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| Description | |

# Fuzzy logics from substructural perspective

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**Abstract**

Hájek's basic logic **BL** is an extension of the substructural logic $\mathbf{FL_{ew}}$, or equivalently, Höhle's monoidal logic. Thus, fuzzy logics can be viewed as a special subclass of substructural logics. On the other hand, their close connections are often overlooked, since these two classes of logics have been motivated by different aims, and so introduced and studied separately.

Here we attempt to bridge this gap. Several topics of substructural logics that are closely related to fuzzy logics are selected and are surveyed briefly. Above all, almost maximal logics, interpolation property, finite model property and decidability are discussed.

**Keywords**: Basic fuzzy logic, monoidal t-norm logic, substructural logics, residuated lattices, interpolation property, finite model property

## 1 Introduction

Substructural logics are typically presented as sequent systems, lacking some (zero or more, as logicians are wont to say) of structural rules of the standard Gentzen systems like **LK** and **LJ**. Since (left) structural rules concern the existence, repetition and order of assumptions, or *resources*, substructural logics have often been dubbed *resource-sensitive logics*. This is one side of the coin. The other side is that from a mathematical point of view, it is more suitable to regard substructural logics as *logics of residuated structures*. Indeed, recent developments in algebraic study of substructural logics confirm this. For a gentle introduction to these topics, we refer the reader to [29], for a comprehensive state-of-the-art handbook, to [10].

In this paper, we give a brief survey of some topics in substructural logics $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$ and their extensions, in relation to fuzzy logics. The bulk of the paper was written in 2005, but most of the results, or at least their preliminary versions, were discovered in years 2000–2002 at Algebraic Logic Seminar at JAIST organized by the authors (cf. [23]). Quite a few of these results have only appeared in master theses of graduate students who participated in the seminar, others circulated by word of mouth. Wherever possible and convenient, we give the newest versions of these results. As for general information on fuzzy logics and many-valued logics, see [13] and [6]. The paper [9] discusses also relations of fuzzy logics to substructural logics.

Substructural logics $\mathbf{FL_{ew}}$ and $\mathbf{FL_e}$ are formalized as a sequent systems obtained from the Gentzen sequent system **LJ** for intuitionistic logic by deleting the *contraction rule* (for $\mathbf{FL_{ew}}$) and also the *weakening rules* (for $\mathbf{FL_e}$), and adding rules for the logical connective *fusion*. Syntactic and algebraic properties of $\mathbf{FL_{ew}}$ are studied comprehensively in Ono-Komori [30], in which it is shown that this logic is characterized by the class of all *commutative, integral residuated lattices* (but under a different name). The same logic is known also as *monoidal logic*, introduced by U. Höhle in [14]. The monoidal t-norm logic **MTL** by Esteva-Godo [8], and a fortiori Hájek's basic logic **BL** [13], are among extensions of $\mathbf{FL_{ew}}$, and hence we can naturally view fuzzy logics and Łukasiewicz's many-valued logics as extensions of the substructural logic $\mathbf{FL_{ew}}$.

In recent years, much work has been done on noncommutative generalizations of fuzzy logic. These dispense with commutativity of fusion, and (typically) also with integrality and prelinearity, but retain divisibility. This paper takes a different route. We try to place classical fuzzy logics against a wider, but still commutative, background.

## 2 Sequent systems for substructural logics

Let $\mathbf{FL_e}$ be the sequent system obtained from **LJ** by deleting the following contraction rule and weakening rules:

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ \text{(contraction)}$$

$$\frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ \text{(left-weakening)} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ \text{(right-weakening)}$$

and then adding rules for a new connective $\cdot$ called *fusion*. More precisely, the system $\mathbf{FL_e}$ consists of the following *initial* sequents;

$$\alpha \Rightarrow \alpha \qquad \Rightarrow 1 \qquad 0 \Rightarrow$$

weakening rules for constants 1 and 0, exchange rule and cut;

$$\frac{\Gamma \Rightarrow \gamma}{1, \Gamma \Rightarrow \gamma} \text{ (1 weakening)} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0} \text{ (0 weakening)}$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma} \text{ (exchange)} \qquad \frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma} \text{ (cut)}$$

and rules for logical connectives;

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha \vee \beta, \Gamma \Rightarrow \gamma} \ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 2)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \ (\Rightarrow \wedge)$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma} \ (\wedge 1 \Rightarrow) \qquad \frac{\beta, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma} \ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta} \ (\Rightarrow \cdot) \qquad \frac{\alpha, \beta, \Gamma \Rightarrow \gamma}{\alpha \cdot \beta, \Gamma \Rightarrow \gamma} \ (\cdot \Rightarrow)$$

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \ (\Rightarrow \rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha \rightarrow \beta, \Gamma, \Delta \Rightarrow \gamma} \ (\rightarrow \Rightarrow)$$

Here, the constant 1 will behave as the unit for fusion, and 0 is used for defining the negation $\neg\alpha$ of a formula $\alpha$ by $\alpha \rightarrow 0$. Sometimes, constants $\top$ and $\bot$ for the top and the bottom are introduced and the following initial sequents are assumed:

$$\Gamma \Rightarrow \top \qquad \bot, \Gamma \Rightarrow \gamma$$

A sequent $\Gamma \Rightarrow \alpha$ is *provable* in $\mathbf{FL_e}$ if it can be obtained from initial sequents of $\mathbf{FL_e}$ by repeated applications of the rules of $\mathbf{FL_e}$. A formula $\alpha$ is provable in $\mathbf{FL_e}$, if the sequent $\Rightarrow \alpha$ is provable in it. The sequent system $\mathbf{FL_{ew}}$ is obtained from $\mathbf{FL_e}$ by adding the above left and right weakening rules to it. Clearly, in $\mathbf{FL_{ew}}$ both 1- and 0-weakening rules are redundant, since they follow from these weakening rules. Also, 1 and 0 become provably equivalent to $\top$ and $\bot$, respectively, in $\mathbf{FL_{ew}}$.

In the same way as the above, we can introduce a sequent system $\mathbf{CFL_e}$ ($\mathbf{CFL_{ew}}$) which is obtained form the sequent system **LK** for classical logic by deleting both contraction and weakening rules (only contraction rules, respectively). Its alternative definition is a sequent system obtained from $\mathbf{FL_e}$ ($\mathbf{FL_{ew}}$) by adding the law of double negation $\neg\neg\alpha \Rightarrow \alpha$ as an initial sequent. To be exact, it means that the sets of provable sequents in these two different systems are the same. The system $\mathbf{CFL_e}$ is term equivalent to the linear logic **MALL** (without exponentials) by J.-Y. Girard and $\mathbf{CFL_{ew}}$ is term equivalent to a system studied by V. Grishin in 70s.

We note here that 0 is not equal to $\perp$ in $\mathbf{FL_e}$. To show this, let us introduce a new negation $\sim \alpha$ by defining it to be $\alpha \to \perp$, and consider the sequent system $\mathbf{C^*FL_e}$ obtained from $\mathbf{FL_e}$ by adding $\sim\sim \alpha \Rightarrow \alpha$ as an initial sequent. Then, we can show that left-weakening rule is derivable in $\mathbf{C^*FL_e}$, and hence the system is equivalent to $\mathbf{CFL_{ew}}$. On the other hand, if 0 is equal to $\perp$ then $\mathbf{C^*FL_e}$ must be equivalent to $\mathbf{CFL_e}$. Obviously, this is a contradiction.

As we want to discuss *logics* over $\mathbf{FL_e}$ (or, *extensions* of $\mathbf{FL_e}$) in general, we need to define them precisely. A set of formulas **L** is a logic over $\mathbf{FL_e}$ (or, simply a logic) if

1. every formula provable in $\mathbf{FL_e}$ belongs to **L**,
2. if both $\alpha$ and $\alpha \to \beta$ are in **L** then $\beta$ is also in **L**,
3. if both $\alpha$ and $\beta$ are in **L** then $\alpha \wedge \beta$ is also in **L**,
4. if $\alpha$ is in **L** then every substitution instance of $\alpha$ is also in **L**.

The following fact will tell us that our definition of logics given above is not ad hoc. A set of formulas **K** is *closed under substitution*, if a formula $\alpha$ is in **K** then every substitution instance of $\alpha$ is also in **K**. A set **K** is *deductively closed* in $\mathbf{FL_e}$ when if a formula $\alpha$ is provable in the sequent system obtained from $\mathbf{FL_e}$ by adding sequents $\Rightarrow \beta$ as initial sequents for all $\beta \in \mathbf{K}$ then $\alpha$ is already in **K**. Then we have the following.

**Proposition 1** *Every logic over* $\mathbf{FL_e}$ *is closed under substitution and deductively closed in* $\mathbf{FL_e}$*. Conversely, if a set of formulas is deductively closed in* $\mathbf{FL_e}$ *is also closed under substitution, it is a logic over* $\mathbf{FL_e}$*.*

For simplicity, we will often identify a sequent system with the set of all formulas that are provable in it, which is obviously a logic in the above sense. Thus, $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$ will stand for both sequent systems and logics, and we will rely on context to distinguish between them. The set of all logics over $\mathbf{FL_e}$ is partially ordered by set inclusion, and moreover forms a complete lattice since the intersection of arbitrary number of logics is always a logic. The smallest logic is $\mathbf{FL_e}$, and the greatest is the set of all formulas. When a logic $\mathbf{L}$ includes another logic $\mathbf{K}$ as a set, we say that $\mathbf{L}$ is an extension of $\mathbf{K}$.

*Cut elimination* for a given sequent system means that every sequent which is provable in it is also provable in the system without using cut rule.

**Theorem 2** *Cut elimination holds for any of* $\mathbf{FL_e}$*,* $\mathbf{FL_{ew}}$*,* $\mathbf{CFL_e}$ *and* $\mathbf{CFL_{ew}}$*.*

We can prove this in a syntactic way. In the usual syntactic proof of cut elimination theorem for **LK** and **LJ**, we first replace cut rules by *mix rules* and then eliminate these mix rules. On the other hand, this replacement is not necessary for the above four systems, since none of them has contraction rules. In fact, the proof becomes much simpler than the standard proof for **LK** or **LJ**. For the details, see [30, 28]. Also, for an algebraic proof, see [2].

From cut elimination theorem for a given system $\mathcal{S}$, we can derive many important logical properties of the logic determined by it. (See e.g. [28].) Here are some examples. We say that a logic $\mathbf{L}$ has *Craig's interpolation property* (CIP) if for all formulas $\alpha$ and $\beta$, if $\alpha \to \beta$ is provable in $\mathbf{L}$ then there exists a formula $\gamma$ such that

1. both $\alpha \to \gamma$ and $\gamma \to \beta$ are provable in $\mathbf{L}$,
2. every propositional variable in $\gamma$ appears in both $\alpha$ and $\beta$.

We say that a logic $\mathbf{L}$ has the *disjunction property* if for all formulas $\alpha$ and $\beta$, if $\alpha \vee \beta$ is provable in $\mathbf{L}$ then either $\alpha$ or $\beta$ is provable in $\mathbf{L}$. By using the standard proof-theoretic arguments, we can show the following results as consequences of cut elimination theorem. The disjunction property of these logics follows from the fact that none of them have right-contraction rule (see [28] for the details).

**Theorem 3** *Logics* $\mathbf{FL_e}$*,* $\mathbf{FL_{ew}}$*,* $\mathbf{CFL_e}$ *and* $\mathbf{CFL_{ew}}$ *are all decidable, and also all have both Craig's interpolation property and the disjunction property.*

Proof-theoretic methods are quite powerful in deriving various logical properties, once a given logic is formalized in a sequent system for which cut elimination theorem holds. Actually, we can see this in the case of some particular fuzzy logics that are successfully formalized as *hypersequent systems*, extensions of original sequent systems. On the other hand, we cannot expect always to find such a sequent system. So, when we focus on general logical properties of fuzzy logics, semantical approach will be more helpful.

## 3 Residuated lattices and substructural logics

We introduce here algebraic structures for extensions of $\mathbf{FL_e}$. A *lattice ordered monoid* is an algebra $\mathbf{A}$ of the form $\langle A; \vee, \wedge, \cdot, 1\rangle$ such that

1. $\langle A; \vee, \wedge\rangle$ is a lattice,
2. $\langle A; \cdot, 1\rangle$ is a monoid with the unit 1,
3. $x \leq y$ implies both $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ for all $x, y, z \in A$.

Sometimes, $x \cdot y$ is written as $xy$. A lattice ordered monoid $\mathbf{A}$ is commutative, if the monoid $\langle A; \cdot, 1\rangle$ is commutative. A commutative lattice ordered monoid $\mathbf{A}$ is *residuated*, if for all elements $y, z \in A$ there exists an element $y \to z \in A$ such that

$x \cdot y \leq z$ if and only if $x \leq y \to z$ holds for all $x, y, z \in A$.

This equivalence is called the *law of residuation*. Commutative, residuated lattice ordered monoids are usually called *commutative residuated lattices* (CRLs). We note here that the third condition (monotonicity) in the above follows from the law of residuation. The law of residuation can be expressed by equations, and hence all the conditions for CRLs are expressed by equations. Thus, the class of all CRLs forms a *variety*. For general information on residuated lattices, including non-commutative case, see [10]. We note here the following.

**Lemma 4** *In each CRL* $\mathbf{A}$*, if the join* $\bigvee_i x_i$ *exists for* $x_i \in A$ *then* $\bigvee_i (x_i \cdot y)$ *exists and is equal to* $(\bigvee_i x_i) \cdot y$ *for every* $y \in A$.

By an $\mathbf{FL_e}$-*algebra* we mean a CRL $\mathbf{A}$ with an (*arbitrary*) element $0 \in A$. A CRL $\mathbf{A}$ is *integral* if the unit 1 is also the greatest element $\top$ of $A$. An integral $\mathbf{FL_e}$-algebra is called an $\mathbf{FL_{ew}}$-*algebra* if 0 is the smallest element $\bot$. Thus, $\mathbf{FL_{ew}}$-algebras are nothing but *bounded* CRLs such that $1 = \top$ and $0 = \bot$. From syntactical point of view, integrality corresponds to *left-weakening rule*, and the assumption that $0 = \bot$ corresponds to *right-weakening rule*. It is easy to see that a $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$ is a Heyting algebra if and only if $x \cdot y = x \wedge y$ for all $x, y \in A$. Thus, all of classes of $\mathbf{FL_e}$-algebras, $\mathbf{FL_{ew}}$-algebras, and Heyting algebras are varieties, that are denoted by $\mathsf{FL_e}, \mathsf{FL_{ew}}$ and $\mathsf{HA}$, respectively. It can be shown that subvarieties of $\mathsf{FL_e}$ form a complete lattice.

Another important class of CRLs is the class of *commutative unital quantales*. An algebra $\mathbf{A} = \langle A; \bigvee, \cdot\rangle$ is a commutative quantale if

1. $\langle A; \bigvee\rangle$ forms a complete lattice (and hence is bounded),

2. $\langle A; \cdot \rangle$ is a commutative semigroup,

3. $(\bigvee_i x_i) \cdot y = \bigvee_i (x_i \cdot y)$ for $x_i, y \in A$.

For more information on quantales, see [31]. A commutative quantale $\mathbf{A}$ is *unital* if $\langle A, \cdot \rangle$ has the unit element. Thus, each commutative unital quantale is a commutative, complete lattice ordered monoid satisfying the above third condition on distributivity of infinite joins, and vice versa. Introducing $\rightarrow$ by $y \rightarrow z = \bigvee \{x : x \cdot y \leq z\}$, we can show the following.

**Lemma 5** *Each commutative unital quantale is residuated.*

From these two lemmas, it follows that:

**Theorem 6** *For each commutative, complete lattice ordered monoid* $\mathbf{A}$*,* $\mathbf{A}$ *is residuated if and only if the equation* $(\bigvee_i x_i) \cdot y = \bigvee_i (x_i \cdot y)$ *holds always for* $x_i, y \in A$*.*

Let us consider *triangular norms* (or, simply, t-norms) in fuzzy logic in this context. A t-norm $T$ is a map from $[0, 1]^2$ to $[0, 1]$, where $[0, 1]$ is the unit interval of reals, which satisfies the following conditions:

1. $T(x, T(y, z)) = T(T(x, y), z)$,
2. $T(x, y) = T(y, x)$,
3. $T(x, 1) = x$,
4. $x \leq y$ implies $T(x, z) \leq T(y, z)$.

Define $*_T$ by $x *_T y = T(x, y)$. It is obvious that $\langle [0, 1]; \max, \min \rangle$ forms a complete lattice. Thus, the above condition means that $\mathbf{C}_T = \langle [0, 1]; \max, \min, *_T, 1 \rangle$ is a commutative, complete lattice ordered monoid. We note here that for the unit interval, the condition $(\bigvee_i x_i) *_T y = \bigvee_i (x_i *_T y)$ can be expressed as

$$T(x, y) = \lim_{z \nearrow x} T(z, y),$$

which means that $T$ is left-continuous. Now, the next corollary is an immediate consequence of Theorem 6. See e.g. [8].

**Corollary 7** *For each t-norm* $T$*,* $T$ *is left-continuous if and only if* $\mathbf{C}_T$ *is a CRL.*

Let $\mathbf{L}$ be a logic over $\mathbf{FL_e}$. We will introduce a relation $\vdash_{\mathbf{L}}$ between sets of formulas and formulas, called the *deducibility relation* of $\mathbf{L}$ as follows. For a set of formulas $\Sigma$ and a formula $\beta$,

$\Sigma \vdash_{\mathbf{L}} \beta$ if and only if $\beta$ belongs to the deductive closure of the set $\mathbf{L} \cup \Sigma$ in $\mathbf{FL_e}$.

Because of compactness, we can always assume that $\Sigma$ is finite. Sometimes, we write $\Sigma, \alpha \vdash_{\mathbf{L}} \beta$ ($\Sigma, \Gamma \vdash_{\mathbf{L}} \beta$) instead of $\Sigma \cup \{\alpha\} \vdash_{\mathbf{L}} \beta$ ($\Sigma \cup \Gamma \vdash_{\mathbf{L}} \beta$) for a formula $\alpha$ (and a set of formulas $\Gamma$, respectively). Each deducibility relation is a consequence relation in the sense of abstract algebraic logic. We can show that the the deducibility relation $\vdash_{\mathbf{FL_e}}$ is *algebraizable* and that its *equivalent algebraic semantics* is the variety $\mathsf{FL_e}$. We can show moreover the following (see [11] for the details). The first statement says that the *local deduction theorem* holds for all logics over $\mathbf{FL_e}$.

**Theorem 8** *Let* $\mathbf{L}$ *be a logic over* $\mathbf{FL_e}$*.*
*1.* $\Gamma, \alpha \vdash_{\mathbf{L}} \beta$ *if and only if* $\Gamma \vdash_{\mathbf{L}} (\alpha \wedge 1)^n \to \beta$ *for some* $n \geq 0$*.*
*2.* $\Gamma \vdash_{\mathbf{L}} \beta$ *if and only if* $1 \leq v(\gamma)$ *holds for all* $\gamma \in \Gamma$ *implies* $1 \leq v(\beta)$ *holds for every valuation* $v$ *on every* $\mathbf{FL_e}$*-algebra* $\mathbf{A}$ *which validates* $\mathbf{L}$*.*

Suppose that $\mathsf{V}$ is a subvariety of $\mathsf{FL_e}$. Then, the set of formulas $\mathbf{L}(\mathsf{V})$, defined by $\mathbf{L}(\mathsf{V}) = \{\alpha\colon 1 \leq v(\alpha)$ holds for each $\mathbf{A}$ in $\mathsf{V}$ and each valuation $v$ of $\mathbf{A}\}$ is a logic over $\mathbf{FL_e}$. Conversely, for each logic $\mathbf{L}$ over $\mathbf{FL_e}$ let $\mathsf{V}(\mathbf{L})$ be the class of $\mathbf{FL_e}$-algebras $\mathbf{A}$ such that the inequation $1 \leq \alpha$ holds for every $\alpha \in \mathbf{L}$. Let $\mathbf{A}$ be a $\mathbf{FL_e}$-algebra and $\mathsf{V}_{\mathbf{A}}$ be the subvariety of $\mathsf{FL_e}$ generated by $\mathbf{A}$. If a logic $\mathbf{L}$ is equal to $\mathbf{L}(\mathsf{V}_{\mathbf{A}})$ then it is said to be *characterized* by $\mathbf{A}$. The above $\mathbf{L}$ can be regarded as a map from the lattice of subvarieties of $\mathsf{FL_e}$ to the lattice of logics over $\mathbf{FL_e}$, and also $\mathsf{V}$ is a map of the opposite direction.

**Theorem 9** *The maps* $\mathbf{L}$ *and* $\mathsf{V}$ *are mutually inverse, dual isomorphisms between the lattice of subvarieties of* $\mathsf{FL_e}$ *and the lattice of logics over* $\mathbf{FL_e}$*.*

# 4 Almost maximal logics — logics just below classical logic

It is easy to see that classical logic $\mathbf{CL}$ is the maximum among consistent logics over $\mathbf{FL_{ew}}$. A natural question is what logics (over $\mathbf{FL_{ew}}$) will come just below $\mathbf{CL}$ and how many? We say that a logic $\mathbf{L}$ over $\mathbf{FL_{ew}}$ is *almost maximal* if $\mathbf{L}$ is strictly weaker than $\mathbf{CL}$ and consistent logic which is strictly stronger than $\mathbf{L}$ is only $\mathbf{CL}$. In other words, $\mathbf{L}$ is almost maximal if and only if the corresponding variety of $\mathbf{FL_{ew}}$-algebras is *almost minimal* in the subvariety lattice of $\mathsf{FL_{ew}}$.

Among logics over intuitionistic logic there exists a single almost maximal logic, which corresponds to the variety generated by the three-element Heyting algebra $\mathbf{H_3}$. Also, Y. Komori [19] gave a complete list of almost maximal logics over Łukasiewicz's infinitely-many valued logic Ł, which are countably many. Another interesting example of almost maximal logics is product logic Π (see [7]).

Let us consider the *n-potency* axiom scheme $\alpha^n \to \alpha^{n+1}$. Note that the 1-potency axiom scheme is no other than the axiom of contraction. In 2000, M. Ueda with the first author showed the following. Note that the basic fuzzy logic **BL** is obtained from **MTL** by adding the following axiom scheme, called *divisibility* $(\alpha \wedge \beta) \to (\alpha \cdot (\alpha \to \beta))$.

**Theorem 10** *There exist exactly six almost maximal extensions of* **MTL** *satisfying the 2-potent axiom scheme, and there exist uncountably many almost maximal extensions of* **MTL** *satisfying the 3-potent axiom scheme.*

In contrast with this, Y. Katou with the first author proved the following in 2001.

**Theorem 11** *Almost maximal logics over* **BL** *consist of* $\mathbf{H_3}$*,* Π *and almost maximal logics over Ł.*

For detailed proofs of related results including the above theorems, see [15]. The situation changes when we shift our attention to logics over $\mathbf{FL_e}$. It can be easily observed that there exists maximal consistent logic other than **CL**. In fact, the following is shown by H. Kihara in 2003.

**Theorem 12** *There exist uncountably many maximal logics over* $\mathbf{FL_e}$*.*

# 5 Algebraic characterization of logical properties

In the following, we will discuss algebraic characterization of some of logical properties. Such a characterization has its own interest, and is at the same time useful for showing that a logical property holds for a given logic.

Since the logic **MTL** is obtained from $\mathbf{FL_{ew}}$ by adding the prelinearity axiom scheme $(\alpha \to \beta) \vee (\beta \to \alpha)$, none of extensions of **MTL** has the disjunction property. Let us consider a weaker property, called *Halldén completeness.* We say that a logic **L** is Halldén complete, if for all formulas $\alpha$ and $\beta$ which have no variables in common, if $\alpha \vee \beta$ is provable in **L** then either $\alpha$ or $\beta$ is provable in **L**.

H. Kihara has showed that a characterization of Halldén complete superintuitionistic logics given by A. Wroński [33] holds also for logics over $\mathbf{FL_{ew}}$. More precisely, the following holds.

**Theorem 13** *The following three conditions are mutually equivalent for any logic* **L** *over* $\mathbf{FL_{ew}}$*.*

1. **L** *is Halldén complete,*

2. **L** *cannot be represented as the intersection of two incomparable logics,*

3. **L** *is characterized by a single well-connected* $\mathbf{FL_{ew}}$*-algebra.*

Here, an $\mathbf{FL_{ew}}$-algebra **A** is *well-connected* if for all $x, y \in A$ if $x \vee y = 1$ then either $x = 1$ or $y = 1$. It is clear that every linearly-ordered $\mathbf{FL_{ew}}$-algebra is well-connected. Thus, we have immediately the following.

**Corollary 14** *Basic logic* **BL***, Łukasiewicz logic Ł, Gödel logic* **G** *and product logic* $\Pi$ *are all Halldén complete.*

In [33], it is shown that for superintuitionistic logics the third condition in Theorem 13 can be replaced by the following.

> **L** can be characterized by a single subdirectly irreducible Heyting-algebra.

It seems that the similar equivalence does not hold between logics over $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ew}}$-algebras. On the other hand, if $n$-potent axiom scheme holds in a given logic **L** for some $n$, the following condition $(*)$ is also equivalent to any of conditions in Theorem 13. This follows from the fact that under this assumption, the subdirect irreducibility becomes first-order definable and thus it is preserved under ultraproducts. For the details, see [16].

> **L** can be characterized by a single subdirectly irreducible $\mathbf{FL_{ew}}$-algebra. $(*)$

Note that an algebraic approach to the disjunction property of substructural logics is given in [32].

In §2, Craig's interpolation property (CIP) of a logic **L** is defined. Also, we say **L** has the *deductive interpolation property* (DIP), if for all formulas $\varphi, \psi$, if $\varphi \vdash_{\mathbf{L}} \psi$, there exists a formula $\sigma$ such that

1. $\varphi \vdash_{\mathbf{L}} \sigma$ and $\sigma \vdash_{\mathbf{L}} \psi$,

2. every propositional variable in $\sigma$ appears in both $\varphi$ and $\psi$.

A subvariety $\mathsf{V}$ of $\mathsf{FL_e}$ has the *amalgamation property* (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}$ and for all embeddings $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$, there exists an algebra **D** in $\mathsf{V}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$, $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$. Then the following can be shown (see e.g. [11]). A comprehensive study of interpolation properties and amalgamation properties is developed in a recent paper [17].

**Theorem 15** *1. CIP implies DIP for every logic over* $\mathbf{FL_e}$.
*2.* $\mathbf{L}$ *has the DIP iff* $\mathsf{V}(\mathbf{L})$ *has the AP.*

Both CIP and DIP of logics over **BL** and AP of corresponding subvarieties of $\mathsf{FL_{ew}}$ are extensively studied in the paper [26] by F. Montagna. Next, consider another property which is closely related to DIP. A logic $\mathbf{L}$ has the *pseudo-relevance property* (PRP), if for all formulas $\varphi$ and $\psi$ which have no variables in common, $\varphi \vdash_{\mathbf{L}} \psi$ implies either $\varphi \vdash_{\mathbf{L}} \bot$ or $\vdash_{\mathbf{L}} \psi$. (Since $\bot$ is used in the definition of PRP, in the rest of this section we assume that our language contains the constant $\bot$, and consider *bounded* $\mathbf{FL_e}$-algebras.) Two algebras $\mathbf{B}, \mathbf{C}$ are *jointly embeddable* into an algebra $\mathbf{D}$, if there exists embeddings $h : \mathbf{B} \to \mathbf{D}$ and $j : \mathbf{C} \to \mathbf{D}$. Then, we have the following (cf. Maksimova [25]).

**Theorem 16** *1. A logic* $\mathbf{L}$ *over* $\mathbf{FL_e}$ *has the PRP iff every pair of subdirectly irreducible algebras in* $\mathsf{V}(\mathbf{L})$ *are jointly embeddable into an algebra in* $\mathsf{V}(\mathbf{L})$.
*2. If a subvariety* $\mathsf{V}$ *of* $\mathsf{FL_{ew}}$ *has the AP, then all pairs of s.i. algebras in* $\mathsf{V}$ *are jointly embeddable into an algebra in* $\mathsf{V}$*. Thus, the DIP implies the PRP for every logic over* $\mathbf{FL_{ew}}$.

Note that the converse of the consequence of the second statement in the above does not hold always. For instance, $\mathbf{FL_e}$ has the CIP and hence the DIP, but it does not have the PRP. We can show the following in the same way as a result by Komori [18], by using an extension of Glivenko's theorem in [12].

**Theorem 17** *Every extension of the logic* $\mathbf{FL_{ew}}$ *with the axiom scheme* $\neg(\alpha \wedge \neg\alpha)$*, called pseudo complementation, has the PRP. Thus, the joint embeddability of subdirectly irreducible algebras holds for each of corresponding variety.*

These results on PRP were obtained by H. Kihara in his PhD thesis in 2006. See also Chapter 5 of [10] for the proofs..

# 6 Finite model property and finite embeddability property

A useful semantical method of showing decidability of a logic $\mathbf{L}$ is to prove the *finite model property* (FMP), i.e. to prove that $\mathbf{L}$ is characterized by a class of *finite* algebras. In other words, $\mathbf{L}$ has the FMP iff the variety $\mathsf{V}(\mathbf{L})$ is generated by its finite members. By Harrop's result, $\mathbf{L}$ is decidable if it is finitely axiomatizable and has the FMP. While showing the FMP via Kripke frames is a standard and powerful technique in proving decidability of modal logics, we do not know much about the FMP of substructural logics. For instance, whether $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$ have the FMP or not remained an open question until the middle of 90s while their decidability was shown already in 80s as an easy consequence of cut elimination. Strangely enough, Lafont [24] and Okada-Terui [27], who have solved the problem affirmatively, used cut elimination results in their proofs. See also [2].

In the following, we discuss the *finite embeddability property* (FEP) which is a promising algebraic method of proving the FMP of substructural logics. After the early works in 40s, the FEP has not been paid much attention until recent works by Blok-Ferreirim [3, 4], Blok-van Alten [5] and also [1]. A class $\mathsf{K}$ of $\mathbf{FL_e}$-algebras has the FEP, if every finite *partial* algebra $\mathbf{B}$ of a member $\mathbf{A}$ of $\mathsf{K}$ can be embedded into some finite member $\mathbf{D}$ of $\mathsf{K}$. The FMP of $\mathbf{L}$ follows from the FEP of $\mathsf{V}(\mathbf{L})$. In fact the FEP is stronger than the FMP. Namely, if a class $\mathsf{K}$ has the FEP then every universal sentence that fails in $\mathsf{K}$ fails also in a finite member of $\mathsf{K}$. So, moreover if $\mathsf{K}$ is finitely axiomatizable, then the universal theory is decidable. In [5], the following is shown.

**Theorem 18** *The variety* $\mathsf{FL_{ew}}$ *has the FEP, while the variety* $\mathsf{FL_e}$ *does not have it.*

Thus, $\mathbf{FL_e}$ is an example of a logic with the FMP, whose corresponding variety does not have the FEP. Now, let us discuss the FEP of subvarieties of $\mathsf{FL_{ew}}$ that correspond to some fuzzy logics. A simple example is the variety $\mathsf{G}$ of Gödel algebras, i.e. Heyting algebras satisfying the equation $(x \to y) \vee (y \to x) = 1$. In this case, it is easily seen that every Gödel algebra is locally finite, i.e. any subalgebra generated by a finite subset is always finite. So, it suffices for us to take the subalgebra of $\mathbf{A}$ generated by $\mathbf{B}$ for $\mathbf{D}$. Therefore, it has obviously the FEP. An important result on the variety $\mathsf{V}(\mathbf{BL})$ of basic algebras is shown in [1].

**Theorem 19** *The variety* $\mathsf{V}(\mathbf{BL})$ *has the FEP, and hence its universal theory is decidable.*

In the following, by a slight modification of the proof of the FEP of $\mathsf{FL_{ew}}$ by Blok and van Alten [5], we show that each of subvarieties of $\mathsf{FL_{ew}}$ defined by any one of equations for the prelinearity $(x \to y) \vee (y \to x) = 1$, the pseudo-complementation $x \wedge \neg x = 0$ and the involution $\neg\neg x = x$, has the FEP. In fact, every subvariety defined by any combination of these three equations has it. (Note that the variety $\mathsf{FL_{ew}}$ is denoted by $\mathcal{RL}$ in [5].) The result was obtained in 2001, and the note [22] was privately circulated.

In the rest of this section, we assume the familiarity with notations and terminology of [5], in particular of its Section 5, which is briefly described here. In our setting, we suppose that $\mathbf{A}$ is a $\mathbf{FL_{ew}}$-algebra, $\mathbf{B}$ is a finite partial subalgebra of $\mathbf{A}$, and $\mathbf{M}$ is the partially ordered submonoid of the *pocrim* reduct of $\mathbf{A}$ generated by the pocrim reduct of $\mathbf{B}$. For each $a \in M$ and $b \in B$, $(a \rightsquigarrow b]$ is a downward closed subset of $M$ defined by $(a \rightsquigarrow b] = \{c \in M \colon ac \leq b\}$ (pocrim is the acronym for *partially ordered commutative residuated integral monoid*). Moreover, $\overline{D}$ is the set of all such downward closed subsets of $M$, $D = \{\bigcap \chi \colon \chi \subseteq \overline{D}\}$, and $C$ is the closure operator on $\wp(M)$ associated with $D$. Then the algebra $\mathbf{D}$ with the universe $D$ forms a finite $\mathbf{FL_{ew}}$-algebra, and the map $\iota$ from $\mathbf{B}$ to $\mathbf{D}$ defined by $\iota(a) = \{x \in M \colon x \leq a\}$ for $a \in B$ is an embedding.

We note the following result in [4]. Here, $\mathsf{V}_{\mathsf{si}}$ denotes the class of all subdirectly irreducible members of a variety $\mathsf{V}$.

**Lemma 20** *If a class $\mathsf{V}_{\mathsf{si}}$ has the FEP for a variety $\mathsf{V}$, then so does $\mathsf{V}$.*

**Theorem 21** *The subvariety $\mathsf{V}_1$ of $\mathsf{FL}_{\mathsf{ew}}$ defined by the equation $(x \to y) \vee (y \to x) = 1$ has the FEP.*

Proof. By the above lemma, it is enough to show that the class $(\mathsf{V}_1)_{\mathsf{si}}$ has the FEP. So, we take an arbitrary algebra $\mathbf{A}$ in $(\mathsf{V}_1)_{\mathsf{si}}$. Then $\mathbf{A}$ is linearly ordered. Since each member of $\overline{D}$ is a downward closed subset of $A$, the set $\overline{D}$ is also linearly ordered by set inclusion. Since $D$ is finite, each member of $D$ is a finite intersection of members of $\overline{D}$. Hence $D$ is equal to $\overline{D}$. Now, take arbitrary members $X$ and $Y$ of $D(= \overline{D})$. Then, either $X \subseteq Y$ or $Y \subseteq X$ holds. Thus, $\mathbf{D}$ satisfies $(x \to y) \vee (y \to x) = 1$.

**Theorem 22** *The subvariety of $\mathsf{FL}_{\mathsf{ew}}$ defined by the equation $x \wedge \neg x = 0$ has the FEP.*

Proof. We note first that the condition $x \wedge \neg x = 0$ is equivalent to $\neg x^2 = \neg x$ in every $\mathbf{FL_{ew}}$-algebra. From the discussion in [5], $0^{\mathbf{D}} = \{0\}$, and hence $\neg^{\mathbf{D}} X = \{y \in M ; X \cdot y = \{0\}\}$ for every $X \subseteq M$. Now, suppose that the equation $x \wedge \neg x = 0$ holds in the original algebra $\mathbf{A}$. Let X be an arbitrary element of $D$. Obviously, $0 \in X \cap \neg^{\mathbf{D}} X$. Take any element $x \in M$ such that $x \in X \cap \neg^{\mathbf{D}} X$. Since both $x \in X$ and $X \cdot x = \{0\}$ hold, $x^2 = 0$. Hence, $\neg x = \neg x^2 = 1$. Thus, $x = x \wedge 1 = x \wedge \neg x = 0$, and therefore $X \cap \neg^{\mathbf{D}} X = 0^{\mathbf{D}}$. This means that the equation $x \wedge \neg x = 0$ holds also in $\mathbf{D}$.

**Theorem 23** *The subvariety of $\mathsf{FL}_{\mathsf{ew}}$ defined by the equation $\neg\neg x = x$ has the FEP.*

Proof. In this case, a small nonessential modification is necessary. Let $\mathbf{B}$ be a finite partial subalgebra of a $\mathbf{FL_{ew}}$-algebra in which the equation $\neg\neg x = x$

holds. Let $B^- = \{\neg b \colon b \in B\}$, and consider a finite partial subalgebra $\mathbf{B}^*$ of $\mathbf{A}$ with the underlying set $B \cup B^-$. It is clear that $c \in B^*$ if and only if $\neg c \in B^*$ for each $c \in A$. Let $\mathbf{D}$ be a finite $\mathbf{FL_{ew}}$-algebra constructed by $\mathbf{B}^*$, instead of $\mathbf{B}$. Since $\mathbf{B} \subseteq \mathbf{B}^*$ and $\mathbf{B}^*$ is embedded into $\mathbf{D}$, $\mathbf{B}$ is also embedded into $\mathbf{D}$. It remains to show that $\neg\neg x = x$ holds in $\mathbf{D}$.

Take an arbitrary element $X \in D$. Suppose that $X = \bigcap_i (u_i \rightsquigarrow c_i]$ where $u_i \in M$ and $c_i \in B^*$ for each $i$. (Of course, this $M$ is constructed from $B^*$.) Now, assume that $x \in \neg^{\mathbf{D}}\neg^{\mathbf{D}} X$. That is,

for all $w \in M$, $X \cdot w = \{0\}$ implies $x \cdot w = 0$.

Take any element $z \in X$. Then for each $j$, $z \cdot u_j \leq c_j$, and hence $z \cdot (u_j \cdot \neg c_j) = 0$. Since $u_j \in M$ and $\neg c_j \in B^*$, $u_j \cdot \neg c_j \in M$. Thus, $X \cdot (u_j \cdot \neg c_j) = 0$ holds for $u_j \cdot \neg c_j \in M$. Then, by our assumption, $x \cdot (u_j \cdot \neg c_j) = 0$, i.e. $x \cdot u_j \leq \neg\neg c_j \leq c_j$. Hence $x \in (u_j \rightsquigarrow c_j]$ for all $j$. Therefore, $x \in X$. So, $\neg^{\mathbf{D}}\neg^{\mathbf{D}} X = X$ holds.

We met $n$-potency axiom, a weak form of contraction, in Section 4. Now, we will introduce a weak form of excluded middle principle, namely, the axiom $\alpha \vee \neg\alpha^n$. Algebraically, it is expressed by the inequality $x \vee \neg x^n \geq 1$. Over $\mathsf{FL_{ew}}$, this inequality characterizes exactly the *semisimple* subvarieties, which by a result from [20], are precisely the *discriminator* subvarieties of $\mathsf{FL_{ew}}$. This equivalence of semisimplicity and discriminator was extended to $\mathsf{FL_e}$, and beyond, by H. Takamura. The theorem below is new, but its proof is obtained by combining the construction from [5] with some observations from [21].

**Theorem 24** *Any subvariety of* $\mathsf{FL_{ew}}$ *defined by the equation* $x \vee \neg x^n = 1$, *for some positive integer* $n$, *has the FEP.*

Proof. By Lemma 20, it suffices to prove FEP for a subdirectly irreducible algebra $\mathbf{A}$. By semisimplicity, all such algebras are simple, and therefore for any element $a < 1$, we have $a^n = 0$. We will show that this property carries over to $\mathbf{D}$. Let $X$ be an element of $D$ with $X \neq 1^{\mathbf{D}}$. Since $X$ is downward closed, we get that $1 \notin X$. By properties of the construction in [5], it also follows that $X^n$ is the downward closure of the set

$$\{x_1 \cdots x_n \colon x_i \in X,\ \text{for } 1 \leq i \leq n\}.$$

Take any $z \in X^n$. We will show that $z = 0$. To this end, observe that $z \leq x_1 \cdots x_n$, for some members of $X$, and if $x_i = 1$ for some $i$, then $1 \in X$, contradicting the assumptions, so we have $x_i \neq 1$ for $1 \leq i \leq n$. Now, since $\mathbf{A}$ is simple, we get that $x_1 \vee \cdots \vee x_n \neq 1$, and therefore $(x_1 \vee \cdots \vee x_n)^n = 0$. But, distributing fusion over join, we see that $(x_1 \vee \cdots \vee x_n)^n$ develops into a join of the form $J_1 \vee x_1 x_2 \cdots x_n \vee J_2$. Therefore, $z \leq x_1 \cdots x_n \leq (x_1 \vee \cdots \vee x_n)^n = 0$, and thus $z = 0$ as claimed. Since $X < 1^{\mathbf{D}}$ and $z \in X$ were arbitrary, we have

$X^n = \{0\}$, for any non-unit element $X$ of $\mathbf{D}$. Thus, $\mathbf{D}$ is simple, and satisfies $x \vee \neg x^n = 1$.

In the proofs of the above results, the construction of $\mathbf{D}$ we used is essentially the same as that in [5]. This implies that the same argument works well also for any subvariety of $\mathsf{FL}_{\mathsf{ew}}$ defined by a combination of the equations above. In particular, we have the FMP of **MTL**, **IMTL** and **SMTL** (cf. [9]).

**Corollary 25** *Every subvariety of* $\mathsf{FL}_{\mathsf{ew}}$ *defined by any combination of prelinearity* $(x \to y) \vee (y \to x) = 1$*, pseudo-complementation* $x \wedge \neg x = 0$*, involution* $\neg\neg x = x$*, and weak excluded-middle* $x \vee \neg^n x$*, for some positive integer* $n$*, has the FEP. Thus, every logic over* $\mathbf{FL_{ew}}$ *obtained by adding any combination of the prelinearity axiom scheme, the pseudo-complementation axiom scheme* $\neg(\alpha \wedge \neg\alpha)$*, the law of double negation, and weak excluded middle principle* $\alpha \vee \neg\alpha^n$*, has the FMP, and hence is decidable.*

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