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Author(s)	Kobayashi, Koichi; Hiraishi, Kunihiko
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Description	

PAPER *Special Section on Theory of Concurrent Systems and its Applications*

MLD-Based Modeling of Hybrid Systems with Parameter Uncertainty

Koichi KOBAYASHI[†a)] *and* **Kunihiko HIRAISHI**[†], *Members*

SUMMARY In this paper, we propose a new modeling method to express discrete-time hybrid systems with parameter uncertainty as a mixed logical dynamical (MLD) model. In analysis and control of hybrid systems, there are problem formulations in which convex polyhedra are computed, but for high-dimensional systems, it is difficult to solve these problems within a practical computation time. The key idea of this paper is to use an interval method, which is one of the classical methods in verified numerical computation, and to regard an interval as an over-approximation of a convex polyhedron. By using the obtained MLD model, analysis and synthesis of robust control systems are formulated.

key words: *MLD model, interval method, parameter uncertainty, hybrid systems*

1. Introduction

In many cases of analysis and control of hybrid systems, one of the technical difficulties is that the computation times to solve the analysis/control problems become too long. For example, in some of the verification problems and the controllability problems of hybrid systems, it is necessary to manipulate convex polyhedra [3]–[5]. However, computing convex polyhedra is difficult for high-dimensional systems, and it will be desirable from the practical viewpoint to compute an approximation of convex polyhedra. Furthermore, as well as linear systems and nonlinear systems, it is important to consider hybrid systems with parameter uncertainty. For example, mechanical systems with friction phenomena are well-known as one of the typical examples in hybrid systems, because it is in general difficult to precisely identify friction phenomena. Although analysis/control of uncertain hybrid systems have been developed (e.g., see [13]), these results are complicated, and it will be desirable to consider a simpler approach.

On the other hand, an interval method [9] is well-known as one of the classical techniques in verified numerical computation, and is based on interval arithmetic. Using the interval method, a convex polyhedron is approximated as an interval (a box), that is, an over-approximation of a convex polyhedron is obtained. Obviously the approximation using the interval method will be conservative, but the computation of intervals is relatively easier than that of convex polyhedra. In applications of the interval method to control theory and theoretical computer science, the trajectory generation problem [12] and the reachability problem [10], [11] have been considered. However, the control problem has not been discussed so far.

In this paper, we propose a new modeling method to express hybrid systems with parameter uncertainty by using the interval method. More precisely, discrete-time piecewise affine systems with parameter uncertainty are approximately expressed as a mixed logical dynamical (MLD) model [6], which is one of the powerful models of hybrid systems. Mathematical techniques used in the proposed method are only basics of interval arithmetic and techniques [6], [8] in the MLD framework. Furthermore, by using the obtained MLD model, the analysis/control problems, e.g., the robust control synthesis satisfying state/input constraints, can be considered. Finally the contribution of this paper are summarized as follows: (i) modeling of hybrid systems with parameter uncertainty based on the interval method and the MLD framework, (ii) analysis and synthesis of robust control systems based on the obtained model.

This paper is organized as follows. In Sect. 2, some basics of interval arithmetic are explained. In Sect. 3, discrete-time linear systems with parameter uncertainty are approximately transformed into the MLD model. In Sect. 4, the result on linear systems is extend to piecewise affine systems. In Sect. 5, some analysis/control problems are formulated by using the obtained model. In Sect. 6, numerical examples are shown. In Sect. 7, we conclude this paper.

Notation: Let $\mathcal{R}$ denote the set of real numbers. Let $\{0, 1\}^{m\times n}$ denote the set of $m \times n$ matrices, which consists of elements 0 and 1. Let I_n, $0_{m\times n}$ and e_n denote the $n \times n$ identity matrix, the $m \times n$ zero matrix and the $n \times 1$ vector whose elements are all one, respectively. For simplicity of notation, we sometimes use the symbol 0 instead of $0_{m\times n}$, and the symbol I instead of I_n. Let the matrix inequality $X \leq Y$ denote that $X_{ij} - Y_{ij}$ is nonpositive, where X_{ij}, Y_{ij} is the (i, j)-th element of X, Y, respectively. For a vector x, let $x^{(i)}$ denote the i-th element of x. For n vectors $x_1, x_2, \ldots, x_n$, let $\max(\min)(x_1, x_2, \ldots, x_n)$ denote a vector such that i-th element is given by a maximum(minimum) value of $x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}$. For a matrix/vector M, let the matrix/vector $|M|$ denote a matrix such that each element is given by an absolute value of each element of M. For a vector $a = [\ a^{(1)}\ \ \cdots\ \ a^{(n)}\ ]^T \in \mathcal{R}^n$, we use the notation

$$\mathrm{diag}(a) := \begin{bmatrix} a^{(1)} & & 0 \\ & \ddots & \\ 0 & & a^{(n)} \end{bmatrix}.$$

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[†]The authors are with the Japan Advanced Institute of Science and Technology, Nomi-shi, 923-1292 Japan.
a) E-mail: k-kobaya@jaist.ac.jp
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2. Interval Arithmetic

In this section, some basics of interval arithmetic are explained. See [9] for further details.

First, an *interval* is defined as the following bounded set of real numbers

$$[\underline{x}, \overline{x}] := \{x \in \mathcal{R} \mid \underline{x} \le x \le \overline{x}\}$$

where $\underline{x} \le \overline{x} \in \mathcal{R}$ holds, and $\underline{x}$, $\overline{x}$ are the infimum and the supremum of the interval, respectively. For simplicity of notation, we may denote $[\underline{x}, \overline{x}]$ as $[x]$.

Suppose that two intervals $[x]$ and $[y]$ are given. Then four operations, addition $+$, multiplication $\times$, subtraction $-$, and division $\div$ of $[x]$ and $[y]$ are given as follows:

$$\begin{aligned}
[x] + [y] &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}], \qquad (1)\\
[x] - [y] &= [\underline{x} - \overline{y}, \overline{x} - \underline{y}],\\
[x] \times [y] &= \Big[\min\{\underline{x}\underline{y}, \overline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\overline{y}\},\\
&\qquad \max\{\underline{x}\underline{y}, \overline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\overline{y}\}\Big], \qquad (2)\\
[x] \div [y] &= [x] \times \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right], \quad 0 \notin [y].
\end{aligned}$$

Next, an interval is extended to an interval matrix (vector). An interval matrix is defined as

$$[X] = [\underline{X}, \overline{X}] := \{X \in \mathcal{R}^{m\times n} \mid \underline{X} \le X \le \overline{X}\}$$

where $\underline{X}, \overline{X} \in \mathcal{R}^{m\times n}$. Also, the center $c([X])$ and the radius $r([X])$ of an interval matrix $[X]$ are defined as

$$c([X]) := (\overline{X} + \underline{X})/2, \quad r([X]) := (\overline{X} - \underline{X})/2,$$

respectively. From the definitions of $c([X])$ and $r([X])$,

$$[\underline{X}, \overline{X}] = [c([X]) - r([X]), c([X]) + r([X])]$$

holds. Then we introduce the result on a multiplication of two interval matrices $[X]$ and $[Y]$ [9].

Lemma 1: Suppose that interval matrices $[X]$ and $[Y]$ are given. Then the following condition holds:

$$\begin{aligned}
[X] \times [Y] &= [c([X]) - r([X]), c([X]) + r([X])]\\
&\quad \times [c([Y]) - r([Y]), c([Y]) + r([Y])]\\
&\subseteq [c([X])c([Y]) - r([X])|c([Y])|\\
&\quad -|c([X])|r([Y]) - r([X])r([Y]),\\
&\quad c([X])c([Y]) + r([X])|c([Y])|\\
&\quad +|c([X])|r([Y]) + r([X])r([Y])]. \qquad (3)
\end{aligned}$$

Note here that in Lemma 1, if $[X]$ or $[Y]$ is given as some point ($r([X]) = 0$ or $r([Y]) = 0$), then the equality in (3) holds. Furthermore, (3) is an over-approximation of $[X] \times [Y]$, but the size of the obtained over-approximation is less than about 1.5 times of the accurate interval [9].

Hereafter, using interval arithmetic, we consider to express discrete-time linear systems with parameter uncertainty as a mixed logical dynamical (MLD) model, which consists of linear state equation and inequality with binary variables and continuous variables [6]. Then one of the technical problems is how to express the multiplication (2) by using the MLD model. As a simple example, for the scalar linear system $x(k+1) = \alpha x(k)$, $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $x(k) \in [\underline{x}(k), \overline{x}(k)]$, consider to express the interval $[\underline{x}(k+1), \overline{x}(k+1)]$ of $x(k+1)$ as a linear form with binary variables and continuous variables. First, binary variables $\delta_{12}(k), \delta_{13}(k), \delta_{14}(k), \delta_{23}(k), \delta_{24}(k), \delta_{34}(k)$ are defined as

$$\begin{aligned}
&[\delta_{12}(k) = 1] \leftrightarrow \left[\underline{\alpha}\underline{x}(k) \ge \overline{\alpha}\overline{x}(k)\right],\\
&[\delta_{13}(k) = 1] \leftrightarrow \left[\underline{\alpha}\underline{x}(k) \ge \underline{\alpha}\overline{x}(k)\right],\\
&[\delta_{14}(k) = 1] \leftrightarrow \left[\underline{\alpha}\underline{x}(k) \ge \overline{\alpha}\underline{x}(k)\right],\\
&[\delta_{23}(k) = 1] \leftrightarrow \left[\overline{\alpha}\overline{x}(k) \ge \underline{\alpha}\overline{x}(k)\right],\\
&[\delta_{24}(k) = 1] \leftrightarrow \left[\overline{\alpha}\overline{x}(k) \ge \overline{\alpha}\underline{x}(k)\right],\\
&[\delta_{34}(k) = 1] \leftrightarrow \left[\underline{\alpha}\overline{x}(k) \ge \overline{\alpha}\underline{x}(k)\right]
\end{aligned}$$

where "$\leftrightarrow$" denotes logical equivalence. Next, binary variables $\underline{\delta}_1(k), \underline{\delta}_2(k), \underline{\delta}_3(k), \underline{\delta}_4(k)$ are defined as

$$\begin{aligned}
\underline{\delta}_1(k) &= (1 - \delta_{12}(k))(1 - \delta_{13}(k))(1 - \delta_{14}(k)),\\
\underline{\delta}_2(k) &= \delta_{12}(k)(1 - \delta_{23}(k))(1 - \delta_{24}(k)),\\
\underline{\delta}_3(k) &= \delta_{13}(k)\delta_{23}(k)(1 - \delta_{34}(k)),\\
\underline{\delta}_4(k) &= \delta_{14}(k)\delta_{24}(k)\delta_{34}(k).
\end{aligned}$$

In a similar way, binary variables $\overline{\delta}_1, \overline{\delta}_2, \overline{\delta}_3, \overline{\delta}_4$ are defined as

$$\begin{aligned}
\overline{\delta}_1(k) &= \delta_{12}(k)\delta_{13}(k)\delta_{14}(k),\\
\overline{\delta}_2(k) &= (1 - \delta_{12}(k))\delta_{23}(k)\delta_{24}(k),\\
\overline{\delta}_3(k) &= (1 - \delta_{13}(k))(1 - \delta_{23}(k))\delta_{34}(k),\\
\overline{\delta}_4(k) &= (1 - \delta_{14}(k))(1 - \delta_{24}(k))(1 - \delta_{34}(k)).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\underline{x}(k+1) &= \min(\underline{\alpha}\underline{x}(k), \overline{\alpha}\overline{x}(k), \underline{\alpha}\overline{x}(k), \overline{\alpha}\underline{x}(k)) = \hat{\alpha}\underline{z}(k),\\
\overline{x}(k+1) &= \max(\underline{\alpha}\underline{x}(k), \overline{\alpha}\overline{x}(k), \underline{\alpha}\overline{x}(k), \overline{\alpha}\underline{x}(k)) = \hat{\alpha}\overline{z}(k)
\end{aligned}$$

where $\hat{\alpha} = [\ \underline{\alpha}\ \ \overline{\alpha}\ \ \underline{\alpha}\ \ \overline{\alpha}\ ]$,

$$\underline{z}(k) = \begin{bmatrix} \underline{\delta}_1(k)\underline{x}(k)\\ \underline{\delta}_2(k)\overline{x}(k)\\ \underline{\delta}_3(k)\overline{x}(k)\\ \underline{\delta}_4(k)\underline{x}(k) \end{bmatrix}, \quad \overline{z}(k) = \begin{bmatrix} \overline{\delta}_1(k)\underline{x}(k)\\ \overline{\delta}_2(k)\overline{x}(k)\\ \overline{\delta}_3(k)\overline{x}(k)\\ \overline{\delta}_4(k)\underline{x}(k) \end{bmatrix}.$$

By using the techniques described in [6], [8] (see also Lemma 4 and Lemma 5 in Appendix), the above conditions can be transformed into the MLD model with 14 binary variables and 8 continuous variables. However, even if a given system is simple such as scalar systems, the obtained MLD model is complicated. Therefore in this paper, by using Lemma 1, discrete-time linear systems with parameter uncertainty is approximately expressed as the MLD model.

In Sect. 3, this fact will be shown. After that, in Sect. 4, we will extend discrete-time linear systems to discrete-time piecewise affine (DT-PWA) systems.

3. Modeling of Discrete-Time Linear Systems with Parameter Uncertainty

In this section, based on Lemma 1, we consider to express discrete-time linear systems with parameter uncertainty as the MLD model.

Consider the following discrete-time linear system with parameter uncertainty

$$\begin{aligned} &x(k+1) = Ax(k) + Bu(k) + a, \\ &A \in \left[\underline{A}, \overline{A}\right], \quad B \in \left[\underline{B}, \overline{B}\right], \quad a \in \left[\underline{a}, \overline{a}\right] \end{aligned} \tag{4}$$

where $x(k) \in \mathcal{X} \subseteq \mathcal{R}^n$, $u(k) \in \mathcal{U} \subseteq \mathcal{R}^m$, $\mathcal{X}$, $\mathcal{U}$ are closed and bounded convex sets, and a is an affine term. In this paper, a set of initial states is given as an interval, i.e., $x(0) \in [\underline{x}(0), \overline{x}(0)] = [\underline{x}_0, \overline{x}_0]$, $\underline{x}_0, \overline{x}_0 \in \mathcal{X}$. Furthermore, for simplicity of notation, we denote $c([\underline{x}(k), \overline{x}(k)])$ and $r([\underline{x}(k), \overline{x}(k)])$ as $x_c(k)$ and $x_r(k)$, respectively. Then the center $x_c(k)$ and the radius $x_r(k)$ of $[\underline{x}(k), \overline{x}(k)]$ are given by

$$\begin{aligned} &\begin{bmatrix} x_c(k) \\ x_r(k) \end{bmatrix} = \Phi \hat{x}(k), \\ &\Phi := \frac{1}{2} \begin{bmatrix} I_n & I_n \\ -I_n & I_n \end{bmatrix}, \quad \hat{x}(k) := \begin{bmatrix} \underline{x}(k) \\ \overline{x}(k) \end{bmatrix} \end{aligned} \tag{5}$$

respectively. Similarly, the center A_c and the radius A_r of $[\underline{A}, \overline{A}]$, and the center B_c and the radius B_r of $[\underline{B}, \overline{B}]$ are given by

$$\begin{aligned} &A_c = (\overline{A} + \underline{A})/2, \quad A_r = (\overline{A} - \underline{A})/2, \\ &B_c = (\overline{B} + \underline{B})/2, \quad B_r = (\overline{B} - \underline{B})/2, \end{aligned}$$

respectively.

Next, consider to compute a set of states at each time. Although a set of initial states is given as an interval, a set of states at each time is in general given as a convex polyhedron. However, for high-dimensional systems, computing a convex polyhedron is difficult. In this paper, instead of a convex polyhedron, the interval of the state $[\underline{x}(k), \overline{x}(k)]$, $\underline{x}(k), \overline{x}(k) \in \mathcal{X}$ is computed as an over-approximation of a convex polyhedron. Then by using Lemma 1, we can approximately compute the interval $[\underline{x}(k+1), \overline{x}(k+1)]$ of the state at time $k+1$ for a given state interval at time k, $[\underline{x}(k), \overline{x}(k)]$. The result is shown by the following lemma.

Lemma 2: Suppose that the system (4) is given. Then the interval $[\underline{x}(k+1), \overline{x}(k+1)]$ of the state at time $k+1$ is approximately derived by

$$\begin{aligned} &[\underline{x}(k+1), \overline{x}(k+1)] \\ &= [A_c - A_r, A_c + A_r][x_c(k) - x_r(k), x_c(k) + x_r(k)] \\ &\quad + [B_c - B_r, B_c + B_r]u(k) + [\underline{a}, \overline{a}] \\ &\subseteq [A_c x_c(k) - A_r|x_c(k)| - |A_c|x_r(k) - A_r x_r(k) \\ &\quad + B_c u(k) - B_r|u(k)| + \underline{a}, \\ &\quad A_c x_c(k) + A_r|x_c(k)| + |A_c|x_r(k) + A_r x_r(k) \\ &\quad + B_c u(k) + B_r|u(k)| + \overline{a}] \\ &=: [\underline{x}'(k+1), \overline{x}'(k+1)]. \end{aligned} \tag{6}$$

Proof : From Lemma 1 and (1), we can obtain Lemma 2 straightforwardly. □

In this paper, we use $[\underline{x}'(k+1), \overline{x}'(k+1)]$ of (6) as an approximate interval of $[\underline{x}(k+1), \overline{x}(k+1)]$, and consider the relation between $[\underline{x}'(k), \overline{x}'(k)]$ and $[\underline{x}'(k+1), \overline{x}'(k+1)]$ for a given $[\underline{x}(0), \overline{x}(0)]$. For simplicity of discussion, we omit the symbol " ′ " in $[\underline{x}'(k), \overline{x}'(k)]$ hereafter. Then from (6), we obtain

$$\begin{aligned} \underline{x}(k+1) &= -A_r|x_c(k)| - (|A_c| + A_r)x_r(k) \\ &\quad -B_r|u(k)| + A_c x_c(k) + B_c u(k) + \underline{a}, \end{aligned} \tag{7}$$

$$\begin{aligned} \overline{x}(k+1) &= A_r|x_c(k)| + (|A_c| + A_r)x_r(k) \\ &\quad +B_r|u(k)| + A_c x_c(k) + B_c u(k) + \overline{a}. \end{aligned} \tag{8}$$

$|x_c(k)|$ and $|u(k)|$ can be transformed into a linear form with continuous variables and binary variables by applying the following lemma.

Lemma 3: For a given vector $w \in \mathcal{W} \subseteq \mathcal{R}^n$ ($\mathcal{W}$ is a closed and bounded set), $|w|$ is rewritten as

$$|w| = 2z - w,$$

$$z^{(i)} = \delta^{(i)} w^{(i)}, \quad i = 1, 2, \ldots, n, \tag{9}$$

$$\left[\delta^{(i)} = 1\right] \leftrightarrow \left[w^{(i)} \geq 0\right], \quad i = 1, 2, \ldots, n \tag{10}$$

where $z \in \mathcal{R}^n$, $\delta \in \{0, 1\}^n$ are auxiliary continuous variables and auxiliary binary variables, respectively.

This lemma can be directly derived from [6], [8], and is well-used in the MLD model framework. The outline of the proof is shown as follows. From (9) and (10), if $w^{(i)} \geq 0$ then, $z^{(i)} = w^{(i)}$ and $2z^{(i)} - w^{(i)} = w^{(i)}$ hold, and if $w^{(i)} < 0$ then, $z^{(i)} = 0$ and $2z^{(i)} - w^{(i)} = -w^{(i)}$ hold. So $2z^{(i)} - w^{(i)} = |w^{(i)}|$ holds. Note that (9) and (10) can be expressed as linear inequalities. See Appendix for further details. Therefore, $|x_c(k)|$ and $|u(k)|$ can be transformed into a linear form with continuous variables and binary variables.

Thus we obtain the following theorem.

Theorem 1: Suppose that the discrete-time linear system with parameter uncertainty (4) is given. Then (4) is approximately expressed by the following representation

$$\begin{cases} \hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}\hat{v}(k) + \hat{a}, \\ \hat{C}\hat{x}(k) + \hat{D}\hat{v}(k) \leq \hat{E} \end{cases} \tag{11}$$

where $\hat{v}(k) = [\ u^T(k) \quad \hat{z}^T(k) \quad \hat{\delta}^T(k)\ ]^T$, and $\hat{z}(k) = [\ z_x^T(k) \quad z_u^T(k)\ ]^T$, $z_x(k) \in \mathcal{R}^n$, $z_u(k) \in \mathcal{R}^m$ $\hat{\delta}(k) = [\ \delta_x^T(k) \ \ \delta_u^T(k)\ ]^T$, $\delta_x(k) \in \{0, 1\}^n$, $\delta_u(k) \in \{0, 1\}^m$. $\hat{A} \in \mathcal{R}^{2n \times 2n}$, $\hat{B} \in \mathcal{R}^{2n \times (m+(n+m)+(n+m))}$ and $\hat{a} \in \mathcal{R}^{2n \times 1}$ are given as

$$\hat{A} = \begin{bmatrix} A_c + A_r & -(|A_c| + A_r) \\ A_c - A_r & |A_c| + A_r \end{bmatrix} \Phi,$$

$$\hat{B} = \left[\begin{array}{c|cc|cc} B_c + B_r & -2A_r & -2B_r & 0 & 0 \\ A_c - A_r & 2A_r & 2B_r & 0 & 0 \end{array} \right],$$

$$\hat{a} = \left[\begin{array}{c} \underline{a} \\ \hline \overline{a} \end{array} \right].$$

In addition, $\hat{C} \in \mathcal{R}^{6(n+m)\times 2n}$, $\hat{D} \in \mathcal{R}^{6(n+m)\times(2n+3m)}$ and $\hat{E} \in \mathcal{R}^{6(n+m)\times 1}$ are given as

$$\hat{C} = \left[\begin{array}{c} C_x \\ 0 \end{array} \right],$$

$$\hat{D} = \left[\begin{array}{c|cc|cc} 0 & D_x^2 & 0 & D_x^3 & 0 \\ D_u^1 & 0 & D_u^2 & 0 & D_u^3 \end{array} \right],$$

$$\hat{E} = \left[\begin{array}{c} E_x \\ E_u \end{array} \right]$$

where

$$C_x = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ I_n & 0 \\ -I_n & 0 \\ I_n & 0 \\ -I_n & 0 \end{array} \right] \Phi, \quad D_x^2 = \left[\begin{array}{c} -I_n \\ I_n \\ -I_n \\ I_n \\ 0 \\ 0 \end{array} \right],$$

$$D_x^3 = \left[\begin{array}{c} \mathrm{diag}(\underline{x}_c) \\ -\mathrm{diag}(\overline{x}_c) \\ \mathrm{diag}(\overline{x}_c) \\ -\mathrm{diag}(\underline{x}_c) \\ -\mathrm{diag}(\underline{x}_c) \\ -(\mathrm{diag}(\overline{x}_c) + \varepsilon I_n) \end{array} \right], \quad E_x = \left[\begin{array}{c} 0 \\ 0 \\ \overline{x}_c \\ -\underline{x}_c \\ -\underline{x}_c \\ -\varepsilon e_n \end{array} \right],$$

$$D_u^1 = \left[\begin{array}{c} 0 \\ 0 \\ I_m \\ -I_m \\ -I_m \\ I_m \end{array} \right], \quad D_u^2 = \left[\begin{array}{c} -I_m \\ I_m \\ -I_m \\ I_m \\ 0 \\ 0 \end{array} \right],$$

$$D_u^3 = \left[\begin{array}{c} \mathrm{diag}(\underline{u}) \\ -\mathrm{diag}(\overline{u}) \\ \mathrm{diag}(\overline{u}) \\ -\mathrm{diag}(\underline{u}) \\ -\mathrm{diag}(\underline{u}) \\ -(\mathrm{diag}(\overline{u}) + \varepsilon I_m) \end{array} \right], \quad E_u = \left[\begin{array}{c} 0 \\ 0 \\ \overline{u} \\ -\underline{u} \\ -\underline{u} \\ -\varepsilon e_m \end{array} \right]$$

and $\underline{x}_c = \min_{x\in\mathcal{X}} x$, $\overline{x}_c = \max_{x\in\mathcal{X}} x$, $\underline{u} = \min_{u\in\mathcal{U}} u$, $\overline{u} = \max_{u\in\mathcal{U}} u$, and ε is a small tolerance.

Proof : Consider (7) and (8). To apply Lemma 3, we replace $|x_c(k)|$ and $|u(k)|$ to $2z_x(k) - x_c(k)$ and $2z_u(k) - u(k)$, respectively. Then $\hat{A}$, $\hat{B}$ and $\hat{a}$ in (11) are obtained from (5), (7) and (8). Furthermore, from (9) and (10) in Lemma 3, we give the following conditions: $z_x^{(i)}(k) = \delta_x^{(i)}(k)x_c^{(i)}(k)$, $[\delta_x^{(i)} = 1] \leftrightarrow [x_c^{(i)}(k) \geq 0]$, $i = 1, 2, \ldots, n$ and $z_u^{(j)}(k) = \delta_u^{(j)}(k)u^{(j)}(k)$, $[\delta_u^{(j)} = 1] \leftrightarrow [u^{(j)}(k) \geq 0]$, $j = 1, 2, \ldots, m$. These conditions are rewritten as linear inequalities by using Lemma 4 and Lemma 5. Thus we obtain $\hat{C}$, $\hat{D}$ and $\hat{E}$ in (11). □

Since (11) is the MLD model, we see that discrete-time linear systems with parameter uncertainty can be approximately expressed as a kind of hybrid systems. Also, for (11), suppose that $\hat{x}(0)$ and the input sequence $u(0), u(1), \ldots, u(f-1)$ are given. Then the problem to find the state sequence $\hat{x}(1), \hat{x}(2), \ldots, \hat{x}(f)$ can be rewritten as a mixed integer feasibility test (MIFT) problem with continuous variables $\hat{z}(k)$ and binary variables $\hat{\delta}(k)$. The MIFT problem can be solved by using an appropriate solver, e.g., ILOG CPLEX [14].

4. Modeling of Discrete-Time Piecewise Affine Systems with Parameter Uncertainty

In this section, the result on discrete-time linear systems is extended to DT-PWA systems, and as well as discrete-time linear systems, a DT-PWA system with parameter uncertainty is transformed into the MLD model.

Consider the following DT-PWA system with parameter uncertainty

$$\begin{cases} x(k+1) = A_{I(k)}x(k) + B_{I(k)}u(k) + a_{I(k)}, \\ I(k+1) = I_+, \ \text{ if } x(k+1) \in \mathcal{S}_{I_+} \end{cases} \tag{12}$$

where

$$\begin{aligned} & x(k) \in \left[\underline{x}(k), \overline{x}(k)\right], \ \underline{x}(k), \overline{x}(k) \in \mathcal{X} \subseteq \mathcal{R}^n, \\ & u(k) \in \mathcal{U} \subseteq \mathcal{R}^m, \\ & A_{I(k)} \in \left[\underline{A}_{I(k)}, \overline{A}_{I(k)}\right], \ B_{I(k)} \in \left[\underline{B}_{I(k)}, \overline{B}_{I(k)}\right], \\ & a_{I(k)} \in \left[\underline{a}_{I(k)}, \overline{a}_{I(k)}\right] \end{aligned}$$

and $I(k) \in \mathcal{M} := \{1, 2, \ldots, M\}$ is the mode of system, M is the number of modes, $\mathcal{X}$, $\mathcal{U}$ are closed and bounded convex sets. Also, $\mathcal{S}_I$, $I = 1, 2, \ldots, M$ is a bounded convex polyhedron satisfying $\bigcup_{I\in\mathcal{M}} \mathcal{S}_I = \mathcal{X}$ and $\mathcal{S}_I \bigcap \mathcal{S}_J = \emptyset$ for all $I \neq J \in \mathcal{M}$. For simplicity of discussion, the following assumption is made for $\mathcal{X}$ and $\mathcal{S}_I$:

Assumption 1: $\mathcal{X}$ and $\mathcal{S}_I$, $I = 1, 2, \ldots, M$ are expressed by an interval.

Consider to express the system (12) as the MLD model. First, we assign a binary variable $\delta_i'(k) \in \{0, 1\}$, $i = 1, 2, \ldots, M$ satisfying $[\delta_i'(k) = 1] \leftrightarrow [x(k) \in \mathcal{S}_i]$. By using auxiliary binary variables, this condition can be expressed as a set of linear inequalities [6]. In the standard DT-PWA systems without parameter uncertainty, $\delta_i'(k)$ is directly used, but in the DT-PWA system with parameter uncertainty, $\delta_i'(k)$ is not directly used. For example, in bimodal PWA systems, we can consider three cases: (i) $\delta_1'(k) = 1, \delta_2'(k) = 0$, (ii) $\delta_1'(k) = 0, \delta_2'(k) = 1$, (iii) $\delta_1'(k) = 1, \delta_2'(k) = 1$ (Two modes are active simultaneously). Three cases are regarded as three modes. Then we assign a binary variable $\delta_i(k)$, $i = 1, 2, 3$ satisfying $\delta_1(k) = \delta_1'(k) - \delta_1'(k)\delta_2'(k)$, $\delta_2(k) = \delta_2'(k) - \delta_1'(k)\delta_2'(k)$, $\delta_3(k) = \delta_1'(k)\delta_2'(k)$. In general, a binary variable $\delta_i(k)$, $i = 1, 2, \ldots, M, M+1, \ldots, M+N$ is given by a polynomial with respect to $\delta_j'(k)$, $j = 1, 2, \ldots, M$, where $\delta_{M+1}(k), \ldots, \delta_{M+N}(k)$ correspond to the cases that multiple modes are active simultaneously.

By using (11) and $\delta_i(k)$, we obtain the following expression to approximately express (12)

$$\left\{\begin{array}{l} \hat{x}(k+1) = \sum_{I=1}^{M+N} \delta_I(k) \left\{ \hat{A}_I \hat{x}(k) + \hat{B}_I \hat{v}_I(k) + \hat{a}_I \right\}, \\ \sum_{I=1}^{M+N} \delta_I(k) \left\{ \hat{C}_I \hat{x}(k) + \hat{D}_I \hat{v}_I(k) \right\} \leq \sum_{I=1}^{M+N} \delta_I(k) \hat{E}_I \end{array}\right. \tag{13}$$

where $\hat{v}_1(k) = \hat{v}_2(k) = \cdots = \hat{v}_M(k) (= \hat{v}(k))$ and

$$\hat{v}_I(k) = \begin{bmatrix} \hat{v}(k) \\ \hat{w}_I(k) \end{bmatrix}, \quad I = M+1, M+2, \ldots, M+N,$$

$\hat{w}_I(k)$ is auxiliary continuous and binary variables. In addition, $\hat{x}(k+1) = \hat{A}_I \hat{x}(k) + \hat{B}_I \hat{v}_I(k) + \hat{a}_I$, $\hat{C}_I \hat{x}(k) + \hat{D}_I \hat{v}_I(k) \leq \hat{E}_I$, $I = M+1, M+2, \ldots, N$ is the state equation in the case that multiple modes are active simultaneously. These state equations are derived as follow. Suppose that p modes are active simultaneously at time k. First, the interval of the state is decomposed to p intervals. Under Assumption 1, it is easy to consider this decomposition. $\underline{x}_i(k)$, $\overline{x}_i(k)$, $i = 1, 2, \ldots, p$ expresses the infimum and the supremum of the interval at mode i, and we define $\hat{x}_i(k) := [\ \underline{x}_i^T(k) \ \ \overline{x}_i^T(k)\ ]^T$. Then we obtain $\hat{x}_i(k+1) = \hat{A}_i \hat{x}_i(k) + \hat{B}_i \hat{v}(k)$. For simplicity, the inequality is omitted. Thus we obtain the state interval at time $k+1$ as follow:

$$\begin{aligned} &\hat{x}(k+1) = \left[\ \underline{x}^T(k+1) \ \ \overline{x}^T(k+1)\ \right]^T, \qquad (14) \\ &\underline{x}(k+1) = \min(\underline{x}_1(k+1), \underline{x}_2(k+1), \ldots, \underline{x}_p(k+1)), \\ &\overline{x}(k+1) = \max(\overline{x}_1(k+1), \overline{x}_2(k+1), \ldots, \overline{x}_p(k+1)). \end{aligned}$$

(14) can be transformed into the MLD model by using the results in [6], [8]. See Example 1 for further details.

Furthermore, since (13) can be expressed by a linear state equation and a linear inequality [6], the DT-PWA system with parameter uncertainty of (12) can be approximately expressed as the following MLD model

$$\left\{\begin{array}{l} x(k+1) = Ax(k) + Bv(k), \\ Cx(k) + Dv(k) \leq E \end{array}\right. \tag{15}$$

where $x(k) = [\underline{x}^T(k) \ \ \overline{x}^T(k)]^T \in \mathcal{R}^{2n}$ is a vector consisting the infimum and the supremum of the interval of the state $x(k)$, and $v(k)$ is given by $v(k) = [\ u^T(k) \ \ z^T(k) \ \ \delta^T(k)\ ]^T$, $u(k) \in \mathcal{R}^m$ is the control input, and $z(k) \in \mathcal{R}^{m_1}$, $\delta(k) \in \{0,1\}^{m_2}$ are auxiliary continuous and binary variables, respectively. A, B, C, D and E are some vector/matrices.

Since DT-PWA systems with parameter uncertainty can be expressed by the MLD model, the controllability problem and the optimal control problem can be solved by using the framework of the MLD model.

Example 1: Consider the following bimodal piecewise linear (PWL) system

$$x(k+1) = \left\{\begin{array}{ll} A_1 x(k) + B_1 u(k), & \text{if } [\ 1 \ \ 0\ ]\, x(k) < 0, \\ A_2 x(k) + B_2 u(k), & \text{if } [\ 1 \ \ 0\ ]\, x(k) \geq 0, \end{array}\right. \tag{16}$$

where $x(k) \in [\underline{x}(k), \overline{x}(k)]$, $\underline{x}(k), \overline{x}(k) \in \mathcal{R}^2$, $u(k) \in \mathcal{R}^1$ and $A_1 \in [\underline{A}_1, \overline{A}_1]$, $A_2 \in [\underline{A}_2, \overline{A}_2]$.

First, by Theorem 1 the continuous dynamics for $[\ 1 \ \ 0\ ]x(k) < 0$ (mode 1) and $[\ 1 \ \ 0\ ]x(k) \geq 0$ (mode 2) can be derived as

$$\hat{x}(k+1) = \left\{\begin{array}{ll} \hat{A}_1 \hat{x}(k) + \hat{B}_1 \hat{v}(k), & \text{if } \overline{x}^{(1)}(k) < 0, \\ \hat{A}_2 \hat{x}(k) + \hat{B}_2 \hat{v}(k), & \text{if } \underline{x}^{(1)}(k) \geq 0, \end{array}\right.$$

respectively. For simplicity, the inequalities are omitted.

Next, consider the continuous dynamics in the case (called here mode 3) that mode 1 and mode 2 are active simultaneously. Then $\hat{x}(k)$ is decomposed as follow:

$$\hat{x}_1(k) = \begin{bmatrix} \underline{x}_1(k) \\ \overline{x}_1(k) \end{bmatrix} = \left[\begin{array}{c} \underline{x}^{(1)}(k) \\ \underline{x}^{(2)}(k) \\ \hline (1-\delta_1(k))\overline{x}^{(1)}(k) \\ \overline{x}^{(2)}(k) \end{array}\right], \tag{17}$$

$$\hat{x}_2(k) = \begin{bmatrix} \underline{x}_2(k) \\ \overline{x}_2(k) \end{bmatrix} = \left[\begin{array}{c} (1-\delta_2(k))\underline{x}^{(1)}(k) \\ \underline{x}^{(2)}(k) \\ \hline \overline{x}^{(1)}(k) \\ \overline{x}^{(2)}(k) \end{array}\right]. \tag{18}$$

$\hat{x}_1(k+1)$ and $\hat{x}_2(k+1)$ are derived as

$$\hat{x}_1(k+1) = \begin{bmatrix} \underline{x}_1(k+1) \\ \overline{x}_1(k+1) \end{bmatrix} = \hat{A}_1 \hat{x}_1(k) + \hat{B}_1 \hat{v}(k),$$

$$\hat{x}_2(k+1) = \begin{bmatrix} \underline{x}_2(k+1) \\ \overline{x}_2(k+1) \end{bmatrix} = \hat{A}_2 \hat{x}_2(k) + \hat{B}_2 \hat{v}(k),$$

respectively. From $\underline{x}_1(k+1)$, $\overline{x}_1(k+1)$, $\underline{x}_2(k+1)$ and $\overline{x}_2(k+1)$, we obtain the continuous dynamics for mode 3

$$\hat{x}(k+1) = \begin{bmatrix} \min(\underline{x}_1(k+1), \underline{x}_2(k+1)) \\ \max(\overline{x}_1(k+1), \overline{x}_2(k+1)) \end{bmatrix}. \tag{19}$$

To express (19) as a linear form, the following logical conditions using binary variables $\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}$ are given:

$$[\delta_{11}(k) = 1] \leftrightarrow \left[\underline{x}_1^{(1)} - \underline{x}_2^{(1)} \geq 0\right], \tag{20}$$

$$[\delta_{12}(k) = 1] \leftrightarrow \left[\underline{x}_1^{(2)} - \underline{x}_2^{(2)} \geq 0\right], \tag{21}$$

$$[\delta_{21}(k) = 1] \leftrightarrow \left[\overline{x}_1^{(1)} - \overline{x}_2^{(1)} \geq 0\right], \tag{22}$$

$$[\delta_{22}(k) = 1] \leftrightarrow \left[\overline{x}_1^{(2)} - \overline{x}_2^{(2)} \geq 0\right]. \tag{23}$$

Then from (19), we obtain

$$\hat{x}(k+1) = \begin{bmatrix} \delta_{11}(k)\underline{x}_2^{(1)}(k+1) + (1-\delta_{11}(k))\underline{x}_1^{(1)}(k+1) \\ \delta_{12}(k)\underline{x}_2^{(2)}(k+1) + (1-\delta_{12}(k))\underline{x}_1^{(2)}(k+1) \\ \delta_{21}(k)\overline{x}_1^{(1)}(k+1) + (1-\delta_{21}(k))\overline{x}_2^{(1)}(k+1) \\ \delta_{22}(k)\overline{x}_1^{(2)}(k+1) + (1-\delta_{22}(k))\overline{x}_2^{(2)}(k+1) \end{bmatrix}. \tag{24}$$

The product of a binary variable and a continuous variable appeared in (17), (18) and (24) can be transformed into linear inequalities by using Lemma 5. (20)–(23) can also be transformed into a linear form by using Lemma 4. Thus the continuous dynamics for mode 3 can be expressed as the MLD model.

Finally, we illustrate the above procedure in Fig. 1. In

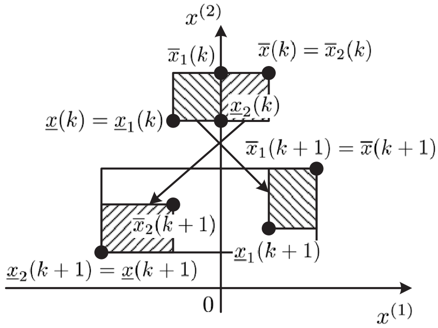

Fig. 1 Derivation of the interval $\hat{x}(k+1)$.

Fig. 1, after the interval $[\underline{x}(k), \overline{x}(k)]$ is decomposed to two intervals $[\underline{x}_1(k), \overline{x}_1(k)]$ and $[\underline{x}_2(k), \overline{x}_2(k)]$, $[\underline{x}_1(k+1), \overline{x}_1(k+1)]$ and $[\underline{x}_2(k+1), \overline{x}_2(k+1)]$ are computed. Thus we obtain $\underline{x}(k+1) = \min(\underline{x}_1(k+1), \underline{x}_2(k+1)) = \underline{x}_2(k+1)$ and $\overline{x}(k+1) = \max(\overline{x}_1(k+1), \overline{x}_2(k+1)) = \overline{x}_1(k+1)$.

Remark 1: In [12], continuous-time piecewise affine systems with parameter uncertainty are discretized with respect to time, using mode transitions in each interval between sampling points. In this paper, we consider the DT-PWA system (12) at first. It is one of future works to consider a behavior between sampling points.

5. Application to Analysis and Control

In this section, using the obtained model (15), the trajectory generation problem, the controllability problem, and the optimal control problem are discussed.

5.1 Preliminaries

As a preparation, some matrices/vectors are defined. Suppose that the MLD model (15) and the finite time f are given. First, a state sequence and an input sequence are denoted by

$$\mathsf{x} := [\ x^T(0)\ \ x^T(1)\ \ \cdots\ \ x^T(f)\ ]^T,$$
$$\mathsf{v} := [\ v^T(0)\ \ v^T(1)\ \ \cdots\ \ v^T(f-1)\ ]^T.$$

Then from the state equation of (15), we obtain

$$\mathsf{x} = \mathsf{A}x(0) + \mathsf{B}\mathsf{v} \tag{25}$$

where

$$\mathsf{A} = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^f \end{bmatrix},\ \ \mathsf{B} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{f-1}B & \cdots & AB & B \end{bmatrix}.$$

Also $\tilde{\mathsf{B}} := \begin{bmatrix} A^{f-1}B & A^{f-2} & \cdots & B \end{bmatrix}$ are defined. Next, from the linear inequality of (15), we obtain

$$\mathsf{C}\mathsf{x} + \mathsf{D}\mathsf{v} \leq \mathsf{E}$$

where

$$\mathsf{C} = \begin{bmatrix} C & & 0 \\ & \ddots & \\ 0 & & C \end{bmatrix},\ \ \mathsf{D} = \begin{bmatrix} D & & 0 \\ & \ddots & \\ 0 & & D \\ 0 & \cdots & 0 \end{bmatrix},$$
$$\mathsf{E} = \begin{bmatrix} E^T & \cdots & E^T \end{bmatrix}^T.$$

5.2 Trajectory Generation Problem

Consider the following trajectory generation problem.

Problem 1: Consider the DT-PWA system with parameter uncertainty (12). Suppose that the terminal time f, the interval of the initial state $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0] \subseteq \mathcal{X}$, and the input sequence $u(0), u(1), \ldots, u(f-1)$ are given. Then find an over-approximation of the interval of the state at each time $[\underline{x}(k), \overline{x}(k)]$, $k = 1, 2, \ldots, f$.

This problem can be solved by using the MLD model (15), i.e., this problem can be transformed into the following MIFT problem

$$\begin{aligned} \text{given}\ \ & x_0, u(0), u(1), \ldots, u(f-1) \\ \text{find}\ \ & \mathsf{v} \\ \text{subject to}\ \ & (\mathsf{D} + \mathsf{C}\mathsf{B})\mathsf{v} \leq \mathsf{E} - \mathsf{C}\mathsf{A}x_0 \end{aligned}$$

where $x_0 := [\ \underline{x}_0^T\ \ \overline{x}_0^T\ ]^T$. By solving this MIFT problem, $z(k), \delta(k)$, $k = 0, 1, \ldots, f-1$ are obtained. Thus we obtain x of (25) as an over-approximation of the interval of the state.

5.3 Controllability Problem

Based on [5], [7], we give the definition of controllability.

Definition 1: Suppose that for the system (12), the terminal time f, the interval of the initial state $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0] \subseteq \mathcal{X}$, and the interval of the terminal state $\mathcal{X}_f = [\underline{x}_f, \overline{x}_f] \subseteq \mathcal{X}$ are given. Then the system (12) is said to be $(f, \mathcal{X}_0, \mathcal{X}_f)$-*controllable*, if for every $x(0) \in \mathcal{X}_0$, there exists an input sequence $u(0), u(1), \ldots, u(f-1)$ such that $[\underline{x}(f), \overline{x}(f)] \subseteq [\underline{x}_f, \overline{x}_f]$.

In general, $\mathcal{X}_0, \mathcal{X}_f$ are given as convex polyhedra, but in this paper, for simplicity of discussion, $\mathcal{X}_0, \mathcal{X}_f$ are given as intervals.

By using the MLD model (15), we can derive a sufficient condition for the system (12) to be $(f, \mathcal{X}_0, \mathcal{X}_f)$-controllable. The result is shown as the following theorem.

Theorem 2: Suppose that the MLD model (15), which approximately expresses the system (12), is given. Then the system (12) is $(f, \mathcal{X}_0, \mathcal{X}_f)$-controllable, if the following MIFT problem

$$\begin{aligned} \text{find}\ \ & \mathsf{v} \\ \text{subject to}\ \ & (\mathsf{D} + \mathsf{C}\mathsf{B})\mathsf{v} \leq \mathsf{E} - \mathsf{C}\mathsf{A}x_0 \\ & \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \tilde{\mathsf{B}}\mathsf{v} \leq \begin{bmatrix} -\underline{x}_f \\ \overline{x}_f \end{bmatrix} \end{aligned}$$

$$-\begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} A^f x_0$$

has a solution, where $x_0 := [\ \underline{x}_0^T\ \ \overline{x}_0^T\ ]^T$.

Proof : In Definition 1, the condition $[\underline{x}(f), \overline{x}(f)] \subseteq [\underline{x}_f, \overline{x}_f]$ is equivalent to $\underline{x}_f \le \underline{x}(f)$ and $\overline{x}(f) \le \overline{x}_f$. Since

$$x(f) = \begin{bmatrix} \underline{x}(f) \\ \overline{x}(f) \end{bmatrix} = A^f x(0) + \tilde{\mathsf{B}}\mathsf{v}$$

holds, the second inequality condition of the MIFT problem is obtained. The first inequality condition corresponds to the inequality of the MLD model (15). Thus we obtain the MIFT problem. If the MIFT problem is infeasible, then there does not exist an input sequence $u(0), u(1), \ldots, u(f-1)$ satisfying $[\underline{x}(f), \overline{x}(f)] \subseteq [\underline{x}_f, \overline{x}_f]$, i.e., the system (12) is not $(f, \mathcal{X}_0, \mathcal{X}_f)$-controllable. This completes the proof. □

By solving the MIFT problem in Theorem 2, we can check the controllability of the system (12).

5.4 Optimal Control Problem

Since the DT-PWA system with parameter uncertainty (12) can be expressed as the MLD model (15), we can consider control problems. For example, we can derive a controller satisfying a kind of temporal logic constraints, e.g., time-varying state/input constraints. Such constraints can be embedded in the MLD model (15). The obtained MLD model is also time-varying, but this complexity does not produce any difficulty. In this paper, for simplicity of discussion, we consider the standard optimal control problem.

Consider the following problem.

Problem 2: Consider the MLD model (15), which approximately expresses the DT-PWA system with parameter uncertainty (12). Suppose that the initial state $x(0) = x_0$ is given. Then find $v^*(k)$, $k = 0, 1, \ldots, f-1$, minimizing the cost function

$$J = \sum_{i=0}^{f-1} \left\{ x^T(i) Q x(i) + v^T(i) R v(i) \right\} + x^T(f) Q_f x(f)$$

where Q, Q_f are semi-positive matrices, and R is a positive matrix.

In Problem 2, as one of methods to give weight matrices Q, Q_f and R, we can consider to minimize a weighted sum of $x_c(k)$ and $x_r(k)$, which are the center and the radius of the interval of the state (see (5)). Then the cost function is given by

$$\begin{aligned} J =& \sum_{i=0}^{f-1} \left\{ x_c^T(i) Q_c x_c(i) + x_r^T(i) Q_r x_r(i) + v^T(i) R v(i) \right\} \\ &+ x_c^T(f) Q_{cf} x_c(f) + x_r^T(f) Q_{rf} x_r(f) \qquad (26) \\ =& \sum_{i=0}^{f-1} \left\{ x^T(i) \begin{bmatrix} (Q_c+Q_r)/4 & (Q_c-Q_r)/4 \\ (Q_c-Q_r)/4 & (Q_c+Q_r)/4 \end{bmatrix} x(i) \right. \\ &\left. + v^T(i) R v(i) \right\} \\ &+ x^T(f) \begin{bmatrix} (Q_{cf}+Q_{rf})/4 & (Q_{cf}-Q_{rf})/4 \\ (Q_{cf}-Q_{rf})/4 & (Q_{cf}+Q_{rf})/4 \end{bmatrix} x(f) \end{aligned}$$

where Q_c, Q_{cf}, Q_r, and Q_{rf} are semi-positive matrices, respectively.

Problem 2 can be transformed into the following mixed integer quadratic programming (MIQP) problem

$$\begin{aligned} \min_{\bar{v} \in \mathcal{V}} \quad & \bar{v}^T M_1 \bar{v} + \bar{v}^T M_2 x_0 \\ \text{subject to} \quad & L_1 \bar{v} \le L_2 x_0 + L_3 \end{aligned}$$

where the input set $\mathcal{V}$ is a set of $(\mathcal{R}^m \times \mathcal{R}^{m_1} \times \{0,1\}^{m_2})^f$, and M_1, M_2, L_1, L_2, L_3 are some matrices/vectors.

Remark 2: Depending on given plant and initial state, there exist cases such that the state interval at each time becomes seriously conservative. Then, to decrease conservativeness, it will be considered to decompose the state interval at each time to some intervals. In Sect. 6.4, a simple example will be shown.

Remark 3: The MIFT problem and the MIQP problem can be solved by using an appropriate solver, e.g. ILOG CPLEX. Unfortunately, solving the MIFT problem and the MIQP problem for large f becomes prohibitive. So it is one of significant works to decrease the computation time to solve these problems.

6. Numerical Example

6.1 Plant

As a numerical example, consider the PWL system (16) where A_1, A_2 and B_1, B_2 are given as

$$A_1 = \alpha \begin{bmatrix} \cos(-\pi/3) & -\sin(-\pi/3) \\ \sin(-\pi/3) & \cos(-\pi/3) \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$A_2 = \alpha \begin{bmatrix} \cos(+\pi/3) & -\sin(+\pi/3) \\ \sin(+\pi/3) & \cos(+\pi/3) \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and α is a uncertain parameter given by $\alpha \in [0.5, 0.6]$. In addition, state and input constraints are given by

$$\underline{x}(k), \overline{x}(k) \in \left[\begin{bmatrix} -10 \\ -10 \end{bmatrix}, \begin{bmatrix} +10 \\ +10 \end{bmatrix} \right], \quad u(k) \in [-1, +1].$$

Using the result of Sect. 4, this PWL system is transformed into the MLD model (15) where $n = 2$, $m = 1$, $m_1 = 25$, $m_2 = 10$. For the obtained MLD model, we will consider the trajectory generation problem and the optimal control problem.

6.2 Example of Trajectory Generation Problem

First, consider a trajectory generation problem. Suppose that the terminal time, the interval of the initial state and the input sequence are given by $f = 5$,

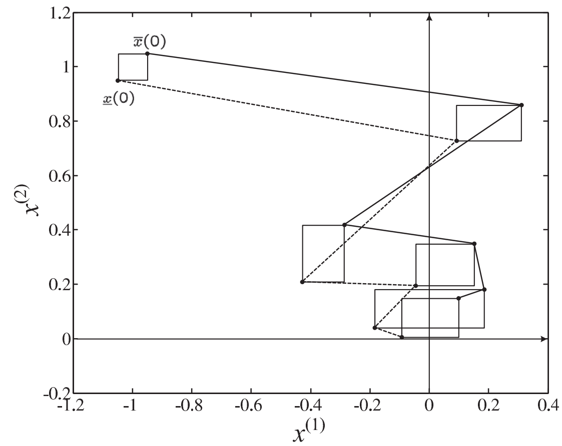

Fig. 2 State trajectory for $u(k) = 0$.

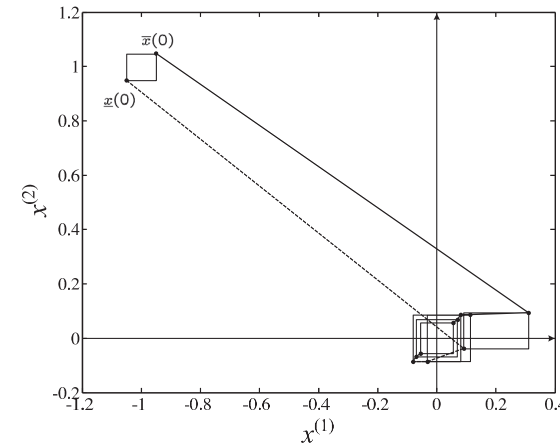

Fig. 3 State trajectory in Case 1.

$$x_0 = \left[\begin{array}{c} \underline{x}_0 \\ \overline{x}_0 \end{array} \right] = \left[\begin{array}{c} -1.05 \\ +0.95 \\ \hline -0.95 \\ +1.05 \end{array} \right], \tag{27}$$

and $u(0) = u(1) = \cdots = u(f-1) = 0$, respectively. Note here that the initial state is given as an interval, not a point. The obtained state trajectory is given by Fig. 2, where we used ILOG CPLEX 11.0 on the computer with the Intel Core 2 Duo 3.0 GHz processor and the 4 GB memory. In Fig. 2, $x^{(i)}$ denotes the i-th element of x. Also, the computation time to solve this trajectory generation problem was 0.02 [sec]. From Fig. 2, we see that the state trajectory converges to a neighborhood of the origin. On the other hand, in the proposed method, an over-approximation of an interval is computed by using Lemma 1. Here, comparing between the obtained approximate trajectory and the accurate trajectory, the error of the infimum and the supremum of the state at each time is less than 10^{-4}. So we see that in this example, the obtained over-approximation is very tight.

6.3 Example of Optimal Control Problem

Next, consider an optimal control problem. In this example, we use the cost function (26), and consider the following two cases, i.e.,

Case 1: $Q_c = Q_{cf} = 100I_2$, $Q_r = Q_{rf} = 0_{2\times2}$,

Case 2: $Q_c = Q_{cf} = 0_{2\times2}$, $Q_r = Q_{rf} = 100I_2$.

In both cases, suppose that the terminal time, the initial state and the input weighting matrix are given by $f = 5$, (27) and $R = \text{block-diag}(1, 0_{35\times35})$, respectively.

Figure 3 shows the state trajectory in Case 1. The computation time to solve the optimal control problem was 2.36 [sec]. From Fig. 3, we see that the state trajectory converges to a neighborhood of the origin faster than that in the case of $u(0) = u(1) = \cdots = u(f-1) = 0$.

Figure 4 shows the state trajectory in Case 2. The computation time to solve the optimal control problem was 20.01 [sec]. Comparing between Fig. 3 and Fig. 4, it seems that the difference is small. Note that the cost function of

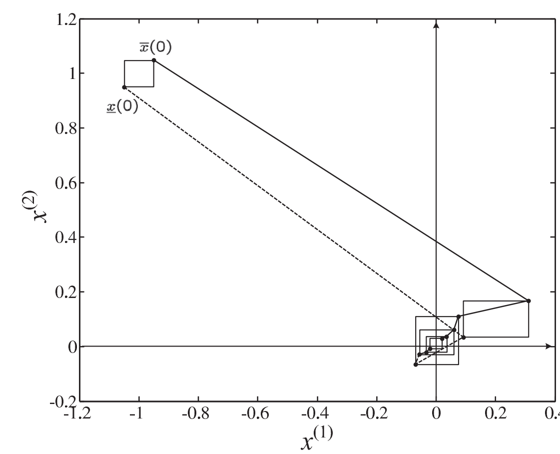

Fig. 4 State trajectory in Case 2.

Case 2 is given for minimizing the radius of an interval of the state at each time. Here, a sum of the size of the state interval at each time is derived as

$$\kappa = \sum_{k=0}^{f} \prod_{i=1}^{n} 2x_r^{(i)}(k), \quad x_r(k) = \frac{1}{2}(-\underline{x}(k) + \overline{x}(k)).$$

Then for each case, κ is calculated as follows:

Case of $u(k) = 0$: $\kappa = 0.1789$,

Case 1: $\kappa = 0.1226$,

Case 2: $\kappa = 0.0799$.

Therefore, we see that in Case 2, the expansion of the size at each time is restrained.

6.4 Extension of the Proposed Method

From the above examples, we see that the proposed method is effective for solving trajectory generation and optimal control problems of hybrid systems with parameter uncertainty. However, as is described in Remark 2, the case such that the state interval at each time becomes seriously conservative is considered. For example, instead of (27), suppose that the initial state is given by

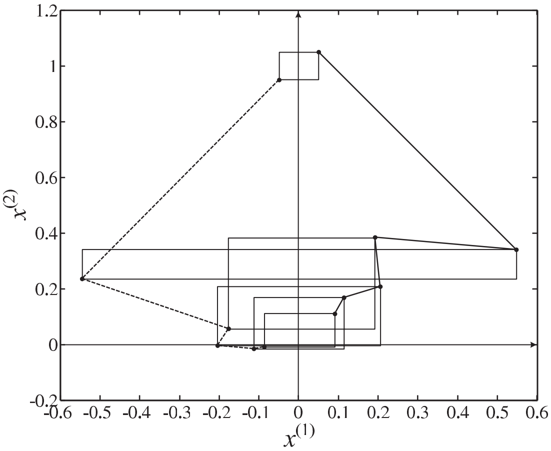

Fig. 5 State trajectory for $u(k) = 0$.

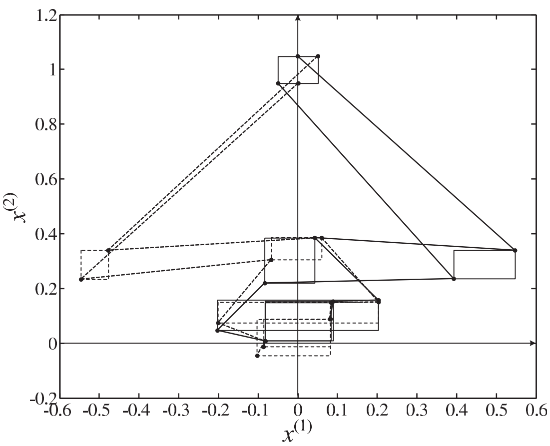

Fig. 6 State trajectory for $u(k) = 0$.

$$x_0 = \left[\begin{array}{c} \underline{x}_0 \\ \hline \overline{x}_0 \end{array}\right] = \left[\begin{array}{c} -0.05 \\ +0.95 \\ \hline +0.05 \\ +1.05 \end{array}\right]. \tag{28}$$

Then the state trajectory for $u(k) = 0$ is obtained as Fig. 5. Comparing Fig. 2 and Fig. 5, we see that the size of the rectangle at each time in Fig. 5 is larger than that in Fig. 2. This is because in the case of the initial state (28), mode 1 and mode 2 are active simultaneously at time $k = 0$.

In such a case, it is desirable to decompose the state interval at each time. The initial state (28) can be decomposed to the following two intervals:

$$x_0^1 = \left[\begin{array}{c} -0.05 \\ +0.95 \\ \hline 0 \\ +1.05 \end{array}\right], \quad x_0^2 = \left[\begin{array}{c} 0 \\ +0.95 \\ \hline +0.05 \\ +1.05 \end{array}\right].$$

The state trajectories ($u(k) = 0$) from x_0^1 and x_0^2 are obtained as Fig. 6. Comparing Fig. 5 and Fig. 6, we see that the state trajectory in Fig. 6 is tighter than that in Fig. 5. In particular, the state interval at time $k = 1$ is significantly different.

Next, consider the optimal control problem. Then consider two same MLD models. For two MLD models, the different initial states x_0^1, x_0^2 are given. However, the control input must be same in two MLD models. By regarding two MLD models as one MLD model, the optimal control problem can be solved. The cost function is given by Case 1 in Sect. 6.3. Figure 7 shows the obtained state trajectory. Comparing Fig. 6 and Fig. 7, we see that the state trajectory is improved.

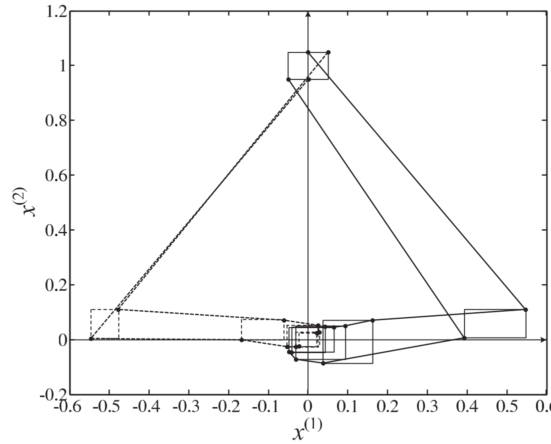

Fig. 7 Controlled State trajectory.

Remark 4: The proposed method is effective for systems such that the state trajectory asymptotically converges to a some point. So the effectiveness of the proposed method depends on stability or stabilizability. It will be easy to derive a sufficient condition for the system to be asymptotically stable. Then it is important to evaluate conservativeness in our framework, and it is one of the significant topics to clarify the effectiveness and the limitation of the proposed method.

7. Conclusion

In this paper, we have proposed a new modeling method to express discrete-time piecewise affine systems with parameter uncertainty as the MLD model. The obtained model is useful for analysis and synthesis of robust control systems. Furthermore, this paper will provide a new viewpoint for the MLD framework.

Some of future works have been already explained (see Remarks 1, 3 and 4). As is described in Remark 2 and Sect. 6.4, it is useful and important to decompose the state interval, but it is necessary to study details. Furthermore, it is interesting to clarify the relation between the proposed method, the concept of box invariance [1], [2], and the predicate abstraction technique [4], which are one of discrete abstraction techniques of hybrid systems [3].

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Appendix: Linear Inequality Expressions of Logical Conditions

In this appendix, we introduce two lemmas on linear inequality expressions of logical conditions. See [6] for further details.

Consider $x \in \mathcal{X} \subseteq \mathcal{R}^n$ ($\mathcal{X}$ is a closed and bounded set), and two functions $h : \mathcal{R}^n \to \mathcal{R}$, $g : \mathcal{R}^n \to \mathcal{R}^m$. Then the following two lemmas hold.

Lemma 4: The logical condition $[\delta = 1] \leftrightarrow [h(x) \geq 0]$ is equivalent to

$$h_{\min}(1-\delta) \leq h(x) \leq h_{\max}\delta + (\delta - 1)\varepsilon$$

where $h_{\min} = \min_{x\in\mathcal{X}} h(x)$, $h_{\max} = \max_{x\in\mathcal{X}} h(x)$, and ε is a small tolerance.

Lemma 5: $z = \delta g(x)$ is equivalent to

$$g_{\min}\delta \leq z \leq g_{\max}\delta,$$
$$g(x) - g_{\max}(1-\delta) \leq z \leq g(x) - g_{\min}(1-\delta)$$

where $g_{\min} = \min_{x\in\mathcal{X}} g(x)$, $g_{\max} = \max_{x\in\mathcal{X}} g(x)$.

Koichi Kobayashi received the M.E. degree from Hosei University in 2000, and the D.E. degree from Tokyo Institute of Technology in 2007. From 2000 to 2004, he worked at Nippon Steel Corporation. He is currently an assistant professor at School of Information Science, Japan Advanced Institute of Science and Technology. His research interests include control of hybrid systems. He is a member of the SICE, ISCIE, and IEEE.

Kunihiko Hiraishi received from the Tokyo Institute of Technology the B.E. degree in 1983, the M.E. degree in 1985, and D.E. degree in 1990. He is currently a professor at School of Information Science, Japan Advanced Institute of Science and Technology. His research interests include discrete event systems and formal verification. He is a member of the IEEE, SICE, and IPSJ.