset{*}{\Rightarrow}$ denotes the reflexive transitive closure of $\Rightarrow$. If $\mathcal{D} \overset{+}{\Rightarrow} \mathcal{D}'$, $\mathcal{D}'$ is called a *reduct* of $\mathcal{D}$. Especially, if $\mathcal{D} \Rightarrow \mathcal{D}'$, $\mathcal{D}'$ is called a *direct reduct* of $\mathcal{D}$.

Suppose that $\mathcal{D} \Rightarrow \mathcal{D}'$. If the major premise of the last inference of $\mathcal{D}$ is a maximum formula of $\mathcal{D}$ and $\mathcal{D}'$ is obtained by reducing this maximum formula, we say that $\mathcal{D}'$ is obtained by *proper reduction* from $\mathcal{D}$.

At the end of this section, we remark a relationship between the reduction steps defined above and Andou's one defined in [1].

Before defining the reduction steps due to Andou, we define the regularity of a proof (see [1]).

An assumption-formula in a proof which is discharged by (RAA) is said to be *regular* if it is the major premise of $(\supset E)$. A proof is said to be *regular* if each assumption-formula discharged by any (RAA) is regular. By Lemma 1 of [1], we can consider only regular proofs.

Then Andou's reduction steps are the same as ours except the case where R is (RAA). This is as follows:

Let $S_1, \ldots, S_n$ be all $(\supset E)$'s whose major premises are discharged by R, if they exist. Then the reduction is carried out as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \dfrac{\dfrac{[\neg C] \quad \overset{\vdots}{C}}{\perp}\,S_i}{}\\ \vdots \\ \perp \end{array}}{C}\,R \qquad \mathcal{E}_1 \quad \mathcal{E}_2}{B} \quad \Rightarrow \quad \dfrac{\begin{array}{c}\dfrac{[\neg B]^1 \qquad \dfrac{\overset{\vdots}{C} \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}}{\perp}\\ \vdots \\ \perp\end{array}}{B}\,(\mathrm{RAA})^1$$

Note that the above translation is applied for all i, $1 \le i \le n$.

Now, if $\mathcal{D}'$ is obtained from $\mathcal{D}$ by the above reduction, then it is easy to see that $\mathcal{D} \overset{*}{\Rightarrow} \mathcal{D}'$ (with respect to our reduction steps). Therefore the strong normalization theorem for our reduction steps implies that the strong normalization theorem for Andou's ones.

Finally, we note that it is not necessary that we assume the regularity of proofs for our reduction steps. Hence our reduction steps are, in the author's opinion, more acceptable.

4.3 Strong normalization theorem

In this section, we shall prove the strong normalization theorem with respect to the reduction procedure defined in the previous section. That is, the aim is the following.

Theorem 4.3.1 *All proofs are strongly normalizable.*

The rest of this chapter is devoted the proof of this theorem.

4.3.1 Preliminaries

In this subsection, we introduce the notion of the validity of proofs and rules and prove some properties for valid proofs as preparations of the proof of the theorem. The validity of proofs defined below is an improvement of the one defined in Chapter 3.

The *reduction tree* of a proof $\mathcal{D}$ is the tree whose paths consist of all the reduction sequences starting from $\mathcal{D}$. Note that if a proof $\mathcal{D}$ is SN, the length of each path of the reduction tree of $\mathcal{D}$ is finite. Therefore, by using König's lemma, there exists the maximum of the lengths of all paths of the reduction tree of $\mathcal{D}$. This is denoted by $l(\mathcal{D})$.

We recall the definition of I-form. A proof of the form

$$\dfrac{\mathcal{D}_1 \cdots \mathcal{D}_n}{A}\,R$$

is said to be in *I-form* if A is not an atomic formula and R is an introduction rule or (RAA).

In the following, we define the notion of validity of proofs. In order to simplify the description, we introduce the notion of eliminating proofs.

As usual, the *degree* of a formula A is defined by the number of logical symbols contained in A. The *degree* of a proof $\mathcal{D}$, which is denoted by $d(\mathcal{D})$, is defined by the degree of the end-formula of $\mathcal{D}$.

Let $\mathcal{D}$ be a proof. Suppose that the validity is already defined for proofs whose degrees are less than $d(\mathcal{D})$. Then an *eliminating proof* (abbreviated to *E-proof*) of $\mathcal{D}$ is defined as follows:

1. If the end-formula of $\mathcal{D}$ is $A_1 \wedge A_2$, then for each $i = 1, 2$, a proof of the form

$$\dfrac{\begin{array}{c} \vdots\, \mathcal{D} \\ A_1 \wedge A_2 \end{array}}{A_i}$$

   is an eliminating proof of $\mathcal{D}$.

2. If the end-formula of $\mathcal{D}$ is $A_1 \vee A_2$, then each proof of the form

$$\dfrac{\begin{array}{c} \\ \vdots\, \mathcal{D} \\ A_1 \vee A_2 \end{array} \quad \begin{array}{c} [A_1] \\ \vdots\, \mathcal{E}_1 \\ B \end{array} \quad \begin{array}{c} [A_2] \\ \vdots\, \mathcal{E}_2 \\ B \end{array}}{B}\, R$$

   is an eliminating proof of $\mathcal{D}$ where for each $i = 1, 2$, $\mathcal{E}_i$ is a proof such that a proof obtained from $\mathcal{E}_i$ by substituting an arbitrary valid proof whose end-formula is A_i for open assumptions which are discharged by R is SN.

3. If the end-formula of $\mathcal{D}$ is $A \supset B$, then each proof of the form

$$\dfrac{\begin{array}{c} \vdots\, \mathcal{D} \\ A \supset B \end{array} \quad \begin{array}{c} \vdots\, \mathcal{E} \\ A \end{array}}{B}$$

   is an eliminating proof of $\mathcal{D}$ where $\mathcal{E}$ is an arbitrary valid proof whose end-formula is A.

4. If the end-formula of $\mathcal{D}$ is $\forall x A(x)$, then each proof of the form

$$\dfrac{\begin{array}{c} \vdots\, \mathcal{D} \\ \forall x A(x) \end{array}}{A(t)}$$

   is an eliminating proof of $\mathcal{D}$ where t is an arbitrary term.

5. If the end-formula of $\mathcal{D}$ is $\exists x A(x)$, then each proof of the form

$$\dfrac{\begin{array}{c} \\ \vdots\, \mathcal{D} \\ \exists x A(x) \end{array} \quad \begin{array}{c} [A(a)] \\ \vdots\, \mathcal{E} \\ B \end{array}}{B}\, R$$

is an eliminating proof of $\mathcal{D}$ where $\mathcal{E}$ is a proof such that a proof obtained from $\mathcal{E}$ by substituting an arbitrary term t for a and substituting an arbitrary valid proof whose end-formula is $A(t)$ for open assumptions which are discharged by R is SN.

A proof $\mathcal{D}$ is said to be *valid* if one of the following conditions is satisfied:

1. $\mathcal{D}$ is in I-form and

   (a) If the end-formula of $\mathcal{D}$ is of the form $A_1 \land A_2$, $A \supset B$ or $\forall x A(x)$, then each E-proof of $\mathcal{D}$ is valid.

   (b) If the last inference of $\mathcal{D}$ is of the form $A_1 \lor A_2$ or $\exists x A(x)$, then each E-proof of $\mathcal{D}$ is SN and if $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ A_i\end{array}}{A_1 \lor A_2}$$

   or

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ A(t)\end{array}}{\exists x A(x)}$$

   then $\mathcal{D}_1$ is valid.

2. $\mathcal{D}$ is not in I-form and each direct reduct of $\mathcal{D}$ is valid and if $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\vdots\\ A_1 \lor A_2\end{array} \quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{array}}{B}$$

or

$$\dfrac{\begin{array}{c}\vdots\\ \exists x A(x)\end{array} \quad \begin{array}{c}[A(a)]\\ \vdots\ \mathcal{E}_1\\ B\end{array}}{B}$$

then $\mathcal{E}_1$ and $\mathcal{E}_2$ are valid.

Remark. The above definition is primarily by induction on $d(\mathcal{D})$, and within this by double induction on $(l(\mathcal{D}), h(\mathcal{D}))$ where $h(\mathcal{D})$ denotes the height of $\mathcal{D}$.

Let $\mathcal{D}$ be a proof and let $\mathcal{D}^*$ be a proof which is obtained from $\mathcal{D}^{**}$ by substituting an arbitrary valid proofs for open assumptions is valid where $\mathcal{D}^{**}$ is obtained from $\mathcal{D}$ by substituting an arbitrary terms for free variables which are not used as eigenvariables For simplicity, we say this " $\mathcal{D}^*$ is obtained from $\mathcal{D}$ by substitution". A proof $\mathcal{D}$ is said to be *valid under substitution* if each proof $\mathcal{D}^*$ which is obtained from $\mathcal{D}$ by substitution is valid.

An inference rule R is said to be *valid* if each proof of the form

$$\frac{\mathcal{D}_1 \cdots \mathcal{D}_n}{A}\; R$$

is valid under substitution where each $\mathcal{D}_i$ is valid under substitution.

Lemma 4.3.1 *Let $\mathcal{D}$ be a valid proof. If $\mathcal{D} \Rightarrow \mathcal{D}'$, then $\mathcal{D}'$ is also valid.*

Proof. This is proved by induction on the definition of validity. If $\mathcal{D}$ is not in I-form, then the claim is obvious by the definition of validity. Therefore we consider the case where $\mathcal{D}$ is in I-form. First, we remark that $\mathcal{D}'$ is also in I-form.

Case 1: The end-formula of $\mathcal{D}$ is of the form $A \supset B$.
We have to show that each E-proof of $\mathcal{D}'$ is valid. Let $\mathcal{C}'$ be an arbitrary E-proof of $\mathcal{D}'$ of the form

$$\frac{\begin{array}{cc} \vdots\, \mathcal{D}' & \vdots\, \mathcal{E} \\ A \supset B & A \end{array}}{B}$$

where $\mathcal{E}$ is an arbitrary valid proof whose end-formula is A. Now, consider the following proof $\mathcal{C}$:

$$\frac{\begin{array}{cc} \vdots\, \mathcal{D} & \vdots\, \mathcal{E} \\ A \supset B & A \end{array}}{B}$$

This is an E-proof of $\mathcal{D}$. Since $\mathcal{D}$ is valid, $\mathcal{C}$ is valid by the definition of validity, and obviously $\mathcal{C} \Rightarrow \mathcal{C}'$. Hence $\mathcal{C}'$ is valid by induction hypothesis.

Case 2: The end-formula of $\mathcal{D}$ is of the form $A_1 \vee A_2$.
We only treat the case where the last inference of $\mathcal{D}$ is $(\vee I)$. Then $\mathcal{D}$ is of the form

$$\frac{\begin{array}{c} \vdots\, \mathcal{D}_1 \\ A_i \end{array}}{A_1 \vee A_2}$$

and $\mathcal{D}'$ is of the form

$$\frac{\begin{array}{c} \vdots\, \mathcal{D}'_1 \\ A_i \end{array}}{A_1 \vee A_2}$$

where $\mathcal{D}'_1$ is a reduct of $\mathcal{D}_1$. By induction hypothesis, $\mathcal{D}'_1$ is valid. Furthermore, we have to show that each E-proof of $\mathcal{D}'$ is SN. Let $\mathcal{C}'$ be an arbitrary E-proof of $\mathcal{D}'$ of the form

$$\frac{\begin{array}{ccc} & [A_1] & [A_2] \\ \vdots\, \mathcal{D}' & \vdots\, \mathcal{E}_1 & \vdots\, \mathcal{E}_2 \\ A_1 \vee A_2 & B & B \end{array}}{B}\; R$$

where for each $i = 1, 2$, $\mathcal{E}_i$ is a proof such that a proof obtained from $\mathcal{E}_i$ by substituting an arbitrary valid proof whose end-formula is A_i for open assumptions which are discharged by R is SN. Now, consider the following proof $\mathcal{C}$:

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ A_1 \vee A_2\end{array} \quad \begin{array}{c}[A_1]\\ \vdots\,\mathcal{E}_1\\ B\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\,\mathcal{E}_2\\ B\end{array}}{B}\,R$$

This is an E-proof of $\mathcal{D}$. Since $\mathcal{D}$ is valid, $\mathcal{C}$ is SN by the definition of validity, and obviously $\mathcal{C} \Rightarrow \mathcal{C}'$. Hence $\mathcal{C}'$ is SN.

Other cases are treated similarly. □

Lemma 4.3.2 *Valid proofs are SN.*

Proof. This is proved by induction on the definition of validity. Let $\mathcal{D}$ be a valid proof.

Case 1: $\mathcal{D}$ is not in I-form.
In order to prove that $\mathcal{D}$ is SN, it suffices to show that each direct reduct of $\mathcal{D}$ is SN. By the definition of validity, each direct reduct of $\mathcal{D}$ is valid and hence SN by induction hypothesis. Therefore $\mathcal{D}$ is SN.

Case 2: $\mathcal{D}$ is in I-form.
Let A be the end-formula of $\mathcal{D}$. By the definition of validity, each E-proof of the form

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ A\end{array} \quad \mathcal{D}' \quad \mathcal{D}''}{B}\,R$$

is valid (if R is $(\wedge I)$, $(\supset I)$ or $(\forall I)$) or SN (if R is $(\vee I)$ or $(\exists I)$). By induction hypothesis, the above proof is also SN when R is $(\wedge I)$, $(\supset I)$ or $(\forall I)$. Therefore $\mathcal{D}$ is obviously SN.
□

Lemma 4.3.3 *Let $\mathcal{D}$ be an SN proof whose end-formula is an atomic formula. Then $\mathcal{D}$ is valid.*

Proof. Notice that $\mathcal{D}$ cannot be an I-form. Hence the claim is proved immediately by induction on the definition of validity. □

4.3.2 Validity of the introduction rules

In order to prove Theorem 4.3.1, we shall prove that all inference rules are valid. First, we consider the introduction rules.

Lemma 4.3.4 (1) *Let $\mathcal{D}$ be a proof whose last inference is $(\wedge I)$ of the form*

$$\frac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ A_1\end{array}\qquad\begin{array}{c}\vdots\ \mathcal{D}_2\\ A_2\end{array}}{A_1 \wedge A_2}$$

If both $\mathcal{D}_1$ and $\mathcal{D}_2$ are valid, then $\mathcal{D}$ is also valid.
(2) *Let $\mathcal{D}$ be a proof whose last inference is $(\supset I)$ of the form*

$$\frac{\begin{array}{c}[A]\\ \vdots\ \mathcal{D}_1\\ B\end{array}}{A \supset B}$$

If each proof of the form

$$\begin{array}{c}\vdots\ \mathcal{E}\\ [A]\\ \vdots\ \mathcal{D}_1\\ B\end{array}$$

is valid for each valid proof $\mathcal{E}$ whose end-formula is A, then $\mathcal{D}$ is also valid.
(3) *Let $\mathcal{D}$ be a proof whose last inference is $(\forall I)$ of the form*

$$\frac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ A(a)\end{array}}{\forall x A(x)}$$

If each proof obtained from $\mathcal{D}_1$ by substituting an arbitrary term for a is valid, then $\mathcal{D}$ is also valid.

Proof. We only prove (2). (1) and (3) are proved similarly.

Now, in order to prove that $\mathcal{D}$ is valid, we have to show that each E-proof of $\mathcal{D}$ is valid. Let $\mathcal{C}$ be an arbitrary E-proof of $\mathcal{D}$ of the form

$$\frac{\dfrac{\begin{array}{c}[A]\\ \vdots\ \mathcal{D}_1\\ B\end{array}}{A \supset B}\qquad\begin{array}{c}\vdots\ \mathcal{E}\\ A\end{array}}{B}$$

where $\mathcal{E}$ is an arbitrary valid proof whose end-formula is A. In order to prove that $\mathcal{C}$ is valid, it suffices to show that each direct reduct of $\mathcal{C}$ is valid. We show this by induction on $l(\mathcal{D}) + l(\mathcal{E})$. Let $\mathcal{C}'$ be an arbitrary direct reduct of $\mathcal{C}$.

Case 1: $\mathcal{C}'$ is obtained by the proper reduction.
$\mathcal{C}'$ is of the form

$$\begin{array}{c}\vdots\ \mathcal{E}\\ [A]\\ \vdots\ \mathcal{D}_1\\ B\end{array}$$

By the assumption, this is valid.

Case 2: $\mathcal{C}'$ is obtained by replacing $\mathcal{D}$ by its direct reduct.
$\mathcal{C}'$ is of the form

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\ \mathcal{D}'_1\\ B\end{matrix}}{A \supset B} \quad \begin{matrix}\vdots\ \mathcal{E}\\ A\end{matrix}}{B}$$

where $\mathcal{D}'_1$ is a direct reduct of $\mathcal{D}_1$. Obviously, $l(\mathcal{D}') < l(\mathcal{D})$. Furthermore, for each valid proof $\mathcal{E}_0$ whose end-formula is A,

$$\begin{matrix}\vdots\ \mathcal{E}_0\\ [A]\\ \vdots\ \mathcal{D}'_1\\ B\end{matrix}$$

is a direct reduct of

$$\begin{matrix}\vdots\ \mathcal{E}_0\\ [A]\\ \vdots\ \mathcal{D}_1\\ B\end{matrix}$$

and hence valid by Lemma 4.3.1. Hence $\mathcal{C}'$ is valid by induction hypothesis.

Case 3: $\mathcal{C}'$ is obtained by replacing $\mathcal{E}$ by its direct reduct.
$\mathcal{C}'$ is of the form

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}\\ A \supset B\end{matrix} \quad \begin{matrix}\vdots\ \mathcal{E}'\\ A\end{matrix}}{B}$$

Obviously, $l(\mathcal{E}') < l(\mathcal{E})$. Hence $\mathcal{C}'$ is valid by induction hypothesis. □

Lemma 4.3.5 (1) *Let $\mathcal{D}$ be a proof whose last inference is $(\vee I)$ of the form*

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ A_i\end{matrix}}{A_1 \vee A_2}$$

If $\mathcal{D}_1$ is valid, then $\mathcal{D}$ is also valid.
(2) *Let $\mathcal{D}$ be a proof whose last inference is $(\exists I)$ of the form*

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ A(t)\end{matrix}}{\exists x A(x)}$$

If $\mathcal{D}_1$ is valid, then $\mathcal{D}$ is also valid.

Proof. We only prove (1). (2) is proved similarly.

By the assumption, $\mathcal{D}_1$ is valid. Hence, in order to prove that $\mathcal{D}$ is valid, we have to show that each E-proof of $\mathcal{D}$ is SN. Let $\mathcal{C}$ be an arbitrary E-proof of $\mathcal{D}$ of the form

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ A_i\end{matrix}}{A_1 \vee A_2} \quad \begin{matrix}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{matrix} \quad \begin{matrix}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{matrix}}{B}\, R$$

where for each $i = 1, 2$, $\mathcal{E}_i$ is a proof such that a proof obtained from $\mathcal{E}_i$ by substituting an arbitrary valid proof whose end-formula is A_i for open assumptions which are discharged by R is SN. In order to prove that $\mathcal{C}$ is SN, it suffices to show that each direct reduct of $\mathcal{C}$ is SN. We show this by induction on $l(\mathcal{D}) + l(\mathcal{E}_1) + l(\mathcal{E}_2)$. Let $\mathcal{C}'$ be an arbitrary direct reduct of $\mathcal{C}$.

Case 1: $\mathcal{C}'$ is obtained by the proper reduction.
$\mathcal{C}'$ is of the form

$$\begin{matrix}\vdots\ D_1\\ A_i\\ \vdots\ \mathcal{E}_i\\ B\end{matrix}$$

By the assumption, this is SN.

Case 2: $\mathcal{C}'$ is obtained by replacing $\mathcal{D}$ by its direct reduct.
$\mathcal{C}'$ is of the form

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1'\\ A_i\end{matrix}}{A_1 \vee A_2} \quad \begin{matrix}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{matrix} \quad \begin{matrix}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{matrix}}{B}\, R$$

where $\mathcal{D}_1'$ is a direct reduct of $\mathcal{D}_1$. Obviously, $l(\mathcal{D}') < l(\mathcal{D})$ and $\mathcal{D}_1'$ is valid by Lemma 4.3.1. Hence $\mathcal{C}'$ is SN by induction hypothesis.

Case 3: $\mathcal{E}'$ is obtained by replacing $\mathcal{E}_1$ by its direct reduct.
$\mathcal{E}'$ is of the form

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}\\ A_1 \vee A_2\end{matrix} \quad \begin{matrix}[A_1]\\ \vdots\ \mathcal{E}_1'\\ B\end{matrix} \quad \begin{matrix}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{matrix}}{B}\, R$$

Obviously, $l(\mathcal{E}_1') < l(\mathcal{E}_1)$. In order to use induction hypothesis, we have to check that each proof

$$\begin{matrix}\vdots\ \mathcal{E}_0\\ A_1\\ \vdots\ \mathcal{E}_1'\\ B\end{matrix}$$

is SN where $\mathcal{E}_0$ is an arbitrary valid proof whose end-formula is A_1. This is obvious since the above proof is a direct reduct of

$$\begin{array}{c} \vdots\ \mathcal{E}_0 \\ A_1 \\ \vdots\ \mathcal{E}_1 \\ B \end{array}$$

which is SN by the assumption. Hence $\mathcal{C}'$ is SN by induction hypothesis.

Case 4: $\mathcal{E}'$ is obtained by replacing $\mathcal{E}_2$ by its direct reduct.
Similar to Case 3. □

Now we obtain the following.

Lemma 4.3.6 *All introduction rules are valid.*

Proof. We only consider $(\supset I)$. Let $\mathcal{D}$ be a proof of the form

$$\dfrac{\begin{array}{c} [A] \\ \vdots\ \mathcal{D}_1 \\ B \end{array}}{A \supset B}$$

Then the claim is that if $\mathcal{D}_1$ is valid under substitution, then so is $\mathcal{D}$. Now, let $\mathcal{D}^*$ be a proof which is obtained from $\mathcal{D}$ by substitution. Then $\mathcal{D}^*$ is of the form

$$\dfrac{\begin{array}{c} [A] \\ \vdots\ \mathcal{D}_1^* \\ B \end{array}}{A \supset B}$$

where $\mathcal{D}_1^*$ is obtained from $\mathcal{D}_1$ by substitution. Consider the following proof

$$\begin{array}{c} \vdots\ \mathcal{E} \\ [A] \\ \vdots\ \mathcal{D}_1^* \\ B \end{array}$$

where $\mathcal{E}$ is an arbitrary valid proof whose end-formula is A. This is also obtained from $\mathcal{D}_1$ by substitution and hence valid by the assumption. Therefore $\mathcal{D}^*$ is valid by Lemma 4.3.4(2).

Other rules are treated similarly. Note that for $(\vee I)$ and $(\exists I)$, use Lemma 4.3.5 instead of Lemma 4.3.4. □

4.3.3 Validity of (RAA)

In this subsection, we consider (RAA).

Lemma 4.3.7 (1) *Let $\mathcal{D}$ be a proof of the form*

$$\dfrac{\begin{array}{c} \\ \vdots\,\mathcal{D}_1 \\ A_1 \vee A_2 \end{array} \quad \begin{array}{c}[A_1]\\ \vdots\,\mathcal{E}_1 \\ B\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\,\mathcal{E}_2\\ B\end{array}}{B}$$

Suppose that both $\mathcal{E}_1$ and $\mathcal{E}_2$ are valid (resp.SN) and if there exists a maximum formula M occurring in $\mathcal{D}_1$ (including the case where M is the end-formula of $\mathcal{D}_1$), then a direct reduct of $\mathcal{D}$ which is obtained by reducing at M is valid (resp.SN). Then $\mathcal{D}$ is valid (resp.SN).

(2) *Let $\mathcal{D}$ be a proof of the form*

$$\dfrac{\begin{array}{c} \\ \vdots\,\mathcal{D}_1 \\ \exists x A(x) \end{array} \quad \begin{array}{c}[A(a)]\\ \vdots\,\mathcal{E} \\ B\end{array}}{B}$$

Suppose that $\mathcal{E}$ is valid (resp.SN) and if there exists a maximum formula M occurring in $\mathcal{D}_1$ (including the case where M is the end-formula of $\mathcal{D}_1$), then a direct reduct of $\mathcal{D}$ which is obtained by reducing at M is valid (resp.SN). Then $\mathcal{D}$ is valid (resp.SN).

Proof. We only prove (1). (2) is proved similarly. We also only prove for validity.

In order to prove that $\mathcal{D}$ is valid, we have to show that (a)each direct reduct of $\mathcal{D}$ is valid and (b)both $\mathcal{E}_1$ and $\mathcal{E}_2$ is valid. But (b) is obvious by the assumption. Therefore, in the following, we show (a) by induction on $l(\mathcal{E}_1)+l(\mathcal{E}_2)$. Let $\mathcal{D}'$ be an arbitrary direct reduct of $\mathcal{D}$.

Case 1: $\mathcal{D}'$ is obtained by reducing at a maximum formula occurring in $\mathcal{D}_1$.
By the assumption, $\mathcal{D}'$ is valid.

Case 2: $\mathcal{D}'$ is obtained by reducing at a maximum formula occurring in $\mathcal{E}_1$.
$\mathcal{D}'$ is of the form

$$\dfrac{\begin{array}{c} \\ \vdots\,\mathcal{D}_1 \\ A_1 \vee A_2 \end{array} \quad \begin{array}{c}[A_1]\\ \vdots\,\mathcal{E}_1' \\ B\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\,\mathcal{E}_2\\ B\end{array}}{B}$$

where $\mathcal{E}_1'$ is a direct reduct of $\mathcal{E}_1$. Obviously, $l(\mathcal{E}_1') < l(\mathcal{E}_1)$. In order to use induction hypothesis, we have to check that each direct reduct of $\mathcal{D}'$ which is obtained by reducing at a maximum formula M occurring in $\mathcal{D}_1$ is valid. Let $\mathcal{D}''$ be such an arbitrary direct reduct of $\mathcal{D}'$. On the other hand, let $\mathcal{D}'''$ be a direct reduct of $\mathcal{D}$ which is obtained by reducing at a maximum formula M. Then one can easily check that $\mathcal{D}''' \stackrel{*}{\Rightarrow} \mathcal{D}''$. By the assumption, $\mathcal{D}'''$ is valid and hence $\mathcal{D}''$ is also valid by Lemma 4.3.1. Therefore $\mathcal{D}'$ is valid by induction hypothesis.

Case 3: $\mathcal{D}'$ is obtained by reducing at a maximum formula occurring in $\mathcal{E}_2$.
Similar to Case 2. □

Lemma 4.3.8 (1) *Let $\mathcal{D}$ be a proof whose end-formula is of the form $A \wedge B$, $A \supset B$ or $\forall x A(x)$. If $\mathcal{D}$ is valid, then each E-proof of $\mathcal{D}$ is also valid.*
(2) *Let $\mathcal{D}$ be a proof whose end-formula is of the form $A \vee B$ or $\exists x A(x)$. If $\mathcal{D}$ is valid, then each E-proof of $\mathcal{D}$ is SN.*

Proof. We only prove (1). (2) is proved similarly. Furthermore, we only treat the case where the end-formula of $\mathcal{D}$ is of the form $A \supset B$.

Let $\mathcal{C}$ be an arbitrary E-proof of $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ A\supset B\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}\\ A\end{array}}{B}$$

where $\mathcal{E}$ is an arbitrary valid proof whose end-formula is A. In order to prove that $\mathcal{C}$ is valid, it suffices to show that each direct reduct of $\mathcal{C}$ is valid. We show this by double induction on $(l(\mathcal{D}) + l(\mathcal{E}), h(\mathcal{D}))$. Let $\mathcal{C}'$ be an arbitrary direct reduct of $\mathcal{C}$.

Case 1: $\mathcal{C}'$ is obtained by replacing $\mathcal{D}$ by its direct reduct or replacing $\mathcal{E}$ by its direct reduct.
The claim is obvious by induction hypothesis.

Case 2: $\mathcal{C}'$ is obtained by the proper reduction.
If $\mathcal{D}$ is in I-form, the claim is obvious by the definition of validity. Hence we consider the case where $\mathcal{D}$ is not in I-form. Then $\mathcal{C}$ is of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_0\\ C_1\vee C_2\end{array}\quad \begin{array}{c}[C_1]\\ \vdots\ \mathcal{D}_1\\ A\supset B\end{array}\quad \begin{array}{c}[C_2]\\ \vdots\ \mathcal{D}_2\\ A\supset B\end{array}}{A\supset B}\qquad \begin{array}{c}\vdots\ \mathcal{E}\\ A\end{array}}{B}$$

or

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \exists x C(x)\end{array}\quad \begin{array}{c}[C(a)]\\ \vdots\\ A\supset B\end{array}}{A\supset B}\qquad \begin{array}{c}\vdots\ \mathcal{E}\\ A\end{array}}{B}$$

We consider the former case. The latter case is treated similarly. Then $\mathcal{C}'$ is of the form

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_0\\ C_1\vee C_2\end{array}\quad \dfrac{\begin{array}{c}[C_1]\\ \vdots\ \mathcal{D}_1\\ A\supset B\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}\\ A\end{array}}{B}\quad \dfrac{\begin{array}{c}[C_2]\\ \vdots\ \mathcal{D}_2\\ A\supset B\end{array}\quad \begin{array}{c}\vdots\ \mathcal{E}\\ A\end{array}}{B}}{B}$$

Note that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are valid by the definition of validity. Therefore, for each $i = 1, 2$,

$$\dfrac{\begin{matrix}[C_i]\\ \vdots\,\mathcal{D}_i\\ A \supset B\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}\\ A\end{matrix}}{B}$$

is valid by induction hypothesis. In order to apply Lemma 4.3.7, we show that if there exists a maximum formula M occurring in $\mathcal{D}_0$, then a direct reduct of $\mathcal{C}'$ which is obtained by reducing at M is valid. Let $\mathcal{C}''$ be a direct reduct of $\mathcal{C}'$ obtained by reducing at M.

Subcase(a): $\mathcal{C}''$ is obtained by replacing $\mathcal{D}_0$ by its direct reduct.
$\mathcal{C}''$ is of the form

$$\dfrac{\begin{matrix}\vdots\,\mathcal{D}'_0\\ C_1 \vee C_2\end{matrix} \quad \dfrac{\begin{matrix}[C_1]\\ \vdots\,\mathcal{D}_1\\ A \supset B\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}\\ A\end{matrix}}{B} \quad \dfrac{\begin{matrix}[C_2]\\ \vdots\,\mathcal{D}_2\\ A \supset B\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}\\ A\end{matrix}}{B}}{B}$$

Now, consider the following proof $\mathcal{C}_0$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}'_0\\ C_1 \vee C_2\end{matrix} \quad \begin{matrix}[C_1]\\ \vdots\,\mathcal{D}_1\\ A \supset B\end{matrix} \quad \begin{matrix}[C_2]\\ \vdots\,\mathcal{D}_2\\ A \supset B\end{matrix}}{A \supset B} \quad \begin{matrix}\vdots\,\mathcal{E}\\ A\end{matrix}}{B}$$

By Case 1, this is valid. Furthermore $\mathcal{C}_0 \Rightarrow \mathcal{C}''$. Hence $\mathcal{C}''$ is valid by Lemma 4.3.1.

Subcase(b): $\mathcal{C}'$ is of the form

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\widetilde{\mathcal{D}}_0\\ C_i\end{matrix}}{C_1 \vee C_2} \quad \dfrac{\begin{matrix}[C_1]\\ \vdots\,\mathcal{D}_1\\ A \supset B\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}\\ A\end{matrix}}{B} \quad \dfrac{\begin{matrix}[C_2]\\ \vdots\,\mathcal{D}_2\\ A \supset B\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}\\ A\end{matrix}}{B}}{B}$$

and $\mathcal{C}''$ is of the form

$$\dfrac{\begin{matrix}\vdots\,\tilde{\mathcal{D}}_0\\ C_i\\ \vdots\,\mathcal{D}_i\\ A \supset B\end{matrix} \quad \begin{matrix}\vdots\,\mathcal{E}\\ A\end{matrix}}{B}$$

In this case, $\mathcal{C}$ is of the form

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \widetilde{\mathcal{D}}_0\\ C_i\end{matrix}}{C_1 \vee C_2}\quad \begin{matrix}[C_1]\\ \vdots\ \mathcal{D}_1\\ A \supset B\end{matrix}\quad \begin{matrix}[C_2]\\ \vdots\ \mathcal{D}_2\\ A \supset B\end{matrix}}{A \supset B}\quad \begin{matrix}\vdots\ \mathcal{E}\\ A\end{matrix}}{B}$$

Consider a direct reduct of $\mathcal{C}$ obtained by reducing at $C_1 \vee C_2$, which is valid by Case 1. This coincides with $\mathcal{C}''$. Hence $\mathcal{C}''$ is valid.

Subcase(c): $\mathcal{C}'$ is of the form

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ D_1 \vee D_2\end{matrix}\quad \begin{matrix}[D_1]\\ \vdots\\ C_1 \vee C_2\end{matrix}\quad \begin{matrix}[D_2]\\ \vdots\\ C_1 \vee C_2\end{matrix}}{C_1 \vee C_2}\quad \dfrac{\begin{matrix}[C_1]\\ \vdots\ \mathcal{D}_1\\ A \supset B\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}\\ A\end{matrix}}{B}\quad \dfrac{\begin{matrix}[C_2]\\ \vdots\ \mathcal{D}_2\\ A \supset B\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}\\ A\end{matrix}}{B}}{B}$$

and $\mathcal{C}''$ is of the form

$$\dfrac{\begin{matrix}\vdots\\ D_1 \vee D_2\end{matrix}\quad \dfrac{\begin{matrix}[D_1]\\ \vdots\\ C_1 \vee C_2\end{matrix}\quad \dfrac{\begin{matrix}[C_1]\\ \vdots\ \mathcal{D}_1\\ A \supset B\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}\\ A\end{matrix}}{B}\quad \dfrac{\begin{matrix}[C_2]\\ \vdots\ \mathcal{D}_2\\ A \supset B\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}\\ A\end{matrix}}{B}}{B}\quad \dfrac{\begin{matrix}[D_2]\\ \vdots\\ C_1 \vee C_2\end{matrix}\quad \dfrac{\begin{matrix}[C_1]\\ \vdots\ \mathcal{D}_1\\ A \supset B\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}\\ A\end{matrix}}{B}\quad \dfrac{\begin{matrix}[C_2]\\ \vdots\ \mathcal{D}_2\\ A \supset B\end{matrix}\quad \begin{matrix}\vdots\ \mathcal{E}\\ A\end{matrix}}{B}}{B}}{B}$$

In this case, $\mathcal{C}$ is of the form

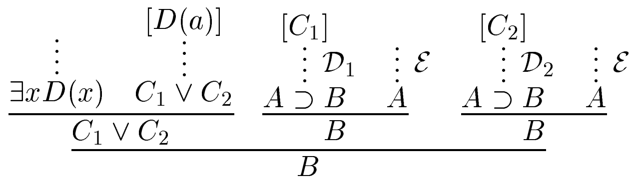
$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ D_1 \vee D_2\end{matrix}\quad \begin{matrix}[D_1]\\ \vdots\\ C_1 \vee C_2\end{matrix}\quad \begin{matrix}[D_2]\\ \vdots\\ C_1 \vee C_2\end{matrix}}{C_1 \vee C_2}\quad \begin{matrix}[C_1]\\ \vdots\ \mathcal{D}_1\\ A \supset B\end{matrix}\quad \begin{matrix}[C_2]\\ \vdots\ \mathcal{D}_2\\ A \supset B\end{matrix}}{A \supset B}\quad \begin{matrix}\vdots\ \mathcal{E}\\ A\end{matrix}}{B}$$

Let $\mathcal{C}'''$ be the following:

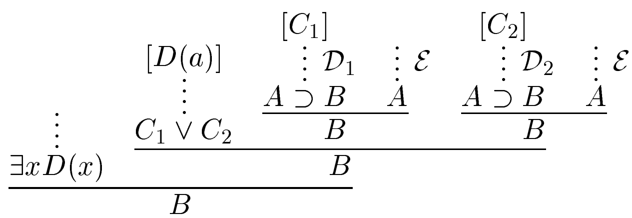
$$\dfrac{\dfrac{\begin{matrix}\vdots\\ D_1 \vee D_2\end{matrix}\quad \dfrac{\begin{matrix}[D_1]\\ \vdots\\ C_1 \vee C_2\end{matrix}\quad \begin{matrix}[C_1]\\ \vdots\ \mathcal{D}_1\\ A \supset B\end{matrix}\quad \begin{matrix}[C_2]\\ \vdots\ \mathcal{D}_2\\ A \supset B\end{matrix}}{A \supset B}\quad \dfrac{\begin{matrix}[D_2]\\ \vdots\\ C_1 \vee C_2\end{matrix}\quad \begin{matrix}[C_1]\\ \vdots\ \mathcal{D}_1\\ A \supset B\end{matrix}\quad \begin{matrix}[C_2]\\ \vdots\ \mathcal{D}_2\\ A \supset B\end{matrix}}{A \supset B}}{A \supset B}\quad \begin{matrix}\vdots\ \mathcal{E}\\ A\end{matrix}}{B}$$

By Case 1, this is valid. Furthermore, it is easy to see that $\mathcal{C}''' \overset{+}{\Rightarrow} \mathcal{C}''$. Hence $\mathcal{C}''$ is valid by Lemma 4.3.1.

Subcase(d): $\mathcal{C}'$ is of the form

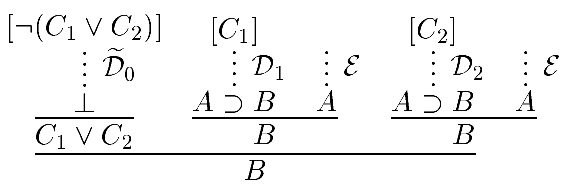


and $\mathcal{C}''$ is of the form

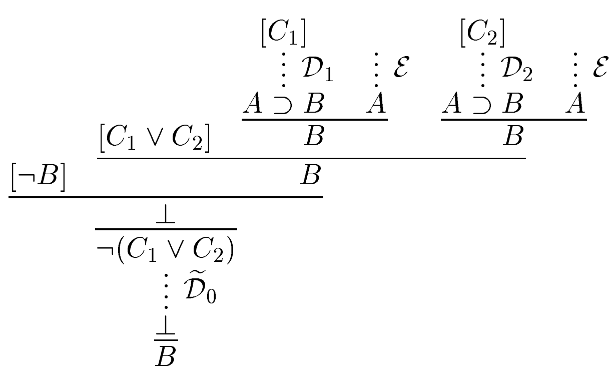


Similar to Subcase(c).

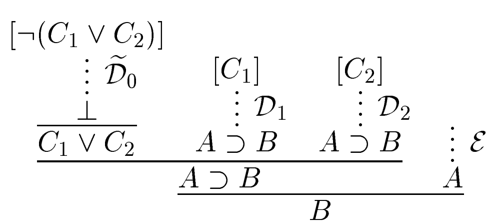
Subcase(e): $\mathcal{C}'$ is of the form



and $\mathcal{C}''$ is of the form



In this case, $\mathcal{C}$ is of the form



Let $\mathcal{C}'''$ be the following:

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg(A\supset B)] \qquad \dfrac{[C_1\lor C_2] \quad \begin{array}{c}[C_1]\\ \vdots\, \mathcal{D}_1\\ A\supset B\end{array} \quad \begin{array}{c}[C_2]\\ \vdots\, \mathcal{D}_2\\ A\supset B\end{array}}{A\supset B}}{\bot}}{\neg(C_1\lor C_2)}\\ \vdots\, \widetilde{\mathcal{D}}_0\\ \dfrac{\bot}{A\supset B}\end{array} \qquad \begin{array}{c}\vdots\,\mathcal{E}\\ A\end{array}}{B}$$

By Case 1, this is valid. Furthermore, $\mathcal{C}''' \overset{+}{\Rightarrow} \mathcal{C}''$ holds. Actually, $\mathcal{C}'''$ is reduced to

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{[\neg B] \quad \dfrac{[A\supset B] \quad \begin{array}{c}\vdots\,\mathcal{E}\\ A\end{array}}{B}}{\bot}}{\neg(A\supset B)} \qquad \dfrac{[C_1\lor C_2] \quad \begin{array}{c}[C_1]\\ \vdots\, \mathcal{D}_1\\ A\supset B\end{array} \quad \begin{array}{c}[C_2]\\ \vdots\, \mathcal{D}_2\\ A\supset B\end{array}}{A\supset B}}{\dfrac{\bot}{\neg(C_1\lor C_2)}}\\ \vdots\, \widetilde{\mathcal{D}}_0\\ \dfrac{\bot}{B}\end{array}$$

and this is reduced to

$$\begin{array}{c}\dfrac{\dfrac{[\neg B] \qquad \dfrac{\dfrac{[C_1\lor C_2] \quad \begin{array}{c}[C_1]\\ \vdots\, \mathcal{D}_1\\ A\supset B\end{array} \quad \begin{array}{c}[C_2]\\ \vdots\, \mathcal{D}_2\\ A\supset B\end{array}}{A\supset B} \quad \begin{array}{c}\vdots\,\mathcal{E}\\ A\end{array}}{B}}{\bot}}{\neg(C_1\lor C_2)}\\ \vdots\, \widetilde{\mathcal{D}}_0\\ \dfrac{\bot}{B}\end{array}$$

and finally this is reduced to $\mathcal{C}''$. Hence $\mathcal{C}''$ is valid by Lemma 4.3.1.

Therefore we conclude that $\mathcal{C}'$ is valid by Lemma 4.3.7. □

Lemma 4.3.9 *Let $\mathcal{D}$ be a proof whose last inference is* (RAA) *of the form*

$$\begin{array}{c}[\neg A]\\ \vdots\,\mathcal{D}_1\\ \dfrac{\bot}{A}\end{array}$$

and A is of the form $A_1 \vee A_2$ or $\exists x A_1(x)$. If each proof of the form

$$\begin{array}{c} \vdots\ \mathcal{E} \\ \neg A \\ \vdots\ \mathcal{D}_1 \\ \bot \end{array}$$

is valid for each valid proof $\mathcal{E}$ whose end-formula is $\neg A$, then $\mathcal{D}$ is also valid.

Proof. We only consider the case where A is of the form $A_1 \vee A_2$.

First, we note that $\mathcal{D}_1$ is SN by Lemma 4.3.2 and hence $\mathcal{D}$ is also SN.

Now, in order to prove that $\mathcal{D}$ is valid, we have to show that each E-proof of $\mathcal{D}$ is SN. Let $\mathcal{C}$ be an arbitrary E-proof of $\mathcal{D}$ of the form

$$\dfrac{\dfrac{\begin{array}{c}[\neg(A_1 \vee A_2)] \\ \vdots\ \mathcal{D}_1 \\ \bot\end{array}}{A_1 \vee A_2} \qquad \begin{array}{c}[A_1] \\ \vdots\ \mathcal{E}_1 \\ B\end{array} \qquad \begin{array}{c}[A_2] \\ \vdots\ \mathcal{E}_2 \\ B\end{array}}{B}\ R$$

where for each $i = 1, 2$, $\mathcal{E}_i$ is a proof such that a proof obtained from $\mathcal{E}_i$ by substituting an arbitrary valid proof whose end-formula is A_i for open assumptions which are discharged by R is SN. In order to prove that $\mathcal{C}$ is SN, it suffices to show that each direct reduct of $\mathcal{C}$ is SN. We show this by induction on $l(\mathcal{D}) + l(\mathcal{E}_1) + l(\mathcal{E}_2)$. Let $\mathcal{C}'$ be an arbitrary direct reduct of $\mathcal{C}$. If $\mathcal{C}'$ is obtained by a reduction which is not proper, then the proof is carried out by the similar way to the one for Lemma 4.3.5 (Case 2 and 3). Therefore we only consider the case where $\mathcal{C}'$ is obtained by proper reduction. In this case, $\mathcal{C}'$ is of the form

$$\begin{array}{c} \dfrac{\dfrac{[\neg B] \qquad \dfrac{[A_1 \vee A_2] \qquad \begin{array}{c}[A_1] \\ \vdots\ \mathcal{E}_1 \\ B\end{array} \qquad \begin{array}{c}[A_2] \\ \vdots\ \mathcal{E}_2 \\ B\end{array}}{B}}{\bot}}{\neg(A_1 \vee A_2)} \\ \vdots\ \mathcal{D}_1 \\ \bot \\ \hline\hline B \end{array}$$

In the following, we prove

$$\dfrac{\neg B \qquad \dfrac{\begin{array}{c}\vdots\ \mathcal{E}_0 \\ A_1 \vee A_2\end{array} \qquad \begin{array}{c}[A_1] \\ \vdots\ \mathcal{E}_1 \\ B\end{array} \qquad \begin{array}{c}[A_2] \\ \vdots\ \mathcal{E}_2 \\ B\end{array}}{B}}{\bot}$$

is valid where $\mathcal{E}_0$ is valid. First, this is SN since

$$\dfrac{\begin{matrix} \\ \vdots\ \mathcal{E}_0 \\ A_1 \vee A_2 \end{matrix} \quad \begin{matrix}[A_1] \\ \vdots\ \mathcal{E}_1 \\ B \end{matrix} \quad \begin{matrix}[A_2] \\ \vdots\ \mathcal{E}_2 \\ B \end{matrix}}{B}$$

is SN by Lemma 4.3.8(2). Therefore

$$\dfrac{\neg B \qquad \dfrac{\begin{matrix} \\ \vdots\ \mathcal{E}_0 \\ A_1 \vee A_2 \end{matrix} \quad \begin{matrix}[A_1] \\ \vdots\ \mathcal{E}_1 \\ B \end{matrix} \quad \begin{matrix}[A_2] \\ \vdots\ \mathcal{E}_2 \\ B \end{matrix}}{B}}{\bot}$$

is valid by Lemma 4.3.3 and hence

$$\dfrac{\dfrac{\neg B \qquad \dfrac{[A_1 \vee A_2] \quad \begin{matrix}[A_1] \\ \vdots\ \mathcal{E}_1 \\ B \end{matrix} \quad \begin{matrix}[A_2] \\ \vdots\ \mathcal{E}_2 \\ B \end{matrix}}{B}}{\bot}}{\neg(A_1 \vee A_2)}$$

valid by Lemma 4.3.4(2). Therefore

$$\begin{matrix} \dfrac{\dfrac{\neg B \qquad \dfrac{[A_1 \vee A_2] \quad \begin{matrix}[A_1] \\ \vdots\ \mathcal{E}_1 \\ B \end{matrix} \quad \begin{matrix}[A_2] \\ \vdots\ \mathcal{E}_2 \\ B \end{matrix}}{B}}{\bot}}{\neg(A_1 \vee A_2)} \\ \vdots\ \mathcal{D}_1 \\ \bot \end{matrix}$$

is valid by the assumption and hence SN by Lemma 4.3.2. Hence $\mathcal{C}'$ is also SN. □

Lemma 4.3.10 *Let $\mathcal{D}$ be a proof whose last inference is* (RAA) *of the form*

$$\dfrac{\begin{matrix}[\neg A] \\ \vdots\ \mathcal{D}_1 \\ \bot \end{matrix}}{A}$$

and A is atomic or of the form $A_1 \wedge A_2$, $A_1 \supset A_2$ or $\forall x A_1(x)$. If each proof of the form

$$\begin{matrix} \vdots\ \mathcal{E} \\ \neg A \\ \vdots\ \mathcal{D}_1 \\ \bot \end{matrix}$$

is valid for each valid proof $\mathcal{E}$ whose end-formula is $\neg A$, then $\mathcal{D}$ is also valid.

Proof. This is proved by (main-)induction on the degree of A.

Base step. By the definition of validity, it suffices to show that each direct reduct of $\mathcal{D}$ is valid. We prove this by (side-)induction on $l(\mathcal{D}_1)$. Let $\mathcal{D}'$ be an arbitrary direct reduct of $\mathcal{D}$. Then $\mathcal{D}'$ is of the form

$$\dfrac{\begin{matrix}[\neg A] \\ \vdots\ \mathcal{D}_1' \\ \bot\end{matrix}}{A}$$

where $\mathcal{D}_1'$ is a direct reduct of $\mathcal{D}_1$. Obviously $l(\mathcal{D}_1') < l(\mathcal{D}_1)$. Furthermore, for each valid proof $\mathcal{E}$ whose end-formula is $\neg A$,

$$\begin{matrix}\vdots\ \mathcal{E} \\ \neg A \\ \vdots\ \mathcal{D}_1' \\ \bot\end{matrix}$$

is valid since this is a direct reduct of

$$\begin{matrix}\vdots\ \mathcal{E} \\ \neg A \\ \vdots\ \mathcal{D}_1 \\ \bot\end{matrix}$$

which is valid by the assumption. Hence $\mathcal{D}'$ is valid by induction hypothesis.

Induction step. We only consider the case where A is of the form $A_1 \supset A_2$. Other cases are treated similarly.

First, we note that $\mathcal{D}_1$ is SN by Lemma 4.3.2 and hence $\mathcal{D}$ is also SN.

Now, in order to prove that $\mathcal{D}$ is valid, we have to show that each E-proof of $\mathcal{D}$ is valid. Let $\mathcal{C}$ be an arbitrary E-proof of $\mathcal{D}$ of the form

$$\dfrac{\dfrac{\begin{matrix}[\neg(A_1 \supset A_2)] \\ \vdots\ \mathcal{D}_1 \\ \bot\end{matrix}}{A_1 \supset A_2} \qquad \begin{matrix}\vdots\ \mathcal{E} \\ A_1\end{matrix}}{A_2}$$

where $\mathcal{E}$ is an arbitrary valid proof whose end-formula is A_1. In order to prove that $\mathcal{C}$ is valid, it suffices to show that each direct reduct of $\mathcal{C}$ is valid. We show this by (side-)induction on $l(\mathcal{D}) + l(\mathcal{E})$. Let $\mathcal{C}'$ be an arbitrary direct reduct of $\mathcal{C}$. If $\mathcal{C}'$ is obtained by a reduction which is not proper, then the proof is carried out by the similar way to the one for Lemma 4.3.4 (Case 2 and 3). Therefore we only consider the case where $\mathcal{C}'$

is obtained by proper reduction. In this case, $\mathcal{C}'$ is of the form

$$\begin{array}{c} \dfrac{\dfrac{[\neg A_2] \qquad \dfrac{[A_1 \supset A_2] \quad \begin{array}{c} \vdots\, \mathcal{E} \\ A_2 \end{array}}{A_2}}{\bot}}{\neg(A_1 \supset A_2)} \\ \vdots\, \mathcal{D}_1 \\ \dfrac{\bot}{A_2} \end{array}$$

First, we prove

$$\dfrac{\begin{array}{c} \vdots\, \mathcal{E}_0 \\ \neg A_2 \end{array} \qquad \dfrac{\begin{array}{c} \vdots\, \mathcal{E}_1 \\ A_1 \supset A_2 \end{array} \quad \begin{array}{c} \vdots\, \mathcal{E} \\ A_2 \end{array}}{A_2}}{\bot}$$

is valid where $\mathcal{E}_0$ and $\mathcal{E}_1$ are valid. This is easy by two applications of Lemma 4.3.8(1). Therefore

$$\dfrac{\dfrac{\begin{array}{c} \vdots\, \mathcal{E}_0 \\ \neg A_2 \end{array} \qquad \dfrac{[A_1 \supset A_2] \quad \begin{array}{c} \vdots\, \mathcal{E} \\ A_2 \end{array}}{A_2}}{\bot}}{\neg(A_1 \supset A_2)}$$

is valid by Lemma 4.3.4(2) and hence

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c} \vdots\, \mathcal{E}_0 \\ \neg A_2 \end{array} \qquad \dfrac{[A_1 \supset A_2] \quad \begin{array}{c} \vdots\, \mathcal{E} \\ A_2 \end{array}}{A_2}}{\bot}}{\neg(A_1 \supset A_2)} \\ \vdots\, \mathcal{D}_1 \\ \bot \end{array}$$

is valid by the assumption. This means that a proof

$$\begin{array}{c} \dfrac{\dfrac{\neg A_2 \qquad \dfrac{[A_1 \supset A_2] \quad \begin{array}{c} \vdots\, \mathcal{E} \\ A_2 \end{array}}{A_2}}{\bot}}{\neg(A_1 \supset A_2)} \\ \vdots\, \mathcal{D}_1 \\ \bot \end{array}$$

satisfies the assumption of the lemma. Hence $\mathcal{C}'$ is valid by Lemma 4.3.9 (if A_2 is of the form $B_1 \vee B_2$ or $\exists x B_1(x)$) or by (main-)induction hypothesis(otherwise). $\square$

Now we obtain the following.

Lemma 4.3.11 *The rule* (RAA) *is valid.*

Proof. Let $\mathcal{D}$ be a proof of the form

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots\ \mathcal{D}_1\\ \bot\end{array}}{A}$$

Then the claim is that if $\mathcal{D}_1$ is valid under substitution, then so is $\mathcal{D}$. Now, let $\mathcal{D}^*$ be a proof which is obtained from $\mathcal{D}$ by substitution. Then $\mathcal{D}^*$ is of the form

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots\ \mathcal{D}_1^*\\ \bot\end{array}}{A}$$

where $\mathcal{D}_1^*$ is obtained from $\mathcal{D}_1$ by substitution. Consider the following proof

$$\begin{array}{c}\vdots\ \mathcal{E}\\ \neg A\\ \vdots\ \mathcal{D}_1^*\\ \bot\end{array}$$

where $\mathcal{E}$ is an arbitrary valid proof whose end-formula is $\neg A$. This is also obtained from $\mathcal{D}_1$ by substitution and hence valid by the assumption. Therefore $\mathcal{D}^*$ is valid by Lemmata 4.3.9 or 4.3.10. □

4.3.4 Validity of the elimination rules

Finally, we consider the elimination rules. For this aim, we need the following which is an improvement of Lemma 4.3.8(2).

Lemma 4.3.12 (1) *Let $\mathcal{D}$ be a proof whose end-formula is of the form $A_1 \vee A_2$. If $\mathcal{D}$ is valid, then each proof of the form*

$$\dfrac{\begin{array}{c}\\ \vdots\ \mathcal{D}\\ A_1 \vee A_2\end{array} \quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{array}}{B}\ R$$

is valid where for each $i = 1, 2$, $\mathcal{E}_i$ is a proof such that a proof obtained from $\mathcal{E}_i$ by substituting an arbitrary valid proof whose end-formula is A_i for open assumptions which are discharged by R is valid.
(2) *Let $\mathcal{D}$ be a proof whose end-formula is of the form $\exists x A(x)$. If $\mathcal{D}$ is valid, then each proof of the form*

$$\dfrac{\begin{array}{c}\\ \vdots\ \mathcal{D}\\ \exists x A(x)\end{array} \quad \begin{array}{c}[A(a)]\\ \vdots\ \mathcal{E}\\ B\end{array}}{B}\ R$$

is valid where $\mathcal{E}$ is a proof such that a proof obtained from $\mathcal{E}$ by substituting t for a and substituting an arbitrary valid proof whose end-formula is $A(t)$ for open assumptions which are discharged by R is valid.

The proof of this lemma will be carried out in section 4.4.

Now we obtain the following.

Lemma 4.3.13 *The elimination rules are valid.*

Proof. For $(\wedge E)$, $(\supset E)$ and $(\forall E)$, the claim is easily proved by using Lemma 4.3.8(1).

Now consider $(\vee E)$. Let $\mathcal{D}$ be a proof of the form

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ A_1 \vee A_2\end{array}\quad \begin{array}{c}[A_1]\\ \vdots\,\mathcal{E}_1\\ B\end{array}\quad \begin{array}{c}[A_2]\\ \vdots\,\mathcal{E}_2\\ B\end{array}}{B}$$

Then the claim is that if $\mathcal{D}_1$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are valid under substitution, then so is $\mathcal{D}$. Now, let $\mathcal{D}^*$ be a proof which is obtained from $\mathcal{D}$ by substitution. Then $\mathcal{D}^*$ is of the form

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1^*\\ A_1 \vee A_2\end{array}\quad \begin{array}{c}[A_1]\\ \vdots\,\mathcal{E}_1^*\\ B\end{array}\quad \begin{array}{c}[A_2]\\ \vdots\,\mathcal{E}_2^*\\ B\end{array}}{B}$$

where $\mathcal{D}_1^*$ obtained from $\mathcal{D}_1$ by substitution and so on. Then for each $i = 1, 2$, each proof obtained from $\mathcal{E}_i^*$ by substituting an arbitrary valid proof whose end-formula is A_i for open assumptions which are discharged by R is also a proof obtained from $\mathcal{E}_i$ by substitution and hence valid by the assumption. Therefore $\mathcal{D}^*$ is valid by Lemma 4.3.12(1).

For $(\exists E)$, use Lemma 4.3.12(2). □

Proof of Theorem 4.3.1. By induction on the height of a proof, one can prove that all proofs are valid under substitution using Lemmata 4.3.6, 4.3.11 and 4.3.13. Especially, all proofs are valid. Hence all proofs are SN by Lemma 4.3.2. □

4.4 Proof of Lemma 4.3.12

In this section, we prove Lemma 4.3.12. For this aim, we need some preparations. First, we introduce the notion of "parallel reduction". This is an analogy to the one which is used in a proof of the Church-Rosser's theorem for β-reduction in λ-calculus.

Now we define the parallel reduction $\Rightarrow_p$ by induction on the structure of a proof as follows:

1.

$$A \Rightarrow_p A$$

where A is a formula.

2.

$$\frac{\mathcal{D}_1 \cdots \mathcal{D}_n}{A} \qquad \Rightarrow_p \qquad \frac{\mathcal{D}'_1 \cdots \mathcal{D}'_n}{A}$$

where $\mathcal{D}_i \Rightarrow_p \mathcal{D}'_i$ for $i = 1, \ldots, n$.

3.

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ A_1\end{matrix} \qquad \begin{matrix}\vdots\,\mathcal{D}_2\\ A_2\end{matrix}}{A_1 \wedge A_2}}{A_i} \qquad \Rightarrow_p \qquad \begin{matrix}\vdots\,\mathcal{D}'_i\\ A_i\end{matrix}$$

where $\mathcal{D}_i \Rightarrow_p \mathcal{D}'_i$ for $i = 1, 2$.

4.

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ A_i\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}[A_1]\\ \vdots\,\mathcal{E}_1\\ B\end{matrix} \qquad \begin{matrix}[A_2]\\ \vdots\,\mathcal{E}_2\\ B\end{matrix}}{B} \qquad \Rightarrow_p \qquad \begin{matrix}\vdots\,\mathcal{D}'\\ A_i\\ \vdots\,\mathcal{E}'_i\\ B\end{matrix}$$

where $\mathcal{D} \Rightarrow_p \mathcal{D}'$ and $\mathcal{E}_i \Rightarrow_p \mathcal{E}'_i$ for $i = 1, 2$.

5.

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\,\mathcal{D}\\ B\end{matrix}}{A \supset B} \qquad \begin{matrix}\vdots\,\mathcal{E}\\ A\end{matrix}}{B} \qquad \Rightarrow_p \qquad \begin{matrix}\vdots\,\mathcal{E}'\\ A\\ \vdots\,\mathcal{D}'\\ B\end{matrix}$$

where $\mathcal{D} \Rightarrow_p \mathcal{D}'$ and $\mathcal{E} \Rightarrow_p \mathcal{E}'$.

6.

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}(a)\\ A(a)\end{matrix}}{\forall x A(x)}}{A(t)} \qquad \Rightarrow_p \qquad \begin{matrix}\vdots\,\mathcal{D}'(t)\\ A(t)\end{matrix}$$

where $\mathcal{D}(a) \Rightarrow_p \mathcal{D}'(a)$.

7.

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}\\ A(t)\end{matrix}}{\exists x A(x)} \qquad \begin{matrix}[A(a)]\\ \vdots\,\mathcal{E}(a)\\ B\end{matrix}}{B} \qquad \Rightarrow_p \qquad \begin{matrix}\vdots\,\mathcal{D}'\\ A(t)\\ \vdots\,\mathcal{E}'(t)\\ B\end{matrix}$$

where $\mathcal{D} \Rightarrow_p \mathcal{D}'$ and $\mathcal{E}(a) \Rightarrow_p \mathcal{E}'(a)$.

8.

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ A_1 \vee A_2\end{matrix} \quad \begin{matrix}[A_1]\\ \vdots\,\mathcal{D}_2\\ C\end{matrix} \quad \begin{matrix}[A_2]\\ \vdots\,\mathcal{D}_3\\ C\end{matrix}}{C} \qquad \mathcal{E}_1 \quad \mathcal{E}_2}{B} \qquad \Rightarrow_p \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_1'\\ A_1 \vee A_2\end{matrix} \quad \dfrac{\begin{matrix}[A_1]\\ \vdots\,\mathcal{D}_2'\\ C\end{matrix} \quad \mathcal{E}_1' \quad \mathcal{E}_2'}{B} \quad \dfrac{\begin{matrix}[A_2]\\ \vdots\,\mathcal{D}_3'\\ C\end{matrix} \quad \mathcal{E}_1' \quad \mathcal{E}_2'}{B}}{B}$$

where $\mathcal{D}_i \Rightarrow_p \mathcal{D}_i'$ for $i = 1, 2, 3$ and $\mathcal{E}_i \Rightarrow_p \mathcal{E}_i'$ for $i = 1, 2$.

9.

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \exists x A(x)\end{matrix} \quad \begin{matrix}[A(a)]\\ \vdots\,\mathcal{D}_2\\ C\end{matrix}}{C} \qquad \mathcal{E}_1 \quad \mathcal{E}_2}{B} \qquad \Rightarrow_p \qquad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_1'\\ \exists x A(x)\end{matrix} \quad \dfrac{\begin{matrix}[A(a)]\\ \vdots\,\mathcal{D}_2'\\ C\end{matrix} \quad \mathcal{E}_1' \quad \mathcal{E}_2'}{B}}{B}$$

where $\mathcal{D}_i \Rightarrow_p \mathcal{D}_i'$ for $i = 1, 2$ and $\mathcal{E}_i \Rightarrow_p \mathcal{E}_i'$ for $i = 1, 2$.

10.

$$\dfrac{\dfrac{\begin{matrix}[\neg C]\\ \vdots\,\mathcal{D}\\ \bot\end{matrix}}{C} \qquad \mathcal{E}_1 \quad \mathcal{E}_2}{B} \qquad \Rightarrow_p \qquad \dfrac{\begin{matrix}\dfrac{\dfrac{[\neg B]^2 \qquad \dfrac{[C]^1 \quad \mathcal{E}_1' \quad \mathcal{E}_2'}{B}}{\bot}}{\neg C}\,(\supset I)^1\\ \vdots\,\mathcal{D}'\\ \bot\end{matrix}}{B}\,(\mathrm{RAA})^2$$

where $\mathcal{D} \Rightarrow_p \mathcal{D}'$ and $\mathcal{E}_i \Rightarrow_p \mathcal{E}_i'$ for $i = 1, 2$.

The following lemma is easily verified by the definition of $\Rightarrow_p$.

Lemma 4.4.1 (1) *$\mathcal{D} \Rightarrow_p \mathcal{D}$ for each proof $\mathcal{D}$.*
(2) *If $\mathcal{D} \Rightarrow \mathcal{D}'$, then $\mathcal{D} \Rightarrow_p \mathcal{D}'$.*
(3) *If $\mathcal{D} \Rightarrow_p \mathcal{D}'$, then $\mathcal{D} \overset{*}{\Rightarrow} \mathcal{D}'$.*
(4) *If $\mathcal{D} \Rightarrow_p \mathcal{D}'$ and $\mathcal{E} \Rightarrow_p \mathcal{E}'$, then*

$$\begin{matrix}\vdots\,\mathcal{E}\\ A\\ \vdots\,\mathcal{D}\\ B\end{matrix} \qquad \Rightarrow_p \qquad \begin{matrix}\vdots\,\mathcal{E}'\\ A\\ \vdots\,\mathcal{D}'\\ B\end{matrix}$$

Next, we define a binary relation on proofs $\Rightarrow_0$. $\mathcal{D} \Rightarrow_0 \mathcal{D}'$ denotes

- $\mathcal{D}$ is of the form

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\,\mathcal{D}\\ B\end{array}}{A \to B} \quad \begin{array}{c}\vdots\,\mathcal{E}\\ A\end{array}}{B}\\ \vdots\,\mathcal{C}\end{array}$$

  and $\mathcal{D}'$ is

$$\begin{array}{c}\vdots\,\mathcal{E}\\ A\\ \vdots\,\mathcal{D}\\ B\\ \vdots\,\mathcal{C}\end{array}$$

  or

- $\mathcal{D}$ is of the form

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\\ A_1 \vee A_2\end{array} \quad \begin{array}{c}[A_1]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\\ C\end{array}}{C} \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}\, R\\ \vdots\end{array}$$

  or

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\\ \exists x A(x)\end{array} \quad \begin{array}{c}[A(a)]\\ \vdots\\ C\end{array}}{C} \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}\, R\\ \vdots\end{array}$$

  or

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}[\neg C]\\ \vdots\\ \bot\end{array}}{C} \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}\, R\\ \vdots\end{array}$$

  which is satisfying that R is an elimination rule and B is not a maximum formula in $\mathcal{D}$, and $\mathcal{D}'$ is obtained from $\mathcal{D}$ by reducing at the major premise C of R.

The following lemma plays the important role in proving Lemma 4.3.12.

Lemma 4.4.2 *If $\mathcal{D} \Rightarrow_p \mathcal{D}'$ and $\mathcal{D} \Rightarrow_0 \mathcal{D}''$, then there exists $\mathcal{D}'''$ such that $\mathcal{D}' \overset{*}{\Rightarrow}_0 \mathcal{D}'''$ and $\mathcal{D}'' \Rightarrow_p \mathcal{D}'''$.*

Proof. This is proved by induction on $h(\mathcal{D})$.

Case 1: The reduction $\mathcal{D} \Rightarrow_0 \mathcal{D}''$ is proper.

Case 1.1: $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\ \mathcal{D}_1\\ B\end{matrix}}{A \supset B} \qquad \begin{matrix}\vdots\ \mathcal{D}_2\\ A\end{matrix}}{B}$$

In this case $\mathcal{D}''$ is

$$\begin{matrix}\vdots\ \mathcal{D}_2\\ A\\ \vdots\ \mathcal{D}_1\\ B\end{matrix}$$

Case 1.1.1: $\mathcal{D}'$ is

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\ \mathcal{D}'_1\\ B\end{matrix}}{A \supset B} \qquad \begin{matrix}\vdots\ \mathcal{D}'_2\\ A\end{matrix}}{B}$$

where $\mathcal{D}_i \Rightarrow_p \mathcal{D}'_i$ for $i = 1, 2$.
In this case, we take

$$\begin{matrix}\vdots\ \mathcal{D}'_2\\ A\\ \vdots\ \mathcal{D}'_1\\ B\end{matrix}$$

as $\mathcal{D}'''$ by Lemma 4.4.1(4).

Case 1.1.2: $\mathcal{D}'$ is

$$\begin{matrix}\vdots\ \mathcal{D}'_2\\ A\\ \vdots\ \mathcal{D}'_1\\ B\end{matrix}$$

where $\mathcal{D}_i \Rightarrow_p \mathcal{D}'_i$ for $i = 1, 2$.
In this case, we take $\mathcal{D}''' := \mathcal{D}'$.

Case 1.2: $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\begin{array}{c}[\neg C]\\ \vdots\\ \bot\end{array}}{C} \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}$$

Similar to Case 1.1.

Case 1.3: $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots \mathcal{D}_1\\ A_1 \vee A_2\end{array} \quad \begin{array}{c}[A_1]\\ \vdots \mathcal{D}_2\\ C\end{array} \quad \begin{array}{c}[A_2]\\ \vdots \mathcal{D}_3\\ C\end{array}}{C} \qquad \mathcal{E}_1 \quad \mathcal{E}_2}{B}$$

In this case, $\mathcal{D}''$ is

$$\dfrac{\begin{array}{c}\vdots \mathcal{D}_1\\ A_1 \vee A_2\end{array} \quad \dfrac{\begin{array}{c}[A_1]\\ \vdots \mathcal{D}_2\\ C\end{array} \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B} \quad \dfrac{\begin{array}{c}[A_2]\\ \vdots \mathcal{D}_3\\ C\end{array} \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}}{B}$$

Case 1.3.1: $\mathcal{D}'$ is

$$\dfrac{\dfrac{\begin{array}{c}\vdots \mathcal{D}'_1\\ A_1 \vee A_2\end{array} \quad \begin{array}{c}[A_1]\\ \vdots \mathcal{D}'_2\\ C\end{array} \quad \begin{array}{c}[A_2]\\ \vdots \mathcal{D}'_3\\ C\end{array}}{C} \qquad \mathcal{E}'_1 \quad \mathcal{E}'_2}{B}$$

where $\mathcal{D}_i \Rightarrow_p \mathcal{D}'_i$ for $i = 1, 2, 3$ and $\mathcal{E}_i \Rightarrow_p \mathcal{E}'_i$ for $i = 1, 2$.
In this case, we take

$$\dfrac{\begin{array}{c}\vdots \mathcal{D}'_1\\ A_1 \vee A_2\end{array} \quad \dfrac{\begin{array}{c}[A_1]\\ \vdots \mathcal{D}'_2\\ C\end{array} \quad \mathcal{E}'_1 \quad \mathcal{E}'_2}{B} \quad \dfrac{\begin{array}{c}[A_2]\\ \vdots \mathcal{D}'_3\\ C\end{array} \quad \mathcal{E}'_1 \quad \mathcal{E}'_2}{B}}{B}$$

as $\mathcal{D}'''$.

Case 1.3.2: $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \mathcal{C}\\ A_i\end{array}}{A_1 \vee A_2} \quad \begin{array}{c}[A_1]\\ \vdots \mathcal{D}_2\\ C\end{array} \quad \begin{array}{c}[A_2]\\ \vdots \mathcal{D}_3\\ C\end{array}}{C} \qquad \mathcal{E}_1 \quad \mathcal{E}_2}{B}$$

and $\mathcal{D}'$ is

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{C}'\\ A_i\\ \vdots\ \mathcal{D}'_{i+1}\\ C\end{array}\qquad \mathcal{E}'_1\quad \mathcal{E}'_2}{B}$$

In this case, we take $\mathcal{D}''' := \mathcal{D}'$.

Case 1.3.3: $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{C}_1\\ D_1 \lor D_2\end{array}\quad \begin{array}{c}[D_1]\\ \vdots\ \mathcal{C}_2\\ A_1 \lor A_2\end{array}\quad \begin{array}{c}[D_2]\\ \vdots\ \mathcal{C}_3\\ A_1 \lor A_2\end{array}}{A_1 \lor A_2}\quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{D}_2\\ C\end{array}\quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{D}_3\\ C\end{array}}{C}\qquad \mathcal{E}_1\quad \mathcal{E}_2}{B}$$

and $\mathcal{D}'$ is

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{C}'_1\\ D_1 \lor D_2\end{array}\quad \dfrac{\begin{array}{c}[D_1]\\ \vdots\ \mathcal{C}'_2\\ A_1 \lor A_2\end{array}\quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{D}'_2\\ C\end{array}\quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{D}'_3\\ C\end{array}}{C}\quad \dfrac{\begin{array}{c}[D_2]\\ \vdots\ \mathcal{C}'_3\\ A_1 \lor A_2\end{array}\quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{D}'_2\\ C\end{array}\quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{D}'_3\\ C\end{array}}{C}}{C}\qquad \mathcal{E}'_1\quad \mathcal{E}'_2}{B}$$

where $\mathcal{C}_i \Rightarrow_p \mathcal{C}'_i$ for $i = 1, 2, 3$, $\mathcal{D}_i \Rightarrow_p \mathcal{D}'_i$ for $i = 2, 3$ and $\mathcal{E}_i \Rightarrow_p \mathcal{E}'_i$ for $i = 1, 2$.
In this case, we take

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{C}'_1\\ D_1 \lor D_2\end{array}\quad \dfrac{\begin{array}{c}[D_1]\\ \vdots\ \mathcal{C}'_2\\ A_1 \lor A_2\end{array}\quad \dfrac{\begin{array}{c}[A_1]\\ \vdots\ \mathcal{D}'_2\\ C\end{array}\quad \mathcal{E}'_1\quad \mathcal{E}'_2}{B}\quad \dfrac{\begin{array}{c}[A_2]\\ \vdots\ \mathcal{D}'_3\\ C\end{array}\quad \mathcal{E}'_1\quad \mathcal{E}'_2}{B}}{B}\quad \dfrac{\begin{array}{c}[D_2]\\ \vdots\ \mathcal{C}'_3\\ A_1 \lor A_2\end{array}\quad \dfrac{\begin{array}{c}[A_1]\\ \vdots\ \mathcal{D}'_2\\ C\end{array}\quad \mathcal{E}'_1\quad \mathcal{E}'_2}{B}\quad \dfrac{\begin{array}{c}[A_2]\\ \vdots\ \mathcal{D}'_3\\ C\end{array}\quad \mathcal{E}'_1\quad \mathcal{E}'_2}{B}}{B}}{B}$$

as $\mathcal{D}'''$.

Case 1.3.4: $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{C}_1\\ \exists x D(x)\end{array}\quad \begin{array}{c}[D(a)]\\ \vdots\ \mathcal{C}_2\\ A_1 \lor A_2\end{array}}{A_1 \lor A_2}\quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{D}_2\\ C\end{array}\quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{D}_3\\ C\end{array}}{C}\qquad \mathcal{E}_1\quad \mathcal{E}_2}{B}$$

and $\mathcal{D}'$ is

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{C}'_1\\ \exists x D(x)\end{array} \quad \dfrac{\begin{array}{c}[D(a)]\\ \vdots\,\mathcal{C}'_2\\ A_1\lor A_2\end{array} \quad \begin{array}{c}[A_1]\\ \vdots\,\mathcal{D}'_2\\ C\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\,\mathcal{D}'_3\\ C\end{array}}{C}}{C} \quad \mathcal{E}'_1 \quad \mathcal{E}'_2}{B}$$

where $\mathcal{C}_i \Rightarrow_p \mathcal{C}'_i$ for $i = 1, 2$, $\mathcal{D}_i \Rightarrow_p \mathcal{D}'_i$ for $i = 2, 3$ and $\mathcal{E}_i \Rightarrow_p \mathcal{E}'_i$ for $i = 1, 2$. In this case, we take

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{C}'_1\\ \exists xD(x)\end{array} \quad \dfrac{\begin{array}{c}[D(a)]\\ \vdots\,\mathcal{C}'_2\\ A_1\lor A_2\end{array} \quad \dfrac{\begin{array}{c}[A_1]\\ \vdots\,\mathcal{D}'_2\\ C\end{array} \quad \mathcal{E}'_1 \quad \mathcal{E}'_2}{B} \quad \dfrac{\begin{array}{c}[A_2]\\ \vdots\,\mathcal{D}'_3\\ C\end{array} \quad \mathcal{E}'_1 \quad \mathcal{E}'_2}{B}}{B}}{B}$$

as $\mathcal{D}'''$.

Case 1.3.5: $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[\neg(A_1\lor A_2)]\\ \vdots\,\mathcal{C}\\ \bot\end{array}}{A_1\lor A_2} \quad \begin{array}{c}[A_1]\\ \vdots\,\mathcal{D}_2\\ C\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\,\mathcal{D}_3\\ C\end{array}}{C} \quad \mathcal{E}_1 \quad \mathcal{E}_2}{B}$$

and $\mathcal{D}'$ is

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg C] \quad \dfrac{[A_1\lor A_2] \quad \begin{array}{c}[A_1]\\ \vdots\,\mathcal{D}'_2\\ C\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\,\mathcal{D}'_3\\ C\end{array}}{C}}{\bot}}{\neg(A_1\lor A_2)}\\ \vdots\,\mathcal{C}'\\ \bot\end{array}}{C} \quad \mathcal{E}'_1 \quad \mathcal{E}'_2}{B}$$

where $\mathcal{C} \Rightarrow_p \mathcal{C}'$, $\mathcal{D}_i \Rightarrow_p \mathcal{D}'_i$ for $i = 2, 3$ and $\mathcal{E}_i \Rightarrow_p \mathcal{E}'_i$ for $i = 1, 2$.

In this case, we take

$$
\dfrac{\begin{matrix}\dfrac{\dfrac{[\neg B] \qquad \dfrac{[A_1 \vee A_2] \qquad \dfrac{\begin{matrix}[A_1]\\ \vdots\ \mathcal{D}_2'\\ C\end{matrix} \quad \mathcal{E}_1' \quad \mathcal{E}_2'}{B} \qquad \dfrac{\begin{matrix}[A_2]\\ \vdots\ \mathcal{D}_3'\\ C\end{matrix} \quad \mathcal{E}_1' \quad \mathcal{E}_2'}{B}}{B}}{\perp}}{\neg(A_1 \vee A_2)}\\ \vdots\ \mathcal{C}'\\ \perp\end{matrix}}{B}
$$

as $\mathcal{D}'''$.

Case 1.3.6: $\mathcal{D}'$ is of the form

$$
\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1'\\ A_1 \vee A_2\end{matrix} \qquad \dfrac{\begin{matrix}[A_1]\\ \vdots\ \mathcal{D}_2'\\ C\end{matrix} \quad \mathcal{E}_1' \quad \mathcal{E}_2'}{B} \qquad \dfrac{\begin{matrix}[A_2]\\ \vdots\ \mathcal{D}_3'\\ C\end{matrix} \quad \mathcal{E}_1' \quad \mathcal{E}_2'}{B}}{B}
$$

In this case, we take $\mathcal{D}''' := \mathcal{D}'$.

Case 1.4: $\mathcal{D}$ is of the form

$$
\dfrac{\dfrac{\begin{matrix}\vdots\\ \exists x A(x)\end{matrix} \qquad \begin{matrix}[A(a)]\\ \vdots\\ C\end{matrix}}{C} \qquad \mathcal{E}_1 \quad \mathcal{E}_2}{B}
$$

Similar to Case 1.3.

Case 2: The reduction $\mathcal{D} \Rightarrow_0 \mathcal{D}''$ is not proper.

Case 2.1: $\mathcal{D}$ is of the form

$$
\dfrac{\mathcal{D}_1 \cdots \mathcal{D}_n}{A}
$$

and $\mathcal{D}'$ is

$$
\dfrac{\mathcal{D}_1' \cdots \mathcal{D}_n'}{A}
$$

where $\mathcal{D}_i \Rightarrow_p \mathcal{D}_i'$ for $i = 1, \ldots, n$.
For instance, we consider the case where $\mathcal{D}''$ is

$$
\dfrac{\mathcal{D}_1'' \mathcal{D}_2 \cdots \mathcal{D}_n}{A}
$$

where $\mathcal{D}_1 \Rightarrow_0 \mathcal{D}_1''$. By induction hypothesis, there exists $\mathcal{D}_1'''$ such that $\mathcal{D}_1' \stackrel{*}{\Rightarrow}_0 \mathcal{D}_1'''$ and $\mathcal{D}_1'' \Rightarrow_p \mathcal{D}_1'''$. Hence we take

$$\frac{\mathcal{D}_1'''\mathcal{D}_2'\cdots\mathcal{D}_n'}{A}$$

as $\mathcal{D}'''$.

Case 2.2: $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\ \mathcal{D}_1\\ B\end{matrix}}{A\supset B}\quad \begin{matrix}\vdots\ \mathcal{D}_2\\ A\end{matrix}}{B}$$

and $\mathcal{D}'$ is

$$\begin{matrix}\vdots\ \mathcal{D}_2'\\ A\\ \vdots\ \mathcal{D}_1'\\ B\end{matrix}$$

where $\mathcal{D}_i \Rightarrow_p \mathcal{D}_i'$ for $i = 1, 2$.
For instance, we consider the case where $\mathcal{D}''$ is

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\ \mathcal{D}_1\\ B\end{matrix}}{A\supset B}\quad \begin{matrix}\vdots\ \mathcal{D}_2''\\ A\end{matrix}}{B}$$

where $\mathcal{D}_2 \Rightarrow_0 \mathcal{D}_2''$. By induction hypothesis, there exists $\mathcal{D}_2'''$ such that $\mathcal{D}_2' \stackrel{*}{\Rightarrow}_0 \mathcal{D}_2'''$ and $\mathcal{D}_2'' \Rightarrow_p \mathcal{D}_2'''$. Hence we take

$$\begin{matrix}\vdots\ \mathcal{D}_2'''\\ A\\ \vdots\ \mathcal{D}_1'\\ B\end{matrix}$$

as $\mathcal{D}'''$.

Case 2.3: $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ A_1\vee A_2\end{matrix}\quad \begin{matrix}[A_1]\\ \vdots\ \mathcal{D}_2\\ C\end{matrix}\quad \begin{matrix}[A_2]\\ \vdots\ \mathcal{D}_3\\ C\end{matrix}}{C}\quad \mathcal{E}_1\quad \mathcal{E}_2}{B}$$

and $\mathcal{D}'$ is

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1'\\ A_1\vee A_2\end{matrix}\quad \dfrac{\begin{matrix}[A_1]\\ \vdots\ \mathcal{D}_2'\\ C\end{matrix}\quad \mathcal{E}_1'\quad \mathcal{E}_2'}{B}\quad \dfrac{\begin{matrix}[A_2]\\ \vdots\ \mathcal{D}_3'\\ C\end{matrix}\quad \mathcal{E}_1'\quad \mathcal{E}_2'}{B}}{B}$$

where $\mathcal{D}_i \Rightarrow_p \mathcal{D}'_i$ for $i = 1,2,3$ and $\mathcal{E}_i \Rightarrow_p \mathcal{E}'_i$ for $i = 1,2$.
Since $\mathcal{D}''$ is not obtained by the reduction at $A_1 \vee A_2$ by the definition of $\Rightarrow_0$, $\mathcal{D}''$ is obtained by replacing $\mathcal{D}_i$ or $\mathcal{E}_i$ by its direct reduct. Hence the proof is similar to Case 2.2.

Other cases are treated similarly. □

Lemma 4.4.3 (1) *If $\mathcal{D} \Rightarrow_p \mathcal{D}'$ and $\mathcal{D} \overset{*}{\Rightarrow}_0 \mathcal{D}''$, then there exists $\mathcal{D}'''$ such that $\mathcal{D}' \overset{*}{\Rightarrow}_0 \mathcal{D}'''$ and $\mathcal{D}'' \Rightarrow_p \mathcal{D}'''$.*
(2) *If $\mathcal{D} \Rightarrow \mathcal{D}'$ and $\mathcal{D} \overset{*}{\Rightarrow}_0 \mathcal{D}''$, then there exists $\mathcal{D}'''$ such that $\mathcal{D}' \overset{*}{\Rightarrow}_0 \mathcal{D}'''$ and $\mathcal{D}'' \overset{*}{\Rightarrow} \mathcal{D}'''$.*

Proof. (1) By applying Lemma 4.4.2 repeatedly.
(2) By (1), Lemma 4.4.1(2) and (3). □

Suppose that $\mathcal{D} \overset{*}{\Rightarrow}_0 \mathcal{E}$ and $\mathcal{E}$ is SN. Let T_0 be the subtree of the reduction tree of $\mathcal{D}$ which consists of all paths containing $\mathcal{E}$. Since $\mathcal{E}$ is SN, the length of each path of T_0 is finite. Hence, by using König's lemma, there exists the maximum of the lengths of all paths of T_0. This is denoted by $l_0(\mathcal{D}, \mathcal{E})$. We remark that in Lemma 4.4.3(2), if $\mathcal{D}''$ is SN and $\mathcal{D}''' = \mathcal{D}''$, then $l_0(\mathcal{D}, \mathcal{D}'') > l_0(\mathcal{D}', \mathcal{D}'')$.

Lemma 4.4.4 *Suppose that $\mathcal{D} \overset{*}{\Rightarrow}_0 \mathcal{D}'$. If $\mathcal{D}'$ is SN, then so is $\mathcal{D}$.*

Proof. Suppose that $\mathcal{D}$ is not SN. Then there exists an infinite reduction sequence starting from $\mathcal{D}$ as follows:

$$\mathcal{D}(= \mathcal{D}_0) \Rightarrow \mathcal{D}_1 \Rightarrow \mathcal{D}_2 \Rightarrow \cdots \Rightarrow \mathcal{D}_n \Rightarrow \cdots$$

By Lemma 4.4.3(2), there exists a sequence

$$\mathcal{D}'(= \mathcal{D}'_0) \overset{*}{\Rightarrow} \mathcal{D}'_1 \overset{*}{\Rightarrow} \mathcal{D}'_2 \overset{*}{\Rightarrow} \cdots \overset{*}{\Rightarrow} \mathcal{D}'_n \overset{*}{\Rightarrow} \cdots$$

where $\mathcal{D}_i \overset{*}{\Rightarrow}_0 \mathcal{D}'_i$ for all $i \in \mathbf{N}$. By the above remark, there exists no $k \in \mathbf{N}$ such that $\mathcal{D}_n = \mathcal{D}_k$ for all $n \geq k$. Therefore we obtain an infinite reduction sequence starting from $\mathcal{D}'$, which contradicts the fact that $\mathcal{D}'$ is SN. Hence $\mathcal{D}$ is SN. □

Lemma 4.4.5 *Let $\mathcal{D}$ be a proof whose last inference is* (RAA) *of the form*

$$\dfrac{\begin{matrix}[\neg A] \\ \vdots\ \mathcal{D}_1 \\ \bot\end{matrix}}{A}$$

and let $\mathcal{C}$ be an arbitrary E-proof of $\mathcal{D}$. If $\mathcal{D}$ is SN and the reduct of $\mathcal{C}$ which is obtained from the proper reduction is valid, then $\mathcal{C}$ is also valid (*and hence $\mathcal{D}$ is valid*).

Proof. Suppose $\mathcal{C}$ is of the form

$$\dfrac{\dfrac{\begin{matrix}[\neg A] \\ \vdots\, \mathcal{D}_1 \\ \bot\end{matrix}}{A} \qquad \mathcal{E}_1 \quad \mathcal{E}_2}{B}$$

In order to prove that $\mathcal{C}$ is valid, it suffices to show that each direct reduct of $\mathcal{C}$ is valid. We show this by induction on $l(\mathcal{D}) + l(\mathcal{E}_1) + l(\mathcal{E}_2)$. Let $\mathcal{C}'$ be an arbitrary direct reduct of $\mathcal{C}$.

Case 1: $\mathcal{C}'$ is obtained by the proper reduction.
The claim is obvious by the assumption.

Case 2: $\mathcal{C}'$ is obtained by replacing $\mathcal{D}$ by its direct reduct $\mathcal{D}'$.
Obviously $l(\mathcal{D}') < l(\mathcal{D})$. Furthermore let $\mathcal{C}_1$ (resp.$\mathcal{C}_1'$) be the reduct of $\mathcal{C}$ (resp.$\mathcal{C}'$) which is obtained by the proper reduction. Then obviously $\mathcal{C}_1 \Rightarrow \mathcal{C}_1'$ holds. Since $\mathcal{C}_1$ is valid by the assumption, $\mathcal{C}_1'$ is valid by Lemma 4.3.1. Hence $\mathcal{C}'$ is valid by induction hypothesis.

Case 3: $\mathcal{C}'$ is obtained by replacing $\mathcal{E}_1$ or $\mathcal{E}_2$ by its direct reduct.
Similar to Case 2. □

Now, we finally reach the goal.

Proof of Lemma 4.3.12. We only prove (1). (2) is treated similarly. The claim is that an arbitrary proof $\mathcal{C}$ of the form

$$\dfrac{\begin{matrix} \\ \vdots\, \mathcal{D} \\ A_1 \vee A_2\end{matrix} \quad \begin{matrix}[A_1] \\ \vdots\, \mathcal{E}_1 \\ B\end{matrix} \quad \begin{matrix}[A_2] \\ \vdots\, \mathcal{E}_2 \\ B\end{matrix}}{B}$$

is valid, where $\mathcal{D}$ is valid and for each $i = 1, 2$, $\mathcal{E}_i$ satisfies the condition mentioned the lemma. For this aim, it suffices to show that each direct reduct of $\mathcal{C}$ is valid. We show this by double induction on $(l(\mathcal{D}) + l(\mathcal{E}_1) + l(\mathcal{E}_2), h(\mathcal{D}))$. Let $\mathcal{C}'$ be an arbitrary direct reduct of $\mathcal{C}$.

Case 1: $\mathcal{C}'$ is obtained by a reduction which is not proper.
The claim is obvious by induction hypothesis.

Case 2: $\mathcal{C}'$ is obtained by the proper reduction.

Case 2.1 The last inference of $\mathcal{D}$ is $(\vee I)$.
Easy by the assumption for $\mathcal{E}_i$, $i = 1, 2$.

Case 2.2 The last inference of $\mathcal{D}$ is $(\vee E)$ or $(\exists E)$.

Similar to the proof of Case 2 of Lemma 4.3.8.

Case 2.3 The last inference of $\mathcal{D}$ is (RAA).
$\mathcal{C}$ is of the form

$$\dfrac{\dfrac{\begin{matrix}[\neg(A_1 \vee A_2)]\\ \vdots\ \mathcal{D}_1\\ \bot\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}[A_1]\\ \vdots\ \mathcal{E}_1\\ B\end{matrix} \qquad \begin{matrix}[A_2]\\ \vdots\ \mathcal{E}_2\\ B\end{matrix}}{B}\ R$$

Let T be a tree which satisfies the following properties:

1. The root of T is $\mathcal{C}$;
2. For each node $\mathcal{T}$ of T, $\mathcal{T}'$ is an immediate successor of $\mathcal{T}$ if and only if $\mathcal{T}'$ is an E-proof of $\mathcal{T}$;
3. If $\mathcal{T}$ is a node of T which is neither the root nor the immediate predecessor of a leaf, then the end-formula of $\mathcal{T}$ is of the form neither $C_1 \vee C_2$ nor $\exists x C(x)$;
4. Each leaf $\mathcal{T}$ of T satisfies one of the following conditions:
    (a) The end-formula of the immediate predecessor of $\mathcal{T}$ is of the form $C_1 \vee C_2$ or $\exists x C(x)$;
    (b) The end-formula of $\mathcal{T}$ is atomic.

Note that the level of each node of T is not greater than $d(B)$. Therefore there exists the maximum N of the levels of all nodes of T.

Let $\mathcal{T}$ be an arbitrary node of T. Then $\mathcal{T}$ is of the form

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[\neg(A_1 \vee A_2)]\\ \vdots\ \mathcal{D}_1\\ \bot\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}[A_1]\\ \vdots\ \mathcal{E}_1\\ B_0\end{matrix} \qquad \begin{matrix}[A_2]\\ \vdots\ \mathcal{E}_2\\ B_0\end{matrix}}{B_0} \qquad \mathcal{E}_1^1}{B_1} \qquad \mathcal{E}_1^2}{B_2}\\ \vdots\\ \dfrac{B_{k-1} \qquad \mathcal{E}_1^k \quad \mathcal{E}_2^k}{B_k}\end{array}$$

where B_0 is identical with B. Note that the above expression includes the case where

$k = 0$, i.e., $\mathcal{T}$ is $\mathcal{C}$. Now, for $i = 1, 2$, let $\mathcal{T}^i$ be a proof of the form

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}[A_i]\\ \vdots\ \mathcal{E}_i\\ B_0\end{array}\quad \mathcal{E}_1^1}{B_1}\quad \mathcal{E}_1^2}{B_2} \\ \vdots \\ \dfrac{B_{k-1}\quad \mathcal{E}_1^k\quad \mathcal{E}_2^k}{B_k}\end{array}$$

and let $\widetilde{\mathcal{T}}$ be a proof of the form

$$\dfrac{\dfrac{\begin{array}{c}[\neg(A_1 \vee A_2)]\\ \vdots\ \mathcal{D}_1\\ \bot\end{array}}{A_1 \vee A_2} \quad \begin{array}{c}[A_1]\\ \vdots\ \mathcal{T}^1\\ B_k\end{array} \quad \begin{array}{c}[A_2]\\ \vdots\ \mathcal{T}^2\\ B_k\end{array}}{B_k}\ R'$$

where the assumption formulae which are discharged by R' are the same as the one for R of $\mathcal{C}$. Then for each $i = 1, 2$, a proof obtained from $\mathcal{T}^i$ by substituting an arbitrary valid proof whose end-formula is A_i for open assumptions which are discharged by R' is valid (by applying Lemma 4.3.8 repeatedly) and hence SN by Lemma 4.3.2. Therefore $\widetilde{\mathcal{T}}$ is SN by the validity of $\mathcal{D}$. Furthermore $\mathcal{T} \overset{*}{\Rightarrow}_0 \widetilde{\mathcal{T}}$ holds. Hence $\mathcal{T}$ is SN by Lemma 4.4.4.

Now, for each node $\mathcal{T}$, we define a proof $\mathcal{T}^\bullet$ by induction on the level of $\mathcal{T}$ as follows:

1. If $\mathcal{T}$ is the root, i.e., $\mathcal{T} = \mathcal{C}$, then $\mathcal{T}^\bullet$ is $\mathcal{C}'$, i.e., a reduct of $\mathcal{C}$ obtained by the proper reduction;

2. If $\mathcal{T}$ is an immediate successor of $\mathcal{T}_0$ of the form

$$\dfrac{\mathcal{T}_0 \quad \mathcal{E}_1^k \quad \mathcal{E}_2^k}{B_k}$$

   then $\mathcal{T}^\bullet$ is the proof obtained from the following proof

$$\dfrac{\mathcal{T}_0^\bullet \quad \mathcal{E}_1^k \quad \mathcal{E}_2^k}{B_k}$$

   by the proper reduction.

Note that the last inference of each $\mathcal{T}^\bullet$ is (RAA) and hence the clause 2 of the above definition is meaningful. It is easy to see that $\mathcal{T} \overset{+}{\Rightarrow} \mathcal{T}^\bullet$. Therefore $\mathcal{T}^\bullet$ is SN for each node $\mathcal{T}$ since $\mathcal{T}$ is SN. In the following, we show that if $\mathcal{T}$ is a node with the level $N-k$ which is not a leaf, then $\mathcal{T}^\bullet$ is valid by induction on k. In order to prove that $\mathcal{T}^\bullet$ is valid, it suffices to show that each E-proof of $\mathcal{T}^\bullet$ is valid since the last inference of $\mathcal{T}^\bullet$

is (RAA). First, we consider the case where $\mathcal{T}$ is the immediate predecessor of a leaf. Let $\mathcal{T}^*$ be an arbitrary E-proof of $\mathcal{T}^\bullet$. Then $\mathcal{T}^*$ is of the form

$$\frac{\mathcal{T}^\bullet \quad \mathcal{E}_1^* \quad \mathcal{E}_2^*}{B_*}$$

Let $\mathcal{T}^{**}$ be a proof of the form

$$\frac{\mathcal{T} \quad \mathcal{E}_1^* \quad \mathcal{E}_2^*}{B_*}$$

Then $\mathcal{T}^{**}$ is a node of T since $\mathcal{T}^{**}$ is an immediate successor of $\mathcal{T}$. Hence $\mathcal{T}^{**}$ is SN. Furthermore $\mathcal{T}^{**} \overset{+}{\Rightarrow} \mathcal{T}^*$ holds. Therefore $\mathcal{T}^*$ is SN, i.e., each E-proof of $\mathcal{T}^\bullet$ is SN. Hence,

- If the end-formula of $\mathcal{T}$ is of the form $C_1 \vee C_2$ or $\exists x C(x)$ then $\mathcal{T}^\bullet$ is valid by the definition of validity;
- Otherwise, i.e., if the end-formula of each immediate successor of $\mathcal{T}$ is atomic, then each E-proof of $\mathcal{T}^\bullet$ is valid by Lemma 4.3.3 and hence $\mathcal{T}^\bullet$ is valid by the definition of validity.

Next, we consider the induction step. Let $\mathcal{T}^*$ be an arbitrary E-proof of $\mathcal{T}$. Then $(\mathcal{T}^*)^\bullet$ is valid by induction hypothesis. Furthermore $\mathcal{T}^\bullet$ is SN. Therefore $\mathcal{T}^\bullet$ is valid by Lemma 4.4.5.

Especially, $\mathcal{C}'$ is valid. This completes the proof. □

4.5 A remark on the Church-Rosser property

The Church-Rosser property for $\Rightarrow$ holds, i.e.,

Theorem 4.5.1 *If $\mathcal{D} \overset{*}{\Rightarrow} \mathcal{D}'$ and $\mathcal{D} \overset{*}{\Rightarrow} \mathcal{D}''$, then there exists $\mathcal{D}'''$ such that $\mathcal{D}' \overset{*}{\Rightarrow} \mathcal{D}'''$ and $\mathcal{D}'' \overset{*}{\Rightarrow} \mathcal{D}'''$.*

By Newman's Lemma ([20]), it suffices to show the following in order to prove the Church-Rosser property since we have already established the strong normalization theorem:

Lemma 4.5.1 *If $\mathcal{D} \Rightarrow \mathcal{D}'$ and $\mathcal{D} \Rightarrow \mathcal{D}''$, then there exists $\mathcal{D}'''$ such that $\mathcal{D}' \overset{*}{\Rightarrow} \mathcal{D}'''$ and $\mathcal{D}'' \overset{*}{\Rightarrow} \mathcal{D}'''$.*

Note that our reduction steps except the one for (RAA) is the same as the Prawitz's ones for the system of the intuitionistic logic in [24], which satisfies the Church-Rosser property. Hence we may assume that $\mathcal{D} \Rightarrow \mathcal{D}'$ (or $\mathcal{D} \Rightarrow \mathcal{D}''$) is the reduction for (RAA). By this restriction, one can easily verify the lemma by the similar argument of the proof of Lemma 4.4.2. Therefore we omit the details.

Chapter 5

Provable well-founded relations of subsystems of the first-order arithmetic

To the last chapter, the main topics we have discussed are "cut-elimination theorem" and "normalization theorem", which are the most important matters in the proof theoretical studies. On the other hand, there exists another main matter in the proof theory: what is called "consistency problem". In this chapter, we treat this topic.

What we discuss below is "provable well-founded relations" (which include provable well-orderings) of subsystems of the first-order arithmetic. "Provable well-orderings" of the first-order arithmetic (**PA**) have been studied firstly by Gentzen, and this result was refined by Takeuti. Furthermore Arai recently extended Takeuti's result to "provable well-founded relations". In this chapter, we discuss Arai's result for subsystems of **PA**. The contents of this chapter are based on [16].

5.1 Introduction

5.1.1 System PA

Through this chapter, we use the system **PA** for Peano arithmetic which is formalized in a sequent calculus. In this subsection, we summarize this system.

First, we assume as usual that **PA** is formalized in a language which includes function constants for all primitive recursive function. We call this language $\mathcal{L}$.

Next, as initial sequents of **PA** except of the form $A \to A$ (which are called logical initial sequents), we have the equality axioms and mathematical initial sequents. Mathematical initial sequents are the defining equations for all primitive recursive functions in quantifier-free styles like

$$\to s + 0 = s$$

and

$$\to s + t' = (s + t)'$$

for $+$, and all sequents $\rightarrow s = t$, where s and t are closed terms which have same values, and all sequents $s = t \rightarrow$, where s and t are closed terms which have different values.

Finally, we add an inference rule

$$\frac{F(a), \Gamma \rightarrow \Delta, F(a')}{F(0), \Gamma \rightarrow \Delta, F(t)} \text{(Ind)}$$

where a is not in $F(0)$, Γ or Δ, t is an arbitrary term (which may be contain a). $F(a)$ is called the *induction formula* and a is called the *eigenvariable* of this inference.

Proposition 5.1.1 *If A is a closed bounded formula of $\mathcal{L}$ then either $\rightarrow A$ or $A \rightarrow$ is* **PA***-provable without an essential cut or induction inference.*

Proof. By induction on the complexity of A. □

If $\rightarrow A$ is provable A is said to be *true*, otherwise A is said to be *false*. A sequent is said to be *true* if there exists a false formula in the antecedent or there exists a true formula in the succedent. Otherwise a sequent is said to be *false*.

5.1.2 History and motivation

In this subsection, we explain history and motivation of the problem treating this chapter. First we define the system $\mathbf{PA}(\varepsilon)$.

Let ε be a new predicate constant. $\mathcal{L}(\varepsilon)$ is the language extending $\mathcal{L}$, formed by admitting $\varepsilon(t)$ as an atomic formula for all terms t. $\mathbf{PA}(\varepsilon)$ is the system **PA** in the language $\mathcal{L}(\varepsilon)$. Hence as mathematical initial sequents we add $s = t, \varepsilon(s) \rightarrow \varepsilon(t)$ for all terms s and t.

As is well-known (see 1.1.4), Gentzen proved that the transfinite induction up to ε_0 (first ε-number) can not be proved in **PA** ([10]). Takeuti refined this result as follows (see section 13 of [33]). Let $\prec$ be a recursive well-ordering. If

$$\forall x(\forall y(y \prec x \supset \varepsilon(y)) \supset \varepsilon(x)) \rightarrow \varepsilon(a)$$

is provable in $\mathbf{PA}(\varepsilon)$, $\prec$ is called a provable well-ordering of **PA**. Takeuti showed that if $\prec$ is a provable well-ordering of **PA**, then there exists a recursive function f such that $a \prec b \Leftrightarrow f(a) <^* f(b)$, where $<^*$ denotes the standard ordering of type ε_0, and there exists an ordinal $\mu < \varepsilon_0$ such that for every a, $f(a) <^* \ulcorner \mu \urcorner$ (where $\ulcorner \mu \urcorner$ denotes the Gödel number of μ). Recently Arai extended the above the case where $\prec$ is well-founded relation ([3, Section1]).

Now, consider that extending the above results to $I\Sigma_k$ for each $k \in \mathbf{N}$ (which is a subsystem of **PA**). Since the proof-theoretical ordinal of $I\Sigma_k$ is ω_{k+1} (Mints[19]), we would like to construct f for provable well-founded relation $\prec$ in $I\Sigma$ such that $a \prec b \Leftrightarrow f(a) <^* f(b)$, where $<^*$ denotes the standard ordering of type ω_{k+1}, and there exists an ordinal $\mu < \omega_{k+1}$ such that for every a, $f(a) <^* \ulcorner \mu \urcorner$. However, there exist two difficulties for this aim. By Gentzen's method, if $\prec$ is provable well-ordering, then there exists an ordinal $\alpha < \varepsilon_0$ such that $|\prec| < \alpha$. This estimate, however, is a little rough. That is to say, while $\omega_n \leq \alpha < \omega_{n+1}$ holds for some n in Gentzen's proof, we can take $\alpha < \omega_n$ in Mints' proof. Takeuti's refinement is deeply depend on Gentzen's proof and hence this is the first difficulty for treating $I\Sigma_k$.

Secondly, in Takeuti and Arai's refinements, the above μ may be greater than $|\prec|$. More precisely, on the construction of f, though $|\prec| < \omega_n$ for some n, μ may be greater than ω_n. Therefore Takeuti and Arai's methods can not apply directly when we consider the case where the base system is $I\Sigma_k$.

We can overcome only the first difficulty. Therefore the result we can obtain is as follows: If $\prec$ is a provable well-founded relation of $I\Sigma_k$, then there exists primitive recursive function f such that $a \prec b \Leftrightarrow f(a) <^* f(b)$, where $<^*$ denotes the standard ordering of type ω_{k+2}, and there exists an ordinal $\mu < \omega_{k+2}$ such that for every a, $f(a) <^* \ulcorner \mu \urcorner$.

5.2 Provable well-founded relations

We define Σ_k- and Π_k-formulae as follows:

1. Σ_0-formulae=Π_0-formulae=bounded formulae,
2. Σ_{k+1}-formulae have the form $\exists x A$ where A is Π_k-formula,
3. Π_{k+1}-formulae have the form $\forall x A$ where A is Σ_k-formula.

$I\Sigma_k$ is obtained from **PA** by restricting induction formulae to Σ_k-formulae. We define $I\Sigma_k(\varepsilon)$ as similar to $\mathbf{PA}(\varepsilon)$.

Remark. According to the extension of the language, Σ_k and Π_k are extended.

$\bar{n}$ denotes the n-th numeral. The following proposition is well-known.

Proposition 5.2.1 *The following sequents are $(I\Sigma(\varepsilon))$-provable without cut or induction inference:*

$$\forall x(x \leq \bar{n} \supset A(x)) \to A(\bar{0}) \wedge A(\bar{1}) \wedge \ldots \wedge A(\bar{n}),$$

$$A(\bar{0}) \wedge A(\bar{1}) \wedge \ldots \wedge A(\bar{n}) \to \forall x(x \leq \bar{n} \supset A(x)),$$

$$\exists x(x \leq \bar{n} \wedge A(x)) \to A(\bar{0}) \vee A(\bar{1}) \vee \ldots \vee A(\bar{n}),$$

$$A(\bar{0}) \vee A(\bar{1}) \vee \ldots \vee A(\bar{n}) \to \exists x(x \leq \bar{n} \wedge A(x)).$$

Proof. cf. e.g. [11, p.55] □

Let $\prec$ be a binary primitive recursive relation of the natural numbers which is well-founded. We use the same symbol $\prec$ in order to denote the formula expressing the relation $\prec$. If the sequent

$$\forall x(\forall y(y \prec x \supset \varepsilon(y)) \supset \varepsilon(x)) \to \varepsilon(a)$$

is provable in $I\Sigma_k$, then $\prec$ is said to be *provable well-founded* in $I\Sigma_k$.

In the following we assume the standard Gödel numbering of the ordinal less than ω_{k+2}. For ordinal $\alpha < \omega_{k+2}$, $\ulcorner \alpha \urcorner$ denotes the Gödel number of α. $<^*$ denotes the order relation of the natural numbers such that $\alpha < \beta \Leftrightarrow \ulcorner \alpha \urcorner <^* \ulcorner \beta \urcorner$.

Then our theorem is the following:

Theorem 5.2.2 *Let $\prec$ is an irreflexive and transitive relation of natural numbers which is provable well-founded in $I\Sigma_k$. Then there exists a primitive recursive function f such that $a \prec b$ if and only if $f(a) <^* f(b)$ and there exists an ordinal $\mu < \omega_{k+2}$ such that $f(a) <^* \ulcorner \mu \urcorner$ for all $a \in \mathbf{N}$.*

The proof of this theorem will be carried out in Section 5.4.

5.3 TJ-proofs

In this section, we define the notion of **TJ**-proofs. In the following we fix $k \in \mathbf{N}$. We use the terminology in [33].

From now on, let $\prec$ be a fixed irreflexive and transitive relation of natural numbers which is provable well-founded in $I\Sigma_k$.

TJ-*proofs* are defined as $I\Sigma_k(\varepsilon)$-proofs with some following modifications:

1. The initial sequents of a **TJ**-proof are those of $I\Sigma_k(\varepsilon)$ and the **TJ**-initial sequents of the following form:
$$\forall x(x \prec t \supset \varepsilon(x)) \to \varepsilon(t)$$
for arbitrary terms t.

2. The end-sequent of a **TJ**-proof must be of the form
$$\to \varepsilon(\bar{n}_1), \varepsilon(\bar{n}_2), \ldots, \varepsilon(\bar{n}_r)$$
or
$$\varepsilon(\bar{m}_1), \varepsilon(\bar{m}_2), \ldots, \varepsilon(\bar{m}_l) \to \varepsilon(\bar{n}_1), \varepsilon(\bar{n}_2), \ldots, \varepsilon(\bar{n}_r),$$
where $\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_l, \bar{n}_1, \bar{n}_2, \ldots, \bar{n}_r$ are numerals such that
$$\{m_1, m_2, \ldots, m_l\} \cap \{n_1, n_2, \ldots, n_r\} = \emptyset.$$

If S is an end-sequent of a **TJ**-proof then S is said to be **TJ**-*provable.*

Let A, B be formulae in a **TJ**-proof. A is called a *direct descendant* of B if A is a descendant of B and A is identical with B. A formula A in a **TJ**-proof is said to be *free* if A is neither a direct descendant of any induction formula nor a direct descendant of a formula which is in mathematical initial sequents or **TJ**-initial sequents.

A cut is said to be *free* if both cut formulae are free. Then the following holds as in [33, p.116]:

Lemma 5.3.1 *If a sequent is* **TJ**-*provable then it is* **TJ**-*provable without free cut.*

The proof is the same as the cut-elimination proof for **LK**, then we omit it.

By the above lemma all the cut formulae occurring a **TJ**-proof without free cut are Σ_k or bounded formulae (include atomic formulae).

Now we define the ordinal assignment for **TJ**-proofs as similar to the case for $\mathbf{PA}_k$ explained in [33, p.116-117] (where $\mathbf{PA}_k$ is, roughly speaking, obtained from **PA** by restricting induction formulae to those which have at most k quantifiers).

We must modify some notions. The *grade* of a formula A is defined by the number of unbounded quantifiers in A minus 1. According to this we modify the grade of a cut and an induction inference and hence the *height* of a sequent in a proof is also modified. The *end-piece* and *boundary inferences* are modified as a boundary inference introduces an unbounded quantifier.

We add some extra inference rules as follows:

1. *term-replacement* for ε:

$$\frac{\Gamma_1, \varepsilon(s), \Gamma_2 \to \Delta}{\Gamma_1, \varepsilon(t), \Gamma_2 \to \Delta}, \qquad \frac{\Gamma \to \Delta_1, \varepsilon(s), \Delta_2}{\Gamma \to \Delta_1, \varepsilon(t), \Delta_2},$$

   where s and t are closed terms which have same values.

2.

$$(\forall n): \quad \frac{\Gamma \to \Delta, A(\bar{0}) \qquad \Gamma \to \Delta, A(\bar{1}) \quad \ldots\ldots \quad \Gamma \to \Delta, A(\bar{n})}{\Gamma \to \Delta, \forall x(x \leq \bar{n} \supset A(x))},$$

$$(\exists n): \quad \frac{A(\bar{0}), \Gamma \to \Delta \qquad A(\bar{1}), \Gamma \to \Delta \quad \ldots\ldots \quad A(\bar{n}), \Gamma \to \Delta}{\exists x(x \leq \bar{n} \land A(x)), \Gamma \to \Delta}$$

   for each $n \in \mathbf{N}$.

It is easy to see that these rules are redundant in the original system (for $(\forall n)$ and $(\exists n)$, use Proposition 5.2.1).

The ordinal assigned to a sequent S in a **TJ**-proof P is denoted by $o(S;P)$ or $o(S)$. For given P we define $o(S;P)$ as follows:

1. If S is an initial sequent in P except a **TJ**-initial sequent, then $o(S;P) = 0$,

2. If S is a **TJ**-initial sequent in P, then $o(S;P) = 2$,

3. If S is the lower sequent of a weak inference or term-replacement for ε or a propositional inference which has one upper sequent or an inference introducing bounded quantifier, then $o(S;P) = o(S';P)$, where S' is the upper sequent,

4. If S is the lower sequent of an inference introducing unbounded quantifier, then $o(S;P) = o(S';P) + \omega$,

5. If S is the lower sequent of a propositional inference which has two upper sequents or cut whose cut formula contains no unbounded quantifier, then $o(S;P) = \max\{o(S_1;P), o(S_2;P)\}$, where S_1 and S_2 are the upper sequents,

6. If S is the lower sequent of $(\forall n)$ or $(\exists n)$, then $o(S;P) = \max\{o(S_1;P), \dots, o(S_n;P)\}$, where $S_1, \dots, S_n$ are the upper sequents,
7. Otherwise $o(S;P)$ is defined precisely as [33, Definition 12.6].

The ordinal of a **TJ**-proof P, denoted by $o(P)$, is defined by the ordinal of its end-sequent. Then it is easy to see that the ordinal of a **TJ**-proof without free cut is less than ω_{k+1}.

Let P be a **TJ**-proof and $\Gamma \to \varepsilon(\bar{n}_1), \varepsilon(\bar{n}_2), \dots, \varepsilon(\bar{n}_r)$ be the end-sequent of P. P is said to be *proper* if there exists $k \in \mathbf{N}$ such that $k \prec n_i$ for all $i \in \{1, 2, \dots, r\}$.

Now we define a reduction procedure for proper **TJ**-proofs as a consistency proof for **PA**(cf.[33, p.105-114]).

Step1. Elimination of free variables which are not used as eigenvariables is carried out as in [33]. But the end-piece may contain an inference introducing a bounded quantifier as follows (we consider the case of $\forall$):

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \to \Delta, a \leq t \supset A(a)\end{array}}{\begin{array}{c}\Gamma \to \Delta, \forall x(x \leq t \supset A(x))\\ \vdots\end{array}}$$

Let n be the value of t. Suppose $m \leq n$. We define the proof $Q(\bar{m})$ as follows:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \to \Delta, \bar{m} \leq t \supset A(\bar{m})\end{array} \quad \dfrac{\to \bar{m} \leq t \quad A(\bar{m}) \to A(\bar{m})}{\bar{m} \leq t \supset A(\bar{m}) \to A(\bar{m})}}{\Gamma \to \Delta, A(\bar{m})}$$

By Lemma 9.6 in [33], $\forall x(x \leq \bar{n} \supset A(x)) \to \forall x(x \leq t \supset A(x))$ is provable (with ordinal 0). Then consider the following proof:

$$\dfrac{\dfrac{\begin{array}{c}Q(\bar{0})\\ \vdots\\ \Gamma \to \Delta, A(\bar{0})\end{array} \quad \begin{array}{c}Q(\bar{1})\\ \vdots\\ \Gamma \to \Delta, A(\bar{1})\end{array} \quad \dots \quad \begin{array}{c}Q(\bar{n})\\ \vdots\\ \Gamma \to \Delta, A(\bar{n})\end{array}}{\Gamma \to \Delta, \forall x(x \leq \bar{n} \supset A(x))} \quad \begin{array}{c}\vdots\\ \forall x(x \leq \bar{n} \supset A(x)) \to \forall x(x \leq t \supset A(x))\end{array}}{\begin{array}{c}\Gamma \to \Delta, \forall x(x \leq t \supset A(x))\\ \vdots\end{array}}$$

By this transformation free variable a is eliminated in the end-piece. It is easy to see that the ordinal does not change by this transformation. The case of $\exists$ is similar. Therefore we can eliminate all free variables which are not used as eigenvariables in the end-piece.

Step2. Suppose that the end-piece contains at least one induction inference. In this case the reduction is carried out as in [33] and the ordinal decreases.

Then we may assume that the end-piece contains no induction inference.

Step3. Before eliminating logical initial sequents from the end-piece, we have to eliminate initial sequents of the form $s = t, \varepsilon(s) \to \varepsilon(t)$. This is done as same as [10]. After this one can eliminate logical initial sequents only those which have unbounded quantifiers.

Step4. Eliminating weakening in the end-piece is carried out as in [33]. But after this reduction the end-sequent may be changed because the end-sequent of a **TJ**-proof is not empty. In this case we add weakenings below the end-sequent of the new proof so that the end-sequent becomes the same as the old one.

Step5. Suppose that there exists a suitable cut in the end-piece. If the grade of the suitable cut is greater than 0, then the essential reduction is carried out precisely as in [33]. Now we consider the case where the grade of the suitable cut is 0. (We call this case *essential reduction of grade 0.*) Consider the proof as follows:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma' \to \Delta', A(a)}}{\Gamma' \to \Delta', \forall x A(x)} \quad \dfrac{\dfrac{\vdots}{A(t), \Pi' \to \Lambda'}}{\forall x A(x), \Pi' \to \Lambda'}}{\vdots}$$

$$\dfrac{\Gamma \to \Delta, \forall x A(x) \qquad \forall x A(x), \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}$$

$$\vdots$$

$$\Phi \to \Psi$$

where $\Phi \to \Psi$ is the end-sequent. Since the grade of $\forall x A(x)$ is 0, $A(t)$ is a bounded formula. Furthermore since there exists no free variable in the end-piece, we may assume that $A(t)$ is a closed formula. By using Proposition 5.2.1, there exists a quantifier-free formula $A^*(t)$ such that $A^*(t) \equiv A(t)$ ($A \equiv B$ is abbreviation for $A \supset B \wedge B \supset A$). Let $\varepsilon(s_1), \varepsilon(s_2), \ldots, \varepsilon(s_n)$ be all the formulae of the form $\varepsilon(s)$ which occur in $A^*(t)$. Let $m_1, m_2, \ldots, m_n$ be the value of $s_1, s_2, \ldots, s_n$ respectively and we set $X = \{m_1, m_2, \ldots, m_n\}$. We define a set $A \subseteq \mathbf{N}$ by

$$A = \{m \in \mathbf{N} \mid \varepsilon(\bar{m}) \text{ occurs in } \Psi\}.$$

Suppose

$$X \cap A = \{m_{i_1}, m_{i_2}, \ldots, m_{i_j}\},$$

$$X \setminus A = \{m_{i_{j+1}}, m_{i_{j+2}}, \ldots, m_{i_n}\}.$$

Then it is easy to see that either $\Phi' \to \Psi', A(t)$ or $A(t), \Phi' \to \Psi'$ is $(I\Sigma_k(\varepsilon)$-)provable without cut or induction inference (and hence with ordinal 0), where Φ' denotes $\varepsilon(\bar{m}_{i_1}), \varepsilon(\bar{m}_{i_2}), \ldots, \varepsilon(\bar{m}_{i_j})$ and Ψ' denotes $\varepsilon(\bar{m}_{i_{j+1}}), \varepsilon(\bar{m}_{i_{j+2}}), \ldots, \varepsilon(\bar{m}_{i_n})$. We consider the case where $\Phi' \to \Psi', A(t)$ is provable. Consider the proof:

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \to \Delta, \forall x A(x)\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots\\ \Phi' \to \Psi', A(t) & A(t), \Pi' \to \Lambda'\end{array}}{\Phi', \Pi' \to \Psi', \Lambda'}}{\dfrac{\text{(some exchanges and a weakening)}}{\forall x A(x), \Pi', \Phi' \to \Psi', \Lambda'}}\\ \vdots\\ \forall x A(x), \Pi, \Phi' \to \Psi', \Lambda\end{array}}{\Gamma, \Pi, \Phi' \to \Delta, \Psi', \Lambda}}{\dfrac{\text{(some exchanges)}}{\Gamma, \Pi, \Phi' \to \Psi', \Delta, \Lambda}}\\
\vdots\\
\dfrac{\Phi, \Phi' \to \Psi', \Psi}{\dfrac{\text{(some exchanges and contractions)}}{\Phi, \Phi' \to \Psi}}
\end{array}
$$

Then it is easy to see that the ordinal decreases. Note that the succedent of the end-sequent does not change. The case where $A(t), \Phi' \to \Psi'$ is provable can be treated similarly.

The case where there exists no suitable cut in the end-piece will be treated later as "critical reduction".

A proper **TJ**-proof P is said to be *non-critical* if one of the following conditions are satisfied:

1. Step2 can be applied to P after Step1,

2. Step5 can be applied to P after Step1, 3 and 4.

Otherwise it is said to be *semi-critical*.

By the above definition of the reduction procedure, if P is a non-critical **TJ**-proof, then there exists another **TJ**-proof P' such that the succedent of the end-sequent of P' is as same as that of P and $o(P') < o(P)$.

P' is called the *non-critical reduct* of P.

Now we consider semi-critical **TJ**-proofs.

Let P be a semi-critical **TJ**-proof. Then we apply Step1, 3 and 4 above to P if necessary, we can obtain another (semi-critical) **TJ**-proof with same end-sequent of P such that $o(P') \leq o(P)$ which satisfies the following conditions:

1. There exists no free variable in the end-piece,

2. There exists no induction inference in the end-piece,

3. There exists no initial sequent of the form $s = t, \varepsilon(s) \to \varepsilon(t)$ in the end-piece,

4. There exists no logical initial sequent which have unbounded quantifiers in the end-piece,

5. There exists no suitable cut in the end-piece,

6. If there exists a weakening I in the end-piece, then any inference below I is a weakening.

A semi-critical **TJ**-proof which satisfies the above conditions is said to be *critical.*

Let A be a closed bounded formula and $B(a)$ be a formula which contains no free variable except a. Then $A(\{x\}B(x))$ denotes the formula obtained from A by replacing all $\varepsilon(t)$ in A by $B(t)$ (if A is a $\mathcal{L}$-formula then $A(\{x\}B(x))$ is A itself).

Lemma 5.3.2 *Let P be a critical* **TJ**-*proof. Then P contains at least one boundary inference or* **TJ**-*initial sequent.*

Proof. Suppose P contains neither boundary inference nor **TJ**-initial sequent. Then P coincides its own end-piece. Hence all initial sequents in P is mathematical initial sequents or logical initial sequent $A \to A$ where A is a closed bounded formula. We choose arbitrary $m \in \mathbf{N}$ such that $\varepsilon(\bar{m})$ appears the antecedent of the end-sequent of P(If the antecedent is empty, we take arbitrary m such that $\varepsilon(\bar{m})$ does not appear in the succedent of the end-sequent). If each formula A in P is replaced by $A(\{x\}(x = m))$ then obviously all initial sequents in P is true. But the end-sequent is false. This is impossible. □

The formula in the succedent of a **TJ**-initial sequent is called the *principal formula* of the **TJ**-initial sequent. A formula A in the end-piece of a **TJ**-proof is called a *principal* **TJ**-*descendant* if A is a descendant of the principal formula of a **TJ**-initial sequent in the end-piece. Similarly a formula A in the end-piece of a **TJ**-proof is called a *principal descendant* if A is a descendant of a principal formula at the boundary.

Lemma 5.3.3 *Let P be a critical* **TJ**-*proof and let S be a sequent in the end-piece of P. If S contains a formula with unbounded quantifiers then there exists a formula A in S or in a sequent above S such that A is a principal descendant or a principal* **TJ**-*descendant.*

Proof. As similar to [33, p.151,13.10]. □

If s, t are closed terms which have same values, then $\varepsilon(s)$ is said to be *equivalent* to $\varepsilon(t)$.

Now we will show the following lemma as similar to [33, p.151,13.11].

Lemma 5.3.4 *Let P be a critical* **TJ**-*proof. Then the succedent of the end-sequent of P contains a formula which is equivalent to the principal formula of one of the* **TJ**-*initial sequents in the end-piece.*

In order to prove this, we show the following first:

Lemma 5.3.5 *Let P be a critical* **TJ***-proof. If there exist no logical inference in the end-piece of P, then the end-sequent of P contains a principal* **TJ***-descendant.*

Remark. Obviously a principal **TJ**-descendant appears the succedent of the end-sequent.

Proof. (cf.[33, p.151,13.11]) For short, 'a sequent is (∗)' means that a sequent contains a principal descendant or a principal **TJ**-descendant. In order to prove the assertion, it suffices to show that the end-sequent of P is (∗). Suppose the end-sequent of P is not (∗). Since the end-piece of P contains a sequent which is (∗), there exists an inference such that at least one of its upper sequents is (∗) while its lower sequent is not (∗) (this property is denoted by (P)). Such inference must be a cut. Let I be an uppermost cut with (P) in the end-piece of P as follows:

$$\frac{\Gamma \to \Delta, A \qquad A, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}.$$

Let S_1 and S_2 be $\Gamma \to \Delta, A$ and $A, \Pi \to \Lambda$ respectively.

Claim 1 *S_1 is not (∗).*

Proof. Suppose that S_1 is (∗). Then A is a principal descendant or a principal **TJ**-descendant. Suppose that A is a principal descendant. Then S_2 contains an unbounded quantifier. Hence S_2 or a sequent above S_2 is (∗) by Lemma 5.3.3. If S_2 is not (∗), then contradicts with the choice of I. Therefore S_2 is (∗). Hence I is a suitable cut, which contradicts with the assumption that P is critical. Therefore A is a principal **TJ**-descendant. By the assumption, A is of the form $\varepsilon(t)$.

Claim 2 *S_2 contains no unbounded quantifier.*

Proof. Suppose that S_2 contains an unbounded quantifier. Then S_2 or a sequent above S_2 is (∗) by Lemma 5.3.3. If S_2 is not (∗), this contradicts with the choice of I. Hence S_2 is (∗). Then the lower sequent of I is also (∗), which contradicts with the assumption of I has (P). (end of Claim 2)

Proof of Claim 1 (continued).
By Claim 2 S_2 is not (∗). By the choice of I, we obtain that each sequent above S_2 is not (∗), i.e., the proof down to S_2 is included in the end-piece. Hence S_2 can not contain ε. This contradicts with A is of the form $\varepsilon(t)$. Therefore S_1 is not (∗). (end of Claim 1)

Claim 3 *S_2 is not (∗).*

Proof. Suppose that S_2 is (∗). Obviously A in S_2 can not be a principal **TJ**-descendant. Hence A is a principal descendant and therefore S_1 contains an unbounded quantifier. By Lemma 5.3.3, S_1 or a sequent above S_1 is (∗). If S_1 is not (∗), this contradicts with the choice of I. Hence S_1 is (∗). Therefore I is a suitable cut, which contradicts with the assumption that P is critical. (end of Claim 3)

Proof of Lemma 5.3.5 (continued).
By Claim 1 and 3 we have a contradiction. Therefore the end-sequent of P is $(*)$. □

Proof of Lemma 5.3.4. If there exists no logical inference, then the assertion holds by Lemma 5.3.5.

Now we consider the case where the end-piece of a critical **TJ**-proof contain logical inferences. Of course these inferences are not the one which introduce unbounded quantifiers. Since these inferences must be implicit, there exist free cuts in the end-piece. First we have to eliminate these free cuts. It is easy to see that in each free-cut-elimination procedure, new **TJ**-initial sequents, induction inferences, free variables and suitable cuts do not appear in the end-piece. Let P' be the **TJ**-proof obtained from P by this free-cut-elimination. Then we can apply the elimination of logical initial sequents and weakenings in the end-piece as the reduction procedure for proper **TJ**-proofs defined before (in this case, we can eliminate all logical initial sequents). Let P'' be the **TJ**-proof obtained from P' in this way. Then we can apply Lemma 5.3.5 to P'' and hence the end-sequent of P'' contains a principal **TJ**-descendant. If this principal **TJ**-descendant is a descendant of a principal formula of a **TJ**-initial sequent S, then S is also contained P. This completes the proof. □

Now we define the notion of critical reduction.

Let P be a semi-critical **TJ**-proof whose end-sequent is $\Gamma \to \varepsilon(\bar{n}_1), \varepsilon(\bar{n}_2), \ldots, \varepsilon(\bar{n}_r)$ and let n_0 be a number such that $n_0 \prec n_i$ for all $i \in \{1, 2, \ldots, r\}$. Then there exists a critical **TJ**-proof P' with same end-sequent of P such that $o(P') \leq o(P)$. By Lemma 5.3.4 for some $j \in \{1, 2, \ldots, r\}$ the formula $\varepsilon(n_j)$ in the end-sequent is equivalent to the principal formula of a **TJ**-initial sequent $\forall x(x \prec t \supset \varepsilon(x)) \to \varepsilon(t)$, where t and n_j have the same value. Hence $\to \bar{n}_0 \prec t$ is an initial sequent. Then we replace the **TJ**-initial sequent

$$\forall x(x \prec t \supset \varepsilon(x)) \to \varepsilon(t)$$

in P by the following proof:

$$\cfrac{\cfrac{\cfrac{\to \bar{n}_0 \prec t \quad \varepsilon(\bar{n}_0) \to \varepsilon(\bar{n}_0)}{\bar{n}_0 \prec t \supset \varepsilon(\bar{n}_0) \to \varepsilon(\bar{n}_0)}}{\forall x(x \prec t \supset \varepsilon(x)) \to \varepsilon(\bar{n}_0)}}{\forall x(x \prec t \supset \varepsilon(x)) \to \varepsilon(\bar{n}_0), \varepsilon(t)}$$

Since the ordinal of this proof is 1, it is less than the ordinal of a **TJ**-initial sequent which is 2. By this transformation and some obvious change, P' is transformed into a **TJ**-proof P'' whose end-sequent is $\Gamma \to \varepsilon(\bar{n}_0), \varepsilon(\bar{n}_1), \varepsilon(\bar{n}_2), \ldots, \varepsilon(\bar{n}_r)$ such that $o(P'') < o(P')$.

P'' is called the *critical reduct* of P at n_0. Obviously $o(P'') < o(P)$.

5.4 Proof of the theorem

In this section, we prove Theorem 5.2.2. First we will show the following lemma.

Lemma 5.4.1 *Let $\prec$ is an irreflexive and transitive relation of natural numbers which is provable well-founded in $I\Sigma_k$. Then there exist an ordinal $\alpha < \omega_{k+2}$ and a primitive recursive function h such that for each k, $h(k)$ is (the Gödel number of) an additive principal number, i.e., of the form ω^β for some β and $h(k) <^* \ulcorner\alpha\urcorner$ for all $k \in \mathbf{N}$ and*

$$\forall k(\forall n < k(k \prec n \Rightarrow h(k) <^* h(n))).$$

Proof. Since $\prec$ is provable well-founded, there exists an $I\Sigma_k(\varepsilon)$-proof of

$$\forall x(\forall y(y \prec x \supset \varepsilon(y)) \supset \varepsilon(x)) \to \varepsilon(a).$$

Consider the following **TJ**-proof:

$$\dfrac{\dfrac{\dfrac{\forall y(y \prec b \supset \varepsilon(y)) \to \varepsilon(b)}{\to \forall y(y \prec b \supset \varepsilon(y)) \supset \varepsilon(b)}}{\to \forall x(\forall y(y \prec x \supset \varepsilon(y)) \supset \varepsilon(x))} \quad \begin{array}{c}\vdots\\ \forall x(\forall y(y \prec x \supset \varepsilon(y)) \supset \varepsilon(x)) \to \varepsilon(a)\end{array}}{\to \varepsilon(a)}$$

Let $P(a)$ be a **TJ**-proof obtained from the above by eliminating free cuts. Then for each $k \in \mathbf{N}$, $P(\bar{k})$ denotes a **TJ**-proof of $\to \varepsilon(\bar{k})$ obtained from $P(a)$ by substituting the numeral $\bar{k}$ for the variable a.

Now we define a **TJ**-proof P_k and a primitive recursive function $\varphi(k)$ for all $k \in \mathbf{N}$ by induction on k satisfying the following conditions:

1. For all $n \in \mathbf{N}$, if $\varepsilon(\bar{n})$ occurs in the succedent of the end-sequent of P_k, then $k \preceq n$,
2. $\forall n(n < k \;\&\; k \prec n \Rightarrow o(P_k) + \varphi(k) < o(P_n) + \varphi(n))$.

For **TJ**-proof P, $F(P)$ denotes the number of formulae which occur in the antecedent of the end-sequent of P.

Case 1: $\neg\exists n < k(k \prec n)$.
In this case, we set $P_k = P(\bar{k})$ and $\varphi(k) = 0$. Obviously the above two conditions hold.

Case 2: $\exists n < k(k \prec n)$.
There exists $n_0 \in \mathbf{N}$ such that $n_0 < k \;\&\; k \prec n_0$ and

$$\forall n < k(k \prec n \Rightarrow o(P_{n_0}) + \varphi(n_0) \le o(P_n) + \varphi(n)).$$

Subcase 1: P_{n_0} is non-critical.
Then P_k is defined to be the non-critical reduct of P_{n_0}. Obviously, condition 1 is satisfied. When we obtain P_k from P_{n_0}, if Step 5 is the essential reduction of grade 0, then we define $\varphi(k) = F(P_k)$. In this case, for all $m \in \mathbf{N}$,

$$o(P_k) + m < o(P_{n_0})$$

holds. (Recall the definition of $o(S;P)$,4.) Hence $o(P_k) + F(P_k) < o(P_{n_0}) + \varphi(n_0)$ and condition 2 holds. Otherwise, we define $\varphi(k) = \varphi(n_0)$.

Subcase 2: P_{n_0} is semi-critical.
(1)If $\varepsilon(\bar{k})$ does not occur in the antecedent of the end-sequent of P_{n_0}.
Then P_k is defined to be the critical reduct of P_{n_0} at k and $\varphi(k) = \varphi(n_0)$.
(2)If $\varepsilon(\bar{k})$ occurs in the antecedent of the end-sequent of P_{n_0}.
Then P_k is equal to P_{n_0} and $\varphi(k) = \varphi(n_0) \dot{-} 1$. (Note that in this case $\varphi(n_0) > 0$.)

It is easy to see that the above two conditions are satisfied.

Hence $h(k) = \ulcorner \omega^{o(P_k)+\varphi(k)} \urcorner$ is a function as required. Finally, we set $\alpha = \omega^{o(P(a))+\omega}$. Then obviously $\alpha < \omega_{k+2}$ and $h(k) <^* \ulcorner \alpha \urcorner$ for all $k \in \mathbf{N}$. □

Proof of Theorem 5.2.2. We define f by

$$f(n) = \max_{<^*}\{h(n_0)\# \cdots \# h(n_r) \mid n_0 \prec \cdots \prec n_r = n \;\&\; n_0, \dots, n_{r-1} < n\},$$

where $\max_{<^*}$ denotes the maximum with respect to $<^*$. Since $\prec$ is irreflexive, the set

$$\{(n_0, \dots, n_r) \mid n_0 \prec \cdots \prec n_r = n \;\&\; n_0, \dots, n_{r-1} < n\}$$

is finite and hence the above definition of f is well-defined. This function f is primitive recursive. Then the claim

$$n \prec k \Rightarrow f(n) <^* f(k)$$

is proved as same as in [3, p.102]. The second point of the theorem is easily seen if one puts $\mu = \omega^{o(P(a))+\omega}$. □

Chapter 6

Conclusions and further studies

In this chapter, we summarize conclusions of this thesis and survey briefly further studies.

In Chapter 2, we have introduced sequent calculi **LBP** for Basic Propositional Logic (BPL) and **LFP** for Formal Propositional Logic (FPL), and have proved the cut-elimination theorems for these by syntactical method. These sequent calculi satisfy the subformula property and hence we can expect many application of the cut-elimination theorem. Typical ones among them are, for example, interpolation property, variable sharing property and variable separation property (For details of these properties, see [21] and [33]). Recently, by Ardeshir and Ruitenburg, some investigations for BPL are published ([5][6]). However, there is almost no study for FPL till now. Therefore, especially for investigation of FPL, our sequent calculi will be expected to be very useful.

In Chapter 4, we have introduced another reduction procedure for the natural deduction system for the classical logic **NK** and have proved the strong normalization theorem and the Church-Rosser property with respect to this new reduction procedure. As thinkable directions of further studies of this problem, there are two directions.

One of them is applications to other systems of our technique to prove strong normalization theorem. For example, an application to second-order classical natural deduction system can be considered. In [22], Parigot present two proofs of strong normalization theorem for second order classical natural deduction. However, the methods he uses are considering $\lambda\mu$-calculus by virtue of "Curry-Howard isomorphism" and hence his system does not have full logical symbols. In the author's opinion, it is difficult to treat the system with symbols $\vee$ and $\exists$ by using term-systems. Therefore, it is an interesting problem to consider whether the technique we use in this thesis can be applied to second-order classical natural deduction system or not.

Second one is simplification of proof in this thesis. The proof we gave in Chapter 4 is very complicated. Especially, proof of Lemma 4.3.12 is quite long and may be boring. The reason of this complication may be in the definition of validity. The definition of validity given in Chapter 4 is not so simple, but Prawitz's one which is used in proof of strong normalization theorem for **NJ** in [24] is more complicated than ours in the case where the last inference of a proof is ($\vee$ E) or ($\exists$ E). However, by this complicated definition, Prawitz's proof is not so complicated. On the other hand, because of simplicity of the definition of validity, our proof need to numerous syntactical argument. Therefore, when one simplifies the proof, one must think of another definition of validity.

In Chapter 5, we have discussed provable well-founded relations of subsystems of the first-order arithmetic and have obtained the following partial result: If $\prec$ is a provable well-founded relation of $I\Sigma_k$, then there exists primitive recursive function f such that $a \prec b \Leftrightarrow f(a) <^* f(b)$, where $<^*$ denotes the standard ordering of type ω_{k+2}, and there exists an ordinal $\mu < \omega_{k+2}$ such that for every a, $f(a) <^* \ulcorner \mu \urcorner$.

As further studies beyond this, we must overcome second difficulty mentioned in the introduction of Chapter 5. As one can understand by reviewing proof of Theorem 5.2.2, the fact that $h(k)$ in Lemma 5.4.1 is an additive principal number is essential. Therefore one have to take $h(k) = \ulcorner \omega^{o(P_k)+\varphi(k)} \urcorner$, and this causes that on the construction of f, though $|\prec| < \omega_n$ for some n, μ may be $> \omega_n$. In order to overcome this difficulty, there may be two directions. One of them is that we change the ordinal assignment such that $o(P_k)$ becomes an additive principal number and the other is that we change the definition of f in proof of Theorem 5.2.2. Though we do not succeed in either case, this directions, in the author's opinion, are hopeful.

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