Weakly-non-overlapping non-collapsing shallow term rewriting systems are confluent

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Abstract

This paper shows that weakly-non-overlapping, non-collapsing and shallow term rewriting systems are confluent, which is a new sufficient condition on confluence for non-left-linear systems.

Key words: Term rewriting systems, confluence, formal languages

1. Introduction

Confluence, which guarantees the uniqueness of a computation, is an important property for term rewriting systems (TRSs). This property is undecidable not only for general TRSs, but also for flat TRSs [Mitsu06] and length-two string rewrite systems [Sakai08]. It becomes decidable if TRSs are either right-linear and shallow [Godoy05], or terminating [KB70].

For left-linear TRSs, many sufficient conditions have been studied: non-overlapping [Rosen73], parallel-closed [Huet80], and their extensions [Toyama87, Oostrom95, Gramlich96, Oyama97, Okui98, Oyama03].

However, the analysis of non-left-linear TRSs is difficult and only few sufficient conditions are known: simple-right-linear TRSs (i.e., right-linear and non-left-linear variables do not appear in the rhs) such that either non-E-overlapping [Ohta95] or its conditional linearizations are weight-decreasing joinable [Toyama95]. Without right-linearity, Gomi, Oyamaguchi, and Ohta showed sufficient conditions: strongly depth-preserving and non-E-overlapping [Gomi96], and strongly depth-preserving and root-E-closed [Gomi98].

This paper shows that weakly-non-overlapping, non-collapsing and shallow TRSs are confluent, which is a new sufficient condition for non-left-linear and non-right-linear systems.

2. Basic notion

We assume that readers are familiar with basic notions of term rewriting systems. The precise definitions are found in [Baader98].

2.1. Abstract reduction system

For a binary relation $\to$, we use $\leftrightarrow$, $\to^+$ and $\to^*$ for the symmetric closure, the transitive closure, and the reflexive and transitive closure of $\to$, respectively. We use $\circ$ for the composition operation of two relations.

An *abstract reduction system* (ARS) G is a pair $\langle V, \to\rangle$ of a set V and a binary relation $\to$ on V. If $\langle u, v\rangle \in \to$ we say that u *is reduced to* v, denoted by $u \to v$. An element u of V is *(G-)normal* if there exists no $v \in V$ such that $u \to v$. We sometimes call a normal element a normal form.

Let $G = \langle V, \to\rangle$ be an ARS. We say G is *finite* if V is finite, *confluent* if $\leftarrow^* \circ \to^* \subseteq \to^* \circ \leftarrow^*$, and *Church-Rosser* (*CR*) if $\leftrightarrow^* \subseteq \to^* \circ \leftarrow^*$. It is well known that confluence and CR are equivalent.

We say G is *terminating* if it does not admit an infinite reduction sequence. We say G is *convergent* if it is confluent and terminating. A *cycle* of G is a reduction sequence $t \to^+ t$. An edge $v \to u$ is called an *out-edge* of v and an *in-edge* of u. Note that a node v having no out-edge is normal. We say G is *connected* if $u \leftrightarrow^* v$ for every $u, v \in G$. We say G' ($\subseteq G$) is a *connected component* of G if G' is connected and $u \not\leftrightarrow^* v$ for any $u \in G'$ and $v \in G \setminus G'$.

2.2. Term rewriting system

Let F be a finite set of function symbols with fixed arity, and X be an enumerable set of variables where $F \cap X = \emptyset$. By $\mathrm{T}(F, X)$, we denote the set of terms constructed from F and X. Terms in $\mathrm{T}(F, \emptyset)$ are said to be *ground*.

The set of *positions* of a term t is the set $\mathrm{Pos}(t)$ of strings of positive integers, which is defined by $\mathrm{Pos}(t) = \{\varepsilon\}$ if t is a variable, and $\mathrm{Pos}(t) = \{\varepsilon\} \cup \{ip \mid p \in \mathrm{Pos}(t_i), 1 \leq i \leq n\}$ if $t = f(t_1, \ldots, t_n)$ $(0 \leq n)$. We call ε the *root* position. For $p \in \mathrm{Pos}(t)$, the subterm of t at position p, denoted by $t|_p$, is defined as $t|_\varepsilon = t$ and $f(t_1, \ldots, t_n)|_{iq} = t_i|_q$. The term obtained from t by replacing its subterm at position p with s, denoted by $t[s]_p$, is defined as $t[s]_\varepsilon = s$ and $f(t_1, \ldots, t_n)[s]_{iq} = f(t_1, \ldots, t_{i-1}, t_i[s]_q, t_{i+1}, \ldots, t_n)$. The *size* $|t|$ of a term t is $|\mathrm{Pos}(t)|$. We use $\mathrm{Args}(t)$ for the set of *direct subterms* (or *arguments*) of a term t defined as $\mathrm{Args}(t) = \emptyset$ if t is a variable and $\mathrm{Args}(t) = \{t_1, \ldots, t_n\}$ if $t = f(t_1, \ldots, t_n)$ $(0 \leq n)$. For a set T of terms, $\mathrm{Args}(T) = \bigcup_{t \in T} \mathrm{Args}(t)$.

A mapping $\theta : X \to \mathrm{T}(F, X)$ is called a *substitution* if its domain $\mathrm{Dom}(\theta) = \{x \mid \theta(x) \neq x\}$ is finite. A substitution θ is naturally extended to the mapping on terms by defining $\theta(f(t_1, \ldots, t_n)) = f(\theta(t_1), \ldots, \theta(t_n))$. The application $\theta(t)$ of a substitution θ to a term t is denoted by $t\theta$.

A *rewrite rule* is a pair $\langle l, r\rangle$ of terms such that $l \notin X$ and every variable in r occurs in l. We write $l \to r$ for the pair. A *term rewriting system* (TRS) is a set R of rewriting rules. The *reduction relation* $\underset{R}{\to}$ on $\mathrm{T}(F, X)$ induced by R is defined as follows; $s \underset{R}{\to} t$ if and only if $s = s[l\sigma]_p$ and $t = s[r\sigma]_p$ for a rewriting rule $l \to r \in R$, a substitution σ, and $p \in \mathrm{Pos}(s)$. We sometimes write $s \overset{p}{\underset{R}{\to}} t$

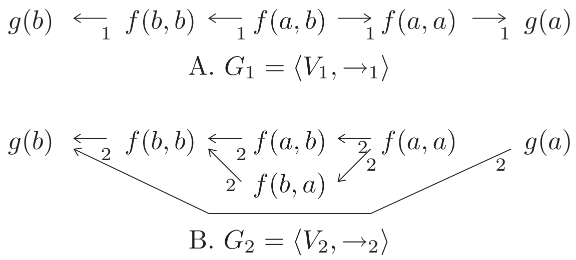

Figure 1: R_1-Reduction graphs

to indicate the *rewrite step* at the position p. Let $s \overset{p}{\underset{R}{\to}} t$. It is a *top reduction* if $p = \varepsilon$. Otherwise it is an *inner reduction*, written as $s \overset{\varepsilon<}{\underset{R}{\to}} t$.

A term is *shallow* if $|p|$ is 0 or 1 for every position p of variables in the term. A rewrite rule $l \to r$ is *shallow* if l and r are shallow, and *collapsing* if r is a variable. A TRS is *shallow* if its rules are all shallow. A TRS is *non-collapsing* if it contains no collapsing rules.

Let $l_1 \to r_1$ and $l_2 \to r_2$ be rewrite rules whose variables have been renamed so that variables in the former rule and those in the latter rule are disjoint. Let p be a position in l_1 such that $l_1|_p$ is not a variable, and let θ be a most general unifier of $l_1|_p$ and l_2. $\langle r_1\theta, (l_1\theta)[r_2\theta]_p \rangle$ is a *critical pair* except that $p = \varepsilon$ and the two rules are identical (up to renaming variables). A TRS is *weakly non-overlapping* if every critical pair consists of the identical terms.

3. Reduction graph

In this section, we introduce the notion of reduction graphs: finite graphs that represent reductions on terms. We will show confluence by a transformation (in Section 4) from a given reduction graph into a connected and confluent reduction graph that contains nodes of the former reduction graph.

Definition 1. Let R be a TRS over $\mathrm{T}(F, X)$. An ARS $G = \langle V, \to \rangle$ is an *R-reduction graph* if V is a finite subset of $\mathrm{T}(F, X)$ and $\to \subseteq \underset{R}{\to}$.

Example 2. Consider a weakly-non-overlapping non-collapsing shallow TRS $R_1 = \{\ f(x, x) \to g(x),\ a \to b,\ b \to a\ \}$. The R_1-reduction graph $G_1 = \langle V_1, \to_1 \rangle$ shown in Figure 1 A. is terminating but is not confluent. The R_1-reduction graph $G_2 = \langle V_2, \to_2 \rangle$ shown in Figure 1 B. is convergent.

We say a mapping $\delta : V \to V$ is a *choice* mapping of $G = \langle V, \to \rangle$ if $v \to^* \delta(v)$ and $v \leftrightarrow^* v' \Rightarrow \delta(v) = \delta(v')$ for all $v, v' \in V$.

Proposition 3. *Let $G = \langle V, \to \rangle$ be an R-reduction graph. Then,*

(1) G is confluent if and only if it has a choice mapping.

(2) G is terminating if and only if it has no cycles.

(3) If G is convergent then it has a unique choice mapping whose range is the set of G-normal forms.

Proof. **(1)** Since "$\Leftarrow$-direction" trivially holds from the definition of choice mappings, we show "$\Rightarrow$-direction". First we show the following claim:

> Let $G = \langle V, \to \rangle$ be a non-empty, connected and confluent reduction graph. Then there exists a node v with $\forall v' \in V. v' \to^* v$.

Let $||v|| = |\{w \mid w \in V,\ w \not\to^* v\}|$, i.e., the number of nodes that cannot reach v. Assume that the claim does not hold. Let v be a minimal node with respect to $||v||$, then $||v|| > 0$ and there exists a node w such that $w \not\to^* v$. There exists a node u such that $w \to^* u \leftarrow^* v$ from confluence. Since every node having a path to v has a path to u, and w has no path to v but a path to u, we obtain $||u|| < ||v||$, which is a contradiction to the minimality of v.

Second we construct a mapping $\delta : V \to V$. By the preceding claim, for every connected component G_i of G there exists a node u_i reachable from all nodes in G_i. Thus it is enough to define δ as $\delta(v) = u_i$ for nodes v of G_i.

(2) The statement follows from the finiteness of V.

(3) Assume that δ_1 and δ_2 are different choice mappings. Then there exists a node u such that $\delta_1(u) \neq \delta_2(u)$. From termination property these terms $\delta_1(u)$ and $\delta_2(u)$ are both normal forms, which contradicts confluence. □

From the previous proposition, if a reduction graph $G = \langle V, \to \rangle$ is convergent, then the choice mapping is equal to the function that returns the G-normal form of a given term. We denote the choice mapping by $\downarrow$; sometimes we also denote $v\downarrow$ instead of $\downarrow(v)$. We use this notation also for substitutions σ: $\sigma\downarrow$ is defined by $x(\sigma\downarrow) = (x\sigma)\downarrow$ for $x \in \mathrm{Dom}(\sigma)$ and $x\sigma \in V$.

Proposition 4. *Let $\langle V, \to_1 \rangle$ be a convergent reduction graph. If $v, v' \in V$ satisfies that v is $\to_1$-normal and $v' \not\to_1^* v$, then $\to_1 \cup \{(v, v')\}$ is convergent.*

Proof. Let $\to_{1'} = \{(v, v')\}$ and $\to_2 = \to_1 \cup \to_{1'}$. First we show the termination. Assume that $\to_1 \cup \to_{1'}$ is not terminating. Since V is finite and $\to_1$ is terminating, any cycle contains the edge (v, v') and hence $v' \to_1^* v$, which is a contradiction to (2).

Second we show the confluence. Let $s \to_2^* t_i$ $(i = 1, 2)$. Each sequence $s \to_2^* t_i$ contains the edge $\to_{1'}$ at most once (from (2)). We can assume that only one sequence contains (v, v') from confluence of $\to_1$; $t_1 \leftarrow_1^* s \to_1^* v \to_2 v' \to_1^* t_2$. Then $t_1 \to_1^* v$ from the confluence of $\to_1$ and (1). Therefore $t_1 \to_2^* t_2$. □

(del):

$$\frac{\to_1;\ \to_2}{\to_1 \setminus \{(l\sigma, r\sigma)\};\ \to_2} \quad \text{if } l \to r \in R,\ (l\sigma, r\sigma) \in \to_1,\ l(\sigma{\downarrow}) \leftrightarrow_2^* r(\sigma{\downarrow})$$

(mov):

$$\frac{\to_1;\ \to_2}{\to_1 \setminus \{(l\sigma, r\sigma)\};\ \to_2 \cup \{(l(\sigma{\downarrow}), r(\sigma{\downarrow}))\}} \quad \text{if } \begin{array}{l} l \to r \in R,\ (l\sigma, r\sigma) \in \to_1, \\ l(\sigma{\downarrow}), r(\sigma{\downarrow}) \in V_2,\ l(\sigma{\downarrow}) \not\leftrightarrow_2^* r(\sigma{\downarrow}) \end{array}$$

Figure 2: Basic-transformation rules

A. $G_{1'} = \langle V_{1'}, \to_{1'} \rangle$ B. $G_{2'} = \langle V_{2'}, \to_{2'} \rangle$

Figure 3: R_1-Reduction graphs in the transformation

4. Confluence of weakly-non-overlapping shallow systems

Theorem 5. *Weakly-non-overlapping, non-collapsing and shallow TRSs are confluent.*

This is the main theorem, which directly follows from the next key lemma proven in Section 5 based on a transformation `Conv`. The transformation gives convergence to a given reduction graph, but neither removes nodes nor divides connected components. (See Example 12)

Lemma 6. *Let R be a weakly-non-overlapping non-collapsing shallow TRS. For any R-reduction graph $G_1 = \langle V_1, \to_1 \rangle$, there exists a convergent R-reduction graph $G_2 = \langle V_2, \to_2 \rangle$ such that $V_2 \supseteq V_1$ and $\leftrightarrow_2^* \supseteq \leftrightarrow_1^*$.*

4.1. Basic transformation

Let $\langle V_1, \to_1 \rangle$ and $\langle V_2, \to_2 \rangle$ be R-reduction graphs, and let $\downarrow$ be a partial function on terms. A *basic transformation* step $[\to_1\ ;\ \to_2] \vdash [\to_{1'}\ ;\ \to_{2'}]$ is an application of a rule shown in Figure 2. We sometimes display the name of a rule at the suffix of $\vdash$.

Example 7. Consider $\to_2$ of G_2 in Figure 1 B. Let $\downarrow$ be the choice mapping of $G_{2'}$ in Figure 3 B. Then

$$\begin{array}{l} [\{(f(a,a), g(a)), (f(b,b), g(b))\}, \to_2 \setminus \{(f(b,b), g(b))\}] \\ \vdash_{(\text{mov})} [\{(f(b,b), g(b))\}, \to_2] \ \vdash_{(\text{del})} [\emptyset, \to_2]. \end{array}$$

Lemma 8. *Let $\langle V_1, \to_1 \rangle$ and $\langle V_2, \to_2 \rangle$ be R-reduction graphs of a TRS R. For a basic transformation $[\to_1\ ;\ \to_2] \vdash [\to_{1'}\ ;\ \to_{2'}]$, the following statements hold.*

*(1) The convergence of $\to_2$ is preserved if the rule (*del*) is applied or $l(\sigma{\downarrow})$ is $\to_2$-normal.*

(2) If $l\sigma\ (\leftrightarrow_{1'} \cup \leftrightarrow_2)^*\ l(\sigma\downarrow)$ *and* $r\sigma\ (\leftrightarrow_{1'} \cup \leftrightarrow_2)^*\ r(\sigma\downarrow)$*, then* $(\leftrightarrow_1 \cup \leftrightarrow_2)^* = (\leftrightarrow_{1'} \cup \leftrightarrow_{2'})^*$*.*

Proof. To prove (1), it is enough to consider an application of the rule (mov). Since $l(\sigma\downarrow)$ is $\rightarrow_2$-normal and $l(\sigma\downarrow) \not\leftrightarrow_2^* r(\sigma\downarrow)$, Proposition 4 implies this claim.

For (2), note that the basic-transformation holds: A. $\rightarrow_1 = \rightarrow_{1'} \cup \{(l\sigma, r\sigma)\}$, B. $\rightarrow_2 \cup \{(l(\sigma\downarrow), r(\sigma\downarrow))\} \supseteq \rightarrow_{2'}$, B'. $\rightarrow_2 \subseteq \rightarrow_{2'}$, and C. $l(\sigma\downarrow) \leftrightarrow_{2'}^* r(\sigma\downarrow)$.

(⊇): We have $\rightarrow_{1'} \cup \rightarrow_{2'} \subseteq \rightarrow_1 \cup \rightarrow_2 \cup \{(l(\sigma\downarrow), r(\sigma\downarrow))\}$ from A. and B. Since $l(\sigma\downarrow)\ (\leftrightarrow_{1'} \cup \leftrightarrow_2)^*\ l\sigma \rightarrow_1 r\sigma\ (\leftrightarrow_{1'} \cup \leftrightarrow_2)^*\ r(\sigma\downarrow)$ from A., we have $l(\sigma\downarrow)\ (\leftrightarrow_1 \cup \leftrightarrow_2)^*\ r(\sigma\downarrow)$ from A. Therefore $(\leftrightarrow_1 \cup \leftrightarrow_2)^* \supseteq (\leftrightarrow_{1'} \cup \leftrightarrow_{2'})^*$.

(⊆): We have $\rightarrow_1 \cup \rightarrow_2 \subseteq \rightarrow_{1'} \cup \{(l\sigma, r\sigma)\} \cup \rightarrow_{2'}$ from A. and B'. Since $l\sigma\ (\leftrightarrow_{1'} \cup \leftrightarrow_2)^*\ l(\sigma\downarrow) \leftrightarrow_{2'}^* r(\sigma\downarrow)\ (\leftrightarrow_{1'} \cup \leftrightarrow_2)^*\ r\sigma$ from C., we have $(l\sigma, r\sigma) \in (\leftrightarrow_{1'} \cup \leftrightarrow_{2'})^*$ from B'. Therefore $(\leftrightarrow_1 \cup \leftrightarrow_2)^* \subseteq (\leftrightarrow_{1'} \cup \leftrightarrow_{2'})^*$. □

4.2. Procedures

For an R-reduction graph $G = \langle V, \rightarrow\rangle$, let $\stackrel{\varepsilon}{\rightarrow} = \rightarrow \cap \stackrel{\varepsilon}{\rightarrow}_R$ and $\stackrel{\varepsilon<}{\rightarrow} = \rightarrow \cap \stackrel{\varepsilon<}{\rightarrow}_R$. Remark that an edge $(s, t) \in \rightarrow$ may belong to both $\stackrel{\varepsilon}{\rightarrow}$ and $\stackrel{\varepsilon<}{\rightarrow}$. For example, consider rules $a \rightarrow b$ and $f(x, x) \rightarrow f(b, a)$, and an edge $(f(a, a), f(b, a))$.

The *monotonic extension* of a reduction graph $G_1 = \langle V_1, \rightarrow_1\rangle$ is a reduction graph $G_2 = \langle V_2, \rightarrow_2\rangle$ where

$$V_2 = \{f(s_1, \ldots, s_n) \mid f \in F,\ s_i \in V_1\},$$
$$\rightarrow_2 = \{(f(\cdots s \cdots), f(\cdots t \cdots)) \mid s, t \in V_1,\ s \rightarrow_1 t\}.$$

Example 9. The monotonic extension of $G_{2'}$ in Figure 3 B. is a subgraph $G_3 = \langle V_2, \rightarrow_2 \setminus \{(f(b, b), g(b))\}\rangle$ of G_2 in Figure 1 (b).

We can easily show the following proposition on a monotonic extension.

Proposition 10. *Let* $G_2 = \langle V_2, \rightarrow_2\rangle$ *be the monotonic extension of a reduction graph* $G_1 = \langle V_1, \rightarrow_1\rangle$*. Then,*

(1) $f(\cdots s \cdots) \in V_2$ *and* $s \rightarrow_1^* t$ *together imply* $f(\cdots t \cdots) \in V_2$*,*
(2) $V_1 \supseteq \mathrm{Args}(V)$ *implies* $V_2 \supseteq V$ *for any* $V \subseteq \mathrm{T}(F, X)$*, and*
(3) both termination and confluence are preserved by this extension.

Procedure `Merge` is shown in Figure 4. If a TRS R is weakly non-overlapping, the output $G_2 = \langle V_2, \rightarrow_2\rangle$ is convergent, $V_2 \supseteq V_1$, and $(\leftrightarrow_1 \cup \leftrightarrow_3)^* = \leftrightarrow_2^*$ (Lemma 14).

Example 11. For a subgraph $G_{1''} = \langle V_1, \stackrel{\varepsilon}{\rightarrow}_1\rangle$ of G_1 in Figure 1 A. and the graph $G_{2'}$ in Figure 3 B., $\mathtt{Merge}_{R_1}(G_{1''}, G_{2'})$ produces G_2 in Figure 1 B. The steps M1 and M2 are demonstrated in Examples 9 and 7, respectively.

Procedure: $\texttt{Merge}_R(G_1, G_{1'})$

Input: A non-collapsing shallow TRS R, an R-reduction graph $G_1 = \langle V_1, \rightarrow_1 \rangle$ and a convergent R-reduction graph $G_{1'} = \langle V_{1'}, \rightarrow_{1'} \rangle$ such that $\rightarrow_1 = \stackrel{\varepsilon}{\rightarrow}_1$ and $V_{1'} \supseteq \mathrm{Args}(V_1)$. Let $\downarrow$ be the choice mapping of $G_{1'}$.

Output: An R-reduction graph G_2.

M1 Compute the monotonic extension $G_3 = \langle V_3, \rightarrow_3 \rangle$ of $G_{1'}$ and set $V_2 := V_3$.

M2 Do basic transformations from $[\rightarrow_1 \ ; \ \rightarrow_3]$ until the first item is empty. Let $[\emptyset \ ; \rightarrow_2]$ be the result.

M3 Output $G_2 = \langle V_2, \rightarrow_2 \rangle$.

Figure 4: Procedure `Merge`

Procedure: $\texttt{Conv}_R(G_1)$

Input: A non-collapsing shallow TRS R and an R-reduction graph $G_1 = \langle V_1, \rightarrow_1 \rangle$.

Output: An R-reduction graph G_2.

C1 If $\stackrel{\varepsilon<}{\rightarrow}_1 = \emptyset$, output the reduction graph $G_2 = \langle V_2, \rightarrow_2 \rangle$ obtained from $\texttt{Merge}_R(G_1, \langle \mathrm{Args}(V_1), \emptyset \rangle)$ and stop.

C2 If $\stackrel{\varepsilon<}{\rightarrow}_1 \neq \emptyset$, construct an R-reduction graph $G_{1'} = \langle V_{1'}, \rightarrow_{1'} \rangle$:

$$V_{1'} = \mathrm{Args}(V_1)$$
$$\rightarrow_{1'} = \{(s_i, t_i) \in V_{1'} \times V_{1'} \mid f(s_1, \ldots, s_n) \stackrel{\varepsilon<}{\rightarrow}_1 f(t_1, \ldots, t_n),\ s_i \neq t_i\}.$$

C3 Invoke $\texttt{Conv}_R(G_{1'})$ recursively. Let $G_{2'}$ be the resulting reduction graph.

C4 Output $G_2 = \langle V_2, \rightarrow_2 \rangle$ obtained from $\texttt{Merge}_R(\langle V_1, \stackrel{\varepsilon}{\rightarrow}_1 \rangle, G_{2'})$ and stop.

Figure 5: Procedure `Conv`

Procedure `Conv` is shown in Figure 5. If a TRS R is weakly non-overlapping, the output $G_2 = \langle V_2, \rightarrow_2 \rangle$ is convergent, $V_2 \supseteq V_1$, and $\leftrightarrow_2^* \supseteq \leftrightarrow_1^*$ (Lemma 6).

Example 12. For G_1 in Figure 1 A., the steps $\texttt{Conv}_{R_1}(G_1)$ are as follows.

1. The step C2 constructs the reduction graph $G_{1'}$ in Figure 3 A..
2. The step C3 produces a convergent R-reduction graph $G_{2'}$ (in Figure 3 B.) from $G_{1'}$ by applying $\texttt{Conv}_{R_1}$ recursively.
3. The step C4 obtains G_2 by $\texttt{Merge}_{R_1}(G_{1''}, G_{2'})$ as shown in Example 11.

5. Proof of Lemma 6

Proposition 13. *Let R be a weakly-non-overlapping shallow TRS, and let $G_3 = \langle V_3, \rightarrow_3 \rangle$ be the monotonic extension of a convergent R-reduction graph $G_{1'} = \langle V_{1'}, \rightarrow_{1'} \rangle$ having the choice mapping $\downarrow$. A node $v \in V_3$ is a G_3-normal form if $v = l(\sigma\downarrow)$ for some $l \rightarrow r \in R$ and a substitution σ such that $l(\sigma\downarrow) \not\rightarrow_3 r(\sigma\downarrow)$.*

Proof. Assume that $l(\sigma{\downarrow})$ is not a G_3-normal form. Since l is shallow and G_3 is a monotonic extension, $t_i \to_{1'} s$ for some ground direct subterm t_i of $l = f(t_1, \ldots, t_n)$ and $s \in V_{1'}$. Since weakly-non-overlapping, we have $l(\sigma{\downarrow}) = f(\cdots t_i \cdots)(\sigma{\downarrow}) \xrightarrow{\varepsilon<}_3 f(\cdots s \cdots)(\sigma{\downarrow}) = r(\sigma{\downarrow})$, contradicting the premise. □

Lemma 14. *Let R be a weakly-non-overlapping non-collapsing shallow TRS. If G_1 and $G_{1'}$ satisfy the input conditions of* `Merge`*, the reduction graph $G_2 = \langle V_2, \to_2\rangle$ obtained by* $\texttt{Merge}_R(G_1, G_{1'})$ *is convergent and satisfies $V_2 \supseteq V_1$ and $(\leftrightarrow_1 \cup \leftrightarrow_3)^* = \leftrightarrow_2^*$, where $G_3 = \langle V_3, \to_3\rangle$ is the monotonic extension of $G_{1'}$.*

Proof. First we have $V_2 \supseteq V_1$, since $V_2 = V_3$ and $V_3 \supseteq V_1$ by Proposition 10 (2).

Second we show that the transformation in Step M2 of `Merge` continues until the first item empty. Since G_1 is an R-reduction graph with $\to_1 = \xrightarrow{\varepsilon}_1$, every pair in $\to_1$ is represented as $(l\sigma, r\sigma)$ for some $l \to r \in R$ and a substitution σ. Thus, it is enough to see that $l(\sigma{\downarrow})$ and $r(\sigma{\downarrow})$ are in V_3 ($= V_2 \supseteq V_1$). This follows from shallowness of l and r, $x\sigma \to_{1'}^* x(\sigma{\downarrow})$, and Proposition 10 (1).

Now we can represent the sequence as $[\to_1 \,;\, \to_3] = [\to_{1_0} \,;\, \to_{2_0}] \vdash [\to_{1_1} \,;\, \to_{2_1}] \vdash \cdots \vdash [\to_{1_k} \,;\, \to_{2_k}] = [\emptyset \,;\, \to_2]$. Note that $V_{1'} \supseteq \mathrm{Args}(V_1)$ and $\to_3 \subseteq \to_{2_i}$.

Third we show the convergence of G_2 and $(\leftrightarrow_1 \cup \leftrightarrow_3)^* = \leftrightarrow_2^*$. By induction on i, we will prove the following claims for each $0 \le i \le k$:

(1) $\to_{2_i}$ is convergent,

(2) $(\leftrightarrow_1 \cup \leftrightarrow_3)^* = (\leftrightarrow_{1_i} \cup \leftrightarrow_{2_i})^*$, and

(3) $\to_{2_i} \setminus \xrightarrow{\varepsilon}_{2_i} \subseteq \to_3 \subseteq \to_{2_i}$.

(Case $i = 0$): $G_3 = \langle V_3, \to_3\rangle$ is convergent by Proposition 10 (3). Thus, the claims (1), (2), and (3) follow from $\to_3 = \to_{2_0}$ and $\to_1 = \to_{1_0}$.
(Case $i > 0$): Let $[\to_{1_{i-1}} \,;\, \to_{2_{i-1}}] \vdash [\to_{1_i} \,;\, \to_{2_i}]$. Then $\to_{2_{i-1}}$ is convergent by induction hypothesis. To prove the claim (1), from Lemma 8 (1) it is enough to consider when (mov) is applied, and show that $l(\sigma{\downarrow})$ is $\to_{2_{i-1}}$-normal. From the side condition of (mov), we have $l(\sigma{\downarrow}) \not\to_{2_{i-1}} r(\sigma{\downarrow})$ and hence

- $l(\sigma{\downarrow})$ has no out-edges in $\xrightarrow{\varepsilon}_{2_{i-1}}$, since R is weakly non-overlapping,
- Since $\to_3 \subseteq \to_{2_{i-1}}$, we have $l(\sigma{\downarrow}) \not\to_3 r(\sigma{\downarrow})$. From Proposition 13, $l(\sigma{\downarrow})$ is G_3-normal. By the induction hypothesis $\to_{2_{i-1}} \setminus \xrightarrow{\varepsilon}_{2_{i-1}} \subseteq \to_3$, $l(\sigma{\downarrow})$ has no out-edges in $\to_{2_{i-1}} \setminus \xrightarrow{\varepsilon}_{2_{i-1}}$.

The claim (2) follows from Lemma 8 (2), if $l\sigma \leftrightarrow_{2_{i-1}}^* l(\sigma{\downarrow})$ and $r\sigma \leftrightarrow_{2_{i-1}}^* r(\sigma{\downarrow})$. Since $x\sigma \to_{1'}^* x(\sigma{\downarrow})$, $\to_3$ is the monotonic extension of $\to_{1'}$, and l and r are shallow, we have $l\sigma \to_3^* l(\sigma{\downarrow})$ and $r\sigma \to_3^* r(\sigma{\downarrow})$. Then, $l\sigma \to_{2_{i-1}}^* l(\sigma{\downarrow})$ and $r\sigma \to_{2_{i-1}}^* r(\sigma{\downarrow})$ follow from the induction hypothesis $\to_3 \subseteq \to_{2_{i-1}}$.

The claim (3) holds if $\rightarrow_{2_i} \setminus \xrightarrow{\varepsilon}_{2_i} \subseteq \rightarrow_{2_{i-1}} \setminus \xrightarrow{\varepsilon}_{2_{i-1}}$ and $\rightarrow_{2_{i-1}} \subseteq \rightarrow_{2_i}$. The former holds, since only top reductions can be added. The latter also holds, since no edges are removed from $\rightarrow_{2_{i-1}}$. □

Proof. (of Lemma 6) It is enough to show that the reduction graph G_2 obtained by invoking $\texttt{Conv}_{R_1}(G_1)$ satisfies $V_2 \supseteq V_1$ and $\leftrightarrow_2^* \supseteq \leftrightarrow_1^*$. This is proved by induction on the total size of terms in V_1.

Case 1. Assume that edges of G_1 are all due to top reductions of R. Then, C1 of `Conv` occurs and we obtain $G_2 = \langle V_2, \rightarrow_2 \rangle$ by invoking $\texttt{Merge}_R(G_1, \langle \mathrm{Args}(V_1), \emptyset \rangle)$. From Lemma 14, G_2 is convergent and $V_2 \supseteq V_1$. Since the monotonic extension of $\langle \mathrm{Args}(V_1), \emptyset \rangle$ has no edges, we have $\leftrightarrow_2^* = \leftrightarrow_1^*$ from Lemma 14.

Case 2. Assume that some edges are due to inner reductions of R. Then, C2-C4 of `Conv` occur. By induction hypothesis $G_{2'} = \langle V_{2'}, \rightarrow_{2'} \rangle$ is convergent and satisfies the conditions that A. $V_{2'} \supseteq V_{1'}$ and B. $\leftrightarrow_{2'}^* \supseteq \leftrightarrow_{1'}^*$. Note that $V_{2'} \supseteq V_{1'} = \mathrm{Args}(V_1)$ from A. From Lemma 14, G_2 is convergent, $V_2 \supseteq V_1$, and $(\overset{\varepsilon}{\leftrightarrow}_1 \cup \leftrightarrow_3)^* = \leftrightarrow_2^*$, where $G_3 = \langle V_3, \rightarrow_3 \rangle$ is the monotonic extension of $G_{2'}$.

Now we show that $\leftrightarrow_3^* \supseteq \overset{\varepsilon<}{\leftrightarrow}_1$. Let $s = f(\cdots, s', \cdots) \overset{\varepsilon<}{\rightarrow}_1 f(\cdots, t', \cdots) = t$. From $s' \rightarrow_{1'} t'$ and B., we have $s' \leftrightarrow_{2'}^* t'$. Thus, we obtain $s \leftrightarrow_3^* t$.

Therefore $\leftrightarrow_1^* = (\overset{\varepsilon}{\leftrightarrow}_1 \cup \overset{\varepsilon<}{\leftrightarrow}_1)^* \subseteq (\overset{\varepsilon}{\leftrightarrow}_1 \cup \leftrightarrow_3^*)^* = (\overset{\varepsilon}{\leftrightarrow}_1 \cup \leftrightarrow_3)^* = \leftrightarrow_2^*$. □

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