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A Semantical Study of Relevant Modal Logics

by

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Abstract

This thesis deals with semantics of relevant modal logics. Originally, modal logics have been developed over classical logic. Recently, the studies of modal logics based on non-classical logic have also been developed. These include relevant modal logics, but many of basic problems on them remain open. In this thesis, we develop a semantical study and show Kripke completeness for a wide class of relevant modal logics in a systematic way.

In general discussion of relevant modal logics, we assume that modal operators $\Box$ and $\Diamond$ are independent. Further, we take regular logics in our sense for basic modal logics.

In this thesis, first we show completeness of our basic relevant logics in terms of both Routley-Meyer frames, Kripke-style semantics, and relevant modal matrices, algebraic semantics. Any regular relevant modal logic is complete with respect to a class of relevant modal matrices, while it is not necessary complete with respect to a class of Routley-Meyer frames. To make any regular relevant modal logic complete by using Routley-Meyer frames, we introduce general frames. Also, we investigate the relationship between general frames and relevant modal matrices as R.Goldblatt developed for classical modal logic.

Our main result in this thesis is a Sahlqvist theorem for relevant modal logics, that is, Kripke completeness of relevant modal logics with Sahlqvist formulas. To obtain it, we show the frame postulate written by a first order sentence corresponding to a given Sahlqvist formula. Also, it is shown that usual Sahlqvist theorem for classical modal logics can be obtained as a special case of our Sahlqvist theorem.

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Contents

Chapter 1

Introduction

In this thesis, we develop a semantical study of propositional relevant modal logics. Relevant modal logics are modal logics over relevant logics. There are two methods of studying mathematical logic: the one is semantical method and the other is syntactical method. In this thesis we develop a systematic study of semantics of relevant modal logics. Completeness is one of important subjects in the semantical study. In the semantical study, we have a question which semantics to be used. This thesis adopts Routley-Meyer semantics, a kind of Kripke semantics, and matrix semantics based on algebraic one. Our main subjects are as follows:

- completeness of relevant modal logics in terms of Routley-Meyer semantics and matrix semantics,
- correspondence between Routley-Meyer semantics and matrix semantics,
- correspondence between modal formulas and frame postulates.

Before discussing formal systems of relevant modal logics, we will give a short review of the motivation of relevant logics and modal logics together with their histories. Also, we will give a short survey of studies of relevant modal logics from the view of semantical method, and will make clear the position of this thesis in the studies of relevant modal logics. Further, we will give a survey of contents of this thesis.

1.1 Relevant logics and modal logics

Classical logic is usually regarded as standard logic. However, it is often noticed that there are several differences between logics in human thinking and classical logic. For bridging them, many logicians have proposed logics which are different from classical logic, and investigated logical properties.

The first formulation of classical logic traces to Boole in the middle of the 19th century. His formulation is known as Boolean algebra now. In early periods of the last century, several formulations of classical logic were introduced by using formulas. At present, the classical propositional logic **Cl** is defined in a Hilbert-style formulation, for instance as follows.

(a) Axioms

(C1) $A \supset (B \supset A)$

(C2) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

(C3) $A \wedge B \supset A$

(C4) $A \wedge B \supset B$

(C5) $(A \supset B) \supset ((A \supset C) \supset (A \supset B \wedge C))$

(C6) $A \supset A \vee B$

(C7) $B \supset A \vee B$

(C8) $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$

(C9) $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$

(C10) $A \supset (\neg A \supset B)$

(C11) $A \vee \neg A$

(b) Rule of inference

$$\frac{A \supset B \quad A}{B}\text{(Modus Ponens)}$$

Here, $\supset$, $\wedge$, $\vee$ and $\neg$ denote (classical) implication, conjunction, disjunction and (classical) negation, respectively. In writing formulas we save on parentheses by assuming that $\neg$ binds more strongly than $\wedge$, $\vee$, and that in turn $\wedge$, $\vee$ bind more strongly than $\supset$. Thus, (C3) is read as $(A \wedge B) \supset A$.

Classical logic is often criticized. Let us consider the following statement:

$$\text{if snow is black then } 2 + 2 = 7. \tag{1.1}$$

In classical logic, the statement (1.1) is true. But it is strange that (1.1) is true, because there is no connection between antecedent and succeedent. So one of criticisms in classical logic is that there are true implicational statements in which there is no connection between antecedent and succeedent.

For avoiding this criticism, alternative formulations have been suggested by paying attention to implication. According to [14], I.E.Orlov proposed alternative formulation in 1928. After that, in 1950's Moh Shaw-Kwei and A.Church also suggested alternative formulation. They can be regarded as forerunners of relevant logics.

In 1956, W.Ackermann's paper [1] is regarded as the first study of relevant logic. In 1960's A.R.Anderson, N.Belnap and J.M.Dunn etc. introduced formal systems and studied algebraic methods, which are mentioned in [4], and further indicated a course of studies of relevant logics. In 1970's L.L.Maksimova, R.K.Meyer, R.Routley and K.Fine etc. got semantical results and proposed several relevant logics. After that, A.Urquhart, S.Giambrone and R.T.Brady etc. showed important logical properties of relevant logics, for example, decision problem and interpolation problem.

At present, relevant logics are understood as follows. The motivation of relevant logic is to exclude 'paradoxes of material implication'. It is known that there are two classes of these. One is 'paradox of relevance' and the other is 'paradox of consistency'. Typical formula of the former is

$$A \to (B \to A) \tag{1.2}$$

and of the latter is

$$A \wedge \sim A \to B. \tag{1.3}$$

Roughly, (1.2) is understood that if A is true then $B \to A$ is also true. So, let A and B denote that 'Beethoven wrote nine symphonies' and 'Fishes are plants', respectively. Then it follows that $B \to A$ is a true sentence, which is strange. To ignore the relevance of implication causes this strangeness. On the other hand, (1.3) insists that a contradiction leads to anything. So, it is permitted that if a contradiction follows then we can argue unrelated topics.

We present typical relevant logics according to [48]. The basic relevant logic **B** is defined as follows.

(a) Axioms

- (B1) $A \to A$
- (B2) $A \wedge B \to A$
- (B3) $A \wedge B \to B$
- (B4) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
- (B5) $A \to A \vee B$
- (B6) $B \to A \vee B$
- (B7) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
- (B8) $A \wedge (B \vee C) \to (A \wedge B) \vee C$
- (B9) $\sim\sim A \to A$

(b) Rules of inference

$$\frac{A \to B \quad A}{B} \qquad \frac{A \quad B}{A \wedge B}$$

$$\frac{A \to B}{(B \to C) \to (A \to C)} \qquad \frac{A \to B}{(C \to A) \to (C \to B)} \qquad \frac{A \to \sim B}{B \to \sim A}.$$

Here, $\to$, $\wedge$, $\vee$ and $\sim$ denote relevant implication, conjunction, disjunction and relevant negation, respectively. Other typical relevant logics are as follows.

- **T**, called ticket entailment, is obtained from **B** by adding the axioms

  - (B10) $(A \to B) \to ((C \to A) \to (C \to B))$
  - (B11) $(A \to B) \to ((B \to C) \to (A \to C))$
  - (B12) $(A \to (A \to B)) \to (A \to B)$
  - (B13) $(A \to \sim A) \to \sim A$
  - (B14) $(A \to \sim B) \to (B \to \sim A)$.

- **E**, called entailment, is obtained from **T** by adding the rule of inference

$$\frac{A}{(A \to B) \to B}.$$

- **R**, called relevant implication, is obtained from **T** by adding the axiom

(B15) $A \to ((A \to B) \to B)$.

It is known that the logic obtained by adding (1.2) as an axiom to the relevant logic **R** is exactly the propositional classical logic **Cl**, and hence (1.3) is derived in this logic. However, the logic obtained by adding (1.3) as an axiom to **R** is not equal to **Cl**. The logic obtained in this way is called **KR**. Also, in relevant logics mentioned above, it is known that if $A \to B$ is a theorem, then A and B have common propositional variables. Further, we know that $A \to B$ is not logically equivalent to $\sim A \vee B$ in relevant logics mentioned above. In relevant logics mentioned above, De Morgan laws hold, so it suffices to define logical connectives $\to$, $\wedge$ (or $\vee$) and $\sim$.

There are some logics between the relevant logic **R** and the classical logic **Cl**. **RM** and **KR** are typical ones. **RM** is obtained from **R** by adding the mingle axiom

$$A \to (A \to A),$$

and **KR** is defined as above. Further, to consider classical negation and relevant negation simultaneously, **CR** is introduced.

Recently, substructural logics have been studied. Substructural logics are logics obtained from Gentzen's **LK** or **LJ** by eliminating some (or all) structural rules. In particular, substructural logics without weakening are regarded as relevant logics.

Roughly speaking, modal logics can be regarded as logics of 'necessity' and 'possibility'. Among true propositions we can distinguish between those which merely happen to be true and those which are bound true. This implies distinction among false propositions. We call a proposition which is bound to be true a 'necessarily true' proposition; a proposition which is bound to be false an 'impossible' proposition; and a non-impossible proposition a 'possible' proposition. For example, let us consider the following proposition.

The sum of interior angles of triangle is 180 degrees. (1.4)

The author is married. (1.5)

Humans can be alive on. (1.6)

Then the sentence (1.4) is a necessarily true proposition, the sentence (1.5) is a possible proposition, and the sentence (1.6) is an impossible proposition.

We notice that the truths of propositions (1.4) and (1.6) above are always decided while the truth of (1.5) depends on the times. That is, the proposition (1.5) is false at the time of submission of this thesis, but may be true in future. Thus, there are cases in which the truth of the formulas are changeable depending on time and situation etc. Now let us consider the following proposition instead of (1.5).

It is possible for the author to be married. (1.7)

The proposition (1.7) is true at the time of submission of this thesis, because the author will be married in future.

In formalization of modal logic, we use symbols $\Box$ and $\Diamond$ as modal operators. $\Box A$ means that A is necessary, and $\Diamond A$ means that A is possible. In classical modal logic, $\Diamond A$ is defined usually by $\neg\Box\neg A$. Suppose that B and C denote the proposition (1.4) and (1.5), respectively. Then the proposition (1.7) is denoted by $\Diamond C$. Further, since B is considered to be a necessarily true proposition, we can consider that $\Box B$ is true.

The history of modal logics will be traced back to Aristotle, and it was studied in the medieval time. Modern modal logics are begun by C.I.Lewis in 1912. He published a series of articles and books expressing dissatisfaction with the notion of material implication found in Whitehead and Russel's "Principia Mathematica". He suggested 'strict implication' as an alternative implication for eliminating paradoxes of material implication, but he could not eliminate all paradoxes of material implication. (As concerns eliminating paradoxes of material implication, we have to wait until the birth of so-called 'relevant implication'.) He also introduced connectives which express necessity and possibility, not appearing classical logic. For Lewis's argument, see [28]. In this stage, J.Łukasiewicz, R.Carnap, etc. studied modal logics in terms of syntactical and algebraic method, independently.

In 1963, S.Kripke published model-theoretic method, and studies of modal logics have been developed by his method. Important notions in Kripke's idea are possible worlds and accessible relations. These can be dealt with by mathematical way. This idea is applied widely, including models for intuitionistic logic, relevant logics, etc.. At present, we have many comprehensive results on modal logics.

Studies of intensional logics, for example, temporal logics, dynamic logics, deontic logics, provability logics and logics of knowledge, are closely derived from modal logics. Further, there are several results on connections between modal logics and intermediate logics.

Below, we give some standard modal logics to which we often refer in this thesis. Every logic includes the following abbreviation:

$$\Diamond A \stackrel{\text{abb}}{=} \neg\Box\neg A$$

- **K**, the least normal modal logic, is obtained from **Cl** by adding the axiom

  (**K**) $\Box(A \supset B) \supset (\Box A \supset \Box B)$.

  and the rule of inference

  $$\frac{A}{\Box A}\text{(Necessitation)}.$$

- **KD** is obtained from **K** by adding the axiom

  (**D**) $\Box A \supset \Diamond A$.

- **KT** is obtained from **K** by adding the axiom

  (**T**) $\Box A \supset A$.

- **KTB** is obtained from **KT** by adding the axiom

  (**B**) $A \supset \Box\Diamond A$.

- **S4** is obtained from **KT** by adding the axiom

  (**4**) $\Box A \supset \Box\Box A$.

- **S5** is obtained from **KT** by adding the axiom

  (**5**) $\Diamond A \supset \Box\Diamond A$.

1.2 Relevant modal logics

To approximate human's thinking, several non-classical logics are suggested. There are two ways to obtain non-classical logics. One is extension of classical logic and the other is alternative in classical logic. Modal logics have been usually developed over classical logic. Recently, the studies of modal logics based on various non-classical logic have also been developed. Since relevant logics are regarded as alternatives in classical logic, we treat relevant modal logics as extensions of relevant logics. In relevant modal logics, we can make clear the relationship between relevant implication and modalities.

From the view of mathematical logic, we are interested in whether some properties in classical modal logics proceed in relevant modal logics. This question comes out the difference between classical modal logics and relevant modal logics and is also interesting from the point how some properties of modal logics depends on classical logic. Recently, we have some results on relevant modal logics, including completeness and incompleteness. See, for instance, [46], [19], [34], [37], [35], [38], [41], [21] and [36]. On the other hand, there are not so many comprehensive results on relevant modal logics. This thesis shows that some basic results on classical modal logic hold for relevant modal logics.

Below, we outline contents of papers on relevant modal logics published so far. In [46], R.Routley and R.K.Meyer showed completeness and γ-admissibility etc. of **NR**, which is **S4**-style relevant modal logic, with respect to reduced model. **NR** is obtained from **R** adding axioms

- $\Box(A \to B) \to (\Box A \to \Box B)$
- $\Box A \to A$
- $\Box A \to \Box\Box A$
- if A is an axiom of **NR** then so is $\Box A$.

(Note that this logic implies the rule of necessitation.)

In [19], A.Fuhrmann developed the frame postulates of the following rules of inference and axioms. (Here we follow names in [19]):

RN. $\dfrac{A}{\Box A}$

RI. for $n \geq 1$, $\dfrac{A_1 \to (\cdots \to (A_n \to A))}{\Box A_1 \to (\cdots \to (\Box A_n \to \Box A))}$

$\Box$T. $\Box A \to A$

$\Box$D. $\Box \sim A \to \sim \Box A$

$\Box$4. $\Box A \to \Box\Box A$

$\Box$B. $A \to \Box\Diamond A$

$\Box$5. $\Diamond A \to \Box\Diamond A$,

where $\Diamond A$ is defined by $\sim \Box \sim A$. Moreover, he showed the incompleteness of the logic weaker than **R.KT4**, which is just **NR**, with respect to reduced models. Here note that reduced models used in [19] are different from those in [46].

In [37], [34], and [38], they showed completeness and γ-admissibility of **R4** (**RK**,in [34]), obtained from **NR** by adding the axiom

$$\Box(A \vee B) \to \Diamond A \vee \Box B.$$

(Note that this is derivable in classical modal logic **K**, but is not in **NR**.) In [41], R.K.Meyer and E.D.Mares extended the argument about **NR** and **R4** to them based on classical relevant logic. Further, E.D.Mares showed completeness and γ-admissibility of relevant analogue of **KD**, **KT**, **KTB**, **S4** and **S5** in [35].

In [10], S.A.Celani showed the representation theorem of classical relevant modal algebras by means of Priestly spaces. Its argument is very close to the completeness argument. As concerns incompleteness, L.Goble showed that of **G** obtained from **NR** by adding the axiom

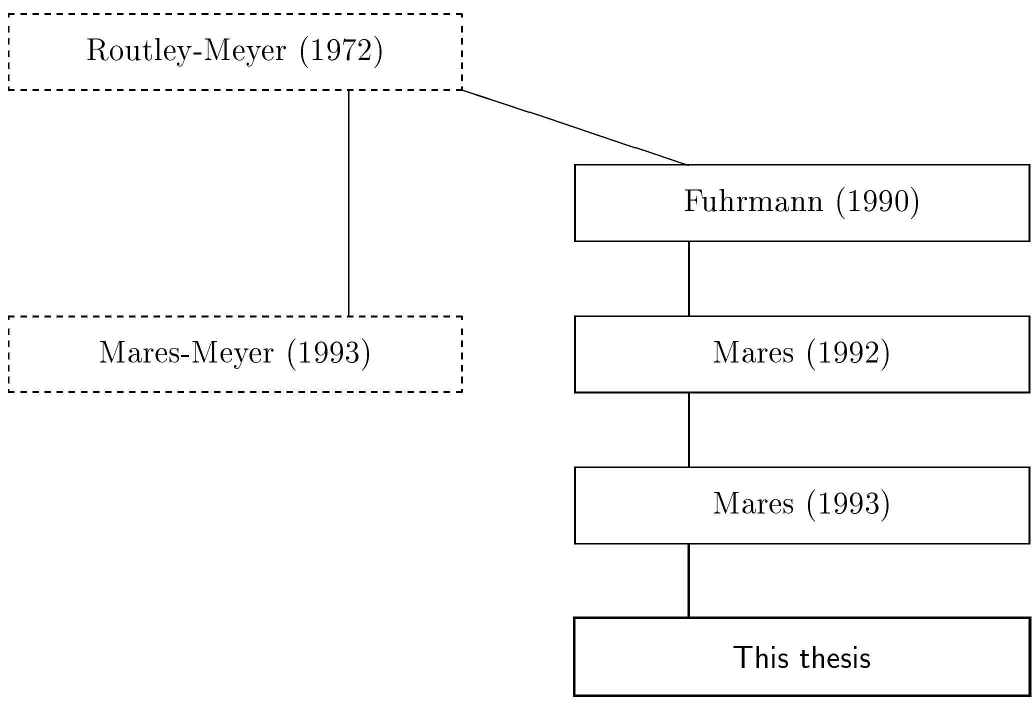
$$\Box A \to (B \to B)$$

in [21], and E.D.Mares showed that of **RGL** obtained from **R.K4** by adding the axiom

$$\Box(\Box A \to A) \to \Box A$$

in [36].

This thesis deals with completeness of of relevant modal logics. This thesis can be located as in the following figure, where devision of left and right parts means a difference of semantics.



Our completeness result includes [19], [34] and [35].

1.3 Organization of this thesis

This thesis shows a comprehensive result on completeness of relevant modal logics, that is, completeness of relevant modal logics with Sahlqvist formulas. Our main result is that frame postulates for Sahlqvist formulas can be expressed by first-order predicate formulas and relevant modal logics with Sahlqvist formulas are complete with respect to the class of the frames satisfying these postulates as well as in classical modal logics.

In Chapter 2, we will summarize semantical results of the relevant logic **R** and (classical) normal modal logics. We assume familiarity with basic results on semantics of **R**. In order to understand the difference (or similarity) between classical modal logics and relevant modal logics, we will summarize basic semantical results of (classical) normal modal logics.

In Chapter 3, we will introduce basic relevant modal logics and prove their completeness in terms of both Routley-Meyer semantics and matrix semantics.

In general discussion of relevant modal logic, several problems arise. First, we have a question how to define the modalities. In modal logics based on classical logic, $\Diamond$ is defined in the following way:

$$\Diamond A \leftrightarrow \sim \Box \sim A. \tag{1.8}$$

Since (1.3) is not a theorem of **R**, negation in relevant logics (called *relevant negation*) differs from that in classical logic (called *classical negation*). Because of this difference, definition (1.8) is problematic. It will be natural to start by assuming that $\Box$ and $\Diamond$ are independent. Definition (1.8), which is adopted in [19], [34], [37] and [38], is regarded as a special case. A relevant modal logic containing (1.8) as an axiom will be called a *dependent extension*.

Next, which logic should we take for the basic modal logics? It is well-known that the normal modal logic **K** is one of the basic classical modal logics. In fact, **K** is shown to be complete with respect to the class of all Kripke frames. Turning to relevant modal logic, we may ask which logic is complete with respect to the class of all relevant modal frames. Our answer points to a regular logic, which is called a conjunctively regular logic in [19], as a suitable basic logic. However, this logic cannot be considered a relevant analogue of **K**. We will call relevant analogues of **K** *normal*, and if an extension of a relevant modal logic is normal, then it will be called a *normal extension*.

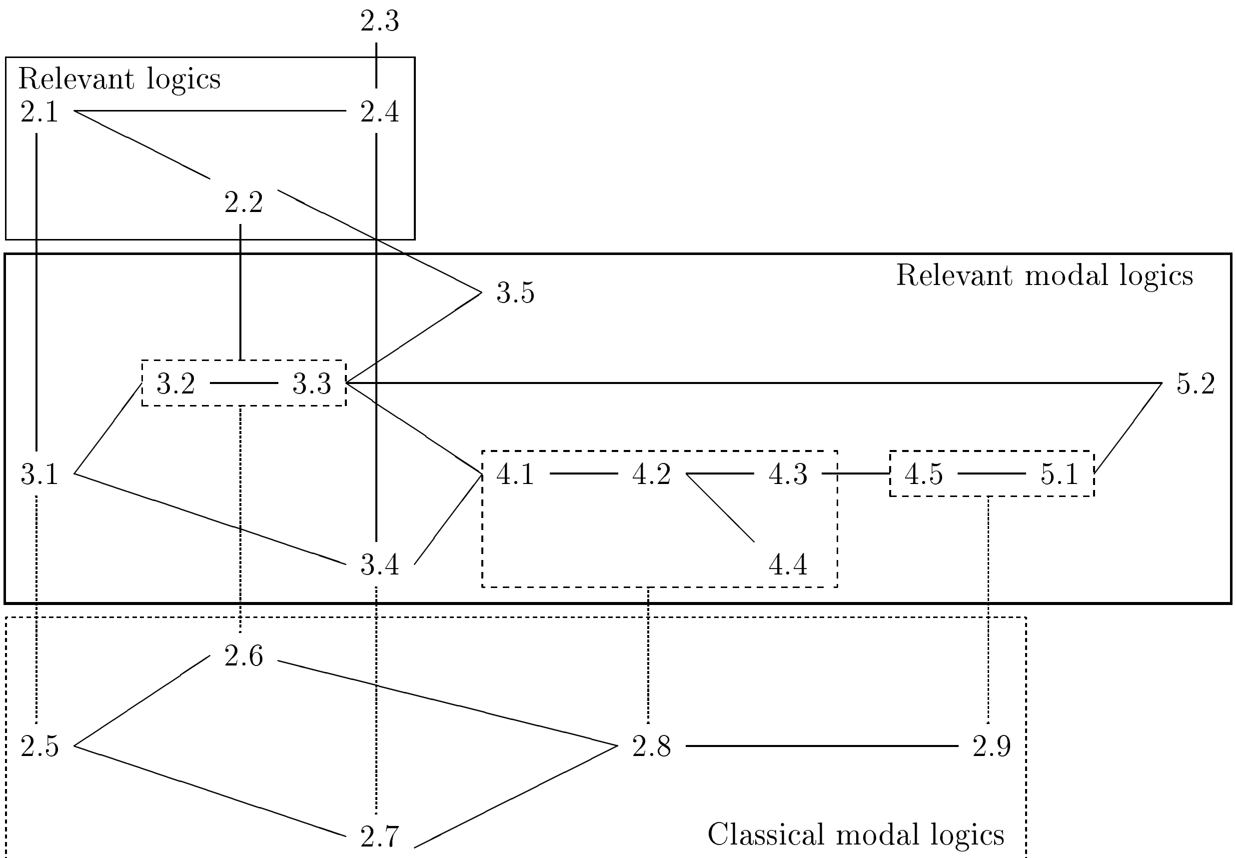
In Chapter 4, we will introduce general frames of relevant modal logics and investigate their fundamental properties.

Studies of general frames have been developed for classical modal logics. General frames can be regarded as combining the merits of both algebras and Kripke frames. That is, each classical modal logic is characterized by the class of all general frames for it as well as by the class of all modal algebras for it. In order to discuss completeness of wider class of relevant modal logics in terms of the idea of Routley-Meyer frames, we introduce general frames for relevant modal logics. We will get the similar results on them to those on classical modal logics. Further, we introduce descriptive frames. As concerning descriptive frames, we define $\mathcal{D}$-persistent logics and $\mathcal{D}^{\star}$-elementary logics.

In Chapter 5, we will show a Sahlqvist theorem for relevant modal logics. The Sahlqvist theorem is one of the most important results on Kripke completeness of classical modal logics. Our Sahlqvist theorem says that for a given modal formula A which has the certain form (called a Sahlqvist formula), we can construct effectively a first order formula characterizing descriptive frames or Routley-Meyer frames which validates A. Also, it follows that every $\mathcal{D}^{\star}$-elementary logic with any set of Sahlqvist formulas as axioms is Kripke complete. Further, it is shown that usual Sahlqvist theorem for classical modal logics can be obtained as a special case of our theorem.

Finally, in Section 6, we will summarize this thesis and state further studies.

The diagram below shows the dependency between sections of this thesis.



Chapter 2

Preliminaries

In this chapter, we survey basic semantical properties of relevant logics and classical modal logics. Further, we present basic properties in lattice theory. They will be referred in later chapters. We assume the familiarity with basic results on classical logic and Boolean algebras.

2.1 Relevant logic R

The language of relevant logics consists of (i) propositional variables $p, q, r, \cdots$, (ii) logical connectives $\to$ (relevant implication), $\wedge$ (and), $\vee$ (or) and $\sim$ (relevant negation).

Formulas are defined in the usual way, and are denoted by capital letters $A, B, C, \cdots$. We write $A \leftrightarrow B$ for $(A \to B) \wedge (B \to A)$. Prop and Wff will denote the set of all propositional variables and of formulas, respectively. Capital Greek letters $\Sigma, \Gamma, \Delta, \cdots$ denote sets of formulas. When necessary, we add $'$ or subscripts to capital letters and capital Greek letters.

The relevant logic $\mathbf{R}$ is defined as follows.

(a) Axioms

(R1) $A \to A$

(R2) $A \to ((A \to B) \to B)$

(R3) $(A \to B) \to ((B \to C) \to (A \to C))$

(R4) $(A \to (A \to B)) \to (A \to B)$

(R5) $A \wedge B \to A$

(R6) $A \wedge B \to B$

(R7) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(R8) $A \to A \vee B$

(R9) $B \to A \vee B$

(R10) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$

(R11) $A \wedge (B \vee C) \to (A \wedge B) \vee C$

(R12) $(A \to \sim B) \to (B \to \sim A)$

(R13) $\sim\sim A \to A$

(b) Rules of inference

$$\frac{A \to B \quad A}{B}(\text{Modus Ponens}) \qquad \frac{A \quad B}{A \wedge B}(\text{Adjunction})$$

It is easy to see the following.

Theorem 2.1 *The following formulas are theorems of* **R***:*

1. $A \wedge (A \to B) \to B$
2. $(A \to (B \to C)) \to (B \to (A \to C))$
3. $((A \to A) \wedge (B \to B) \to C) \to C$
4. $(A \to B) \to (\sim B \to \sim A)$
5. $A \to \sim\sim A$.

Moreover, **RM** and **KR** are well-known logics as extensions of **R**. **RM** is the logic obtained from **R** by adding the *mingle* axiom

(RM) $A \to (A \to A)$.

KR is known as *superclassical relevant logic* and is obtained from **R** by adding the axiom

(KR) $A \wedge \sim A \to B$.

Further, if we add classical negation $\neg$ to our language, then we can obtain a *classical relevant logic* **CR**. **CR** is obtained from **R** by adding the axiom

(CR) $\neg\neg A \to A$

and the rule of inference

$$\frac{A \wedge B \to \neg C}{A \wedge C \to \neg B}(\text{Antilogism}).$$

In **KR**, $\neg$ is identified with $\sim$.

Note that the logic obtained from **R** by adding the axiom $A \to (B \to A)$ is exactly classical logic **Cl**.

2.2 Routley-Meyer semantics for R

In this section, we introduce a semantics for **R**. The semantics introduced in this section is often called *Routley-Meyer semantics* (or *three-termed relational semantics*). Our model is an unreduced **R**-model in the sense of [48].

An **R**-*frame* is a quadruple $\langle O, W, R, ^* \rangle$ where (a) W is a non-empty set of all worlds, (b) O is a non-empty subset of W, (c) R is a ternary relation on W, and (d) * is an unary operation on W. To simplify the notation, we define a binary relation $\leq$ on W as follows. For all $a, b \in W$:

$$a \leq b \overset{\text{def}}{\Longleftrightarrow} \text{there exists } c \in O \text{ such that } Rcab.$$

An **R**-frame $\langle O, W, R, ^* \rangle$ satisfies the following postulates for all $a, b, c, a', b' \in W$:

(p1) $a \leq a$

(p2) $Raaa$

(p3) if $Rabc$ and $Rca'b'$ then there exists $d \in W$ such that $Raa'd$ and $Rdbb'$

(p4) if $a \leq b$ and $Rbcd$ then $Racd$

(p5) if $Rabc$ then Rac^*b^*

(p6) $a^{**} = a$.

Notice that each **R**-frame $\langle O, W, R, ^* \rangle$ has the following properties for all $a, b, c, d \in W$:

(t1) if $Rabc$ then $Rbac$,

(t2) $\leq$ is transitive,

(t3) if $a \leq b$ then $b^* \leq a^*$,

(t4) if $Rabc$ and $c \leq d$ then $Rabd$,

(t5) Raa^*a.

We call a quintuple $\langle O, W, R, ^*, V\rangle$ an **R**-*model* on an **R**-frame $\mathcal{F} = \langle O, W, R, ^* \rangle$ (we simply say an **R**-model), where $\mathcal{F}$ is an **R**-frame and V is a mapping from Prop to 2^W, called a *valuation* on $\mathcal{F}$, which satisfies the following *hereditary condition*. For all $a, b \in W$ and all $p \in \mathsf{Prop}$:

$$\text{if } a \leq b \text{ and } a \in V(p) \text{ then } b \in V(p).$$

Given an **R**-model $\langle O, W, R, ^*, V\rangle$, for $a \in W$ and $A \in \mathsf{Wff}$, a relation $\models$ between W and Wff is defined inductively as follows:

i. for any $p \in \mathsf{Prop}$, $a \models p$ iff $a \in V(p)$

ii. $a \models A \wedge B$ iff $a \models A$ and $a \models B$

iii. $a \models A \vee B$ iff $a \models A$ or $a \models B$

iv. $a \models A \rightarrow B$ iff for all $b, c \in W$, if $Rabc$ and $b \models A$, then $c \models B$

v. $a \models \sim A$ iff $a^* \not\models A$

where $a \not\models A$ means that $a \models A$ does not hold.

Then by induction on the length of the formula A, we can show the following "hereditary lemma".

Lemma 2.2 *Let $\langle O, W, R, ^*, V\rangle$ be an **R**-model. For all $a, b \in W$ and all $A \in \mathsf{Wff}$, if $a \leq b$ and $a \models A$ then $b \models A$.*

Let $\mathcal{M} = \langle O, W, R, ^*, V\rangle$ be an **R**-model on an **R**-frame $\mathcal{F} = \langle O, W, R, ^* \rangle$, $A \in \mathsf{Wff}$, and $\mathcal{C}$ be a class of **R**-frames. Then we say

(a) A *holds in* $\mathcal{M}$ iff $a \models A$ for every world $a \in O$,

(b) A is *valid* in an **R**-frame $\mathcal{F}$ iff A holds in every **R**-model $\mathcal{M}$ on $\mathcal{F}$,

(c) **R** is *sound with respect to* $\mathcal{C}$ iff A is valid in every $\mathcal{F} \in \mathcal{C}$ for all theorems A of **R**,

(d) **R** is *complete with respect to* $\mathcal{C}$ iff A is a theorem of **R** for every formula A valid in all $\mathcal{F} \in \mathcal{C}$.

By induction on the length of proof in $\mathbf{R}$, we can show soundness of $\mathbf{R}$ easily.

Theorem 2.3 $\mathbf{R}$ *is sound with respect to the class of* $\mathbf{R}$*-frames.*

For the converse, the proof basically goes in the same way as in Section 4.6 of [48]. Here, we introduce some notions and study properties of them. (Note that the terminology is somewhat different from that in [48].)

Key notions are as follows.

- Let $\Sigma \neq \emptyset$ and $\Delta \neq \emptyset$. $\mathbf{R} \vdash \Sigma \to \Delta$ iff there exist $A_1, \cdots, A_m \in \Sigma$ $(m > 0)$ and $B_1, \cdots, B_n \in \Delta$ $(n > 0)$ such that

$$\mathbf{R} \vdash A_1 \wedge \cdots \wedge A_m \to B_1 \vee \cdots \vee B_n.$$

- (Σ, Δ) is an $\mathbf{R}$-*pair* iff (a) $\mathbf{R} \nvdash \Sigma \to \Delta$ and (b) $\Sigma \cup \Delta = \mathsf{Wff}$.
- Σ is an $\mathbf{R}$-*theory* iff (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$, and (b) if $\mathbf{R} \vdash A \to B$ and $A \in \Sigma$ then $B \in \Sigma$.
- For an $\mathbf{R}$-theory Σ,
  - Σ is *regular* iff Σ contains all theorems of $\mathbf{R}$.
  - Σ is *prime* iff $A \vee B \in \Sigma$ implies either $A \in \Sigma$ or $B \in \Sigma$.
- Let $\mathsf{Th}(\mathbf{R})$ be the set of all $\mathbf{R}$-theories. Then a ternary relation R on $\mathsf{Th}(\mathbf{R})$ is defined by

$$R\Sigma\Gamma\Delta \quad \text{iff} \quad \text{for any } A, B \in \mathsf{Wff}, \text{ if } A \to B \in \Sigma \text{ and } A \in \Gamma \text{ then } B \in \Delta\ .$$

A few comments on the definitions above. It is clear that $\Sigma \cap \Delta = \emptyset$ whenever $\mathbf{R} \nvdash \Sigma \to \Delta$. Hence we have that if (Σ, Δ) is an $\mathbf{R}$-pair, then for all $A \in \mathsf{Wff}$, either $A \in \Sigma$ or $A \in \Delta$ but not both. It is clear that the set of all theorems of $\mathbf{R}$ is an $\mathbf{R}$-theory.

The following lemmas are essentially proved in [48] (pp.305-318). So, we omit the proof.

Lemma 2.4

1. *If* (Σ, Δ) *is an* $\mathbf{R}$*-pair, then* Σ *is a prime* $\mathbf{R}$*-theory.*
2. *If* $\mathbf{R} \nvdash \Sigma \to \Delta$, *then there exist* $\Sigma' \supseteq \Sigma$ *and* $\Delta' \supseteq \Delta$ *such that* (Σ', Δ') *is an* $\mathbf{R}$*-pair.*
3. *Suppose that* Σ *is an* $\mathbf{R}$*-theory and* Δ *is a set of formulas closed under disjunction such that* $\Sigma \cap \Delta = \emptyset$. *Then there exists a prime* $\mathbf{R}$*-theory* $\Sigma' \supseteq \Sigma$ *such that* $\Sigma' \cap \Delta = \emptyset$.
4. *If* A *is not a theorem of* $\mathbf{R}$, *then there exists a regular prime* $\mathbf{R}$*-theory* Π *such that* $A \notin \Pi$.
5. *Suppose that* Σ *and* Γ *are* $\mathbf{R}$*-theories and* Δ *is a prime* $\mathbf{R}$*-theory such that* $R\Sigma\Gamma\Delta$. *Then there exists a prime* $\mathbf{R}$*-theory* $\Sigma' \supseteq \Sigma$ *such that* $R\Sigma'\Gamma\Delta$.

6. *Suppose that Σ and Γ are* **R***-theories and Δ is a prime* **R***-theory such that $R\Sigma\Gamma\Delta$. Then there exists a prime* **R***-theory $\Gamma' \supseteq \Gamma$ such that $R\Sigma\Gamma'\Delta$.*

7. *Suppose that $R\Sigma\Gamma\Delta$ and $D \notin \Delta$ for a prime* **R***-theory Σ and* **R***-theories Γ and Δ. Then there exist prime* **R***-theories Γ' and Δ' such that $\Gamma \subseteq \Gamma', D \notin \Delta'$ and $R\Sigma\Gamma'\Delta'$.*

8. *If Σ is a prime* **R***-theory such that $C \to D \notin \Sigma$, then there exist prime* **R***-theories Γ' and Δ' such that $R\Sigma\Gamma'\Delta', C \in \Gamma'$ and $D \notin \Delta'$.*

Using this lemma, we will be able to prove completeness theorem for **R**. First, we define the *canonical* **R***-model* $\langle O_c, W_c, R_c, g_c, V_c\rangle$ as follows:

- W_c is the set of all prime **R**-theories
- O_c is the set of all regular prime **R**-theories
- R_c is the ternary relation R restricted to W_c
- g_c is the unary operation on W_c defined by $g_c(\Sigma) = \{A \mid \sim A \notin \Sigma\}$
- V_c is defined by

$$\text{for all } p \in \mathsf{Prop} \text{ and } \Sigma \in W_c, \quad \Sigma \in V_c(p) \text{ iff } p \in \Sigma.$$

We call $\langle O_c, W_c, R_c, g_c\rangle$ the *canonical* **R***-frame*. The relation $\leq_c$ is defined as in the definition of $\leq$. Note that for $\Sigma \in W_c$, $g_c(\Sigma) \in W_c$.

The following lemma is proved in [48].

Lemma 2.5

1. *Let $\langle O_c, W_c, R_c, g_c\rangle$ be the canonical* **R***-frame. Then for all $\Sigma, \Gamma \in W_c$, $\Sigma \leq_c \Gamma$ iff $\Sigma \subseteq \Gamma$.*

2. *The canonical* **R***-frame $\langle O_c, W_c, R_c, g_c\rangle$ is an* **R***-frame.*

3. *Let $\langle O_c, W_c, R_c, g_c, V_c\rangle$ be the canonical* **R***-model. For all $A \in \mathsf{Wff}$ and $\Sigma \in W_c$,*

$$\Sigma \models_c A \quad \textit{iff} \quad A \in \Sigma.$$

Thus, we can state completeness of **R**.

Theorem 2.6 **R** *is complete with respect to the class of* **R***-frames.*

In the following, we present Routley-Meyer semantics for **KR**. A **KR**-*frame* in the usual sense is an **R**-frame obtained by assuming that

(p1)' there exists $a \in O$ such that $Rabc$ iff $b = c$

(p6)' $a^* = a$

instead of (p1) and (p6), respectively. The definition of **KR**-models (on **KR**-frames) is the same as that of **R**-models. Note that $a \models \sim A$ iff $a \not\models A$ in any **KR**-model.

Completeness of **KR** is shown in the same way as that of **R** except that **KR**-theories are non-empty.

As another semantics for **KR**, it is necessary here to discuss the adequacy of including both empty set $\emptyset$ and Wff among prime **KR**-theories. As a consequence of our Sahlqvist theorem in Section 5.1, we can show that some of superclassical relevant modal logics, i.e., relevant modal logics over superclassical relevant logic **KR**, are complete. To make this possible, we need to include $\emptyset$ and Wff among **KR**-theories and make explicit use of them. In fact, in Section 5.5 of [48], the notion of enlarged frames is introduced in order to treat this problem.

An *enlarged* **KR**-frame $\mathcal{F} = \langle O, W, R, ^* \rangle$ is a **KR**-frame with elements e, called the *null world*, and u, called the *universal world*, in W which satisfy the following definition and postulates for all $a, b \in W$:

(du) $u \overset{\text{def}}{=} e^*$, $u \in O$

(ep1) if $Ruab$, then $a = e$ or $b = u$

(ep2) $e \neq u$

Every enlarged **KR**-frame satisfies the following postulates:

$$(1) \quad Reue \qquad \text{and} \qquad (2) \quad e \leq a \leq u, \quad \text{for all} \;\; a \in W.$$

A *valuation* V on $\mathcal{F}$ must satisfy also the following conditions for all $p \in \mathsf{Prop}$:

$$e \notin V(p) \qquad \text{and} \qquad u \in V(p).$$

Then by induction on the length of A, we see that $e \not\models A$ and $u \models A$, for all $A \in \mathsf{Wff}$.

In the canonical enlarged **KR**-frame $\langle O_c, W_c, R_c, g_c \rangle$, prime **KR**-theories $\emptyset$ and Wff are taken for e_c and u_c, respectively.

Of course, **KR** is complete with respect to the class of enlarged **KR**-frames.

2.3 Algebraic preliminaries of distributive lattices

Since algebraic studies mentioned below are based on distributive lattices, we present results on distributive lattices in this section. Distributive lattices are defined as usual. A Boolean algebra is one of examples of distributive lattices.

First, we define the notions of filter. Given a lattice $\langle M, \cap, \cup \rangle$, a non-empty subset ∇ of M is a *filter* if the following postulates hold for all $x, y \in M$:

(F1) if $x, y \in \nabla$ then $x \cap y \in \nabla$,

(F2) if $x \in \nabla$ and $x \leq y$ then $y \in \nabla$,

where $\leq$ denotes the lattice-order, i.e., $x \leq y$ is defined by $x \cap y = x$.

Moreover, we say that a filter ∇ is *prime*, if $x \cup y \in \nabla$ implies $x \in \nabla$ or $y \in \nabla$, for all $x, y \in M$. Further, the smallest filter containing a given non-empty subset M' of M is called the *filter generated by M'*.

Next, we define the notion of ideals. Given a lattice $\langle M, \cap, \cup \rangle$, a non-empty subset Δ of M is said to be an *ideal* if the following postulates hold for all $x, y \in M$:

(I1) if $x, y \in \Delta$ then $x \cup y \in \Delta$,

(I2) if $x \in \Delta$ and $y \leq x$ then $y \in \Delta$.

Moreover, we say that an ideal Δ is *prime*, if $x \cap y \in \Delta$ implies $x \in \Delta$ or $y \in \Delta$, for all $x, y \in M$. Further, the smallest ideal containing a given non-empty subset I of Δ is called the *ideal generated by* I.

Since each result is well-known in lattice theory, we omit the proof.

Lemma 2.7 *Let $\langle M, \cap, \cup \rangle$ be a distributive lattice.*

1. *Suppose that ∇ is a filter and Δ is an ideal such that $\nabla \cap \Delta = \emptyset$. Then there exists a prime filter $\nabla' \supseteq \nabla$ such that $\nabla' \cap \Delta = \emptyset$.*

2. *If ∇ is a filter such that $x \notin \nabla$, then there exists a prime filter $\nabla' \supseteq \nabla$ such that $x \notin \nabla'$.*

3. *If $x, y \in M$ satisfies $x \not\leq y$, then there exists a prime filter ∇ such that $x \in \nabla$ and $y \notin \nabla$.*

4. *Suppose that $\nabla \subseteq M$ and $\Delta \subseteq M$ satisfy $\nabla \cap \Delta = \emptyset$ and $\nabla \cup \Delta = M$. Then ∇ is a prime filter iff Δ is a prime ideal.*

5. *Let $M_1, M_2 \subseteq M$ satisfy*

   (i) for any $y_1, \cdots, y_n \in M_1$ and $z \in M_2$, $y_1 \cap \cdots \cap y_n \not\leq z$,

   (ii) for any $z_1, z_2 \in M_2$, there exists $z \in M_2$ such that $z_1 \cup z_2 \leq z$.

   Then there exists a prime filter ∇' in M such that $M_1 \subseteq \nabla'$ and $M_2 \cap \nabla' = \emptyset$.

We say that $\langle M', \cap, \cup \rangle$ is a *sublattice* of $\langle M, \cap, \cup \rangle$ if (a) M' is a non-empty subset of M, (b) for every $x, y \in M'$, $x \cap y \in M'$ and $x \cup y \in M'$.

Lemma 2.8 *Suppose that $\langle M', \cap, \cup \rangle$ is a sublattice of distributive lattice $\langle M, \cap, \cup \rangle$. Then for every prime filter $\nabla \subseteq M'$, there exists a prime filter $\nabla' \subseteq M$ such that $\nabla \subseteq \nabla'$ and $\nabla = \nabla' \cap M'$.*

Proof.
Let $\Delta = M' - \nabla$. By 4 of Lemma 2.7, Δ is a prime ideal. By 5 of Lemma 2.7, there exists a prime filter ∇' in M such that $\nabla \subseteq \nabla'$ and $\Delta \cap \nabla' = \emptyset$. Then we see that $\nabla \subseteq \nabla' \cap M'$ easily, so it suffices to show the converse inclusion. Suppose that $x \in \nabla' \cap M'$. Then $x \notin \Delta$, so $x \in \nabla$. Hence $\nabla = \nabla' \cap M'$. ∎

Let $\mathbf{M} = \langle M, \cap, \cup \rangle$ and $\mathbf{M'} = \langle M', \cap, \cup \rangle$ be lattices. Then we say that f is a *homomorphism* of $\mathbf{M}$ in $\mathbf{M'}$ if $f : M \to M'$ satisfies the following equalities: for any $x, y \in M$, (i) $f(x \cap y) = f(x) \cap f(y)$ and (ii) $f(x \cup y) = f(x) \cup f(y)$.

Lemma 2.9 *Let f be a homomorphism of distributive lattice $\mathbf{M}$ in $\mathbf{M'}$ and ∇ be a prime filter in $\mathbf{M'}$. Then the set $f^{-1}(\nabla) = \{x \mid f(x) \in \nabla\}$ is a prime filter in $\mathbf{M}$.*

Proof.

First, suppose that $x, y \in f^{-1}(\nabla)$. Then $f(x), f(y) \in \nabla$. Since ∇ is a filter, $f(x) \cap f(y) \in \nabla$, and hence $f(x \cap y) \in \nabla$. So, we have $x \cap y \in f^{-1}(\nabla)$.

Next, suppose that $x \in f^{-1}(\nabla)$ and $x \leq y$. Then $f(x) \in \nabla$ and $f(x) \leq f(y)$. Since ∇ is a filter, $f(y) \in \nabla$. Hence, $y \in f^{-1}(\nabla)$.

Finally, suppose that $x \cup y \in f^{-1}(\nabla)$. Then $f(x \cup y) \in \nabla$. Since ∇ is prime, we have $f(x) \in \nabla$ or $f(y) \in \nabla$. So, $x \in f^{-1}(\nabla)$ or $y \in f^{-1}(\nabla)$.

Therefore, $f^{-1}(\nabla)$ is a prime filter. ■

2.4 Matrix semantics for R

In this section we define relevant matrices. First, we will present relevant matrices in terms of modified form of Font-Rodríguez's ones (see [18]). A structure $\langle M, \cap, \cup, - \rangle$ is a *De Morgan lattice* if it satisfies the following postulates for all $x, y \in M$:

(DML1) $\langle M, \cap, \cup \rangle$ is a distributive lattice,

(DML2) $x \leq -y$ implies $y \leq -x$,

(DML3) $- - x \leq x$.

A structure $\langle M, \cap, \cup, \rightarrow, - \rangle$ is a *De Morgan semigroup* if it satisfies the following postulates for all $x, y, z \in M$:

(DMS1) $\langle M, \cap, \cup, - \rangle$ is a De Morgan lattice,

(DMS2) $x \rightarrow (y \rightarrow z) \leq y \rightarrow (x \rightarrow z)$,

(DMS3) $x \leq (x \rightarrow y) \cap z \rightarrow y$,

(DMS4) $x \rightarrow -x \leq -x$,

(DMS5) $x \rightarrow y \leq -y \rightarrow -x$.

Basic properties of a De Morgan semigroup are shown in [17].

Further, a De Morgan semigroup $\langle M, \cap, \cup, \rightarrow, - \rangle$ satisfying

(DMS6) $(x \rightarrow x) \cap (y \rightarrow y) \rightarrow z \leq z$, for all $x, y, z \in M$,

is called an **R**-*algebra*.

We call $\langle \mathbf{M}, E \rangle$ a *relevant matrix* (**R**-*matrix*) if (a) $\mathbf{M} = \langle M, \cap, \cup, \rightarrow, - \rangle$ is a De Morgan semigroup (**R**-algebra), and (b) E is the filter generated by $\{x \rightarrow x \mid x \in M\}$.

In particular, an **R**-matrix has the following postulate.

Lemma 2.10 *Let $\langle \mathbf{M}, E \rangle$ be an* **R**-*matrix. Then for all $x, y \in M$, $x \leq y$ iff $x \rightarrow y \in E$.*

Proof.

The proof is given in [17]. The 'if' part is proved as follows. Before proving this, note that the following fact holds:

$$\text{if } x_i \rightarrow x_i \leq x_i \text{ for } i = 1, \cdots, n, \text{ then } x_1 \cap \cdots \cap x_n \rightarrow x_1 \cap \cdots \cap x_n \leq x_1 \cap \cdots \cap x_n.$$

Suppose that $x \to y \in E$. Then there exist $z_1, \cdots, z_m$ such that $(z_1 \to z_1) \cap \cdots \cap (z_m \to z_m) \leq x \to y$. By (RA6), for each i, $(z_i \to z_i) \cap (z_i \to z_i) \to (z_i \to z_i) \leq z_i \to z_i$, so $(z_i \to z_i) \to (z_i \to z_i) \leq z_i \to z_i$. Let $z = (z_1 \to z_1) \cap \cdots \cap (z_m \to z_m)$. By above fact, $z \to z \leq z$. Since $z \leq x \to y$, we have $z \to z \leq x \to y$, and hence $x \leq (z \to z) \to y$. Since $(z \to z) \to y = (z \to z) \cap (z \to z) \to y$, we have $x \leq y$ by (DMS6).

The 'only if' part is proved as follows. Suppose that $x \leq y$. Then $x \to x \leq x \to y$. Since $x \to x \in E$, we have $x \to y \in E$. ■

Note that the 'if' part does not hold for relevant matrices.

In the following, we prove that a special **R**-matrix characterizes **R**.

For any **R**-algebra $\mathbf{M} = \langle M, \cap, \cup, \to, - \rangle$, a mapping v from Prop to M is called a *valuation* on **M**. Further, given a valuation v on **M**, a mapping I from Wff to M, called the *interpretation associated with* v, is defined as follows:

i. for $p \in$ Prop, $I(p) = v(p)$
ii. $I(A \wedge B) = I(A) \cap I(B)$
iii. $I(A \vee B) = I(A) \cup I(B)$
iv. $I(A \to B) = I(A) \to I(B)$
v. $I(\sim A) = -I(A)$.

Let $\langle \mathbf{M}, E \rangle$ be an **R**-matrix, v be a valuation on **M** and I be the interpretation associated with v. Then we say that (a) A is *valid* in v iff $I(A) \in E$, and (b) A is *valid* in $\langle \mathbf{M}, E \rangle$ iff A is valid in any v.

By induction on the length of proof in **R**, we can show soundness of **R** easily.

Theorem 2.11 *If A is a theorem of* **R**, *then A is valid in any* **R**-*matrix* $\langle \mathbf{M}, E \rangle$.

For the converse, we use well-known Lindenbaum's method. Let $[A] = \{B \mid \mathbf{L} \vdash A \leftrightarrow B\}$ and $M_L = \{[A] \mid A \in \mathsf{Wff}\}$. Further, define the operations $\cap$, $\cup$, $\to$, $-$ on M_L by

$$[A] \cap [B] = [A \wedge B], \quad [A] \cup [B] = [A \vee B], \quad [A] \to [B] = [A \to B], \quad -[A] = [\sim A].$$

Then it is clear that $\cap, \cup, \to$ and $-$ are well-defined. The *Lindenbaum algebra for* **R** is the algebra $\mathbf{M}_L = \langle M_L, \cap, \cup, \to, - \rangle$ defined above. Note that for all $[A], [B] \in M_L$, $[A] \leq [B]$ iff $\mathbf{R} \vdash A \to B$.

Next, let $E_L = \{[A] \mid \mathbf{R} \vdash A\}$. Of course, E_L is also well-defined. Then the *Lindenbaum matrix for* **R** is a matrix $\langle \mathbf{M}_L, E_L \rangle$ defined by that $\mathbf{M}_L$ is the Lindenbaum algebra for **R** and that E_L is as above.

Lemma 2.12 *The Lindenbaum matrix* $\langle \mathbf{M}_L, E_L \rangle$ *for* **R** *is an* **R**-*matrix.*

Proof.

Here, we will show only that E_L is the filter generated by $\{[A] \to [A] \mid [A] \in M_L\}$. It is easy to see that E_L is a filter. It remains to show that E_L is the least filter containing $\{[A] \to [A] \mid [A] \in M_L\}$. Let F be any filter containing $\{[A] \to [A] \mid [A] \in M_L\}$. Suppose that $[A] \in E_L$. Then A is a theorem of **R**, and hence $(A \to A) \to A$ is also a theorem of **R**. Thus $[A \to A] \leq [A]$. Since $[A \to A] \in E_L$, we have $[A] \in E_L$. ■

The *canonical valuation* v_c is defined by

$$\text{for all } p \in \mathsf{Prop}, \qquad v_c(p) = [p].$$

Further, let I_c be the interpretation associated with v_c. By induction on the length of A, we can show the following easily.

Lemma 2.13 *For any $A \in \mathsf{Wff}$, $I_c(A) = [A]$.*

Thus, we have the completeness result.

Theorem 2.14 *If A is valid in any* **R***-matrix, then A is a theorem of* **R**.

Proof.
Suppose that A is not a theorem of **R**. Then in the Lindenbaum matrix $\langle \mathbf{M}_L, E_L \rangle$ for **R**, $[A] \notin E_L$, and hence $I_c(A) \notin E_L$ by Lemma 2.13. This means that A is not valid in $\langle \mathbf{M}_L, E_L \rangle$. By Lemma 2.12, there exists an **R**-matrix in which A is not valid. ∎

2.5 Classical normal modal logics

The language of classical modal logics consists of (i) propositional variables $p, q, r, \cdots$; (ii) logical connectives $\supset$ (classical implication), $\wedge$ (and), $\vee$ (or) and $\neg$ (classical negation); (iii) modal operator $\Box$ (necessity) and (iv) propositional constants $\top$ (truth) and $\bot$ (falsehood). Formulas are defined in the usual way, and are denoted by capital letters $A, B, C, \cdots$.

A *classical modal logic* is a set of formulas including the following axioms and rule of inference.

(a) Axioms

(**Cl**) All axioms of classical logic **Cl**.

(b) Rule of inferences

$$\frac{A \supset B \quad A}{B}\text{(Modus Ponens)}.$$

For short, we often call it a modal logic.

A *normal* modal logic is a modal logic together with the following.

(a) Abbreviations

$$A \equiv B \stackrel{\text{abb}}{=} (A \supset B) \wedge (B \supset A) \qquad \Diamond A \stackrel{\text{abb}}{=} \neg\Box\neg A$$

(b) Axiom

(**K**) $\Box(A \supset B) \supset (\Box A \supset \Box B)$.

(c) Rule of inferences

$$\frac{A}{\Box A}\text{(Necessitation)}$$

The least normal modal logic is called **K**.

Any normal modal logic is obtained by adding some axiom schemes to **K**. Typical axiom schemes are listed in Table 2.1, where formulas $\Box^n A$ and $\Diamond^n A$ stand for

$$\underbrace{\Box \cdots \Box}_{n} A \text{ and } \underbrace{\Diamond \cdots \Diamond}_{n} A,$$

respectively. (Their names follow from [12] and [11], and they are used throughout this thesis.)

Table 2.1: Typical axiom schemes

Name	Axiom scheme
D	$\Box A \supset \Diamond A$
T	$\Box A \supset A$
B	$A \supset \Box\Diamond A$
4	$\Box A \supset \Box\Box A$
5	$\Diamond A \supset \Box\Diamond A$
$\mathbf{G}(k,l,m,n)$	$\Diamond^k \Box^l A \supset \Box^m \Diamond^n A$
Dir	$\Diamond(\Box A \wedge B) \supset \Box(\Diamond A \vee B)$
U	$\Box(\Box A \supset A)$
SC	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$
Con	$\Box(A \wedge \Box A \supset B) \vee \Box(B \wedge \Box B \supset A)$
$\mathbf{Tra}(n)$	$A \wedge \Box A \wedge \cdots \wedge \Box^n A \supset \Box^{n+1} A$
$\mathbf{Alt}(n)$	$\Box A_1 \vee \Box(A_1 \supset A_2) \vee \cdots \vee \Box(A_1 \wedge \cdots \wedge A_n \supset A_{n+1})$

Here we give some examples of normal modal logics. **KT** is the logic obtained from **K** by adding the axiom **T**. **S4** is the logic obtained from **KT** by adding the axiom **4**. **S5** is the logic obtained from **KT** by adding the axiom **5** (equivalently, the logic obtained from **S4** by adding the axiom **B**).

2.6 Kripke semantics for normal modal logics

In this section, we introduce a semantics for normal modal logics. This semantics is often called *Kripke semantics*, introduced by S.Kripke. A *Kripke frame* is a pair $\langle W, S\rangle$ where (a) W is a set of all worlds, and (b) S is a binary relation on W.

We call a triple $\langle W, S, V\rangle$ a *Kripke model* on a Kripke frame $\mathcal{F} = \langle W, S\rangle$, where $\mathcal{F}$ is a Kripke frame and V is a mapping from Prop to 2^W, called a *valuation* on $\mathcal{F}$. Given a Kripke model $\langle W, S, V\rangle$, for $a \in W$ and $A \in \mathsf{Wff}$, a relation $\models$ between W and Wff is defined inductively as follows:

i. for any $p \in \mathsf{Prop}$, $a \models p$ iff $a \in V(p)$

ii. $a \models \top$

iii. $a \not\models \bot$

iv. $a \models A \wedge B$ iff $a \models A$ *and* $a \models B$

v. $a \models A \vee B$ iff $a \models A$ *or* $a \models B$

vi. $a \models A \supset B$ iff $a \not\models A$ *or* $a \models B$

vii. $a \models \neg A$ iff $a \not\models A$

viii. $a \models \Box A$ iff for any $b \in W$, if Sab then $b \models A$.

Note that

$$a \models \Diamond A \text{ iff } \text{ there exists } b \in W \text{ such that } Sab \text{ and } b \models A.$$

Let us consider axioms in Table 2.1. When A is one of axioms in Table 2.1, to make $a \models A$ for all $a \in W$, a Kripke frame must satisfy some condition. Table 2.2 shows the condition which a Kripke frame should satisfy corresponding to each axiom in Table 2.1, where $\bigwedge$ and $\bigvee$ denote conjunction and disjunction in the metalanguage.

If $\mathbf{L}_N$ be the normal logic obtained from $\mathbf{K}$ by adding some left-hand side axioms, then we call the Kripke frame satisfying the corresponding right-side postulates an $\mathbf{L}_N$-*frame*. The definition of $\mathbf{L}_N$-model is as above.

Table 2.2: Frame postulates

Axiom	Frame postulate
D	$\exists b(Sab)$
T	Saa
B	$Sab \Rightarrow Sba$
4	$Sab \ \&\ Sbc \Rightarrow Sac$
5	$Sab \ \&\ Sac \Rightarrow Sbc$
$\mathbf{G}(k,l,m,n)$	$S^k ab \ \&\ S^m ac \Rightarrow \exists d(S^l bd \ \&\ S^n cd)$
Dir	$Sab \ \&\ Sac \ \&\ b \neq c \Rightarrow \exists d(Sbd \ \&\ Scd)$
U	$Sab \Rightarrow Sbb$
SC	$Sab \ \&\ Sac \Rightarrow Sbc \text{ or } Scb$
Con	$Sab \ \&\ Sac \ \&\ b \neq c \Rightarrow Sbc \text{ or } Scb$
$\mathbf{Tra}(n)$	$S^{n+1}ab \Rightarrow Sab \text{ or } \cdots \text{ or } S^n ab$
$\mathbf{Alt}(n)$	$\bigwedge_{i=1}^{n+1} Sab_i \Rightarrow \bigvee_{i \neq j} b_i = b_j$

Here S^n ($n \geq 0$) is a binary relation on W and is defined as follows. For all $a, b \in W$:

(i) $S^0 ab$ iff $a = b$,

(ii) for $n > 0$, $\quad S^n ab$ iff there exists $c \in W$ such that $S^{n-1}ac$ and Scb.

Suppose that $\mathbf{L}_N$ is a normal modal logic. Let $\mathcal{M} = \langle W, S, V \rangle$ be an $\mathbf{L}_N$-model on an $\mathbf{L}_N$-frame $\mathcal{F} = \langle W, S \rangle$, $A \in \mathsf{Wff}$, and $\mathcal{C}$ be a class of $\mathbf{L}_N$-frames. Then we say

(a) A *holds in* $\mathcal{M}$ iff $a \models A$ for every world $a \in W$,

(b) A is *valid* in an $\mathbf{L}_N$-frame $\mathcal{F}$ iff A holds in every $\mathbf{L}_N$-model $\mathcal{M}$ on $\mathcal{F}$,

(c) $\mathbf{L}_N$ is *sound with respect to* $\mathcal{C}$ iff A is valid in every $\mathcal{F} \in \mathcal{C}$ for all theorems A of $\mathbf{L}_N$,

(d) $\mathbf{L}_N$ is *complete with respect to* $\mathcal{C}$ iff A is a theorem of $\mathbf{L}_N$ for every formula A valid in all $\mathcal{F} \in \mathcal{C}$,

(e) $\mathbf{L}_N$ is *determined by* $\mathcal{C}$ iff $\mathbf{L}_N$ is both sound and complete with respect to $\mathcal{C}$.

Then the following results are well-known.

Proposition 2.15

1. **K** *is determined by the class of Kripke frames.*
2. *Let* $\mathbf{L}_N$ *be the logic obtained from* **K** *by adding some of axioms listed in Table 2.2. Then* $\mathbf{L}_N$ *is determined by the class of* $\mathbf{L}_N$*-frames.*

Next, we present the basic properties of the truth-preserving operations. A frame $\mathcal{F}' = \langle W', S' \rangle$ is called a *subframe* of a frame $\mathcal{F} = \langle W, S \rangle$ if (a) W' is a non-empty subset of W which satisfies the following condition:

$$\text{if } a \in W' \text{ and } S'ab, \text{ then } b \in W',$$

and (b) S' is the restriction of S to W'.

As concerns subframes, the following result is known.

Proposition 2.16 *Let* $\mathcal{F}'$ *be a subframe of* $\mathcal{F}$*. If* A *is valid in* $\mathcal{F}$*, then* A *is valid in* $\mathcal{F}'$*.*

Let $\mathcal{F} = \langle W, S \rangle$ and $\mathcal{F}' = \langle W', S' \rangle$ be frames. We call a surjection $f : W \to W'$ *p-morphism* from $\mathcal{F}$ to $\mathcal{F}'$ if f satisfies the following conditions:

(i) for all $a, b \in W$, if Sab then $S'f(a)f(b)$

(ii) for all $a \in W$ and $b' \in W'$, if $S'f(a)b'$ then there exists $b \in W$ such that Sab and $f(b) = b'$.

As concerns p-morphisms, the following result is known.

Proposition 2.17 *Let* $\mathcal{F}$ *and* $\mathcal{F}'$ *be frames. Suppose that there exists a p-morphism from* $\mathcal{F}$ *to* $\mathcal{F}'$*. If* A *is valid in* $\mathcal{F}$*, then* A *is valid in* $\mathcal{F}'$*.*

Let $\{\mathcal{F}_i = \langle W_i, S_i \rangle \mid i \in I\}$ be a family of frames such that $W_i \cap W_j = \emptyset$, for all $i \neq j$. The *disjoint union of the family* $\{\mathcal{F}_i \mid i \in I\}$ is the frame $\sum_{i \in I} \mathcal{F}_i = \langle \bigcup_{i \in I} W_i, \bigcup_{i \in I} R_i \rangle$.

As concerns disjoint unions, the following result is known.

Proposition 2.18 *Let* $\sum_{i \in I} \mathcal{F}_i$ *be the disjoint union of a family* $\{\mathcal{F}_i \mid i \in I\}$ *of frames. Then* A *is valid in* $\sum_{i \in I} \mathcal{F}_i$ *iff* A *is valid in* $\mathcal{F}_i$ *for all* $i \in I$*.*

2.7 Algebraic semantics for normal modal logics

In this section, we define an algebraic semantics for normal modal logics. An algebra $\mathbf{M} = \langle M, \cap, \cup, ', \top, \bot, \Box \rangle$ is called a *modal algebra* if

(a) $\langle M, \cap, \cup, ', \top, \bot \rangle$ is a Boolean algebra,

(b) for all $x, y \in M$, $\quad \Box(x \cap y) = \Box x \cap \Box y$,

(c) $\Box\top = \top$.

For any modal algebra $\mathbf{M} = \langle M, \cap, \cup, ', \top, \bot, \Box \rangle$, we define a binary operation $\rightarrow$ and an unary operation $\Diamond$ on M as follows. For $x, y \in M$,

$$x \rightarrow y \stackrel{\text{def}}{=} x' \cup y, \qquad \Diamond x \stackrel{\text{def}}{=} (\Box x')'.$$

For any modal algebra $\mathbf{M} = \langle M, \cap, \cup, ', \top, \bot, \Box \rangle$, a mapping v from Prop to M is called a *valuation* on $\mathbf{M}$. Further, given a valuation v on $\mathbf{M}$, a mapping I from Wff to M, called the *interpretation associated with* v, is defined as follows:

i. for $p \in \mathsf{Prop}$, $I(p) = v(p)$
ii. $I(\top) = \top$
iii. $I(\bot) = \bot$
iv. $I(A \wedge B) = I(A) \cap I(B)$
v. $I(A \vee B) = I(A) \cup I(B)$
vi. $I(A \supset B) = I(A) \rightarrow I(B)$
vii. $I(\neg A) = (I(A))'$
viii. $I(\Box A) = \Box I(A)$.

Note that $I(\Diamond A) = \Diamond I(A)$.

Let $\mathbf{M} = \langle M, \cap, \cup, ', \top, \bot, \Box \rangle$ be a modal algebra, v be a valuation on $\mathbf{M}$ and I be the interpretation associated with v. Then we say (a) A is *valid* in v iff $I(A) = \top$, and (b) A is *valid* in $\mathbf{M}$ iff A is valid in any v.

By Lindenbaum's method, we can show the following algebraic completeness result.

Proposition 2.19 *A is a theorem of* $\mathbf{K}$ *iff A is valid in any modal algebra.*

In the following, we present the truth-preserving operations on modal algebras.

Let $\mathbf{M} = \langle M, \cap, \cup, ', \top, \bot, \Box \rangle$ be a modal algebra. A model algebra $\mathbf{M}' = \langle M', \cap, \cup, ', \top, \bot, \Box \rangle$ is called a *subalgebra* of $\mathbf{M}$ if (a) $M' \subseteq M$ and (b) each operation of $\mathbf{M}'$ is closed.

Then the following result holds.

Proposition 2.20 *If A is valid in a modal algebra* $\mathbf{M}$*, then A is valid in every subalgebra of* $\mathbf{M}$.

If f is a homomorphism of $\mathbf{M} = \langle M, \cap, \cup, ', \top, \bot, \Box \rangle$ in $\mathbf{M}' = \langle M', \cap, \cup, ', \top, \bot, \Box \rangle$, then the set $f(M)$ is clearly closed under the operations in $\mathbf{M}'$ and hence $\langle f(M), \cap, \cup, ', \top, \bot, \Box \rangle$ is a subalgebra of $\mathbf{M}'$. We call it the *homomorphic image* of $\mathbf{M}$.

Then the following result holds.

Proposition 2.21 *If A is valid in a modal algebra* $\mathbf{M}$*, then A is valid in every homomorphic image of* $\mathbf{M}$.

Given a family $\{\mathbf{M}_i = \langle M_i, \cap, \cup, ', \top, \bot, \Box \rangle \mid i \in I\}$ of modal algebras, the *direct product* of $\{\mathbf{M}_i \mid i \in I\}$ is the algebra

$$\prod_{i \in I} \mathbf{M}_i = \langle \prod_{i \in I} M_i, \cap, \cup, ', \top, \bot, \Box \rangle,$$

where (a) $\prod_{i \in I} M_i$ is the set of all functions f from I into $\bigcup_{i \in I} M_i$ such that $f(i) \in M_i$ and (b) for every $f_1, f_2 \in \prod_{i \in I} M_i$ and every $i \in I$,

i. $(f_1 \cap f_2)(i) = f_1(i) \cap f_2(i)$

ii. $(f_1 \cup f_2)(i) = f_1(i) \cup f_2(i)$

iii. $f_1'(i) = (f_1(i))'$

iv. $(\Box f_1)(i) = \Box f_1(i)$.

Then the following result holds.

Proposition 2.22 *Suppose that $\prod_{i \in I} \mathbf{M}_i$ is the direct product of $\{\mathbf{M}_i \mid i \in I\}$. Then A is valid in every $\mathbf{M}_i$ iff A is valid in $\prod_{i \in I} \mathbf{M}_i$.*

2.8 General frames for normal modal logics

In this section, we present basic properties of general frames for normal modal logics. For more details, see Chapter 8 of [11] (except slight differences on notations).

A *modal general frame* is a triple $\mathfrak{F} = \langle W, S, P \rangle$ where (a) $\langle W, S \rangle$ is a Kripke frame, written by $\kappa\mathfrak{F}$, and (b) P, *a set of possible values* in $\mathfrak{F}$, is a subset of 2^W containing $\emptyset$ and closed under $\cap, \cup, '$ and the operation $\Box$ which is defined as follows. For $X \subseteq W$,

$$\Box X = \{a \in W \mid \forall b \in W(Sab \Rightarrow b \in X)\}.$$

The algebra $\langle P, \cap, \cup, ', W, \emptyset, \Box \rangle$ is called the *dual* of $\mathfrak{F}$, and is denoted by $\mathfrak{F}^+$.

Then the following fact holds.

Proposition 2.23 *The dual of every modal general frame is a modal algebra.*

Modal general frames $\mathfrak{F} = \langle W, S, P \rangle$ and $\mathfrak{F}' = \langle W', S', P' \rangle$ are *isomorphic* if there is an isomorphism f of $\langle W, S \rangle$ onto $\langle W', S' \rangle$ such that for every $X \subseteq W$, $X \in P$ iff $f(X) \in P'$.

Let $\mathfrak{F} = \langle W, S, P \rangle$ be a modal general frame. A *model* on $\mathfrak{F}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where V, a *valuation* on $\mathfrak{F}$, is a map from Prop in P, i.e., $V(p) \in P$ for every $p \in \mathsf{Prop}$. The truth-relation $\models$ in $\mathfrak{M}$ is defined in exactly the same way as in Section 2.6 for Kripke models. We write $V(A) = \{a \in W \mid a \models A\}$.

The definitions of truth, validity, etc. given in Section 2.6, can be extended to general frames without any change.

Proposition 2.24 *Let $\mathfrak{F}$ be a modal general frame. Then A is valid in $\mathfrak{F}$ iff A is valid in $\mathfrak{F}^+$.*

Next, we present the duality theory. Let $\mathbf{M} = \langle M, \cap, \cup, ', \top, \bot, \Box \rangle$ be a modal algebra. Then

- $W_{\mathbf{M}}$ is the set of all prime filters in $\mathbf{M}$
- $S_{\mathbf{M}} \nabla_1 \nabla_2$ iff $\forall x \in M(\Box x \in \nabla_1 \Rightarrow x \in \nabla_2)$, for $\nabla_1, \nabla_2 \in W_{\mathbf{M}}$
- $P_{\mathbf{M}} = \{f_{\mathbf{M}}(x) \mid x \in M\}$, where $f_{\mathbf{M}} : M \to 2^{W_{\mathbf{M}}}$ defined by $f_{\mathbf{M}}(x) = \{\nabla \in W_{\mathbf{M}} \mid x \in \nabla\}$

We say that $\langle W_{\mathbf{M}}, S_{\mathbf{M}}, P_{\mathbf{M}} \rangle$ is the *dual* of $\mathbf{M}$ and is denoted by $\mathbf{M}_+$. Note that prime filters coincide with maximal filters in Boolean algebras.

Then the following results hold.

Proposition 2.25 *Let* $\mathbf{M}$ *be a modal algebra.*

1. *The dual* $\mathbf{M}_+$ *of* $\mathbf{M}$ *is a modal general frame.*
2. A *is valid in* $\mathbf{M}$ *iff* A *is valid in* $\mathbf{M}_+$.

The following "representation theorem" holds.

Proposition 2.26 *Every modal algebra* $\mathbf{M}$ *is isomorphic to its bidual* $(\mathbf{M}_+)^+$ *under the isomorphism* $f_{\mathbf{M}}$.

Next, we present descriptive frames. Here the definition of descriptive frames follows [11].

In general, a modal algebra $\mathbf{M}$ is isomorphic to its bidual $(\mathbf{M}_+)^+$, but a modal general frame $\mathfrak{F}$ is not always isomorphic to its bidual $(\mathfrak{F}^+)_+$. We call a modal general frame $\mathfrak{F}$ *descriptive* if $\mathfrak{F}$ is isomorphic to its bidual $(\mathfrak{F}^+)_+$.

Let $\mathfrak{F} = \langle W, S, P\rangle$ be a modal general frame. Then we say that

(a) $\mathfrak{F}$ is *differentiated* if for any $a, b \in W$,

$$a = b \ \text{ iff } \ \forall X \in P(a \in X \Leftrightarrow b \in X),$$

(b) $\mathfrak{F}$ is *tight* if for any $a, b \in W$,

$$Sab \ \text{ iff } \ \forall X \in P(a \in \Box X \Rightarrow b \in X),$$

(c) $\mathfrak{F}$ is *compact* if, for any families $\mathcal{X} \subseteq P$ and $\mathcal{Y} \subseteq \overline{P} = \{W - X \mid X \in P\}$,

$$\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$$

whenever $\bigcap(\mathcal{X}' \cup \mathcal{Y}') \neq \emptyset$ for all finite subfamilies $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$.

Note that $\mathfrak{F}$ is tight if for any $a, b \in W$,

$$Sab \text{ iff } \forall X \in P(b \in X \Rightarrow a \in \Diamond X).$$

Then, descriptive frames are characterized by the following.

Proposition 2.27 *A modal general frame* $\mathfrak{F} = \langle W, S, P\rangle$ *is descriptive iff* $\mathfrak{F}$ *is differentiated, tight and compact.*

Finally, we present truth-preserving operations on general frames.

A modal general frame $\mathfrak{F}' = \langle W', S', P'\rangle$ is a *generated subframe* of $\mathfrak{F} = \langle W, S, P\rangle$ if (a) $\kappa\mathfrak{F}'$ is a generated subframe of $\kappa\mathfrak{F}$ and (b) $P' = \{X \cap W' \mid X \in P\}$.

As concerns generated subframes, the following facts hold.

Proposition 2.28

1. *If* h *is an isomorphism of* $\mathfrak{F}' = \langle W', S', P'\rangle$ *onto a generated subframe of* $\mathfrak{F} = \langle W, S, P\rangle$*, then the map* h^+ *defined by*

$$h^+(X) = h^{-1}(X) = \{a \in W' \mid h(a) \in X\}, \quad \textit{for every } X \in P,$$

is a homomorphism of $\mathfrak{F}^+$ *onto* $\mathfrak{F}'^+$.

2. *Suppose h is a homomorphism of a modal algebra $\mathbf{M}$ onto a modal algebra $\mathbf{M}'$. Then the map h_+ defined by*

$$h_+(\nabla) = h^{-1}(\nabla), \quad \textit{for every prime filter } \nabla \textit{ in } \mathbf{M}',$$

is an isomorphism of $\mathbf{M}'_+$ onto a generated subframe of $\mathbf{M}_+$.

Given frames $\mathfrak{F} = \langle W, S, P\rangle$ and $\mathfrak{F}' = \langle W', S', P'\rangle$, we say a map f from W onto W' is a *p-morphism* of $\mathfrak{F}$ to $\mathfrak{F}'$ if the following three conditions hold:

(pM1) for all $a, b \in W$, Sab implies $S'f(a)f(b)$

(pM2) for all $a \in W$ and $b' \in W'$, $S'f(a)b'$ implies there exists $b \in W$ such that Sab and $f(b) = b'$

(pM3) for all $X \in P'$, $f^{-1}(X) \in P$.

As concerns p-morphisms, the following facts hold.

Proposition 2.29

1. *If f is a p-morphism of $\mathfrak{F} = \langle W, S, P\rangle$ to $\mathfrak{F}' = \langle W', S', P'\rangle$, then the map f^+ defined by*
$$f^+(X) = f^{-1}(X), \quad \textit{for every } X \in P',$$
is an isomorphism of $\mathfrak{F}'^+$ in $\mathfrak{F}$.

2. *If f is an isomorphism of a modal algebra $\mathbf{M}'$ in $\mathbf{M}$, then the map f_+ defined by*

$$f_+(\nabla) = f^{-1}(\nabla), \quad \textit{for every } \nabla \in W_{\mathbf{M}},$$

is a p-morphism of $\mathbf{M}_+$ to $\mathbf{M}'_+$.

The *disjoint union* of a family $\{\mathfrak{F}_i = \langle W_i, S_i, P_i\rangle \mid i \in I\}$ of pairwise disjoint frames is the frame $\sum_{i\in I} \mathfrak{F}_i = \langle W, S, P\rangle$ where $W = \bigcup_{i\in I} W_i$, $S = \bigcup_{i\in I} S_i$, $P = \{\bigcup_{i\in I} X_i \mid X_i \in P_i, \text{for all } i \in I\}$.

As concerns disjoint unions, the following facts hold.

Proposition 2.30

1. *Let $\{\mathfrak{F}_i \mid i \in I\}$ be a family of modal general frames and $\sum_{i\in I} \mathfrak{F}_i = \langle W, S, P\rangle$ be its disjoint union. Then the map f defined by*

$$f(X)(i) = X \cap W_i, \qquad \textit{for every } X \in P \textit{ and } i \in I,$$

is an isomorphism of $(\sum_{i\in I} \mathfrak{F}_i)^+$ onto $\prod_{i\in I} \mathfrak{F}_i^+$.

2. *Suppose that $\mathbf{M}$ and $\mathbf{M}'$ are modal algebras. Then the map f defined by*

$$f(\nabla) = \{\langle x, y\rangle \in M \times M' \mid x \in \nabla, y \in M'\}, \quad \textit{for every } \nabla \in W_{\mathbf{M}}$$

and

$$f(\nabla') = \{\langle x, y\rangle \in M \times M' \mid x \in M, y \in \nabla'\}, \quad \textit{for every } \nabla' \in W_{\mathbf{M}'}$$

is an isomorphism of $\mathbf{M}_+ + \mathbf{M}'_+$ onto $(\mathbf{M} \times \mathbf{M}')_+$.

2.9 Sahlqvist theorem for normal modal logic

In this section, we present Sahlqvist theorem for normal modal logic which are connected with Chapter 5, without proof.

Before presenting a Sahlqvist theorem for normal modal logics, we introduce the following terminologies.

Let $\mathbf{L}$ be any normal modal logic. Below, $\mathfrak{F} \models \mathbf{L}$ denotes that $\mathfrak{F} \models A$ for all theorems A of $\mathbf{L}$. Let $\mathcal{D}$ be the class of descriptive $\mathbf{L}$-frames. Then let $\kappa\mathcal{D} = \{\kappa\mathfrak{F} \mid \mathfrak{F} \in \mathcal{D}\}$, and $\mathcal{D}^\star = \mathcal{D} \cup \kappa\mathcal{D}$. Then we say that

- $\mathbf{L}$ is *$\mathcal{D}$-complete* (*$\kappa\mathcal{D}$-complete*) if A is a theorem of $\mathbf{L}$ whenever it is valid in every descriptive $\mathbf{L}$-frame (every $\mathbf{L}$-frame).
- $\mathbf{L}$ is *$\mathcal{D}$-persistent* if for any descriptive $\mathbf{L}$-frame $\mathfrak{F}$, $\mathfrak{F} \models \mathbf{L}$ implies $\kappa\mathfrak{F} \models \mathbf{L}$.
- $\mathbf{L}$ is *$\mathcal{D}^\star$-elementary* if there exists a set Φ of first-order sentences in the predicates S and $=$ such that for every $\mathfrak{F} \in \mathcal{D}^\star$,

$$\mathfrak{F} \text{ is an } \mathbf{L}\text{-frame iff } \mathfrak{F} \text{ is a model for } \Phi.$$

For these notions, the following facts are known.

Proposition 2.31 *Let* $\mathbf{L}$ *be any normal logic.*

1. *If* $\mathbf{L}$ *is both $\mathcal{D}$-complete and $\mathcal{D}$-persistent, then it is $\kappa\mathcal{D}$-complete.*
2. *If* $\mathbf{L}$ *is $\mathcal{D}^\star$-elementary, then it is $\mathcal{D}$-persistent.*
3. *If* $\mathbf{L}$ *is both $\mathcal{D}$-complete and $\mathcal{D}^\star$-elementary, then it is $\kappa\mathcal{D}$-complete.*

Since $\mathbf{L}$ is $\mathcal{D}$-complete, 1 of Proposition 2.31 insists that if $\mathbf{L}$ is $\mathcal{D}$-persistent then it is $\kappa\mathcal{D}$-persistent. Then we consider whether the converses of these propositions hold. It is known that the converse of 1 does not hold. For example, we may take the logic obtained from $\mathbf{K}$ by adding the *Löb's* axiom $\Box(\Box A \supset A) \supset \Box A$. On the other hand, it is an open problem whether the converse of 2 holds.

Now we present a Sahlqvist theorem for normal modal logic.

A formula A is *positive* if A contains only $\top, \bot, \wedge, \vee, \Box$ and $\Diamond$. A modal formula of the form $\Box^{m_1} p_1 \wedge \cdots \wedge \Box^{m_k} p_k$ with not necessarily distinct propositional variables $p_1, \cdots, p_k$ is called a *strongly positive formula*. A given formula A is *negative* (in $\mathbf{L}$) if A is built from the negations of variables with the help of $\top$, $\bot$, $\wedge$, $\vee$, $\Box$ and $\Diamond$. A modal formula A is *untied* (in $\mathbf{L}$) if it can be constructed from negative formulas and strongly positive formulas using only $\wedge$ and $\Diamond$.

Theorem 2.32 (Sahlqvist) *Suppose that A is a formula which is equivalent in $\mathbf{K}$ to a conjunction of formulas of the form $\Box^k(B \supset C)$, where $k \geq 0$, B is untied and C is positive. Then there exists a first order formula $\phi(a)$ in the predicates S and $=$ having a as its only free variable and such that the following holds for every $\mathfrak{F} \in \mathcal{D}^\star$ and every $a \in W$,*

$$(\mathfrak{F}, a) \models A \quad \textit{iff} \quad \mathfrak{F} \textit{ satisfies } \phi(a),$$

where $(\mathfrak{F}, a) \models A$ means that $a \models A$ under any valuation on $\mathfrak{F}$.

Any formula A of the form described in above theorem is called a *Sahlqvist formula.* From Theorem 2.32, we have $\mathfrak{F} \models A$ iff $\mathfrak{F}$ satisfies $\forall a \in W(\phi(a))$.

Theorem 2.33 *Let* $\mathbf{L}$ *be a logic obtained from* $\mathbf{K}$ *by adding a set of Sahlqvist formulas as axioms. Then* $\mathbf{L}$ *is* $\mathcal{D}^\star$*-elementary, and hence* $\mathcal{D}$*-persistent. Hence,* $\mathbf{L}$ *is* $\kappa\mathcal{D}$*-complete.*

As concerns these notions, we note some remarks. It is clear that the *McKinsey axiom* $\Box\Diamond A \supset \Diamond\Box A$ is not a Sahlqvist formula. But it is known that the logic **K4M** which is obtained from **K** by adding the axioms **4** and the McKinsey axiom is $\mathcal{D}^\star$-elementary. Because an **K4M**-frame $\langle W, S\rangle$ satisfies the following frame postulates:

- for all $a, b, c \in W$, if Sab and Sbc, then Sac
- for all $a \in W$, there exists $b \in W$ such that for all $c \in W$, Sbc implies $b = c$.

This fact shows that the Sahlqvist theorem does not cover all $\mathcal{D}^\star$-elementary logics.

2.10 Notes

The relevant logic **R** was first formulated by N.D.Belnap,Jr. in [6]. In relevant logic, Routley-Meyer semantics is well-known Kripke-style semantics. This was published first in [47] for **R**. The key idea of Routley-Meyer semantics is introducing ternary relation and Routley's star operation in order to interpret relevant implication and relevant negation, respectively. An **R**-frame which we adopt here is called an *unreduced* frame. An **R**-frame is usually a reduced frame, so we present its definition below following [47].

A *reduced* **R**-frame is a quadruple $\langle 0, W, R, ^* \rangle$ where W, R and * are as in our definition of **R**-frame, and $0 \in W$. Further, a binary relation $<$ is defined by

$$a < b \ \text{ iff } \ R0ab.$$

A reduced **R**-model is a quintuple $\langle 0, W, R, ^*, V\rangle$ where $\mathcal{F} = \langle 0, W, R, ^* \rangle$ is a reduced **R**-frame and V, a valuation on $\mathcal{F}$, is defined as in Section 2.3. As concerns validity, A holds in a reduced **R**-model $\mathcal{M}$ iff $0 \models A$. Also, the proof of completeness with respect to reduced **R**-models is slightly complicated. That is, we must consider both **R**-theories and T-theories. A set of formulas Σ is called a *T-theory* if (a) if $A, B \in \Sigma$, then $A \wedge B \in \Sigma$, and (b) if $A \to B \in T$ and $A \in \Sigma$, then $B \in \Sigma$. Consequently, the canonical model $\langle 0_c, W_c, R_c, g_c, V_c\rangle$ is as follows:

- $0_c = T$, where T is a fixed regular **R**-theory
- W_c is the set of all prime T-theories
- R_c, g_c and V_c are defined as in Section 2.2.

In the same way as this, this semantics is extended to several relevant logics weaker than **R**, for example **E**, **T** and **B**, and stronger than **R**, for example **RM**, **CR**, **KR** and their modal extensions. For more details, see [48].

For algebraic models for **R**, De Morgan monoid is a standard one. We call $\langle M, \cap, \cup, \to, \cdot, -, e\rangle$ a *De Morgan monoid* if it satisfies the following. For all $x, y, z \in M$,

(DMM1) $\langle M, \cap, \cup, -\rangle$ is a De Morgan lattice, (DMM2) $\langle M, \cdot, e \rangle$ is a commutative monoid,

(DMM3) $x \cdot y \leq z$ iff $x \leq y \rightarrow z$

(DMM4) $x \leq x \cdot x$

(DMM5) $x \cdot y \leq z$ iff $x \cdot -z \leq -y$.

This can be used also as a semantics for $\mathbf{R^t}$ which is obtained from $\mathbf{R}$ by adding the axioms

- $\mathbf{t}$
- $\mathbf{t} \rightarrow (A \rightarrow A)$.

If e and $\cdot$ interpret $\mathbf{t}$ and $\circ$, which is defined by

$$A \circ B \stackrel{\text{def}}{=} \sim (A \rightarrow \sim B),$$

respectively. Since $\mathbf{R^t}$ is a conservative extension of $\mathbf{R}$, De Morgan monoid is regarded as an algebraic model for $\mathbf{R}$. Here we will present a matrix model for $\mathbf{R}$ according to [17] and [18]. An advantage of matrix model is that we need not consider constant.

For modal logics, we will present normal modal logics and their fundamental properties on semantics following [11]. Kripke semantics and algebraic semantics are familiar ones. In Kripke semantics, it is most characteristic to introduce possible worlds and accessible relations. They give intuitive understanding of validity on modal formulas. An algebraic semantics is obtained from Boolean algebra by adding modal operators.

Chapter 3

Basic relevant modal logics and their completeness

In this chapter, we introduce basic relevant modal logics and show their completeness by using both Routley-Meyer semantics and matrix semantics. To prove completeness, we use the method of using canonical models for the former and Lindenbaum's method for the latter. Introducing models on frames and matrices, we are interested in the truth-preserving operations of frames and matrices. In relevant modal logics, as in classical modal logic, we can consider subframes, relevant p-morphisms and disjoint unions as the truth-preserving operations of frames, and submatrices, homomorphic images and direct products as the truth-preserving operations of matrices. Further, using completeness with respect to Routley-Meyer semantics, we show that our relevant modal logics are conservative extensions of **R**.

3.1 Basic relevant modal logics

Our language $\mathcal{L}$ of relevant modal logics consists of (i) propositional variables $p, q, r, \cdots$; (ii) logical connectives $\to$ (relevant implication), $\wedge$ (and), $\vee$ (or) and $\sim$ (relevant negation); and (iii) modal operators $\Box$ (necessity) and $\Diamond$ (possibility).

Formulas are defined in the usual way, and are denoted by capital letters $A, B, C, \cdots$. We write $A \leftrightarrow B$ for $(A \to B) \wedge (B \to A)$. Prop and Wff will denote the set of all propositional variables and of formulas, respectively. Capital Greek letters $\Sigma, \Gamma, \Delta, \cdots$ denote non-empty sets of formulas. When necessary, we add $'$ or subscripts to these capital letters and capital Greek letters.

The relevant logic **R** is defined as in Section 2.1. Now we define modal logics based on **R**. A set of formulas is called a *regular relevant modal logics over* **R** (a regular logic over **R**, for short), if it includes the following axioms and rules of inference.

(a) Axioms

(Ax) all axioms of **R**.

(R14) $\Box A \wedge \Box B \to \Box(A \wedge B)$

(R15) $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$

(b) Rules of inference

$$\frac{A \to B \quad A}{B}(\text{Modus Ponens}) \qquad \frac{A \quad B}{A \wedge B}(\text{Adjunction})$$

$$\frac{A \to B}{\Box A \to \Box B}(\Box\text{-monotonicity}) \qquad \frac{A \to B}{\Diamond A \to \Diamond B}(\Diamond\text{-monotonicity})$$

The least regular logic over **R** is called $\mathbf{R.C}_{\Box\Diamond}$. In [19], A.Fuhrmann calls the logic obtained from $\mathbf{R.C}_{\Box\Diamond}$ by deleting $\Diamond$ 'conjunctively regular'; here we will call it 'regular'. It is known that classical modal logics containing (R14) and ($\Box$-monotonicity) are regular (see [12] (p.236)).

We call a logic, a *normal relevant modal logics over* **R** (a normal logics over **R**, for short), if it includes the following axioms and rules of inference.

(a) Axioms

- (Ax) all axioms of $\mathbf{R.C}_{\Box\Diamond}$.
- (R16) $\Box(A \to B) \to (\Box A \to \Box B)$
- (R17) $\Box(A \to B) \to (\Diamond A \to \Diamond B)$

(b) Rules of inference

$$\frac{A \to B \quad A}{B}(\text{Modus Ponens}) \qquad \frac{A \quad B}{A \wedge B}(\text{Adjunction}) \qquad \frac{A}{\Box A}(\text{Necessitation})$$

The least normal logic over **R** is called $\mathbf{R.K}_{\Box\Diamond}$. As a special case, if $\mathbf{R.C}_{\Box\Diamond}$ ($\mathbf{R.K}_{\Box\Diamond}$) satisfies

(R18) $\Diamond A \leftrightarrow \sim \Box \sim A$,

then we call it **R.C** (**R.K**). For **R.C**, we may omit the axiom (R15) and the ($\Diamond$-monotonicity) rule. For **R.K**, we may omit both the axioms (R15) and (R17).

Formulas $\Box^n A$ and $\Diamond^n A$ stand for

$$\underbrace{\Box \cdots \Box}_{n} A \text{ and } \underbrace{\Diamond \cdots \Diamond}_{n} A,$$

respectively.

It is easy to see that every normal logic over **R** is a regular logic over **R**. The following fact is easily shown.

Theorem 3.1

1. *The following formulas are theorems of any regular logic over* **R**, *for* $n \geq 0$*:*

$$\Box^n A \wedge \Box^n B \to \Box^n (A \wedge B), \qquad \Diamond^n (A \vee B) \to \Diamond^n A \vee \Diamond^n B.$$

2. *The following inferences are admissible in any regular logic over* **R**, *for* $m > 0, n \geq 0$*:*

$$\frac{A_1 \wedge \cdots \wedge A_m \to B}{\Box^n A_1 \wedge \cdots \wedge \Box^n A_m \to \Box^n B} \qquad \frac{A \to B_1 \vee \cdots \vee B_m}{\Diamond^n A \to \Diamond^n B_1 \vee \cdots \vee \Diamond^n B_m}.$$

In the following, we assume that $\mathbf{L}_B$ denotes always any of $\mathbf{R.C}_{\Box\Diamond}$, $\mathbf{R.K}_{\Box\Diamond}$, $\mathbf{R.C}$ and $\mathbf{R.K}$, and $\mathbf{L}$ denotes any regular logic over $\mathbf{R}$. Further, we write $\mathbf{L} \vdash A$ when A is a theorem of $\mathbf{L}$.

These four basic logics are pairwise distinct. Since there is no theorem of the form $\Box A$ in $\mathbf{R.C}_{\Box\Diamond}$, it follows that $\mathbf{R.C}_{\Box\Diamond}$ and $\mathbf{R.K}_{\Box\Diamond}$ are distinct. We can say the same thing between $\mathbf{R.C}$ and $\mathbf{R.K}$. It is clear that $\Diamond A \to\sim \Box \sim A$ is not a theorem of $\mathbf{R.K}_{\Box\Diamond}$, so $\mathbf{R.K}_{\Box\Diamond}$ and $\mathbf{R.K}$ are distinct. Similarly, we see that $\mathbf{R.C}_{\Box\Diamond}$ and $\mathbf{R.C}$ are distinct.

3.2 Models

In this section we present models for our logics and prove soundness. Further, we consider the truth-preserving operations of frames. Our models are obtained by extending the $\mathbf{R}$-model introduced by R.Routley and R.K.Meyer ([47]).

An $\mathbf{R}$-*frame* is defined as in Section 2.2. We define frames for relevant modal logics introduced in Section 3.1. An $\mathbf{R.C}_{\Box\Diamond}$-*frame* is a 6-tuple $\mathcal{F} = \langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$, where $\langle O, W, R, ^* \rangle$ is an $\mathbf{R}$-frame, and both $S_\Box$ and $S_\Diamond$ are binary relations on W. An $\mathbf{R.C}_{\Box\Diamond}$-*frame* $\langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ satisfies the following postulates for all $a, b, c \in W$:

(p7) if $a \leq b$ and $S_\Box bc$ then $S_\Box ac$

(p8) if $a \leq b$ and $S_\Diamond ac$ then $S_\Diamond bc$.

An $\mathbf{R.K}_{\Box\Diamond}$-frame $\langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ is an $\mathbf{R.C}_{\Box\Diamond}$-frame satisfying the following postulates for all $a, b, c, d \in W$:

(p9) if $a \in O$ and $S_\Box ab$ then $b \in O$

(p10) if $Rabd$ and $S_\Box dc$ then there exist $a', b' \in W$ such that $S_\Box aa'$, $S_\Box bb'$ and $Ra'b'c$

(p11) if $Rabc$ and $S_\Diamond ad$ then there exist $b', c' \in W$ such that $Rdb'c'$, $S_\Box bb'$ and $S_\Diamond cc'$.

An $\mathbf{R.C}$-frame ($\mathbf{R.K}$-frame) $\langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ is an $\mathbf{R.C}_{\Box\Diamond}$-frame ($\mathbf{R.K}_{\Box\Diamond}$-frame) satisfying the following postulates for all $a, b \in W$:

(p12) $S_\Diamond ab$ iff $S_\Box a^* b^*$.

When a frame satisfies postulate (p12), $S_\Diamond$ can be expressed by using $S_\Box$ and *.

For a given $\mathbf{R.C}_{\Box\Diamond}$-frame $\langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$, define binary relations $S^n_\Box$ and $S^n_\Diamond$ on W for each $n \geq 0$ as follows. For all $a, b \in W$:

($\Box$i) $S^0_\Box ab$ iff $a \leq b$,

($\Box$ii) for $n > 0$, $S^n_\Box ab$ iff there exists $c \in W$ such that $S^{n-1}_\Box ac$ and $S_\Box cb$,

($\Diamond$i) $S^0_\Diamond ab$ iff $b \leq a$,

($\Diamond$ii) for $n > 0$, $S^n_\Diamond ab$ iff there exists $c \in W$ such that $S^{n-1}_\Diamond ac$ and $S_\Diamond cb$.

We call a 7-tuple $\mathcal{M} = \langle O, W, R, S_\Box, S_\Diamond, ^*, V \rangle$ an $\mathbf{L}_B$-*model* on an $\mathbf{L}_B$-frame $\mathcal{F} = \langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ (we simply say $\mathbf{L}_B$-model), where $\mathcal{F}$ is an $\mathbf{L}_B$-frame and V is a mapping from Prop to 2^W, called a *valuation* on $\mathcal{F}$, which satisfies the following *hereditary condition*. For all $a, b \in W$ and all $p \in \mathsf{Prop}$:

$$\text{if } a \leq b \text{ and } a \in V(p) \text{ then } b \in V(p).$$

Given an $\mathbf{L}_B$-model $\langle O, W, R, S_\Box, S_\Diamond, ^*, V \rangle$, for $a \in W$ and $A \in \mathsf{Wff}$, a relation $\models$ between W and Wff is defined inductively as follows:

i. for any $p \in \mathsf{Prop}$, $a \models p$ iff $a \in V(p)$

ii. $a \models A \wedge B$ iff $a \models A$ and $a \models B$

iii. $a \models A \vee B$ iff $a \models A$ or $a \models B$

iv. $a \models A \to B$ iff for all $b, c \in W$, if $Rabc$ and $b \models A$, then $c \models B$

v. $a \models \sim A$ iff $a^* \not\models A$

vi. $a \models \Box A$ iff for any $b \in W$, if $S_\Box ab$, then $b \models A$

vii. $a \models \Diamond A$ iff there exists $b \in W$ such that $S_\Diamond ab$ and $b \models A$,

where $a \not\models A$ means that $a \models A$ does not hold. Note that in each model on a frame satisfying (p12) we have $a \models \Diamond A$ iff there exists $b \in W$ such that $S_\Box a^* b^*$ and $b \models A$. It is easy to see that for $n \geq 0$,

1. $a \models \Box^n A$ iff for any $b \in W$, if $S^n_\Box ab$ then $b \models A$
2. $a \models \Diamond^n A$ iff there exists $b \in W$ such that $S^n_\Diamond ab$ and $b \models A$

Then by induction on the length of the formula A, we can show the following "hereditary lemma".

Lemma 3.2 *Let $\langle O, W, R, S_\Box, S_\Diamond, ^*, V\rangle$ be an $\mathbf{L}_B$-model. For all $a, b \in W$ and all $A \in$ Wff, if $a \leq b$ and $a \models A$ then $b \models A$.*

Let $\mathcal{M} = \langle O, W, R, S_\Box, S_\Diamond, ^*, V\rangle$ be an $\mathbf{L}_B$-model on an $\mathbf{L}_B$-frame $\mathcal{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*\rangle$, $A \in \mathsf{Wff}$, and $\mathcal{C}_\mathcal{F}$ be a class of $\mathbf{L}_B$-frames. Then we say

(a) A *holds in* $\mathcal{M}$ iff $a \models A$ for every world $a \in O$,

(b) A is *valid* in an $\mathbf{L}_B$-frame $\mathcal{F}$ iff A holds in every $\mathbf{L}_B$-model $\mathcal{M}$ on $\mathcal{F}$,

(c) $\mathbf{L}_B$ is *sound with respect to* $\mathcal{C}_\mathcal{F}$ iff A is valid in every $\mathcal{F} \in \mathcal{C}_\mathcal{F}$ for all theorems A of $\mathbf{L}_B$,

(d) $\mathbf{L}_B$ is *complete with respect to* $\mathcal{C}_\mathcal{F}$ iff A is a theorem of $\mathbf{L}_B$ for every formula A valid in all $\mathcal{F} \in \mathcal{C}_\mathcal{F}$,

(e) $\mathbf{L}_B$ *is determined by* $\mathcal{C}_\mathcal{F}$ iff $\mathbf{L}_B$ is both sound and complete with respect to $\mathcal{C}_\mathcal{F}$.

By induction on the length of proof in $\mathbf{L}_B$, we can show soundness of $\mathbf{L}_B$ easily.

Theorem 3.3 $\mathbf{L}_B$ *is sound with respect to the class of* $\mathbf{L}_B$*-frames.*

An $\mathbf{L}_B$-frame in which all theorems of $\mathbf{L}$ are valid is called an $\mathbf{L}$-*frame.*

Next, we deal with the truth-preserving operations on relevant modal frames: subframes, relevant p-morphisms and disjoint unions. Below, let $\mathcal{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*\rangle$ and $\mathcal{F}' = \langle O', W', R', S'_\Box, S'_\Diamond, ^{*\prime}\rangle$ be $\mathbf{L}$-frames.

$\mathcal{F}'$ is called a *subframe* of $\mathcal{F}$ if (a) $W' \subseteq W$ satisfies

1. $a \in W'$ & $Rabc \Rightarrow b, c \in W'$

2. $c \in W'$ & $Rabc \Rightarrow a, b \in W'$
3. $a \in W'$ & $S_\Box ab \Rightarrow b \in W'$
4. $a \in W'$ & $S_\Diamond ab \Rightarrow b \in W'$
5. $a \in W' \Rightarrow a^* \in W'$

for all $a, b, c \in W$, (b) $R', S'_\Box, S'_\Diamond$ and $^{*\prime}$ are the restrictions of $R, S_\Box, S_\Diamond$ and *, respectively, to W', and (c) $O' \subseteq W'$. A subframe $\mathcal{F}'$ is a *generated subframe* of $\mathcal{F}$ if $W' \subseteq W$ is upward closed.

Theorem 3.4 *Let $\mathcal{F}'$ be a generated subframe of an* **L***-frame $\mathcal{F}$.*

1. *Suppose that a valuation V on $\mathcal{F}$ and a valuation V' on $\mathcal{F}'$ satisfies $a \in V(p)$ iff $a \in V'(p)$, for all $p \in \mathsf{Prop}$ and $a \in W'$. Then*

$$a \models A \quad \textit{iff} \quad a \models' A, \qquad \textit{for all } a \in W' \textit{ and } A \in \mathsf{Wff}.$$

2. *If A is valid in $\mathcal{F}$, then A is valid in $\mathcal{F}'$.*

Proof.

1. By induction on the length of A. Take any $a \in W'$.
   (a) A is of the form p $(p \in \mathsf{Prop})$. It is clear.
   (b) A is of the form $B \wedge C$.

$$\begin{array}{lll} a \models B \wedge C & \text{iff} & a \models B \ \& \ a \models C \\ & \text{iff} & a \models' B \ \& \ a \models' C \quad \text{(induction hypotheses)} \\ & \text{iff} & a \models' B \wedge C. \end{array}$$

   (c) A is of the form $B \vee C$. Similar to (b).
   (d) A is of the form $B \to C$.
   The 'if' part is proved as follows. Suppose that $Rabc$ and $b \models B$. By the definition of subframes, $b, c \in W'$. So, we have $R'abc$ and by the hypothesis of induction, $b \models' B$. Thus, $c \models' C$. By the hypothesis of induction again, $c \models C$, which is the desired result.
   The 'only if' part is proved as follows. Suppose that $R'abc$ and $b \models' B$. By the hypothesis of induction, $b \models B$. It is obvious that $Rabc$, so $c \models C$. Since $c \in W'$, $c \models' C$ by the hypothesis of induction. This is the desired result.
   (e) A is of the form $\sim B$.
   By the definition of subframes and the hypothesis of induction,

$$a \models \sim B \quad \text{iff} \quad a^* \not\models B \quad \text{iff} \quad a^* \not\models' B \quad \text{iff} \quad a \models' \sim B.$$

   (f) A is of the form $\Box B$.
   The 'if' part is proved as follows. Suppose that $S_\Box ab$. By the definition of subframes, $b \in W'$, and so $S'_\Box ab$. Then we have $b \models' B$. By the hypothesis of induction, $b \models B$, which is the desired result.
   The 'only if' part is proved as follows. Suppose that $S'_\Box ab$. Then it is clear that $S_\Box ab$. So we have $b \models B$. Since $b \in W'$, $b \models' B$ by the hypothesis of induction. This is the desired result.

(g) A is of the form $\Diamond B$.

The 'if' part is proved as follows. From the assumption, there exists $b \in W'$ such that $S'_\Diamond ab$ and $b \models' B$. By the hypothesis of induction, $b \models B$. It is clear that $S_\Diamond ab$, so $a \models \Diamond B$.

The 'only if' part is proved as follows. From the assumption, there exists $b \in W$ such that $S_\Diamond ab$ and $b \models B$. By the definition of subframes, $b \in W'$, and so $S'_\Diamond ab$. By the hypothesis of induction, $b \models' B$. Thus, $a \models' \Diamond B$.

2. Suppose that A is not valid in $\mathcal{F}'$. Then there exists a valuation V' on $\mathcal{F}'$ and $a \in O'$ such that $a \not\models' A$. Now define a valuation V on $\mathcal{F}$ by $b \in V(p)$ iff $b \in V'(p)$, for all $p \in \mathsf{Prop}$ and $b \in W'$, and by $b \notin V(p)$, for all $p \in \mathsf{Prop}$ and $b \notin W'$. By 1 and the definition of subframes, we have $a \in O$ and $a \not\models A$. Therefore, A is not valid in $\mathcal{F}$. ■

Next, we introduce relevant p-morphisms. For non-modal part, we refer to [58]. A mapping $f : W \to W'$ is a *relevant p-morphism* from $\mathcal{F}$ to $\mathcal{F}'$ if it is a surjection satisfying the following conditions. For all $a, b, c \in W$ and $a', b', c' \in W'$,

(m1) $Rabc \Rightarrow R' f(a) f(b) f(c)$

(m2) $R'a'b'f(c) \Rightarrow \exists a \in W \exists b \in W (Rabc \;\&\; a' \leq' f(a) \;\&\; b' \leq' f(b))$

(m3) $R' f(a) b' c' \Rightarrow \exists b \in W \exists c \in W (Rabc \;\&\; b' \leq' f(b) \;\&\; f(c) \leq' c')$

(m4) $S_\Box ab \Rightarrow S'_\Box f(a) f(b)$

(m5) $S'_\Box f(a) b' \Rightarrow \exists b \in W (S_\Box ab \;\&\; f(b) \leq' b')$

(m6) $S_\Diamond ab \Rightarrow S'_\Diamond f(a) f(b)$

(m7) $S'_\Diamond f(a) b' \Rightarrow \exists b \in W (S_\Diamond ab \;\&\; b' \leq' f(b))$

(m8) $f(a^*) = (f(a))^{*'}$

(m9) $f^{-1}(O') = O$.

Theorem 3.5 *Let f be a relevant p-morphism from $\mathcal{F}$ to $\mathcal{F}'$.*

1. Suppose that a valuation V on $\mathcal{F}$ and a valuation V' on $\mathcal{F}'$ satisfies $a \in V(p)$ iff $f(a) \in V'(p)$, for all $p \in \mathsf{Prop}$ and $a \in W$. Then

$$a \models A \quad \textit{iff} \quad f(a) \models' A, \qquad \textit{for all } A \in \mathsf{Wff} \textit{ and } a \in W.$$

2. If A is valid in $\mathcal{F}$, then A is valid in $\mathcal{F}'$.

Proof.

1. By induction on the length of A.

(a) A is of the form p ($p \in \mathsf{Prop}$). It is clear.

(b) A is of the form $B \wedge C$.

$$\begin{array}{lll} a \models B \wedge C & \text{iff} & a \models B \;\&\; a \models C \\ & \text{iff} & f(a) \models' B \;\&\; f(a) \models' C \quad \text{(induction hypotheses)} \\ & \text{iff} & f(a) \models' B \wedge C. \end{array}$$

(c) A is of the form $B \vee C$. As in (b).

(d) A is of the form $B \to C$.

The 'if' part is proved as follows. Suppose that $Rabc$ and $b \models B$. By (m1), we have $R'f(a)f(b)f(c)$. Further, $f(b) \models' B$ by the hypothesis of induction. From the assumption, $f(c) \models' C$. By the hypothesis of induction again, we have $c \models C$, which is the desired result.

The 'only if' part is proved as follows. Suppose that $R'f(a)b'c'$ and $b' \models' B$. By (m3), there exist $b, c \in W$ such that $Rabc$, $b' \leq' f(b)$ and $f(c) \leq' c'$. By the hereditariness, $f(b) \models' B$, so $b \models B$ by the hypothesis of induction. From the hypothesis, $c \models C$, so $f(c) \models' C$ by the hypothesis of induction again. Again, by the hereditariness, we have $c' \models' C$, which is the desired result.

(e) A is of the form $\sim B$.

$$\begin{array}{rll} a \models \sim B & \text{iff} \;\; a^* \not\models B & \\ & \text{iff} \;\; f(a^*) \not\models' B & \text{(induction hypothesis)} \\ & \text{iff} \;\; (f(a))^{*'} \not\models' B & \text{(by (m8))} \\ & \text{iff} \;\; f(a) \models' \sim B & . \end{array}$$

(f) A is of the form $\Box B$.

The 'if' part is proved as follows. Suppose that $S_\Box ab$. Then $S'_\Box f(a)f(b)$ by (m4). From the assumption, we have $f(b) \models' B$. By the hypothesis of induction, $b \models B$, which is the desired result.

The 'only if' part is proved as follows. Suppose that $S'_\Box f(a)b'$. By (m5), there exists $b \in W$ such that $S_\Box ab$ and $f(b) \leq' b'$. From the assumption, $b \models B$, so $f(b) \models' B$ by the hypothesis of induction. By the hereditariness, we have $b' \models' B$, which is the desired result.

(g) A is of the form $\Diamond B$.

The 'if' part is proved as follows. From the assumption, there exists $b' \in W'$ such that $S'_\Diamond f(a)b'$ and $b' \models' B$. By (m7), there exists $b \in W$ such that $S_\Diamond ab$ and $b' \leq' f(b)$. By the hereditariness, $f(b) \models' B$. By the hypothesis of induction, we have $b \models B$. Hence $a \models \Diamond B$.

The 'only if' part is proved as follows. From the assumption, there exists $b \in W$ such that $S_\Diamond ab$ and $b \models B$. By (m6), $S'_\Diamond f(a)f(b)$, and by the hypothesis of induction, $f(b) \models' B$. Hence $f(a) \models' \Diamond B$.

2. Suppose that A is not valid in $\mathcal{F}'$. Then there exists a valuation V' on $\mathcal{F}'$ and $a \in O'$ such that $a \not\models' A$. By (m9), there exists $b \in O$ such that $a = f(b)$. Now define a valuation V on $\mathcal{F}$ by $c \in V(p)$ iff $f(c) \in V'(p)$, for all $p \in \mathsf{Prop}$ and $c \in W$. Then by 1, we have $b \not\models A$. Therefore, A is not valid in $\mathcal{F}$. ∎

The *disjoint union* of a family $\{\mathcal{F}_i = \langle O_i, W_i, R_i, S_{\Box i}, S_{\Diamond i}, g_i \rangle \mid i \in I\}$ of pairwise disjoint frames is the frame $\sum_{i \in I} \mathcal{F}_i = \langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ where

$$O = \bigcup_{i \in I} O_i, \; W = \bigcup_{i \in I} W_i, \; R = \bigcup_{i \in I} R_i, \; S_\Box = \bigcup_{i \in I} S_{\Box i}, \; S_\Diamond = \bigcup_{i \in I} S_{\Diamond i}, \; ^* = \bigcup_{i \in I} g_i.$$

Note that every $\mathcal{F}_i$ is a generated subframe of $\sum_{i \in I} \mathcal{F}_i$.

Proposition 3.6 *$\sum_{i\in I}\mathcal{F}_i = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ defined above is a frame.*

Proof.

We consider the case of $\mathbf{R.C}_{\Box\Diamond}$. Other cases are proved similarly. The way of proof is similar, so we prove (p3). Suppose that $Rabc$ and $Rca'b'$. Then there exist $i, j \in I$ such that $R_i abc$ and $R_j ca'b'$. Since each frame is disjoint, $i = j$, say i. Then there exists $d \in W_i$ such that $R_i aa'd$ and $R_i dbb'$. Thus there exists $d \in W$ such that $Raa'd$ and $Rdbb'$.

■

Theorem 3.7 *Let $\sum_{i\in I}\mathcal{F}_i$ be the disjoint union of a family $\{\mathcal{F}_i \mid i \in I\}$ of* **L***-frames.*

1. *Suppose that a valuation V_i on $\mathcal{F}_i$ for all $i \in I$ and a valuation V on $\sum_{i\in I}\mathcal{F}_i$ satisfy $a \in V_i(p)$ iff $a \in V(p)$, for all $p \in \mathsf{Prop}$ and $a \in W_i, i \in I$. Then*

$$a \models_i A \quad \textit{iff} \quad a \models A, \qquad \textit{for all } A \in \mathsf{Wff} \textit{ and } a \in W_i, i \in I.$$

2. *A is valid in $\mathcal{F}_i$ for all $i \in I$ iff A is valid in $\sum_{i\in I}\mathcal{F}_i$.*

Proof.

Let $\sum_{i\in I}\mathcal{F}_i = \langle O, W, R, S_\Box, S_\Diamond, ^*\rangle$.

1. By induction on the length of A. Take any $a \in \bigcup_{i\in I} W_i$.
   (a) A is of the form p $(p \in \mathsf{Prop})$. It is clear.
   (b) A is of the form $B \wedge C$.

$$\begin{aligned} a \models_i B \wedge C \ &\text{iff} \ \ a \models_i B \ \&\ a \models_i C \\ &\text{iff} \ \ a \models B \ \&\ a \models C \qquad \text{(induction hypotheses)} \\ &\text{iff} \ \ a \models B \wedge C. \end{aligned}$$

   (c) A is of the form $B \vee C$. Similar to (b).
   (d) A is of the form $B \to C$.
   The ‘if’ part is proved as follows. Suppose that $R_i abc$ and $b \models_i B$. Then it is clear that $Rabc$, and we have $b \models B$ by the hypothesis of induction. Thus, we have $c \models C$, so $c \models_i C$ by the hypothesis of induction again. This is the desired result.
   The ‘only if’ part is proved as follows. Suppose that $Rabc$ and $b \models B$. Since each $\mathcal{F}_i$ is pairwise disjoint, $a \in W_j$ for some $j \in I$. By the definition of disjoint unions, we have $b, c \in W_j$ and $R_j abc$. By the hypothesis of induction, $b \models_j B$, so we have $c \models_j C$. By the hypothesis of induction again, $c \models C$, which is the desired result.
   (e) A is of the form $\sim B$.
   By the hypothesis of induction,

$$a \models_i \sim B \quad \text{iff} \quad g_i(a) \not\models_i B \quad \text{iff} \quad a^* \not\models B \quad \text{iff} \quad a \models \sim B.$$

   (f) A is of the form $\Box B$. Similar to (d).

(g) A is of the form $\Diamond B$.

$$\begin{array}{rll} a \models_i \Diamond B & \text{iff} & \exists b \in W_i(S_{\Diamond i}ab \ \& \ b \models_i B) \\ & \text{iff} & \exists b \in W(S_\Diamond ab \ \& \ b \models B) \qquad \text{(induction hypothesis)} \\ & \text{iff} & a \models \Diamond B. \end{array}$$

2. The 'if' part is proved as follows. Suppose that A is not valid in $\sum_{i\in I}\mathcal{F}_i$. Then there exists a valuation V on $\sum_{i\in I}\mathcal{F}_i$ and $a \in \bigcup_{i\in I} O_i$ such that $a \not\models A$. Since $\sum_{i\in I}\mathcal{F}_i$ is a disjoint union, there exists $j \in I$ such that $a \in O_j$ uniquely. Now define a valuation V_j on $\mathcal{F}_j$ by $b \in V_j(p)$ iff $b \in V(p)$, for all $p \in \mathsf{Prop}$ and $b \in W_j$. Then by 1, we have $a \not\models_j A$. Therefore, A is not valid in $\mathcal{F}_j$.

   The 'only if' part is proved as follows. Suppose that A is not valid in $\mathcal{F}_j$ for some $j \in I$. Then there exists a valuation V_j on $\mathcal{F}_j$ and $a \in O_j$ such that $a \not\models_j A$. Now we define a valuation V on $\sum_{i\in I}\mathcal{F}_i$ by $b \in V(p)$ iff $b \in V_i(p)$, for all $p \in \mathsf{Prop}$ and $b \in W_i, i \in I$. Then by 1, we have $a \not\models A$. Further, it is clear that $a \in \bigcup_{i\in I} O_i$, so A is not valid in $\sum_{i\in I}\mathcal{F}_i$. ■

3.3 Completeness

In this section we will prove completeness of $\mathbf{R.C}_{\Box\Diamond}$, $\mathbf{R.K}_{\Box\Diamond}$, $\mathbf{R.C}$ and $\mathbf{R.K}$. Basically, the proof goes in the same way as in section 4.6 of [48]. Here, we introduce some notions and study properties of them. (Note that the terminology is somewhat different from that in [48].)

To define the canonical frame for $\mathbf{L}$, we introduce the following key notions.

- Let $\Sigma \neq \emptyset$ and $\Delta \neq \emptyset$. $\mathbf{L} \vdash \Sigma \to \Delta$ iff there exist $A_1, \cdots, A_m \in \Sigma$ $(m > 0)$ and $B_1, \cdots, B_n \in \Delta$ $(n > 0)$ such that

$$\mathbf{L} \vdash A_1 \wedge \cdots \wedge A_m \to B_1 \vee \cdots \vee B_n.$$

- (Σ, Δ) is an $\mathbf{L}$-*pair* iff (a) $\mathbf{L} \nvdash \Sigma \to \Delta$ and (b) $\Sigma \cup \Delta = \mathsf{Wff}$.
- Σ is an $\mathbf{L}$-*theory* iff (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$, and (b) if $\mathbf{L} \vdash A \to B$ and $A \in \Sigma$ then $B \in \Sigma$.
- For an $\mathbf{L}$-theory Σ,
  - Σ is *regular* iff Σ contains all theorems of $\mathbf{L}$.
  - Σ is *prime* iff $A \vee B \in \Sigma$ implies either $A \in \Sigma$ or $B \in \Sigma$.
- Let $\mathsf{Th}(\mathbf{L})$ be the set of all $\mathbf{L}$-theories. Then a ternary relation R on $\mathsf{Th}(\mathbf{L})$, and binary relations on $S_\Box$ and $S_\Diamond$ on $\mathsf{Th}(\mathbf{L})$, are defined by

$$R\Sigma\Gamma\Delta \quad \text{iff} \quad \text{for any } A, B \in \mathsf{Wff}, \text{ if } A \to B \in \Sigma \text{ and } A \in \Gamma \text{ then } B \in \Delta\ ,$$

$$S_\Box\Sigma\Gamma \quad \text{iff} \quad \text{for any } A \in \mathsf{Wff}, \text{ if } \Box A \in \Sigma \text{ then } A \in \Gamma,$$

$$S_\Diamond\Sigma\Gamma \quad \text{iff} \quad \text{for any } A \in \mathsf{Wff}, \text{ if } A \in \Gamma \text{ then } \Diamond A \in \Sigma.$$

A few comments on the definitions above. It is clear that $\Sigma \cap \Delta = \emptyset$ whenever $\mathbf{L} \not\vdash \Sigma \to \Delta$. Hence we have that if (Σ, Δ) is an $\mathbf{L}$-pair, then for all $A \in \mathsf{Wff}$ either $A \in \Sigma$ or $A \in \Delta$ but not both. It is clear that the set of all theorems of $\mathbf{L}$ is an $\mathbf{L}$-theory. As for binary relations $S^n_\Box$ $(n \geq 0)$ and $S^n_\Diamond$ $(n \geq 0)$ on $\mathbf{L}$-theories, it is easy to see the following:

$$S^n_\Box \Sigma\Gamma \quad \text{iff} \quad \text{for any } A \in \mathsf{Wff}, \text{ if } \Box^n A \in \Sigma \text{ then } A \in \Gamma,$$

$$S^n_\Diamond \Sigma\Gamma \quad \text{iff} \quad \text{for any } A \in \mathsf{Wff}, \text{ if } A \in \Gamma \text{ then } \Diamond^n A \in \Sigma.$$

The following lemmas are essentially proved in [48] (pp.305-318).

Lemma 3.8 *Let* $\mathbf{L}$ *be any regular logic over* $\mathbf{R}$.

1. *If* (Σ, Δ) *is an* $\mathbf{L}$*-pair, then* Σ *is a prime* $\mathbf{L}$*-theory.*
2. *If* $\mathbf{L} \not\vdash \Sigma \to \Delta$*, then there exist* $\Sigma' \supseteq \Sigma$ *and* $\Delta' \supseteq \Delta$ *such that* (Σ', Δ') *is an* $\mathbf{L}$*-pair.*
3. *Suppose that* Σ *is an* $\mathbf{L}$*-theory and* Δ *is a set of formulas closed under disjunction such that* $\Sigma \cap \Delta = \emptyset$*. Then there exists a prime* $\mathbf{L}$*-theory* $\Sigma' \supseteq \Sigma$ *such that* $\Sigma' \cap \Delta = \emptyset$*.*
4. *If* A *is not a theorem of* $\mathbf{L}$*, then there exists a regular prime* $\mathbf{L}$*-theory* Π *such that* $A \notin \Pi$*.*
5. *Suppose that* Σ *and* Γ *are* $\mathbf{L}$*-theories and* Δ *is a prime* $\mathbf{L}$*-theory such that* $R\Sigma\Gamma\Delta$*. Then there exists a prime* $\mathbf{L}$*-theory* $\Sigma' \supseteq \Sigma$ *such that* $R\Sigma'\Gamma\Delta$*.*
6. *Suppose that* Σ *and* Γ *are* $\mathbf{L}$*-theories and* Δ *is a prime* $\mathbf{L}$*-theory such that* $R\Sigma\Gamma\Delta$*. Then there exists a prime* $\mathbf{L}$*-theory* $\Gamma' \supseteq \Gamma$ *such that* $R\Sigma\Gamma'\Delta$*.*
7. *Suppose that there are a prime* $\mathbf{L}$*-theory* Σ *and* $\mathbf{L}$*-theories* Γ *and* Δ *such that* $R\Sigma\Gamma\Delta$ *and* $D \notin \Delta$*. Then there exist prime* $\mathbf{L}$*-theories* Γ' *and* Δ' *such that* $\Gamma \subseteq \Gamma', D \notin \Delta'$ *and* $R\Sigma\Gamma'\Delta'$*.*
8. *If* Σ *is a prime* $\mathbf{L}$*-theory such that* $C \to D \notin \Sigma$*, then there exist prime* $\mathbf{L}$*-theories* Γ' *and* Δ' *such that* $R\Sigma\Gamma'\Delta', C \in \Gamma'$ *and* $D \notin \Delta'$*.*

Lemma 3.9 *Suppose that* $S_\Box\Sigma\Gamma$ *and* $C \notin \Gamma$ *for a prime* $\mathbf{L}$*-theory* Σ *and an* $\mathbf{L}$*-theory* Γ*. Then there exists a prime* $\mathbf{L}$*-theory* Γ' *such that* $S_\Box\Sigma\Gamma'$ *and* $C \notin \Gamma'$*.*

Proof.
Let Ψ be the closure of $\{C\}$ under disjunction. Then it is clear that $\Gamma \cap \Psi = \emptyset$. By 3 of Lemma 3.8, there exists a prime $\mathbf{L}$-theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \cap \Psi = \emptyset$. Then it is easy to check that $S_\Box\Sigma\Gamma'$ and $C \notin \Gamma'$. ■

Lemma 3.10 *If* Σ *is a prime* $\mathbf{L}$*-theory such that* $\Box C \notin \Sigma$*, then there exist prime* $\mathbf{L}$*-theory* Γ' *such that* $S_\Box\Sigma\Gamma'$ *and* $C \notin \Gamma'$*.*

Proof.

Let $\Gamma = \{A \mid \Box A \in \Sigma\}$. First we check that Γ is an **L**-theory. Suppose that $A, B \in \Gamma$. Then $\Box A, \Box B \in \Sigma$. By the hypothesis, we can see that $\Box(A \wedge B) \in \Sigma$. Thus $A \wedge B \in \Gamma$. Next suppose that $\mathbf{L} \vdash A \to B$ and $A \in \Gamma$. Then $\mathbf{L} \vdash \Box A \to \Box B$ and $\Box A \in \Sigma$. By the hypothesis, we see that $\Box B \in \Sigma$, so that $B \in \Gamma$. It is obvious that $S_{\Box}\Sigma\Gamma$ and $C \notin \Gamma$. By Lemma 3.9, there exists a prime **L**-theory Γ' such that $S_{\Box}\Sigma\Gamma'$ and $C \notin \Gamma'$. ■

Using from Lemmas 3.8 through 3.10, we will be able to prove completeness theorem for $\mathbf{L}_B$. First, we define the *canonical* **L**-*model* $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, V_c\rangle$ as follows:

- W_c is the set of all prime **L**-theories
- O_c is the set of all regular prime **L**-theories
- R_c is the ternary relation R restricted to W_c
- $S_{\Box c}$ is the binary relation $S_{\Box}$ restricted to W_c
- $S_{\Diamond c}$ is the binary relation $S_{\Diamond}$ restricted to W_c
- g_c is the unary operation on W_c defined by $g_c(\Sigma) = \{A \mid \ \sim A \notin \Sigma\}$
- V_c is defined by

$$\text{for all } p \in \mathsf{Prop} \text{ and } \Sigma \in W_c, \quad \Sigma \in V_c(p) \ \text{ iff } \ p \in \Sigma.$$

As concerns V_c, $\models_c$ denotes the relation determined uniquely from V_c.

We call $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c\rangle$ the *canonical* **L**-*frame*. The relations $\leq_c$, $S^n_{\Box c}$ and $S^n_{\Diamond c}$ are defined as in Section 3.2. Note that if $\Sigma \in W_c$ then $g_c(\Sigma) \in W_c$.

Lemma 3.11 *Let $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c\rangle$ be the canonical* **L**-*frame. Then for all $\Sigma, \Gamma \in W_c$ and $n \geq 0$,*

1. $\Sigma \leq_c \Gamma$ *iff* $\Sigma \subseteq \Gamma$,
2. $S^n_{\Box c}\Sigma\Gamma$ *iff for all* $A \in \mathsf{Wff}$, *if* $\Box^n A \in \Sigma$ *then* $A \in \Gamma$,
3. $S^n_{\Diamond c}\Sigma\Gamma$ *iff for all* $A \in \mathsf{Wff}$, *if* $A \in \Gamma$ *then* $\Diamond A \in \Sigma$.

Proof.

1. The 'only if' part is proved easily, so we prove the 'if' part. Let Θ be the set of all theorems of **L**. It is clear that Θ is an **L**-theory. Suppose that $A \to B \in \Theta$ and $A \in \Sigma$. Since Σ is an **L**-theory, $B \in \Sigma$, and hence $B \in \Gamma$ by the hypothesis. So $R\Theta\Sigma\Delta$. By 5 of Lemma 3.8, there exists a prime **L**-theory $\Pi \supseteq \Theta$ such that $R\Pi\Sigma\Delta$. It is obvious that Π is regular.

2. The proof is by induction on n.

   For $n = 0$, the claim trivially holds by the definition of $S_{\Box c}$ and 1. So, we suppose that $n > 0$. The 'if' part is proved as follows. Let $\Delta = \{A \mid \Box^{n-1}A \in \Sigma\}$ and

$\Xi = \{\Box A \mid A \notin \Gamma\}$. To show that $\mathbf{L} \nvdash \Delta \to \Xi$, assume the contrary. Then there exist $\Box^{n-1}A_1, \cdots, \Box^{n-1}A_k \in \Sigma$ and $B_1, \cdots, B_l \notin \Gamma$ such that

$$\mathbf{L} \vdash A_1 \wedge \cdots \wedge A_k \to \Box B_1 \vee \cdots \vee \Box B_l.$$

It follows that

$$\mathbf{L} \vdash \Box^{n-1}A_1 \wedge \cdots \wedge \Box^{n-1}A_k \to \Box^n(B_1 \vee \cdots \vee B_l).$$

Since Σ is an $\mathbf{L}$-theory, $\Box^n(B_1 \vee \cdots \vee B_l) \in \Sigma$. By the assumption, we see that $B_1 \vee \cdots \vee B_l \in \Gamma$, which contradicts the primeness of Γ. Therefore $\mathbf{L} \nvdash \Delta \to \Xi$. By 2 of Lemma 3.8, there exist $\Delta' \supseteq \Delta$ and $\Xi' \supseteq \Xi$ such that (Δ', Ξ') is an $\mathbf{L}$-pair. By 1 of Lemma 3.8, Δ' is a prime $\mathbf{L}$-theory. Further, by the hypothesis of induction, we see that $S_\Box^{n-1}\Sigma\Delta$, and thus $S_\Box^{n-1}{}_c\Sigma\Delta'$. Also, it is clear that $S_{\Box c}\Delta'\Gamma$. Therefore $S_{\Box c}^n\Sigma\Gamma$ by the definition of $S_{\Box c}^n$.

The 'only if' part is proved as follows. Suppose that $S_{\Box c}^n\Sigma\Gamma$ and $\Box^n A \in \Sigma$. From the definition of $S_{\Box c}^n$, there exists $\Delta \in W_c$ such that $S_\Box^{n-1}{}_c\Sigma\Delta$ and $S_{\Box c}\Delta\Gamma$. By the hypothesis of induction, we see that $\Box A \in \Delta$. Hence $A \in \Gamma$.

3. Analogous to 2. ■

Lemma 3.12 *The canonical $\mathbf{L}_B$-frame $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c\rangle$ is an $\mathbf{L}_B$-frame.*

Proof.

1. Case in which $\mathbf{L}_B$ is $\mathbf{R.C}_{\Box\Diamond}$.

   To prove this lemma, it is sufficient to show that $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c\rangle$ satisfies all the postulates for an $\mathbf{R.C}_{\Box\Diamond}$-frame. For (p1), (p4), (p7) and (p8), it is obvious by 1 of Lemma 3.11, so we show other postulates.

   (p2) Suppose that $A \to B \in \Sigma$ and $A \in \Sigma$. Since Σ is an $\mathbf{R.C}_{\Box\Diamond}$-theory, $B \in \Sigma$.

   (p3) Suppose that $R_c\Sigma\Gamma\Lambda$ and $R_c\Lambda\Delta\Xi$. Let $\Psi = \{B \mid \exists A \in \Delta(A \to B \in \Sigma)\}$. First, suppose that $B_1, B_2 \in \Psi$. Then there exist $A_1, A_2 \in \Delta$ such that $A_1 \to B_1 \in \Sigma$ and $A_2 \to B_2 \in \Sigma$. Since Σ is an $\mathbf{R.C}_{\Box\Diamond}$-theory, $A_1 \wedge A_2 \to B_1 \wedge B_2 \in \Sigma$. Hence we have $B_1 \wedge B_2 \in \Psi$. Next, suppose that $B \in \Psi$ and $\mathbf{R.C}_{\Box\Diamond} \vdash B \to C$. Then there exists $A \in \Delta$ such that $A \to B \in \Sigma$. Since Σ is an $\mathbf{R.C}_{\Box\Diamond}$-theory, $A \to C \in \Sigma$. Hence we have $C \in \Psi$. Thus, Ψ is an $\mathbf{R.C}_{\Box\Diamond}$-theory.

   To see that $R\Psi\Gamma\Xi$, so suppose that $A \to B \in \Psi$ and $A \in \Gamma$. Then there exists $C \in \Delta$ such that $C \to (A \to B) \in \Sigma$. Since Σ is an $\mathbf{R.C}_{\Box\Diamond}$-theory, $A \to (C \to B) \in \Sigma$, so $C \to B \in \Lambda$. Hence $B \in \Xi$.

   By 5 of Lemma 3.8, there exists a prime $\mathbf{R.C}_{\Box\Diamond}$-theory $\Psi' \supseteq \Psi$ such that $R\Psi'\Gamma\Xi$. Then it is obvious that $R_c\Sigma\Delta\Psi'$ and $R_c\Psi'\Gamma\Xi$.

   (p5) Suppose that $R_c\Sigma\Gamma\Delta$. To see $R_c\Sigma g_c(\Delta)g_c(\Gamma)$, suppose that $A \to B \in \Sigma$ and $A \in g_c(\Delta)$. Then $\sim B \to \sim A \in \Sigma$ and $\sim A \notin \Delta$, so $\sim B \notin \Gamma$. Hence $B \in g_c(\Gamma)$, which is the desired result.

(p6) By (R13) and 5 of Theorem 2.1,

$$A \in g_c(g_c(\Sigma)) \text{ iff } \sim A \notin g_c(\Sigma) \text{ iff } \sim\sim A \in \Sigma \text{ iff } A \in \Sigma.$$

2. Case in which $\mathbf{L}_B$ is $\mathbf{R.K}_{\Box\Diamond}$.

   It is sufficient to show that $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c\rangle$ satisfies all the postulates (p9) through (p11) for an $\mathbf{R.K}_{\Box\Diamond}$-frame.

   (p9) Suppose that there exists $\Pi \in O_c$ such that $S_{\Box c}\Pi\Gamma$. It is sufficient to show that Γ is regular. Let A be a theorem of $\mathbf{R.K}_{\Box\Diamond}$. Then $\mathbf{R.K}_{\Box\Diamond} \vdash \Box A$. Since Π is regular, $\Box A \in \Pi$. By the assumption $A \in \Gamma$, and thus Γ is a regular prime $\mathbf{R.K}_{\Box\Diamond}$-theory.

   (p10) Suppose that $R_c\Sigma\Gamma\Xi$ and $S_{\Box c}\Xi\Delta$. Let $\Lambda = \{A \mid \Box A \in \Sigma\}$ and $\Psi = \{A \mid \Box A \in \Gamma\}$. First we have to show that Λ and Ψ are $\mathbf{R.K}_{\Box\Diamond}$-theories. This is similar to the proof of Lemma 3.10. Then, it is clear that $S_{\Box}\Sigma\Lambda$ and $S_{\Box}\Gamma\Psi$. To show $R\Lambda\Psi\Delta$, suppose that $A \to B \in \Lambda$ and $A \in \Psi$. Then $\Box(A \to B) \in \Sigma$ and $\Box A \in \Gamma$. Since Σ is an $\mathbf{R.K}_{\Box\Diamond}$-theory, $\Box A \to \Box B \in \Sigma$. By the assumption $\Box B \in \Xi$, hence $B \in \Delta$ by the assumption again.

   By 5 and 6 of Lemma 3.8, there exist prime $\mathbf{R.K}_{\Box\Diamond}$-theories $\Lambda' \supseteq \Lambda$ and $\Psi' \supseteq \Psi$ such that $R\Lambda'\Psi'\Delta$. Then it is clear that $S_{\Box}\Sigma\Lambda'$ and $S_{\Box}\Gamma\Psi'$. Hence there exist $\Lambda', \Psi' \in W_c$ such that $S_{\Box c}\Sigma\Lambda'$, $S_{\Box c}\Gamma\Psi'$ and $R_c\Lambda'\Psi'\Delta$.

   (p11) Suppose that $R_c\Sigma\Gamma\Delta$ and $S_{\Diamond c}\Sigma\Lambda$. Let $\Psi = \{A \mid \Box A \in \Gamma\}$. As in Lemma 3.10, Ψ is an $\mathbf{R.K}_{\Box\Diamond}$-theory such that $S_{\Box}\Gamma\Psi$.

   Next, let $\Xi = \{B \mid \exists A \in \Psi(A \to B \in \Lambda)\}$ and $\Xi_1 = \{A \mid \Diamond A \notin \Delta\}$. As in 1, Ξ is an $\mathbf{R.K}_{\Box\Diamond}$-theory such that $R\Lambda\Psi\Xi$. To see that Ξ_1 is closed under disjunction, suppose that $A, B \in \Xi_1$. Then $\Diamond A, \Diamond B \notin \Delta$. By the primeness of Δ, $\Diamond A \vee \Diamond B \notin \Delta$. Since Δ is an $\mathbf{R.K}_{\Box\Diamond}$-theory, $\Diamond(A \vee B) \notin \Delta$, so $A \vee B \in \Xi_1$. This is the desired result.

   Moreover, assume that $B \in \Xi \cap \Xi_1$. Then there exists A such that $A \to B \in \Lambda$, and $\Diamond B \notin \Delta$. By the assumption, we see that $\Box A \in \Gamma$ and $\Diamond(A \to B) \in \Sigma$. Because $\mathbf{R.K}_{\Box\Diamond} \vdash \Diamond(A \to B) \to (\Box A \to \Diamond B)$ and Σ is an $\mathbf{R.K}_{\Box\Diamond}$-theory, $\Box A \to \Diamond B \in \Sigma$. We obtain that $\Diamond B \in \Delta$, which is a contradiction. Hence $\Xi \cap \Xi_1 = \emptyset$.

   By 3 of Lemma 3.8, there exists a prime $\mathbf{R.K}_{\Box\Diamond}$-theory $\Xi' \supseteq \Xi$ such that $\Xi' \cap \Xi_1 = \emptyset$. It is clear that $R\Lambda\Psi\Xi'$, so there exists a prime $\mathbf{R.K}_{\Box\Diamond}$-theory $\Psi' \supseteq \Psi$ such that $R_c\Lambda\Psi'\Xi'$ by 6 of Lemma 3.8. Then it is clear that $S_{\Box c}\Gamma\Psi'$. Further, if $A \in \Xi'$ then $A \notin \Xi_1$, so $\Diamond A \in \Delta$. Hence $S_{\Diamond c}\Delta\Xi'$.

3. Case in which $\mathbf{L}_B$ is $\mathbf{R.C}$ and $\mathbf{R.K}$.

   It is sufficient to show that $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c\rangle$ satisfies the postulate (p12) for an $\mathbf{R.C}$-frame ($\mathbf{R.K}$-frame). First, suppose that $S_{\Diamond c}\Sigma\Gamma$. Moreover suppose that $\Box A \in g_c(\Sigma)$. Then $\sim \Box A \notin \Sigma$, so $\Diamond \sim A \notin \Sigma$ since Σ is an $\mathbf{R.C}$-theory ($\mathbf{R.K}$-theory). From the assumption we see that $\sim A \notin \Gamma$, so that $A \in g_c(\Gamma)$. Therefore $S_{\Box c}g_c(\Sigma)g_c(\Gamma)$.

   Now suppose that $S_{\Box c}g_c(\Sigma)g_c(\Gamma)$. Further suppose that $A \in \Gamma$. Then $\sim A \notin g_c(\Gamma)$, so $\Box \sim A \notin g_c(\Sigma)$ from the assumption. We now see that $\sim \Box \sim A \in \Sigma$, so $\Diamond A \in \Sigma$ since Σ is an $\mathbf{R.C}$-theory ($\mathbf{R.K}$-theory). Therefore $S_{\Diamond c}\Sigma\Gamma$. ∎

Note that the canonical **L**-frame is not necessary an **L**-frame.

Lemma 3.13 *Let* $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, V_c \rangle$ *be the canonical* **L**-*model. For all* $A \in \mathsf{Wff}$ *and* $\Sigma \in W_c$,

$$\Sigma \models_c A \quad \textit{iff} \quad A \in \Sigma.$$

Proof.

We proceed by induction on the length of A.

1. A is of the form p ($p \in \mathsf{Prop}$). It is clear by the definition of V_c.

2. A is of the form $B \wedge C$.

$$\begin{array}{lll} \Sigma \models_c B \wedge C & \text{iff} & \Sigma \models_c B \text{ and } \Sigma \models_c C \\ & \text{iff} & B \in \Sigma \quad \text{and} \quad C \in \Sigma \quad \text{(induction hypotheses)} \\ & \text{iff} & B \wedge C \in \Sigma \qquad\qquad (\Sigma \text{ is an } \mathbf{L}\text{-theory}). \end{array}$$

3. A is of the form $B \vee C$.

$$\begin{array}{lll} \Sigma \models_c B \vee C & \text{iff} & \Sigma \models_c B \text{ or } \Sigma \models_c C \\ & \text{iff} & B \in \Sigma \quad \text{or} \quad C \in \Sigma \quad \text{(induction hypotheses)} \\ & \text{iff} & B \vee C \in \Sigma \qquad (\Sigma \text{ is a prime } \mathbf{L}\text{-theory}). \end{array}$$

4. A is of the form $B \to C$.

   The 'if' part is proved as follows. Suppose that $B \to C \in \Sigma$. Moreover we suppose that $R_c\Sigma\Gamma\Delta$ and $\Gamma \models_c B$. By the hypothesis of induction, we see that $B \in \Gamma$, so $C \in \Delta$. Hence $\Delta \models_c C$ by the hypothesis of induction again. It follows that $\Sigma \models_c B \to C$.

   The 'only if' part is proved as follows. Suppose that $B \to C \notin \Sigma$. Then there exist $\Gamma, \Delta \in W_c$ such that $R_c\Sigma\Gamma\Delta, B \in \Gamma$ and $C \notin \Delta$ by 8 of Lemma 3.8. By the hypotheses of induction, $\Gamma \models_c B$ and $\Delta \not\models_c C$. Hence $\Sigma \not\models_c B \to C$.

5. A is of the form $\sim B$.

$$\begin{array}{lll} \Sigma \models' \sim B & \text{iff} & g_c(\Sigma) \not\models_c B \\ & \text{iff} & B \notin g_c(\Sigma) \quad \text{(induction hypothesis)} \\ & \text{iff} & \sim B \in \Sigma. \end{array}$$

6. A is of the form $\Box B$.

   The 'if' part is proved as follows. Suppose that $\Box B \in \Sigma$. In order to show $\Sigma \models_c \Box B$, suppose that $S_{\Box c}\Sigma\Gamma$. Then $B \in \Gamma$ by the first assumption. By the hypothesis of induction, $\Gamma \models_c B$. This holds for any Γ such that $S_{\Box c}\Sigma\Gamma$. Thus $\Sigma \models\, _c\Box B$.

   The 'only if' part is proved as follows. Suppose that $\Box B \notin \Sigma$. Then there exists a prime **L**-theory Γ such that $S_{\Box c}\Sigma\Gamma$ and $B \notin \Gamma$ by Lemma 3.10. By the hypothesis of induction, $\Gamma \not\models_c B$. Hence $\Sigma \not\models_c \Box B$.

7. A is of the form $\Diamond B$.

   The 'if' part is proved as follows. Suppose that $\Diamond B \in \Sigma$. Let $\Gamma = \{A \mid \mathbf{L} \vdash B \to A\}$ and $\Delta = \{A \mid \Diamond A \notin \Sigma\}$. First we show that Γ is an $\mathbf{L}$-theory. Suppose that $A_1, A_2 \in \Gamma$. Then $\mathbf{L} \vdash B \to A_1$ and $\mathbf{L} \vdash B \to A_2$, so $\mathbf{L} \vdash B \to A_1 \wedge A_2$. Hence $A_1 \wedge A_2 \in \Gamma$. Further suppose that $A_1 \in \Gamma$ and $\mathbf{L} \vdash A_1 \to A_2$. Then $\mathbf{L} \vdash B \to A_1$, so $\mathbf{L} \vdash B \to A_2$. Hence $A_2 \in \Gamma$.

   As in Lemma 3.12, we see that Δ is closed under disjunction. Moreover, assume that $A \in \Gamma \cap \Delta$. Then $\mathbf{L} \vdash B \to A$ and $\Diamond A \notin \Sigma$. Because $\mathbf{L} \vdash \Diamond B \to \Diamond A$ and Σ is an $\mathbf{L}$-theory, $\Diamond B \notin \Sigma$, which contradicts the assumption. Now, by 3 of Lemma 3.8, there exists $\Gamma' \supseteq \Gamma$ such that $\Gamma' \cap \Delta = \emptyset$. Suppose $A \in \Gamma'$. Then $A \notin \Delta$, so $\Diamond A \in \Sigma$. Hence $S_{\Diamond c}\Sigma\Gamma'$. Further it is clear that $B \in \Gamma$. Thus $B \in \Gamma'$. By the hypothesis of induction, $\Gamma' \models_c B$. Therefore $\Sigma \models_c \Diamond B$.

   The 'only if' part is proved as follows. Suppose that $\Sigma \models_c \Diamond B$. Then there exists $\Gamma \in W_c$ such that $S_{\Diamond c}\Sigma\Gamma$ and $\Gamma \models_c B$. By the hypothesis of induction, $B \in \Gamma$. By the definition of $S_{\Diamond c}$, $\Diamond B \in \Sigma$. ■

Now we can state completeness of $\mathbf{L}_B$.

Theorem 3.14 $\mathbf{L}_B$ *is complete with respect to the class of* $\mathbf{L}_B$*-frames.*

Proof.
Suppose that A is not a theorem of $\mathbf{L}_B$, then there exists a regular prime $\mathbf{L}_B$-theory Π such that $A \notin \Pi$ by 4 of Lemma 3.8. By Lemmas 3.12 and 3.13, the canonical $\mathbf{L}_B$-model $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, V_c \rangle$ is an $\mathbf{L}_B$-model. In this model, $\Pi \not\models_c A$ for some $\Pi \in O_c$, which means that A is not valid in an $\mathbf{L}_B$-frame. ■

3.4 Relevant modal matrices

In this section we define relevant modal matrices and show that they characterize relevant modal logics. Further, we consider the truth-preserving operations of matrices. First, we will present relevant matrices in terms of modified form of the ones by Font-Rodríguez (see [17] and [18]).

Relevant matrices and $\mathbf{R}$-matrices have been defined in Section 2.4. In the following, we define relevant modal matrices. A structure $\langle M, \cap, \cup, \to, -, \Box, \Diamond \rangle$ is a *De Morgan modal semigroup* ($\mathbf{R.C}_{\Box\Diamond}$*-algebra*) if the following postulates hold for all $x, y \in M$:

- (A1) $\langle M, \cap, \cup, \to, - \rangle$ is a De Morgan semigroup ($\mathbf{R}$-algebra),
- (A2) $\Box(x \cap y) = \Box x \cap \Box y$,
- (A3) $\Diamond(x \cup y) = \Diamond x \cup \Diamond y$.

It is clear that any De Morgan modal semigroup has the following postulates.

Proposition 3.15 *Let* $\langle M, \cap, \cup, \to, -, \Box, \Diamond \rangle$ *be an* $\mathbf{R.C}_{\Box\Diamond}$*-algebra. Then the following properties hold for all* $x, y \in M$*:*

- (P1) *if* $x \leq y$ *then* $\Box x \leq \Box y$

(P2) *if* $x \leq y$ *then* $\Diamond x \leq \Diamond y$.

Let $\mathbf{M} = \langle M, \cap, \cup, \rightarrow, -, \Box, \Diamond \rangle$. Then we say that

(a) $\langle \mathbf{M}, E \rangle$ is an $\mathbf{R.C}_{\Box\Diamond}$-*matrix* if $\mathbf{M}$ is an $\mathbf{R.C}_{\Box\Diamond}$-algebra and E is as in the definition of relevant matrix.

(b) $\langle \mathbf{M}, E \rangle$ is an $\mathbf{R.K}_{\Box\Diamond}$-*matrix* if $\mathbf{M}$ is a De Morgan modal semigroup satisfying the following. For all $x, y \in M$,

(A4) $\Box(x \rightarrow y) \leq \Box x \rightarrow \Box y$,

(A5) $\Box(x \rightarrow y) \leq \Diamond x \rightarrow \Diamond y$;

and if E is as in the definition of relevant matrix and satisfies the following:

(F4) if $x \in E$ then $\Box x \in E$, for all $x \in M$.

(c) $\langle \mathbf{M}, E \rangle$ is an $\mathbf{R.C}$-*matrix* ($\mathbf{R.K}$-*matrix*) if $\langle \mathbf{M}, E \rangle$ is an $\mathbf{R.C}_{\Box\Diamond}$-matrix ($\mathbf{R.K}_{\Box\Diamond}$-matrix) satisfying

(A6) $\Diamond x = -\Box - x$, for all $x \in M$

In the following, we consider the matrix characterizing $\mathbf{L}$.

For any De Morgan semigroup ($\mathbf{R.C}_{\Box\Diamond}$-algebra) $\mathbf{M} = \langle M, \cap, \cup, \rightarrow, -, \Box, \Diamond \rangle$, a mapping $v : \mathsf{Prop} \rightarrow M$ is called a *valuation* on $\mathbf{M}$. Further, given a valuation v on $\mathbf{M}$, a mapping $I : \mathsf{Wff} \rightarrow M$, called the *interpretation associated with* v, is defined as follows:

i. for $p \in \mathsf{Prop}$, $I(p) = v(p)$

ii. $I(A \wedge B) = I(A) \cap I(B)$

iii. $I(A \vee B) = I(A) \cup I(B)$

iv. $I(A \rightarrow B) = I(A) \rightarrow I(B)$

v. $I({\sim} A) = -I(A)$

vi. $I(\Box A) = \Box I(A)$

vii. $I(\Diamond A) = \Diamond I(A)$.

Let $\langle \mathbf{M}, E \rangle$ be an $\mathbf{L}_B$-matrix, v be a valuation on $\mathbf{M}$ and I be the interpretation associated with v. Then we say (a) A is *valid* in v iff $I(A) \in E$, and (b) A is *valid* in $\langle \mathbf{M}, E \rangle$ iff A is valid in any v.

An $\mathbf{L}_B$-matrix $\langle \mathbf{M}, E \rangle$ is called an $\mathbf{L}$-*matrix* if all theorems of $\mathbf{L}$ are valid in $\langle \mathbf{M}, E \rangle$.

When $\mathcal{C}_{\mathcal{M}}$ is a class of matrices, we say that

(a) $\mathbf{L}$ is *sound with respect to* $\mathcal{C}_{\mathcal{M}}$ iff all theorems of $\mathbf{L}$ are valid in any matrix belonging to $\mathcal{C}_{\mathcal{M}}$,

(b) $\mathbf{L}$ is *complete with respect to* $\mathcal{C}_{\mathcal{M}}$ iff all formulas valid in any matrix belonging to $\mathcal{C}_{\mathcal{M}}$ are theorems of $\mathbf{L}$,

(c) $\mathbf{L}$ is *characterized by* $\mathcal{C}_{\mathcal{M}}$ iff $\mathbf{L}$ is both sound and complete with respect to $\mathcal{C}_{\mathcal{M}}$.

By the definition of $\mathbf{L}$-matrices, we have the following.

Theorem 3.16 **L** *is sound with respect to the class of* **L***-matrices.*

In considering soundness of $\mathbf{L}_B$, we should note the following. To show that ($\Box$-monotonicity) preserves the validity, for example, we must show that $I(A \to B) \in E$ implies $I(\Box A \to \Box B) \in E$. Suppose that $I(A \to B) \in E$. Then $I(A) \leq I(B)$, so $I(\Box A) \leq I(\Box B)$ by (P1). By Lemma 2.10, we have that $I(\Box A \to \Box B) \in E$. When **L** is a normal logic, to show that (Necessitation) preserves the validity, we must show that $I(A) \in E$ implies $I(\Box A) \in E$. This follows from (F4).

For the converse, we use well-known Lindenbaum's method. Let $[A] = \{B \mid \mathbf{L} \vdash A \leftrightarrow B\}$ and $M_L = \{[A] \mid A \in \mathsf{Wff}\}$. Further, define the operations $\cap, \cup, \to, -, \Box, \Diamond$ on M_L by

$$[A] \cap [B] = [A \wedge B], \quad [A] \cup [B] = [A \vee B], \quad [A] \to [B] = [A \to B],$$

$$-[A] = [\sim A], \quad \Box[A] = [\Box A], \quad \Diamond[A] = [\Diamond A].$$

Then it is clear that $\cap, \cup, \to, -, \Box$ and $\Diamond$ are well-defined. The *Lindenbaum algebra for* **L** is the algebra $\mathbf{M}_L = \langle M_L, \cap, \cup, \to, -, \Box, \Diamond \rangle$ defined above. Note that for all $[A], [B] \in M_L$, $[A] \leq [B]$ iff $\mathbf{L} \vdash A \to B$.

Next, let $E_L = \{[A] \mid \mathbf{L} \vdash A\}$. Of course, E_L is also well-defined. Then the *Lindenbaum matrix for* **L** is the matrix $\langle \mathbf{M}_L, E_L \rangle$ defined by that $\mathbf{M}_L$ is the Lindenbaum algebra for **L** and that E_L is as above. Then it is easy to see the following.

Lemma 3.17 *The Lindenbaum matrix* $\langle \mathbf{M}_L, E_L \rangle$ *for* **L** *is an* **L***-matrix.*

The *canonical valuation* v_c is defined by

$$v_c(p) = [p], \qquad \text{for all } p \in \mathsf{Prop}.$$

Further, let I_c be the interpretation associated with v_c. By induction on the length of A, we have the following easily.

Lemma 3.18 *For any* $A \in \mathsf{Wff}$, $I_c(A) = [A]$.

As in Theorem 2.14, we have the completeness result.

Theorem 3.19 **L** *is complete with respect to the class of* **L***-matrices.*

Next, we deal with the truth-preserving operations on relevant modal matrices, including submatrices, homomorphic images and subdirect products. Below, let $\langle \mathbf{M}, E \rangle$ and $\langle \mathbf{M}', E' \rangle$ be **L**-matrices, where $\mathbf{M} = \langle M, \cap, \cup, \to, -, \Box, \Diamond \rangle$ and $\mathbf{M}' = \langle M', \cap, \cup, \to, -, \Box, \Diamond \rangle$.

We say that $\langle \mathbf{M}', E' \rangle$ is a *submatrix* of $\langle \mathbf{M}, E \rangle$ if (a) M' is closed under each operation of $\mathbf{M}$, and (b) $E' = E \cap M'$.

Theorem 3.20 *Let* $\langle \mathbf{M}', E' \rangle$ *be a submatrix of* $\langle \mathbf{M}, E \rangle$*. If* A *is valid in* $\langle \mathbf{M}, E \rangle$*, then* A *is valid in* $\langle \mathbf{M}', E' \rangle$*.*

Proof.

Suppose that A is not valid in $\langle \mathbf{M}', E' \rangle$. Then there exists a valuation v' on $\mathbf{M}'$ such that $I'(A) \notin E'$. For this v', define a valuation v on $\mathbf{M}$ by

$$v(p) = v'(p), \qquad \text{for all } p \in \mathsf{Prop}.$$

Then it is clear that $I(B) = I'(B)$ for all $B \in \mathsf{Wff}$. So, we have $I(A) \notin E'$, and hence $I(A) \notin E$ or $I(A) \notin M'$. Since $I(A) \in M'$, we have $I(A) \notin E$. Therefore, A is not valid in $\langle \mathbf{M}, E \rangle$. ■

We say that f is a *homomorphism* of $\langle \mathbf{M}, E \rangle$ in $\langle \mathbf{M}', E' \rangle$ if $f : M \to M'$ satisfies the following equalities: for any $x, y \in M$,

i. $f(x \cap y) = f(x) \cap f(y)$
ii. $f(x \cup y) = f(x) \cup f(y)$
iii. $f(x \to y) = f(x) \to f(y)$
iv. $f(-x) = -f(x)$
v. $f(\Box x) = \Box f(x)$
vi. $f(\Diamond x) = \Diamond f(x)$.

Note that if f is a homomorphism of $\langle \mathbf{M}, E \rangle$ in $\langle \mathbf{M}', E' \rangle$, then

$$(1) \quad f^{-1}(E') \text{ is a filter in } \mathbf{M}, \quad \text{and} \quad (2) \quad E \subseteq f^{-1}(E').$$

Since it is easy to see (1), we see only (2). Suppose that $x \in E$. Then $(y_1 \to y_1) \cap \cdots \cap (y_n \to y_n) \leq x$, for $y_1, \cdots, y_n \in M$. Since f is a homomorphism, $(f(y_1) \to f(y_1)) \cap \cdots \cap (f(y_n) \to f(y_n)) \leq f(x)$, so we have $f(x) \in E'$. Therefore, $x \in f^{-1}(E')$.

In particular, let f is a homomorphism of $\langle \mathbf{M}, E \rangle$ in $\langle \mathbf{M}', E' \rangle$. Then it is easy to see that $\langle f(\mathbf{M}), E' \cap f(M) \rangle$ is a submatrix of $\langle \mathbf{M}', E' \rangle$. So, $\langle f(\mathbf{M}), E' \cap f(M) \rangle$ is called the *homomorphic image of* $\langle \mathbf{M}, E \rangle$ (under the homomorphism f).

Homomorphic images have the following logical meaning.

Theorem 3.21 *Let f be a homomorphism of $\langle \mathbf{M}, E \rangle$ in $\langle \mathbf{M}', E' \rangle$. If A is valid in $\langle \mathbf{M}, E \rangle$, then A is valid in the homomorphic image of $\langle \mathbf{M}, E \rangle$ under f.*

Proof.

Let f be a homomorphism of $\langle \mathbf{M}, E \rangle$ in $\langle \mathbf{M}', E' \rangle$. Then a homomorphic image of $\langle \mathbf{M}, E \rangle$ under f is $\langle f(\mathbf{M}), E' \cap f(M) \rangle$. Suppose that A is not valid in $\langle f(\mathbf{M}), E' \cap f(M) \rangle$. Then there exists a valuation v' on $\mathbf{M}'$ such that $I'(A) \notin E' \cap f(M)$. Since $I'(A) \in f(M)$, we have $I'(A) \notin E'$. Now define a valuation v on $\mathbf{M}$ by

$$v(p) \in f^{-1}(v'(p)), \qquad \text{for all } p \in \mathsf{Prop}.$$

Since f is a homomorphism, it is easy to see that $I(B) \in f^{-1}(I'(B))$, for all $B \in \mathsf{Wff}$. Then we have $f(I(A)) \notin E'$, that is, $I(A) \notin f^{-1}(E')$. Since f is a homomorphism, we have $I(A) \notin E$. Hence A is not valid in $\langle \mathbf{M}, E \rangle$. ■

A homomorphism f of $\langle \mathbf{M}, E \rangle$ in $\langle \mathbf{M}', E' \rangle$ is called an *isomorphism* (or *embedding*) of $\langle \mathbf{M}, E \rangle$ into $\langle \mathbf{M}', E' \rangle$ if f is injective. Note that a homomorphism f is an isomorphism iff $f^{-1}(E') \subseteq E$.

Theorem 3.22 *Suppose that there exists an isomorphism of $\langle \mathbf{M}, E\rangle$ into $\langle \mathbf{M}', E'\rangle$. If A is valid in $\langle \mathbf{M}', E'\rangle$, then A is valid in $\langle \mathbf{M}, E\rangle$.*

Proof.
Let f be an isomorphism of $\langle \mathbf{M}, E\rangle$ into $\langle \mathbf{M}', E'\rangle$. Suppose that A is not valid in $\langle \mathbf{M}, E\rangle$. Then there exists a valuation v on $\mathbf{M}$ such that $I(A) \notin E$. Since f is an isomorphism, we have $I(A) \notin f^{-1}(E')$, i.e., $f(I(A)) \notin E'$. Now define a valuation v' on $\mathbf{M}'$ by

$$v'(p) = f(v(p)), \quad \text{for all } p \in \mathsf{Prop}.$$

Since f is an isomorphism, it is easy to see that $I'(B) = f(I(B))$, for all $B \in \mathsf{Wff}$. Hence we have $I'(A) \notin E'$. Therefore, A is not valid in $\langle \mathbf{M}', E'\rangle$. ■

Further, if an isomorphism f of $\langle \mathbf{M}, E\rangle$ into $\langle \mathbf{M}', E'\rangle$ is surjective, then it is called an *isomorphism of* $\langle \mathbf{M}, E\rangle$ onto $\langle \mathbf{M}', E'\rangle$. In this case, we say that $\langle \mathbf{M}, E\rangle$ is *isomorphic* to $\langle \mathbf{M}', E'\rangle$ (under an isomorphism f). Note that f is bijective.

Lemma 3.23 *$\langle \mathbf{M}, E\rangle$ is isomorphic to $\langle \mathbf{M}', E'\rangle$ iff the homomorphic image of $\langle \mathbf{M}, E\rangle$ under the homomorphism f of $\langle \mathbf{M}, E\rangle$ in $\langle \mathbf{M}', E'\rangle$ which is injective is exactly $\langle \mathbf{M}', E'\rangle$.*

Proof.
The 'if' part is proved as follows. Let f be a homomorphism of $\langle \mathbf{M}, E\rangle$ in $\langle \mathbf{M}', E'\rangle$ which is injective. First, suppose that $x \in f^{-1}(E')$. Then $f(x) \in E'$, and hence $f(x) \in E' \cap f(M)$ since $E' = E' \cap f(M)$. By the assumption $M' = f(M)$, so for $y_1, \cdots, y_n \in M$, $(f(y_1) \to f(y_1)) \cap \cdots \cap (f(y_n) \to f(y_n)) \leq f(x)$. Since f is injective, we have $(y_1 \to y_1) \cap \cdots \cap (y_n \to y_n) \leq x$. Hence $x \in E$, which implies $f^{-1}(E') \subseteq E$. Further, since $M' = f(M)$, f is surjective. Thus, f is an isomorphism of $\langle \mathbf{M}, E\rangle$ onto $\langle \mathbf{M}', E'\rangle$. Hence, $\langle \mathbf{M}, E\rangle$ is isomorphic to $\langle \mathbf{M}', E'\rangle$.
The 'only if' part is proved as follows. Let f be an isomorphism of $\langle \mathbf{M}, E\rangle$ onto $\langle \mathbf{M}', E'\rangle$. Since f is surjective, we have $f(\mathbf{M}) = \mathbf{M}'$ and $E' \subseteq f(M)$. So, $E' = E' \cap f(M)$. From above remark, it is clear that f is injective. Therefore, the homomorphic image of $\langle \mathbf{M}, E\rangle$ under the homomorphism f of $\langle \mathbf{M}, E\rangle$ in $\langle \mathbf{M}', E'\rangle$ which is injective is exactly $\langle \mathbf{M}', E'\rangle$. ■

In the light of Lemma 3.23, by Theorems 3.21 and 3.22 it is easy to see the following.

Theorem 3.24 *Suppose that $\langle \mathbf{M}, E\rangle$ is isomorphic to $\langle \mathbf{M}', E'\rangle$. Then A is valid in $\langle \mathbf{M}, E\rangle$ iff it is valid in $\langle \mathbf{M}', E'\rangle$.*

Given a family $\{\mathbf{M}_i = \langle M_i, \cap, \cup, \to, -, \Box, \Diamond\rangle \mid i \in I\}$ of De Morgan modal semigroups, the *direct product* of $\{\mathbf{M}_i \mid i \in I\}$ is the matrix

$$\prod_{i\in I} \mathbf{M}_i = \langle \prod_{i\in I} M_i, \cap, \cup, \to, -, \Box, \Diamond\rangle,$$

where (a) $\prod_{i\in I} M_i$ is the set of all functions f from I into $\bigcup_{i\in I} M_i$ such that $f(i) \in M_i$ and (b) for every $f, f' \in \prod_{i\in I} M_i$ and every $i \in I$,

i. $(f \cap f')(i) = f(i) \cap f'(i)$
ii. $(f \cup f')(i) = f(i) \cup f'(i)$

iii. $(f \to f')(i) = f(i) \to f'(i)$

iv. $(-f)(i) = -f(i)$

v. $(\Box f)(i) = \Box f(i)$

vi. $(\Diamond f)(i) = \Diamond f(i)$.

Given a family $\{\langle \mathbf{M}_i, E_i\rangle \mid i \in I\}$ of $\mathbf{L}$-matrices, the *direct product* of $\{\langle \mathbf{M}_i, E_i\rangle \mid i \in I\}$ is the matrix

$$\langle \prod_{i\in I} \mathbf{M}_i, \prod_{i\in I} E_i\rangle,$$

where $\prod_{i\in I} \mathbf{M}_i = \langle \prod_{i\in I} M_i, \cap, \cup, \to, -, \Box, \Diamond\rangle$.

Proposition 3.25 *$\prod_{i\in I} E_i$ is the filter generated by $\{f \to f \mid f \in \prod_{i\in I} M_i\}$.*

Proof.

First, suppose that $f, f' \in \prod_{i\in I} E_i$. Then we have $f(i), f'(i) \in E_i$ for each $i \in I$. Since E_i is a filter, $f(i) \cap f'(i) \in E_i$, so $(f \cap f')(i) \in E_i$. Thus, $f \cap f' \in \prod_{i\in I} E_i$.

Next, suppose that $f \in \prod_{i\in I} E_i$ and $f \leq f'$. Then $f(i) \in E_i$ and $f(i) \leq f'(i)$ for each $i \in I$. Since E_i is a filter, $f'(i) \in E_i$. Thus, $f' \in \prod_{i\in I} E_i$.

It remains to show that $\prod_{i\in I} E_i$ is the least filter containing $\{f \to f \mid f \in \prod_{i\in I} M_i\}$. Let F be any filter containing $\{f \to f \mid f \in \prod_{i\in I} M_i\}$. Suppose that $f' \in \prod_{i\in I} E_i$. Then $f'(i) \in E_i$ for each $i \in I$. Since E_i is the filter generated by $\{x \to x \mid x \in M_i\}$, we have $(x_1 \to x_1) \cap \cdots \cap (x_n \to x_n) \leq f'(i)$. Now putting $x_j = f_j(i)$, we see that $(f_1 \to f_1) \cap \cdots \cap (f_n \to f_n) \leq f'$ for $f_1, \cdots, f_n \in \prod_{i\in I} M_i$. So, we have $f' \in F$, which is the desired result. ■

Theorem 3.26 *Suppose that $\langle \prod_{i\in I} \mathbf{M}_i, \prod_{i\in I} E_i\rangle$ is the direct product of $\{\langle \mathbf{M}_i, E_i\rangle \mid i \in I\}$. Then A is valid in every $\langle \mathbf{M}_i, E_i\rangle$ iff A is valid in $\langle \prod_{i\in I} \mathbf{M}_i, \prod_{i\in I} E_i\rangle$.*

Proof.

The 'only if' part is proved as follows. Suppose that A is not valid in $\langle \prod_{i\in I} \mathbf{M}_i, \prod_{i\in I} E_i\rangle$. Then there exists a valuation v on $\prod_{i\in I} \mathbf{M}_i$ such that $I(A) \notin \prod_{i\in I} E_i$. So, there exists $j \in I$ such that $(I(A))(j) \notin E_j$. Now we define a valuation v_j on $\mathbf{M}_j$ by $v_j(p) = (v(p))(j)$ for all $p \in \mathsf{Prop}$. Then it is clear that $I_j(B) = (I(B))(j)$ for all $B \in \mathsf{Wff}$. Thus, we have $I_j(A) \notin E_j$. Therefore, A is not valid in $\langle \mathbf{M}_j, E_j\rangle$.

The 'if' part is proved as follows. Suppose that there exists $j \in I$ such that A is not valid in $\langle \mathbf{M}_j, E_j\rangle$. Then there exists a valuation v_j on $\mathbf{M}_j$ such that $I_j(A) \notin E_j$. Now we define a valuation v on $\prod_{i\in I} \mathbf{M}_i$. by $v(p) = \prod_{i\in I} v_i(p)$ for all $p \in \mathsf{Prop}$. Then it is clear that $I(B)(i) = I_i(B)$ for all $B \in \mathsf{Wff}$ and $i \in I$. Thus, we have $I(A)(j) \notin E_j$, and hence $I(A) \notin \prod_{i\in I} E_i$. Therefore, A is not valid in $\langle \prod_{i\in I} \mathbf{M}_i, \prod_{i\in I} E_i\rangle$. ■

3.5 Conservative extensions

On closing this chapter, we show that any of our basic relevant modal logic is a conservative extension of $\mathbf{R}$. We have the following result.

Theorem 3.27 *Any of $\mathbf{R.C}_{\Box\Diamond}$, $\mathbf{R.K}_{\Box\Diamond}$, $\mathbf{R.C}$ or $\mathbf{R.K}$ is a conservative extension of $\mathbf{R}$.*

Proof.

We show that $\mathbf{R.C}_{\Box\Diamond}$ is a conservative extension of $\mathbf{R}$ as follows. Suppose that a non-modal formula A is not a theorem of $\mathbf{R}$. Then there exists an $\mathbf{R}$-model $\langle O, W, R, ^*, V\rangle$ and $a \in O$ such that $a \not\models A$ by Theorem 2.6. We define $\langle O', W', R', S'_\Box, S'_\Diamond, ^{*\prime}, V'\rangle$ by

- $\langle O', W', R, ^{*\prime}, V'\rangle$ is just $\langle O, W, R, ^*, V\rangle$.
- for $a, b \in W'$, $S'_\Box ab$ iff $b \in O$ or $a \leq b$
- for $a, b \in W'$, $S'_\Diamond ab$ iff $b^* \in O$ or $b \leq a$

Then it is easy to see that $\langle O', W', R', S'_\Box, S'_\Diamond, ^{*\prime}, V'\rangle$ is an $\mathbf{R.C}_{\Box\Diamond}$-model, and thus $a \in O'$ and $a \not\models' A$. Hence A is not a theorem of $\mathbf{R.C}_{\Box\Diamond}$. For other case, the proof is similar. ■

3.6 Notes

For Section 3.1, $\mathbf{R.C}_{\Box\Diamond}$ is the generalization of a conjunctively regular modal logic in [19]. Dependent relevant modal logics over $\mathbf{R}$ have been often discussed as in Sections 3.2 and 3.3. However, there is no work on independent relevant modal logics over $\mathbf{R}$. For Section 3.4, our algebraic argument using matrices is modal extensions of [17] and [18]. On the other hand, relevant modal algebra was dealt in [10].

Also, truth-preserving operations of frames and matrices in relevant modal logics have not been discussed before. However, the idea of relevant modal p-morphisms essentially appeared in [58] and [10]. A.Urquhart introduces this notion to (non-modal) relevant logics in order to show the duality between relevant algebras and relevant spaces, which is quite similar to frames for relevant logics. After that, it was extended to relevant modal logics by S.A.Celani ([10]).

Chapter 4

General frames

In this chapter, we introduce general frames for relevant modal logics and investigate their basic properties, including the duality theory, descriptive frames and truth-preserving operations. This chapter shows that in most cases general frames for relevant modal logics have the same properties as those of classical modal logics. Further, we introduce $\mathcal{D}^\star$-elementary and $\mathcal{D}$-persistent logics, which will be discussed in the next section.

Below, $\&, \Rightarrow, \Leftrightarrow, \forall$ and $\exists$ are used to denote respectively conjunction, implication, equivalence, universal and existential quantifiers in the metalanguage. Terminologies and notations follow those in [11].

4.1 General frames

Before introducing general frames, it is necessary here to discuss the adequacy of including both empty set $\emptyset$ and Wff among prime **L**-theories. As a consequence of our Sahlqvist theorem in Section 5.1, we can show that some of superclassical relevant modal logics, i.e., relevant modal logics over superclassical relevant logic **KR**, are complete. To make this possible, we need to include $\emptyset$ and Wff among **L**-theories and make explicit use of them. In fact, in Section 5.5 of [48], the notion of *enlarged frames* is introduced in order to treat this problem.

Following [48], in the rest of the present thesis, we assume that for every **L**-frame $\mathcal{F} = \langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$, there exist elements e, called the *null world*, and u, called the *universal world*, in W which satisfy the following definition and postulates for all $a, b \in W$:

(du) $u \overset{\text{def}}{=} e^*$, $u \in O$

(ep1) if $Ruab$, then $a = e$ or $b = u$

(ep2) $e \neq u$

(ep3) $S_\Box ee$

(ep4) if $S_\Box ua$ then $a = u$

(ep5) if $S_\Diamond ea$ then $a = e$

(ep6) $S_\Diamond uu$.

Every **L**-frame satisfies the following postulates:

$$(1) \quad Reue \qquad \text{and} \qquad (2) \quad e \leq a \leq u, \quad \text{for all} \quad a \in W.$$

A *valuation* V on $\mathcal{F}$ must satisfy also the following conditions: for all $p \in \mathsf{Prop}$,

$$e \notin V(p) \qquad \text{and} \qquad u \in V(p).$$

Then by induction on the length of A, we see that $e \not\models A$ and $u \models A$, for all $A \in \mathsf{Wff}$.

In the canonical **L**-frame $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c \rangle$, prime **L**-theories $\emptyset$ and Wff are taken for e_c and u_c, respectively.

For a given **L**-frame $\langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$, let $Up(W)^+ = \{X \subseteq W \mid X \neq \emptyset \ \& \ X \neq W \ \& \ \forall a \forall b (a \in X \ \& \ a \leq b \Rightarrow b \in X)\}$. Then note that in the definition of $Up(W)^+$, conditions $X \neq \emptyset$ and $X \neq W$ are equivalent to conditions $u \in X$ and $e \notin X$, respectively. A *general* **L**-*frame* is a 7-tuple $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P \rangle$ where

(a) $\langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ is an **L**-frame, written by $\boldsymbol{\kappa}\mathfrak{F}$,

(b) P, called a *set of possible values* in $\mathfrak{F}$, is a non-empty subset of $Up(W)^+$ closed under $\cap$, $\cup$ and the operations $\rightarrow$, $-$, $\Box$ and $\Diamond$ defined as follows: for all $X, Y \subseteq W$,

- $X \rightarrow Y = \{a \in W \mid \forall b \forall c (Rabc \ \& \ b \in X \Rightarrow c \in Y)\}$
- $-X = \{a \in W \mid a^* \notin X\}$
- $\Box X = \{a \in W \mid \forall b (S_\Box ab \Rightarrow b \in X)\}$
- $\Diamond X = \{a \in W \mid \exists b (S_\Diamond ab \ \& \ b \in X)\}$.

Let $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P \rangle$ be a general **L**-frame. We call a 8-tuple $\langle O, W, R, S_\Box, S_\Diamond, ^*, P, V \rangle$ an **L**-*model* on $\mathfrak{F}$, where (a) $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P \rangle$ and (b) V is a mapping from Prop to P, called a *valuation* on $\mathfrak{F}$, i.e., $V(p) \in P$ for all $p \in \mathsf{Prop}$. Further, a relation $\models$ between W and Wff is defined as in Section 3.2. Thus, a general **L**-frame $\mathfrak{F}$ with $P = Up(W)^+$ is essentially equal to $\kappa\mathfrak{F}$.

Then we write $V(A) = \{a \mid a \models A\}$ for all $A \in \mathsf{Wff}$. Further, we write $\mathfrak{F} \models A$ if for any valuation V on a frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P \rangle$ and for all $a \in O$, $a \models A$. Also, we write $\mathfrak{F} \models \mathbf{L}$ if for all theorems A of **L**, $\mathfrak{F} \models A$.

The matrix $\langle P, \cap, \cup, \rightarrow, -, \Box, \Diamond, O^+ \rangle$ is called the *dual* of $\mathfrak{F}$, which is denoted by $\mathfrak{F}^+$, where $\langle P, \cap, \cup, \rightarrow, -, \Box, \Diamond \rangle$ is defined above and $O^+ = \{X \in P \mid O \subseteq X\}$.

Then we have the following.

Theorem 4.1 *The dual of a general* $\mathbf{L}_B$*-frame is an* $\mathbf{L}_B$*-matrix.*

Proof.

1. Case that $\mathbf{L}_B$ is $\mathbf{R.C}_{\Box\Diamond}$

   It is sufficient to see (A1) through (A3), and (F1) through (F3). For (A1), it is sufficient to see (DMS1) through (DMS6).

(DMS1) We must see (DML1) through (DML3). (DML1) is obvious. To see (DML2), suppose that $X \subseteq -Y$ and $a \in Y$. Then $a^* \notin -Y$, so $a^* \notin X$. Hence $a \in -X$. Therefore, $Y \subseteq -X$. Next, to see (DML3), suppose that $a \in --X$. Then $a^{**} \in X$, so $a \in X$ by (p6).

(DMS2) Suppose that $a \in X \to (Y \to Z)$. To see that $a \in Y \to (X \to Z)$, suppose that $Rabc$, $b \in Y$, $Rca'b'$ and $a' \in X$. By (p3), there exists $d \in W$ such that $Raa'd$ and $Rdbb'$. Then we have $b' \in Z$, which is the desired result.

(DMS3) Suppose that $a \in X$. To see that $a \in (X \to Y) \cap Z \to Y$, suppose that $Rabc$ and $b \in (X \to Y) \cap Z$. Then we have $b \in X \to Y$. Further, $Rbac$ by (t1), so $c \in Y$, which is the desired result.

(DMS4) Suppose that $a \notin -X$. Then $a^* \in X$. By (t5), Raa^*a, so $a \notin X \to -X$. This is the desired result.

(DMS5) Suppose that $a \in X \to Y$. To see that $a \in -Y \cdot -X$, suppose that $Rabc$ and $b \in -Y$. Then we have $b^* \notin Y$ and Rac^*b^* by (p5). Thus $c^* \notin X$, so $c \in -X$. This is the desired result.

(DMS6) Suppose that $a \notin Z$. If $b \in O$, $Rbcd$ and $c \in X$, then we have $d \in X$. So, $b \in X \to X$. Similarly, we have $b \in Y \to Y$, for any $b \in O$. Hence $b \in (X \to X) \cap (Y \to Y)$, for any $b \in O$. Since $Raxa$, we have $a \notin (X \to X) \cap (Y \to Y) \to Z$.

For (A2) and (A3), the proofs are done in the same way, so we prove only (A2). The proof is as follows:

$$\begin{array}{rl} a \in \Box(X \cap Y) & \text{iff} \;\; \forall b(S_\Box ab \Rightarrow b \in X \cap Y) \\ & \text{iff} \;\; \forall b(S_\Box ab \Rightarrow b \in X) \;\&\; \forall b(S_\Box ab \Rightarrow x \in Y) \\ & \text{iff} \;\; a \in \Box X \;\&\; a \in \Box Y \\ & \text{iff} \;\; a \in \Box X \cap \Box Y. \end{array}$$

In the following, we show that O^+ is the filter generated by $\{X \to X \mid X \in P\}$.

First, suppose that $X, Y \in P$. Then $O \subseteq X$ and $O \subseteq Y$, so $O \subseteq X \cap Y$. Hence $X \cap Y \in O^+$. Moreover, suppose that $X \in O^+$ and $X \subseteq Y$. Then $O \subseteq X$, so $O \subseteq Y$. Hence $Y \in O^+$.

Finally, let F be any filter containing $\{X \to X \mid X \in P\}$. Suppose that $Y \in O^+$. Then $O \subseteq Y$. Further, since $Y \subseteq (Y \to Y) \to Y$, we have $O \subseteq (Y \to Y) \to Y$. Now assume that $Y \to Y \not\subseteq Y$. Then there exists $b \in W$ such that $b \in Y \to Y$ and $b \notin Y$. By (p1), there exists $a \in O$ such that $Rabb$, so $a \notin (Y \to Y) \to Y$. This is a contradiction, so $Y \to Y \subseteq Y$. Since $Y \to Y \in F$, we have $Y \in F$. This shows that O^+ is the least filter containing $\{X \to X \mid X \in P\}$.

2. Case that $\mathbf{L}_B$ is $\mathbf{R.K}_{\Box\Diamond}$.

   It is sufficient to check (A4) and (A5) and (F4). For (A4), suppose that $a \in \Box(X \to Y)$. To show $a \in \Box X \to \Box Y$, suppose that $Rabc$ and $b \in \Box X$. Moreover suppose that $S_\Box cd$. By (p10), there exist $a', b' \in W$ such that $S_\Box aa'$, $S_\Box bb'$ and $Ra'b'd$. Then $a' \in X \to Y$ and $b' \in X$, so $d \in Y$. This is a desired result. Hence $\Box(X \to Y) \subseteq \Box X \to \Box Y$. For (A5), we may argue in a similar way.

   For (F4), suppose that $X \in O^+$. To show $\Box X \in O^+$, suppose that $a \in O$ and $S_\Box ab$. Then $O \subseteq X$ and $b \in O$ by (p9), so $b \in X$. Hence $a \in \Box X$. Therefore, $O \subseteq \Box X$, which is the desired result.

3. Case that $\mathbf{L}_B$ is **R.C** and **R.K**.

   It is sufficient to check only (A6). The proof is as follows:

$$\begin{array}{rll} a \in \Box X & \text{iff} & \forall b(S_\Box ab \Rightarrow b \in X) \\ & \text{iff} & \forall b(S_\Diamond a^* b^* \Rightarrow b^* \notin -X) \\ & \text{iff} & a^* \notin \Diamond - X \\ & \text{iff} & a \in -\Diamond - X. \end{array}$$

■

Theorem 4.2 *Let $\mathfrak{F}$ be a general $\mathbf{L}_B$-frame. Then, A is valid in $\mathfrak{F}$ iff A is valid in $\mathfrak{F}^+$.*

Proof.

Let $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$. The 'if' part is proved as follows. Suppose that A is not valid in $\mathfrak{F}$. Take a model on $\mathfrak{F}$ such that $a \not\models A$ for some $a \in O$. Now define a valuation v^+ on $\mathfrak{F}^+$ by

$$v^+(p) = \{a \in W \mid a \models p\}, \qquad \text{for all } p \in \mathsf{Prop}.$$

By the hereditary condition, $v^+(p)$ is upward closed. Further, by induction on the length of B, we see easily that $a \in I^+(B)$ iff $a \models B$, for all $a \in W$ and $B \in \mathsf{Wff}$. Then we see that $a \notin I^+(A)$. So, we have $O \not\subseteq I^+(A)$, which means that $I^+(A) \notin O^+$. Thus, A is not valid in $\mathfrak{F}^+$.

The 'only if' part is proved as follows. Suppose that A is not valid in $\mathfrak{F}^+$. Take a valuation v^+ on $\mathfrak{F}^+$ such that $I^+(A) \notin O^+$. Then there exists $a \in O$ such that $a \notin I^+(A)$. Now define a valuation V on $\mathfrak{F}$ by

$$V(p) = v^+(p), \qquad \text{for all } p \in \mathsf{Prop}.$$

Since $v^+(p)$ is upward closed, V satisfies the hereditary condition. Further, by induction on the length of B, we see that $V(B) = I^+(B)$, for all $B \in \mathsf{Wff}$. Then we have $a \not\models A$, so A is not valid in $\mathfrak{F}$. ■

By Theorem 4.2, we have the following.

Theorem 4.3 *The dual of a general $\mathbf{L}$-frame is an $\mathbf{L}$-matrix.*

General $\mathbf{L}$-frames $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ and $\mathfrak{F}' = \langle O', W', R', S'_\Box, S'_\Diamond, ^{*\prime}, P'\rangle$ are *isomorphic* if there is a bijection f from W to W' such that

1) $Rabc$ iff $R' f(a) f(b) f(c)$
2) $S_\Box ab$ iff $S'_\Box f(a) f(b)$
3) $S_\Diamond ab$ iff $S'_\Diamond f(a) f(b)$
4) $f(a^*) = (f(a))^{*\prime}$
5) $f(e) = e'$
6) $a \in O$ iff $f(a) \in O'$
7) $X \in P$ iff $f(X) \in P'$,

for all $a, b, c \in W$, for the null worlds $e \in W$ and $e' \in W'$, and for all $X \subseteq W$.

Given an **L**-model $\mathcal{M} = \langle O, W, R, S_\Box, S_\Diamond, {}^*, V\rangle$, the general **L**-frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, {}^*, P\rangle$ with

$$P = \{V(A) \mid A \in \mathsf{Wff}\}$$

is called the *general* **L**-*frame associated with* $\mathcal{M}$.

The general frame associated with the canonical **L**-model $\mathcal{M}_c = \langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, V_c\rangle$, is denoted by $\gamma\mathfrak{F}_c = \langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, P_c\rangle$. We will call $\gamma\mathfrak{F}_c$ the *universal* **L**-*frame*. The canonical **L**-frame is obtained from the universal **L**-frame $\gamma\mathfrak{F}_c$ by omitting P_c.

Theorem 4.4 *The Lindenbaum matrix* $\langle \mathbf{M}_L, E_L\rangle$ *for* **L** *is isomorphic to* $\gamma\mathfrak{F}_c^+$*, where the map* f *defined by*

$$f([A]) = V_c(A), \qquad \text{for every } A \in \mathsf{Wff}$$

is an isomorphism.

Proof.

First of all, we will show that f is bijective. It is clear that f is surjective by the definition of P_c. So, we show that f is injective. Suppose that $[A] \neq [B]$. Then $\mathbf{L} \not\vdash A \leftrightarrow B$. By 4 of Lemma 3.8, there exists $\Pi \in O_c$ such that $A \leftrightarrow B \notin \Pi$. Then $\Pi \not\models_c A \leftrightarrow B$ by Lemma 3.13. This implies that $V_c(A) \neq V_c(B)$, that is $f([A]) \neq f([B])$.

Next, we will show that f preserves each operations of $\mathbf{M}_L$. Let $\Sigma \in W_c$.

1.
$$\begin{aligned}
\Sigma \in f([A] \cap [B]) \quad &\text{iff} \quad \Sigma \in V_c(A \wedge B)\\
&\text{iff} \quad \Sigma \models_c A \wedge B\\
&\text{iff} \quad \Sigma \models_c A \;\&\; \Sigma \models_c B\\
&\text{iff} \quad \Sigma \in V_c(A) \;\&\; \Sigma \in V_c(B)\\
&\text{iff} \quad \Sigma \in f([A]) \cap f([B]).
\end{aligned}$$

2. As in 1, we see that $f([A] \cup [B]) = f([A]) \cup f([B])$.

3. First, suppose that $\Sigma \in f([A] \to [B])$. To show that $\Sigma \in f([A]) \to f([B])$, suppose that $R_c\Sigma\Gamma\Delta$ and $\Gamma \in f([A])$. Then $\Sigma \models_c A \to B$ and $\Gamma \models_c A$, so $\Delta \models_c B$. This means that $\Delta \in f([B])$, which is the desired result.

   Next, suppose that $\Sigma \notin f([A] \to [B])$. Then $\Sigma \not\models_c A \to B$, so there exist $\Gamma, \Delta \in W_c$ such that $R_c\Sigma\Gamma\Delta$, $\Gamma \models_c A$ and $\Delta \not\models_c B$. Hence we have $\Gamma \in V_c(A)$ and $\Delta \notin V_c(B)$, which mean that $\Gamma \in f([A])$ and $\Delta \notin f([B])$. Therefore, $\Sigma \notin f([A]) \to f([B])$.

4. $\Sigma \in f(-[A])$ iff $\Sigma \models_c \sim A$ iff $g_c(\Sigma) \not\models_c A$ iff $g_c(\Sigma) \notin V_c(A)$ iff $\Sigma \in -f([A])$.

5. First, suppose that $\Sigma \in f(\Box[A])$. To show that $\Sigma \in \Box f([A])$, suppose that $S_{\Box c}\Sigma\Gamma$. Then $\Sigma \models_c \Box A$, so $\Gamma \models_c A$. This means that $\Gamma \in f([A])$, which is the desired result.

   Next, suppose that $\Sigma \notin f(\Box[A])$. Then $\Sigma \not\models_c \Box A$, so there exists $\Gamma \in W_c$ such that $S_{\Box c}\Sigma\Gamma$ and $\Gamma \not\models_c A$. Then we have $\Gamma \notin V_c(A)$, and hence $\Sigma \notin \Box f([A])$.

6.

$$\begin{array}{rll} \Sigma \in f(\Diamond[A]) & \text{iff} & \Sigma \models_c \Diamond A \\ & \text{iff} & \exists\Gamma(S_{\Diamond c}\Sigma\Gamma \ \& \ \Gamma \models_c A) \\ & \text{iff} & \exists\Gamma(S_{\Diamond c}\Sigma\Gamma \ \& \ \Gamma \in V_c(A)) \\ & \text{iff} & \Sigma \in \Diamond f([A]). \end{array}$$

■

Applying our arguments on duality to frames, we get the similar results by changing P to $Up(W)^+$.

4.2 Duality of matrices

In this section, we consider the dual of matrices. For non-modal part with constants, A.Urquhart have introduced relevant spaces as the dual of relevant algebra in [58]. Further, S.A.Celani investigates modal extension in [10]. Here we show the similar result on relevant modal matrices to Urquhart's one on relevant algebras. In general, the argument in dual algebra uses the notion of prime filters, which we follow here. To do so, we introduce the notion of the relations of filters and investigate their properties.

As concerning the introductions of e and u in Section 4.1, we assume that both $\emptyset$ and M are regarded as prime filters and prime ideals.

Let $\langle \mathbf{M}, E\rangle$ be an **L**-matrix, where $\mathbf{M} = \langle M, \cap, \cup, \rightarrow, -, \Box, \Diamond\rangle$. Let $\mathsf{F}_\mathbf{M}$ be the set of all filters in **M**. Now, we define a ternary relation R on $\mathsf{F}_\mathbf{M}$, and binary relations $S_\Box$ and $S_\Diamond$ on $\mathsf{F}_\mathbf{M}$, as follows. For all $\nabla_1, \nabla_2, \nabla_3 \in \mathsf{F}_\mathbf{M}$:

$$R\nabla_1\nabla_2\nabla_3 \quad \text{iff} \quad \text{for all } x, y \in M, \text{ if } x \rightarrow y \in \nabla_1 \text{ and } x \in \nabla_2 \text{ then } y \in \nabla_3$$

$$S_\Box\nabla_1\nabla_2 \quad \text{iff} \quad \text{for all } x \in M, \text{ if } \Box x \in \nabla_1 \text{ then } x \in \nabla_2$$

$$S_\Diamond\nabla_1\nabla_2 \quad \text{iff} \quad \text{for all } x \in M, \text{ if } x \in \nabla_2 \text{ then } \Diamond x \in \nabla_1.$$

In the following, we see that relations $R, S_\Box, S_\Diamond$ on $\mathsf{F}_\mathbf{M}$ are restricted to those on a class of prime filters.

Lemma 4.5 *Suppose that ∇_1 and ∇_2 are filters and ∇_3 is a prime filter such that $R\nabla_1\nabla_2\nabla_3$. Then there exists a prime filter $\nabla_1' \supseteq \nabla_1$ such that $R\nabla_1'\nabla_2\nabla_3$.*

Proof.

Let $\Delta = \{x \mid \exists y \in \nabla_2 \exists z \notin \nabla_3(x \leq y \rightarrow z)\}$. First, suppose that $x_1, x_2 \in \Delta$. Then there exist $y_1, y_2 \in \nabla_2$ and $z_1, z_2 \notin \nabla_3$ such that $x_1 \leq y_1 \rightarrow z_1$ and $x_2 \leq y_2 \rightarrow z_2$. It follows that $x_1 \cup x_2 \leq y_1 \cap y_2 \rightarrow z_1 \cup z_2$. Since ∇_2 is a filter, $y_1 \cap y_2 \in \nabla_2$. Since ∇_3 is prime, $z_1 \cup z_2 \notin \nabla_3$. Hence $x_1 \cup x_2 \in \Delta$. Next, suppose that $x_2 \in \Delta$ and $x_1 \leq x_2$. Then there exist $y \in \nabla_2$ and $z \notin \nabla_3$ such that $x_2 \leq y \rightarrow z$. It follows that $x_1 \leq y \rightarrow z$, so $x_1 \in \Delta$. Therefore, Δ is an ideal.

Assume that $x \in \nabla_1 \cap \Delta$. Then there exist $y \in \nabla_2$ and $z \notin \nabla_3$ such that $x \leq y \rightarrow z$. By the assumption, $y \rightarrow z \notin \nabla_1$. Since ∇_1 is a filter, $x \notin \nabla_1$. This is a contradiction. Hence $\nabla_1 \cap \Delta = \emptyset$.

By 1 of Lemma 2.7, there exists a prime filter $\nabla'_1 \supseteq \nabla_1$ such that $\nabla'_1 \cap \Delta = \emptyset$. Now suppose that $x \to y \in \nabla'_1$ and $x \in \nabla_2$. Then $x \to y \notin \Delta$, so $y \in \nabla_3$. Therefore, $R\nabla'_1\nabla_2\nabla_3$. ∎

Lemma 4.6 *Suppose that ∇_1 and ∇_2 are filters and ∇_3 is a prime filter such that $R\nabla_1\nabla_2\nabla_3$. Then there exists a prime filter $\nabla'_2 \supseteq \nabla_2$ such that $R\nabla_1\nabla'_2\nabla_3$.*

Proof.

Let $\Delta = \{x \mid \exists y \notin \nabla_3(x \to y \in \nabla_1)\}$. First, suppose that $x_1, x_2 \in \Delta$. Then there exist $y_1, y_2 \notin \nabla_3$ such that $x_1 \to y_1, x_2 \to y_2 \in \nabla_1$. Since ∇_1 is a filter, $x_1 \cup x_2 \to y_1 \cup y_2 \in \nabla_1$. Since ∇_3 is prime, $y_1 \cup y_2 \notin \nabla_3$. Hence $x_1 \cup x_2 \in \Delta$. Next, suppose that $x_2 \leq x_1$ and $x_1 \in \Delta$. Then $x_1 \to y \leq x_2 \to y$, and there exists $y \notin \nabla_3$ such that $x_1 \to y \in \nabla_1$. Since ∇_1 is a filter, $x_2 \to y \in \nabla_1$. Hence $x_2 \in \Delta$. Therefore, Δ is an ideal.

Assume that $x \in \nabla_2 \cap \Delta$. Then there exists $y \notin \nabla_3$ such that $x \to y \in \nabla_1$. By the assumption, we have $x \notin \nabla_2$, which is a contradiction. Hence $\nabla_2 \cap \Delta = \emptyset$.

By 1 of Lemma 2.7, there exists a prime filter $\nabla'_2 \supseteq \nabla_2$ such that $\nabla'_2 \cap \Delta = \emptyset$. Now suppose that $x \to y \in \nabla_1$ and $x \in \nabla'_2$. Then $x \notin \Delta$, so $y \in \nabla_3$. Therefore, $R\nabla_1\nabla'_2\nabla_3$. ∎

Lemma 4.7

1. *Suppose that ∇ is a prime filter and ∇_1, ∇_2 are filters such that $R\nabla\nabla_1\nabla_2$ and $y \notin \nabla_2$. Then there exist prime filters ∇'_1 and ∇'_2 such that $R\nabla\nabla'_1\nabla'_2$, $\nabla_1 \subseteq \nabla'_1$ and $y \notin \nabla'_2$.*

2. *For a prime filter ∇ such that $x \to y \notin \nabla$, there exist prime filters ∇_1 and ∇_2 such that $R\nabla\nabla_1\nabla_2$, $x \in \nabla_1$ and $y \notin \nabla_2$.*

Proof.

1. Let Δ be the ideal generated by $\{y\}$. Assume that $x \in \nabla_2 \cap \Delta$. Then $x \leq y$. Since ∇_2 is a filter, $y \in \nabla_2$. This is a contradiction. Hence $\nabla_2 \cap \Delta = \emptyset$. By 1 of Lemma 2.7, there exists a prime filter $\nabla'_2 \supseteq \nabla_2$ such that $\nabla'_2 \cap \Delta = \emptyset$. Since $y \in \Delta$, we have $y \notin \nabla'_2$. It is obvious that $R\nabla\nabla_1\nabla'_2$, so there exists a prime filter $\nabla'_1 \supseteq \nabla_1$ such that $R\nabla\nabla'_1\nabla'_2$ by Lemma 4.6.

2. Let ∇_1 be the filter generated by $\{x\}$, and let $\nabla_2 = \{y \mid \exists z \in \nabla_1(z \to y \in \nabla)\}$. To show that ∇_2 is a filter, suppose first that $y_1, y_2 \in \nabla_2$. Then there exist $z_1, z_2 \in \nabla_1$ such that $z_1 \to y_1 \in \nabla$ and $z_2 \to y_2 \in \nabla$. Since ∇ and ∇_1 are filters, $z_1 \cap z_2 \in \nabla_1$ and $z_1 \cap z_2 \to y_1 \cap y_2 \in \nabla$. Hence, $y_1 \cap y_2 \in \nabla$. Secondly, suppose that $y_1 \in \nabla_2$ and $y_1 \leq y_2$. Then there exists $z \in \nabla_1$ such that $z \to y_1 \in \nabla$. Since ∇ is a filter and $z \to y_1 \leq z \to y_2$, $z \to y_2 \in \nabla$. Hence, $y_2 \in \nabla_2$. Therefore, ∇_2 is a filter. Further, it is clear that ∇_2 satisfies $R\nabla\nabla_1\nabla_2$. Assume that $y \in \nabla_2$. Then there exists $z \in \nabla_1$ such that $z \to y \in \nabla$. So, we have $x \leq z$, and hence $z \to y \leq x \to y$. Since ∇ is a filter, $x \to y \in \nabla$. This contradicts the assumption. Hence $y \notin \nabla_2$.

   By 1, there exist prime filters ∇'_1 and ∇'_2 such that $R\nabla\nabla'_1\nabla'_2$, $\nabla_1 \subseteq \nabla'_1$ and $y \notin \nabla'_2$. Then it is obvious that $x \in \nabla'_1$. ∎

Lemma 4.8

1. *Suppose that* ∇ *is a prime filter and* ∇_1 *is a filter such that* $S_\Box\nabla\nabla_1$ *and* $x \notin \nabla_1$. *Then there exists a prime filter* ∇_1' *such that* $S_\Box\nabla\nabla_1'$ *and* $x \notin \nabla_1$.

2. *For a prime filter* ∇ *such that* $\Box x \notin \nabla$, *there exists a prime filter* ∇_1 *such that* $S_\Box\nabla\nabla_1$ *and* $x \notin \nabla_1$.

Proof.

1. Let Δ be the ideal generated by $\{x\}$. Assume that $y \in \nabla_1 \cap \Delta$. Then $y \leq x$. Since ∇_1 is a filter, $x \in \nabla_1$. This contradicts the assumption. Hence $\nabla_1 \cap \Delta = \emptyset$. By 1 of Lemma 2.7, there exists a prime filter $\nabla_1' \supseteq \nabla_1$ such that $\nabla_1' \cap \Delta = \emptyset$. It is obvious that $S_\Box\nabla\nabla_1'$. Further, we have $x \notin \nabla_1'$ since $x \in \Delta$.

2. Let $\nabla_1 = \{y \mid \Box y \in \nabla\}$. First, suppose that $y, z \in \nabla_1$. Then $\Box y, \Box z \in \nabla$. Since ∇ is a filter, $\Box y \cap \Box z \in \nabla$. By (A2), $\Box(y \cap z) \in \nabla$. Hence $y \cap z \in \nabla_1$. Next, suppose that $y \in \nabla_1$ and $y \leq z$. Then $\Box y \in \nabla$ and $\Box y \leq \Box z$. Since ∇ is a filter, we have $\Box z \in \nabla$. Hence $z \in \nabla_1$. Therefore, ∇_1 is a filter. It is obvious that $S_\Box\nabla\nabla_1$ and $x \notin \nabla_1$. By 1, there exists a prime filter $\nabla_1' \supseteq \nabla_1$ such that $S_\Box\nabla\nabla_1'$ and $x \notin \nabla_1'$. ■

Lemma 4.9 *For a prime filter* ∇ *such that* $\Diamond x \in \nabla$, *there exists a prime filter* ∇_1 *such that* $S_\Diamond\nabla\nabla_1$ *and* $x \in \nabla_1$.

Proof.

Let ∇_1 be the filter generated by $\{x\}$, and let $\Delta = \{y \mid \Diamond y \notin \nabla\}$. First, suppose that $y, z \in \Delta$. Then $\Diamond y, \Diamond z \notin \nabla$. Since ∇ is prime, $\Diamond y \cup \Diamond z \notin \nabla$. By (A3), $\Diamond(y \cup z) \notin \nabla$. Hence $y \cup z \in \Delta$. Next, suppose that $y \in \Delta$ and $z \leq y$. Then $\Diamond y \notin \nabla$ and $\Diamond z \leq \Diamond y$. Since ∇ is a filter, $\Diamond z \notin \nabla$. Hence $z \in \Delta$. Therefore, Δ is an ideal.

Assume that $y \in \nabla_1 \cap \Delta$. Then $x \leq y$ and $\Diamond y \notin \nabla$, so $\Diamond x \leq \Diamond y$. Since ∇ is a filter, $\Diamond x \notin \nabla$. This contradicts the assumption. Hence $\nabla_1 \cap \Delta = \emptyset$.

By 1 of Lemma 2.7, there exists a prime filter $\nabla_1' \supseteq \nabla_1$ such that $\nabla_1' \cap \Delta = \emptyset$. Now suppose that $y \in \nabla_1'$. Then $y \notin \Delta$, so $\Diamond y \in \nabla$. Therefore, $S_\Diamond\nabla\nabla_1'$. It is obvious that $x \in \nabla_1'$. ■

For an **L**-matrix $\langle \mathbf{M}, E\rangle$, where $\mathbf{M} = \langle M, \cap, \cup, \rightarrow, -, \Box, \Diamond\rangle$,

$$\langle \mathbf{M}, E\rangle_+ = \langle O_\mathbf{M}, W_\mathbf{M}, R_\mathbf{M}, S_{\Box\mathbf{M}}, S_{\Diamond\mathbf{M}}, g_\mathbf{M}, P_\mathbf{M}\rangle$$

is, called the *dual* of $\langle \mathbf{M}, E\rangle$, defined as follows:

(a) $W_\mathbf{M}$ is the set of all prime filters in $\mathbf{M}$,

(b) $O_\mathbf{M} = \{\nabla \in W_\mathbf{M} \mid E \subseteq \nabla\}$

(c) $R_\mathbf{M}$ is the restriction of R to $W_\mathbf{M}$

(d) $S_{\Box\mathbf{M}}$ is the restriction of $S_\Box$ to $W_\mathbf{M}$

(e) $S_{\Diamond\mathbf{M}}$ is the restriction of $S_\Diamond$ to $W_\mathbf{M}$

(f) $g_\mathbf{M}(\nabla) = \{x \in M \mid -x \notin \nabla\}$, for $\nabla \in W_\mathbf{M}$.

(g) $e_{\mathbf{M}} = \emptyset$.

(h) $u_{\mathbf{M}} = M$.

(i) $P_{\mathbf{M}} = \{f_{\mathbf{M}}(x) \mid x \in M\}$, where $f_{\mathbf{M}} : M \to Up(W_{\mathbf{M}})^+$ is defined by $f_{\mathbf{M}}(x) = \{\nabla \in W_{\mathbf{M}} \mid x \in \nabla\}$.

Of course, $\leq_{\mathbf{M}}$ is defined by

$$\nabla_1 \leq_{\mathbf{M}} \nabla_2 \quad \text{iff} \quad \text{there exists } \nabla \in O_{\mathbf{M}} \text{ such that } R_{\mathbf{M}}\nabla\nabla_1\nabla_2.$$

Lemma 4.10 *For each $\mathbf{L}_B$-matrix $\langle \mathbf{M}, E\rangle$, $\langle O_{\mathbf{M}}, W_{\mathbf{M}}, R_{\mathbf{M}}, S_{\Box\mathbf{M}}, S_{\Diamond\mathbf{M}}, g_{\mathbf{M}}\rangle$ is an $\mathbf{L}_B$-frame.*

Proof.

1. Case in which $\mathbf{L}_B$ is $\mathbf{R.C}_{\Box\Diamond}$.

   Before we check all postulates, we show that $\leq_{\mathbf{M}}$ is regarded as $\subseteq$. First, suppose that for any prime filters ∇_1 and ∇_2,$\nabla_1 \subseteq \nabla_2$. To show that $RE\nabla_1\nabla_2$, suppose that $x \to y \in E$ and $x \in \nabla_1$. Then $x \leq y$. Since ∇_1 is a filter, $y \in \nabla_1$. So, we have the desired result. By Lemma 4.5, there exists a prime filter $\nabla \supseteq E$ such that $R\nabla\nabla_1\nabla_2$. Therefore, there exists $\nabla \in O_{\mathbf{M}}$ such that $R_{\mathbf{M}}\nabla\nabla_1\nabla_2$.

   For the converse, suppose that there exists $\nabla \in O_{\mathbf{M}}$ such that $R_{\mathbf{M}}\nabla\nabla_1\nabla_2$. Further, suppose that $x \in \nabla_1$. Then $E \subseteq \nabla$, so $x \to x \in \nabla$, and hence $x \in \nabla_2$. Therefore, $\nabla_1 \subseteq \nabla_2$.

   In the following, we will check postulates (p1) through (p8). For (p1), (p4), (p7) and (p8), it is obvious from above argument.

   (p2) Suppose that $x \to y, x \in \nabla$. Since $(x \to y) \cap x \leq y$, we have $y \in \nabla$. Therefore, $R_{\mathbf{M}}\nabla\nabla\nabla$.

   (p3) Suppose that $R_{\mathbf{M}}\nabla_1\nabla_2\nabla_3$ and $R_{\mathbf{M}}\nabla_3\nabla_4\nabla_5$. Let $\nabla = \{y \mid \exists x \in \nabla_4(x \to y \in \nabla_1)\}$. As in 2 of Lemma 4.7, ∇ is a filter such that $R\nabla_1\nabla_4\nabla$. Now, suppose that $x \to y \in \nabla$ and $x \in \nabla_2$. Then there exists $z \in \nabla_4$ such that $z \to (x \to y) \in \nabla_1$, and hence $x \to (z \to y) \in \nabla_1$. By the hypothesis, $z \to y \in \nabla_3$, so $y \in \nabla_5$. Hence $R\nabla\nabla_2\nabla_5$. By Lemma 4.5, there exists a prime filter $\nabla' \supseteq \nabla$ such that $R\nabla'\nabla_2\nabla_5$. Therefore, there exists $\nabla' \in W_{\mathbf{M}}$ such that $R_{\mathbf{M}}\nabla_1\nabla_4\nabla'$ and $R_{\mathbf{M}}\nabla'\nabla_2\nabla_5$.

   (p5) Suppose that $R_{\mathbf{M}}\nabla_1\nabla_2\nabla_3$. Further, suppose that $x \to y \in \nabla_1$ and $x \in g_{\mathbf{M}}(\nabla_3)$. Then $-y \to -x \in \nabla_1$ and $-x \notin \nabla_3$, so $-y \notin \nabla_2$. Hence $y \in g_{\mathbf{M}}(\nabla_2)$. Therefore, $R_{\mathbf{M}}\nabla_1 g_{\mathbf{M}}(\nabla_3) g_{\mathbf{M}}(\nabla_2)$.

   (p6) For all $x \in M$, $x \in g_{\mathbf{M}}(g_{\mathbf{M}}(\nabla))$ iff $-x \in g_{\mathbf{M}}(\nabla)$ iff $--x \in \nabla$ iff $x \in \nabla$.

2. Case in which $\mathbf{L}_B$ is $\mathbf{R.K}_{\Box\Diamond}$.

   It is sufficient to check postulates (p9) through (p11).

   (p9) Suppose that $\nabla_1 \in O_{\mathbf{M}}$ and $S_{\Box\mathbf{M}}\nabla_1\nabla_2$. Further, suppose that $x \in E$. Then $\Box x \in E$. Since $E \subseteq \nabla_1$, $\Box x \in \nabla_1$. So, we have $x \in \nabla_2$, and hence $E \subseteq \nabla_2$. Therefore, $\nabla_2 \in O_{\mathbf{M}}$.

(p10) Suppose that $R_{\mathbf{M}}\nabla_1\nabla_2\nabla_3$ and $S_{\Box\mathbf{M}}\nabla_3\nabla_4$. Let $\nabla_5 = \{x \mid \Box x \in \nabla_1\}$ and $\nabla_6 = \{x \mid \Box x \in \nabla_2\}$. As in 2 of Lemma 4.8, ∇_5 and ∇_6 are filters such that $S_\Box\nabla_1\nabla_5$ and $S_\Box\nabla_2\nabla_6$, respectively. Now, suppose that $x \to y \in \nabla_5$ and $x \in \nabla_6$. Then $\Box(x \to y) \in \nabla_1$ and $\Box x \in \nabla_2$. By (A4), $\Box x \to \Box y \in \nabla_1$, so $\Box y \in \nabla_3$, and hence $y \in \nabla_4$. Hence $R\nabla_5\nabla_6\nabla_4$. By Lemmas 4.5 and 4.6, there exist prime filters $\nabla'_5 \supseteq \nabla_5$ and $\nabla'_6 \supseteq \nabla_6$ such that $R\nabla'_5\nabla'_6\nabla_4$. Therefore, there exist $\nabla'_5, \nabla'_6 \in W_{\mathbf{M}}$ such that $S_{\Box\mathbf{M}}\nabla_1\nabla'_5$, $S_{\Box\mathbf{M}}\nabla_2\nabla'_6$ and $R_{\mathbf{M}}\nabla'_5\nabla'_6\nabla_4$.

(p11) Suppose that $R_{\mathbf{M}}\nabla_1\nabla_2\nabla_3$ and $S_{\Diamond\mathbf{M}}\nabla_1\nabla_4$. Let $\nabla_5 = \{x \mid \Box x \in \nabla_2\}$, $\nabla_6 = \{y \mid \exists x \in \nabla_5(x \to y \in \nabla_4)\}$, and $\Delta = \{x \mid \Diamond x \notin \nabla_3\}$. As in Lemmas 4.8, 4.7 and 4.9, ∇_5 is a filter such that $S_\Box\nabla_2\nabla_5$, ∇_6 is a filter such that $R\nabla_4\nabla_5\nabla_6$ and Δ is an ideal.

Assume that $y \in \nabla_6 \cap \Delta$. Then there exists $x \in \nabla_5$ such that $x \to y \in \nabla_4$, and $\Diamond y \notin \nabla_3$. Then $\Box x \in \nabla_2$ and $\Diamond(x \to y) \in \nabla_1$. Since $\Diamond(x \to y) \leq \Box x \to \Diamond y$, $\Diamond y \in \nabla_3$. This is a contradiction. Hence $\nabla_6 \cap \Delta = \emptyset$.

By 1 of Lemma 2.7, there exists a prime filter $\nabla'_6 \supseteq \nabla_6$ such that $\nabla'_6 \cap \Delta = \emptyset$. It is clear that $R\nabla_4\nabla_5\nabla'_6$, so there exists a prime filter $\nabla'_5 \supseteq \nabla_5$ such that $R\nabla_4\nabla'_5\nabla'_6$. It follows that $S_\Box\nabla_2\nabla'_5$. Now suppose that $x \in \nabla'_6$. Then $x \notin \Delta$, so $\Diamond x \in \nabla_3$. Hence $S_\Diamond\nabla_3\nabla'_6$. Therefore, there exist $\nabla'_5, \nabla'_6 \in W_{\mathbf{M}}$ such that $R_{\mathbf{M}}\nabla_4\nabla'_5\nabla'_6$, $S_{\Box\mathbf{M}}\nabla_2\nabla'_5$ and $S_{\Diamond\mathbf{M}}\nabla_3\nabla'_6$.

3. Case in which $\mathbf{L}_B$ is **R.C** and **R.K**.

   It is sufficient to check postulate (p12). First, suppose that $S_{\Diamond\mathbf{M}}\nabla_1\nabla_2$. Further, suppose that $\Box x \in g_{\mathbf{M}}(\nabla_1)$. Then $-\Box x \notin \nabla_1$, so $\Diamond - x \notin \nabla_1$. Hence $-x \notin \nabla_2$, so $x \in g_{\mathbf{M}}(\nabla_2)$. Therefore, $S_{\Box\mathbf{M}}g_{\mathbf{M}}(\nabla_1)g_{\mathbf{M}}(\nabla_2)$.

   For the converse, suppose that $S_{\Box\mathbf{M}}g_{\mathbf{M}}(\nabla_1)g_{\mathbf{M}}(\nabla_2)$. Further, suppose that $x \in \nabla_2$. Then $-x \notin g_{\mathbf{M}}(\nabla_2)$, so $\Box - x \notin g_{\mathbf{M}}(\nabla_1)$. Hence $-\Box - x \in \nabla_1$, so $\Diamond x \in \nabla_1$. Therefore, $S_{\Diamond\mathbf{M}}\nabla_1\nabla_2$. ■

Lemma 4.11 *Every set $X \in P_{\mathbf{M}}$ is upward closed in $\langle O_{\mathbf{M}}, W_{\mathbf{M}}, R_{\mathbf{M}}, S_{\Box\mathbf{M}}, S_{\Diamond\mathbf{M}}, g_{\mathbf{M}}\rangle$, i.e., if $\nabla \in X$ and $\nabla \leq_{\mathbf{M}} \nabla'$, then $\nabla' \in X$.*

Proof.
Let $X \in P_{\mathbf{M}}$. Then $X = f_{\mathbf{M}}(x)$ for some $x \in M$. Suppose that $\nabla \in X$ and $\nabla \leq_{\mathbf{M}} \nabla'$. Then $x \in \nabla$, so $x \in \nabla'$. This is just $\nabla' \in f_{\mathbf{M}}(x)$, which is the desired result. ■

Lemma 4.12 *$P_{\mathbf{M}}$ is closed under $\to$, $-$, $\Box$ and $\Diamond$.*

Proof.
Let $f_{\mathbf{M}}(x), f_{\mathbf{M}}(y) \in P_{\mathbf{M}}$ for some $x, y \in M$.

1. First, suppose that $\nabla \in f_{\mathbf{M}}(x \to y)$. To show that $\nabla \in f_{\mathbf{M}}(x) \to f_{\mathbf{M}}(y)$, suppose that $R_{\mathbf{M}}\nabla\nabla_1\nabla_2$ and $\nabla_1 \in f_{\mathbf{M}}(x)$. Then $x \to y \in \nabla$ and $x \in \nabla_1$, so $y \in \nabla_2$. This is just $\nabla_2 \in f_{\mathbf{M}}(y)$, which is the desired result.

   Next, suppose that $\nabla \notin f_{\mathbf{M}}(x \to y)$. Then $x \to y \notin \nabla$. By 2 of Lemma 4.7, there exist $\nabla_1, \nabla_2 \in W_{\mathbf{M}}$ such that $R_{\mathbf{M}}\nabla\nabla_1\nabla_2$, $x \in \nabla_1$ and $y \notin \nabla_2$. Then $\nabla_1 \in f_{\mathbf{M}}(x)$ and $\nabla_2 \notin f_{\mathbf{M}}(y)$. Hence $\nabla \notin f_{\mathbf{M}}(x) \to f_{\mathbf{M}}(y)$.

2. $\nabla \in f_{\mathbf{M}}(-x)$ iff $-x \in \nabla$ iff $x \notin g_{\mathbf{M}}(\nabla)$ iff $g_{\mathbf{M}}(\nabla) \notin f_{\mathbf{M}}(x)$ iff $\nabla \in -f_{\mathbf{M}}(x)$.

3. First, suppose that $\nabla \in f_{\mathbf{M}}(\Box x)$. To show that $\nabla \in \Box f_{\mathbf{M}}(x)$, suppose that $S_{\Box\mathbf{M}}\nabla\nabla_1$. Then $\Box x \in \nabla$, so $x \in \nabla_1$. This means that $\nabla_1 \in f_{\mathbf{M}}(x)$, which is the desired result.

   Next, suppose that $\nabla \notin f_{\mathbf{M}}(\Box x)$. Then $\Box x \notin \nabla$, so there exists $\nabla_1 \in W_{\mathbf{M}}$ such that $S_{\Box\mathbf{M}}\nabla\nabla_1$ and $x \notin \nabla_1$ by 2 of Lemma 4.8. Hence we have $\nabla_1 \notin f_{\mathbf{M}}(x)$, so $\nabla \notin \Box f_{\mathbf{M}}(x)$.

4. Using Lemma 4.9,

$$\begin{aligned}
\nabla \in f_{\mathbf{M}}(\Diamond x) \quad &\text{iff} \quad \Diamond x \in \nabla \\
&\text{iff} \quad \exists \nabla_1 \in W_{\mathbf{M}}(S_{\Diamond\mathbf{M}}\nabla\nabla_1 \ \&\ x \in \nabla_1) \\
&\text{iff} \quad \exists \nabla_1 \in W_{\mathbf{M}}(S_{\Diamond\mathbf{M}}\nabla\nabla_1 \ \&\ \nabla_1 \in f_{\mathbf{M}}(x)) \\
&\text{iff} \quad \nabla \in \Diamond f_{\mathbf{M}}(x).
\end{aligned}$$

■

From Lemmas 4.10 through 4.12, we have the following.

Theorem 4.13 *Let $\langle \mathbf{M}, E\rangle$ be an $\mathbf{L}_B$-matrix. Then the dual $\langle \mathbf{M}, E\rangle_+$ of $\langle \mathbf{M}, E\rangle$ is a general $\mathbf{L}_B$-frame.*

Then we have the representation theorem.

Theorem 4.14 *Every $\mathbf{L}$-matrix $\langle \mathbf{M}, E\rangle$ is isomorphic to $(\langle \mathbf{M}, E\rangle_+)^+$ under the isomorphism $f_{\mathbf{M}}$.*

Proof.
First of all, we will show that $f_{\mathbf{M}}$ is bijective. It is clear that $f_{\mathbf{M}}$ is surjective, so we see that $f_{\mathbf{M}}$ is injective. Suppose that $x \neq y$. Then either $x \not\leq y$ or $y \not\leq x$. Without loss of generality, we discuss only the case in which $x \not\leq y$. By 3 of Lemma 2.7, there exists a prime filter ∇ such that $x \in \nabla$ and $y \notin \nabla$. So, we have $\nabla \in f_{\mathbf{M}}(x)$ and $\nabla \notin f_{\mathbf{M}}(y)$, and hence $f_{\mathbf{M}}(x) \neq f_{\mathbf{M}}(y)$.

It remains to show that $f_{\mathbf{M}}$ preserves each operation of $\mathbf{M}$. Since any element of $f_{\mathbf{M}}(x)$ must be a prime filter, it is easy to see that $f_{\mathbf{M}}(x \cap y) = f_{\mathbf{M}}(x) \cap f_{\mathbf{M}}(y)$ and $f_{\mathbf{M}}(x \cup y) = f_{\mathbf{M}}(x) \cup f_{\mathbf{M}}(y)$. For other operations, we have proved in Lemma 4.12. ■

In general, we note that there exists an isomorphism f of $\langle \mathbf{M}, E\rangle$ into $\langle Up(W_{\mathbf{M}})^+, \cap, \cup, \rightarrow, -, \Box, \Diamond, (O_{\mathbf{M}})^+\rangle^+$. In this case, f is not necessary surjective. So, when we take general frames for the dual of matrices, a matrix is isomorphic to its bidual.

From Theorems 4.14 and 4.2, we have the following.

Theorem 4.15 *Let $\langle \mathbf{M}, E\rangle$ be an $\mathbf{L}$-matrix. Then, A is valid in $\langle \mathbf{M}, E\rangle$ iff A is valid in $\langle \mathbf{M}, E\rangle_+$.*

Proof.
By Theorem 4.13, $\langle \mathbf{M}, E\rangle_+$ be a general $\mathbf{L}_B$-frame. Further, by Theorem 4.2, A is valid in $\langle \mathbf{M}, E\rangle_+$ iff A is valid in $(\langle \mathbf{M}, E\rangle_+)^+$. From Theorem 4.14, we have the desired result.

■

Note that for an $\mathbf{L}$-matrix $\langle \mathbf{M}, E\rangle$, if A is valid in $\langle O_{\mathbf{M}}, W_{\mathbf{M}}, R_{\mathbf{M}}, S_{\Box\mathbf{M}}, S_{\Diamond\mathbf{M}}, g_{\mathbf{M}}\rangle$, then A is valid in $\langle \mathbf{M}, E\rangle$.

From Theorems 4.13 through 4.15, we have the following.

Theorem 4.16 *Let $\langle \mathbf{M}, E\rangle$ be an $\mathbf{L}$-matrix. Then the dual $\langle \mathbf{M}, E\rangle_+$ of $\langle \mathbf{M}, E\rangle$ is a general $\mathbf{L}$-frame.*

There is the relationship between the dual of the Lindenbaum matrices and the universal frames.

Theorem 4.17 *The dual $\langle \mathbf{M}_L, E_L\rangle_+$ of the Lindenbaum matrix for $\mathbf{L}$ is isomorphic to the universal $\mathbf{L}$-frame $\gamma\mathfrak{F}_c$.*

Proof.

Define a mapping $f : W_{\mathbf{M}_L} \to W_c$ by

$$f(\nabla) = \{A \in \mathsf{Wff} \mid [A] \in \nabla\}.$$

First, we will show that f is bijective. Taking $\Sigma \in W_c$, Σ is a prime $\mathbf{L}$-theory. Let $\nabla = \{[A] \in M_L \mid A \in \Sigma\}$. Then it is easy to show that ∇ is a prime filter in $\mathbf{M}_L$, i.e., $\nabla \in W_{\mathbf{M}_L}$. Further, we have $f(\nabla) = \Sigma$. Hence, f is surjective. To show that f is injective, take $\nabla, \nabla' \in W_{\mathbf{M}_L}$ such that $\nabla \neq \nabla'$. Then there exists $A \in \mathsf{Wff}$ such that $[A] \in \nabla$ and $[A] \notin \nabla'$. So, we have $A \in f(\nabla)$ and $A \notin f(\nabla')$, which imply that $f(\nabla) \neq f(\nabla')$.

It remains to show the following 1 through 6.

1. $R_{\mathbf{M}_L}\nabla_1\nabla_2\nabla_3$ iff $R_c f(\nabla_1)f(\nabla_2)f(\nabla_3)$, for all $\nabla_1, \nabla_2, \nabla_3 \in W_{\mathbf{M}_L}$.

   The 'if' part is proved as follows. Suppose that $R_c f(\nabla_1)f(\nabla_2)f(\nabla_3)$. To show that $R_{\mathbf{M}_L}\nabla_1\nabla_2\nabla_3$, suppose that $[A] \to [B] \in \nabla_1$ and $[A] \in \nabla_2$. Then $A \to B \in f(\nabla_1)$ and $A \in f(\nabla_2)$, so we have $B \in f(\nabla_3)$. Hence $[B] \in \nabla_3$, which is the desired result.

   The 'only if' part is proved as follows. Suppose that $R_{\mathbf{M}_L}\nabla_1\nabla_2\nabla_3$. To show that $R_c f(\nabla_1)f(\nabla_2)f(\nabla_3)$, suppose that $A \to B \in f(\nabla_1)$ and $A \in f(\nabla_2)$. Then $[A] \to [B] \in \nabla_1$ and $[A] \in \nabla_2$, so we have $[B] \in \nabla_3$. Hence $B \in f(\nabla_3)$, which is the desired result.

2. $S_{\Box\mathbf{M}_L}\nabla_1\nabla_2$ iff $S_{\Box c} f(\nabla_1)f(\nabla_2)$, for all $\nabla_1, \nabla_2 \in W_{\mathbf{M}_L}$. Similar to 1.

3. $S_{\Diamond\mathbf{M}_L}\nabla_1\nabla_2$ iff $S_{\Diamond c} f(\nabla_1)f(\nabla_2)$, for all $\nabla_1, \nabla_2 \in W_{\mathbf{M}_L}$. Similar to 1.

4. $f(g_{\mathbf{M}_L}(\nabla)) = g_c(f(\nabla))$, for all $\nabla \in W_{\mathbf{M}_L}$.

   Taking $A \in \mathsf{Wff}$, we have

$$\begin{array}{rll} A \in f(g_{\mathbf{M}_L}(\nabla)) & \text{iff} & [A] \in g_{\mathbf{M}_L}(\nabla) \\ & \text{iff} & -[A] \notin \nabla \\ & \text{iff} & \sim A \notin f(\nabla) \\ & \text{iff} & A \in g_c(f(\nabla)). \end{array}$$

5. $\nabla \in O_{\mathbf{M}_L}$ iff $f(\nabla) \in O_c$, for all $\nabla \in W_{\mathbf{M}_L}$.

The 'if' part is proved as follows. Suppose that $\nabla \notin O_{\mathbf{M}_L}$. Then $E_L \not\subseteq \nabla$, so there exists $A \in \mathsf{Wff}$ such that $[A] \in E_L$ and $[A] \notin \nabla$. Hence $\mathbf{L} \vdash A$ and $A \notin f(\nabla)$, which imply that $f(\nabla)$ is not regular. Therefore, $f(\nabla) \notin O_c$.

The 'only if' part is proved as follows. Suppose that $\nabla \in O_{\mathbf{M}_L}$. Then $E_L \subseteq \nabla$. Since $f(\nabla)$ is a prime $\mathbf{L}$-theory, it is sufficient to see that $f(\nabla)$ is regular. Suppose that $\mathbf{L} \vdash A$. Then $[A] \in E_L$, so $[A] \in \nabla$. Thus, we have $A \in f(\nabla)$, which is the desired result.

6. $X \in P_{\mathbf{M}_L}$ iff $f(X) \in P_c$, for all $X \subseteq W_{\mathbf{M}_L}$.

The 'if' part is proved as follows. By the assumption, there exists $A \in \mathsf{Wff}$ such that $V_c(A) = f(X)$. Taking first $\nabla \in W_{\mathbf{M}_L}$ such that $\nabla \in f_{\mathbf{M}_L}([A])$, $[A] \in \nabla$, so we have $A \in f(\nabla)$. By Lemma 3.13, $f(\nabla) \models_c A$, so we have $f(\nabla) \in V_c(A)$. Thus $f(\nabla) \in f(X)$, which implies that $\nabla \in X$. Hence $f_{\mathbf{M}_L}([A]) \subseteq X$. For the converse inclusion, suppose that $\nabla \in X$. Then $f(\nabla) \in f(X)$, so $f(\nabla) \in V_c(A)$. We have $f(\nabla) \models_c A$, and hence $A \in f(\nabla)$ by Lemma 3.13. Thus, we have $[A] \in \nabla$, which implies $\nabla \in f_{\mathbf{M}_L}([A])$. This is the desired result. Therefore there exists $A \in \mathsf{Wff}$ such that $f_{\mathbf{M}_L}([A]) = X$, and hence $X \in P_{\mathbf{M}_L}$.

The 'only if' part is proved as follows. By the assumption, there exists $A \in \mathsf{Wff}$ such that $f_{\mathbf{M}_L}([A]) = X$. Taking first $\Sigma \in W_c$ such that $\Sigma \in f(X)$, there exists $\nabla \in W_{\mathbf{M}_L}$ such that $f(\nabla) = \Sigma$ since f is surjective. Suppose that $A \in f(\nabla)$. Then $A \in \Sigma$, so $\Sigma \models_c A$. Thus, we have $\Sigma \in V_c(A)$. Hence $f(X) \subseteq V_c(A)$. For the converse inclusion, suppose that $\Sigma \in V_c(A)$. Then $\Sigma \models_c A$, so $A \in \Sigma$ by Lemma 3.13. Since f is surjective, there exists $\nabla \in W_{\mathbf{M}_L}$ such that $f(\nabla) = \Sigma$, so $A \in f(\nabla)$. Then $[A] \in \nabla$, so we have $\nabla \in f_{\mathbf{M}_L}([A])$,which is $\nabla \in X$. Hence $f(\nabla) \in f(X)$, which is $\Sigma \in f(X)$. This is the desired result. Therefore there exists $A \in \mathsf{Wff}$ such that $V_c(A) = f(X)$, and hence $f(X) \in P_c$. ■

Further, general frames have the following property.

Corollary 4.18 *Any regular logic* $\mathbf{L}$ *over* $\mathbf{R}$ *is complete with respect to the class of all general* $\mathbf{L}$*-frames.*

Proof.
Suppose that A is not a theorem of $\mathbf{L}$. Then by Theorem 3.19 there exists an $\mathbf{L}$-matrix $\langle \mathbf{M}, E \rangle$ in which A is not valid. By Corollary 4.15, A is not valid in $\langle \mathbf{M}, E \rangle_+$, which is a general $\mathbf{L}$-frame by Theorem 4.16. Hence, there exists a general $\mathbf{L}$-frame in which A is not valid. ■

4.3 Descriptive frames

We first introduce some notions for defining descriptive frame. Given a general $\mathbf{L}$-frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P \rangle$, we say that

(a) $\mathfrak{F}$ is *differentiated* if for any $a, b \in W$,

$$a = b \ \text{ iff } \ \forall X \in P(a \in X \Leftrightarrow b \in X),$$

(b) $\mathfrak{F}$ is *r-tight* if for any $a, b, c \in W$,

$$Rabc \text{ iff } \forall X \in P \forall Y \in P(a \in X \to Y \ \&\ b \in X \Rightarrow c \in Y),$$

(c) $\mathfrak{F}$ is $\Box$-*tight* if for any $a, b \in W$,

$$S_{\Box}ab \text{ iff } \forall X \in P(a \in \Box X \Rightarrow b \in X),$$

(d) $\mathfrak{F}$ is $\Diamond$-*tight* if for any $a, b \in W$,

$$S_{\Diamond}ab \text{ iff } \forall X \in P(b \in X \Rightarrow a \in \Diamond X),$$

(e) $\mathfrak{F}$ is *compact* if, for any families $\mathcal{X} \subseteq P$ and $\mathcal{Y} \subseteq \overline{P} = \{W - X \mid X \in P\}$,

$$\bigcap(\mathcal{X} \cup \mathcal{Y}) = \{a \mid \forall X \in \mathcal{X} \forall Y \in \mathcal{Y}(a \in X \ \&\ a \in Y)\} \neq \emptyset$$

whenever $\bigcap(\mathcal{X}' \cup \mathcal{Y}') \neq \emptyset$ for all finite subfamilies $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$.

A general **L**-frame $\mathfrak{F}$ is called *descriptive* if $\mathfrak{F}$ is differentiated, r-tight, $\Box$-tight, $\Diamond$-tight, compact and

$$O = \bigcap\{X \in P \mid O \subseteq X\}.$$

In the following, we will investigate the properties of these notions. For a general **L**-frame $\mathfrak{F} = \langle O, W, R, S_{\Box}, S_{\Diamond}, {}^*, P\rangle$ and $a \in W$, define

$$Pa = \{X \in P \mid a \in X\}, \qquad \overline{P}a = \{X \in \overline{P} \mid a \in X\}.$$

Proposition 4.19 *For every general* **L**-*frame* $\mathfrak{F} = \langle O, W, R, S_{\Box}, S_{\Diamond}, {}^*, P\rangle$ *and every* $a \in W$*,* Pa *is a prime filter in* $\mathfrak{F}^+$*.*

Proof.

First, suppose that $X, Y \in Pa$. Then we have $a \in X$ and $a \in Y$, so $a \in X \cap Y$. Hence $X \cap Y \in Pa$. Next, suppose that $X \in Pa$ and $X \subseteq Y$. Then we have $a \in X$, so $a \in Y$. Hence $Y \in Pa$. Finally, suppose that $X, Y \notin Pa$. Then $a \notin X$ and $a \notin Y$, so $a \notin X \cup Y$. Hence $X \cup Y \notin Pa$. ■

Proposition 4.20 *A general* **L**-*frame* $\mathfrak{F} = \langle O, W, R, S_{\Box}, S_{\Diamond}, {}^*, P\rangle$ *is differentiated iff, for any* $a \in W$*,*

$$\bigcap(Pa \cup \overline{P}a) = \{a\}.$$

Proof.

The 'if' part is proved as follows. It is clear that if $b = c$, then $\forall X \in P(b \in X \Leftrightarrow c \in X)$, so we see the converse. Suppose that $b \neq c$. Then $c \notin \{b\}$, so $c \notin \bigcap(Pb \cup \overline{P}b)$ by the assumption. Then there exists $X \in P$ or $X \in \overline{P}$ such that $b \in X$ and $c \notin X$. For the former, we have the desired result. For the latter, there exists $Y \in P$ such that $X = W - Y$. Then we have $b \notin Y$ and $c \in Y$, which imply the desired result.

The 'only if' part is proved as follows. First, suppose that $b \in \bigcap(Pa \cup \overline{P}a)$. Then for all $X \in P$ and $X \in \overline{P}$ satisfying $a \in X$, $b \in X$. This means that for all $X \in P$, $a \in X$ iff $b \in X$. By the assumption, we have $a = b$, which means that $b \in \{a\}$. Next, suppose that $b \in \{a\}$. Then $a = b$, so for any $X \in P$, $a \in X$ iff $b \in X$ by the assumption. To show that $b \in \bigcap(Pa \cup \overline{P}a)$, take any $X \in P$ and $X \in \overline{P}$. If $X \in Pa$, then $a \in X$ and hence $b \in X$. If $X \in \overline{P}a$, then $a \in X$ and hence $b \in X$. Thus, we have the desired result. ■

For $a \in W$, we write $a\uparrow = \{b \in W \mid S_{\Box}ab\}$. Note that $a\uparrow \subseteq X$ iff $a \in \Box X$.

Proposition 4.21 *A general* **L***-frame* $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ *is* $\Box$*-tight iff for any* $a \in W$*,*

$$a\uparrow = \bigcap\{X \in P \mid a\uparrow \subseteq X\}.$$

Proof.

The 'if' part is proved as follows. First, suppose that $S_\Box ab$. Then $b \in a\uparrow$, so $a\uparrow \subseteq X$ implies $b \in X$ for any $X \in P$. Here $a\uparrow \subseteq X$ means that $\forall b \in W(S_\Box ab \Rightarrow b \in X)$, that is, $a \in \Box X$. Therefore, $\forall X \in P(a \in \Box X \Rightarrow b \in X)$. Next, suppose that $\forall X \in P(a \in \Box X \Rightarrow b \in X)$. By the assumption, we have $b \in a\uparrow$, which means that $S_\Box ab$.

The 'only if' part is proved as follows. First, suppose that $b \in a\uparrow$. Taking any $X \in P$ such that $a\uparrow \subseteq X$, we have $b \in X$. Hence $b \in \bigcap\{X \in P \mid a\uparrow \subseteq X\}$. For the converse inclusion, suppose that $b \in \bigcap\{X \in P \mid a\uparrow \subseteq X\}$. Then $\forall X \in P(a\uparrow \subseteq X \Rightarrow b \in X)$, so $\forall X \in P(a \in \Box X \Rightarrow b \in X)$. By the assumption, we see that $S_\Box ab$, which is just $b \in a\uparrow$. ∎

Proposition 4.22 *A general* **L***-frame* $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ *is compact iff every prime filter* ∇ *in* $\mathfrak{F}^+$ *is of the form* Pa *for some* $a \in W$*.*

Proof.

The 'if' part is proved as follows. Suppose that $\mathcal{X} \subseteq P$, $\mathcal{Y} \subseteq \overline{P}$ and $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property. Let ∇ be the filter in $\mathfrak{F}^+$ generated by $\mathcal{X}$ and Δ be the ideal generated by $\{W - Y \mid Y \in \mathcal{Y}\}$. Assume that $Z \in \nabla \cap \Delta$. Then there exist $X_1, \cdots, X_m \in \mathcal{X}$ and $Y_1, \cdots, Y_n \in \mathcal{Y}$ such that $X_1 \cap \cdots \cap X_m \subseteq Z$ and $Z \subseteq (W - Y_1) \cup \cdots \cup (W - Y_n)$. That is, there exist finite $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$ such that $\bigcap \mathcal{X}' \subseteq Z$ and $Z \subseteq W - \bigcap \mathcal{Y}'$. Then $\bigcap \mathcal{X}' \subseteq W - \bigcap \mathcal{Y}'$, so $(\bigcap \mathcal{X}') \cap (\bigcap \mathcal{Y}') = \bigcap(\mathcal{X}' \cup \mathcal{Y}') = \emptyset$. This contradicts the assumption. Therefore, $\nabla \cap \Delta = \emptyset$. By 1 of Lemma 2.7, there exists a prime filter $\nabla' \supseteq \nabla$ such that $\nabla' \cap \Delta = \emptyset$.

By the assumption, we can write $\nabla' = Pa$ for some $a \in W$. Taking any $Z \in \nabla$, $Z \in \nabla'$, so $a \in Z$. Further, taking any $Z \in \Delta$, $Z \notin \nabla'$, so $a \notin Z$. Then for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, $a \in X$ and $a \notin W - Y$, i.e., $a \in Y$. Therefore, $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

The 'only if' part is proved as follows. Take a prime filter ∇ in $\mathfrak{F}^+$. Let $\Delta = P - \nabla$. Then Δ is a prime ideal by 4 of Lemma 2.7. Let $\mathcal{X} = \nabla$ and $\mathcal{Y} = \{W - X \mid X \in \Delta\}$. Consider a set $(\bigcap \mathcal{X}') \cap (\bigcap \mathcal{Y}')$, where $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$ are finite. Since $\mathcal{X}$ is a filter, $\bigcap \mathcal{X}' \in \mathcal{X}$. We have $\bigcap \mathcal{Y}' = \bigcap_i \{W - Y_i \mid Y_i \in \Delta\} = W - \bigcup_i Y_i$ with $\bigcup_i Y_i \in \Delta$, so $\bigcap \mathcal{Y}' \in \mathcal{Y}$. Assume that $(\bigcap(\mathcal{X}' \cup \mathcal{Y}') = \emptyset$. Then it is clear that $\bigcap \mathcal{X}' \subseteq W - \bigcap \mathcal{Y}'$. Since $\mathcal{X}$ is a filter, $W - \bigcap \mathcal{Y}' \in \mathcal{X}$. This is a contradiction. Therefore, $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property, so there exists $a \in \bigcap(\mathcal{X} \cup \mathcal{Y})$ by the assumption.

Now we will show that $\nabla = Pa$, where a is as above. First, suppose that $X \in \nabla$. Then $a \in X$ since $X \in \mathcal{X}$. This means that $X \in Pa$. Hence $\nabla \subseteq Pa$. For the converse inclusion, suppose that $X \in Pa$, that is, $a \in X$ for some $X \in P$. Then either $X \in \nabla$ or $X \in \Delta$. Assume that $X \in \Delta$. Then $W - X \in \mathcal{Y}$, so $a \notin \bigcap \mathcal{Y}$. But this contradicts the way of taking a. Therefore, we must have $X \in \nabla$. ∎

Theorem 4.23 *A general* **L***-frame* $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ *is descriptive iff it is isomorphic to* $(\mathfrak{F}^+)_+$*.*

Proof.

The 'if' part is proved as follows. Let $\nabla_1, \nabla_2, \nabla_3 \in W_{\mathfrak{F}^+}$, i.e., $\nabla_1, \nabla_2, \nabla_3$ be prime filters in P.

1. We will show that $(\mathfrak{F}^+)_+$ is differentiated. It is obvious to see the 'only if' part, so we will prove only the 'if' part. Suppose that $\nabla_1 \neq \nabla_2$. Then there exists $X \in P$ such that $X \in \nabla_1$ and $X \notin \nabla_2$. Then we have $\nabla_1 \in f_{\mathfrak{F}^+}(X)$, $\nabla_2 \notin f_{\mathfrak{F}^+}(X)$ and $f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$.

2. We will show that $(\mathfrak{F}^+)_+$ is r-tight. The 'if' part is proved as follows. Suppose that $R_{\mathfrak{F}^+}\nabla_1\nabla_2\nabla_3$ does not hold. Then there exist $X, Y \in P$ such that $X \to Y \in \nabla_1$, $X \in \nabla_2$ and $Y \notin \nabla_3$. Then we have $f_{\mathfrak{F}^+}(X), f_{\mathfrak{F}^+}(Y) \in P_{\mathfrak{F}^+}$ satisfying $\nabla_1 \in f_{\mathfrak{F}^+}(X) \to f_{\mathfrak{F}^+}(Y)$, $\nabla_2 \in f_{\mathfrak{F}^+}(X)$ and $\nabla_3 \notin f_{\mathfrak{F}^+}(Y)$.

   The 'only if' part is proved as follows. Suppose that $R_{\mathfrak{F}^+}\nabla_1\nabla_2\nabla_3$. Further, take any $f_{\mathfrak{F}^+}(X), f_{\mathfrak{F}^+}(Y) \in P_{\mathfrak{F}^+}$ satisfying $\nabla_1 \in f_{\mathfrak{F}^+}(X) \to f_{\mathfrak{F}^+}(Y)$ and $\nabla_2 \in f_{\mathfrak{F}^+}(X)$. Then we have $X \to Y \in \nabla_1$ and $X \in \nabla_2$, so $Y \in \nabla_3$, and hence $\nabla_3 \in f_{\mathfrak{F}^+}(Y)$.

3. We will show that $(\mathfrak{F}^+)_+$ is $\Box$-tight. The 'if' part is proved as follows. Suppose that $S_{\Box\mathfrak{F}^+}\nabla_1\nabla_2$ does not hold. Then there exists $X \in P$ such that $\Box X \in \nabla_1$ and $X \notin \nabla_2$. Then we have $f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$ satisfying $\nabla_1 \in \Box f_{\mathfrak{F}^+}(X)$ and $\nabla_2 \notin f_{\mathfrak{F}^+}(X)$.

   The 'only if' part is proved as follows. Suppose that $S_{\Box\mathfrak{F}^+}\nabla_1\nabla_2$. Further, take any $f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$ satisfying $\nabla_1 \in \Box f_{\mathfrak{F}^+}(X)$. Then we have $\Box X \in \nabla_1$, so $X \in \nabla_2$, and hence $\nabla_2 \in f_{\mathfrak{F}^+}(X)$.

4. We will show that $(\mathfrak{F}^+)_+$ is $\Diamond$-tight. The 'if' part is proved as follows. Suppose that $S_{\Diamond\mathfrak{F}^+}\nabla_1\nabla_2$ does not hold. Then there exists $X \in P$ such that $X \in \nabla_2$ and $\Diamond X \notin \nabla_1$ Then we have $f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$ satisfying $\nabla_2 \in f_{\mathfrak{F}^+}(X)$ and $\nabla_1 \notin \Diamond f_{\mathfrak{F}^+}(X)$.

   The 'only if' part is proved as follows. Suppose that $S_{\Diamond\mathfrak{F}^+}\nabla_1\nabla_2$. Further, take any $f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$ satisfying $\nabla_2 \in f_{\mathfrak{F}^+}(X)$. Then we have $X \in \nabla_2$, so $\Diamond X \in \nabla_1$, and hence $\nabla_1 \in \Diamond f_{\mathfrak{F}^+}(X)$.

5. We will show that $(\mathfrak{F}^+)_+$ is compact. By Theorem 4.14, $\mathfrak{F}^+$ is isomorphic to $((\mathfrak{F}^+)_+)^+$ under an isomorphism $f_{\mathfrak{F}^+}$. Take X, any prime filter in $((\mathfrak{F}^+)_+)^+$. Since $f_{\mathfrak{F}^+}$ is surjective, there exists $\nabla \in W_{\mathfrak{F}^+}$ such that $X = f_{\mathfrak{F}^+}(\nabla)$. Also, we have $f_{\mathfrak{F}^+}(\nabla) = \{f_{\mathfrak{F}^+}(X) \mid X \in \nabla\} = \{f_{\mathfrak{F}^+}(X) \mid \nabla \in f_{\mathfrak{F}^+}(X)\} = P_{\mathfrak{F}^+}\nabla$. Therefore, $(\mathfrak{F}^+)_+$ is compact by Proposition 4.22.

6. We will show that

   $$O_{\mathfrak{F}^+} = \bigcap\{f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+} \mid O_{\mathfrak{F}^+} \subseteq f_{\mathfrak{F}^+}(X)\}.$$

   First, suppose that $\nabla \in O_{\mathfrak{F}^+}$. Take any $f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$ such that $O_{\mathfrak{F}^+} \subseteq f_{\mathfrak{F}^+}(X)$. Then it is clear that $\nabla \in f_{\mathfrak{F}^+}(X)$. Therefore, $\nabla \in \bigcap\{f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+} \mid O_{\mathfrak{F}^+} \subseteq f_{\mathfrak{F}^+}(X)\}$.

   Next, suppose that $\nabla \notin O_{\mathfrak{F}^+}$. Then $O^+ \not\subseteq \nabla$, so there exists $X \in P$ such that $X \in O^+$ and $X \notin \nabla$. Then we have $f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$ and $\nabla \notin f_{\mathfrak{F}^+}(X)$. Moreover, suppose that $\nabla' \in O_{\mathfrak{F}^+}$. Then $O^+ \subseteq \nabla'$, so $X \in \nabla'$, which implies that $\nabla' \in f_{\mathfrak{F}^+}(X)$. Hence $O_{\mathfrak{F}^+} \subseteq f_{\mathfrak{F}^+}(X)$. Therefore, $\nabla \notin \bigcap\{f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+} \mid O_{\mathfrak{F}^+} \subseteq f_{\mathfrak{F}^+}(X)\}$.

Thus, we see that $(\mathfrak{F}^+)_+$ is descriptive.

The 'only if' part is proved as follows. By Proposition 4.19, for each $a \in W$, Pa is a prime filter in $\mathfrak{F}^+$. We define the mapping $f_{\mathfrak{F}} : W \to W_{\mathfrak{F}^+}$ by

$$f_{\mathfrak{F}}(a) = Pa, \quad \text{for } a \in W.$$

By Proposition 4.22, it is possible to write $W_{\mathfrak{F}^+} = \{Pa \mid a \in W\}$. Then it is clear that $f_{\mathfrak{F}}$ is surjective. Further, take $a, b \in W$ such that $a \neq b$. Since $\mathfrak{F}$ is differentiated, there exists $X \in P$ such that $a \in X$ and $b \notin X$. So, we have $X \in Pa$ and $X \notin Pb$. Hence $f_{\mathfrak{F}}(a) \neq f_{\mathfrak{F}}(b)$. Therefore, $f_{\mathfrak{F}}$ is injective.

It remains to show the following claims 1 through 6, where $a, b, c \in W$.

1. $Rabc$ iff $R_{\mathfrak{F}^+} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b) f_{\mathfrak{F}}(c)$.

$$\begin{array}{lll} & Rabc & \\ \text{iff} & \forall X \in P \forall Y \in P(a \in X \to Y \ \& \ b \in X \Rightarrow c \in Y) & (\mathfrak{F} \text{ is r-tight}) \\ \text{iff} & \forall X \in P \forall Y \in P(X \to Y \in Pa \ \& \ X \in Pb \Rightarrow Y \in Pc) & (\text{definition of } Pa) \\ \text{iff} & R_{\mathfrak{F}^+} PaPbPc & (\text{definition of } R_{\mathfrak{F}^+}) \\ \text{iff} & R_{\mathfrak{F}^+} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b) f_{\mathfrak{F}}(c) & (\text{definition of } f_{\mathfrak{F}}). \end{array}$$

2. $S_\Box ab$ iff $S_{\Box\mathfrak{F}^+} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b)$.

$$\begin{array}{llll} S_\Box ab & \text{iff} & \forall X \in P(a \in \Box X \Rightarrow b \in X) & (\mathfrak{F} \text{ is } \Box\text{-tight}) \\ & \text{iff} & \forall X \in P(\Box X \in Pa \Rightarrow X \in Pb) & \\ & \text{iff} & S_{\Box\mathfrak{F}^+} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b). & \end{array}$$

3. $S_\Diamond ab$ iff $S_{\Diamond\mathfrak{F}^+} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b)$.

$$\begin{array}{llll} S_\Diamond ab & \text{iff} & \forall X \in P(b \in X \Rightarrow a \in \Diamond X) & (\mathfrak{F} \text{ is } \Diamond\text{-tight}) \\ & \text{iff} & \forall X \in P(X \in Pb \Rightarrow \Diamond X \in Pa) & \\ & \text{iff} & S_{\Diamond\mathfrak{F}^+} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b). & \end{array}$$

4. $f_{\mathfrak{F}}(a^*) = g_{\mathfrak{F}^+}(f_{\mathfrak{F}}(a))$.

$$X \in f_{\mathfrak{F}}(a^*) \ \text{ iff } \ a^* \in X \ \text{ iff } \ a \notin -X \ \text{ iff } \ -X \notin f_{\mathfrak{F}}(a) \ \text{ iff } \ X \in g_{\mathfrak{F}^+}(f_{\mathfrak{F}}(a)).$$

5. $f_{\mathfrak{F}}(O) = O_{\mathfrak{F}^+}$.

   First, suppose that $\nabla \in f_{\mathfrak{F}}(O)$. Then there exists $a \in O$ such that $\nabla = f_{\mathfrak{F}}(a)$. Since $\mathfrak{F}$ is descriptive, $a \in \bigcap\{X \in P \mid O \subseteq X\}$. It means that $\forall X \in P(X \in O_+ \Rightarrow X \in Pa)$, that is, $O_+ \subseteq Pa$. Hence we have $\nabla \in O_{\mathfrak{F}^+}$.

   For the converse inclusion, suppose that $\nabla \in O_{\mathfrak{F}^+}$. Since ∇ is a prime filter in $\mathfrak{F}^+$, $\nabla = Pa$ for some $a \in W$ by Proposition 4.22. Then $\forall X \in P(X \in O_+ \Rightarrow X \in Pa)$, so $\forall X \in P(O \subseteq X \Rightarrow a \in X)$. This means that $a \in \bigcap\{X \in P \mid O \subseteq X\}$. Since $\mathfrak{F}$ is descriptive, $a \in O$. Therefore, $\nabla \in f_{\mathfrak{F}}(O)$.

6. $X \in P$ iff $f_{\mathfrak{F}}(X) \in P_{\mathfrak{F}^+}$.

   The 'if' part is proved as follows. By the assumption, there exists $Y \in P$ such that $f_{\mathfrak{F}}(X) = f_{\mathfrak{F}}(Y)$. Since $f_{\mathfrak{F}}$ is bijective, $X = Y$. Therefore, we have $X \in P$.

   The 'only if' part is proved as follows. By the assumption, $f_{\mathfrak{F}}(X) = \{f_{\mathfrak{F}}(a) \mid a \in X\} = f_{\mathfrak{F}^+}(a)$ Since $f_{\mathfrak{F}^+}(a) \in P_{\mathfrak{F}^+}$, we have $f_{\mathfrak{F}}(X) \in P_{\mathfrak{F}^+}$.

■

Theorem 4.24 **L** *is characterized by the class of descriptive* **L***-frames.*

Proof.

Let $\mathfrak{F}$ be any descriptive **L**-frames. By the definition of **L**-frames, if A is a theorem of **L**, then $\mathfrak{F} \models A$. On the other hand, if A is not a theorem of **L**, then A is not valid in the Lindenbaum matrix $\langle \mathbf{M}_L, E_L \rangle$ for **L**. By Theorem 4.15, A is not valid in $\langle \mathbf{M}_L, E_L \rangle_+$. By Theorem 4.17, A is not valid in the universal **L**-frame $\gamma\mathfrak{F}_c$. Further, we see that $\gamma\mathfrak{F}_c$ is descriptive by Theorems 4.4, 4.17 and 4.23. Therefore, there exists a descriptive **L**-frame in which A is not valid. ■

4.4 Truth-preserving operations

In this section, we consider the truth-preserving operations of general frames and the correspondence between general frames and matrices. Throughout this section, let $\langle \mathbf{M}, E \rangle$ and $\langle \mathbf{M}', E' \rangle$ be **L**-matrices, where $\mathbf{M} = \langle M, \cap, \cup, \rightarrow, -, \Box, \Diamond \rangle$ and $\mathbf{M}' = \langle M', \cap, \cup, \rightarrow, -, \Box, \Diamond \rangle$. Further, let $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P \rangle$ and $\mathfrak{F}' = \langle O', W', R', S'_\Box, S'_\Diamond, ^{*\prime}, P' \rangle$ be general **L**-frames.

A general **L**-frame $\mathfrak{F}' = \langle O', W', R', S'_\Box, S'_\Diamond, ^{*\prime}, P' \rangle$ is a *generated subframe* of $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P \rangle$ if (a) $W' \subseteq W$ is upward closed, and satisfies

1. $a \in W'$ & $Rabc \Rightarrow b, c \in W'$
2. $c \in W'$ & $Rabc \Rightarrow a, b \in W'$
3. $a \in W'$ & $S_\Box ab \Rightarrow b \in W'$
4. $a \in W'$ & $S_\Diamond ab \Rightarrow b \in W'$
5. $a \in W' \Rightarrow a^* \in W'$

for all $a, b, c \in W$, (b) $R', S'_\Box, S'_\Diamond$ and $^{*\prime}$ are the restrictions of $R, S_\Box, S_\Diamond$ and *, respectively, to W', (c) $O' \subseteq W'$, and (d) $P' = \{X \cap W' \mid X \in P\}$.

Theorem 4.25 *Let $\mathfrak{F}' = \langle O', W', R', S'_\Box, S'_\Diamond, ^{*\prime}, P' \rangle$ be a generated subframe of $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P \rangle$. Then the mapping h^+ defined by*

$$h^+(X) = X \cap W', \quad \textit{for every } X \in P,$$

is a homomorphism, and $\mathfrak{F}'^+$ is the homomorphic image of $\mathfrak{F}^+$ under h^+.

Proof.

First, we show that h^+ is a homomorphism. Let $X, Y \in P$.

1. $h^+(X \cap Y) = (X \cap Y) \cap W' = (X \cap W') \cap (Y \cap W') = h^+(X) \cap h^+(Y)$.
2. $h^+(X \cup Y) = (X \cup Y) \cap W' = (X \cap W') \cup (Y \cap W') = h^+(X) \cup h^+(Y)$.

3. First, suppose that $a \in h^+(X \to Y)$. To show that $a \in h^+(X) \to h^+(Y)$, suppose that $R'abc$ and $b \in h(X)$. Then $a \in X \to Y$ and $b \in X$, so $c \in Y$. Further, since $c \in W'$, we have $c \in h^+(Y)$, which is the desired result.

   For the converse inclusion, suppose that $a \in h^+(X) \to h^+(Y)$. Then it is clear that $a \in W'$, so it is sufficient to show that $a \in X \to Y$. Suppose that $Rabc$ and $b \in X$. Since $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$, we have $b, c \in W'$. Thus we have $b \in h^+(X)$, so $c \in h^+(Y)$. Hence we have $x \in Y$, which is the desired result.

4. $$\begin{array}{rll} a \in h^+(-X) & \text{iff} & a \in -X \ \&\ a \in W' \\ & \text{iff} & a^* \notin X \ \&\ a^* \in W' \\ & \text{iff} & a^* \notin h^+(X) \ \&\ a \in W' \\ & \text{iff} & a \in -h^+(X). \end{array}$$

5. First, suppose that $a \in h^+(\Box X)$. To show that $a \in \Box h^+(X)$, suppose that $S'_\Box ab$. Then $a \in \Box X$, so $b \in X$. Since $b \in W'$, we have $b \in h^+(X)$, which is the desired result.

   For the converse inclusion, suppose that $a \in \Box h^+(X)$. Then it is clear that $a \in W'$, so it is sufficient to show that $a \in \Box X$. Suppose that $S_\Box ab$. Then we have $b \in h^+(X)$, so $b \in X$, which is the desired result.

6. First, suppose that $a \in h^+(\Diamond X)$. Then we have $a \in W'$ and there exists $b \in X$ satisfying $S_\Diamond ab$. Since $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$, $b \in W'$. Thus we have $b \in h^+(X)$. Therefore, $a \in \Diamond h^+(X)$.

   For the converse inclusion, suppose that $a \in \Diamond h^+(X)$. Then there exists $b \in h^+(X)$ satisfying $S'_\Diamond ab$. So, we have $b \in X$, and hence $a \in \Diamond X$. Since $a \in W'$, we have $a \in h^+(\Diamond X)$.

It remains to show that $h^+(P) = P'$ and $O'^+ \cap h^+(P) = O'^+$. For the former, suppose first that $X \in h^+(P)$. Then there exists $Y \in P$ such that $X = h^+(Y)$, i.e., $X = Y \cap W'$. By the definition of generated subframes, we have $Y \cap W' \in P'$, which is just $X \in P'$. Next suppose that $X \in P'$. Then there exists $Y \in P$ such that $X = Y \cap W'$, i.e., $X = h^+(Y)$. So we have $X \in h^+(P)$.

For the latter, it is obvious that $O'^+ \cap h^+(P) \subseteq O'^+$. So, we see the converse, which is easy to see because $O'^+ \subseteq P' = h^+(P)$ by the former. ■

Theorem 4.26 *Let h be a surjective homomorphism of $\langle \mathbf{M}, E\rangle$ in $\langle \mathbf{M}', E'\rangle$. Then the map h_+ defined by*

$$h_+(\nabla) = h^{-1}(\nabla), \qquad \textit{for every prime filter } \nabla \textit{ in } \mathbf{M}',$$

is an isomorphism of $\langle \mathbf{M}', E'\rangle_+$ onto a generated subframe of $\langle \mathbf{M}, E\rangle_+$.

Proof.

Let $W = \{\nabla' \in W_{\mathbf{M}} \mid h^{-1}(E') \subseteq \nabla'\}$. Then it is clear that W is upward closed. Further, let $O = \{\nabla' \in O_{\mathbf{M}} \mid h^{-1}(E') \subseteq \nabla'\}$. Let $\langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ is the subframe of $\langle \mathbf{M}, E\rangle_+$ generated by W. We show that h_+ is an isomorphism of $\langle \mathbf{M}', E'\rangle_+$ onto $\langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$.

First, we see that h_+ is bijective. We should see that

$$x \in \nabla' \quad \text{iff} \quad h(x) \in h(\nabla'), \qquad \text{for all } x \in M \text{ and } \nabla' \in W.$$

Since the 'only if' part is clear, it is sufficient to show the 'if' part. Suppose that $h(x) \in h(\nabla')$. Then there exists $y \in \nabla'$ such that $h(x) = h(y)$. So, we have $h(y) \to h(x) \in E'$, and hence $h(y \to x) \in E'$ since h is a homomorphism. Also, we have $y \to x \in h^{-1}(E')$, so $y \to x \in \nabla'$ by the way of taking W. Since ∇' is a filter, we have $x \in \nabla'$. It follows that $h(\nabla')$ is a prime filter in $\mathbf{M}'$, for each $\nabla' \in W$. Also,

$$x \in h_+(h(\nabla')) \quad \text{iff} \quad h(x) \in h(\nabla') \quad \text{iff} \quad x \in \nabla'.$$

Hence $h_+(h(\nabla)) = \nabla$, so h_+ is surjective. Suppose that $\nabla_1 \neq \nabla_2$, for $\nabla_1, \nabla_2 \in W_{\mathbf{M}'}$. Then there exists $x \in M'$ such that $x \in \nabla_1$ and $x \notin \nabla_2$. Since h is surjective, there exists $y \in M$ such that $h(y) = x$. So, we have $y \in h_+(\nabla_1)$ and $y \notin h_+(\nabla_2)$, and hence $h_+(\nabla_1) \neq h_+(\nabla_2)$. Thus, h_+ is injective.

Next, we see that h_+ preserves the relations. Note that for $\nabla \in W_{\mathbf{M}}$, $h_+(\nabla)$ is a prime filter by Lemma 2.9. Let $\nabla_1, \nabla_2, \nabla_3 \in W_{\mathbf{M}'}$.

1. Suppose first that $R_{\mathbf{M}'}\nabla_1\nabla_2\nabla_3$. To show that $Rh_+(\nabla_1)h_+(\nabla_2)h_+(\nabla_3)$, suppose that $x \to y \in h_+(\nabla_1)$ and $x \in h_+(\nabla_2)$. Then $h(x) \to h(y) \in \nabla_1$ and $h(x) \in \nabla_2$, so $h(y) \in \nabla_3$. Hence $y \in h_+(\nabla_3)$, which is the desired result.

   Suppose next that $Rh_+(\nabla_1)h_+(\nabla_2)h_+(\nabla_3)$. To show that $R_{\mathbf{M}'}\nabla_1\nabla_2\nabla_3$, suppose that $x \to y \in \nabla_1$ and $x \in \nabla_2$, for every $x, y \in M'$. Since h is surjective, there exist $z, w \in M$ such that $h(z) = x$ and $h(w) = y$. Since h is a homomorphism, we have $h(z \to w) \in \nabla_1$ and $h(z) \in \nabla_2$. Then $z \to w \in h_+(\nabla_1)$ and $z \in h_+(\nabla_2)$, so $w \in h_+(\nabla_3)$. Hence $h(w) \in \nabla_3$, which is the desired result.

2. Suppose first that $S_{\Box\mathbf{M}'}\nabla_1\nabla_2$. To show that $S_\Box h_+(\nabla_1)h_+(\nabla_2)$, suppose that $\Box x \in h_+(\nabla_1)$. Then $\Box h(x) \in \nabla_1$, so $h(x) \in \nabla_2$. Hence $x \in h_+(\nabla_2)$, which is the desired result.

   Suppose next that $S_\Box h_+(\nabla_1)h_+(\nabla_2)$. To show that $S_{\Box\mathbf{M}'}\nabla_1\nabla_2$, suppose that $\Box x \in \nabla_1$, for every $x \in M'$. Since h is surjective, there exists $y \in M$ such that $h(y) = x$. Since h is a homomorphism, $h(\Box y) \in \nabla_1$. Then $\Box y \in h_+(\nabla_1)$, so $y \in h_+(\nabla_2)$. Hence $h(y) \in \nabla_2$, which is the desired result.

3. Suppose first that $S_{\Diamond\mathbf{M}'}\nabla_1\nabla_2$. To show that $S_\Diamond h_+(\nabla_1)h_+(\nabla_2)$, suppose that $x \in h_+(\nabla_2)$. Then $h(x) \in \nabla_2$, so $h(\Diamond x) \in \nabla_1$. Hence $\Diamond x \in h_+(\nabla_1)$, which is the desired result.

   Suppose next that $S_\Diamond h_+(\nabla_1)h_+(\nabla_2)$. To show that $S_{\Diamond\mathbf{M}'}\nabla_1\nabla_2$, suppose that $x \in \nabla_2$, for $x \in M'$. Since h is surjective, there exists $y \in M$ such that $h(y) = x$. Then $y \in h_+(\nabla_2)$, so $\Diamond y \in h_+(\nabla_1)$. Since h is a homomorphism, $\Diamond h(y) \in \nabla_1$, which is the desired result.

4. $x \in h_+(\nabla^*)$ iff $h(x) \in \nabla^*$ iff $-h(x) \notin \nabla$ iff $-x \notin h_+(\nabla)$ iff $x \in g_{\mathbf{M}'}(h_+(\nabla))$.

5. Suppose first that $\nabla \in O_{\mathbf{M}'}$. Then $E' \subseteq \nabla$, so $h^{-1}(E') \subseteq h^{-1}(\nabla)$. This means that $h_+(\nabla) \in O$.

   Suppose next that $\nabla \notin O_{\mathbf{M}'}$. Then $E' \nsubseteq \nabla$, so there exists $x \in M'$ such that $x \in E'$ and $x \notin \nabla$. Since h is surjective, there exists $y \in M$ such that $h(y) = x$. Then we have $y \in h^{-1}(E')$ and $y \notin h^{-1}(\nabla)$, which imply that $h^{-1}(E') \nsubseteq h_+(\nabla)$. This means that $h_+(\nabla) \notin O$.

6. Suppose first that $h_+(X) \in P$. Then there exists $x \in M$ such that $h_+(X) = \{\nabla' \in W \mid x \in \nabla'\}$. Now $x \in \nabla'$ iff $h(x) \in h(\nabla')$, for all $x \in M$ and all $\nabla' \in M$, so

$$\nabla' \in h_+(X) \quad \text{iff} \quad h(x) \in h(\nabla'), \qquad \text{for all } x \in M \text{ and all } \nabla' \in M.$$

   This means that $X = \{h(\nabla') \mid h(x) \in h(\nabla')\}$. We have seen that $h(\nabla') \in W_{\mathbf{M}'}$, and hence $X \in P_{\mathbf{M}'}$.

   Suppose next that $X \in P_{\mathbf{M}'}$. Then there exists $x \in M'$ such that $X = \{\nabla \in W_{\mathbf{M}'} \mid x \in \nabla\}$. Since h is surjective, there exists $y \in M$ such that $x = h(y)$. So,

$$\nabla \in X \quad \text{iff} \quad y \in h^{-1}(\nabla), \qquad \text{for all } \nabla \in W_{\mathbf{M}'}.$$

   We have seen that h_+ is bijective, so this can be written by $h_+(X) = \{h_+(\nabla) \mid y \in h_+(\nabla)\}$, and $h_+(\nabla) \in h_+(W_{\mathbf{M}'}) = W$. Therefore, $h_+(X) \in P$.

■

The category of **L**-matrices is the category whose objects are the **L**-matrices and the maps lattice homomorphisms preserving the operations of the matrix. In the dual category of **L**-frames, we can define a relevant p-morphism. This idea comes from relevant maps defined for relevant spaces in [58].

Let $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, {}^*, P\rangle$ and $\mathfrak{F}' = \langle O', W', R', S'_\Box, S'_\Diamond, {}^{*\prime}, P'\rangle$ be general **L**-frames. Then a mapping $f : W \to W'$ is a *relevant p-morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ if it is a surjection satisfying the following conditions. For all $a, b, c \in W$, $a', b', c' \in W'$ and $X \in P'$,

(m1) $Rabc \Rightarrow R'f(a)f(b)f(c)$

(m2) $R'a'b'f(c) \Rightarrow \exists a \in W \exists b \in W(Rabc \;\&\; a' \leq' f(a) \;\&\; b' \leq' f(b))$

(m3) $R'f(a)b'c' \Rightarrow \exists b \in W \exists c \in W(Rabc \;\&\; b' \leq' f(b) \;\&\; f(c) \leq' c')$

(m4) $S_\Box ab \Rightarrow S'_\Box f(a)f(b)$

(m5) $S'_\Box f(a)b' \Rightarrow \exists b \in W(S_\Box ab \;\&\; f(b) \leq' b')$

(m6) $S_\Diamond ab \Rightarrow S'_\Diamond f(a)f(b)$

(m7) $S'_\Diamond f(a)b' \Rightarrow \exists b \in W(S_\Diamond ab \;\&\; b' \leq' f(b))$

(m8) $f(a^*) = (f(a))^{*\prime}$

(m9) $f^{-1}(O') = O$

(m10) $f^{-1}(X) \in P$.

Theorem 4.27 *If f is a relevant p-morphism from $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ to $\mathfrak{F}' = \langle O', W', R', S'_\Box, S'_\Diamond, ^{*\prime}, P'\rangle$, then f^+, defined by*

$$f^+(X) = f^{-1}(X), \qquad \text{for every } X \in P',$$

is an isomorphism of $\mathfrak{F}'^+$ into $\mathfrak{F}^+$.

Proof.
First, we show that f^+ is injective. Taking $X, Y \in P'$ such that $X \neq Y$, there exists $a \in W'$ such that $a \in X$ and $a \notin Y$. By (m10), we have $f^{-1}(X), f^{-1}(Y) \in P$. Since f is surjective, there exists $b \in W$ such that $a = f(b)$. Hence we have $b \in f^{-1}(X)$ and $b \notin f^{-1}(Y)$, which imply that $f^+(X) \neq f^+(Y)$.

Next, we show that f^+ is a homomorphism. Let $X, Y \in P'$.

1.
$$\begin{array}{rll} a \in f^+(X \cap Y) & \text{iff} & f(a) \in X \cap Y \\ & \text{iff} & f(a) \in X \ \& \ f(a) \in Y \\ & \text{iff} & a \in f^+(X) \ \& \ a \in f^+(Y) \\ & \text{iff} & a \in f^+(X) \cap f^+(Y). \end{array}$$

2. As in 1, we see that $f^+(X \cup Y) = f^+(X) \cup f^+(Y)$.

3. First, suppose that $a \in f^+(X \to Y)$. To show that $a \in f^+(X) \to f^+(Y)$, suppose that $Rabc$ and $b \in f^+(X)$. Then we have $f(a) \in X \to Y$ and $f(b) \in X$. By (m1), $R'f(a)f(b)f(c)$, so $f(c) \in Y$. This means that $c \in f^+(Y)$, which is the desired result.

   For the converse, suppose that $a \in f^+(X) \to f^+(Y)$. To show that $a \in f^+(X \to Y)$, suppose that $R'f(a)b'c'$ and $b' \in X$. By (m3), there exist $b, c \in W$ such that $Rabc$ and $b' \leq' f(b)$ and $f(c) \leq' c'$. Since X is upward closed, $f(b) \in X$, so we have $b \in f^+(X)$. Hence $c \in f^+(Y)$, so $c' \in Y$. This is the desired result.

4. By (m8), $a \in f^+(-X)$ iff $f(a) \in -X$ iff $(f(a))^{*\prime} \notin X$ iff $f(a^*) \notin X$ iff $a^* \notin f^+(X)$ iff $a \in -f^+(X)$.

5. First, suppose that $a \in f^+(\Box X)$. To show that $a \in \Box f^+(X)$, suppose that $S_\Box ab$. Then $f(a) \in \Box X$, and $S'_\Box f(a)f(b)$ by (m4). So, we have $f(b) \in X$, and hence $b \in f^+(X)$. This is the desired result.

   For the converse, suppose that $a \in \Box f^+(X)$. To show that $a \in f^+(\Box X)$, suppose that $S'_\Box f(a)b'$. By (m5), there exists $b \in W_1$ such that $S_\Box ab$ and $f(b) \leq' b'$. Then $b \in f^+(X)$, so we have $f(b) \in X$. Since X is upward closed, we have $b' \in X$, which is the desired result.

6. First, suppose that $a \in f^+(\Diamond X)$. Then $f(a) \in \Diamond X$, so there exists $b' \in X$ such that $S'_\Diamond f(a)b'$. By (m7), there exists $b \in W$ such that $S_\Diamond ab$ and $b' \leq' f(b)$. Since X is upward closed, we have $f(b) \in X$, so $b \in f^+(X)$. Therefore, we have $a \in \Diamond f^+(X)$.

   For the converse, suppose that $a \in \Diamond f^+(X)$. Then there exists $b \in f^+(X)$ such that $S_\Diamond ab$. Then $f(b) \in X$, and $S'_\Diamond f(a)f(b)$ by (m6). So, we have $f(a) \in \Diamond X$, and hence $a \in f^+(\Diamond X)$. ∎

Theorem 4.28 *If an* $\mathbf{L}$*-matrix* $\langle \mathbf{M}', E' \rangle$ *is a submatrix of* $\langle \mathbf{M}, E \rangle$*, then the map* f_+ *defined by*

$$f_+(\nabla) = \nabla \cap M', \text{ for every } \nabla \in W_{\mathbf{M}},$$

is a relevant p-morphism from $\langle \mathbf{M}, E \rangle_+$ *to* $\langle \mathbf{M}', E' \rangle_+$*.*

Proof.

For every $\nabla \in W_{\mathbf{M}}'$, there exists $\nabla' \in W_{\mathbf{M}}$ such that $\nabla \subseteq \nabla'$ and $\nabla = \nabla' \cap M'$ by Lemma 2.8. Hence, f_+ is surjective.

It remains to show that f_+ is a relevant p-morphism from $\langle \mathbf{M}, E \rangle_+$ to $\langle \mathbf{M}', E' \rangle_+$. It is sufficient to check (m1) through (m10).

(m1) Suppose that $R_{\mathbf{M}} \nabla_1 \nabla_2 \nabla_3$. To show $R_{\mathbf{M}'} f_+(\nabla_1) f_+(\nabla_2) f_+(\nabla_3)$, suppose that $x \to y \in f_+(\nabla_1)$ and $x \in f_+(\nabla_2)$ for $x, y \in M'$. Then we have $x \to y \in \nabla_1$ and $x \in \nabla_2$, so $y \in \nabla_3$. Hence, $y \in f_+(\nabla_3)$, which is the desired result.

(m2) Suppose that $R_{\mathbf{M}'} \nabla_1 \nabla_2 f_+(\nabla_3)$. Let $\nabla_4 = \{y \in M \mid \exists x \in \nabla_1 (x \leq y)\}$ and $\nabla_5 = \{y \in M \mid \exists x \in \nabla_2 (x \leq y)\}$.

Suppose that $y_1, y_2 \in \nabla_4$. Then there exist $x_1, x_2 \in \nabla_1$ such that $x_1 \leq y_1$ and $x_2 \leq y_2$. Since ∇_1 is a filter, $x_1 \cap x_2 \in \nabla_1$. Also, $x_1 \cap x_2 \leq y_1 \cap y_2$, so $y_1 \cap y_2 \in \nabla_4$. Further, suppose that $y \in \nabla_4$ and $y \leq z$. Then there exists $x \in \nabla_1$ such that $x \leq y$. It follows that $x \leq z$, so $z \in \nabla_4$. Therefore, ∇_4 is a filter. Similarly, we see that ∇_5 is a filter.

Now suppose that $y \to z \in \nabla_4$ and $y \in \nabla_5$. Then there exists $x_1 \in \nabla_1$ such that $x_1 \leq y \to z$, and there exists $x_2 \in \nabla_2$ such that $x_2 \leq y$. Since ∇_1 and ∇_2 are filters, $y \to z \in \nabla_1$ and $y \in \nabla_2$. So, we have $z \in f_+(\nabla_3)$. Thus $z \in \nabla_3$. Therefore, $R \nabla_4 \nabla_5 \nabla_3$.

By Lemmas 4.5 and 4.6, there exist prime filters $\nabla_4' \supseteq \nabla_4$ and $\nabla_5' \supseteq \nabla_5$ such that $R_{\mathbf{M}} \nabla_4' \nabla_5' \nabla_3$. Now suppose that $x \in \nabla_1$. Since $x \leq x$, $x \in \nabla_4$. So, we have $x \in \nabla_4'$, and hence $x \in f_+(\nabla_4')$. Therefore, $\nabla_1 \subseteq f_+(\nabla_4')$. Similarly, we see that $\nabla_2 \subseteq f_+(\nabla_5')$.

(m3) Suppose that $R_{\mathbf{M}'} f_+(\nabla_1) \nabla_2 \nabla_3$. Let $\nabla_4 = \{y \in M \mid \exists x \in \nabla_2 (x \leq y)\}$. As in (m2), we see that ∇_4 is a filter. Further, let $\nabla_5 = \{y \in M \mid \exists x \in \nabla_4 (x \to y \in \nabla_1)\}$ and $\Delta = \{x \in M' \mid x \notin \nabla_3\}$.

As in 2 of Lemma 4.7, we see that ∇_5 is a filter satisfying $R \nabla_1 \nabla_4 \nabla_5$. Now we will show that Δ is an ideal satisfying that $\nabla_5 \cap \Delta = \emptyset$. First, suppose that $x, y \in \Delta$. Then $x, y \notin \nabla_3$. Since ∇_3 is a prime filter, $x \cup y \notin \nabla_3$. Therefore, $x \cup y \in \Delta$. Secondly, suppose that $x \in \Delta$ and $y \leq x$. Then $x \notin \nabla_3$. Since ∇_3 is a filter, $y \notin \nabla_3$, which implies that $y \in \Delta$. To show the remainder, assume that $y \in \nabla_5 \cap \Delta$. Then there exists $x \in \nabla_4$ such that $x \to y \in \nabla_1$ and that $y \notin \nabla_3$. So, there exists $z \in \nabla_2$ such that $z \leq x$. Since ∇_1 is a filter and $x \to y \leq z \to y$, $z \to y \in \nabla_1$. Since $z, y \in M'$, $z \to y \in M'$, and hence $z \to y \in f_+(\nabla_1)$. Thus we have a contradiction, so $\nabla_5 \cap \Delta = \emptyset$.

By 1 of Lemma 2.7, there exists a prime filter $\nabla_5' \supseteq \nabla_5$ such that $\nabla_5' \cap \Delta = \emptyset$. It follows that $R \nabla_1 \nabla_4 \nabla_5'$, so that there exists a prime filter $\nabla_4' \supseteq \nabla_4$ such that $R_{\mathbf{M}'} \nabla_1 \nabla_4' \nabla_5'$ by Lemma 4.6. Finally, we show that $\nabla_2 \subseteq f_+(\nabla_4')$ and $f_+(\nabla_5') \subseteq \nabla_3$.

For the former, suppose that $x \in \nabla_2$. Since $x \leq x$, we have $x \in \nabla_4$, and hence $x \in \nabla'_4$. Since $x \in M'$, $x \in f_+(\nabla'_4)$. For the latter, suppose that $x \in f_+(\nabla'_5)$. Then $x \in \nabla'_5$, so $x \notin \Delta$. Hence $x \in \nabla_3$.

(m4) Suppose that $S_{\Box\mathbf{M}}\nabla_1\nabla_2$. To show that $S_{\Box\mathbf{M}'}f_+(\nabla_1)f_+(\nabla_2)$, suppose that $\Box x \in f_+(\nabla_1)$ for $x \in M'$. Then $\Box x \in \nabla_1$, so $x \in \nabla_2$. We have $x \in f_+(\nabla_2)$, which is the desired result.

(m5) Suppose that $S_{\Box\mathbf{M}'}f_+(\nabla_1)\nabla_2$. Let $\nabla_3 = \{x \in M \mid \Box x \in \nabla_1\}$ and $\Delta = \{x \in M' \mid x \notin \nabla_2\}$. As in 2 of Lemma 4.8, we see that ∇_3 is a filter satisfying $S_\Box\nabla_1\nabla_3$. Also, we see that Δ is an ideal as in (m3).

Now assume that $x \in \nabla_3 \cap \Delta$. Then $x \in M'$,$\Box x \in \nabla_1$ and $x \notin \nabla_2$, so we have $\Box x \in f_+(\nabla_1)$ and $x \notin \nabla_2$. This contradicts the assumption, so $\nabla_3 \cap \Delta = \emptyset$. By 1 of Lemma 2.7, there exists a prime filter $\nabla'_3 \supseteq \nabla_3$ such that $\nabla'_3 \cap \Delta = \emptyset$. Then it is clear that $S_{\Box\mathbf{M}'}\nabla_1\nabla'_3$. Moreover, suppose that $x \in f_+(\nabla'_3)$. Then $x \in \nabla'_3$, so $x \notin \Delta$. Hence we have $x \in \nabla_2$. Therefore, $f_+(\nabla'_3) \subseteq \nabla_2$.

(m6) Suppose that $S_{\Diamond\mathbf{M}}\nabla_1\nabla_2$. To show that $S_{\Diamond\mathbf{M}'}f_+(\nabla_1)f_+(\nabla_2)$, suppose that $x \in f_+(\nabla_2)$. Then $x \in \nabla_2$ and $x \in M'$. By the assumption, we have $\Diamond x \in \nabla_1$. Further, $\Diamond x \in M'$, so $\Diamond x \in f_+(\nabla_1)$. This is the desired result.

(m7) Suppose that $S_{\Diamond\mathbf{M}'}f_+(\nabla_1)\nabla_2$. Let $\nabla_3 = \{y \in M \mid \exists x \in \nabla_2(x \leq y)\}$ and $\Delta = \{x \in M' \mid \Diamond x \notin \nabla_1\}$. As in (m2), we see that ∇_3 is a filter. Moreover, we will see that Δ is an ideal as in Lemma 4.9. Now assume that $y \in \nabla_3 \cap \Delta$. Then there exists $x \in \nabla_2$ such that $x \leq y$, and that $y \in M'$ and $\Diamond y \notin \nabla_1$. Since ∇_1 is a filter and $\Diamond x \leq \Diamond y$, $\Diamond x \notin \nabla_1$. So, we have $\Diamond x \notin f_+(\nabla_1)$. This contradicts the hypothesis, and hence $\nabla_3 \cap \Delta = \emptyset$.

By 1 of Lemma 2.7, there exists a prime filter $\nabla'_3 \supseteq \nabla_3$ such that $\nabla'_3 \cap \Delta = \emptyset$. Now suppose that $x \in \nabla'_3$. Then $x \notin \Delta$, so $\Diamond x \in \nabla_1$. Hence $S_{\Diamond\mathbf{M}'}\nabla_1\nabla'_3$. Finally, suppose that $x \in \nabla_2$. Since $x \leq x$, we have $x \in \nabla_3$, and hence $x \in \nabla'_3$. Since $x \in M'$, $x \in f_+(\nabla'_3)$. Therefore, $\nabla_2 \subseteq f_+(\nabla'_3)$.

(m8)

$$\begin{array}{rll} x \in f_+(g_{\mathbf{M}}(\nabla)) & \text{iff} & x \in g_{\mathbf{M}}(\nabla) \ \&\ x \in M' \\ & \text{iff} & -x \notin \nabla \ \&\ -x \in M' \\ & \text{iff} & -x \notin f_+(\nabla) \\ & \text{iff} & x \in g_{\mathbf{M}'}(f_+(\nabla)). \end{array}$$

(m9)

$$\begin{array}{rll} \nabla \in f_+^{-1}(O_{\mathbf{M}'}) & \text{iff} & f_+(\nabla) \in O_{\mathbf{M}'} \\ & \text{iff} & E' \subseteq f_+(\nabla) \\ & \text{iff} & E \cap M' \subseteq \nabla \cap M' \\ & \text{iff} & E \subseteq \nabla \\ & \text{iff} & \nabla \in O_{\mathbf{M}}. \end{array}$$

(m10) Taking $X \in P_{\mathbf{M}'}$, there exists $x \in M'$ such that $X = \{\nabla \in W_{\mathbf{M}'} \mid x \in \nabla\}$. Then for all $\nabla' \in W_{\mathbf{M}}$,

$$\nabla' \in f_+^{-1}(X) \ \text{ iff } \ f_+(X) \in X \ \text{ iff } \ \exists x \in \nabla(\nabla = \nabla' \cap M') \ \text{ iff } \ x \in \nabla'.$$

Thus, $f_+^{-1}(X) = \{\nabla' \in W_{\mathbf{M}} \mid x \in \nabla'\}$. Therefore, $f_+^{-1}(X) \in P_{\mathbf{M}}$. ■

The *disjoint union* of a family $\{\mathfrak{F}_i = \langle O_i, W_i, R_i, S_{\Box i}, S_{\Diamond i}, g_i, P_i\rangle \mid i \in I\}$ of pairwise disjoint frames is the frame $\sum_{i\in I}\mathfrak{F}_i = \langle O, W, R, S_\Box, S_\Diamond, {}^*, P\rangle$ where $O = \bigcup_{i\in I} O_i$, $W = \bigcup_{i\in I} W_i$, $R = \bigcup_{i\in I} R_i$, $S_\Box = \bigcup_{i\in I} S_{\Box i}$, $S_\Diamond = \bigcup_{i\in I} S_{\Diamond i}$, ${}^* = \bigcup_{i\in I} g_i$, $P = \{\bigcup_{i\in I} X_i \mid X_i \in P_i, \text{for all } i \in I\}$.

Note that every $\mathfrak{F}_i$ is a generated subframe of $\sum_{i\in I}\mathfrak{F}_i$.

Proposition 4.29 $\sum_{i\in I}\mathfrak{F}_i = \langle O, W, R, S_\Box, S_\Diamond, {}^*, P\rangle$ *defined above is a general frame.*

Proof.

We have shown $\langle O, W, R, S_\Box, S_\Diamond, {}^*\rangle$ is a frame in Proposition 3.6. It remains to show that P is closed under $\cap$, $\cup$, $\to$, $-$, $\Box$ and $\Diamond$. Let $\bigcup_{i\in I} X_i, \bigcup_{i\in I} Y_i \in P$, and let denote $\bigcup X_i, \bigcup Y_i$, respectively.

1. First of all, we see that $\bigcup X_i \cap \bigcup Y_i = \bigcup(X_i \cap Y_i)$.

$$\begin{array}{ll} & a \in \bigcup X_i \cap \bigcup Y_i \\ \text{iff} & \exists m \in I(a \in X_m) \ \&\ \exists n \in I(a \in Y_n) \\ \text{iff} & \exists n \in I(a \in X_n \ \&\ a \in Y_n) \quad \text{(Note that each frame is disjoint.)} \\ \text{iff} & \exists n \in I(a \in X_n \cap Y_n) \\ \text{iff} & a \in \bigcup(X_i \cap Y_i). \end{array}$$

   Suppose that $\bigcup X_i, \bigcup Y_i \in P$. Then for all $i \in I$, $X_i, Y_i \in P_i$, so $X_i \cap Y_i \in P_i$. Thus, we have $\bigcup(X_i \cap Y_i) \in P$, and hence $\bigcup X_i \cap \bigcup Y_i \in P$ by above observation.

2. As in 1, we see that if $\bigcup X_i, \bigcup Y_i \in P$, then $\bigcup X_i \cup \bigcup Y_i \in P$.

3. First of all, we see that $\bigcup X_i \to \bigcup Y_i = \bigcup(X_i \to Y_i)$. Suppose that $a \notin \bigcup(X_i \to Y_i)$. Then for all $i \in I$, $a \notin X_i \to Y_i$, i.e., there exists $b, c \in W_i$ such that $R_i abc$, $b \in X_i$ and $c \notin Y_i$. So, there exists $b, c \in W$ such that $Rabc$, $b \in \bigcup X_i$ and $c \notin \bigcup Y_i$. Thus, $a \notin \bigcup X_i \to \bigcup Y_i$. Next suppose that $a \in \bigcup(X_i \to Y_i)$. To see that $a \in \bigcup X_i \to \bigcup Y_i$, suppose that $Rabc$ and $b \in \bigcup X_i$. Then there exist $l, m, n \in I$ such that $a \in X_l \to Y_l$, $R_m abc$ and $b \in X_n$. Since each frame is disjoint, $l = m = n$, say n. Then we have $c \in Y_n$. So, $c \in \bigcup Y_i$, which is the desired result.

   As in the last sentences of 1, we show that if $\bigcup X_i, \bigcup Y_i \in P$, then $\bigcup X_i \to \bigcup Y_i \in P$.

4. Suppose first that $a \in -\bigcup X_i$. Then $a^* \notin \bigcup X_i$, so $a^* \notin X_i$ for all $i \in I$. By the definition of *, there exists $n \in I$ such that $g_n(a) \notin X_n$. So, we have $a \in \bigcup -X_i$.

   Next suppose that $a \notin -\bigcup X_i$. Then $a^* \in \bigcup X_i$, so there exists $n \in I$ such that $a^* \in X_n$. By the definition of *, $g_i(a) \in X_i$ for all $i \in I$. Thus, $a \notin \bigcup(-X_i)$.

   As in the last sentences of 1, we show that if $\bigcup X_i \in P$, then $-\bigcup X_i \in P$.

5. As in 3, we see that if $\bigcup X_i \in P$, then $\Box \bigcup X_i \in P$.

6. As in 4, we see that if $\bigcup X_i \in P$, then $\Diamond \bigcup X_i \in P$. ■

Theorem 4.30 *Let $\{\mathfrak{F}_i \mid i \in I\}$ be a family of general* **L***-frames and $\sum_{i\in I} \mathfrak{F}_i = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ be its disjoint union. Then the mapping f defined by*

$$f(X)(i) = X \cap W_i, \qquad \text{for every } X \in P \text{ and } i \in I,$$

is an isomorphism of $(\sum_{i\in I} \mathfrak{F}_i)^+$ onto $\prod_{i\in I} \mathfrak{F}_i^+$.

Proof.

By the definition, $f(X) \in \prod_{i\in I} \mathfrak{F}_i^+$, i.e., $f(X)$ is a function from I into $\bigcup_{i\in I} P_i$ with $f(X)(i) \in P_i$, for all $i \in I$. Taking any $h \in \prod_{i\in I} \mathfrak{F}_i^+$, we can write $h = f(X)$, for $X \in P$. This means that f is surjective. Suppose that $f(X) = f(Y)$, for $X, Y \in P$. Then $f(X)(i) = f(Y)(i)$, i.e., $X \cap W_i = Y \cap W_i$, for all $i \in I$. Since X and Y are of the forms $\bigcup_{i\in I} X_i$ and $\bigcup_{i\in I} Y_i$, respectively, we have $X_i = Y_i$, for all $i \in I$. Hence $X = Y$. Thus f is injective.

It remains to show that f is a homomorphism. Let $X, Y \in P$.

1. For all $i \in I$,
$$\begin{aligned} a \in f(X \cap Y)(i) \ &\text{iff}\ \ a \in (X \cap Y) \cap W_i \\ &\text{iff}\ \ a \in f(X)(i) \cap f(Y)(i) \\ &\text{iff}\ \ a \in (f(X) \cap f(Y))(i). \end{aligned}$$
Hence, $f(X \cap Y) = f(X) \cap f(Y)$.

2. As in 1, we see that $f(X \cup Y) = f(X) \cup f(Y)$.

3. First, suppose that $a \in f(X \to Y)(i)$, for all $i \in I$. To show that $a \in (f(X) \to f(Y))(i)$, for all $i \in I$, suppose that $Rabc$ and $b \in f(X)(i)$. Then $a \in (X \to Y) \cap W_i$, R_iabc and $b \in X \cap W_i$. So we have $c \in Y \cap W_i = f(Y)(i)$, which is the desired result.

   Next, suppose that $a \in (f(X) \to f(Y))(i)$, for all $i \in I$. To show that $a \in f(X \to Y)(i)$, for all $i \in I$, suppose that R_iabc and $b \in X$. Then $a \in f(X)(i) \to f(Y)(i)$, $Rabc$ and $b \in f(X)(i)$. So we have $c \in f(Y)(i) = Y \cap W_i$. Thus, $c \in Y$, which is the desired result.

   Therefore, $f(X \to Y) = f(X) \to f(Y)$.

4. For all $i \in I$,
$$\begin{aligned} a \in f(-X)(i) \ &\text{iff}\ \ a \in -X \cap W_i \\ &\text{iff}\ \ g_i(a) \notin X \\ &\text{iff}\ \ a^* \notin X \cap W_i \\ &\text{iff}\ \ a \in -f(X)(i). \end{aligned}$$
Hence, $f(-X) = -f(X)$.

5. First, suppose that $a \in f(\Box X)(i)$, for all $i \in I$. To show that $a \in (\Box f(X))(i)$, for all $i \in I$, suppose that $S_\Box ab$. Then $a \in \Box X \cap W_i$ and $S_{\Box i}ab$, so we have $b \in X \cap W_i = f(X)(i)$. This is the desired result.

   Next, suppose that $a \in (\Box f(X))(i)$, for all $i \in I$. To show that $a \in f(\Box X)(i)$, for all $i \in I$, suppose that $S_{\Box i}ab$. Then $a \in \Box f(X)(i)$ and $S_\Box ab$, so $b \in f(X)(i)$. And hence $b \in X$, so we have $a \in \Box X$. Further, $a \in W_i$, so $a \in f(\Box X)(i)$. This is the desired result.

   Therefore, $f(\Box X) = \Box f(X)$.

6. For all $i \in I$,

$$\begin{array}{rll} a \in f(\Diamond X)(i) & \text{iff} & a \in \Diamond X \cap W_i \\ & \text{iff} & \exists b \in W(S_\Diamond ab \;\&\; b \in X) \;\&\; a \in W_i \\ & \text{iff} & \exists b(S_{\Diamond i}ab \;\&\; b \in X \cap W_i) \\ & \text{iff} & a \in \Diamond f(X)(i) \\ & \text{iff} & a \in (\Diamond f(X))(i). \end{array}$$

Hence, $f(\Diamond X) = \Diamond f(X)$. ■

Theorem 4.31 *Suppose that $\langle \mathbf{M}, E\rangle$ and $\langle \mathbf{M}', E'\rangle$ are $\mathbf{L}$-matrices. Then the mapping f defined by*

$$f(\nabla) = \{\langle x, y\rangle \in M \times M' \mid x \in \nabla, y \in M'\}, \quad \textit{for every } \nabla \in W_{\mathbf{M}}$$

and

$$f(\nabla') = \{\langle x, y\rangle \in M \times M' \mid x \in M, y \in \nabla'\}, \quad \textit{for every } \nabla' \in W_{\mathbf{M}'}$$

is an isomorphism of $\langle \mathbf{M}, E\rangle_+ + \langle \mathbf{M}', E'\rangle_+$ onto $(\langle \mathbf{M}, E\rangle \times \langle \mathbf{M}', E'\rangle)_+$.

Proof.

First, we show that f is bijective. It is easy to show that f is injective, so it suffices to show that f is surjective. Take any $\nabla \in W_{\mathbf{M}\times\mathbf{M}'}$. Let $\nabla_1 = \{x \in M \mid \langle x, y\rangle \in \nabla\}$ and $\nabla_2 = \{y \in M' \mid \langle x, y\rangle \in \nabla\}$. Then it is easy to see that $\nabla_1 \in W_{\mathbf{M}}$ and $\nabla_2 \in W_{\mathbf{M}'}$. Further, we see that $\nabla = f(\nabla_1)$ and $\nabla = f(\nabla_2)$. Hence, f is surjective.

Next, we show that f preserves each operation, i.e., the following 1 through 6 hold. Let $\langle \mathbf{M}, E\rangle_+ + \langle \mathbf{M}', E'\rangle_+ = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$.

1. $R\nabla_1\nabla_2\nabla_3$ iff $R_{\mathbf{M}\times\mathbf{M}'}f(\nabla_1)f(\nabla_2)f(\nabla_3)$, for all $\nabla_1, \nabla_2, \nabla_3 \in W$.

   The 'if' part is proved as follows. Suppose that $R_{\mathbf{M}\times\mathbf{M}'}f(\nabla_1)f(\nabla_2)f(\nabla_3)$. To see that $R\nabla_1\nabla_2\nabla_3$, suppose that $x \to y \in \nabla_1$ and $x \in \nabla_2$. Without loss of generality, let $\nabla_1, \nabla_2 \in W_{\mathbf{M}}$. Then $\langle x \to y, z \to w\rangle \in f(\nabla_1)$ and $\langle x, z\rangle \in f(\nabla_2)$, for all $z, w \in M'$, so we have $\langle y, w\rangle \in f(\nabla_3)$. Thus, $y \in \nabla_3$, which is the desired result.

   The 'only if' part is proved as follows. Suppose that $R\nabla_1\nabla_2\nabla_3$. To see that $R_{\mathbf{M}\times\mathbf{M}'}f(\nabla_1)f(\nabla_2)f(\nabla_3)$, suppose that $\langle x, z\rangle \to \langle y, w\rangle \in f(\nabla_1)$ and $\langle x, z\rangle \in f(\nabla_2)$. Then either (i) $x \to y \in \nabla_1$ and $x \in \nabla_2$, or (ii) $z \to w \in \nabla_1$ and $z \in \nabla_2$. For (i), since $R_{\mathbf{M}}\nabla_1\nabla_2\nabla_3$, $y \in \nabla_3$, and hence $\langle y, w\rangle \in f(\nabla_3)$. For (ii), similarly, $\langle y, w\rangle \in f(\nabla_3)$.

2. $S_\Box\nabla_1\nabla_2$ iff $S_{\Box\mathbf{M}\times\mathbf{M}'}f(\nabla_1)f(\nabla_2)$, for all $\nabla_1, \nabla_2 \in W$.

   The 'if' part is proved as follows. Suppose that $S_{\Box\mathbf{M}\times\mathbf{M}'}f(\nabla_1)f(\nabla_2)$. To see that $S_\Box\nabla_1\nabla_2$, suppose that $\Box x \in \nabla_1$. Without loss of generality, let $\nabla_1 \in W_{\mathbf{M}}$. Then $\langle \Box x, \Box y\rangle \in f(\nabla_1)$, for all $y \in M'$, so we have $\langle x, y\rangle \in f(\nabla_2)$. Thus, $x \in \nabla_2$, which is the desired result.

   The 'only if' part is proved as follows. Suppose that $S_\Box\nabla_1\nabla_2$. To see that $S_{\Box\mathbf{M}\times\mathbf{M}'}f(\nabla_1)f(\nabla_2)$, suppose that $\Box\langle x, y\rangle \in f(\nabla_1)$. Then either $\Box x \in \nabla_1$ or $\Box y \in \nabla_1$. For the former, since $S_{\Box\mathbf{M}}\nabla_1\nabla_2$, we have $x \in \nabla_2$, and hence $\langle x, y\rangle \in f(\nabla_2)$. For the latter, similarly, $\langle x, y\rangle \in f(\nabla_2)$.

3. $S_\diamond \nabla_1 \nabla_2$ iff $S_{\diamond \mathbf{M}\times\mathbf{M}'} f(\nabla_1) f(\nabla_2)$, for all $\nabla_1, \nabla_2 \in W$.

   The 'if' part is proved as follows. Suppose that $S_{\diamond \mathbf{M}\times\mathbf{M}'} f(\nabla_1) f(\nabla_2)$. To see that $S_\diamond \nabla_1 \nabla_2$, suppose that $x \in \nabla_2$. Without loss of generality, let $\nabla_2 \in W_{\mathbf{M}}$. Then $\langle x, y\rangle \in f(\nabla_2)$, for all $y \in M'$, so we have $\diamond\langle x, y\rangle \in f(\nabla_1)$. Thus, $\diamond x \in \nabla_1$, which is the desired result.

   The 'only if' part is proved as follows. Suppose that $S_\diamond \nabla_1 \nabla_2$. To see that $S_{\diamond \mathbf{M}\times\mathbf{M}'} f(\nabla_1) f(\nabla_2)$, suppose that $\langle x, y\rangle \in f(\nabla_2)$. Then either $x \in \nabla_2$ or $y \in \nabla_2$. For the former, since $S_{\diamond \mathbf{M}} \nabla_1 \nabla_2$, we have $\diamond x \in \nabla_1$, and hence $\diamond\langle x, y\rangle \in f(\nabla_1)$. For the latter, similarly, $\diamond\langle x, y\rangle \in f(\nabla_1)$.

4. $\nabla^* = g_{\mathbf{M}\times\mathbf{M}'}(f(\nabla))$, for all $\nabla \in W$.

   First, suppose that $\langle x, y\rangle \in \nabla^*$. Then either $x \in g_{\mathbf{M}}(\nabla)$ or $x \in g_{\mathbf{M}'}(\nabla)$. Without loss of generality, we consider the case in which $x \in g_{\mathbf{M}}(\nabla)$. Then $-x \notin \nabla$, and hence $\langle -x, -y\rangle \notin f(\nabla)$, for all $y \in M'$. Thus, $\langle x, y\rangle \in g_{\mathbf{M}\times\mathbf{M}'}(f(\nabla))$.

   Next, suppose that $\langle x, y\rangle \in g_{\mathbf{M}\times\mathbf{M}'}(f(\nabla))$. Then $-\langle x, y\rangle \notin f(\nabla)$, so either $-x \notin \nabla$ or $-y \notin \nabla$. Then either $x \in g_{\mathbf{M}}(\nabla)$ or $y \in g_{\mathbf{M}'}(\nabla)$, so we have $\langle x, y\rangle \in \nabla^*$.

5. $\nabla \in O$ iff $f(\nabla) \in O_{\mathbf{M}\times\mathbf{M}'}$.

   The 'if' part is proved as follows. Suppose that $\nabla \notin O$. This means that $E \not\subseteq \nabla$ and $E' \not\subseteq \nabla$. Then there exists $x \in M$ and $y \in M'$ such that $x \in E$, $x \notin \nabla$, $y \in E'$ and $y \notin \nabla$. So, we have $\langle x, y\rangle \in E \times E'$ and $\langle x, y\rangle \notin f(\nabla)$. Thus, $f(\nabla) \notin O_{\mathbf{M}\times\mathbf{M}'}$.

   The 'only if' part is proved as follows. Suppose that $\nabla \in O$. Then $\nabla \in O_{\mathbf{M}}$ or $\nabla \in O_{\mathbf{M}'}$. To see that $f(\nabla) \in O_{\mathbf{M}\times\mathbf{M}'}$, suppose that $\langle x, y\rangle \in E \times E'$. For the former, we have $x \in \nabla$, so $\langle x, y\rangle \in f(\nabla)$. For the latter, similarly, $\langle x, y\rangle \in f(\nabla)$.

6. $X \in P$ iff $f(X) \in P_{\mathbf{M}\times\mathbf{M}'}$.

   The 'if' part is proved as follows. By the assumption, there exists $\langle x, y\rangle \in M \times M'$ such that $f(X) = \{\nabla \in W_{\mathbf{M}\times\mathbf{M}'} \mid \langle x, y\rangle \in \nabla\}$. Since f is bijective, there exists $\nabla' \in W$ such that $\nabla = f(\nabla')$. If $\nabla' \in W_{\mathbf{M}}$, then $x \in \nabla'$, and let $Y = \{\nabla' \in W_{\mathbf{M}} \mid x \in \nabla'\}$. If $\nabla' \in W'_{\mathbf{M}}$, then $y \in \nabla'$, and let $Z = \{\nabla' \in W'_{\mathbf{M}} \mid y \in \nabla'\}$. We have $f(X) = f(Y \cup Z)$, so $X = Y \cup Z$ since f is bijective.

   The 'only if' part is proved as follows. By the assumption, there exist $Y \in P_{\mathbf{M}}$ and $Z \in P_{\mathbf{M}'}$ such that $X = Y \cup Z$. That is, there exist $x \in M$ and $y \in M'$ such that $Y = \{\nabla \in W_{\mathbf{M}} \mid x \in \nabla\}$ and $Z = \{\nabla' \in W_{\mathbf{M}'} \mid y \in \nabla'\}$. We have $\langle x, y\rangle \in f(\nabla)$ and $\langle x, y\rangle \in f(\nabla')$, so $f(X) = f(Y) \cup f(Z) = \{f(\nabla) \in W_{\mathbf{M}\times\mathbf{M}'} \mid \langle x, y\rangle \in f(\nabla)\} \cup \{f(\nabla') \in W_{\mathbf{M}\times\mathbf{M}'} \mid \langle x, y\rangle \in f(\nabla')\}$. Therefore, $f(X) \in P_{\mathbf{M}\times\mathbf{M}'}$. ■

4.5 $\mathcal{D}^\star$-elementary logics and $\mathcal{D}$-persistent logics

Let $\mathbf{L}$ be any regular logic over $\mathbf{R}$. Let $\mathcal{D}$ be the class of descriptive $\mathbf{L}$-frames. Then let $\kappa\mathcal{D} = \{\kappa\mathfrak{F} \mid \mathfrak{F} \in \mathcal{D}\}$, and $\mathcal{D}^\star = \mathcal{D} \cup \kappa\mathcal{D}$. Then we say that

- $\mathbf{L}$ is *$\mathcal{D}$-complete* (*$\kappa\mathcal{D}$-complete*) if A is a theorem of $\mathbf{L}$ whenever it is valid in every $\mathbf{L}$-frame in $\mathcal{D}$ (in $\kappa\mathcal{D}$).

- $\mathbf{L}$ is *$\mathcal{D}$-persistent* if for every $\mathfrak{F} \in \mathcal{D}$, $\mathfrak{F} \models \mathbf{L}$ implies $\kappa\mathfrak{F} \models \mathbf{L}$.

- $\mathbf{L}$ is $\mathcal{D}^{\star}$*-elementary* if there exists a set Φ of first-order sentences in the predicates O, R, $S_{\Box}$, $S_{\Diamond}$, * and the constant u such that for every $\mathfrak{F} \in \mathcal{D}^{\star}$,

$$\mathfrak{F} \text{ is an } \mathbf{L}\text{-frame} \text{ iff } \mathfrak{F} \text{ satisfies each sentence in } \Phi.$$

From Theorem 4.24, it follows that $\mathbf{L}$ is $\mathcal{D}$-complete.

Proposition 4.32 *Let* $\mathbf{L}$ *be any regular logic over* $\mathbf{R}$.

1. *If* $\mathbf{L}$ *is both* $\mathcal{D}$*-complete and* $\mathcal{D}$*-persistent, then it is* $\kappa\mathcal{D}$*-complete.*
2. *If* $\mathbf{L}$ *is* $\mathcal{D}^{\star}$*-elementary, then it is* $\mathcal{D}$*-persistent.*
3. *If* $\mathbf{L}$ *is both* $\mathcal{D}$*-complete and* $\mathcal{D}^{\star}$*-elementary, then it is* $\kappa\mathcal{D}$*-complete.*

Proof.
Since 3 follows from 1 and 2, we will give proofs of 1 and 2.

1. Suppose that A is not a theorem of $\mathbf{L}$. Since $\mathbf{L}$ is $\mathcal{D}$-complete, there exists an $\mathbf{L}$-model on a descriptive $\mathbf{L}$-frame $\mathfrak{F}$ such that $\mathfrak{F} \models \mathbf{L}$ and $\mathfrak{F} \not\models A$. Since $\mathbf{L}$ is $\mathcal{D}$-persistent, $\kappa\mathfrak{F} \models \mathbf{L}$. Further, it is clear that $\kappa\mathfrak{F} \not\models A$. Thus, there exists an $\mathbf{L}$-model on an $\mathbf{L}$-frame $\kappa\mathfrak{F} \in \kappa\mathcal{D}$ that falsifies A.
2. Note that $\mathfrak{F}$ satisfies each sentence in Φ iff $\kappa\mathfrak{F}$ satisfies each sentence in Φ, by the assumption. Take a descriptive $\mathbf{L}$-frame $\mathfrak{F}$ such that $\mathfrak{F} \models \mathbf{L}$. Since $\mathbf{L}$ is $\mathcal{D}^{\star}$-elementary, $\mathfrak{F}$ satisfies each sentence in Φ, for some set Φ of the first-order sentences in the predicates O, R, $S_{\Box}$, $S_{\Diamond}$, * and the constant u. Since $\mathbf{L}$ is $\mathcal{D}^{\star}$-elementary again, we have $\kappa\mathfrak{F} \models \mathbf{L}$. ■

Note that $\mathbf{L}_B$ is $\mathcal{D}$-complete, $\mathcal{D}^{\star}$-elementary, and $\kappa\mathcal{D}$-complete. The last fact has been proved in Theorem 3.14.

4.6 Notes

The notion of general frames was explicitly introduced by D.Makinson ([33]).
S.K.Thomason proved completeness for tense logics with respect to the semantics of general frames (which he called *first order frames*) in [54]. After that, an extensive and systematical study of the semantics of general frames for classical modal logics was developed by R.I.Goldblatt ([23] and [24], which are included in [26]). He showed most of properties of general frames for classical modal logics, which we mentioned in Section 2.8. But general frames for relevant logics have not discussed so far.

Duality theory is one of the most familiar topics in the semantical study. In this chapter, we showed the representation theorem for relevant modal logics and the correspondence between frame-theoretic notions and algebraic ones. In classical modal logics, the usage of general frames makes them refined. In relevant logics, the representation of non-modal $\mathbf{R}^{\neg}$-algebra shown by C.Brink ([8]) is a remarkable result, where $\mathbf{R}^{\neg}$ is the logic obtained from $\mathbf{R}$ by adding the classical negation $\neg$, and an $\mathbf{R}^{\neg}$-algebra is the algebraic counter part of $\mathbf{R}^{\neg}$. This is a Stone-style representation result. On the other hand, A.Urquhart studies duality theory both of relevant algebras and relevant spaces in [58]. Further, S.A.Celani extends Urquhart's result to some relevant modal algebras in [10]. These are Priestley-style representation results.

Chapter 5

Sahlqvist theorem for relevant modal logics

Kripke-completeness of every classical modal logic with Sahlqvist formulas is one of the basic general results on completeness of classical modal logics. In this chapter, we consider relevant modal logics with Sahlqvist formulas. Sahlqvist theorem is proved first by H.Sahlqvist in [49]. Our completeness result covers most of completeness results of relevant modal logics so far. It is shown that usual Sahlqvist theorem for classical modal logics can be obtained as a special case of our theorem. Further, we give some comments on completeness of relevant modal logics with non-Sahlqvist formulas.

5.1 A Sahlqvist theorem

In this section, we consider a Sahlqvist theorem for relevant modal logics. Here we follow [52], which gives the proof of a Sahlqvist theorem for the modal logic over classical logic. Our proof follows [11].

Below, $A[p_1, \cdots, p_n]$ denotes a formula A whose variables are listed among $p_1, \cdots, p_n$. Given a formula $A[p_1, \cdots, p_n]$, a general $\mathbf{R.C}_{\Box\Diamond}$-frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ and $X_1, \cdots, X_n \in P$, we denote by $A[X_1, \cdots, X_n]$ the set of points in $\mathfrak{F}$ at which A is true under the valuation V on $\mathfrak{F}$ defined by $V(p_i) = X_i$, for $i = 1, \cdots, n$, i.e., $A[X_1, \cdots, X_n] = V(A)$. Thus, we have the following.

Proposition 5.1 *Let $A[p_1, \cdots, p_n]$ be a formula and $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ be a general $\mathbf{R.C}_{\Box\Diamond}$-frame. For $a \in W$,*

$$(\mathfrak{F}, a) \models A[p_1, \cdots, p_n] \quad \textit{iff} \quad \forall X_1 \in P \cdots \forall X_n \in P(a \in A[X_1, \cdots, X_n]),$$

where $(\mathfrak{F}, a) \models A$ means that $a \models A$ under any valuation on $\mathfrak{F}$.

Proof.

It is enough to show that $a \models A[p_1, \cdots, p_n]$ iff $a \in A[X_1, \cdots, X_n]$ for $V(p_i) = X_i$ by induction on the length of A.

1. A is of the form p_i $(i = 1, \cdots, n)$. $\quad$ $a \models p_i$ iff $a \in V(p_i)$ iff $a \in X_i$.

2. A is of the form $B \wedge C$.

   By the hypotheses of induction,

$$\begin{array}{rll} a \models (B \wedge C)[p_1, \cdots, p_n] & \text{iff} & a \models B[p_1, \cdots, p_n] \ \& \ a \models C[p_1, \cdots, p_n] \\ & \text{iff} & a \in B[X_1, \cdots, X_n] \ \& \ a \in C[X_1, \cdots, X_n] \\ & \text{iff} & a \in B[X_1, \cdots, X_n] \cap C[X_1, \cdots, X_n] \\ & \text{iff} & a \in (B \wedge C)[X_1, \cdots, X_n]. \end{array}$$

3. A is of the form $B \vee C$. Similar to 2.

4. A is of the form $B \to C$.

   By the hypotheses of induction,

$$\begin{array}{ll} & a \models (B \to C)[p_1, \cdots, p_n] \\ \text{iff} & \forall b \in W \forall c \in W (Rabc \ \& \ b \models B[p_1, \cdots, p_n] \Rightarrow c \models C[p_1, \cdots, p_n]) \\ \text{iff} & \forall b \in W \forall c \in W (Rabc \ \& \ b \in B[X_1, \cdots, X_n] \Rightarrow c \in C[X_1, \cdots, X_n]) \\ \text{iff} & a \in (B \to C)[X_1, \cdots, X_n]. \end{array}$$

5. A is of the form $\sim B$.

   By the hypothesis of induction,

$$\begin{array}{rll} a \models (\sim B)[p_1, \cdots, p_n] & \text{iff} & a^* \not\models B[p_1, \cdots, p_n] \\ & \text{iff} & a^* \notin B[X_1, \cdots, X_n] \\ & \text{iff} & a \in -(B[X_1, \cdots, X_n]) \\ & \text{iff} & a \in (\sim B)[X_1, \cdots, X_n]. \end{array}$$

6. A is of the form $\Box B$. Similar to 4.

7. A is of the form $\Diamond B$.

   By the hypothesis of induction,

$$\begin{array}{rll} a \models (\Diamond B)[p_1, \cdots, p_n] & \text{iff} & \exists b \in W (S_\Diamond ab \ \& \ b \models B[p_1, \cdots, p_n]) \\ & \text{iff} & \exists b \in W (S_\Diamond ab \ \& \ b \in B[X_1, \cdots, X_n]) \\ & \text{iff} & a \in (\Diamond B)[X_1, \cdots, X_n]. \end{array}$$

■

For $a \in W$, we write $a \uparrow^n = \{b \in W \mid S_\Box^n ab\}$. The following proposition is regarded as the generalization of Proposition 4.21.

Proposition 5.2 *For every $\Box$-tight general* $\mathbf{R.C}_{\Box\Diamond}$*-frame* $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$, *every* $a_1, \cdots, a_k \in W$ *and every* $n_1, \cdots, n_k \geq 0$,

$$\bigcap\{X \in P \mid a_1 \uparrow^{n_1} \cup \cdots \cup a_k \uparrow^{n_k} \subseteq X\} = a_1 \uparrow^{n_1} \cup \cdots \cup a_k \uparrow^{n_k}.$$

Proof.

It is clear that $a_1 \uparrow^{n_1} \cup \cdots \cup a_k \uparrow^{n_k} \subseteq \bigcap\{X \in P \mid a_1 \uparrow^{n_1} \cup \cdots \cup a_k \uparrow^{n_k} \subseteq X\}$, so we see the converse inclusion. Suppose that $b \notin a_1 \uparrow^{n_1} \cup \cdots \cup a_k \uparrow^{n_k}$. Then $b \notin a_i \uparrow^{n_i}$, for all $i = 1, \cdots, k$. Since $\mathfrak{F}$ is $\Box$-tight, there exists $X_i \in P$ such that $a_i \in \Box^{n_i} X_i$ and $b \notin X_i$, for all $i = 1, \cdots, k$. Let $X = X_1 \cup \cdots \cup X_k$. Then $X \in P$, $a_i \uparrow^{n_i} \subseteq X$ and $b \notin X$, for all $i = 1, \cdots, k$. Hence, there exists $X \in P$ such that $a_1 \uparrow^{n_1} \cup \cdots \cup a_k \uparrow^{n_k} \subseteq X$ and $b \notin X$, so $b \notin \bigcap\{X \in P \mid a_1 \uparrow^{n_1} \cup \cdots \cup a_k \uparrow^{n_k} \subseteq X\}$. ■

A frame-theoretic term $a_1 \uparrow^{n_1} \cup \cdots \cup a_k \uparrow^{n_k}$ with (not necessarily distinct) $a_1, \cdots, a_k \in W$ will be called an $S_\Box$-*term* for brevity. In the rest of this section, the letter T and T' with subscripts denote arbitrary $S_\Box$-terms.

Lemma 5.3 *Suppose that $A[p_1, \cdots, p_n]$ is a modal formula and $T_1, \cdots, T_n$ are $S_\Box$-terms. Then the relation $a \in A[T_1, \cdots, T_n]$ can be expressed by a first order formula (in the predicates O, R, $S_\Box$, $S_\Diamond$, * and the constant u) having a as its only free variable.*

Proof.

By induction on the length of A. We will show the following two cases. Other cases are shown similarly.

1. A is of the form p_i $(i = 1, \cdots, n)$.

   We may consider $a \in T_i$. Since T_i is a $S_\Box$-term, it can be expressed by $a_1 \uparrow^{n_1} \cup \cdots \cup a_k \uparrow^{n_k}$. We further use induction on k. When $k = 0$, we can write $a \not\leq a$ since $T_i = \emptyset$. When $k > 0$, $a \in T_i$ can be rewritten in $S_\Box^{n_1} a_1 a$ *or* $\cdots$ *or* $S_\Box^{n_k} a_k a$, so it is possible to be expressed by a first order formula in the predicate(s) $S_\Box$ (and O and R, in case that $n_i = 0$).

2. A is of the form $B \wedge C$.

   $a \in A[T_1, \cdots, T_n]$ can be rewritten in $a \in B[T_1, \cdots, T_n]$ & $a \in C[T_1, \cdots, T_n]$. By the hypotheses of induction, both $a \in B[T_1, \cdots, T_n]$ and $a \in C[T_1, \cdots, T_n]$ can be expressed by a first order formula. Therefore, $a \in A[T_1, \cdots, T_n]$ can be expressed by a first order formula.

3. A is of the form $B \vee C$. Similar to 2.

4. A is of the form $B \to C$.

   $a \in A[T_1, \cdots, T_n]$ can be rewritten in $\forall b \forall c(Rabc$ & $b \in B[T_1, \cdots, T_n] \Rightarrow c \in C[T_1, \cdots, T_n])$. By the hypotheses of induction, both $a \in B[T_1, \cdots, T_n]$ and $a \in C[T_1, \cdots, T_n]$ can be expressed by a first order formula. Therefore, $a \in A[T_1, \cdots, T_n]$ can be expressed by a first order formula.

5. A is of the form $\sim B$.

   $a \in A[T_1, \cdots, T_n]$ can be rewritten in $a^* \notin B[T_1, \cdots, T_n]$. By the hypothesis of induction, $a^* \notin B[T_1, \cdots, T_n]$ can be expressed by a first order formula. Therefore, $a \in A[T_1, \cdots, T_n]$ can be expressed by a first order formula.

6. A is of the form $\Box B$.

   $a \in A[T_1, \cdots, T_n]$ can be rewritten in $\forall b(S_\Box ab \Rightarrow b \in B[T_1, \cdots, T_n])$. By the hypothesis of induction, $b \in B[T_1, \cdots, T_n]$ can be expressed by a first order formula. Therefore, $a \in A[T_1, \cdots, T_n]$ can be expressed by a first order formula.

7. A is of the form $\Diamond B$.

   $a \in A[T_1, \cdots, T_n]$ can be rewritten in $\exists b(S_\Diamond ab \;\&\; b \in B[T_1, \cdots, T_n])$. By the hypothesis of induction, $b \in B[T_1, \cdots, T_n]$ can be expressed by a first order formula. Therefore, $a \in A[T_1, \cdots, T_n]$ can be expressed by a first order formula. ■

A modal formula of the form $\Box^{m_1} p_1 \wedge \cdots \wedge \Box^{m_k} p_k$ with not necessarily distinct propositional variables $p_1, \cdots, p_k$ is called a *strongly positive formula.*

Lemma 5.4 *Suppose that $A[p_1, \cdots, p_n]$ is a strongly positive formula containing all the variables $p_1, \cdots, p_n$ and $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ is a general* **R.C**$_{\Box\Diamond}$*-frame. Then there exist $S_\Box$-terms $T_1, \cdots, T_n$ (of one variable a) such that for any $a \in W$ and any $X_1, \cdots, X_n \in P$,*

$$a \in A[X_1, \cdots, X_n] \;\; \textit{iff} \;\; T_1 \subseteq X_1 \;\&\; \cdots \;\&\; T_n \subseteq X_n.$$

Proof.
By induction on the number k of conjuncts of A.

(I) When $k = 1$, A is of the form $\Box^m p$.

Take $a \uparrow^m$ for T_i . Then, suppose first $a \in \Box^m p_i[X_1, \cdots, X_n]$, i.e., $a \in \Box^m X_i$. To show that $T_i \subseteq X_i$, suppose that $b \in T_i$. Then $S_\Box^m ab$, so $b \in X_i$, which is the desired result.

Next, suppose that $T_i \subseteq X_i$. This means that $a \in \Box^m X_i$.

(II) When $k > 1$, $A[p_1, \cdots, p_n]$ can be written by $B \wedge \Box^m p_i$, where B is a strongly positive formula which has conjuncts less than k, and $1 \leq i \leq n$. By the hypothesis of induction and (I), there exist $S_\Box$-terms $T_1, \cdots, T_n$ such that

$$\begin{aligned} a \in A[X_1, \cdots, X_n] \;\; &\text{iff} \;\; T_1 \subseteq X_1 \;\&\; \cdots \;\&\; T_n \subseteq X_n \;\&\; a \uparrow^m \subseteq X_i \\ &\text{iff} \;\; T_1 \subseteq X_1 \;\&\; \cdots \;\&\; T_i \cup a \uparrow^m \subseteq X_i \;\&\; \cdots \;\&\; T_n \subseteq X_n. \end{aligned}$$

Of course, $T_i \cup a \uparrow^m$ is a $S_\Box$-term. ■

A modal formula A is *positive* if A contain only $\wedge, \vee, \Box$ and $\Diamond$ as connectives. The following proposition is known as *monotonicity*, and is easily proved.

Proposition 5.5 *Let $A[\cdots, p, \cdots]$ be an arbitrary positive formula and $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ be a general* **R.C**$_{\Box\Diamond}$*-frame. For all $X, Y \subseteq W$, $X \subseteq Y$ implies $A[\cdots, X, \cdots] \subseteq A[\cdots, Y, \cdots]$.*

Proposition 5.6 *For every general* **R.C**$_{\Box\Diamond}$*-frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$, every positive $A_i[\cdots, p, \cdots]$, $i = 1, \cdots, n$, and every $a_1, \cdots, a_n \in W$,*

$$\forall X \in P(Y \subseteq X \Rightarrow \bigvee_{i \leq n} a_i \in A_i[\cdots, X, \cdots]) \;\; \textit{iff} \;\; \bigvee_{i \leq n} a_i \in \bigcap\{A_i[\cdots, X, \cdots] \mid Y \subseteq X \in P\}.$$

Proof.

The 'if' part is a logical consequence, so we prove the 'only if' part by using the contraposition. Suppose that $\bigwedge_{i\leq n} a_i \notin \bigcap\{A_i[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$. That is, for every i $(1 \leq i \leq n)$, there exists $X_i \in P$ such that $Y \subseteq X_i$ and $a_i \notin A_i[\cdots, X_i, \cdots]$. Taking $X = X_1 \cap \cdots \cap X_n$, we have $X \in P$ and $Y \subseteq X$. Further, by Proposition 5.5, $a_i \notin A_i[\cdots, X, \cdots]$ for each i $(1 \leq i \leq n)$. Thus, we have $\exists X \in P(Y \subseteq X \;\&\; \bigwedge_{i\leq n} a_i \notin A_i[\cdots, X, \cdots])$. ∎

A family $\mathcal{X}$ of non-empty subsets of W is called *downward directed* if for every $X, Y \in \mathcal{X}$, there is $Z \in \mathcal{X}$ such that $Z \subseteq X \cap Y$. The following lemma is an analogue of *Esakia's lemma* in [11].

Lemma 5.7 *Suppose that* $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, {}^*, P\rangle$ *is a descriptive general* $\mathbf{R.C}_{\Box\Diamond}$*-frame. Then for every downward directed family* $\mathcal{X} \subseteq P$*,*

$$\Diamond(\bigcap_{X\in\mathcal{X}} X) = \bigcap_{X\in\mathcal{X}} (\Diamond X).$$

Proof.

First, suppose that $a \in \Diamond(\bigcap_{X\in\mathcal{X}} X)$. Then there exists $b \in \bigcap_{X\in\mathcal{X}} X$ satisfying $S_\Diamond ab$. We have $b \in X$ for every $X \in \mathcal{X}$, so $a \in \Diamond X$ for every $X \in \mathcal{X}$. Hence $a \in \bigcap_{X\in\mathcal{X}}(\Diamond X)$.

Next, suppose that $a \in \bigcap_{X\in\mathcal{X}}(\Diamond X)$. Then for every $X \in \mathcal{X}$, there exists $b \in X$ such that $S_\Diamond ab$. Below, let $a\uparrow_\Diamond = \{c \in W \mid S_\Diamond ac\}$. We have

$$\begin{array}{lll} S_\Diamond ac & \text{iff} & \forall X \in P(c \in X \Rightarrow a \in \Diamond X) \qquad\qquad (\mathfrak{F} \text{ is } \Diamond\text{-tight}) \\ & \text{iff} & \forall X \in P(a \notin \Diamond X \Rightarrow c \notin X) \\ & \text{iff} & \forall X(W - X \in \overline{P} \;\&\; a \notin \Diamond X \Rightarrow c \in W - X) \\ & \text{iff} & c \in \bigcap\{W - X \in \overline{P} \mid a \notin \Diamond X\} \\ & \text{iff} & c \in \bigcap\{X \in P \mid a \notin \Diamond X\}, \end{array}$$

so $a\uparrow_\Diamond = \bigcap\{X \in P \mid a \notin \Diamond X\}$. Then $a\uparrow_\Diamond \cap X \neq \emptyset$ for every $X \in \mathcal{X}$, so $\{a\uparrow_\Diamond\} \cup \mathcal{X}$ has the finite intersection property. Since $\mathfrak{F}$ is compact, $a\uparrow_\Diamond \cap(\bigcap_{X\in\mathcal{X}} X) \neq \emptyset$. This means that there exists $b \in a\uparrow_\Diamond$ such that $b \in \bigcap_{X\in\mathcal{X}} X$, i.e., $a \in \Diamond(\bigcap_{X\in\mathcal{X}} X)$. ∎

Using Lemma 5.7, we have the following lemma, called an *intersection lemma.*

Lemma 5.8 *Suppose that* $A[p, \cdots, q, \cdots, r]$ *is a positive formula and* $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, {}^*, P\rangle$ *is a descriptive* $\mathbf{R.C}_{\Box\Diamond}$*-frame or an* $\mathbf{R.C}_{\Box\Diamond}$*-frame. Then for every* $Y \subseteq W$ *and all* $U, \cdots, V \in P$*,*

$$\bigcap\{A[U, \cdots, X, \cdots, V] \mid Y \subseteq X \in P\} = A[U, \cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots, V].$$

Proof.

We write simply $A[\cdots, X, \cdots]$ for $A[U, \cdots, X, \cdots, V]$. If $\mathfrak{F}$ is an $\mathbf{R.C}_{\Box\Diamond}$-frame, the proof is as follows. In this case, we note that $P = Up(W)^+$. First, suppose that $a \in \bigcap\{A[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$. By Proposition 5.6, $\forall X \in P(Y \subseteq X \Rightarrow a \in A[\cdots, X, \cdots])$. It is clear that $Y \subseteq \bigcap\{X \in P \mid Y \subseteq X\}$ and $\bigcap\{X \in P \mid Y \subseteq X\} \in P$, so we have $a \in A[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]$.

For the converse inclusion, suppose that $a \in A[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]$. Further, take any $X \in P$ satisfying that $Y \subseteq X$. Since $a \notin \bigcap\{X \in P \mid Y \subseteq X\}$ whenever $a \notin X$, we have $\bigcap\{X \in P \mid Y \subseteq X\} \subseteq X$. By Proposition 5.5, we have $a \in A[\cdots, X, \cdots]$, which is the desired result.

If $\mathfrak{F}$ is a descriptive $\mathbf{R.C}_{\Box\Diamond}$-frame, the proof is by induction on the length of the positive formula A.

1. A is of the form p_i $(i = 1, \cdots, n)$. It is clear.

2. A is of the form $B \wedge C$.

   By the hypotheses of induction,

$$\begin{array}{ll} & a \in \bigcap\{(B \wedge C)[\cdots, X, \cdots] \mid Y \subseteq X \in P\} \\ \text{iff} & \forall X \in P(Y \subseteq X \Rightarrow a \in (B \wedge C)[\cdots, X, \cdots]) \\ \text{iff} & \forall X \in P(Y \subseteq X \Rightarrow a \in B[\cdots, X, \cdots]) \\ & \ \& \ \forall X \in P(Y \subseteq X \Rightarrow a \in C[\cdots, X, \cdots]) \\ \text{iff} & a \in \bigcap\{B[\cdots, X, \cdots] \mid Y \subseteq X \in P\} \ \& \ a \in \bigcap\{C[\cdots, X, \cdots] \mid Y \subseteq X \in P\} \\ \text{iff} & a \in B[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots] \ \& \ a \in C[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots] \\ \text{iff} & a \in (B \wedge C)[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]. \end{array}$$

3. A is of the form $B \vee C$.

   The 'if' part is proved as follows. Suppose that $a \in (B \vee C)[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]$. Then $a \in B[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]$ or $a \in C[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]$. By the hypotheses of induction, $a \in \bigcap\{B[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$ or $a \in \bigcap\{C[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$. Taking any $X \in P$ such that $Y \subseteq X$, we have $a \in B[\cdots, X, \cdots]$ or $a \in C[\cdots, X, \cdots]$, which implies that $a \in (B \vee C)[\cdots, X, \cdots]$. Hence $a \in \bigcap\{(B \vee C)[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$.

   The 'only if' part is proved as follows. Suppose that $a \notin (B \vee C)[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]$. Then $a \notin B[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]$ and $a \notin C[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]$. By the hypotheses of induction, $a \notin \bigcap\{B[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$ and $a \notin \bigcap\{C[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$. Then there exist $X_1, X_2 \in P$ such that $Y \subseteq X_1$, $a \notin B[\cdots, X_1, \cdots]$, $Y \subseteq X_2$ and $a \notin C[\cdots, X_2, \cdots]$. Since $X_1 \cap X_2 \subseteq X_1$ and $X_1 \cap X_2 \subseteq X_2$, we have $a \notin B[\cdots, X_1 \cap X_2, \cdots]$ and $a \notin C[\cdots, X_1 \cap X_2, \cdots]$ by Proposition 5.5. Further, $Y \subseteq X_1 \cap X_2$ and $X_1 \cap X_2 \in P$, so $a \notin \bigcap\{A[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$.

4. A is of the form $\Box B$.

   By the fact that $\bigcap(\Box X) = \Box(\bigcap X)$ and hypothesis of induction,

$$\begin{array}{rcl} \bigcap\{(\Box B)[\cdots, X, \cdots] \mid Y \subseteq X \in P\} & = & \Box(\bigcap\{B[\cdots, X, \cdots] \mid Y \subseteq X \in P\}) \\ & = & (\Box B)[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]. \end{array}$$

5. A is of the form $\Diamond B$.

   If there is some $X \in P$ such that $Y \subseteq X$ and $B[\cdots, X, \cdots] = \emptyset$, it is clear. So, we assume that $\emptyset \notin \{B[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$. Taking $B[\cdots, Z_1, \cdots]$ and $B[\cdots, Z_2, \cdots]$ in $\{B[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$, we have $Y \subseteq Z_1$ and $Y \subseteq Z_2$. Then $Y \subseteq Z_1 \cap Z_2$ and $Z_1 \cap Z_2 \in P$. Further, by Proposition 5.5, $B[\cdots, Z_1 \cap Z_2, \cdots] \subseteq B[\cdots, Z_1, \cdots] \cap B[\cdots, Z_2, \cdots]$. Thus, we see that $\{B[\cdots, X, \cdots] \mid Y \subseteq X \in P\}$ is downward directed. Hence, by Lemma 5.7 and the hypothesis of induction,

$$\begin{array}{rcl} \bigcap\{(\Diamond B)[\cdots, X, \cdots] \mid Y \subseteq X \in P\} & = & \Diamond(\bigcap\{B[\cdots, X, \cdots] \mid Y \subseteq X \in P\}) \\ & = & (\Diamond B)[\cdots, \bigcap\{X \in P \mid Y \subseteq X\}, \cdots]. \end{array}$$

■

A given formula A is *negative* (in a regular logic $\mathbf{L}$ over $\mathbf{R}$) if A is equivalent to $\sim B$ for a positive formula B in $\mathbf{L}$. When $\mathbf{L}$ is either $\mathbf{R.C}$ or $\mathbf{R.K}$, we may define negative formulas in $\mathbf{L}$ to be formulas built from the negations of variables with the help of $\wedge$, $\vee$, $\Box$ and $\Diamond$ (see [11]). In the following, we say simply negative formulas, assuming that a regular logic $\mathbf{L}$ over $\mathbf{R}$ is fixed.

A modal formula A is *untied* (in $\mathbf{L}$) if it can be constructed from negative formulas (in $\mathbf{L}$) and strongly positive formulas using only $\wedge$ and $\Diamond$.

Lemma 5.9 *Let $A[p_1, \cdots, p_n]$ be an untied formula and $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ be a general $\mathbf{L}$-frame. Then for every $a \in W$ and all $X_1, \cdots, X_n \in P$,*

$$a \in A[X_1, \cdots, X_n] \;\; \textit{iff} \;\; \exists b_1 \cdots \exists b_t (D \;\&\; \bigwedge_{i \le n} T_i \subseteq X_i \;\&\; \bigwedge_{j \le m} c_j \in N_j[X_1, \cdots, X_n]),$$

where the formula in the right-hand side, effectively constructed from A, has only one free individual variable a, D is a conjunction of formulas of the form $S_\Diamond bc$, T_i are suitable $S_\Box$-terms and $N_j[p_1, \cdots, p_n]$ are negative formulas in $\mathbf{L}$.

Proof.

By induction on the length of the untied formula A.

1. When A is a negative formula,

$$a \in A[X_1, \cdots, X_n] \;\text{ iff }\; \bigwedge_{i \le n} u \uparrow^0 \subseteq X_i \;\&\; a \in A[X_1, \cdots, X_n]$$

   since $u \in X$ always holds.

2. When A is a strongly positive formula, it is clear by Lemma 5.4.

3. When A is of the form $B \wedge C$, by the hypotheses of induction,

$$\begin{aligned} & a \in (B \wedge C)[X_1, \cdots, X_n] \\ \text{iff} \;\; & \exists b_1 \cdots \exists b_l (D_1 \;\&\; \bigwedge_{i \le n} T_i \subseteq X_i \;\&\; \bigwedge_{j \le s} c_j \in N_j[X_1, \cdots, X_n]) \\ & \;\&\; \exists b_{l+1} \cdots \exists b_t (D_2 \;\&\; \bigwedge_{i \le n} T_i' \subseteq X_i \;\&\; \bigwedge_{s+1 \le j \le m} c_j \in N_j[X_1, \cdots, X_n]) \\ \text{iff} \;\; & \exists b_1 \cdots \exists b_t (D_1 \;\&\; D_2 \;\&\; \bigwedge_{i \le n} T_i \cup T_i' \subseteq X_i \;\&\; \bigwedge_{j \le m} c_j \in N_j[X_1, \cdots, X_n]). \end{aligned}$$

4. When A is of the form $\Diamond B$, by the hypothesis of induction,

$$\begin{aligned} & a \in (\Diamond B)[X_1, \cdots, X_n] \\ \text{iff} \;\; & \exists b (S_\Diamond ab \;\&\; b \in B[X_1, \cdots, X_n]) \\ \text{iff} \;\; & \exists b \exists b_1 \cdots \exists b_t (S_\Diamond ab \;\&\; D \;\&\; \bigwedge_{i \le n} T_i \subseteq X_i \;\&\; \bigwedge_{j \le m} c_j \in N_j[X_1, \cdots, X_n]). \end{aligned}$$

∎

Note that up to this points, the elements e and u introduced in Chapter 4 are not used essentially. Now, we show a *Sahlqvist theorem*, where these elements become necessary.

Theorem 5.10 *Let* $\mathbf{L}$ *be regular logic over* $\mathbf{R}$*. Suppose that* A *is a formula which is equivalent in* $\mathbf{L}$ *to a conjunction of formulas of the form* $\Box^k(B \to C)$*, where* $k \geq 0$*,* B *is untied in* $\mathbf{L}$ *and* C *is positive. Then there exists a first order formula* $\phi(a)$ *in the predicates* $O, R, S_\Box, S_\Diamond, ^*$ *and the constant* u *having* a *as its only free variable and such that the following holds for every* $\mathbf{L}$*-frame* $\mathfrak{F} \in \mathcal{D}^\star$ *and every* $a \in W$*,*

$$(\mathfrak{F}, a) \models A \quad \textit{iff} \quad \mathfrak{F} \textit{ satisfies } \phi(a).$$

Proof.

The proof goes essentially in the same way as the case of classical modal logics. Since $A \leftrightarrow \Box^{k_1}(B_1 \to C_1) \wedge \cdots \wedge \Box^{k_s}(B_s \to C_s)$, it is sufficient to consider $\Box^{k_i}(B_i \to C_i)$ for $i = 1, \cdots, s$. Below, we fix i and omit the subscripts.

Enumerating all variables appearing in $\Box^k(B \to C)$, let $q_1, \cdots, q_l$ be all variables appearing only in C and $p_1, \cdots, p_n$ be remaining variables. Let $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, P\rangle$ be a descriptive $\mathbf{L}$-frame or $\mathbf{L}$-frame. Below, let $X_1, \cdots, X_n, Y_1, \cdots, Y_l \in P$ and $b_i \in W$. Further, $\overrightarrow{X}$ and $\overrightarrow{Y}$ denote $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_l$, respectively. By Proposition 5.1,

$$(\mathfrak{F}, a) \models \Box^k(B \to C)$$

is equivalent to

$$\forall\overrightarrow{X}\forall\overrightarrow{Y}(a \in \Box^k(B \to C)[\overrightarrow{X}, \overrightarrow{Y}]),$$

and hence equivalent to

$$\forall\overrightarrow{X}\forall\overrightarrow{Y}\forall b_1\forall b_2\forall b_3(S^k_\Box ab_1 \ \& \ Rb_1b_2b_3 \ \& \ b_2 \in B[\overrightarrow{X}] \Rightarrow b_3 \in C[\overrightarrow{X}, \overrightarrow{Y}]).$$

By Lemma 5.9 and the fact that $u \in Y$ holds for every $Y \in P$, this is equivalent to

$$\forall\overrightarrow{X}\forall\overrightarrow{Y}\forall b_1\forall b_2\forall b_3(S^k_\Box ab_1 \ \& \ Rb_1b_2b_3 \ \& \ \exists b_4 \cdots \exists b_t(D \ \& \bigwedge_{i \leq n} T_i \subseteq X_i \ \& \bigwedge_{j \leq m} c_j \in N_j[\overrightarrow{X}])$$
$$\& \bigwedge_{h \leq l} u \in Y_h \Rightarrow b_3 \in C[\overrightarrow{X}, \overrightarrow{Y}]),$$

where D is a conjunction of formulas of the form $S_\Diamond bc$, T_i are suitable $S_\Box$-terms, $N_j[p_1, \cdots, p_n]$ are negative formulas. For each j $(1 \leq j \leq m)$, there exists a positive formula such that N_j is equivalent to $\sim K_j$. Since $c_j \in N_j$ iff $c^*_j \notin K_j$, this is equivalent to

$$\forall b_1 \cdots \forall b_t(D' \Rightarrow \forall\overrightarrow{X}\forall\overrightarrow{Y}(\bigwedge_{i \leq n} T_i \subseteq X_i \ \& \bigwedge_{h \leq l} u\uparrow^0 \subseteq Y_h \Rightarrow \bigvee_{j \leq m+1} d_j \in K_j[\overrightarrow{X}, \overrightarrow{Y}])),$$

where D' denotes $S^k_\Box ab_1 \ \& \ Rb_1b_2b_3 \ \& \ D$, and $d_j = c^*_j$ for j $(1 \leq j \leq m)$, $d_{m+1} = b_3$ and K_{m+1} is C. By Proposition 5.6, this is equivalent to

$$\forall b_1 \cdots \forall b_t(D' \Rightarrow \bigvee_{j \leq m+1} d_j \in \bigcap\{K_j[\overrightarrow{X}, \overrightarrow{Y}] \mid T_i \subseteq X_i, \text{ for } i \leq n; u\uparrow^0 \subseteq Y_h, \text{ for } h \leq l\}).$$

By Lemma 5.8, this is equivalent to

$$\forall b_1 \cdots \forall b_t(D' \Rightarrow \bigvee_{j \leq m+1} d_j \in K_j[X'_1, \cdots, X'_n, Y'_1, \cdots, Y'_l]),$$

where $X_i' = \bigcap\{X_i \mid T_i \subseteq X_i\}$ for $1 \leq i \leq n$ and $Y_h' = \bigcap\{Y_h \mid u\uparrow^0 \subseteq Y_h\}$ for $1 \leq h \leq l$. By Proposition 5.2, this is equivalent to

$$\forall b_1 \cdots \forall b_t (D' \Rightarrow \bigvee_{j \leq m+1} d_j \in K_j[T_1, \cdots, T_n, u\uparrow^0, \cdots, u\uparrow^0]).$$

Then using Lemma 5.3, each $d_j \in K_j[T_1, \cdots, T_n, u\uparrow^0, \cdots, u\uparrow^0]$ can be expressed by a first order formula (in the predicates O, R, $S_\Box$, $S_\Diamond$, *, and the constant u) having a as its only free variable. ■

Any formula A of the form described in above theorem is called a *Sahlqvist formula.* From Theorem 5.10, we have $\mathfrak{F} \models A$ iff $\mathfrak{F}$ satisfies $\forall a \in O(\phi(a))$.

Theorem 5.11 *Let* $\mathbf{L}_0$ *be a* $\mathcal{D}^\star$*-elementary regular logic over* $\mathbf{R}$*, and* $\mathbf{L}$ *be a logic obtained from* $\mathbf{L}_0$ *by adding a set of Sahlqvist formulas as axioms. Then* $\mathbf{L}$ *is* $\mathcal{D}^\star$*-elementary, and hence* $\mathcal{D}$*-persistent. Further,* $\mathbf{L}$ *is* $\kappa\mathcal{D}$*-complete.*

Proof.
Since $\mathbf{L}$ is a regular logic over $\mathbf{R}$, $\mathbf{L}$ is $\mathcal{D}$-complete by Theorem 4.24. By Theorem 5.10, $\mathbf{L}$ is also $\mathcal{D}^\star$-elementary. Thus, $\mathbf{L}$ is $\kappa\mathcal{D}$-complete by 3 of Proposition 4.32. ■

We can see that Theorem 5.11 covers all completeness results of relevant modal logics in [19], [34] and [35]. Note that each of $\mathbf{R.K}_{\Box\Diamond}$, $\mathbf{R.C}$ and $\mathbf{R.K}$ is $\mathcal{D}^\star$-elementary extension of $\mathbf{R.C}_{\Box\Diamond}$ (see postulates from (p9) to (p12) in Section 3.2). Thus, we can apply Theorem 29 to any logic obtained from one of these logics by adding a set of Sahlqvist formulas as axioms.

Moreover, the classical modal logic $\mathbf{K}$ is a $\mathcal{D}^\star$-elementary extension of $\mathbf{R.K}$. In fact, to obtain frames for $\mathbf{K}$, it suffices to add the following three postulates to those of $\mathbf{R.K}$-frames
$\langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ (see Section 5.5 of [48]), which can be represented by first order formulas for all $a, b, c \in W$,

- if $b \neq e$ and $Rabc$, then $a \leq c$
- if $a \neq e$, $b \neq u$ and $a \leq b$, then $a = b$
- $e \notin O$.

As long as we neglect elements e and u, frames for $\mathbf{K}$ thus obtained are exactly usual Kripke frames for $\mathbf{K}$ since we have the following:

$$(1) \quad O = W, \quad (2) \quad \text{if } Rabc \text{ then } a = b = c, \quad (3) \quad a^* = a, \quad (4) \quad S_\Box = S_\Diamond.$$

Therefore, Theorem 29 covers Sahlqvist theorem for classical modal logics.

As a consequence of Theorem 5.11, we have the following.

A *Lemmon-Scott axiom* is of the form

$$\Diamond^{m_1}\Box^{n_1}p_1 \wedge \cdots \wedge \Diamond^{m_k}\Box^{n_k}p_k \to A[p_1, \cdots, p_k],$$

where $A[p_1, \cdots, p_k]$ is a positive formula. Since each Lemmon-Scott axiom is a Sahlqvist formula, we have completeness and $\mathcal{D}$-persistency of relevant modal logics with Lemmon-Scott axioms from Theorem 5.11.

Corollary 5.12 *Let* $\mathbf{L}_0$ *be a* $\mathcal{D}^\star$*-elementary regular logic over* $\mathbf{R}$*, and* $\mathbf{L}$ *be a logic obtained from* $\mathbf{L}_0$ *by adding a set of Lemmon-Scott axioms. Then* $\mathbf{L}$ *is* $\mathcal{D}$*-persistent and* $\kappa\mathcal{D}$*-complete.*

Another interesting consequence of Theorem 5.11 is completeness of superclassical relevant modal logics, i.e., relevant modal logics over **KR**. **KR** is obtained from **R** by adding an axiom $p\wedge \sim p \to q$, which is a Sahlqvist formula in our sense, and is discussed in Section 5.4 of [48]. Thus, all of $\mathbf{KR.C}_{\Box\Diamond}$, $\mathbf{KR.K}_{\Box\Diamond}$, **KR.C** and **KR.K** are complete, where $\mathbf{KR.C}_{\Box\Diamond}$, $\mathbf{KR.K}_{\Box\Diamond}$, **KR.C** and **KR.K** are obtained from $\mathbf{R.C}_{\Box\Diamond}$, $\mathbf{R.K}_{\Box\Diamond}$, **R.C** and **R.K**, respectively, by adding above axiom.

This result shows the necessity of introducing *enlarged frames* with the null world e and the universal world u. In fact, they are incomplete with respect to the frames without these e and u, i.e., frames in the usual sense.

We will give here a proof of these incompleteness results. A $\mathbf{KR.C}_{\Box\Diamond}$-frame in the usual sense is an $\mathbf{R.C}_{\Box\Diamond}$-frame obtained by assuming that

(p1)' there exists $a \in O$ such that $Rabc$ iff $b = c$

(p6)' $a^* = a$

instead of (p1) and (p6), respectively. $\mathbf{KR.K}_{\Box\Diamond}$-frames, **KR.C**-frames and **KR.K**-frames are defined similarly. Note that $a \models \sim A$ iff $a \not\models A$ in any model on these frames. We can easily see that $\Box(A\vee \sim A)$ is valid in every $\mathbf{KR.C}_{\Box\Diamond}$-frame. But it is not a theorem of $\mathbf{KR.C}_{\Box\Diamond}$. Because $\mathbf{KR.C}_{\Box\Diamond}$ is contained in the classically-based (non-normal) classical modal logic **EMC** (cf. [12], section 8.2) and $\Box(A\vee \sim A)$ is not a theorem of **EMC**. This argument can be applied to **KR.C**.

Also, $A\vee \sim A \to \Box(A\vee \sim A)$ is valid in any $\mathbf{KR.K}_{\Box\Diamond}$-frame. Using the following truth table with the designated values **T** and **t**, we see that $A\vee \sim A \to \Box(A\vee \sim A)$ is not a theorem of $\mathbf{KR.K}_{\Box\Diamond}$.

$A \to B$	**T**	**t**	**f**	**F**
T	**T**	**F**	**F**	**F**
t	**T**	**t**	**f**	**F**
f	**T**	**F**	**t**	**F**
F	**T**	**T**	**T**	**T**

$A \vee B$	**T**	**t**	**f**	**F**
T	**T**	**T**	**T**	**T**
t	**T**	**t**	**t**	**t**
f	**T**	**t**	**f**	**f**
F	**T**	**t**	**f**	**F**

A	$\sim A$	$\Box A$
T	**F**	**t**
t	**f**	**t**
f	**t**	**F**
F	**T**	**F**

This argument can be applied to **KR.K**.

Finally, we consider some Sahlqvist formulas from the viewpoint of correspondence. Let us consider

$$\mathbf{Dir} \qquad \Diamond(\Box p \wedge q) \to \Box(\Diamond p \vee q).$$

Then using (p1) and (p8), for all $x \in O$,

$$
\begin{array}{ll}
& x \models \Diamond(\Box p \wedge q) \to \Box(\Diamond p \vee q) \\
\text{iff} & \forall X, Y \in P \forall a_1 \forall a_2 \forall a (S^0_\Box x a_1 \ \&\ R a_1 a_2 a \ \&\ a_2 \in \Diamond(\Box X \cap Y) \Rightarrow a \in \Box(\Diamond X \cup Y)) \\
\text{iff} & \forall X, Y \in P \forall a (a \in \Diamond(\Box X \cap Y) \Rightarrow a \in \Box(\Diamond X \cup Y)) \\
\text{iff} & \forall X, Y \in P \forall a (\exists b (S_\Diamond ab \ \&\ b \in \Box X \ \&\ b \in Y) \Rightarrow a \in \Box(\Diamond X \cup Y)) \\
\text{iff} & \forall X, Y \in P \forall a \forall b (S_\Diamond ab \ \&\ b \uparrow \subseteq X \ \&\ b \uparrow^0 \subseteq Y) \Rightarrow a \in \Box(\Diamond X \cup Y)) \\
\text{iff} & \forall X, Y \in P \forall a \forall b (S_\Diamond ab \Rightarrow a \in \{\Box(\Diamond X \cup Y) \mid b \uparrow \subseteq X \ \&\ b \uparrow^0 \subseteq Y\}) \\
\text{iff} & \forall a \forall b (S_\Diamond ab \Rightarrow \forall X, Y \in P (a \in \Box\{\Diamond X \cup Y \mid b \uparrow \subseteq X \ \&\ b \uparrow^0 \subseteq Y\})) \\
\text{iff} & \forall a \forall b \forall c (S_\Diamond ab \ \&\ S_\Box ac \Rightarrow \forall X, Y \in P (c \in \{\Diamond X \mid b \uparrow \subseteq X \ \&\ b \uparrow^0 \subseteq Y\}) \\
& \quad \text{or } \forall X, Y \in P (c \in \{Y \mid b \uparrow \subseteq X \ \&\ b \uparrow^0 \subseteq Y\})) \\
\text{iff} & \forall a \forall b \forall c (S_\Diamond ab \ \&\ S_\Box ac \Rightarrow \forall X, Y \in P (c \in \Diamond\{X \mid b \uparrow \subseteq X \ \&\ b \uparrow^0 \subseteq Y\}) \\
& \quad \text{or } \forall X, Y \in P (c \in \{Y \mid b \uparrow \subseteq X \ \&\ b \uparrow^0 \subseteq Y\})) \\
\text{iff} & \forall a \forall b \forall c (S_\Diamond ab \ \&\ S_\Box ac \Rightarrow \exists d (S_\Diamond cd \ \&\ d \in b \uparrow) \text{ or } c \in b \uparrow^0) \\
\text{iff} & \forall a \forall b \forall c (S_\Diamond ab \ \&\ S_\Box ac \Rightarrow \exists d (S_\Diamond cd \ \&\ S_\Box bd) \text{ or } b \leq c).
\end{array}
$$

Note that **Dir** is a Sahlqvist formula but not a Lemmon-Scott axiom. Besides,

$$\mathbf{Tra}(n) \qquad p \wedge \Box p \wedge \cdots \Box^n p \to \Box^{n+1} p$$

is also an examples of a Sahlqvist formula but not a Lemmon-Scott axiom.

Finally, we present an example using essentially enlarged frames. We notice that $p \wedge \sim p \to q$ is also a Sahlqvist formula. By Theorem 5.10, we have the following. For all $a \in O$,

$$
\begin{array}{ll}
& (\mathfrak{F}_e, a) \models p \wedge \sim p \to q \\
\text{iff} & \forall X, Y \in P \forall a_1 \forall a_2 \forall b (S^0_\Box a a_1 \ \&\ R a_1 a_2 b \ \&\ a_2 \in X \cap -X \Rightarrow b \in Y) \\
\text{iff} & \forall X, Y \in P \forall b (b \in X \ \&\ b^* \notin X \ \&\ u \in Y \Rightarrow b \in Y) \\
\text{iff} & \forall X, Y \in P \forall b (b \uparrow^0 \subseteq X \ \&\ u \uparrow^0 \subseteq Y \Rightarrow b^* \in X \ \text{ or } \ b \in Y) \\
\text{iff} & \forall b (b^* \in b \uparrow^0 \ \text{ or } \ b \in u \uparrow^0) \\
\text{iff} & \forall b (b \leq b^* \ \text{ or } b = u) \\
\text{iff} & \forall b (b \neq u \Rightarrow b \leq b^*).
\end{array}
$$

5.2 Non-Sahlqvist formulas and completeness

In this section, we consider completeness of relevant modal logics with non-Sahlqvist formulas. This result is shown by using the canonical model.

Let us consider the following axioms.

SC	$\Box(\Box A \to B) \vee \Box(\Box B \to A)$
Con	$\Box(A \wedge \Box A \to B) \vee \Box(B \wedge \Box B \to A)$
DG(k,l,m,n)	$\sim \Diamond^k \Box^l A \vee \Box^m \Diamond^n A$
Alt(n)	$\Box A_1 \vee \Box(A_1 \to A_2) \vee \cdots \vee \Box(A_1 \wedge \cdots \wedge A_n \to A_{n+1})$

Theorem 5.13 *The logic* **L** *obtained from* $\mathbf{R.C}_{\Box\Diamond}$ *by adding the left-hand side axiom is determined by a class of* $\mathbf{R.C}_{\Box\Diamond}$*-frames satisfying the right-hand side postulate. For all* $x \in O$ *and other variables in* W*:*

SC	$S_\Box xa \ \&\ Rabc \ \&\ S_\Box xa' \ \&\ Ra'b'c' \Rightarrow S_\Box bc' \text{ or } S_\Box b'c$
Con	$S_\Box xa \ \&\ Rabc \ \&\ S_\Box xa' \ \&\ Ra'b'c' \Rightarrow b \leq c' \text{ or } b' \leq c \text{ or } S_\Box bc' \text{ or } S_\Box b'c$
DG(k,l,m,n)	$S^k_\Diamond x^* a \ \&\ S^m_\Box xb \Rightarrow \exists y \in W (S^l_\Box ay \ \&\ S^n_\Diamond by)$
Alt(n)	$S_\Box xc_0 \ \&\ \bigwedge_{i=1}^n (S_\Box xa_i \ \&\ Ra_ib_ic_i) \ \&\ c_n \neq u \Rightarrow \bigvee_{i=0}^{n-1} \bigvee_{j=i+1}^n b_j \leq c_i,$

where $\bigwedge$ and $\bigvee$ denotes conjunction and disjunction, respectively, in the metalanguage.

Proof.

It is easy to prove the soundness part. So we will prove only the completeness part. Let $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c\rangle$ be the canonical **L**-frame.

1. Case that **L** is obtained by adding **SC**.

   Assume that for $\Pi \in O_c$ and $\Sigma, \Gamma, \Delta, \Sigma', \Gamma', \Delta' \in W_c$, $S_{\Box c}\Pi\Sigma$, $R_c\Sigma\Gamma\Delta$, $S_{\Box c}\Pi\Sigma'$ and $R_c\Sigma'\Gamma'\Delta'$ hold and that $S_{\Box c}\Gamma\Delta'$ does not hold. From the last assumption, there exists $B \in \mathsf{Wff}$ such that $\Box B \in \Gamma$ and $B \notin \Delta'$.

   It is sufficient to show $S_{\Box c}\Gamma'\Delta$. Suppose that $\Box A \in \Gamma'$. Then $\Box A \to B \notin \Sigma'$ by the assumption $R_c\Sigma'\Gamma'\Delta'$, so we have $\Box(\Box A \to B) \notin \Pi$ by the assumption $S_{\Box c}\Pi\Sigma'$. Since $\Pi \in O_c$, $\Box(\Box A \to B) \vee \Box(\Box B \to A) \in \Pi$, and hence we have $\Box(\Box B \to A) \in \Pi$. By the assumption $S_{\Box c}\Pi\Sigma$, we have $\Box B \to A \in \Sigma$, so that $A \in \Delta$ by the assumption $R_c\Sigma\Gamma\Delta$. Hence we see that $S_{\Box}\Gamma'\Delta$, which is just $S_{\Box c}\Gamma'\Delta$ since both Γ' and Δ are prime.

   Therefore if $S_{\Box c}\Pi\Sigma$, $R_c\Sigma\Gamma\Delta$, $S_{\Box c}\Pi\Sigma'$ and $R_c\Sigma'\Gamma'\Delta'$ then $S_{\Box c}\Gamma\Delta'$ or $S_{\Box c}\Gamma'\Delta$.

2. Case that **L** is obtained by adding **Con**.

   Assume that there are $\Pi \in O_c$ and $\Sigma, \Gamma, \Delta, \Sigma', \Gamma', \Delta' \in W_c$ such that $S_{\Box c}\Pi\Sigma$, $R_c\Sigma\Gamma\Delta$, $S_{\Box c}\Pi\Sigma'$, $R_c\Sigma'\Gamma'\Delta'$, $\Gamma \not\subseteq \Delta'$ and $\Gamma' \not\subseteq \Delta$ hold and that $S_{\Box c}\Gamma\Delta'$ and $S_{\Box c}\Gamma'\Delta$ do not hold. Then there exist $A_1, A_2, B_1, B_2 \in \mathsf{Wff}$ such that $A_1 \in \Gamma$, $A_1 \notin \Delta'$, $\Box A_2 \in \Gamma$, $A_2 \notin \Delta'$, $B_1 \in \Gamma'$, $B_1 \notin \Delta$, $\Box B_2 \in \Gamma'$ and $B_2 \notin \Delta$. Now we put $A = A_1 \vee A_2$ and $B = B_1 \vee B_2$. Since both Γ and Γ' are **L**-theories, $A \wedge \Box A \in \Gamma$ and $B \wedge \Box B \in \Gamma'$. Further we have $B \notin \Delta$ and $A \notin \Delta'$ since both Δ and Δ' are prime. So we get $\Box(A \wedge \Box A \to B) \notin \Pi$ and $\Box(B \wedge \Box B \to A) \notin \Pi$.

   Since $\Pi \in O_c$, we have $\Box(A \wedge \Box A \to B) \vee \Box(B \wedge \Box B \to A) \notin \Pi$, which is a contradiction.

3. Case that **L** is obtained by adding $\mathbf{DG}(k, l, m, n)$.

   Suppose that ${S_{\Diamond c}}^k g_c(\Pi)\Sigma$ and ${S_{\Box c}}^m\Pi\Gamma$ for $\Pi \in O_c$ and $\Sigma, \Gamma \in W_c$. Let $\Delta = \{A \mid \Box^l A \in \Sigma\}$ and $\Xi = \{A \mid \Diamond^n A \notin \Gamma\}$. Then it is clear that Δ is an **L**-theory satisfying $S^l_{\Box}\Sigma\Delta$ and that Ξ is closed under disjunction. Assume that there is $A \in \mathsf{Wff}$ such that $A \in \Delta \cap \Xi$. Then $\Box^l A \in \Sigma$ and $\Diamond^n A \notin \Gamma$, so $\sim \Diamond^k \Box^l A \notin \Pi$ and $\Box^m \Diamond^n \notin \Pi$ by the assumptions. Since $\Pi \in O_c$, we have $\sim \Diamond^k \Box^l A \vee \Box^m \Diamond^n A \notin \Pi$, which is a contradiction. Hence $\Delta \cap \Xi = \emptyset$. By 3 of Lemma 3.8, there exists a prime **L**-theory $\Delta' \supseteq \Delta$ such that $\Delta' \cap \Xi = \emptyset$. It is clear that ${S_{\Box c}}^l\Sigma\Delta'$. And if $A \in \Delta'$, then $A \notin \Xi$, so $\Diamond^n A \in \Gamma$. Hence ${S_{\Diamond c}}^n\Gamma\Delta'$.

4. Case that **L** is obtained by adding $\mathbf{Alt}(n)$.

   Assume that there exist $\Pi \in O_c$ and $\Delta_0, \Sigma_p, \Gamma_p, \Delta_p \in W_c (1 \le p \le n)$ such that $S_{\Box c}\Pi\Delta_0$, $\bigwedge_{p=1}^n (S_{\Box c}\Pi\Sigma_p \;\&\; R_c\Sigma_p\Gamma_p\Delta_p)$, $\Delta_n \neq \mathsf{Wff}$ and $\bigwedge_{j=0}^{n-1}\bigwedge_{k=j+1}^n \Gamma_k \not\subseteq \Delta_j$. Then there exist $A_i^j \in \mathsf{Wff}$ $(1 \le i \le n, 1 \le j \le n+1-i)$ such that $A_i^j \in \Gamma_{i+j-1}$ and $A_i^j \notin \Delta_{i-1}$.

   Let A_i be $A_i^1 \wedge \cdots \wedge A_i^{n+1-i}$. Since each Γ_p is an **L**-theory, $A_i \in \Gamma_p (1 \le i \le p)$. Moreover since Δ_{i-1} is a prime **L**-theory, $A_i \notin \Delta_{i-1}$. Further, there exists $A_{n+1} \notin \Delta_n$

by the assumption $\Delta_n \neq \mathsf{Wff}$. Then $\Box A_1 \notin \Pi$ and $\bigwedge_{i=1}^{n} \Box(A_1 \wedge \cdots \wedge A_i \to A_{i+1}) \notin \Pi$. Since $\Pi \in O_c$, we have $\Box A_1 \vee \Box(A_1 \to A_2) \vee \cdots \vee \Box(A_1 \wedge \cdots \wedge A_n \to A_{n+1}) \notin \Pi$, which is a contradiction. ■

5.3 Notes

As mentioned at the beginning of this chapter, Sahlqvist theorem was first shown by H.Sahlqvist in 1975 ([49]). After that some improved proof of this theorem have been studied by G.Sambin ([50]), J.A.F.K.van Benthem ([59]), G.Sambin and V.Vaccaro ([52]) and B.Jónsson ([30]). Further, an intuitionistic analog of Sahlqvist theorem was proved by S.Ghilardi and G.Meloni in [20]. Also, Sahlqvist theorem for intuitionistic modal logic was proved by C.Grefe in [27].

We showed completeness and $\mathcal{D}$-persistency of relevant modal logics with Lemmon-Scott axioms in Corollary 5.12. For these results for classical modal logic, R.I.Goldblatt showed in [22] and [24] without using Sahlqvist theorem.

Chapter 6

Summary and Further studies

In this chapter, we summarize our results in this thesis and state further studies concerning relevant modal logics.

6.1 Summary

This thesis deals with semantics of relevant modal logics. Our main subject is completeness of wider class of modal logics over the relevant logic **R**.

We introduce four basic relevant modal logics $\mathbf{R.C}_{\Box\Diamond}$, $\mathbf{R.K}_{\Box\Diamond}$, **R.C** and **R.K** in Chapter 3. The characteristic point in this thesis is that a modal operator $\Diamond$ is not defined by $\sim$ and $\Box$ in $\mathbf{R.C}_{\Box\Diamond}$ and $\mathbf{R.K}_{\Box\Diamond}$, while it is defined as usual in **R.C** and **R.K**. Their semantics are obtained from the semantics of the relevant logic **R** by adding some conditions for discussing modalities. The one is the extension based on Routley-Meyer semantics, and the other is the extension of relevant matrices based on De Morgan semigroups. Our semantics characterize these relevant modal logics. In particular, this thesis shows that $\mathbf{R.C}_{\Box\Diamond}$ is the logic characterizing all relevant modal frames.

In Chapter 4, we consider the notion of general frames. Further, we introduce the dual of general frames and the dual of matrices. Then we see that each relevant modal matrix is isomorphic to its bidual as in classical modal algebras. We define descriptive frames to make each general frame become isomorphic to its bidual. We know the correspondences between general frames and matrices in classical modal logics, and we expect them to hold for relevant modal logics. We investigate them from the view of the truth-preserving operations. In relevant modal logics, as well as in classical modal logics, we see that the following correspondences between matrices and general frames also hold:

1. submatrices — relevant p-morphisms
2. homomorphic images — (generated) subframes
3. direct products — disjoint unions.

As concerns descriptive frames, we introduce $\mathcal{D}$-persistent logics and $\mathcal{D}^\star$-elementary logics.

In Chapter 5, we give a proof of the completeness of relevant modal logics with Sahlqvist formulas following the Sambin and Vaccaro's method. Our results are quite similar to classical modal logics. Given a Sahlqvist formula, we can get the frame postulate written by a first order sentence. This fact means that $\mathcal{D}^\star$-elementary relevant modal logics with Sahlqvist formulas are also $\mathcal{D}^\star$-elementary. Therefore, this implies Kripke completeness of relevant modal logics with Sahlqvist formulas. Our result includes a Sahlqvist theorem for modal logics over classical modal logics. Further, we also consider completeness of some relevant modal logics with non-Sahlqvist formulas.

6.2 Further studies

This thesis develops basic semantical results on relevant modal logics. The following research points of relevant modal logics based on the results obtained in this thesis are considered.

Concerning our Sahlqvist theorem for relevant modal logics, there is a place for its extension. It seems to apply a Sahlqvist theorem to the formula in which implications are nested in modalities. We have already a Sahlqvist theorem for (non-modal) relevant logics, which will be discussed in other occasion.

In the proof of a Sahlqvist theorem, we see that it is possible to get a first order sentence corresponding to a Sahlqvist formula. On the other hand, M.Kracht developed the characterization of a class of first order sentences corresponding to Sahlqvist formulas for classical modal logics in [31]. This result is called the Kracht theorem. So, the Kracht theorem for relevant modal logics is one of interesting research topics.

In modal logics, concerning the relationship between $\mathcal{D}$-persistent logics and elementary logics, the Fine-van Benthem theorem is well-known result. The Fine-van Benthem theorem says that if a logic is characterized by an elementary class of Kripke frames then it is $\mathcal{D}$-persistent, is one of the most important results. It is interesting to consider the Fine-van Benthem theorem for relevant modal logics.

In this thesis, we adopt Routley-Meyer semantics, matrix semantics and general frame semantics. Our frames are unreduced frames. In relevant logics, the conditions for using reduced models are known. According to [48], reduced models can be constructed in relevant logics including theorems

- $A \wedge (A \to B) \to B$,
- $(A \to B) \to ((B \to C) \to (A \to C))$
- $(A \to\sim B) \to (B \to\sim A)$.

Further, J.K.Slaney ([53]) considered reduced models for relevant logics of which $A\wedge(A \to B) \to B$ is not a theorem. The problem to be clear is which conditions make it possible to use reduced models in relevant modal logics. In [19], A.Fuhrmann shows that if $\leq$ is defined in our way, then no logic weaker than **R.KT4** is complete with respect to some class of reduced frames, where **R.KT4** is the **S4**-style relevant modal logic. On the other hand, R.Routley and R.K.Meyer shows that **R.KT4** (they called **NR**) is complete with respect to the class of reduced frames if $\leq$ is defined by

$$a \leq b \overset{\text{def}}{=} \exists c \in W(S_\Box 0c \ \&\ Rcab).$$

In the study of relevant logic, γ-admissibility is one of important problems. A relevant logic $\mathbf{L}$ is called γ-*admissible* if B is a theorem of $\mathbf{L}$ whenever both $\sim A \vee B$ and A are theorems of $\mathbf{L}$. [39] and [47] show that $\mathbf{R}$ is γ-admissible by using an algebra and a frame, respectively. Further, it is known when γ-admissibility fails ([40]). For relevant modal logic, [46] and [37] deal with γ-admissibility. But we have no comprehensive result on γ-admissibility.

In algebraic studies, the notion of varieties are often dealt. For a class $\mathcal{C}$ of algebras, $H\mathcal{C}$, $S\mathcal{C}$ and $P\mathcal{C}$ denote the class of all homomorphic images of algebras in $\mathcal{C}$, the class of all subalgebras of algebras in $\mathcal{C}$ and the class of all possible direct products of $\mathcal{C}$'s subclasses, respectively. Then a variety is a class closed under H, S and P. But we meet the problem how to define given equality to be satisfied in a given matrix.

In this thesis, relevant modal logics contain no propositional constant. For this, we must use matrices as algebraic models. So, we extend our results to relevant modal logics with propositional constants. It is known that there are four propositional constants $\mathbf{t}$, $\mathbf{f}$, $\top$ and $\bot$ in relevant logics. If we use them, algebraic models can be defined by algebra. Already, S.A.Celani has studied relevant modal logics with these four propositional constants from the algebraic view (see [10]). In our impression, it is problematic to consider the dual of algebras as general frames. Further, we wonder whether $\Box\mathbf{t}$ and $\Box\top$ are identified with $\mathbf{t}$ and $\top$, respectively, in relevant modal logics.

There are some criticisms for Routley-Meyer semantics, so several semantics for relevant logics are suggested. For example, Urquhart's semilattice semantics (see [55] or Section 47 of [5]), Fine semantics (see [16] or Section 51 of [5]) and American-plan semantics (see Section 4.7 of [48] and [45]). It may be interesting to extend these semantics for discussing relevant modal logics.

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