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Towards Combined Systems of Modal Logics
— a syntactic and semantic study

by

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Abstract

This thesis discusses mainly multimodal logics constructed by combining two modal logics. Many monomodal logics have been investigated well. Recently, combined logics have been developed and applied in various fields of computer science. Here we focus our attention on multimodal logics obtained by *fusion* of modal logics. Main results on the present thesis are as follows.

First, through Kripke type semantics, we give a complete answer to inclusion relationship between pseudo-Euclidean logics $\mathbf{K} \oplus \{\Diamond^k\varphi \to \Box^m\Diamond^n\varphi\}$ where m and n are fixed non-negative integers, and $k \leq 0$.

Next, we discuss fusions of well-known modal logics. We take up especially presevations of proof theoretic properties for sequent system of fusions without interdependent axioms, and those with some interdependent axioms. Attempt to derive general results on fusions with interdependent axioms, in particular the finite model property of fusions with more generalized interdependent axioms will be made.

Finally, we consider fusions of epistemic logics and temporal logics, which is called *temporal epistemic logics*. Then the subformula property of sequent systems for their basic temporal epistemic logics is shown by the cut-restriction theorem and restricting inference rules on temporal notions. Also, Craig's interpolation property and decidability as the proof-search procedure are obtained as the consequence.

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Contents

Chapter 1

Introduction

1.1 Backgrounds

We have two main goals in this thesis. The first is to show Kripke completeness and finite model property for some bimodal logics with interdependent axioms. The second is a proof-theoretic study of temporal epistemic logics; we show the subformula property for the sequent calculi for their logics, Craig's interpolation theorem for them, and give an effective decision procedure. For the former, the finite model property will be shown for bimodal logics obtained by adding some interdependent axioms to fusion of two monomodal logics, using filtration method. As for the later, sequent calculi of some temporal epistemic logics are introduced, and the subformula property and the interpolation property for them will be shown.

Many monomodal logics have been investigated extensively since the early time of 20th century, and a lot of interesting properties of them have been shown. On the other hand, we have certain technical difficulties when we try to develop study of multimodal logics to the extent of monomodal logics. Moreover, it will be quite necessary to develop such a study since we need to introduce many modalities in many cases of applications. Multimodal logics combined more then two propositional modal logics by means of some constructions have been studied semantically since 1990's. Various semantical logical properties for them were made clear. However, many properties for multimodal logics with some interdependent axioms are left unanswered. So far, bimodal logics have been discussed as a fundamental research to general multimodal logics. The following ways of construction of multimodal logics have been proposed.

(1) *fusion* $\cdots$ the least modal logics containing two different modal logics

(2) *product* $\cdots$ the modal logics validated in products of modal frames.

In the construction (1), the correlations among each modalities are out of consideration. In this construction, in order to obtained some dependently axiomatized multimodal logics, it is necessary to add some interdependent axioms to fusions. As for dependently axiomatized logics like this, the introduction of appropriate axioms, the logical properties for logics with them as axioms and so on are unsolved problems.

Study of multimodal logics will be put in practice not only theoretically but also in paying attention to application of applied fields. Especially, it has been hoped to apply multimodal logics into various fields of information science. As application of multimodal logics, they will be useful in artificial intelligence; for instance, the formalization of multi-agent systems, information flow, the representations of natural languages, and so on. The several formalizations of multi-agent systems have been proposed since 1980. Multimodal logics combined epistemic logics and temporal logics are useful for their formalizations. By the way, both epistemic logics and temporal logics are regarded as multimodal logics. They are fusions, or fusions with some interdependent axioms. As to temporal logics, axiom schemes $\varphi \to \langle F \rangle [P]\varphi$ and $\varphi \to \langle P \rangle [F]\varphi$ representing conversions of future and past are interdependent axioms, and connected two modalities.

1.2 Outline of this thesis

This thesis discusses combined modal logics defined by either Kripke semantics or Gentzen's sequent systems. Main contents of the present thesis consist of a study of general preservation results in combined logics, and a study of temporal epistemic logics as important examples of combined modal logics. We introduce sequent systems for these logics and develop both syntactic and semantical study of them. A key result is the subformula property of these sequent systems, which is obtained from the cut restriction property. The subformula property implies not only Craig's interpolation theorem for these logics but also the existence of an efficient proof-search procedure for these logics.

Chapter 2 reviews propositional monomodal logics. As tools used later, Kripke type semantics and Gentzen style sequent systems for modal logics are introduced and discussed in Section 2.2 and in Section 2.3, respectively. A brief survey of the finite model property by filtration method is given in Section 2.2. Cut elimination property and cut restriction property are discussed in Section 2.3.

Chapter 3 is devoted to a study of inclusion relation among logics, called pseudo-Euclidean logics. Here, one will see a good example which shows the usefulness of semantical methods. Pseudo-Euclidean logics E_k obtained from $\mathbf{K}$ by adding the axiom $\Diamond^k \varphi \to \Box^m \Diamond^n \varphi$ for $k \geq 0$ and fixed non-negative integers m and n, are introduced. In Kripke semantics, a binary relation on a set W of possible worlds is *k-pseudo-Euclidean* if for any $u, v, w \in W$, $uR^k v$ and $uR^m w$ imply $wR^n v$. If $m = n = 1$, then 1-pseudo-Euclidean relations coincide with usual Euclidean relations. Then, inclusion relationship among them is threw light on. When the class of all Kripke frames (W, R) where R is a

k-pseudo-Euclidean relation on W is denoted by $\mathcal{PE}_k$,

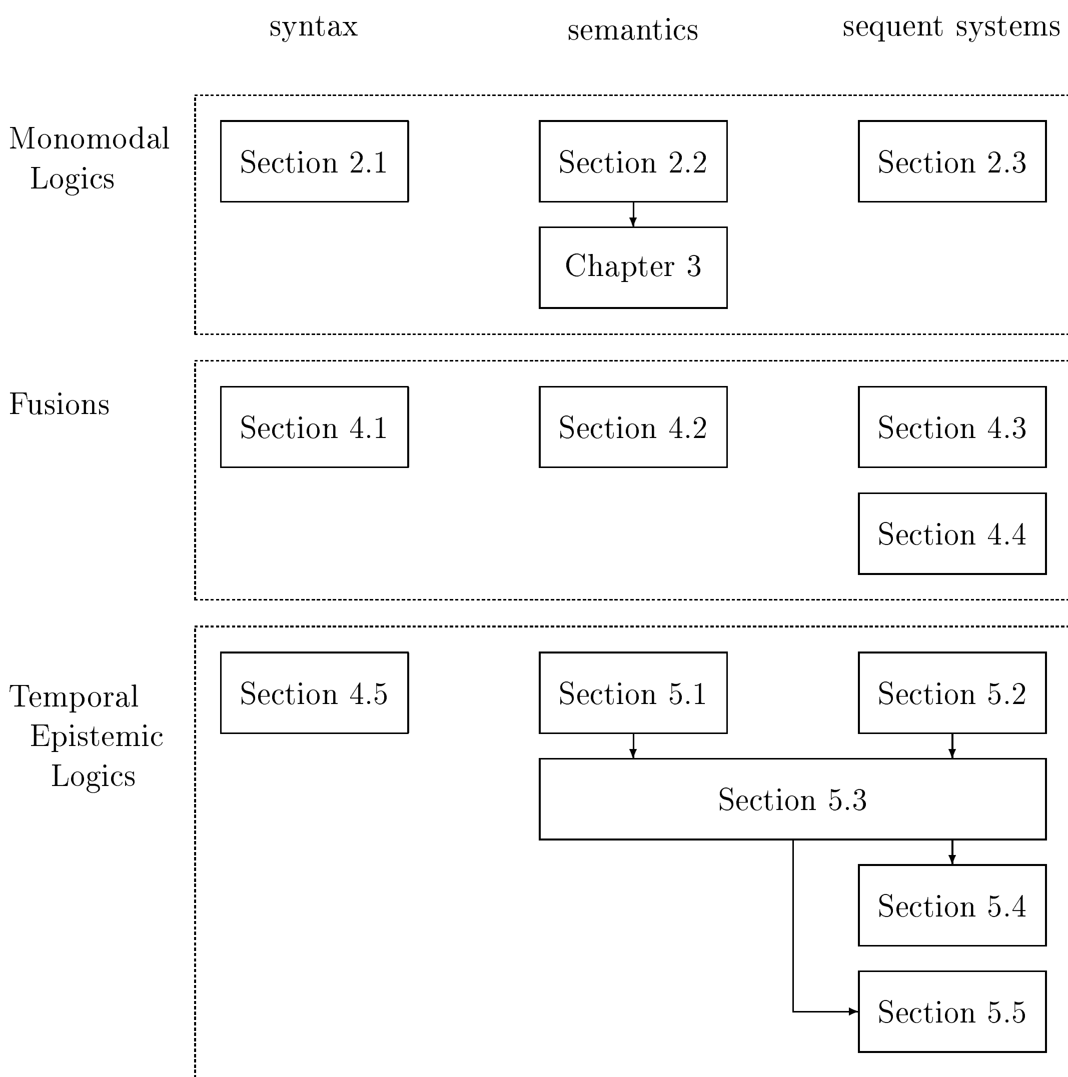
$$E_k \supseteq E_{k'} \iff \mathcal{PE}_k \subseteq \mathcal{PE}_{k'}.$$

Then, a complete figure of inclusion relation among pseudo-Euclidean logics is given.

Chapter 4 discusses how to construct combined logics and what kind of logical properties are preserved under these constructions. Section 4.1 presents two standard ways of combining modal logics, called *fusions* and *products* respectively. Then, we concentrate mainly on fusions, and give a survey of preservation results on Kripke completeness, the finite model property and so on. In Section 4.2, the finite model property for fusions with "weak" interdependent axioms is shown by means of filtrations. Proof-theoretical properties of Gentzen style sequent systems for fusions of modal logics including axioms T, D, 4, 5 or B is discussed in Section 4.3. We introduce a sequent system $\mathcal{S}(L)$ for a logic L under consideration. Then we show that for monomodal logics L_1 and L_2 under consideration, if both $\mathcal{S}(L_1)$ and $\mathcal{S}(L_2)$ have the cut elimination property then $\mathcal{S}(L_1 \otimes L_2)$ has also the cut elimination property. On the other hand, since sequent systems for **KB**, **KTB**, **KDB**, **KB4** and **S5** given there lacks the cut elimination property but has the cut restriction property, we can show that any fusion where one of components is among them has also the cut restriction property and therefore the subformula property. When one of components of a fusion is **K5** or **KD5** we can show an extended subformula property, by using the similar technique. In Section 4.4, we consider two interdependent axioms $\Box\varphi \to \blacksquare\varphi$ and $\Box\varphi \to \blacksquare\Box\varphi$ where $\Box$ and $\blacksquare$ are deferent modal operators, and introduced some sequent systems obtained from those for fusions of modal logics including axioms T or 4 by adding such axioms. For some of their logics, we show the cut elimination theorem for them.

Main results of the present thesis are given in Chapter 5. We introduce two basic temporal epistemic logics and discuss their logical properties. They are obtained from the temporal logic K_t by combining either the logic of knowledge or the logic of belief. They will offer logical bases of formalizing multi-agent systems. In Section 5.1, we introduce a Kripke type semantics for these temporal epistemic logics. In Section 5.2, we introduce sequent systems for them. Since the logic of knowledge and the logic of belief correspond to **S5** and **KD45**, respectively, we cannot expect that cut elimination theorem holds for these sequent systems. To overcome this difficulty, we try to show the cut restriction property instead. But, two rules of inferences of K_t violate the cut restriction property. The idea which we introduce here is to restrict applications of these rules so that the cut restriction property holds also for them. Now it remains to show that each of these restricted systems determines the same logic as one which the original one determines. This can be done in Section 5.3 by showing the completeness of these restricted systems.

As mentioned before, the cut restriction property implies the subformula property. Thus, Craig's interpolation theorem is shown in Section 5.4 by using Maehara's Method. In Section 5.5, a proof-search procedure for these logics, based on the subformula property, is described in details.



Chapter 2

Preliminaries

In this chapter, we introduce notations and basic notions of monomodal logics, and give a brief survey. This chapter take up (i) syntax of monomodal logics, (ii) Kripke type semantics and (iii) proof theory of Gentzen style sequent systems.

2.1 Fundamentals of modal logics

The language $\mathcal{L}$ of propositional monomodal logic consists of propositional variables, denoted by p, q, r, $\cdots$ and logical symbols $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\Box$. When we want to express a particular modality $\Box$ explicitly, the language is denoted by $\mathcal{L}_\Box$. Formulas, denoted by φ, ψ, χ, $\cdots$, are constructed in the usual way from propositional variables and logical symbols. In particular, $\Box\varphi$ is a formula when φ is a formula. The set of all subformulas of φ is denoted by $Sub(\varphi)$. Sometimes, we use propositional variables and formulas with subscripts. A set L of formulas in $\mathcal{L}$ is a *modal logic*, if the following conditions are satisfied:

- all tautologies belong to L,
- if $\varphi, \varphi \rightarrow \psi \in L$, then $\psi \in L$,
- if $\varphi \in L$, then $\Box\varphi \in L$,

Let L be a modal logic of $\mathcal{L}_\Box$, and Q be a set of formulas in the language $\mathcal{L}_\Box$. Then the least modal logic containing the set $L \cup Q$ is denoted by $L \oplus Q$. The symbol $\mathbf{K}$ denotes the least modal logic containing the axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$. Any modal logic with the axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ is called a *normal* modal logic. Historical names for some well-known axiom schemes are

$$D:\ \Box\varphi \rightarrow \Diamond\varphi, \qquad T:\ \Box\varphi \rightarrow \varphi, \qquad 4:\ \Box\varphi \rightarrow \Box\Box\varphi,$$

$$B:\ \varphi \rightarrow \Box\Diamond\varphi, \qquad 5:\ \Diamond\varphi \rightarrow \Box\Diamond\varphi,$$

where $\Diamond$ is the abbreviation of $\neg\Box\neg$ in propositional modal logic. Then the following modal logics can be constructed from their axiom schemes :

$\mathbf{K4} = \mathbf{K} \oplus \{4\}$	$\mathbf{KD4} = \mathbf{K} \oplus \{D, 4\}$	$\mathbf{KB} = \mathbf{K} \oplus \{B\}$
$\mathbf{K5} = \mathbf{K} \oplus \{5\}$	$\mathbf{KD5} = \mathbf{K} \oplus \{D, 5\}$	$\mathbf{KTB} = \mathbf{K} \oplus \{T, B\}$
$\mathbf{K45} = \mathbf{K} \oplus \{4, 5\}$	$\mathbf{KD45} = \mathbf{K} \oplus \{D, 4, 5\}$	$\mathbf{KDB} = \mathbf{K} \oplus \{D, B\}$
$\mathbf{KD} = \mathbf{K} \oplus \{D\}$	$\mathbf{S4} = \mathbf{K} \oplus \{T, 4\}$	$\mathbf{KB4} = \mathbf{K} \oplus \{B, 4\}$
$\mathbf{KT} = \mathbf{K} \oplus \{T\}$	$\mathbf{S5} = \mathbf{K} \oplus \{T, 5\}$	

Proposition 2.1 *The following equivalences hold.*

1. *If either $\{T, 5\}$, $\{T, 4, B\}$, $\{D, 4, B\}$ or $\{D, 5, B\}$ is the subset of Q_1, then $\mathbf{K} \oplus Q_1 = \mathbf{S5}$,*
2. $\mathbf{KB4} = \mathbf{K} \oplus \{5, B\} = \mathbf{K} \oplus \{4, 5, B\}$,
3. $\mathbf{KT} = \mathbf{K} \oplus \{T, D\}$,
4. $\mathbf{KT4} = \mathbf{K} \oplus \{T, D, 4\}$,
5. $\mathbf{KTB} = \mathbf{K} \oplus \{T, D, B\}$.

By our definition, modal logics are sets of formulas, and therefore set inclusion defines a partial order in the set of all modal logics. For logics introduced above, the following inclusion relationship holds.

Other typical axiom schemes are as follows:

$$(\mathrm{ax1}): \quad \Diamond\varphi \to \Box\varphi \qquad (\mathrm{ax2}): \quad \Diamond\varphi \leftrightarrow \Box\varphi \qquad (\mathrm{ax3}): \quad \Box\Box\varphi \to \Box\varphi$$

$$(\mathrm{ax4}): \quad \Diamond^k\Box^l\varphi \to \Box^m\Diamond^n\varphi \qquad (\mathrm{ax5}): \quad \Box(\varphi \wedge \Box\varphi \to \psi) \vee \Box(\psi \wedge \Box\psi \to \varphi)$$

where formulas $\Box^n\varphi$ and $\Diamond^{n'}\varphi$ denote formulas $\Box\cdots\Box\varphi$ with n boxes and $\Diamond\cdots\Diamond\varphi$ with n' diamonds. Note that all of axioms D, T, 4, 5 and B introduced before are of the form (ax4). Moreover, axioms $\Diamond^k\varphi \to \Box^m\Diamond^n\varphi$ for pseudo-Euclidean logics in Chapter 3 are also of the form (ax4).

2.2 Kripke type semantics and completeness

Completeness and finite model property

In this Section, we introduce Kripke type semantics, and present semantical properties for modal logics. Let W be a nonempty set, and $R_\Box$ be a binary relation on W; i.e.

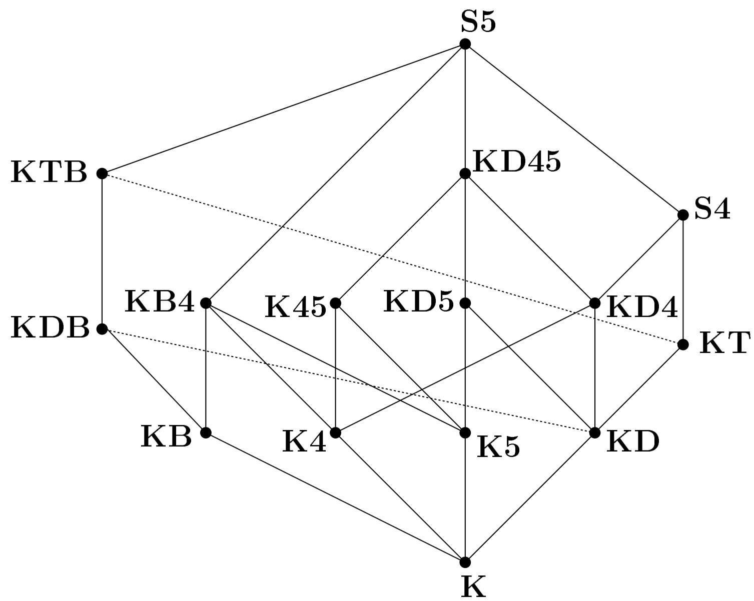


Figure 2.1: inclusion relation of modal logics

$R \subseteq W \times W$. Then a *Kripke frame* is a pair (W, R). Here W is called the set of *possible worlds*, and R is called *accessibility relations*. When we express the modality $\Box$ explicitly, the relation is denoted by $R_\Box$. Let $\mathcal{F} = (M, R)$ be a frame, and V be a mapping such that $V(p) \subseteq W$ for each propositional variable p. Then V is called a *valuation* on $\mathcal{F}$. The triple (W, R, V) is called a *Kripke model*. Sometimes, (W, R, V) is denoted by $(\mathcal{F}, V)$. For a given Kripke model (W, R, V), a binary relation $\models$ between $u \in W$ and formulas is defined inductively on the length of formulas as follows:

$$
\begin{array}{lll}
u \models p & \iff & u \in V(p) \\
u \models \varphi \wedge \psi & \iff & u \models \varphi \text{ and } u \models \psi \\
u \models \varphi \vee \psi & \iff & u \models \varphi \text{ or } u \models \psi \\
u \models \varphi \to \psi & \iff & u \models \varphi \text{ implies } u \models \psi \\
u \models \neg\varphi & \iff & \text{not } u \models \varphi \\
u \models \Box\varphi & \iff & \text{for any } v \in W,\ uR_\Box v \text{ implies } v \models \varphi
\end{array}
$$

The relation $\models$ is defined uniquely by the valuation V. So $\models$ and $(W, R, \models)$ are also called a valuation and a Kripke model, respectively, when no confusions will occur. A formula φ is *true in a model* $\mathcal{M} = (W, R, \models)$, written as $\mathcal{M} \models \varphi$, if $u \models \varphi$ for any $u \in W$. A formula φ is *valid in frame* $\mathcal{F} = (W, R)$, written as $\mathcal{F} \models \varphi$, if $\mathcal{M} \models \varphi$ for any model $\mathcal{M} = (W, R, \models)$.

Suppose that $\mathcal{F} = (W, R)$ is a frame. Then, the binary relation R satisfies one of the following first-order conditions if and only if the axiom schemes corresponding the condition is valid in $\mathcal{F}$. This property is called *a corresponding theory.*

D :	$\forall u \exists v(uRv)$	(serial)
T :	$\forall u(uRu)$	(reflexive)
4 :	$\forall u \forall v \forall w(uRv \wedge vRw \rightarrow uRw)$	(transitive)
5 :	$\forall u \forall v \forall w(uRv \wedge uRw \rightarrow vRw)$	(Euclidean)
B :	$\forall u \forall v(uRv \rightarrow vRu)$	(symmetric)
(ax1) :	$\forall u \forall v(uRv \wedge uRw \rightarrow v = w)$	(partially functional)
(ax2) :	$\forall u \forall v(u = v)$	(functional)
(ax3) :	$\forall u \forall v(uRv \rightarrow \exists w(uRw \wedge wRv))$	(weakly dense)
(ax4) :	$\forall u \forall v \forall w(uR^k v \wedge uR^m w \rightarrow \exists t(vR^l t \wedge wR^n t))$	(Church-Rosser)
(ax5) :	$\forall u \forall v \forall w(uRv \wedge uRw \rightarrow vRw \vee v = w \vee wRv)$	(weakly connected)

For a logic L, every frame with the conditions corresponding to the axioms is called *L-frame.* For example, every **S4**-frame is reflexive and transitive. Then the following Proposition 2.2 can be shown.

Proposition 2.2

1. *The lease normal modal logic* **K** *is determined by the class of all Kripke frames; i.e. for any frame $\mathcal{F}$, $\mathcal{F} \models \varphi$ iff $\varphi \in$* **K**.

2. *Let L be any of logics introduced in previous section. Then the logic L is determined by the class of all L-frames; i.e. for any L-frame $\mathcal{F}$, $\mathcal{F} \models \varphi$ iff $\varphi \in L$.*

Proposition 2.2 is often shown by constructing their canonical models (See [11]). If $\varphi \in L$ then there exists a model $\mathcal{M}$, for example the canonical model of L, such that $\mathcal{M} \not\models \varphi$ by the above completeness theorem. But it would be quite useful if we could get a finite model in which a given unprovable formula is false. Because a consequence of the finite model property is the decidability. A concrete finite procedure which decides to be provable or not for any formula in a system is called a *decision procedure.* If there exists a decision procedure, the system is said to be *decidable.* By the Harrop's theorem, that is *if a finitely axiomatizable logic has the finite model property, then it is decidable* (cf. [11]). A logic L has the finite model property if the following condition is satisfied,

$$\text{if } \varphi \notin L, \text{ then there is a finite } L\text{-model } \mathcal{W} \text{ such that } \mathcal{M} \not\models \varphi.$$

Let $(W, R, \models)$ be a model, and $\Psi(\varphi)$ a finite set which contains $Sub(\varphi)$. Now we introduce the filtration method. We define a binary relation $\sim$ on W as follows:

$$u \sim v \iff \text{for any } \chi \in \Psi(\varphi),\ u \models \chi \text{ iff } v \models \chi.$$

Clearly $\sim$ is an equivalence relation. Let $[u]$ denote the equivalence class of u, i.e. $[u] = \{x \in W | u \sim x\}$. Then *filtration of* $\mathcal{M} = (W, R, \models)$ *through* $\Psi(\varphi)$ is any model $\mathcal{N} = (W/\sim, S, \models^*)$ such that

1. if uRv, then $[u]S_\alpha[v]$,
2. if $[u]S[v]$, then for any $\Box\psi \in \Psi(\varphi)$, $u \models \Box\psi$ implies $v \models \psi$,
3. $[u] \models^* p \iff u \models p$.

Proposition 2.3 *Let $\mathcal{N} = (W/\sim, S, \models^*)$ be a filtration of a model $\mathcal{M} = (W, R, \models)$ through $\Psi(\varphi)$. Then for every point $u \in W$ and every $\psi \in \Psi(\varphi)$, $u \models \psi$ iff $[u] \models^* \psi$.*

In general, the conditions of the filtration do not determine S uniquely. The following are two extreme ways of choosing S satisfying the above 1 and 2.

- Coarsest filtration:

$$[u]S[v] \iff \text{for any } \Box\psi \in \Psi(\varphi),\ u \models \Box\psi \text{ implies } v \models \psi.$$

- Finest filtration:

$$[u]S[v] \iff \text{there exist } u', v' \text{ such that } u \sim u',\ v \sim v' \text{ and } u'Rv'.$$

Using filtration method, we can show the following.

Proposition 2.4 *Let Q be a subset of $\{T, D, 4, 5, B\}$. Then the logic $\mathbf{K} \oplus Q$ has the finite model property.*

By the Harrop's theorem, the next corollary can be derived since the logic $L \oplus Q$ is finitely axiomatizable.

Corollary 2.5 *Let Q be a subset of $\{T, D, 4, 5, B\}$. Then the logic $\mathbf{K} \oplus Q$ is decidable.*

Algebraic semantics and general frame

An algebra $\mathcal{A} = \langle A, \wedge, \vee, ', \top, \bot, \Box \rangle$ is called a *modal algebra* if it satisfies the following conditions:

(i) $\langle A, \wedge, \vee, ', \top, \bot \rangle$ is a Boolean algebra,

(ii) for every $a, b \in A$, $\Box(a \wedge b) = \Box a \wedge \Box b$,

(iii) $\Box\top = \top$

For any modal algebra $\mathcal{A} = \langle A, \wedge, \vee, {}', \bot, \Box \rangle$, then operations $\to$ and $\Diamond$ on A are defined as follows.

$$a \to b = a' \vee b, \qquad \Diamond a = (\Box a')'$$

For any modal algebra $\mathcal{A} = \langle A, \wedge, \vee, {}', \top, \bot, \Box \rangle$, a mapping v from the set of all propositional variables to A is called *a valuation* on $\mathcal{A}$. A given valuation v on $\mathcal{A}$ is extended to a mapping from all formulas to A by defining inductively as follows;

$$\begin{aligned} v(\varphi \wedge \psi) &= v(\varphi) \wedge v(\psi), & v(\neg\varphi) &= v(\varphi)', \\ v(\varphi \vee \psi) &= v(\varphi) \vee v(\psi), & v(\Box\varphi) &= \Box v(\varphi), \\ v(\varphi \to \psi) &= v(\varphi) \to v(\psi). \end{aligned}$$

A formula φ is *valid in modal algebra* $\mathcal{A}$, if $v(\varphi) = \top$ for any valuation v on $\mathcal{A}$. The following is the list of conditions corresponding to axioms discussed before. For any modal algebra, the identity $\varphi = \psi$ is true in $\mathcal{A}$ for every modal formula φ and ψ such that $(\varphi \to \psi) \wedge (\psi \to \varphi) \in \mathbf{K}$. If we replace in this condition $\mathbf{K}$ with a normal modal logic L then $\mathcal{A}$ is called L-algebra. In particular, algebras for logics containing axioms D, T, 4, 5 and B satisfy the following conditions, respectively.

$$D\colon \Box a \leq \Diamond a, \quad T\colon \Box a \leq a, \quad 4\colon \Box a \leq \Box\Box a, \quad B\colon a \leq \Box\Diamond a, \quad 5\colon \Diamond a \leq \Box\Diamond a.$$

By using Lindenbaum algebra, it can be shown that any normal modal logic L is complete with respect to L-algebra. Then Proposition 2.6 holds.

Proposition 2.6 *For each normal modal logic L and each formula φ, $\varphi \in L$ iff φ is valid in every modal algebra.*

A *modal general frame* is a triple $\mathcal{F} = (W, R, P)$ in which (W, R) is an ordinary Kripke frame and P is a subset of 2^W containing $\emptyset$ and closed under $\cap$, $\cup$, ^{c} and $\Box$ which is defined as follows: for every $X, Y \subseteq W$,

$$\Box X = \{x \in W \mid \forall y \in W(xRy \to y \in X)\}.$$

The subset P of 2^W is called *a set of possible values* in $\mathcal{F}$. Let $\mathcal{F} = (W, R, P)$ be a modal general frame. A *model* on $\mathcal{F}$ is a pair $(\mathcal{F}, V)$ in which *valuation* V is a map from all propositional variables into P; i.e. $V(p) \in P$ for every variable p. The relation $\models$ is defined in exactly the same way of ordinary Kripke models. It is easy to see that a modal general frame (W, R, P) is essentially equal to a Kripke frame (W, R) when $P = 2^W$.

For a given modal general frame $\mathcal{F} = (W, R, P)$, we denote by $\mathcal{F}^+$ the algebra $\langle P, \cap, \cup, {}^c, W, \emptyset, \Box \rangle$ and call it the *dual* of $\mathcal{F}$. We can easily show that the dual of every modal general frame is a modal algebra.

Proposition 2.7 *Let $\mathcal{F}$ be a modal general frame.*

1. *The dual $\mathcal{F}^+$ is a modal algebra.*
2. *φ is valid in $\mathcal{F}$ iff φ is valid in $\mathcal{F}^+$.*

Conversely, suppose that a modal algebra $\mathcal{A} = \langle A, \wedge, \vee, ', \top, \bot, \Box \rangle$ is given. Then the *dual* of $\mathcal{A}$, denoted by $\mathcal{A}_+$, is defined by the triple $(W_\mathcal{A}, R_\mathcal{A}, P_\mathcal{A})$ where (i) $W_\mathcal{A}$ is the set of all prime filters in $\mathcal{A}$, (ii) $R_\mathcal{A}$ is a binary relation on $W_\mathcal{A}$ satisfying that $\nabla_1 R_\mathcal{A} \nabla_2$ iff $\forall a \in A(\Box a \in \nabla_1 \Rightarrow a \in \nabla_2)$, and (iii) $P_\mathcal{A} = \{f_\mathcal{A}(a) \mid a \in A\}$ where $f_\mathcal{A}(a) = \{\nabla \in W_\mathcal{A} \mid a \in \nabla\}$.

Proposition 2.8 *Let $\mathcal{A}$ be a modal algebra.*

1. *The dual $\mathcal{A}_+$ is a modal general frame.*
2. *φ is valid in $\mathcal{A}$ iff φ is valid in $\mathcal{A}_+$.*

Indeed, every modal algebra $\mathcal{A}$ is isomorphic to its bidual $(\mathcal{A}_+)^+$, in symbol $\mathcal{A} \cong (\mathcal{A}_+)^+$. On the other hand, there are modal general frames $\mathcal{F}$ which are not isomorphic to its bidual $(\mathcal{F}^+)_+$. So, we consider the condition that $\mathcal{F} \cong (\mathcal{F}^+)_+$ holds. Any general frame satisfying the following (GF1), (GF2) and (GF3) is said to be *differentiated*, *tight* and *compact*, respectively.

$$
\begin{array}{ll}
\text{(GF1)} & \forall u, v \in W(u = v \text{ iff } \forall X \in P(u \in X \text{ iff } v \in X)) \\
\text{(GF2)} & \forall u, v \in W(uRv \text{ iff } \forall X \in P(u \in \Box X \text{ implies } v \in X)) \\
\text{(GF3)} & \forall \mathcal{X} \subseteq P, \forall \mathcal{Y} \subseteq \bar{P} \\
& (\ \bigcap(\mathcal{X}' \cup \mathcal{Y}') \neq \emptyset, \mathcal{X}' \subseteq_{fin} \mathcal{X}, \mathcal{Y}' \subseteq_{fin} \mathcal{Y} \text{ imply } \bigcap(\ \mathcal{X} \cup \mathcal{Y}) \neq \emptyset\ )
\end{array}
$$

A modal general frame $(W, R, , P)$ is called *refined* if the frame is both differentiated and tight, and *descriptive* if the frame is both refined and compact. The classes of refined frames and descriptive frames are denoted by $\mathcal{R}$ and $\mathcal{D}$, respectively. Let $\mathcal{C}$ be any class of modal general frames. Then L is $\mathcal{C}$-persistent if, for all $(W, R, P) \in \mathcal{C}$, $(W, R, P) \models L$ implies $(W, R) \models L$.

Proposition 2.9 *For any descriptive frame $\mathcal{F}$, $\mathcal{F} \cong (\mathcal{F}^+)_+$.*

For a class $\mathcal{C}$ of modal general frames, a logic L is called $\mathcal{C}$-complete if φ is in L whenever it is valid in L-frame which belongs to $\mathcal{C}$. Then the following holds.

Proposition 2.10 *Suppose that $\mathcal{C}$ is a class of bimodal general frames. If L is both $\mathcal{C}$-complete and $\mathcal{C}$-persistent, then L is Kripke complete.*

We discuss here *Sahlqvist's schemes.* A formula φ is *positive* if it can be constructed using no connectives other than $\wedge$, $\vee$, $\Box$ and $\Diamond$. Sahlqvist's scheme is of the form $\Box^n(\varphi \to \psi)$, where $n \geq 0$, ψ is positive, and φ is constructed from propositional variables and their negations using at most $\wedge$, $\vee$, $\Box$ and $\Diamond$, in such a way that no occurrence of $\wedge$, $\vee$ or $\Diamond$ is inside the scope of a $\Box$.

Proposition 2.11 *Suppose that L is a $\mathcal{D}$-persistent modal logic and Q is any set of Sahlqvist's schemes. Then the logic $L \oplus Q$ is also $\mathcal{D}$-persistent.*

2.3 Gentzen style sequent systems for modal logics

In this section, we introduce Gentzen style sequent systems for some modal logics and give a survey of their proof-theoretical properties. They are obtained from Gentzen's sequent system **LK** for classical propositional logic by adding some rules for modal operator $\Box$. Greek capital letters Γ, Δ, Π, Σ, Θ and Ξ denote sequences (maybe sets) of formulas. The sequence $\Box\Gamma$ denotes $\Box\varphi_1, \Box\varphi_2, \cdots, \Box\varphi_n$, when Γ is $\varphi_1, \varphi_2, \cdots, \varphi_n$. The set $\{Sub(\varphi) \mid \varphi \in \Gamma\}$ is denoted by $Sub(\Gamma)$. Any expression of the form $\Gamma \Rightarrow \Delta$ is called a *sequent*, the left hand side Γ *the succedent* and the right hand side Δ *the antecedent*. An *inference rule* is of the form

$$\text{either} \quad \frac{S_1}{S} \quad \text{or} \quad \frac{S_2 \quad S_3}{S}$$

where S_1, S_2, S_3 and S are sequents. In the inference, S_1, S_2 and S_3 are called the *upper sequents*, and S the *lower sequent*. In particular, S_2 and S_3 is called the *left* and *right upper sequents* of the inference, respectively. The sequent system **LK** consists of the following initial sequents and inference rules.

【 Initial sequents 】

- the sequents of the form $\varphi \Rightarrow \varphi$

【 Inference rules 】

- Structural rules:

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,(w \Rightarrow) \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,(c \Rightarrow) \qquad \frac{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta}{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}\,(e \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\,(\Rightarrow w) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\,(\Rightarrow c) \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi, \Sigma}{\Gamma \Rightarrow \Delta, \varphi, \psi, \Sigma}\,(\Rightarrow e)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,(cut)$$

- Logical rules:

$$\frac{\varphi\Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow) \qquad \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ (\Rightarrow \wedge)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ (\Rightarrow \vee) \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ (\Rightarrow \vee) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Sigma}{\varphi \to \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\to\Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\ (\neg \Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\ (\Rightarrow \neg)$$

Inference rules $(w \Rightarrow)$ and $(\Rightarrow w)$ are called *weakening rules*, $(c \Rightarrow)$ and $(\Rightarrow c)$ *contraction rules*, and $(e \Rightarrow)$ and $(\Rightarrow e)$ *exchange rules*. Weakening, contraction and exchange rules are called *weak inferences*. The formula φ in cut rule is called the *cut formula* of the rule.

In a sequent system $\mathcal{S}$, *proofs* of $\mathcal{S}$ and *end sequents* are defined inductively as follows:

1) Each initial sequent is a proof of $\mathcal{S}$, and the end sequent of the proof is itself,
2) Let P_1 and P_2 be proofs of $\mathcal{S}$ with the end sequents S_1 and S_2, respectively. If

$$\frac{S_1}{S} \quad \text{or} \quad \frac{S_2 \quad S_3}{S}$$

is one of the inferences in the system of $\mathcal{S}$, then

$$\frac{P_1}{S} \quad \text{or} \quad \frac{P_2 \quad P_3}{S}$$

is a proof of $\mathcal{S}$, and the end sequent is S. A sequent S is provable in $\mathcal{S}$ if there exists a proof of $\mathcal{S}$ whose end sequent is S.

If a sequent S is provable in a system $\mathcal{S}$, then it is often denoted by $\mathcal{S} \vdash S$. For a formula φ, if the sequent $\Rightarrow \varphi$ is provable in a sequent system, then it is often said that the formula φ is provable in the system.

Cut-elimination theorem for a given sequent system $\mathcal{S}$ says that any sequent S which is provable in $\mathcal{S}$ has a proof of S containing no applications of cut rule. Such a proof is called a cut-free proof. When cut-elimination theorem holds for $\mathcal{S}$, sometimes we say that $\mathcal{S}$ has the cut-elimination property. Then the following holds.

Theorem 2.12 (Cut-elimination theorem for LK) *The system* **LK** *has the cut elimination property. In fact, every proof in* **LK** *can be transformed, without changing the end-sequent, into cut-free one.*

As a corollary of Theorem 2.12, the following can be shown by checking all inference rules except cut rule (outlines in later).

Corollary 2.13 (Subformula property) *For any provable sequent in* **LK***, proofs of it can be consists only of subformulas of formulas in the sequent.*

One of most important consequences of the cut elimination theorem is the decidability. In general, the decision procedure by using cut-elimination theorem goes as follows:

1. First, we show cut-elimination theorem.
2. Then, we derive subformula property. In many cases, subformula property follows from cut-elimination theorem. This is shown by checking that in each inference rule except cut rule every formula in an upper sequent of the rule is a subformula of some formulas in the lower sequent.
3. We show the finiteness of proof-search procedure. That, for a given sequent $\Gamma \Rightarrow \Delta$ we show that the number of "candidates" of proofs of $\Gamma \Rightarrow \Delta$ is finite. If we succeed to show this, we make an exhaustive search of these candidates and check whether some of them are "correct proofs" of $\Gamma \Rightarrow \Delta$ or not. This gives us a decision procedure. To show the finiteness of proof-search procedure, a standard strategy is as follows;

   (1) restriction to *reduced* sequents:
   We show that it is enough to consider sequents of a special form. For example, in **LK** we need to consider only sequents such that each formula occurs at most three times in the antecedent and the succedent.

   (2) restriction to proofs without repetitions:
   Apparently, if a proof contains the same sequent in a different place of one of its branches, this proof is redundant, and hence such a repetition can be eliminated.

   If we succeed to show both (1) and (2), we can also the finiteness of proof-search procedure.

In the following, we will consider sequent systems for modal logics with some of axioms T, D, 4,5 and B. Their sequent systems are obtained from **LK** by adding the following rules for the modal operator $\Box$.

$$\frac{\Gamma \Rightarrow \Theta}{\Box\Gamma \Rightarrow \Box\Theta}\,(SR1) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\,(SR2) \qquad \frac{\Box\Gamma, \Gamma \Rightarrow \Theta}{\Box\Gamma \Rightarrow \Box\Theta}\,(SR3)$$

$$\frac{\Gamma \Rightarrow \Box\Delta, \Theta}{\Box\Gamma \Rightarrow \Box\Delta, \Box\Theta}\,(SR4) \qquad \frac{\Box\Gamma, \Gamma \Rightarrow \Box\Delta, \Theta}{\Box\Gamma \Rightarrow \Box\Delta, \Box\Theta}\,(SR5)$$

$$\frac{\Gamma \Rightarrow \Box\Pi, \Theta}{\Box\Gamma \Rightarrow \Pi, \Box\Theta}\,(SR6) \qquad \frac{\Box\Gamma, \Gamma \Rightarrow \Box\Delta, \Box\Omega, \varphi}{\Box\Gamma \Rightarrow \Box\Delta, \Omega, \Box\varphi}\,(SR7)$$

In rules $(SR6)$ and $(SR7)$, $\Box\Pi \subseteq Sub(\Gamma \cup \{\varphi\})$ and $\Box\Omega \subseteq Sub(\Box\Gamma \cup \Delta \cup \{\varphi\})$, respectively. Also we assume that Θ consists of a single formula. When we relax this condition on Θ and assume that Θ consists of *at most one formula*, we will add the superscript D to these rules, like $(SR1)^D$. This relaxation is necessary when a modal logic under consideration includes the axiom D. Also, when a rule (SRi) is assumed for a particular modal operator $\Box$, we write it as $(SRi)_\Box$, if necessary. In the following, the Gentzen style sequent system for a modal logic L is denoted by $\mathcal{S}(L)$. Here we will introduce sequent systems for some of well-known modal logics.

Systems	Rules
$\mathcal{S}(\mathbf{K})$	$(SR1)$
$\mathcal{S}(\mathbf{K4})$	$(SR3)$
$\mathcal{S}(\mathbf{K5})$	$(SR4)$
$\mathcal{S}(\mathbf{K45})$	$(SR5)$
$\mathcal{S}(\mathbf{KD})$	$(SR1)^D$

Systems	Rules
$\mathcal{S}(\mathbf{KT})$	$(SR1)$, $(SR2)$
$\mathcal{S}(\mathbf{S4})$	$(SR3)$, $(SR2)$
$\mathcal{S}(\mathbf{KD4})$	$(SR3)^D$
$\mathcal{S}(\mathbf{KD5})$	$(SR4)^D$
$\mathcal{S}(\mathbf{KD45})$	$(SR5)^D$

Systems	Rules
$\mathcal{S}(\mathbf{KB})$	$(SR6)$
$\mathcal{S}(\mathbf{KTB})$	$(SR2)$, $(SR6)$
$\mathcal{S}(\mathbf{KDB})$	$(SR6)^D$
$\mathcal{S}(\mathbf{KB4})$	$(SR7)$
$\mathcal{S}(\mathbf{S5})$	$(SR5)$, $(SR2)$

We can show Proposition 2.14 easily.

Proposition 2.14 *Let L be any of* **K**, **K4**, **K5**, **K45**, **KT**, **S4**, **S5**, **KD**, **KD4**, **KD5**, **KD45**, **KB**, **KTB**, **KDB** *and* **KB4**. *Then $L \vdash \varphi$ iff $\mathcal{S}(L) \vdash \Rightarrow \varphi$.*

In deriving the decidability of a logic by using the sequent system, we show usually the cut elimination theorem for the system in order to obtain the subformula property. But some of the above systems lacks the cut elimination property. However, if we can restrict any application of cut rule to the following way,

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

where $\varphi \in Sub(\Gamma \cup \Pi \cup \Delta \cup \Sigma)$

then we can obtain the subformula property. We call such cut rule, *acceptable cut*. It is called that the sequent system $\mathcal{S}(L)$ has the *cut restriction property*, if every proof in $\mathcal{S}(L)$ can be transformed, without changing the end sequent, into the proof in which every cut rule applied in it is acceptable. By [42, 43, 44], we can divide their logics into the following three groups.

(G1) **K**, **K4**, **K45**, **KT**, **S4**, **KD**, **KD4**, **KD45**

(G2) **KB**, **KTB**, **KDB**, **KB4**, **S5**

(G3) **K5**, **KD5**

The group (G1) is a collection of logics satisfying the cut elimination theorem. For a logic in (G2), even if the sequent system for the logic is restricted the cut rule into the acceptable cut, provable sequents are same in both systems. Therefore, for every logic L in (G1) and (G2), sequent system $\mathcal{S}(L)$ has the subformula property. But in the case of a logic in (G3), some formulas which are provable in the logic are unprovable in the restricted systems by such restriction. Indeed, every logic in (G3) lacks the subformula property. For example, the cut rule of the lowest inference in the following proof can not be eliminated.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box p \Rightarrow \Box p}{\Rightarrow \Box p, \neg\Box p}}{\Rightarrow \Box p, \Box\neg\Box p}}{\Rightarrow \Box\Box p, \Box\neg\Box p} \qquad \dfrac{\dfrac{\dfrac{\Box p \Rightarrow \Box p}{\overline{\neg\Box p, \Box p \Rightarrow q}} \quad \dfrac{q \Rightarrow q}{\neg\Box p, q \Rightarrow q}}{\neg\Box p, \Box p \vee q \Rightarrow q}}{\Box\neg\Box p, \Box(\Box p \vee q) \Rightarrow \Box q}}{\Box(p \vee q) \Rightarrow \Box\Box p, \Box q}$$

Now consider that the cut rule restricted the cut formula into an element of the set

$$Sub(\Gamma \cup \Pi \cup \Delta \cup \Sigma) \cup \Box\neg\Box Sub(\Gamma \cup \Pi \cup \Delta \cup \Sigma) \cup \neg\Box Sub(\Gamma \cup \Pi \cup \Delta \cup \Sigma).$$

For every logic in (G3), the restricted system is equivalent to the former system. By this restriction, it can be shown that every proof in $\mathcal{S}(L)$ where L is in (G3) consists only of elements in a finite set of formulas which depends on formulas in the lowest sequent. Here, *extended acceptable cut rule* is defined as

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

where $\varphi \in Q$, and Q is a finite set of formulas which is determined uniformly and effectively by $\Gamma, \Pi, \Delta, \Sigma$. It is called that a sequent system has the *extended cut restriction property*, if every proof in the system can be transformed, without changing the end sequent, into the proof in which every cut rule applied in it is extended acceptable. Then a logic in (G3) has the extended cut restriction property.

Chapter 3

Pseudo-Euclidean logics

In the following, to show the inclusion relation $L_1 \supseteq L_2$ for logics L_1 and L_2, we will prove that any L_1-frame is also a L_2-frame, instead. Of course, this is possible only when both L_1 and L_2 are Kripke complete.

In this section, we discuss inclusion relationship among pseudo-Euclidean logics. For fixed non-negative integers m and n, let E_k be the logic which is obtained from the smallest normal propositional modal logic **K** by adding the pseudo-Euclidean axiom $\Diamond^k\varphi \to \Box^m\Diamond^n\varphi$, where $k \geq 0$. We will then give a complete description of the inclusion relationship among these logics. By doing so, we can show how semantical method works effectively.

3.1 Inclusion relationship among pseudo-Euclidean logics

Inclusion relationships among various propositional modal logics have been found out since the early works of them. For example, the inclusion relationship among a class of logics over **K45** is shown in [39]. Our work throws light on the power of proof among pseudo-Euclidean logics.

Throughout this chapter, m and n are fixed non-negative integers. Let E_k be the logic which is obtained from the smallest normal modal logic **K** by adding the axiom $\Diamond^k\varphi \to \Box^m\Diamond^n\varphi$, where $k \geq 0$. Here, $\Diamond^k\varphi$ and $\Box^{k'}\varphi$ denote formulas $\Diamond\cdots\Diamond\varphi$ with k diamonds and $\Box\cdots\Box\varphi$ with k' boxes, respectively. We call any logic of the form E_k, a *pseudo-Euclidean* logic. Since each axiom $\Diamond^k\varphi \to \Box^m\Diamond^n\varphi$ is a Sahlqvist formula, we can show that the logic E_k is Kripke complete for each k. In fact, let us say that a binary relation R on a set W is *k-pseudo-Euclidean* if for any $x, y, z \in W$, xR^ky and xR^mz imply zR^ny. Then, it is easy to see that E_k is Kripke complete with respect to the class of all Kripke frames of the form (W, R) with a k-pseudo-Euclidean relation R on W. Note that when $m = n = 1$, R is 1-pseudo-Euclidean if and only if it is Euclidean. Let $\mathcal{PE}_k$ be the class of all Kripke frames of the form (W, R), where R is a k-pseudo-Euclidean relation on W. Then it is easy to see that $E_k \supseteq E_{k'}$ if and only if $\mathcal{PE}_k \subseteq \mathcal{PE}_{k'}$. In the rest of this chapter, we identify the axiom system E_k with the set of all formulas provable in E_k. Our main goal of this chapter is to make a through study of when the inclusion relation $E_k \supseteq E_{k'}$ holds. Our result is summarized in the following theorem. Note that $E_k \supseteq E_{k'}$ trivially holds when $k = k'$. So, we assume $k \neq k'$ in the following.

Theorem 3.1 *1. For $k > k'$:* $E_k \supseteq E_{k'}$ *iff* $m = 0$ *and* $k' = n$.

2. For$k' > k$:

2a. *If $m = 0$ and $n = k'$ then $E_k \supseteq E_{k'}$.*

2b. *Suppose that either $m > 0$ and $n \neq k'$.*

If one of the following (1), (2), (3) holds

(1) $k \geq m + n$,

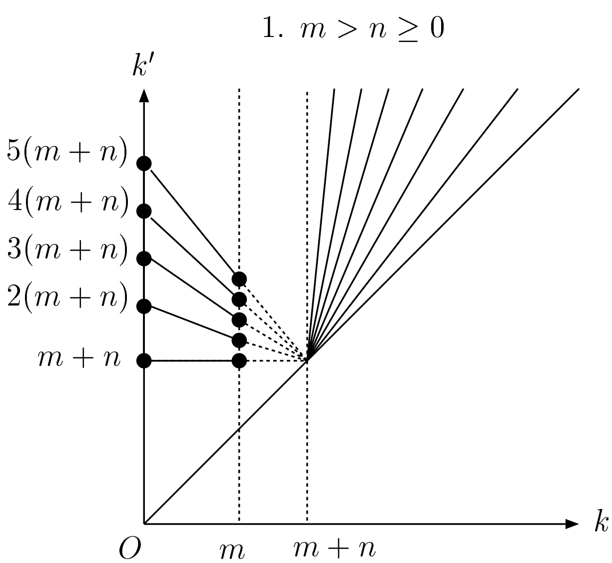
(2) $m \geq k$ *and* $m > n$,

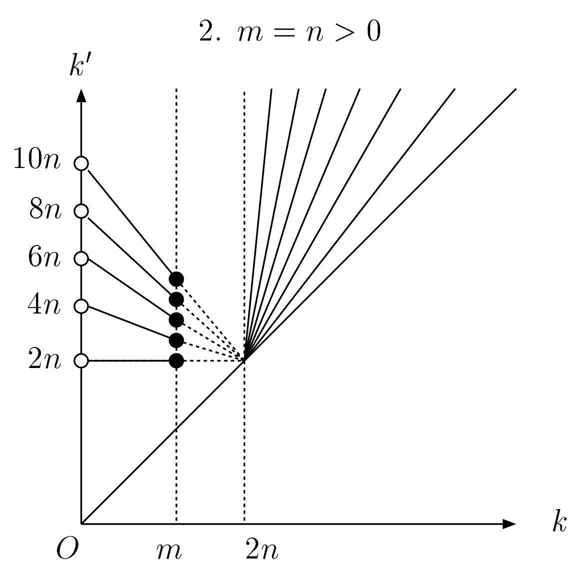
(3) $m = n \geq k > 0$,

then

$E_k \supseteq E_{k'}$ *iff* $(k - m - n) \mid (k' - m - n)$.

2c. *Otherwise,* $E_k \not\supseteq E_{k'}$.





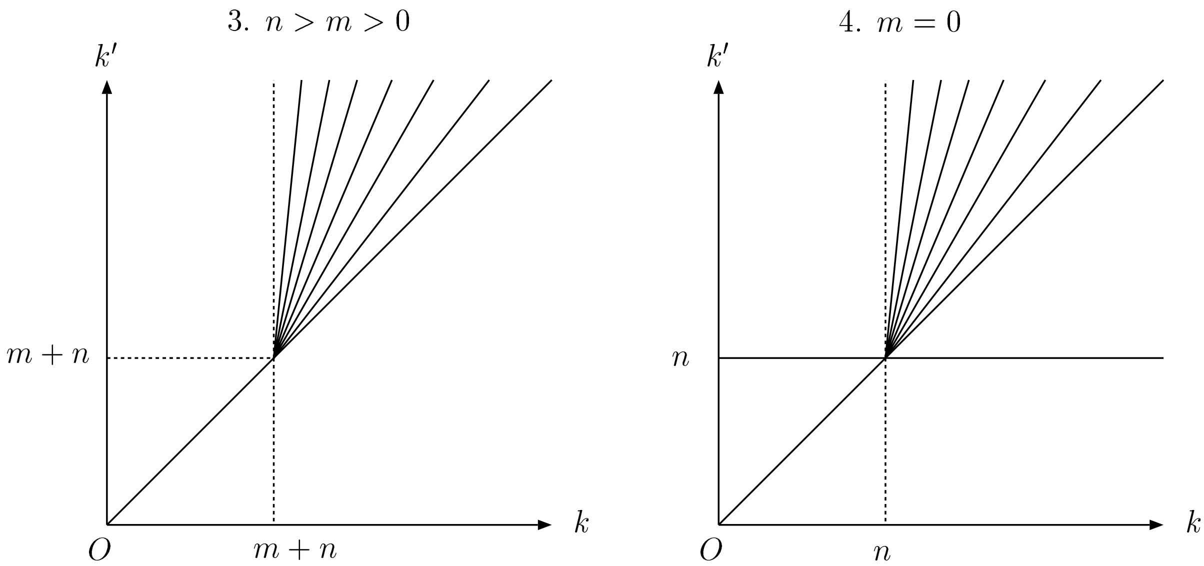


Figure 3.1: $E_k \supseteq E_{k'}$ iff (k, k') in the graphs

3.2 Proof of the theorem

The rest of the chapter will be devoted to an outline of the proof of Theorem 3.1. It is obvious that $E_k = E_{k'}$ when $k = k'$. Henceforth, we assume $k \neq k'$. Also, when $m = 0$ and $k' = n$, the axiom $\Diamond^{k'}\varphi \to \Box^m\Diamond^n\varphi$ becomes $\Diamond^n\varphi \to \Diamond^n\varphi$, which is obviously provable in **K**. That is, $E_{k'}$ coincides with **K**. Hence, we have the following.

Lemma 3.2 *If $m = 0$ and $n = k'$ then $E_k \supseteq E_{k'} = \mathbf{K}$.*

When $k > k'$, the converse of Lemma 3.2 holds as shown below.

Lemma 3.3 *If $k > k'$ and either $m > 0$ or $n \neq k'$ then $E_k \not\supseteq E_{k'}$.*

Proof. Suppose first that $k > m$. We define a frame $\mathcal{F} = (W, R)$ as follows: $W = \{w_i \mid 0 \leq i \leq k' + m\}$, and the binary relation R is defined by 1) $w_i R w_{i-1}$ for each $i = 1, \cdots, m$, and 2) $w_i R w_{i+1}$ for each $i = m, m+1, \cdots, k' + m - 1$.

Then, we can show that both $w_m R^{k'} w_{k'+m}$ and $w_m R^m w_0$ hold, while $w_0 R^n w_{k'+m}$ doesn't, since either $m > 0$ or $k' \neq n$. Thus, if $w_i \models \varphi$ only for $i = k' + m$ then $w_m \not\models E_{k'}$. Therefore $\mathcal{F} \notin \mathcal{PE}_{k'}$. On the other hand, for each $x \in W$, there is no $y \in W$ such that $xR^k y$ since $k > k'$ and $k > m$. Therefore $\mathcal{F} \in \mathcal{PE}_k$.

Suppose next that $k \leq m$. Let us take a frame $\mathcal{G} = (V, S)$ defined as follows: $V = \{w_i \mid 0 \leq i \leq k' + 1\}$, and the binary relation S is defined by 1) $w_0 S w_0$, 2) $w_1 S w_0$, and

3) $w_i S w_{i+1}$ for each $i = 1, \cdots, k'$. Similar to the above, we can show that $\mathcal{G} \in \mathcal{PE}_k$ but $\mathcal{G} \notin \mathcal{PE}_{k'}$. ∎

Thus, we have proved the first part of Theorem 3.1. The following lemma holds for arbitrary k and k'.

Lemma 3.4 *If $E_k \supseteq E_{k'}$ then $(k - m - n) \Big| (k' - m - n)$.*

Proof. Suppose that $E_k \supseteq E_{k'}$ but $(k - m - n) \Big| (k' - m - n)$ doesn't hold. Let $a = k - m - n$ and define a frame $\mathcal{F} = (W, R)$ as follows: $W = \{w_i \mid \bar{i} \in \boldsymbol{Z}/a\boldsymbol{Z}\}$, and $w_i R w_j$ iff $j \equiv i + 1 \pmod{a}$.

By the assumption, since $k' - m \neq n + h(k - m - n)$ for any $h \in \boldsymbol{Z}$, i.e. $k' - m \not\equiv n \pmod{a}$, $w_m R^n w_{k'}$ doesn't hold. On the other hand, both $w_0 R^{k'} w_{k'}$ and $w_0 R^m w_m$ hold. Thus $\mathcal{F} \notin \mathcal{PE}_{k'}$. Next, suppose that $w_i R^k w_j$ and $w_i R^m w_s$. Then, $j - i \equiv k \pmod{a}$ and $s - i \equiv m \pmod{a}$. Hence $j - s \equiv k - m \pmod{a}$. But $k - m \equiv n \pmod{a}$ since $a = k - m - n$. Thus $j - s \equiv n \pmod{a}$, i.e. $w_s R^n w_j$. Hence $\mathcal{F} \in \mathcal{PE}_k$. This contradicts that $E_k \supseteq E_{k'}$. ∎

In the following, we will find sufficient conditions by which the converse of Lemma 3.4 holds. We can assume that $k' > k$, and moreover that either $m > 0$ or $n \neq k'$, by Lemma 3.2.

Lemma 3.5 *If $k' > k \geq m + n$ and $(k - m - n) \Big| (k' - m - n)$ then $E_k \supseteq E_{k'}$.*

Proof. By the assumption, $k' - m - n = h(k - m - n)$, that is $k' = k + (h-1)(k - m - n)$, for a certain number $h \in \boldsymbol{Z}$. Since $k' \geq k$ and $k - m - n \geq 0$, we can assume that $k' = k + (h - 1)(k - m - n)$ with $h > 1$. To show that $E_k \supseteq E_{k'} = E_{k+(h-1)(k-m-n)}$, it is enough to show that every $(W, R) \in \mathcal{PE}_k$ belongs also to $\mathcal{PE}_{k+(h-1)(k-m-n)}$ for any $h > 1$. This can be shown by the induction on h.

The base step, that is the case of $h = 2$, can be ascertain in similar way of the induction step. So, we assume that this holds for h. To show that (W, R) belongs to $\mathcal{PE}_{k+h(k-m-n)}$, we assume that $xR^{k+h(k-m-n)}y$ and $xR^m z$. Then, for some $w \in W$, $xR^{k+(h-1)(k-m-n)}w$ and $wR^{k-m-n}y$, since $k + (h - 1)(k - m - n) \geq 0$ and $k - m - n \geq 0$. Since (W, R) belongs to $\mathcal{PE}_{k+(h-1)(k-m-n)}$ by the hypothesis of induction, $xR^{k+(h-1)(k-m-n)}w$ and $xR^m z$ imply $zR^n w$. Since $xR^m z$, $zR^n w$ and $wR^{k-m-n}y$ hold, $xR^k y$. But since (W, R) is in $\mathcal{PE}_k$, $zR^n y$. Thus, we have shown that (W, R) belongs to $\mathcal{PE}_{k+h(k-m-n)}$. ∎

Lemma 3.6 *Suppose that $m \geq k$ and either 1) $m > n$ or 2) $m = n$ and $k > 0$. Let (W, R) be in $\mathcal{PE}_k$. Then for any $l \geq 0$ and any $M \geq \max(m-n-1, k-1)$, if $xR^{n+l}y$, $xR^l z$ and $x'R^M x$ then $zR^n y$.*

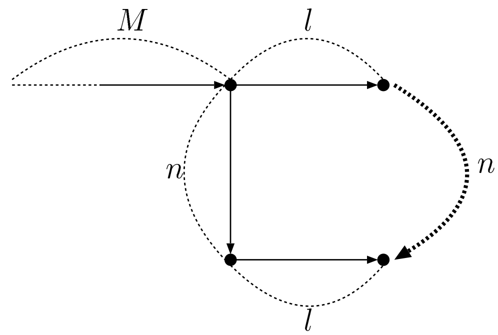


Figure 3.2:

Proof. We will show by the induction on l. If $l = 0$, this is trivial. When $l = 1$, we will divide the case into two. First, suppose that $k \geq m - n$. Then, for some $w, u \in W$, $x'R^{M-(k-1)}w$, $wR^{k-1}x$, $xR^{m-k+1}u$ and $uR^{k+n-m}y$, since $M \geq k-1 \geq 0$, $m-k+1 > 0$ and $k+n-m \geq 0$. Since $wR^{k-1}x$ and xRz hold, $wR^k z$. Also, since $wR^{k-1}x$ and $xR^{m-k+1}u$ hold, $wR^m u$. Since (W, R) is in $\mathcal{PE}_k$, $uR^n z$. Then, for some $v \in W$, $x'R^{M+m-k+1-(m-n)}v$ and $vR^{m-n}u$, since $M + m - k + 1 - (m - n) \geq 0$ and $m - n \geq 0$. Since $vR^{m-n}u$ and $uR^{k+n-m}y$ hold, $vR^k y$. Also, since $vR^{m-n}u$ and $uR^n z$ hold, $vR^m z$. Therefore $zR^n y$ since (W, R) is in $\mathcal{PE}_k$.

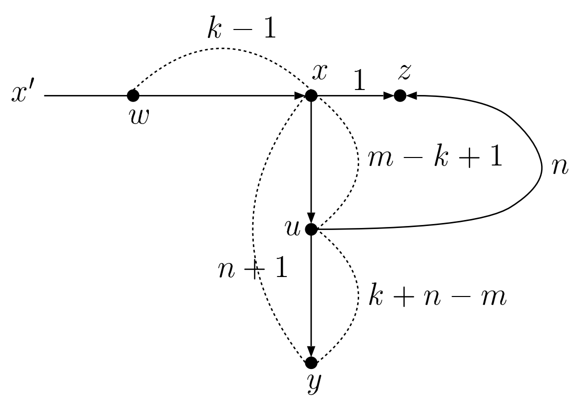


Figure 3.3:

When $k < m - n$, for some $w, u \in W$, $x'R^{M-k-(m-n-k-1)}w$, $wR^k u$ and $uR^{m-n-k-1}x$, since $M \geq k + (m - n - k - 1)$, $k \geq 0$ and $m - n - k - 1 \geq 0$. Since $wR^k u$, $uR^{m-n-k-1}x$

and $xR^{n+1}y$ hold, $wR^m y$. Since (W, R) is in $\mathcal{PE}_k$, $yR^n u$. Then, for some $v \in W$, $wR^{m-k}v$ and $vR^k y$, since $m - k \geq 0$ and $k \geq 0$. Since $vR^k y$, $yR^n u$, $uR^{m-n-k-1}x$ and xRz hold, $vR^m z$. Since (W, R) is in $\mathcal{PE}_k$, $zR^n y$. Therefore, we have shown, when $l = 1$.

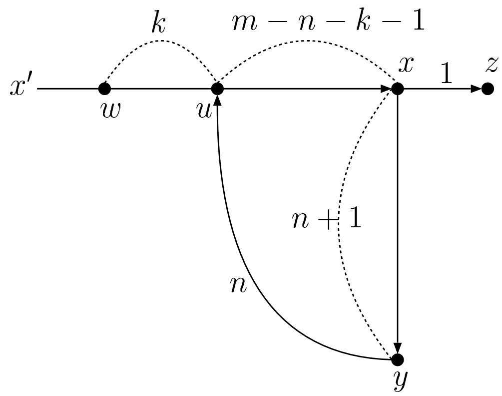


Figure 3.4:

Now, we assume that this holds for l. To show for $l + 1$, we assume that $xR^{n+l+1}y$, $xR^{l+1}z$ and $x'R^M x$. Then, for some $y', z' \in W$, $xR^{n+1}y'$, $y'R^l y$, xRz' and $z'R^l z$. Hence $z'R^n y'$ by the result when $l = 1$. Since $z'R^n y'$ and $y'R^l y$ hold, $z'R^{n+l}y$. Since $x'R^M x$ and xRz' hold, $x'R^{M+1}z'$. Since $z'R^{n+l}y$, $z'R^l z$, $x'R^{M+1}z'$ and $M + 1 \geq M \geq \max(m - n - 1, k - 1)$, $zR^n y$ by the hypothesis of induction. ∎

Lemma 3.7 *Suppose $k' > k$ and $m \geq k$. Moreover suppose that either 1) $m > n$ or 2) $m = n$ and $k > 0$. Then $(k - m - n) \Big| (k' - m - n)$ implies $E_k \supseteq E_{k'}$.*

Proof. By the assumption, $k' - m - n = h(m + n - k)$, that is $k' = k + (h+1)(m+n-k)$, for a certain number $h \in \boldsymbol{Z}$. Since $k' > k$ and $m + n - k \geq 0$, we can assume that $k' = k + (h + 1)(m + n - k)$ with $h \geq 0$. To show that $E_k \supseteq E_{k'} = E_{k+(h+1)(m+n-k)}$, it is enough to show that every $(W, R) \in \mathcal{PE}_k$ belongs also to $\mathcal{PE}_{k+(h+1)(m+n-k)}$ for any $h \geq 0$. This can be shown by the induction on h.

If $h = 0$ then $k' = m + n$. we assume that $(W, R) \in \mathcal{PE}_k$, and also that $xR^{m+n}y$ and $xR^m z$ for $x, y, z \in W$. Then, for some $w \in W$, $xR^k w$ and $wR^{m+n-k}y$, since $m + n - k \geq 0$. Then $zR^n w$ and $wR^{m+n-k}y$ by the assumption, so $zR^{m-k+2n}y$. Then, for some $u, v \in W$, $xR^{m-k}u$, $uR^k z$, $zR^{m-k}v$ and $vR^{2n}y$, since $m - k \geq 0$. Since $uR^m v$ and $uR^k z$, $vR^n z$. But by using Lemma 3.6, $zR^n y$ by taking $l = n$. Hence $\mathcal{PE}_{m+n}$.

Since the essence in the proof is involved in the base step, we can check the induction step in the similar way of the base step. ∎

Thus, combining Lemma 3.7 with Lemma 3.4 and 3.5 we have the following.

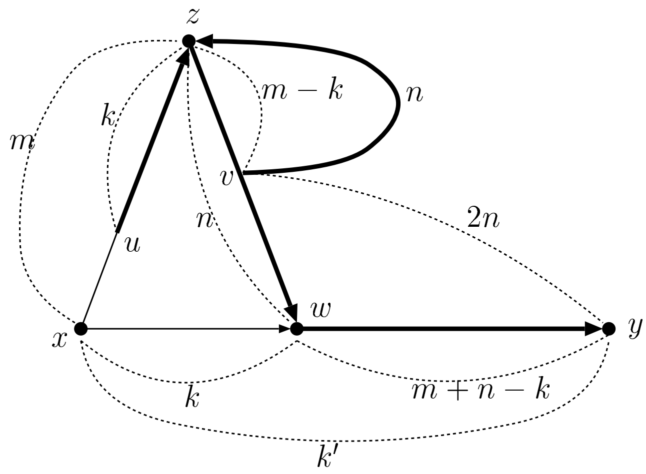


Figure 3.5:

Corollary 3.8 *Suppose that $k' > k$ and that either $m > 0$ or $k' \neq n$. If one of the following (1), (2), (3) holds*

(1) $k \geq m+n$

(2) $m \geq k$ and $m > n$

(3) $m \geq k$, $m = n$ and $k > 0$

then $E_k \supseteq E_{k'}$ iff $(k-m-n) \mid (k'-m-n)$.

In the last place, we will show that $E_k \supseteq E_{k'}$ never hold in the remaining cases. So, we assume that none of (1), (2) and (3) in the above corollary holds.

First, suppose that $m > 0$. Suppose moreover that $k > m$. Note that $m+n > k$ holds, because (1) of Corollary 3.8 doesn't hold.

Lemma 3.9 *If $k' > k$ and $m+n \geq k > m > 0$ then $E_k \not\supseteq E_{k'}$.*

Proof. Define a frame $\mathcal{F} = (W, R)$ as follows: $W = \{w_i \mid 0 \leq i \leq m+n+1\}$, and 1) $w_i R w_i$ for each $i = m+1, \cdots, m+n+1$, 2) $w_i R w_{i+1}$ for each $i = 0, \cdots, m+n$, 3) $w_i R w_{i-1}$ for each $i = m+2, \cdots, m+n+1$, 4) $w_0 R w_{m+n+1-k}$.

First, we will show that $\mathcal{F} \in \mathcal{PE}_k$. If $i \geq 1$, $w_i R^k w_j$ and $w_i R^m w_{j'}$ then both w_j and $w_{j'}$ are between w_{m+1} and w_{m+n+1} since $i+k \geq m+1$ and $i+m \geq m+1$. Thus $w_{j'} R^n w_j$. If $w_0 R^k w_j$ and $w_0 R^m w_{j'}$ then $w_{j'} R^n w_j$ since $m+1 \leq j \leq m+n$ and $m \leq j' < m+n$. Hence $\mathcal{F} \in \mathcal{PE}_k$. On the other hand, $w_m R^n w_{m+n+1}$ doesn't hold since $m \neq 0$, while both $w_0 R^{k'} w_{m+n+1}$ and $w_0 R^m w_m$ hold. (Note here that $w_0 R^{k+1} w_{m+n+1}$ and $k+1 \leq k'$.) Hence $\mathcal{F} \notin \mathcal{PE}_{k'}$. ∎

Suppose next that $m \geq k$. Because (2) of Corollary 3.8 doesn't hold, $n \geq m$. We assume first that $n > m > 0$. Then we have the following.

Lemma 3.10 *If $k' > k$, $m \geq k \geq 0$ and $n > m > 0$ then $E_k \not\supseteq E_{k'}$.*

Proof. If $k' < m+n$ then $m+n-k > m+n-k' > 0$, so $(k-m-n) \mid (k'-m-n)$ doesn't hold. Thus, we can derive our conclusion by using Lemma 3.4. It is therefore sufficient to consider the case where $k' \geq m+n$. We will divide the case into two.

For $n \geq k+m$, we define a frame $\mathcal{F} = (W, R)$ as follows: $W = \{w_i \mid 0 \leq i \leq m+n\}$, and $w_i R w_j$ iff $|i-j| \leq 1$. Since $m+n > n$ by $m > 0$, $w_0 R^n w_{m+n}$ doesn't hold while both $w_0 R^m w_0$ and $w_0 R^{k'} w_{m+n}$ hold for $k' \geq m+n$. Therefore $\mathcal{F} \notin \mathcal{PE}_{k'}$. We will next show that $\mathcal{F} \in \mathcal{PE}_k$. We first note that $w_i R^t w_j$ holds if and only if $|i-j| \leq t$. Now, suppose that $w_i R^k w_j$ and $w_i R^m w_s$. Then, $|i-j| \leq k$ and $|i-s| \leq m$. Therefore, $|s-j| \leq |s-i| + |i-j| \leq m+k \leq n$. Hence, $w_s R^n w_j$

For $n < k+m$, define a frame $\mathcal{G} = (V, S)$ as follows (see Figure 3.6): $V = \{v_i \mid 0 \leq i \leq k+m+1\}$, and

$v_i S v_j \Leftrightarrow$ either

1) $|i-j| \leq 1$ if $0 \leq i, j \leq k+m+1$ or

2) $j = k+m-n+2$ if $1 \leq i < k+m-n+2$ or

3) $j = n-1$ if $n-1 \leq j \leq k+m$.

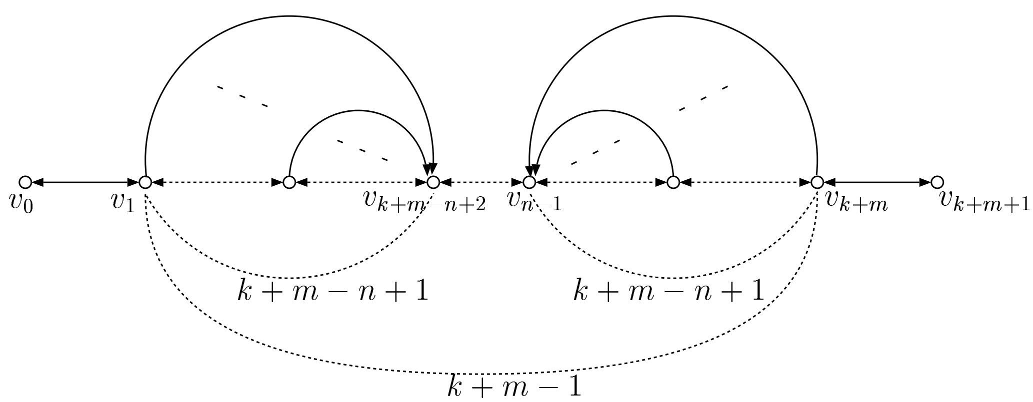


Figure 3.6:

Note that the frame takes at least $n+1$ steps from v_0 to v_{k+m+1} by the relation S. Thus $v_0 S^n v_{k+m+1}$ doesn't hold. But both $v_m S^{k'} v_{k+m+1}$ and $v_m S^m v_0$ hold because of $k+m+1 \leq k'+m$. Thus $\mathcal{G} \notin \mathcal{PE}_{k'}$.

Assume that $xS^k y$ and $xS^m z$ for any $x, y, z \in V$. Then both y and z must be either between v_0 and v_{k+m}, or between v_1 and v_{k+m+1}, depending on x. For each case, y is accessible from z by n steps, i.e. $zS^n y$. Therefore $\mathcal{G} \in \mathcal{PE}_k$. ∎

Next, assume that $n = m > 0$. Since (3) on Corollary 3.8 doesn't hold, k must be equal to 0. Then, we have the following.

Lemma 3.11 *If $k' > k$, $m = n > 0$ and $k = 0$ then $E_0 \not\supseteq E_{k'}$.*

Proof. We define a frame $\mathcal{F} = (W, R)$ as follows;

$$\begin{aligned} W &= \{w_i \mid 0 \le i \le m+1\}, \\ w_i R w_j &\Leftrightarrow |i-j| \le 1. \end{aligned}$$

Then $w_0 R^n w_{m+1}$ doesn't hold while both $w_1 R^{k'} w_0$ and $w_1 R^m w_{m+1}$ hold. Hence $\mathcal{F} \notin \mathcal{PE}_{k'}$. On the other hand, $xR^m y$ implies $yR^n x$ since the frame R is symmetric. Thus $\mathcal{F} \in \mathcal{PE}_0$. ∎

Finally suppose that $m = 0$. Then by our assumption, $n \neq k'$. Since the condition (1) $k \ge m + n = n$ on Corollary 3.8 doesn't hold, $n > k$. Then, we have the following.

Lemma 3.12 *If $k' > k$, $m = 0$, $n > k$ and $k' \neq n$ then $E_k \not\supseteq E_{k'}$.*

Proof. Similarly to Lemma 3.10, we can show our lemma easily when $k' < n$. So, suppose that $k' > n$. If $k' < 2n - k$ then $n - k > k' - n > 0$, so $(k-n) \mid (k'-n)$ doesn't hold. This case has been discussed already in Lemma 3.4. It is therefore sufficient to consider the case $k' \ge 2n - k$. Then we define a frame $\mathcal{F} = (W, R)$ as follows;

$$\begin{aligned} W &= \{w_i \mid 0 \le i \le 2n-k\}, \\ w_i R w_j &\Leftrightarrow |i-j| \le 1. \end{aligned}$$

Since $2n - k > n$ by $n - k > 0$, $w_0 R^n w_{2n-k}$ doesn't hold while $w_0 R^{k'} w_{2n-k}$ hold, therefore $\mathcal{F} \notin \mathcal{PE}_{k'}$. On the other hand, if $xR^k y$ then $xR^n y$ for any $x, y \in W$, since $n > k$. Thus $\mathcal{F} \in \mathcal{PE}_k$. ∎

3.3 Note

For non-negative integers m and n, we have shown when $E_k \supseteq E_{k'}$ holds. As generalization of our results, it is interested in what happen if we allow both m and n to change. More precisely, let $E_k^{m,n}$ be the logic which is obtained from the smallest normal logic **K** by adding the axiom $\Diamond^k \varphi \to \Box^m \Diamond^n \varphi$, where $k, m, n \ge 0$. Then it is to see when $E_k^{m,n} \supseteq E_{k'}^{m',n'}$ holds.

The content of this chapter is also discussed in [14], and is compiled as a joint paper with Y. Hasimoto, which is under submission.

Chapter 4

Combination of modal logics

Most of studies of modal logics until recent years are concerned with *monomodal logics*, i.e. modal logics with a single modal operator. We have already had many of strong and general results on monomodal logics. On the other hand, it is quite natural and necessary to introduce modal logics with many modal operators when we want to use modal logics as frameworks for describing problems in philosophy, linguistics and computer science. Such modal logics, called now *multimodal logics*, have been one of most important topics of modal logics for these 20 years.

Here, we will give two important examples of multimodal logics that have been paid much attention. The first example is dynamic logic. Dynamic logic is introduced as a logical framework of verifications of programs. For each "program" π, $[\pi]\varphi$ means "φ holds after the execution of π". Since π runs over programs, dynamic logic is considered to be a modal logic with infinitely many modal operators. The interdependency between these modal operators are determined by basic operations of programs.

Multimodal logics in the second example are temporal logics. The basic temporal logic K_t has two operators, past operator $[P]$ and future operator $[F]$, and other temporal logics are usually defined to be extensions of K_t, which sometimes have some additional operators like "until" and "next time". The interdependency between $[P]$ and $[F]$ can be described in an obvious way. That is , the present is a future of any past, and is also a past of any future.

These two classes of multimodal logics arose from application side, and therefore interdependency of modal operators are determined by their intuitive meaning.

Now let us consider multimodal logics that are obtained from some of modal logics by *combining* them, in a general setting. Then, a question is how to combine them. A natural way of combining modal logics, say L_1 and L_2, is just to consider the least modal logic containing both L_1 and L_2 (assuming that there is no modal operators common to them). The logic thus obtained is called the *fusion* of L_1 and L_2, and is denoted by

$L_1 \otimes L_2$. For instance, take a logic of knowledge and a logic of belief. Then, the fusion of them becomes a logic of knowledge and belief.

The notion of fusions of modal logics is originally introduced by S. K. Thomason [47]. Then, in the early 90's M. Kracht and F. Wolter [23] developed an extensive study of fusions. Among others, they succeeded to show many important preservation theorems, i.e. to clarify what kind of (semantical) properties is preserved by the fusion. In spite of these strong results on fusions, we have only a little knowledge on combination of modal logics. When there are some dependency between modal operators even if it is quite weak.

There is another interesting way of combining modal logics. The *product* of modal logics L_1 and L_2 is defined to be the modal logic characterized by the product of Kripke frames for L_1 and those for L_2. The study is started by V. Shehtman, and has been developed mainly by D. Gabbay and V. Shehtman in recent years (see e.g. [9], [10]). A mathematically clearer, but logically a bit more complicated notion of products, called *normal products* is introduced and studied by Y. Hasimoto [15].

In this chapter, we will discuss fusions of modal logics. First, we will give a brief survey of fusions and preservation results mainly by M. Kracht and F. Wolter [23]. In Section 4.2, as an attempt to develop a study of fusions with some interdependency, we will discuss the finite model property. As for a fusion with some interdependency, not to mention the proof-theoretical property, even the finite model property is not so easy. In Section 4.3, we show our results on cut elimination theorem and consequences of fusion. Moreover, Section 4.4 discusses the cut elimination theorem for some fusions with an interdependent axioms. In the last section, we will introduce temporal epistemic logics, the combination of temporal logic K_t and some of epistemic logics, as an concrete example of combinations of modal logics. Temporal epistemic logics introduced there are fully discussed in the next chapter.

4.1 Composite of modal logics

In this section, we will explain two important way of combining modal logics. The first is fusion, and second is products. Then, we will give a brief survey of recent results on fusions, since this paper mainly discusses multimodal logics obtained by fusions.

Construction of multimodal logics

Let L_1 and L_2 be two modal (maybe monomodal) logics formulated in languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. We assume that there is no modal operators common to them. Then the *fusion* of L_1 and L_2, denoted by $L_1 \otimes L_2$, is the least multimodal logic in $\mathcal{L}_1 \cup \mathcal{L}_2$ containing $L_1 \cup L_2$. The fusion $L_1 \otimes L_2 \otimes \cdots \otimes L_n$ of n modal logics can be defined in the same way, for any natural number $n \geq 2$. As an abbreviation, the fusion of logics L_1, L_2, $\cdots$, L_n is denoted as $\bigotimes_{i=1}^{n} L_i$. Besides, Kripke type semantics for fusions can be introduced naturally. For a logic L, $\mathbf{F}(L)$ denotes the class of all frames which validate L. Suppose that $(W, R_1, \cdots, R_m) \in \mathbf{F}(L_1)$ and $(W, S_1, \cdots, S_n) \in \mathbf{F}(L_1)$ are L_1- and L_2-frames, respectively. Then the frame formed $(W, R_1, \cdots, R_m, S_1, \cdots, S_n)$ which satisfies first-order conditions for each fragment is a frame for the combined modal logic $L_1 \otimes L_2$, that is $(W, R_1, \cdots, R_m, S_1, \cdots, S_n) \in \mathbf{F}(L_1 \otimes L_2)$. From this point of view, worlds in frames for fusions are same as those for fragments. It can be ascertained that all frames in $\mathbf{F}(L_1 \otimes L_2)$ validate the fusion $L_1 \otimes L_2$ whenever $\mathbf{F}(L_1)$ and $\mathbf{F}(L_2)$ determine logics L_1 and L_2, respectively.

Here we consider fusions of axiomatizable modal logics L_1 and L_2 in $\mathcal{L}_1$ and $\mathcal{F}_2$, respectively. Then we can see that it is not necessary to use formulas containing both modal operators in L_1 and L_2 in the axiomatization of the fusion. In this sense, modal operators in L_1 and L_2 remain independent in the fusion. According to this construction, the fusion is axiomatizable whenever the fragments are.

Another way of combining modal logics is to introduce a multimodal logic by using *products* of modal frames. That is introduced in [9]. This is a natural way of introducing interactions among modal operators. There have been many applications of products in computer science, and artificial intelligence and so on since 1980. Here, we introduce only two-dimensional case of products. The product of a frame $\mathcal{F}_1 = (W, R_1, \cdots, R_m)$ and $\mathcal{F}_2 = (V, S_1, \cdots, S_n)$, denoted by $\mathcal{F}_1 \times \mathcal{F}_2$, is defined as $(W \times V, \tilde{R_1}, \cdots, \tilde{R_m}, \tilde{S_1}, \cdots, \tilde{S_n})$ where $\tilde{R_i} = \{((x, z), (y, z)) \mid xR_iy\}$ and $\tilde{S_j} = \{((z, x), (z, y)) \mid xS_jy\}$. For classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of frames, their product is the class of $\mathcal{C}_1 \times \mathcal{C}_2 = \{\mathcal{F}_1 \times \mathcal{F}_2 \mid \mathcal{F}_1 \in \mathcal{C}_1, \mathcal{F}_2 \in \mathcal{C}_2\}$. Let $\mathbf{L}(\mathcal{C})$ be the set of all formulas which are valid in every modal frame in $\mathcal{C}$ for a class $\mathcal{C}$ of modal frames. Then the *product* of modal logics L_1 and L_2, denoted by $L_1 \times L_2$, are defined as $\mathbf{L}(\mathbf{F}(L_1) \times \mathbf{F}(L_2))$.

In both of fusions and products, while semantical properties for these logics have been studied considerably, proof-theoretical properties have not been fully discussed yet. In the following, we first present previous results on fusions briefly. Then we will discuss proof-theoretical properties of fusions.

Semantical properties of fusions

Fusions of modal logics have been studied since the later half of 20th century. The first explicit result about fusions was obtained by Thomason [47]. In that paper, it is shown that fusions of consistent modal logics turn out to be a conservative extensions of each of monomodal fragments. Many of preservation results under fusions are studied by M. Kracht and F. Wolter [23], K. Fine and G. Schurz [8], V. Goranko and S. Passy

[12], etc. All of these papers are technically based on Kripke semantics and prove general results only on logics which are complete with respect to Kripke frames. For instance, transfer theorem of decidability and interpolation property of Kripke complete logics is proved independently in [8] and [23].

A structure $\langle A, \wedge, \vee, ', \top, \bot, \Box \rangle$ is a modal algebra if $\langle A, \wedge, \vee, ', \top, \bot \rangle$ is a Boolean algebra and $\Box$ is an operator on A satisfying $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Box\top = \top$. A structure $\langle A, \wedge, \vee, ', \top, \bot, \Box, \blacksquare \rangle$ is a *bimodal algebra*, if both $\langle A, \wedge, \vee, ', \top, \bot, \Box \rangle$ and $\langle A, \wedge, \vee, ', \top, \bot, \blacksquare \rangle$ are modal algebra. Bimodal algebras can be represented by *bimodal general frames* $(W, R_\Box, R_\blacksquare, P)$, where W is a set, $R_\Box$ and $R_\blacksquare$ are binary relations on W and $P \subseteq 2^W$ is a system of sets closed under complementation, intersection, and $\Box X = \{u \mid \forall v(uR_\Box v \text{ implies } v \in X)\}$ and $\blacksquare X = \{u \mid \forall v(uR_\blacksquare v \text{ implies } v \in X)\}$. If $P = 2^W$, then $(W, R_\Box, R_\blacksquare, P)$ is equivalent to a bimodal frame $(W, R_\Box, R_\blacksquare)$. Let $\mathcal{C}$ be a class of bimodal general frames. Then L is $\mathcal{C}$-persistent if, for all $(W, R_\Box, R_\blacksquare, P) \in \mathcal{C}$, $(W, R_\Box, R_\blacksquare, P) \models L$ implies $(W, R_\Box, R_\blacksquare) \models L$.

Proposition 4.1 *Suppose that $\mathcal{C}$ is a class of general bimodal frames. Let $\mathcal{C}_\Box$ and $\mathcal{C}_\blacksquare$ be sets $\{(W, R_\Box, P) \mid (W, R_\Box, R_\blacksquare, P) \in \mathcal{C}\}$ and $\{(W, R_\blacksquare, P) \mid (W, R_\Box, R_\blacksquare, P) \in \mathcal{C}\}$, respectively. Then if L_1 is $\mathcal{C}_\Box$-persistent and L_2 is $\mathcal{C}_\blacksquare$-persistent, $L_1 \otimes L_2$ is $\mathcal{C}$-persistent.*

The general frames satisfying the following (GF1), (GF2), (GF3) and (GF4) are called *differentiated*, *$\Box$-tight*, *$\blacksquare$-tight* and *compact*, respectively.

(GF1) $\forall u, v \in W(u = v \text{ iff } \forall X \in P(u \in X \text{ iff } v \in X))$

(GF2) $\forall u, v \in W(uR_\Box v \text{ iff } \forall X \in P(u \in \Box X \text{ implies } v \in X))$

(GF3) $\forall u, v \in W(uR_\blacksquare v \text{ iff } \forall X \in P(u \in \blacksquare X \text{ implies } v \in X))$

(GF4) $\forall \mathcal{X} \subseteq P, \forall \mathcal{Y} \subseteq \bar{P}$
$(\bigcap(\mathcal{X}' \cup \mathcal{Y}') \neq \emptyset, \mathcal{X}' \subseteq_{fin} \mathcal{X}, \mathcal{Y}' \subseteq_{fin} \mathcal{Y} \text{ imply } \bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset)$

A bimodal general frame $(W, R_\Box, R_\blacksquare, P)$ is called *refined* if the frame is differentiated, $\Box$-tight and $\blacksquare$-tight, and *descriptive* if the frame is refined and compact. The classes of refined frames and descriptive frames are denoted by $\mathcal{R}$ and $\mathcal{D}$, respectively. Descriptive frames are frames which are representations of modal algebras; i.e. for any descriptive frame $\mathcal{F}$, $\mathcal{F} \cong (\mathcal{F}^+)_+$. A frame $\mathcal{F}$ is called *canonical* for L if it is the representation of a generated L-algebra. A logic L is *canonical* if it is persistent with respect to its canonical frames. Then, the following theorem on the persistency and the fundamental properties is shown in [23].

Theorem 4.2 *Suppose that $\bot \notin L_1, L_2$. Then followings hold.*

(1) *$L_1 \otimes L_2$ is axiomatizable iff both L_1 and L_2 are axiomatizable.*

(2) *$L_1 \otimes L_2$ is $\mathcal{R}$-persistent iff both L_1 and L_2 are $\mathcal{R}$-persistent.*

(3) *$L_1 \otimes L_2$ is $\mathcal{D}$-persistent iff both L_1 and L_2 are $\mathcal{D}$-persistent.*

(4) *$L_1 \otimes L_2$ is canonical iff both L_1 and L_2 are canonical.*

(5) *$L_1 \otimes L_2$ is complete iff both L_1 and L_2 are complete.*

(6) *$L_1 \otimes L_2$ has the finite model property iff both L_1 and L_2 have their finite model property.*

By Theorem 4.2, fusions of axiomatizable modal logics with the finite model property are decidable. By [23], whenever both modal logics L_1 and L_2, not containing $\bot$, are complete, the fusion $L_1 \otimes L_2$ is decidable if both L_1 and L_2 are decidable. As shown in the present section, many of semantical properties are preserved under fusions.

4.2 Finite model property of fusions with some interdependent axioms

Some preservation results on cut elimination and cut restriction properties of the sequent systems for fusions of modal logics are discussed above. As for multimodal logics with interdependent axioms, however, it is not so easy to find sequent systems either with cut elimination property or with cut restriction property. In this section, we make an attempt to study bimodal logics defined as fusions of monomodal logics by adding interdependent axioms of the form $\tau_i\varphi \to \sigma_i\varphi$ where both τ_i and σ_i are sequences of modalities for $i \in I$.

Proposition 4.3 *If both L_1 and L_2 are canonical monomodal logics, then bimodal logics of the form $L_1 \otimes L_2 \oplus \{\tau_i p \to \sigma_i p \mid i \in I\}$, where both τ_i and σ_i are sequences of their modalities, are Kripke complete.*

For a sequent τ of modalities, the kth modality from the left is denoted by $\tau^{(k)}$. Then for any frame $\mathcal{F} = (W, R)$, the following holds.

$$\mathcal{F} \models \tau\varphi \to \sigma\varphi \iff \forall u \forall v (u R_{\sigma^{(1)}} \circ \cdots \circ R_{\sigma^{(n)}} v \to u R_{\tau^{(1)}} \circ \cdots \circ R_{\tau^{(m)}} v) \qquad \text{(C1)}$$

Interdependent axioms cause a lot of difficulties, and therefore the study of them has not been developed well. As a small attempt to the study in this direction,we will show the finite model property of bimodal logics discussed in the above theorem when both L_1 and L_2 are **S4**. For the logic which is a normal extension of **S4**, a formula obtained by increasing or decreasing some sequences of same modalities is equivalent to the original one. As an example, $\mathbf{S4}_{\square} \otimes \mathbf{S4}_{\blacksquare} \oplus \{\square\square\blacksquare\varphi \to \blacksquare\blacksquare\square\varphi\} = \mathbf{S4}_{\square} \otimes \mathbf{S4}_{\blacksquare} \oplus \{\square\blacksquare\varphi \to \blacksquare\square\varphi\}$.

To show the finite model property, we will use the filtration method. Suppose that a model $(W, R_{\square}, R_{\blacksquare}, \models)$ and a finite set $\Psi(\varphi)$ of formulas which contains $Sub(\varphi)$ are given. We define a binary relation $\sim$ on W as follows:

$$u \sim v \quad \text{iff} \quad \text{for any } \chi \in \Psi(\varphi),\ u \models \chi \text{ iff } v \models \chi.$$

Clearly $\sim$ is equivalence relation. The set $W/\sim$ of all equivalence classes with respect to $\sim$ is finite, since $\Psi(\varphi)$ is finite. Let $[u]$ denote the equivalence class of u, i.e. $[u] = \{x \in W | u \sim x\}$. Let $\alpha \in \{\square, \blacksquare\}$. Here, we prove the finite model property by means of adopting the following coarsest filtration.

Coarsest filtration: $[u]S_{\alpha}[v]$ iff for any $\alpha\psi \in \Psi(\varphi)$, $u \models \alpha\psi$ implies $v \models \psi$.

Theorem 4.4 *Each bimodal logic of the form* $\mathbf{S4}_{\square} \otimes \mathbf{S4}_{\blacksquare} \oplus \{\tau_i p \to \sigma_i p \mid i \in L\}$, *where both* τ_i *and* σ_i *is sequences of* $\square$ *and* $\blacksquare$, *has the finite model property.*

Let $L = \mathbf{S4}_{\square} \otimes \mathbf{S4}_{\blacksquare} \oplus \{\tau p \to \sigma p\}$ where τ and σ are m and n modalities, respectively. We note that L is complete with respect to the frame $(W, R_{\square}, R_{\blacksquare})$ where both $R_{\square}$ and $R_{\blacksquare}$ are reflexive and transitive and satisfies the condition (C1). Suppose $\varphi \notin L$. Then by the completeness theorem, there exists a L-model $(W, R_{\square}, R_{\blacksquare}, \models)$ such that for some $u_0 \in W$, $u_0 \not\models \varphi$. Now we define $\Psi(\varphi)$ as follows:

$$\Psi_1 = Sub(\varphi),$$
$$\Psi_2 = \{\square\square\psi \mid \square\psi \in \Psi_1\},$$
$$\Psi_3 = \{\blacksquare\blacksquare\psi \mid \blacksquare\psi \in \Psi_1\},$$
$$\Psi_4 = Sub(\{\sigma\chi | \tau\chi \in \Psi_1\}),$$
$$\Psi_5 = \{\tau^{(1)}\tau\psi \mid \tau\psi \in \Psi_4\},$$

$$\Psi(\varphi) = \bigcup_{i=1}^{5} \Psi_i.$$

It is easily seen that $\Psi(\varphi)$ is finite.

$$[u]S_{\square}[v] \quad \text{iff} \quad \text{for any } \square\psi \in \Psi(\varphi), u \models \square\psi \text{ implies } v \models \psi,$$
$$[u]S_{\blacksquare}[v] \quad \text{iff} \quad \text{for any } \blacksquare\psi \in \Psi(\varphi), u \models \blacksquare\psi \text{ implies } v \models \psi,$$
$$[u] \models^* p \quad \text{iff} \quad u \models p.$$

Now let us, consider the following model:

$(W/\sim, S_{\square}, S_{\blacksquare}, \models^*)$.

If the number of formulas in $\Psi(\varphi)$ is m, then the number of elements of $W/\sim$ is at most 2^m. So $W/\sim$ is a finite set. Moreover, we can also the following.

Lemma 4.5

(1) For any $u, v \in W$, $uR_{\Box}v$ implies $[u]S_{\Box}[v]$.

(2) For any $u, v \in W$, $uR_{\blacksquare}v$ implies $[u]S_{\blacksquare}[v]$.

(3) For any $u \in W$, $[u]S_{\Box}[u]$

(4) For any $u, v, w \in W$, $[u]S_{\Box}[v]$ and $[v]S_{\Box}[w]$ implies $[u]S_{\Box}[w]$.

(5) For any $u \in W$, $[u]S_{\blacksquare}[u]$

(6) For any $u, v, w \in W$, $[u]S_{\blacksquare}[v]$ and $[v]S_{\blacksquare}[w]$ implies $[u]S_{\blacksquare}[w]$.

(7) For any $u, v \in W$, $[u]S_{\sigma}[v]$ implies $[u]S_{\tau}[v]$.

Proof. Since we can ascertain that both $S_{\Box}$ and $S_{\blacksquare}$ are reflexive and transitive easily, we show here that $S_{\Box}$ and $S_{\blacksquare}$ satisfy the condition (C1). Assume that $uS_{\sigma^{(1)}} \circ \cdots \circ S_{\sigma^{(n)}}v$ and, for all $\tau\psi \in \Psi(\varphi)$, $u \models \tau\psi$.

The case of $\tau\psi \in \Psi_1$. If $u \models \tau\psi$, then $u \models \sigma\psi$. By the assumption $uS_{\sigma^{(1)}} \circ \cdots \circ S_{\sigma^{(n)}}v$, $v \models \psi$ since $\sigma\psi \in \Psi_4$.

The case of $\tau\psi \in \Psi_4$. This case arises if $\tau^{(1)} = \sigma^{(1)}$. If $m > n$ then $\tau\psi$ is of the form $\sigma\tau'\psi$ for a sequence τ' of modalities. Since $uS_{\sigma^{(1)}} \circ \cdots \circ S_{\sigma^{(n)}}v$ and $\sigma\tau'\psi \in \Psi_4$, $v \models \tau'\psi$. Thus $v \models \psi$ since both $S_{\Box}$ and $S_{\blacksquare}$ are reflexive. If $m < n$ then $\tau\psi$ is of the form $\tau'\sigma\alpha\psi$ where τ' is a sequence of modalities and α is $\sigma^{(n-1)}$ or null sequence. If $u \models \tau'\sigma\alpha\psi$ then $u \models \sigma\alpha\psi$ by reflexivity of $S_{\Box}$ and $S_{\blacksquare}$. Then $u \models \alpha\psi$ since $uS_{\sigma^{(1)}} \circ \cdots \circ S_{\sigma^{(n)}}v$ and $\sigma\alpha\psi \in \Psi_4$. Therefore $u \models \psi$ by reflexivity for α. ∎

We can see that the model $(W/\sim, S_{\Box}, S_{\blacksquare}, \models^*)$ is L-model by Lemma 4.5. Now, the finite model property of L is derived by combining the following lemma.

Lemma 4.6 *If $\psi \in \Psi(\varphi)$, then for any $u \in W$, $u \models \psi$ iff $[u] \models^* \psi$.*

Proof. We will prove this by induction on the formation of ψ.
The case where ψ is a propositional variables is given by the definition of $\models^*$. The case where ψ is of the form $\chi_1 \wedge \chi_2$, $\chi_1 \vee \chi_2$ or $\chi_1 \supset \chi_2$ is straightforward. The case where $\psi = \Box\chi$. [$\Rightarrow$] Suppose that $u \models \Box\chi$. If $[u]S_{\Box}[v]$ then $v \models \chi$ since $\Box\chi \in \Psi(\varphi)$. By the induction hypothesis $v \models^* \chi$. Hence $[u] \models^* \Box\chi$. [$\Leftarrow$] Suppose $[u] \models^* \Box\chi$. If $uR_{\Box}v$ then $[u]S_{\Box}[v]$ by Lemma 4.5 (1). Since $[u]S_{\Box}[v]$ and $[u] \models^* \Box\chi$, and so $[v] \models^* \chi$. By induction hypothesis $v \models \chi$. Thus $u \models \Box\chi$.

The case where $\psi = \blacksquare\chi$ can be shown in the similar way. This time we use Lemma 4.5 (2) instead of (1). ∎

By the above, therefore $u_0 \models \varphi$ iff $[u_0] \models^* \varphi$. Thus, there exists a finite L-model $(W/\sim, S_{\Box}, S_{\blacksquare}, \models^*)$ such that for some $[u_0] \in W/\sim$, $[u_0] \not\models^* \varphi$.

4.3 Cut elimination and cut restriction properties of fusions

In this section, proof-theoretical properties of Gentzen type sequent systems for fusions of well-known monomodal logics are discussed.

While semantical studies of fusions have been developed much since the later half of the 20th century, not so much work have been done for proof-theoretical studies of fusions. From the point of view of constructing theorem provers and implementing them, proof-theoretical approach to fusions will be also desirable. As a proof-theoretical approach, here we will be mainly concerned with Gentzen type sequent systems for fusions. In the following, we will consider fusions of modal logics with some of axioms T, D, 4, 5 or B. Their sequent systems can be obtained from the sequent system **LK** for classical propositional logic simply by adding both of the rules for components of the fusion. For example, the system $\mathcal{S}(\mathbf{KD45}_{\Box} \otimes \mathbf{S5}_{\blacksquare})$ consists of rules of **LK**,

$$\frac{\Box\Gamma, \Gamma \Rightarrow \Box\Delta, \Theta}{\Box\Gamma \Rightarrow \Box\Delta, \Box\Theta}\,(SR5)^{D}_{\Box}, \quad \frac{\blacksquare\Gamma, \Gamma \Rightarrow \blacksquare\Delta, \varphi}{\blacksquare\Gamma \Rightarrow \blacksquare\Delta, \blacksquare\varphi}\,(SR5)_{\blacksquare} \quad \text{and} \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\blacksquare\varphi, \Gamma \Rightarrow \Delta}\,(SR2)_{\blacksquare},$$

where Θ consists of at most one formula. Other sequent systems for fusions can be also constructed in the similar way. We can show easily that the following Proposition 4.7 holds.

Proposition 4.7 *Let L_1 and L_2 be of the form $\mathbf{K} \oplus Q$ where $Q \subseteq \{T, D, 4, 5, B\}$. Then $L_1 \otimes L_2 \vdash \varphi$ iff $\mathcal{S}(L_1 \otimes L_2) \vdash \Rightarrow \varphi$.*

The rest of this section presents the proof-theoretical property for fusions. Then the following holds.

Theorem 4.8 (cut-elimination theorem for fusions) *Let L_1 and L_2 be any of* **K**, **K4**, **K45**, **KT**, **S4**, **KD**, **KD4** *and* **KD45**. *Then the sequent system $\mathcal{S}(L_1 \otimes L_2)$ for each fusion of their logics has the cut elimination property.*

This theorem can be proved by Gentzen's method for any logics of them, and similarly to the case with the each logic since each modality is independent of another one. Some difficulties occur when a logic contains the axiom 5. The cut elimination theorem for **K45** and **KD45** is proved in [41]. The sequent system $\mathcal{S}(\mathbf{S5})$ defined above lacks the cut elimination property, but M. Sato introduced a cut-free Gentzen type sequent system for the modal logic S5 by means of eliminating the visible cut one by one [38]. As to the logics **K5** and **KD5**, no cut-free systems are known yet.

An important consequence of cut elimination theorem for a sequent system $\mathcal{S}$ is the subformula property. That is, if a sequent $\Gamma \Rightarrow \Delta$ is provable in $\mathcal{S}$ then it has a proof which consists only of sequents containing subformulas of formulas in $\Gamma \cup \Delta$. In fact, most of outcomes of cut elimination theorem, including decidability, can be derived from the subformula property. In standard sequent systems, only cut rule violates the subformula property, i.e. the cut formula in a given application of cut rule may not appear in the lower sequent. In other words, if we can restrict any application of cut rule to the following way,

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$\text{where } \varphi \in Sub(\Gamma \cup \Pi \cup \Delta \cup \Sigma)$$

then we can obtain the subformula property. We call such cut rule, *acceptable cut*. Takano succeeded to show that for some of sequent systems for modal logics, including **KB**, **KTB**, **KDB**, **KB4**, **S5**, every provable sequent has a proof in which every rule is acceptable [42, 43]. In [44], he show the cut restriction property for **K5** and **KD5** by using extension of the acceptable cut rule. It is called that the sequent system $\mathcal{S}(L)$ has the *cut restriction property*, if every proof in $\mathcal{S}(L)$ can be transformed, without changing the end sequent, into the proof in which every cut rule applied in it is acceptable. Note that every system with the cut elimination property has the cut restriction property. By means of some method of derivation of the cut restriction property for **KB**, **KTB**, **KDB**, **KB4**, **S5**, **K5** and **KD5** which was shown by Takano, the following cut restriction theorem can be shown.

Theorem 4.9 (cut-restriction theorem for fusions) *Let L_1 and L_2 be of the form $\mathbf{K} \oplus Q$ where $Q \subseteq \{T, D, 4, 5, B\}$. Then the sequent system $\mathcal{S}(L_1 \otimes L_2)$ for each fusion of these logics has the (extended) cut restriction property, and therefore has the (extended) subformula property.*

This theorem can be shown through Kripke semantics; i.e. for logics L_1 and L_2 which are of the form $\mathbf{K} \oplus Q$ where $Q \subseteq \{T, D, 4, 5, B\}$, the restricted system for $L_1 \otimes L_2$ is determined by $L_1 \otimes L_2$-frame, and similarly to the case with the each logic since each modality is independent of another one. As important corollaries of the subformula property of sequent systems for modal logics discussed above, we can show Craig's interpolation property and decidability of them. For Craig's interpolation property, we can use Maehara's method (see [34] for details).

Corollary 4.10 (Craig's interpolation property) *Let L_1 and L_2 be of the form* $\mathbf{K}\oplus Q$ *where* $Q \subseteq \{T, D, 4, 5, B\}$*. If* $\varphi \to \psi$ *is provable in* $\mathcal{S}(L_1 \otimes L_2)$*, then there exists a formula* χ *such that*

1) $\varphi \to \chi$ *and* $\chi \to \psi$ *are both provable in* $\mathcal{S}(L_1 \otimes L_2)$
2) $V(\chi) \subseteq V(\varphi) \cap V(\psi)$.

Corollary 4.11 (decidability) *Let L_1 and L_2 be of the form* $\mathbf{K}\oplus Q$ *where* $Q \subseteq \{T, D, 4, 5, B\}$*. Then the system* $\mathcal{S}(L_1 \otimes L_2)$ *is decidable.*

These results can be obviously extended to the system $\mathcal{S}(\bigotimes_{i=1}^{n} L_i)$ $(n > 2)$ as long as each L_i is among modal logics discussed above.

4.4 Sequent systems for fusions with some interdependent axioms

Due to T. Shimura's suggestions, we introduce here sequent systems for fusions with an interdependent axiom either of the form $\Box\varphi \to \blacksquare\varphi$ or of the form $\Box\varphi \to \blacksquare\Box\varphi$ and show cut elimination theorem. Some of the applications to temporal epistemic logics will be discussed at the end of this section. Here, we consider fusions of modal logics **K**, **KT**, **K4** and **S4**. The next proposition is useful, while it can be easily shown.

Proposition 4.12

1. *Let L be a normal bimodal logics* $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\varphi\}$*. If* $\blacksquare\psi \to \psi \in L$*, then* $\Box\psi \to \psi \in L$*. In other words, if* $\blacksquare\psi \to \psi \in L_{2\blacksquare}$*, then* $\Box\psi \to \psi \in L_{1\Box}$*.*
2. *Let L be a normal bimodal logics* $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\Box\varphi\}$*. If* $\Box\psi \to \psi \in L_1$*, then* $\Box\psi \to \blacksquare\psi \in L$*.*
3. *Let L be a normal bimodal logics* $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\varphi\}$*. If* $\Box\psi \to \Box\Box\psi \in L_1$*, then* $\Box\psi \to \blacksquare\Box\psi \in L$*.*

To get cut-free sequent systems for fusions with axioms $\Box\varphi \to \blacksquare\varphi$ and $\Box\varphi \to \blacksquare\Box\varphi$, we introduce following rules.

$$\frac{\Xi, \Gamma \Rightarrow \Theta}{\Box\Xi, \blacksquare\Gamma \Rightarrow \blacksquare\Theta}\ (SR1)^{\sharp}_{\blacksquare} \qquad \frac{\Xi, \blacksquare\Gamma, \Gamma \Rightarrow \Theta}{\Box\Xi, \blacksquare\Gamma \Rightarrow \blacksquare\Theta}\ (SR3)^{\sharp}_{\blacksquare}$$

$$\frac{\Box\Xi, \Gamma \Rightarrow \Theta}{\Box\Xi, \blacksquare\Gamma \Rightarrow \blacksquare\Theta}\ (SR1)^{\sharp\sharp}_{\blacksquare} \qquad \frac{\Box\Xi, \blacksquare\Gamma, \Gamma \Rightarrow \Theta}{\Box\Xi, \blacksquare\Gamma \Rightarrow \blacksquare\Theta}\ (SR3)^{\sharp\sharp}_{\blacksquare}$$

$$\frac{\Box\Xi, \Xi, \Gamma \Rightarrow \Theta}{\Box\Xi, \blacksquare\Gamma \Rightarrow \blacksquare\Theta}\ (SR1)^{\sharp\sharp\sharp}_{\blacksquare} \qquad \frac{\Box\Xi, \Xi, \blacksquare\Gamma, \Gamma \Rightarrow \Theta}{\Box\Xi, \blacksquare\Gamma \Rightarrow \blacksquare\Theta}\ (SR3)^{\sharp\sharp\sharp}_{\blacksquare}$$

For $L \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}\}$, let $\mathcal{S}^I(L_\blacksquare)$ be the sequent system obtained from $\mathcal{S}(L_\blacksquare)$ by replacing $(SR1)_\blacksquare$ and $(SR3)_\blacksquare$ in rules of $\mathcal{S}(L_\blacksquare)$ by $(SR1)^I_\blacksquare$ and $(SR3)^I_\blacksquare$, respectively, where I is any of $\sharp$, $\sharp\sharp$ and $\sharp\sharp\sharp$. Then $\mathcal{S}^I(L_{1\Box} \otimes L_{2\blacksquare})$ is the sequent system obtained from the sequent system **LK** by adding rules for modal operators in both of $\mathcal{S}(L_{1\Box})$ and $\mathcal{S}^I(L_{2\blacksquare})$. For example, the system $\mathcal{S}^{\sharp\sharp}(\mathbf{K4}_\Box \otimes \mathbf{S4}_\blacksquare)$ consists of rules of **LK**,

$$\frac{\Box\Gamma, \Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\,(SR3)_\Box, \quad \frac{\Box\Xi\blacksquare\Gamma, \Gamma \Rightarrow \varphi}{\Box\Xi, \blacksquare\Gamma \Rightarrow \blacksquare\varphi}\,(SR3)^{\sharp\sharp}_\blacksquare \quad \text{and} \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\blacksquare\varphi, \Gamma \Rightarrow \Delta}\,(SR2)_\blacksquare,$$

I. Fusions with $\Box\varphi \to \blacksquare\Box\varphi$

First, we consider normal bimodal logics of the form $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\Box\varphi\}$. By Proposition 4.12 *3.*, we can show the following equivalences.

$$\mathbf{KT}_\Box \otimes \mathbf{K4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\Box\varphi\} = \mathbf{S4}_\Box \otimes \mathbf{K4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\Box\varphi\}$$

$$\mathbf{KT}_\Box \otimes \mathbf{S4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\Box\varphi\} = \mathbf{S4}_\Box \otimes \mathbf{S4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\Box\varphi\}$$

The next proposition can be shown easily.

Proposition 4.13 *Let L_1 and L_2 be any of* **K**, **KT**, **K4** *and* **S4**. *Then $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\Box\varphi\} \vdash \psi$ iff $\mathcal{S}^{\sharp\sharp}(L_{1\Box} \otimes L_{2\blacksquare}) \vdash \Rightarrow \psi$.*

Theorem 4.14 (cut elimination theorem) *Let L_1 and L_2 be any of* **K**, **KT**, **K4** *and* **S4**. *Then the sequent system $\mathcal{S}^{\sharp\sharp}(L_{1\Box} \otimes L_{2\blacksquare})$ has the cut elimination property.*

This theorem can be shown in usual way by Gentzen's method, which is summarized in following table. In the table, ○ means that cut elimination theorem holds, and show that cut elimination theorem follows from the above equivalence.

	$\mathbf{K}_\blacksquare$	$\mathbf{KT}_\blacksquare$	$\mathbf{K4}_\blacksquare$	$\mathbf{S4}_\blacksquare$
$\mathbf{K}_\Box$	○	○	○	○
$\mathbf{KT}_\Box$	○	○	○	○
$\mathbf{K4}_\Box$	○	○	○	○
$\mathbf{S4}_\Box$	○	○	○	○

II. Fusions with $\Box\varphi \to \blacksquare\varphi$

Next, we consider normal bimodal logics with the axiom $\Box\varphi \to \blacksquare\varphi$. By Proposition 4.12, we can show the following eqivalences.

$$\mathbf{KT}_\Box \otimes \mathbf{KT}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\} = \mathbf{K}_\Box \otimes \mathbf{KT}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\}$$

$$\mathbf{KT}_\Box \otimes \mathbf{S4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\} = \mathbf{K}_\Box \otimes \mathbf{S4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\}$$

$$\mathbf{KT}_\Box \otimes \mathbf{K}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\Box\varphi\} = \mathbf{S4}_\Box \otimes \mathbf{K}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\}$$

$$\begin{aligned}\mathbf{KT}_\Box \otimes \mathbf{KT}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\Box\varphi\} &= \mathbf{S4}_\Box \otimes \mathbf{KT}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\} \\ &= \mathbf{K4}_\Box \otimes \mathbf{KT}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\}\end{aligned}$$

$$\mathbf{KT}_\Box \otimes \mathbf{K4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\Box\varphi\} = \mathbf{S4}_\Box \otimes \mathbf{K4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\}$$

$$\begin{aligned}\mathbf{KT}_\Box \otimes \mathbf{S4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\Box\varphi\} &= \mathbf{S4}_\Box \otimes \mathbf{S4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\} \\ &= \mathbf{K4}_\Box \otimes \mathbf{S4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\}\end{aligned}$$

Proposition 4.15

1. *Let L_1 and L_2 be any of* **K** *and* **KT**. *Then,* $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\varphi\} \vdash \psi$ *iff* $\mathcal{S}^\sharp(L_{1\Box} \otimes L_{2\blacksquare}) \vdash\Rightarrow \psi$
2. *Let L_3 be either* **K** *or* **K4**. *Then,* $\mathbf{K4}_\Box \otimes L_{3\blacksquare} \oplus \{\Box\varphi \to \blacksquare\varphi\} \vdash \psi$ *iff* $\mathcal{S}^{\sharp\sharp\sharp}(\mathbf{K4}_\Box \otimes L_{3\blacksquare}) \vdash\Rightarrow \psi$.

Theorem 4.16 (cut elimination theorem) *Let L_1 and L_2 be any of* **K** *and* **KT**, *and L_3 either* **K** *or* **K4**. *Then,* $\mathcal{S}^\sharp(L_{1\Box}\otimes L_{2\blacksquare})$ *and* $\mathcal{S}^{\sharp\sharp\sharp}(\mathbf{K4}_\Box\otimes L_{3\blacksquare})$ *have the cut elimination property.*

As for bimodal logics $\mathbf{K}_\Box \otimes \mathbf{K4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\}$, $\mathbf{KT}_\Box \otimes \mathbf{K4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\}$ and $\mathbf{KT}_\Box \otimes \mathbf{S4}_\blacksquare \oplus \{\Box\varphi \to \blacksquare\varphi\}$, any of sequent systems introduced similarly as above lack the cut elimination property. Thus, we can summarize these results as shown in the next table. Here, ☆ means that we don't have cut-free sequent systems with subformula property for it yet.

	$\mathbf{K}_{\blacksquare}$	$\mathbf{KT}_{\blacksquare}$	$\mathbf{K4}_{\blacksquare}$	$\mathbf{S4}_{\blacksquare}$
$\mathbf{K}_{\Box}$	○	○	☆	- -
$\mathbf{KT}_{\Box}$	○	○	☆	☆
$\mathbf{K4}_{\Box}$	○	○	○	○
$\mathbf{S4}_{\Box}$	○	○	○	○

As for bimodal logics $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\varphi,\ \Box\varphi \to \blacksquare\Box\varphi\}$ where $L_1, L_2 \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}\}$, we can show cut elimination property by using rules $(SR1)^{\sharp\sharp}_{\blacksquare}$, $(SR1)^{\sharp\sharp\sharp}_{\blacksquare}$, $(SR3)^{\sharp\sharp}_{\blacksquare}$ and $(SR3)^{\sharp\sharp\sharp}_{\blacksquare}$.

Some interdependent axioms is useful for formalization in many applied fields. Now, we refer to applications of logics in this section into temporal epistemic logics. As account in next section, K_α and K_β are interpreted as "agent α knows φ" and "agent β knows φ", and $[F]$ is interpreted as "at all future times, φ". Then, $\mathsf{K}_\alpha\varphi \to \mathsf{K}_\beta\varphi$ means that "agent β knows what agent α knows", and $\mathsf{K}_\alpha\varphi \to [F]\mathsf{K}_\alpha\varphi$ says that "in the future agent α will know always what α knows now".

4.5 Fusions of temporal and epistemic logics

Among many applications of fusions, epistemic logics incorporated with temporal notions are useful, since informations (knowledge or belief) which an agent possesses change in the flow of time. These logics are in fact used for formalizing frameworks of various fields of computer science, in particular of artificial intelligence.

Formalization of multi-agent systems using logical framework has served as an important bridge between communities of distributed artificial intelligence; one is the symbolic logician's camp and the other is social behavioral camp [48]. As to the former, BDI model has been studied as modal logics with three modalities for *intention*, *belief* and *desire* [36, 40, 25]. These logics have been studied especially as models of recognizing outside world for an agent embedded in a situation [4], and have been applied in *situation* (*cannel*) theory in information science [1]. If we try to build a computer systems for multi-agent systems, the most important issue would be how we formalize the change of states, how we synchronize the actions of agents on linear or branching time, and how we assure soundness and consistency of their systems. In such a situation, it will be necessary to consider logics that are obtained by combining two kinds of modal logics; epistemic logics which are logics of knowledge and belief and temporal logics which handle changes of time. In this section, we will introduce two temporal epistemic logics. They are logics of knowledge and belief over the temporal logic K_t.

To formalize multi-agent models, we need to introduce such modal operators as:

$\mathsf{B}_\alpha, \mathsf{B}_\beta, \cdots$	belief of $\alpha, \beta, \cdots$
$\mathsf{K}_\alpha, \mathsf{K}_\beta, \cdots$	knowledge of $\alpha, \beta, \cdots$
$[F]$ and $[P]$	future and past,

where α, β etc. run over agents. When knowledge and belief of one agent do not directly affect those of the other agents, modal logic of a single agent $\{\mathsf{B}_\alpha, \mathsf{K}_\alpha, [F], [P]\}$ can be easily extended to that of multi-agents with modal operators $\{\mathsf{B}_\alpha, \mathsf{B}_\beta, \cdots, \mathsf{K}_\alpha, \mathsf{K}_\beta, \cdots [F], [P]\}$, simply adding axioms of different modal operators. Therefore, we will discuss mainly temporal epistemic logic with modal operators $\{\mathsf{B}_\alpha, [F], [P]\}$ or $\{\mathsf{K}_\alpha, [F], [P]\}$ in this chapter.

Epistemic logic

Epistemic logics (logics of knowledge and those of belief) have been set to work in philosophy. The aim of employment of their logics was to analyze formal properties of reasoning about knowledge and belief. The possible world semantics for epistemic logics originated in Hintikka [17]. For epistemic logics, recently, their semantics using techniques developed by Kripke [24], that is called Kripke type semantics, is often adopted. Epistemic logics are usually formulated as normal modal logics. It is natural since every agent is wanted to have minimum inferential ability.

A finite set of agent identifiers is denoted by *Agent*. The language of our epistemic logics consists of propositional epistemic operators B_α and K_β where $\alpha, \beta \in Agent$. Epistemic formulas, denoted by Greek small letters, φ, ψ, χ, $\cdots$, are constructed inductively from propositional variables, logical connectives and modal operators in the usual way. Formulas with epistemic operators are read as follows:

$\mathsf{B}_\alpha\varphi$	agent α believes φ
$\mathsf{K}_\beta\varphi$	agent β knows φ

The minimal logic of belief K_B and the minimal logic of knowledge K_K are the least normal modal logics containing moreover following axioms, respectively:

K_B		K_K	
(Ab1)	$\mathsf{B}_\alpha\varphi \to \neg\mathsf{B}_\alpha\neg\varphi$	(Ak1)	$\mathsf{K}_\beta\varphi \to \neg\mathsf{K}_\beta\neg\varphi$
(Ab2)	$\mathsf{B}_\alpha\varphi \to \mathsf{B}_\alpha\mathsf{B}_\alpha\varphi$	(Ak2)	$\mathsf{K}_\beta\varphi \to \mathsf{K}_\beta\mathsf{K}_\beta\varphi$
(Ab3)	$\neg\mathsf{B}_\alpha\varphi \to \mathsf{B}_\alpha\neg\mathsf{B}_\alpha\varphi$	(Ak3)	$\neg\mathsf{K}_\beta\varphi \to \mathsf{K}_\beta\neg\mathsf{K}_\beta\varphi$
		(Ak4)	$\mathsf{K}_\beta\varphi \to \varphi$

where $\alpha, \beta \in Agent$. In epistemic logics, axioms (Ab1) and (Ak1) represent consistency of belief and knowledge, (Ab2) and (Ak2) positive introspective axioms, (Ab3) and (Ak3) negative introspective axioms, and (Ak4) correctness of knowledge. In logics of knowledge with the temporal notion, both (Ak1) and (Ak2) follow from (Ak3) and (Ak4) in the least normal modal logic. Therefore the logic K_K can be defined also as the least normal modal logics containing (Ak3) and (Ak4).

Temporal logic

In applying modal logics, reasoning about time is the most natural and intuitive. By possible world semantics, the flow of time is represented as a frame (W, R), where W is a set of *moments* of time and R a binary relation on W. This semantics can easily represent transitivity, reflexive/irreflexive, connected/non-connected. Here, we adopt two modal operators, called *necessity* operators, which are the future operator $[F]$ and the past operator $[P]$. These necessity operators in such frames are interpreted in the following.

$[F]\varphi$	at all future times, φ
$[P]\varphi$	at all past times, φ

Then the *possibility* operators, denoted by $\langle F\rangle$ and $\langle P\rangle$, are $\neg[F]\neg$ and $\neg[P]\neg$, and represent "some time in the future" and "some time in the past", respectively. The minimal temporal logic K_t is the least normal modal logics containing following axioms:

(At1)	$[F]\varphi \to [F][F]\varphi$	(At3)	$\varphi \to [F]\langle P\rangle\varphi$
(At2)	$[P]\varphi \to [P][P]\varphi$	(At4)	$\varphi \to [P]\langle F\rangle\varphi$

Axioms (At1) and (At2) represent transitivity of time, (At3) and (At4) conversion of future and past.

Temporal epistemic logics

Recently, temporal epistemic logics, i.e. epistemic logics with temporal operators, have been used in the axiomatization and specification of some multi-agent systems. For example, AGENT0 system, METATEM processes are discussed in [48]. In this section, we introduced two temporal epistemic logics as Hilbert-type systems, which will serve as basic systems in formalizing multi-agent systems.

Syntax of temporal epistemic logics

First, we fix a finite set *Agent* of agent identifiers. The language of our temporal epistemic logics consists of

- propositional variables: $p, q, r, \cdots$;
- logical connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$;
- modal operators: $[F]$, $[P]$, B_α, K_β (where $\alpha, \beta \in Agent$).

The set of all propositional variables is denoted by $Prop$. Formulas, denoted by φ, ψ, χ, $\cdots$, are constructed inductively from propositional variables, logical connectives and modal operators in the usual way. Formulas $\langle F\rangle\varphi$ and $\langle P\rangle\varphi$ are abbreviations of $\neg[F]\neg\varphi$ and $\neg[P]\neg\varphi$, respectively. The set of all subformulas of a formula φ is denoted by $Sub(\varphi)$.

Using this language, we can construct such formulas as $[P]\mathsf{B}_\alpha\varphi$, $\mathsf{K}_\alpha\langle F\rangle\psi$, $[F]\mathsf{B}_\beta\langle P\rangle\mathsf{B}_\alpha\chi$, etc. These formulas mean intuitively "in the past, an agent α has always believed that φ", "an agent α knows that φ in some future" and "in the future, an agent β will always believe that once an agent α believed φ", respectively. As an example, consider a temporal epistemic state described in Figure 4.1, in which we assume that time is linear and discrete. We will show a difference between $[P]\mathsf{B}_\alpha\varphi$ and $\mathsf{B}_\alpha[P]\varphi$.

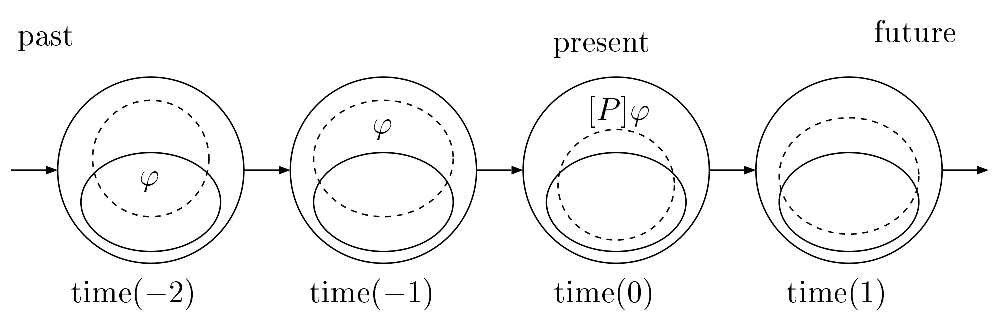


Figure 4.1: Difference in the order of modalities

At each time, the outermost circle represents the set of all propositions, the dotted circle the belief set of the agent α, and the inside oval the set of facts (i.e. true propositions), respectively. Assume that φ is a proposition which is always in the belief set of α for every time(i) when $i < 0$. Then $[P]\mathsf{B}_\alpha\varphi$ is true at time 0. On the other hand, when α realizes that φ is not always true (for example, φ is false at time(-1) as in the figure), the proposition $\mathsf{B}_\alpha[P]\varphi$ becomes false at time 0. The following gives a concrete example which $[P]\mathsf{B}_\alpha\varphi$ is true but $\mathsf{B}_\alpha[P]\varphi$ is not.

α : I
φ : She is faithful to me.

> " I have always believed she is faithful to me. But now that I've seen this photo I no longer believe she always was. "

The interpretations of formulas can be not only represented by figures like above but also discussed explicitly by using mathematical structures like Kripke frames. As is often the case with mathematical logic, we identify a given logic L with the set of all formulas which are provable in L. Following this identification, if a set L of formulas satisfies the following (A1), (A2), (R1) and (R2) for a modal operator $\Box$, then L is called a *normal modal logic*.

(A1) $\{\varphi \mid \varphi \text{ is a propositional tautology }\} \subseteq L$
(A2) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \in L$
(R1) if $\varphi \in L$ and $\varphi \to \psi \in L$, then $\psi \in L$ (modus ponens)
(R2) if $\varphi \in L$, then $\Box\varphi \in L$ (rule of necessitation)

It is natural to assume that temporal and epistemic logics are normal. Each of the minimal temporal logic K_t, the minimal logic K_B of belief and the minimal logic K_K of knowledge is defined to be the least normal modal logics containing each of the following axioms, respectively:

K_t		K_B		K_K	
(At1)	$[F]\varphi \to [F][F]\varphi$	(Ab1)	$\mathsf{B}_\alpha\varphi \to \neg\mathsf{B}_\alpha\neg\varphi$	(Ak1)	$\mathsf{K}_\beta\varphi \to \neg\mathsf{K}_\beta\neg\varphi$
(At2)	$[P]\varphi \to [P][P]\varphi$	(Ab2)	$\mathsf{B}_\alpha\varphi \to \mathsf{B}_\alpha\mathsf{B}_\alpha\varphi$	(Ak2)	$\mathsf{K}_\beta\varphi \to \mathsf{K}_\beta\mathsf{K}_\beta\varphi$
(At3)	$\varphi \to [F]\langle P\rangle\varphi$	(Ab3)	$\neg\mathsf{B}_\alpha\varphi \to \mathsf{B}_\alpha\neg\mathsf{B}_\alpha\varphi$	(Ak3)	$\neg\mathsf{K}_\beta\varphi \to \mathsf{K}_\beta\neg\mathsf{K}_\beta\varphi$
(At4)	$\varphi \to [P]\langle F\rangle\varphi$			(Ak4)	$\mathsf{K}_\beta\varphi \to \varphi$

(Since each of these logics is a normal modal logics, each $[F]$, $[P]$, B_α and K_β must satisfy the above condition (A2).) Through the next section, we will discus temporal epistemic logics $K_t \otimes K_B$ and $K_t \otimes K_K$, which are the least normal modal logics containing K_t and K_B, and K_t and K_K, respectively. Next chapter gives us derivations of logical properties for those temporal epistemic logics.

4.6 Note

For multimodal logics with interdependent axiom, the general arguments are extremely complicated. By [23], the following generalization of Sahlqvist's theorem holds.

> **Theorem.** Let φ be a conjunction of formulas of the form $\tau(\psi_1 \to \psi_2)$, where τ is a sequence of modalities, ψ_1 is positive, and ψ_2 is obtained from propositional variables and constants in such a way that no positive occurrence of a variable is in a subformula of the form $\psi_1 \vee \psi_2$ or $\Diamond_i \psi_1$ within the scope of some $\Box_j$. Then $\mathbf{K} \oplus \{\varphi\}$ is $\mathcal{D}$-persistent.

This theorem gives a general result on Kripke completeness of modal logics with interdependent axioms. On the other hand, we don't have such general results yet on the finite model property and the decidability. Section 4.2 is a trial towards the finite model property for multi modal logics with more general interdependent axioms.

In Section 4.4, we considered proof-theoretical property for fusions with interdependent axioms. Though we discussed the cut elimination theorem for fusions with one of interdependent axioms $\Box\varphi \to \blacksquare\varphi$ and $\Box\varphi \to \blacksquare\Box\varphi$ in that section, we can show that with both of their interdependent axioms.

Let L_1 be either **K** or **KT**, and L_2 either **K4** or **S4**. Then we can not construct sequent systems with subformula property, for $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\varphi\}$ and $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\Box\varphi\}$. For their logics, constructing sequent systems with subformula property is left as future work. As one of methods to do so, cut-free sequent systems for them will be required.

Chapter 5

Temporal epistemic logics as combined systems

In this chapter, for temporal epistemic logics defined in previous section, we show their completeness, finite model property, Craig's interpolation theorem and decidability. Key method for derivation of each properties is construction of sequent systems with their subformula property. In discussion of decidability, we will give an efficient proof-search procedure, which not only decide the provability of a given formula but also give a proof when it is provable.

This chapter is organized as follows. In Section 5.1, we introduce a Kripke type semantics for temporal epistemic logics. Sequent systems for those logics are introduced in Section 5.2. They are naturally obtained from Hilbert-style systems in previous Section. But, cut elimination theorem, from which we usually derive the decidability, does not hold for either of them. Instead of looking for cut-free sequent systems for them, we modify original sequent systems in such a way that the subformula property holds, by restricting cut rule and rules for temporal operators. In Section 5.3, we show the completeness of these restricted systems in a stronger form. That is, any formula which is not provable in one of our sequent systems is not valid in a *finite* model for it. Since the subformula property holds in these systems, we can show Craig's interpolation theorem for them in Section 5.4 by using Maehara's method. Then we give a proof-search procedure for these logics in Section 5.5. In Section 5.6, some concluding remarks will be given.

5.1 Kripke semantics for temporal epistemic logics

Let *Agent* be a set with n elements. A Kripke model for logics of belief with a temporal notion is defined as a $n+4$-tuple $(W, \{R_{\mathsf{B}_\alpha} \mid \alpha \in Agent\}, R_F, R_P, \models)$, where W is a non-empty set, and $R_{\mathsf{B}_\alpha}, R_F$ and R_P are binary relations on W for $\alpha \in Agent$, i.e. $R_{\mathsf{B}_\alpha}, R_F, R_P \subseteq W \times W$, and $\models$ is defined inductively as follows:

(M1)	$u \models \varphi \wedge \psi$	iff	$u \models \varphi$ and $u \models \psi$
(M2)	$u \models \varphi \vee \psi$	iff	$u \models \varphi$ or $u \models \psi$
(M3)	$u \models \varphi \rightarrow \psi$	iff	$u \models \varphi$ implies $u \models \psi$
(M4)	$u \models \neg\varphi$	iff	$u \not\models \varphi$
(M5)	$u \models \mathsf{B}_\alpha\varphi$	iff	for all $v \in W$, $uR_{\mathsf{B}_\alpha}v$ implies $v \models \varphi$
(M6)	$u \models [F]\varphi$	iff	for all $v \in W$, $uR_F v$ implies $v \models \varphi$
(M7)	$u \models [P]\varphi$	iff	for all $v \in W$, $uR_P v$ implies $v \models \varphi$

Since the number of agents doesn't play an essential role in the following discussions, we consider only the case where there is a single agent in the rest of the present chapter. Therefore we can take a model for the logic $K_t \otimes K_B$ as a quintuple $(W, R_{\mathsf{B}_\alpha}, R_F, R_P, \models)$. A Kripke model for temporal logic $K_t \otimes K_K$ of knowledge $(W, R_{\mathsf{K}_\alpha}, R_F, R_P, \models)$ is defined similarly. In this case, it suffices to replace (M5) with the following (M5)′:

(M5)′ $u \models \mathsf{K}_\alpha\varphi$ iff for all $v \in W$, $uR_{\mathsf{K}_\alpha}v$ implies $v \models \varphi$

For $\Box \in \{\mathsf{B}_\alpha, \mathsf{K}_\alpha\}$, a formula φ is *true in model* $\mathcal{M} = (W, R_\Box, R_F, R_P, \models)$, denoted by $\mathcal{M} \models \varphi$, if $u \models \varphi$ for every $u \in W$. Now, the following hold.

Proposition 5.1 (Correspondence theory) *Let $\mathcal{M} = (W, R_{\mathsf{B}_\alpha}, R_F, R_P, \models)$ be a model for temporal logics of belief. Then the following holds.*

(1)	$\mathcal{M} \models \mathsf{B}_\alpha\varphi \rightarrow \neg\mathsf{B}_\alpha\neg\varphi$	*iff*	$\forall u \exists v (uR_{\mathsf{B}_\alpha}v)$	(CR1)
(2)	$\mathcal{M} \models \mathsf{B}_\alpha\varphi \rightarrow \mathsf{B}_\alpha\mathsf{B}_\alpha\varphi$	*iff*	$\forall u, v, w(uR_{\mathsf{B}_\alpha}v \wedge vR_{\mathsf{B}_\alpha}w \rightarrow uR_{\mathsf{B}_\alpha}w)$	(CR2)
(3)	$\mathcal{M} \models \neg\mathsf{B}_\alpha\varphi \rightarrow \mathsf{B}_\alpha\neg\mathsf{B}_\alpha\varphi$	*iff*	$\forall u, v, w(uR_{\mathsf{B}_\alpha}v \wedge uR_{\mathsf{B}_\alpha}w \rightarrow vR_{\mathsf{B}_\alpha}w)$	(CR3)
(4)	$\mathcal{M} \models [F]\varphi \rightarrow [F][F]\varphi$	*iff*	$\forall u, v, w(uR_F v \wedge vR_F w \rightarrow uR_F w)$	(CR4)
(5)	$\mathcal{M} \models [P]\varphi \rightarrow [P][P]\varphi$	*iff*	$\forall u, v, w(uR_P v \wedge vR_P w \rightarrow uR_P w)$	(CR5)
(6)	$\mathcal{M} \models \varphi \rightarrow [F]\langle P\rangle\varphi$	*iff*	$\forall u, v(uR_F v \rightarrow vR_P u)$	(CR6)
(7)	$\mathcal{M} \models \varphi \rightarrow [P]\langle F\rangle\varphi$	*iff*	$\forall u, v(uR_P v \rightarrow vR_F u)$	(CR7)

Proposition 5.1′ (Correspondence theory) *Let $\mathcal{M} = (W, R_{\mathsf{K}_\alpha}, R_F, R_P, \models)$ be a model for temporal logics of knowledge. Then the following holds.*

(1)′	$\mathcal{M} \models \mathsf{K}_\alpha\varphi \rightarrow \varphi$	*iff*	$\forall u(uR_{\mathsf{K}_\alpha}u)$	(CR1)′
(2)′	$\mathcal{M} \models \neg\mathsf{K}_\alpha\varphi \rightarrow \mathsf{K}_\alpha\neg\mathsf{K}_\alpha\varphi$	*iff*	$\forall u, v, w(uR_{\mathsf{K}_\alpha}v \wedge uR_{\mathsf{K}_\alpha}w \rightarrow vR_{\mathsf{K}_\alpha}w)$	(CR2)′

If R_{B_α}, R_F and R_P satisfy all of conditions from (CR1) to (CR7) for $\mathcal{M} = (W, R_{\mathsf{B}_\alpha}, R_F, R_P, \models)$, $\mathcal{M}$ is called a $K_t + K_B$-model. If R_{K_α}, R_F and R_P satisfy all of conditions (CR1)′, (CR2)′ and from (CR4) to (CR7) for $\mathcal{M} = (W, R_{\mathsf{K}_\alpha}, R_F, R_P, \models)$, $\mathcal{M}$ is called a $K_t + K_K$-model.

By Proposition 5.1 (6) and (7), R_F is a converse relation of R_P and thus it is enough to take either of R_F and R_P. Hence, by taking $R_F = R_P$, a model for temporal epistemic logics can be defined as a quadruple $(W, R_{\Box}, R_T, \models)$ for $\Box \in \{\mathsf{B}_\alpha, \mathsf{K}_\alpha\}$, where $\models$ is defined by replacing (M6) and (M7) with (M6)′ and (M7)′, respectively.

(M6)′ $u \models [F]\varphi$ iff for all $v \in W$, uR_Tv implies $v \models \varphi$

(M7)′ $u \models [P]\varphi$ iff for all $v \in W$, vR_Tu implies $v \models \varphi$

It is not hard to show the completeness theorem for $L \in \{K_t \otimes K_B, K_t \otimes K_K\}$.

Proposition 5.2 *Let L be any of $K_t \otimes K_B$ and $K_t \otimes K_K$. For any formula φ, $\varphi \notin L$ iff there exists L-model $\mathcal{M}$ such that $\mathcal{M} \not\models \varphi$.*

In fact, soundness and completeness can be proved by induction on the length of the proof figure and by constructing the canonical model, respectively.

5.2 Sequent systems for temporal epistemic logics and their restricted systems

In this section, we introduce sequent systems for temporal epistemic logics discussed in the previous section. Cut elimination theorem holds for neither of them and therefore subformula property doesn't hold. But we restrict our rules without changing the provability so that subformula property holds in restricted systems. This enables us to show the decidability of these logics, since the subformula property is essential in the decidability proof. In the following, uppercase Greek letters $\Gamma, \Delta, \Pi, \Sigma, \Theta$ and Ξ denote finite sets of formulas, and $\Box\Gamma$ denotes $\{\Box\varphi \mid \varphi \in \Gamma\}$ for $\Box \in \{[F], [P], \mathsf{B}_\alpha, \mathsf{K}_\beta\}$. Also, $Sub(\Gamma)$, Γ_* and Δ^* denote $\bigcup\{Sub(\psi) \mid \psi \in \Gamma\}$, $\bigwedge\{\varphi \mid \varphi \in \Gamma\}$ and $\bigvee\{\varphi \mid \varphi \in \Delta\}$, respectively.

The sequent system $\mathcal{S}(K_t \otimes K_B)$ is obtained from the sequent system **LK** for the classical propositional logic by adding the following three rules:

$$\text{(B)}\quad \frac{\mathsf{B}_\alpha\Sigma, \Sigma \Rightarrow \mathsf{B}_\alpha\Lambda, \Theta}{\mathsf{B}_\alpha\Sigma \Rightarrow \mathsf{B}_\alpha\Lambda, \mathsf{B}_\alpha\Theta} \qquad \text{(T1)}\quad \frac{[F]\Sigma, \Sigma \Rightarrow [P]\Lambda, [P]\Theta, \varphi}{[F]\Sigma \Rightarrow [P]\Lambda, \Theta, [F]\varphi}$$

$$\text{(T2)}\quad \frac{[P]\Sigma, \Sigma \Rightarrow [F]\Lambda, [F]\Theta, \varphi}{[P]\Sigma \Rightarrow [F]\Lambda, \Theta, [P]\varphi}$$

where Θ in the rule (B) contains not more than one formula. The sequent system $\mathcal{S}(K_t \otimes K_K)$ is obtained from **LK** by adding the following four rules:

$$\text{(K1)} \quad \frac{\varphi, \Sigma \Rightarrow \Lambda}{\mathsf{K}_\beta\varphi, \Sigma \Rightarrow \Lambda} \qquad \text{(T1)} \quad \frac{[F]\Sigma, \Sigma \Rightarrow [P]\Lambda, [P]\Theta, \varphi}{[F]\Sigma \Rightarrow [P]\Lambda, \Theta, [F]\varphi}$$

$$\text{(K2)} \quad \frac{\mathsf{K}_\beta\Sigma \Rightarrow \mathsf{K}_\beta\Lambda, \varphi}{\mathsf{K}_\beta\Sigma \Rightarrow \mathsf{K}_\beta\Lambda, \mathsf{K}_\beta\varphi} \qquad \text{(T2)} \quad \frac{[P]\Sigma, \Sigma \Rightarrow [F]\Lambda, [F]\Theta, \varphi}{[P]\Sigma \Rightarrow [F]\Lambda, \Theta, [P]\varphi}$$

Rules (T1) and (T2) are introduced by Nishimura in [31].

Proposition 5.3 *For $L \in \{K_t \otimes K_B, K_t \otimes K_K\}$, $\Gamma_* \to \Delta^* \in L$ iff $\mathcal{S}(L) \vdash \ \Gamma \Rightarrow \Delta$.*

The sequent system $\mathcal{S}(L)$, however, lacks the cut-elimination property for $L \in \{K_t \otimes K_B, K_t \otimes K_K\}$; i.e. there exists a $\mathcal{S}(L)$-provable sequent which is not provable in $\mathcal{S}(L)$ without cut rule. For example, a sequent $p \Rightarrow [F]\neg[P]\neg p$ is provable as shown below, but the sequent is not provable without applying the cut rule.

$$\dfrac{\dfrac{\dfrac{[P]\neg p \Rightarrow [P]\neg p}{\Rightarrow [P]\neg p, \neg[P]\neg p}}{\Rightarrow \neg p, [F]\neg[P]\neg p} \qquad \dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}}{p \Rightarrow [F]\neg[P]\neg p} \ (cut)$$

To overcome the difficulty, we will use Takano's method introduced in [42]. In the paper, Takano discussed the standard sequent system for modal logic **S5**, which was introduced by Ohnishi-Matsumoto [32, 33]. It is known that cut elimination theorem doesn't hold for this sequent system. On the other hand, Takano proved that in this sequent system, it is possible to restrict cut rule in such a way that any of cut formula is a subformula of a formula in the lower sequent. From this, it follows that if a sequent $\Gamma \Rightarrow \Delta$ is provable in **S5** then it has a proof with the *subformula property*, i.e. a proof in which every formula is a subformula of a formula in $\Gamma \Rightarrow \Delta$. On the other hand, a sequent system for **KD45** satisfies cut elimination theorem [41]. This subformula property has some important consequences. The first is the decidability, using Gentzen's method, and the second is Craig's interpolation theorem, applying Maehara's method.

Since our sequent systems $\mathcal{S}(K_t \otimes K_B)$ and $\mathcal{S}(K_t \otimes K_K)$ are natural extensions of modal logics **KD45** and **S5**, respectively. So, we can expect that by applying Takano's method and restricting cut rule, we have sequent systems with the subformula property. This is partly true as shown below. But we need to take care also of rules (T1) and (T2), which apparently disturb the subformula property.

Taking these points into consideration, we will introduce systems $\mathcal{S}(K_t \otimes K_B)^-$ and $\mathcal{S}(K_t \otimes K_K)^-$, which is obtained from $\mathcal{S}(K_t \otimes K_B)$ and $\mathcal{S}(K_t \otimes K_K)$, respectively, by restricting cut, (T1) and (T2) as follows;

$$\text{(AC)} \quad \frac{\Sigma \Rightarrow \Lambda, \varphi \quad \varphi, \Pi \Rightarrow \Theta}{\Sigma, \Pi \Rightarrow \Lambda, \Theta} \qquad \text{where } \varphi \in Sub(\Sigma \cup \Lambda \cup \Pi \cup \Theta),$$

$$\text{(T1)}' \quad \frac{[F]\Sigma, \Sigma \Rightarrow [P]\Lambda, [P]\Theta, \varphi}{[F]\Sigma \Rightarrow [P]\Lambda, \Theta, [F]\varphi} \qquad \text{where } [P]\Theta \subseteq Sub(\Sigma \cup \Lambda \cup \{\varphi\}),$$

$$\text{(T2)}' \quad \frac{[P]\Sigma, \Sigma \Rightarrow [F]\Lambda, [F]\Theta, \varphi}{[P]\Sigma \Rightarrow [F]\Lambda, \Theta, [P]\varphi} \qquad \text{where } [F]\Theta \subseteq Sub(\Sigma \cup \Lambda \cup \{\varphi\}).$$

It is clear that, for $L \in \{K_t \otimes K_B, K_t \otimes K_K\}$, every sequent which is provable in $\mathcal{S}(L)^-$ is provable in $\mathcal{S}(L)$. The restricted cut rule is sometimes called *acceptable cut* (AC). In any of (AC), (T1)′ and (T2)′, we can see easily that every formula occurring in upper sequents consists of a subformula of a formula in lower sequent by conditions attached to these rules. This thesis is first restricting (T1) and (T2) introduced by H. Nishimura into rules (T1)′ and (T2)′. The novelty of our system is to extend the idea of acceptable cut and to introduce restrictions on all of these three rules, in order to obtain a system with cut-restriction property.

5.3 Completeness theorem of restricted systems and some consequences

Now, we should ascertain that the restricted systems $\mathcal{S}(L)^-$ is equivalent to $\mathcal{S}(L)$, i.e. any sequent which is provable in $\mathcal{S}(L)$ is also provable in $\mathcal{S}(L)^-$. Because of Proposition 5.3 and completeness of the logic L, to show this, it is enough to show that the sequent system $\mathcal{S}(L)^-$ is complete with respect to Kripke type semantics for L, that is, for any sequent $\Gamma \Rightarrow \Delta$ there exists a L-model $\mathcal{M}$ such that $\mathcal{M} \not\models \Gamma_* \to \Delta^*$ if the sequent $\Gamma \Rightarrow \Delta$ is not provable. Our main Theorem 5.4 states a result stronger than mere completeness. Our proof of Theorem 5.4 goes as follows. First, the notion of partial valuations will be introduced. Proposition 5.6 shows a basic result on partial valuations. Using it, we will show Theorem 5.4 for each of temporal epistemic systems $\mathcal{S}(K_t \otimes K_B)^-$ and $\mathcal{S}(K_t \otimes K_K)^-$.

Theorem 5.4 *Let $L \in \{K_t \otimes K_B, K_t \otimes K_K\}$. If a sequent $\Gamma \Rightarrow \Delta$ is not provable in $\mathcal{S}(L)^-$, then there is a finite L-model $\mathcal{M}$ such that $\mathcal{M} \not\models \Gamma_* \to \Delta^*$.*

Here we will introduce an important notion of partial valuations to prove the above theorem.

Definition 5.5 (Ξ-partial valuation) *Let $L \in \{K_t \otimes K_B, K_t \otimes K_K\}$, and Ξ be a set of formulas which is closed under subformulas. A sequent $\Sigma \Rightarrow \Lambda$ is a Ξ-*partial valuation *in* *a system $\mathcal{S}(L)^-$, if the following three conditions are satisfied;* (i) $\mathcal{S}(L)^- \not\vdash \Sigma \Rightarrow \Lambda$; (ii) $\Sigma \cup \Lambda = Sub(\Sigma \cup \Lambda)$; (iii) $Sub(\Sigma \cup \Lambda) \subseteq \Xi$.

That is, $\Sigma \Rightarrow \Lambda$ is a Ξ-partial valuation if and only if $\Sigma \cup \Lambda$ is a subset of Ξ which is closed under subformulas such that $\Sigma \Rightarrow \Lambda$ is not provable in $\mathcal{S}(L)^-$. Ξ-partial valuations are denoted by u, v, w, $\cdots$, and $a(u)$ and $s(u)$ denote the antecedent of u and succedent of u, respectively; i.e. $a(\Sigma \Rightarrow \Lambda) = \Sigma$ and $s(\Sigma \Rightarrow \Lambda) = \Lambda$. Thus both a and s can be regarded as functions from the set of sequents to the collection of sets of formulas. The following Proposition 5.6 says a key fact on partial valuations.

Proposition 5.6 *Let L be any of $K_t \otimes K_B$ and $K_t \otimes K_K$. Suppose that a sequent $\Sigma \Rightarrow \Lambda$ is not provable in $\mathcal{S}(L)$ and Ξ is any set of formulas closed under subformulas, which includes $Sub(\Sigma \cup \Lambda)$. Then there exists a Ξ-partial valuation u such that $\Sigma \subseteq a(u) \subseteq Sub(\Sigma \cup \Lambda)$ and $\Lambda \subseteq s(u) \subseteq Sub(\Sigma \cup \Lambda)$.*

PROOF. Let $\chi_1, \chi_2, \chi_3, \ldots, \chi_n$ be an enumeration of formulas in the set $Sub(\Sigma \cup \Lambda)$. Now we define Σ_k and Λ_k for $0 \leq k \leq n$ inductively so that 1) $\Sigma_k \Rightarrow \Lambda_k$ is unprovable in $\mathcal{S}(L)^-$ and 2) $\Sigma \subseteq \Sigma_{k-1} \subseteq \Sigma_k$ and $\Lambda \subseteq \Lambda_{k-1} \subseteq \Lambda_k$ for $0 \leq k \leq n$. First, let $\Sigma_0 = \Sigma$ and $\Lambda_0 = \Lambda$. Obviously $\Sigma_0 \Rightarrow \Lambda_0$ is unprovable. Suppose that Σ_i and Λ_i are already defined and that $\Sigma_i \Rightarrow \Lambda_i$ is unprovable in $\mathcal{S}(L)^-$. Then either $\Sigma_i \Rightarrow \Lambda_i, \chi_{i+1}$ or $\Sigma_i, \chi_{i+1} \Rightarrow \Lambda_i$ is unprovable in $\mathcal{S}(L)^-$. For, if otherwise then both of these sequents are provable. Moreover, $\chi_{i+1} \in Sub(\Sigma \cup \Lambda) \subseteq Sub(\Sigma_i \cup \Lambda_i)$. Thus, $\Sigma_i \Rightarrow \Lambda_i$ is provable in $\mathcal{S}(L)^-$ as shown below. But this is a contradiction. (Note that χ_{i+1} is a subformula of a formula in $\Sigma_i \Rightarrow \Lambda_i$.)

$$\dfrac{\dfrac{\vdots}{\Sigma_i \Rightarrow \Lambda_i, \chi_{i+1}} \qquad \dfrac{\vdots}{\chi_{i+1}, \Sigma_i \Rightarrow \Lambda_i}}{\Sigma_i \Rightarrow \Lambda_i} \text{ (AC)}$$

When $\Sigma_i \Rightarrow \Lambda_i, \chi_{i+1}$ is unprovable in $\mathcal{S}(L)^-$, then let $\Sigma_{i+1} = \Sigma_i$ and $\Lambda_{i+1} = \Lambda_i \cup \{\chi_{i+1}\}$. Otherwise, let $\Sigma_{i+1} = \Sigma_i \cup \{\chi_{i+1}\}$ and $\Lambda_{i+1} = \Lambda_i$ since $\Sigma_i, \chi_{i+1} \Rightarrow \Lambda_i$ is unprovable in $\mathcal{S}(L)^-$. Now let u be $\Sigma_{n-1} \Rightarrow \Lambda_{n-1}$. Clearly u is a Ξ-partial valuation. ∎

I. Proof of Theorem 5.4 for $K_t \otimes K_B$

We suppose here that $\Gamma \Rightarrow \Delta$ is not provable in $\mathcal{S}(K_t \otimes K_B)^-$. We define the model $(W, R_T, R_{\mathsf{B}_\alpha}, \models)$ for the temporal logic of belief as follows:

$$W \quad = \quad \{u \mid u \text{ is a } Sub(\Gamma \cup \Delta)\text{-partial valuation }\},$$

$$uR_T v \quad \text{iff} \quad \text{for all } \psi,\ [F]\psi \in a(u) \text{ implies } (\ [F]\psi \in a(v) \ \text{ and } \psi \in a(v)) \\ \text{and } [P]\psi \in a(v) \text{ implies } (\ [P]\psi \in a(u) \ \text{ and } \psi \in a(u)),$$

$$uR_{\mathsf{B}_\alpha} v \quad \text{iff} \quad \text{for all } \psi, \mathsf{B}_\alpha\psi \in a(u) \text{ implies } (\mathsf{B}_\alpha\psi \in a(v) \text{ and } \psi \in a(v)) \\ \text{and } \mathsf{B}_\alpha\psi \in a(v) \text{ implies } \mathsf{B}_\alpha\psi \in a(u),$$

$$u \models p \quad \text{iff} \quad p \in a(u),\ \text{where } p \in Prop.$$

The set W is non-empty, since by Proposition 5.6 there exists a $Sub(\Gamma \cup \Delta)$-partial valuation as the sequent $\Gamma \Rightarrow \Delta$ is unprovable in $\mathcal{S}(K_t \otimes K_B)^-$. Moreover the set W is finite since $Sub(\Gamma \cup \Delta)$ is finite. In fact, if the number of formulas in $Sub(\Gamma \cup \Delta)$ is k, then the number of elements of W is at most 2^k.

Proposition 5.7 *The model defined above is a $K_t \otimes K_B$-model.*

PROOF. We will give a proof here only (CR1) i.e. $\forall u \exists v (uR_{\mathsf{B}_\alpha} v)$, since conditions from (CR2) to (CR7) are straightforward.

Take an arbitrary $u \in W$. Now define Σ and Λ as follows:

$\Sigma := \{\psi \mid \mathsf{B}_\alpha\psi \in a(u)\}$
$\Lambda := \{\psi \mid \mathsf{B}_\alpha\psi \in s(u)\}$

Then the sequent $\mathsf{B}_\alpha\Sigma, \Sigma \Rightarrow \mathsf{B}_\alpha\Lambda$ is unprovable in $\mathcal{S}(K_t \otimes K_B)^-$. Otherwise, by following proof figure, $a(u) \Rightarrow s(u)$ becomes provable, which is a contradiction.

$$\dfrac{\dfrac{\dfrac{\vdots}{\mathsf{B}_\alpha\Sigma, \Sigma \Rightarrow \mathsf{B}_\alpha\Lambda}}{\mathsf{B}_\alpha\Sigma \Rightarrow \mathsf{B}_\alpha\Lambda}\,(\mathrm{B})}{a(u) \Rightarrow s(u)}$$

Let $\Pi = Sub(\mathsf{B}_\alpha\Sigma \cup \mathsf{B}_\alpha\Lambda)$. Then $\Pi \subseteq Sub(a(u) \cup s(u)) \subseteq Sub(\Gamma \cup \Delta)$. Hence there exists a $Sub(\Gamma \cup \Delta)$-partial valuation v such that $\mathsf{B}_\alpha\Sigma \cup \Sigma \subseteq a(v) \subseteq \Pi$ and $\mathsf{B}_\alpha\Lambda \subseteq s(v) \subseteq \Pi$ by Proposition 5.6.

Then, we will show that $uR_{\mathsf{B}_\alpha} v$. If $\mathsf{B}_\alpha\psi \in a(u)$, then $\psi \in \Sigma$ and $\mathsf{B}_\alpha\psi \in \mathsf{B}_\alpha\Sigma$. Hence $\psi \in a(v)$ and $\mathsf{B}_\alpha\psi \in a(v)$. If $\mathsf{B}_\alpha\psi \in a(v)$, then $\mathsf{B}_\alpha\psi \in \Pi$, and hence $\mathsf{B}_\alpha\psi \in Sub(a(u) \cup s(u)) = a(u) \cup s(u)$. If $\mathsf{B}_\alpha\psi \in s(u)$, then $\mathsf{B}_\alpha\psi \in \mathsf{B}_\alpha\Lambda$, so $\mathsf{B}_\alpha\psi \in s(v)$. This contradicts $\mathsf{B}_\alpha\psi \in a(v)$. Therefore $\mathsf{B}_\alpha\psi \in a(u)$. ∎

Proposition 5.8 *The followings hold for every $u \in W$.*

(1) *If $\varphi \wedge \psi \in a(u)$, then $\varphi \in a(u)$ and $\psi \in a(u)$.*

(2) *If $\varphi \wedge \psi \in s(u)$, then $\varphi \in s(u)$ or $\psi \in s(u)$.*

(3) *If $\varphi \vee \psi \in a(u)$, then $\varphi \in a(u)$ or $\psi \in a(u)$.*

(4) *If $\varphi \vee \psi \in s(u)$, then $\varphi \in s(u)$ and $\psi \in s(u)$.*

(5) *If $\varphi \to \psi \in a(u)$, then $\varphi \in s(u)$ or $\psi \in a(u)$.*

(6) *If $\varphi \to \psi \in s(u)$, then $\varphi \in a(u)$ and $\psi \in s(u)$.*

(7) *If $\neg\varphi \in a(u)$, then $\varphi \in s(u)$.*

(8) *If $\neg\varphi \in s(u)$, then $\varphi \in a(u)$.*

(9) *If $\mathsf{B}_\alpha\varphi \in a(u)$, then for every $v \in W$, $uR_{\mathsf{B}_\alpha}v$ implies $\varphi \in a(v)$.*

(10) *If $\mathsf{B}_\alpha\varphi \in s(u)$, then for some $v \in W$, $uR_{\mathsf{B}_\alpha}v$ and $\varphi \in s(v)$.*

(11) *If $[F]\varphi \in a(u)$, then for every $v \in W$, uR_Tv implies $\varphi \in a(v)$.*

(12) *If $[F]\varphi \in s(u)$, then for some $v \in W$, uR_Tv and $\varphi \in s(v)$.*

(13) *If $[P]\varphi \in a(u)$, then for every $v \in W$, vR_Tu implies $\varphi \in a(v)$.*

(14) *If $[P]\varphi \in s(u)$, then for some $v \in W$, vR_Tu and $\varphi \in s(v)$.*

PROOF. (1) Suppose that $\varphi \notin a(u)$ or $\psi \notin a(u)$. Since $\varphi \wedge \psi \in a(u) \subseteq Sub(a(u) \cup s(u))$, $\varphi \in Sub(a(u) \cup s(u))$ and $\psi \in Sub(a(u) \cup s(u))$, so $\varphi \in a(u) \cup s(u)$ and $\psi \in a(u) \cup s(u)$. If $\varphi \notin a(u)$, then $\varphi \in s(u)$, so the following proof figure gives us a contradiction since u is a $Sub(\Gamma \cup \Delta)$-partial valuation.

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi}\ (\wedge \Rightarrow)}{\overline{a(u) \Rightarrow s(u)}}$$

Thus $\varphi \in a(u)$. Similarly, we can derive a contradiction if $\psi \notin a(u)$. Thus $\psi \in a(u)$.

(2) - (8), (9), (11), (13): In the similar way to (1), we can show (2) - (8) by each of rules and the definition of $Sub(\Gamma \cup \Delta)$-partial valuations. (9), (11) and (13) can be easily shown by the definition of R_{B_α}, R_F and R_P, respectively.

(10) Suppose that $\mathsf{B}_\alpha\varphi \in s(u)$. Define Σ and Λ as follows:

$\Sigma := \{\psi \mid \mathsf{B}_\alpha\psi \in a(u)\}$
$\Lambda := \{\psi \mid \mathsf{B}_\alpha\psi \in s(u)\}$

Then the sequent $\mathsf{B}_\alpha\Sigma, \Sigma \Rightarrow \mathsf{B}_\alpha\Lambda, \varphi$ is unprovable in $\mathcal{S}(K_t \otimes K_B)^-$. For, otherwise we can derive a contradiction by using the following proof figure.

$$\dfrac{\dfrac{\dfrac{\ddots\vdots\cdot^{\cdot}}{\mathsf{B}_\alpha\Sigma, \Sigma \Rightarrow \mathsf{B}_\alpha\Lambda, \varphi}}{\mathsf{B}_\alpha\Sigma \Rightarrow \mathsf{B}_\alpha\Lambda, \mathsf{B}_\alpha\varphi}\,(\mathrm{B})}{a(u) \Rightarrow s(u)}$$

Let $\Pi = Sub(\mathsf{B}_\alpha\Sigma \cup \mathsf{B}_\alpha\Lambda \cup \{\varphi\})$. Then $\Pi \subseteq Sub(a(u) \cup s(u)) \subseteq Sub(\Gamma \cup \Delta)$. Hence there exists a $Sub(\Gamma \cup \Delta)$-partial valuation v such that $\mathsf{B}_\alpha\Sigma \cup \Sigma \subseteq a(v) \subseteq \Pi$ and $\mathsf{B}_\alpha\Lambda \cup \{\varphi\} \subseteq s(v) \subseteq \Pi$ by Proposition 5.6. It is clear that $\varphi \in s(v)$.

Now, we show that $uR_{\mathsf{B}_\alpha}v$. If $\mathsf{B}_\alpha\psi \in a(u)$, then $\psi \in \Sigma$ and hence $\mathsf{B}_\alpha\psi \in \mathsf{B}_\alpha\Sigma$. Thus $\psi \in a(v)$ and $\mathsf{B}_\alpha\psi \in a(v)$. If $\mathsf{B}_\alpha\psi \in a(v)$, then $\mathsf{B}_\alpha\psi \in \Pi$. So $\mathsf{B}_\alpha\psi \in Sub(a(u) \cup s(u)) = a(u) \cup s(u)$. If $\mathsf{B}_\alpha\psi \in s(u)$, then $\mathsf{B}_\alpha\psi \in \mathsf{B}_\alpha\Lambda$, and hence $\mathsf{B}_\alpha\psi \in s(v)$. This contradicts $\mathsf{B}_\alpha\psi \in a(v)$. Therefore $\mathsf{B}_\alpha\psi \in a(u)$.

(12) Suppose that $[F]\varphi \in s(u)$. Define Σ, Λ and Θ as follows:

$\Sigma := \{\psi \mid [F]\psi \in a(u)\}$
$\Lambda := \{\psi \mid [P]\psi \in s(u)\}$
$\Theta := \{\psi \mid \psi \in s(u) \text{ and } [P]\psi \in Sub(\Sigma \cup \Lambda \cup \{\varphi\})\}$

Then the sequent $[F]\Sigma, \Sigma \Rightarrow [P]\Lambda, [P]\Theta, \varphi$ is unprovable in $\mathcal{S}(K_t \otimes K_B)^-$. For, otherwise we can derive a contradiction by using the following proof figure. Note here that $[P]\Theta$ satisfies the condition for applying $(T1)'$.

$$\dfrac{\dfrac{\dfrac{\ddots\vdots\cdot^{\cdot}}{[F]\Sigma, \Sigma \Rightarrow [P]\Lambda, [P]\Theta, \varphi}}{[F]\Sigma \Rightarrow [P]\Lambda, \Theta, [F]\varphi}\,(\mathrm{T1})'}{a(u) \Rightarrow s(u)}$$

Let $\Pi = Sub([F]\Sigma \cup [P]\Lambda \cup [P]\Theta \cup \{\varphi\})$, then $\Pi \subseteq Sub(a(u) \cup s(u)) \subseteq Sub(\Gamma \cup \Delta)$. Hence there exists a $Sub(\Gamma \cup \Delta)$-partial valuation v such that $[F]\Sigma \cup \Sigma \subseteq a(v) \subseteq \Pi$ and $[P]\Lambda \cup [P]\Theta \cup \{\varphi\} \subseteq s(v) \subseteq \Pi$ by Proposition 5.6. It is clear that $\varphi \in s(v)$.

We show, next, that uR_Tv. If $[F]\psi \in a(u)$, then $\psi \in \Sigma$ and $[F]\psi \in [F]\Sigma$. Hence $\psi \in a(v)$ and $[F]\psi \in a(v)$. If $[P]\psi \in a(v)$, then $[P]\psi \in \Pi$. So $[P]\psi, \psi \in Sub(a(u) \cup s(u)) = a(u) \cup s(u)$. If $[P]\psi \in s(u)$, then $[P]\psi \in [P]\Lambda$, and hence $[P]\psi \in s(v)$. This contradicts $[P]\psi \in a(v)$. Therefore $[P]\psi \in a(u)$. If $\psi \in s(u)$, then $[P]\psi \in [P]\Theta$ since $[P]\psi \in Sub(\Sigma \cup \Lambda \cup \{\varphi\})$. Thus $[P]\psi \in s(v)$. This contradicts $[P]\psi \in a(v)$. Therefore $[P]\psi \in a(u)$.

(14) can be proved similarly to (12). ∎

Recall that $\models$ is defined by the condition $u \models p$ iff $p \in a(u)$ for $p \in Prop$. We can show the following.

Proposition 5.9 *Suppose $u \in W$ and $\chi \in Sub(\Gamma \cup \Delta)$.*

(0) *if $p \in s(u)$, then $u \not\models p$ for every $p \in Prop$,*
(1) *if $\chi \in a(u)$, then $u \models \chi$,*
(2) *if $\chi \in s(u)$, then $u \not\models \chi$.*

PROOF. For (0), if $p \in s(u)$, then $p \notin a(u)$, and hence $u \not\models p$. (1) and (2) can be proved by simultaneous induction on the length of χ, using Proposition 5.8. ∎

When $\Gamma \Rightarrow \Delta$ is unprovable in $\mathcal{S}(K_t \otimes K_B)^-$, there exists a $Sub(\Gamma \cup \Delta)$-partial valuation u_0 such that $\Gamma \subseteq a(u_0) \subseteq Sub(\Gamma \cup \Delta)$ and $\Delta \subseteq s(u_0) \subseteq Sub(\Gamma \cup \Delta)$. If $\varphi \in \Gamma$, then $\varphi \in a(u_0)$, and hence $u_0 \models \varphi$. Therefore $u_0 \models \Gamma_*$. If $\varphi \in \Delta$, then $\varphi \in s(u_0)$, and hence $u_0 \not\models \varphi$. Therefore $u_0 \not\models \Delta^*$. Thus $u_0 \not\models \Gamma_* \to \Delta^*$. This completes the proof of Theorem 5.4 for the system $\mathcal{S}(K_t \otimes K_B)^-$.

II. Proof of Theorem 5.4 for $K_t \otimes K_K$

Next we suppose that $\Gamma \Rightarrow \Delta$ is not provable in $\mathcal{S}(K_t \otimes K_K)^-$. Now we define the model $(W, R_T, R_{\mathsf{K}_\alpha}, \models)$ for the temporal logic of knowledge as follows:

$$uR_{\mathsf{K}_\alpha}v \;\Leftrightarrow\; \text{for all } \psi,\ \mathsf{K}_\alpha\psi \in a(u) \text{ iff } \mathsf{K}_\alpha\psi \in a(v)$$

The set W is non-empty since there exists a $Sub(\Gamma \cup \Delta)$-partial valuation by $\mathcal{S}(K_t \otimes K_K)^-$ -unprovability of $\Gamma \Rightarrow \Delta$, and the set W is finite since $Sub(\Gamma \cup \Delta)$ is finite set. Similarly to Proposition 5.7, we have the following.

Proposition 5.10 *The model defined above is $K_t \otimes K_B$-model.*

Proposition 5.11 *The followings hold for every $u \in W$.*

(1) *If $\mathsf{K}_\alpha\varphi \in a(u)$, then for every $v \in W$, $uR_{\mathsf{K}_\alpha}v$ implies $\varphi \in a(v)$.*

(2) *If $\mathsf{K}_\alpha\varphi \in s(u)$, then for some $v \in W$, $uR_{\mathsf{K}_\alpha}v$ and $\varphi \in s(v)$.*

PROOF. (1) Suppose that $\mathsf{K}_\alpha\varphi \in a(u)$ and $uR_{\mathsf{K}_\alpha}v$. Then $\varphi \in a(v) \cup s(v)$ since $\mathsf{K}_\alpha\varphi \in a(v)$ by definition of R_{K_α}. If $\varphi \in s(v)$, then a contradiction follows by the following proof figure.

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\mathsf{K}_\alpha\varphi \Rightarrow \varphi}\,(\mathrm{K1})}{\overline{a(v) \Rightarrow s(v)}}$$

Therefore $\varphi \in a(v)$.

(2) Suppose that $K_\alpha\varphi \in s(u)$. Define Σ and Λ as follows:

$\Sigma := \{\psi \mid \mathsf{K}_\alpha\psi \in a(u)\}$
$\Lambda := \{\psi \mid \mathsf{K}_\alpha\psi \in s(u)\}$

Then the sequent $\mathsf{K}_\alpha\Sigma \Rightarrow \mathsf{K}_\alpha\Lambda, \varphi$ is unprovable in $\mathcal{S}(K_t \otimes K_K)^-$. Otherwise we have a contradiction by following proof figure.

$$\dfrac{\dfrac{\dfrac{\ddots\,\vdots\,\cdot^{\cdot^{\cdot}}}{\mathsf{K}_\alpha\Sigma \Rightarrow \mathsf{K}_\alpha\Lambda, \varphi}}{\mathsf{K}_\alpha\Sigma \Rightarrow \mathsf{K}_\alpha\Lambda, \mathsf{K}_\alpha\varphi}\,(\mathrm{K2})}{\overline{a(u) \Rightarrow s(u)}}$$

Let $\Pi = Sub(\mathsf{K}_\alpha\Sigma \cup \mathsf{K}_\alpha\Lambda \cup \{\varphi\})$, then $\Pi \subseteq Sub(a(u) \cup s(u)) \subseteq Sub(\Gamma \cup \Delta)$. Hence there exists a $Sub(\Gamma \cup \Delta)$-partial valuation v such that $\mathsf{K}_\alpha\Sigma \subseteq a(v) \subseteq \Pi$ and $\mathsf{K}_\alpha\Lambda \cup \{\varphi\} \subseteq s(v) \subseteq \Pi$ by Proposition 5.6. It is clear that $\varphi \in s(v)$.

Next, we show that $uR_{\mathsf{K}_\alpha}v$. If $\mathsf{K}_\alpha\psi \in a(u)$, then $\mathsf{K}_\alpha\psi \in \mathsf{K}_\alpha\Sigma$ and hence $\mathsf{K}_\alpha\psi \in a(v)$. If $\mathsf{K}_\alpha\psi \in a(v)$, then $\mathsf{K}_\alpha\psi \in \Pi$. Hence $\mathsf{K}_\alpha\psi \in Sub(a(u) \cup s(u)) = a(u) \cup s(u)$. If $\mathsf{K}_\alpha\psi \in s(u)$, then $\mathsf{K}_\alpha\psi \in \mathsf{K}_\alpha\Lambda$ and hence $\mathsf{K}_\alpha\psi \in s(v)$. This contradicts $\mathsf{K}_\alpha\psi \in a(v)$. Therefore $\mathsf{K}_\alpha\psi \in a(u)$. ∎

Recall the definition of $\models$. Similarly to Proposition 5.9, we can show the following by the induction on the formation of χ, using Proposition 5.8 from (1) to (8), from (11) to (14) and 5.11 (1) and (2).

Proposition 5.12 *Suppose $u \in W$ and $\chi \in Sub(\Gamma \cup \Delta)$.*

(0) *if $p \in s(u)$, then $u \not\models p$ for every $p \in Prop$,*
(1) *if $\chi \in a(u)$, then $u \models \chi$,*
(2) *if $\chi \in s(u)$, then $u \not\models \chi$.*

Thus, Theorem 5.4 for the system $\mathcal{S}(K_t \otimes K_K)^-$ can be shown similarly to $\mathcal{S}(K_t \otimes K_B)^-$.

Here, we summarize the above results. Our results can be summarized as follows.

① $\Gamma_* \to \Delta^* \notin L$

② $\mathcal{S}(L) \nvdash \Gamma \Rightarrow \Delta$

③ $\mathcal{S}(L)^- \nvdash \Gamma \Rightarrow \Delta$

④ there is a L-model $\mathcal{M}$ s.t. $\mathcal{M} \not\models \Gamma_* \to \Delta^*$

⑤ there is a finite L-model $\mathcal{M}$ s.t. $\mathcal{M} \not\models \Gamma_* \to \Delta^*$

$$\begin{array}{ccc} \textcircled{1} & \Leftrightarrow & \textcircled{4} \\ \Updownarrow & & \Uparrow \\ \textcircled{2} & & \| \\ \Downarrow & & \| \\ \textcircled{3} & \Longrightarrow & \textcircled{5} \end{array}$$

Therefore, statements from ① to ⑤ are equivalent. From this, the temporal epistemic systems $\mathcal{S}(L)^-$ which has the subformula property is a formal system for temporal epistemic logic L. Moreover, an important corollary follows.

Corollary 5.13 *For $L \in \{K_t \otimes K_B, K_t \otimes K_K\}$, the temporal epistemic logic L has the finite model property; i.e. if $\varphi \notin L$, then there exists a finite L-model $\mathcal{M}$ such that $\mathcal{M} \not\models \varphi$.*

As an application of the finite model property which was seen in the above, the decidability for our temporal epistemic logics is follows by using the following Harrop's theorem (cf. [11]):

if a finitely axiomatizable logic has the finite model property, then it is decidable.

It has already known that only of epistemic logics K_B and K_K and temporal logic K_t has the finite model property. Since both temporal epistemic logics presented in this chapter are finitely axiomatizable, they are decidable. By [23], moreover, the fusions of modal logics with finite model property have the finite model property. Besides that, for complete modal logics L_1 and L_2 not containing $\bot$, the fusion $L_1 \otimes L_2$ is decidable if both components L_1 and L_2 are decidable. Anyway, the decision procedure by Harrop's theorem is extremely inefficient, and not feasible realistically. If a logic has the finite model property, a finite model which invalidates unprovable formula exists. Therefore, if finite models are searched effectively, a decision procedure different from the procedure in this chapter can be obtained.

Decision procedure by using restricted sequent system is more effective procedure. This is discussed in Section 5.5.

5.4 Craig's interpolation theorem of temporal epistemic logics

Then the following is another corollary of the diagram in the previous section.

Corollary 5.14 *Let L be any of $K_t \otimes K_B$ and $K_t \otimes K_K$. Then, for any formula φ, $\mathcal{S}(L) \vdash \varphi$ iff $\mathcal{S}(L)^- \vdash \varphi$.*

Here, for the sequent system $\mathcal{S}(L)$ where L is any of $K_t \otimes K_B$ or $K_t \otimes K_K$, the system obtained from $\mathcal{S}(L)$ by replacing only the cut rule to acceptable cut rule is denoted by $\mathcal{S}(L)^*$. Then, it is clearly that, for any formula φ, $\mathcal{S}(L) \vdash \varphi \Rightarrow \mathcal{S}(L)^* \vdash \varphi \Rightarrow \mathcal{S}(L)^- \vdash \varphi$ by the definition. By the above corollary, we have therefore, for any formula φ,

$$\mathcal{S}(L) \vdash \varphi \ \Leftrightarrow\ \mathcal{S}(L)^* \vdash \varphi \ \Leftrightarrow\ \mathcal{S}(L)^- \vdash \varphi \ .$$

In this section, by using system $\mathcal{S}(L)^*$, Craig's interpolation theorem for temporal epistemic logics is shown syntactically using Maehara's method. Usually, Maehara's method is application to cut-free sequent systems. But, it can be applied also to sequent systems with acceptable cut (see e.g. [42]). In the following, we will given a outline of proof of Craig's interpolation theorem for the temporal epistemic logic $K_t \otimes K_B$. Craig's interpolation theorem for the temporal epistemic logic $K_t \otimes K_K$ can be shown similarly to $K_t \otimes K_B$.

For technical reasons, we introduce the constant symbol $\top$, which denotes "true statement" and admit $\Rightarrow \top$ as an initial sequent. The notation $\langle\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}\rangle$ is called a *partition* of sequent $\Gamma \Rightarrow \Delta$, if Γ and Δ are a disjoint union of Γ_1 and Γ_2, and Δ_1 and Δ_2, respectively. The set of all propositional variables which occur in a formula φ is denoted by $V(\varphi)$.

Lemma 5.15 *Suppose that a sequent $\Gamma \Rightarrow \Delta$ is provable in the temporal epistemic system $\mathcal{S}(K_t \otimes K_B)^*$, and also that $\langle\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}\rangle$ is an arbitrary partition of $\Gamma \Rightarrow \Delta$. Then there exists a formula χ, called an* interpolant, *such that*

1) both $\Gamma_1 \Rightarrow \Delta_1, \chi$ and $\chi, \Gamma_2 \Rightarrow \Delta_2$ are provable in $\mathcal{S}(K_t \otimes K_B)^*$,
2) $V(\chi) \subseteq V(\Gamma_1 \cup \Delta_1) \cap V(\Gamma_2 \cup \Delta_2)$.

Proof. This lemma is proved by induction on the length of a s proof of $\Gamma \Rightarrow \Delta$ in which every cut in it is acceptable. We will give a proof only the cases where $\Gamma \Rightarrow \Delta$ is the lower sequent of one of rules (B), (T1) and (AC)

Case 1. The last inference is

$$\frac{\mathsf{B}_\alpha\Sigma, \Sigma \Rightarrow \mathsf{B}_\alpha\Lambda, \Theta}{\mathsf{B}_\alpha\Sigma \Rightarrow \mathsf{B}_\alpha\Lambda, \mathsf{B}_\alpha\Theta}\ (B) \quad .$$

where Θ in the rule (B) contains not more than one formula.

1.1. The partition is of the form $\langle\{\mathsf{B}_\alpha\Sigma_1; \mathsf{B}_\alpha\Lambda_1, \mathsf{B}_\alpha\Theta\}, \{\mathsf{B}_\alpha\Sigma_2; \mathsf{B}_\alpha\Lambda_2\}\rangle$. By taking the partition $\langle\{\mathsf{B}_\alpha\Sigma_1, \Sigma_1; \mathsf{B}_\alpha\Lambda_1, \Theta\}, \{\mathsf{B}_\alpha\Sigma_2, \Sigma_2; \mathsf{B}_\alpha\Lambda_2\}\rangle$ of the upper sequent and applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant χ such that both $\mathsf{B}_\alpha\Sigma_1, \Sigma_1 \Rightarrow \mathsf{B}_\alpha\Lambda_1, \Theta, \chi$ and $\chi, \mathsf{B}_\alpha\Sigma_2, \Sigma_2 \Rightarrow \mathsf{B}_\alpha\Lambda_2$ are provable in $\mathcal{S}(K_t \otimes K_B)^*$. Then we can obtain following two proofs:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\mathsf{B}_\alpha\Sigma_1, \Sigma_1 \Rightarrow \mathsf{B}_\alpha\Lambda_1, \Theta, \chi}}{\neg\chi, \mathsf{B}_\alpha\Sigma_1, \Sigma_1 \Rightarrow \mathsf{B}_\alpha\Lambda_1, \Theta}}{\mathsf{B}_\alpha\neg\chi, \neg\chi, \mathsf{B}_\alpha\Sigma_1 \Rightarrow \mathsf{B}_\alpha\Lambda_1, \Theta}}{\mathsf{B}_\alpha\neg\chi, \mathsf{B}_\alpha\Sigma_1 \Rightarrow \mathsf{B}_\alpha\Lambda_1, \mathsf{B}_\alpha\Theta}}{\mathsf{B}_\alpha\Sigma_1 \Rightarrow \mathsf{B}_\alpha\Lambda_1, \mathsf{B}_\alpha\Theta, \neg\mathsf{B}_\alpha\neg\chi} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\chi, \mathsf{B}_\alpha\Sigma_2, \Sigma_2 \Rightarrow \mathsf{B}_\alpha\Lambda_2}}{\mathsf{B}_\alpha\Sigma_2, \Sigma_2 \Rightarrow \mathsf{B}_\alpha\Lambda_2, \neg\chi}}{\mathsf{B}_\alpha\Sigma_2 \Rightarrow \mathsf{B}_\alpha\Lambda_2, \mathsf{B}_\alpha\neg\chi}}{\neg\mathsf{B}_\alpha\neg\chi, \mathsf{B}_\alpha\Sigma_2 \Rightarrow \mathsf{B}_\alpha\Lambda_2} \quad .$$

Hence $\neg\mathsf{B}_\alpha\neg\chi$ serves as an interpolant of the present partition of the lower sequent.

1.2. The partition is of the form $\langle\{\mathsf{B}_\alpha\Sigma_1; \mathsf{B}_\alpha\Lambda_1\}, \{\mathsf{B}_\alpha\Sigma_2; \mathsf{B}_\alpha\Lambda_2, \mathsf{B}_\alpha\Theta\}\rangle$. By taking the partition $\langle\{\mathsf{B}_\alpha\Sigma_1, \Sigma_1; \mathsf{B}_\alpha\Lambda_1\}, \{\mathsf{B}_\alpha\Sigma_2, \Sigma_2; \mathsf{B}_\alpha\Lambda_2, \Theta\}\rangle$ of the upper sequent and applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant χ such that both $\mathsf{B}_\alpha\Sigma_1, \Sigma_1 \Rightarrow \mathsf{B}_\alpha\Lambda_1, \chi$ and $\chi, \mathsf{B}_\alpha\Sigma_2, \Sigma_2 \Rightarrow \mathsf{B}_\alpha\Lambda_2, \mathsf{B}_\alpha\Theta$ are provable in $\mathcal{S}(K_t \otimes K_B)^*$. Then we can obtain following two proofs:

$$\dfrac{\dfrac{\vdots}{\mathsf{B}_\alpha\Sigma_1, \Sigma_1 \Rightarrow \mathsf{B}_\alpha\Lambda_1, \chi}}{\mathsf{B}_\alpha\Sigma_1 \Rightarrow \mathsf{B}_\alpha\Lambda_1, \mathsf{B}_\alpha\chi} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\chi, \mathsf{B}_\alpha\Sigma_2, \Sigma_2 \Rightarrow \mathsf{B}_\alpha\Lambda_2, \Theta}}{\mathsf{B}_\alpha\chi, \chi, \mathsf{B}_\alpha\Sigma_2, \Sigma_2 \Rightarrow \mathsf{B}_\alpha\Lambda_2, \Theta}}{\mathsf{B}_\alpha\chi, \mathsf{B}_\alpha\Sigma_2 \Rightarrow \mathsf{B}_\alpha\Lambda_2, \mathsf{B}_\alpha\Theta} \quad .$$

Hence $\mathsf{B}_\alpha\chi$ serves as an interpolant of the present partition of the lower sequent.

Case 2. The last inference is

$$\dfrac{[F]\Sigma, \Sigma \Rightarrow [P]\Lambda, [P]\Theta, \varphi}{[F]\Sigma \Rightarrow [P]\Lambda, \Theta, [F]\varphi}\ (T1) \quad .$$

2.1. The partition is of the form $\langle\{[F]\Sigma_1; [P]\Lambda_1, \Theta_1, [F]\varphi\}, \{[F]\Sigma_2; [P]\Lambda_2, \Theta_2\}\rangle$. By taking the partition $\langle\{[F]\Sigma_1, \Sigma_1; [P]\Lambda_1, [P]\Theta_1, \varphi\}, \{[F]\Sigma_2, \Sigma_2; [P]\Lambda_2, [P]\Theta_2\}\rangle$ of the upper sequent and applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant χ such that both $[F]\Sigma_1, \Sigma_1 \Rightarrow [P]\Lambda_1, [P]\Theta_1, \varphi, \chi$ and $\chi, [F]\Sigma_2, \Sigma_2 \Rightarrow [P]\Lambda_2, [P]\Theta_2$ are provable in $\mathcal{S}(K_t \otimes K_B)^*$. Then we can obtain following two proofs:

$$
\begin{array}{c}
\ddots\vdots\iddots \\ \hline
[F]\Sigma_1, \Sigma_1 \Rightarrow [P]\Lambda_1, [P]\Theta_1, \varphi, \chi \\ \hline
\neg\chi, [F]\Sigma_1, \Sigma_1 \Rightarrow [P]\Lambda_1, [P]\Theta_1, \varphi \\ \hline
[F]\neg\chi, \neg\chi, [F]\Sigma_1, \Sigma \Rightarrow [P]\Lambda_1, [P]\Theta_1, \varphi \\ \hline
[F]\neg\chi, [F]\Sigma_1 \Rightarrow [P]\Lambda_1, \Theta_1, [F]\varphi \\ \hline
[F]\Sigma_1 \Rightarrow [P]\Lambda_1, \Theta_1, [F]\varphi, \neg[F]\neg\chi
\end{array}
\qquad
\begin{array}{c}
\ddots\vdots\iddots \\ \hline
\chi, [F]\Sigma_2, \Sigma_2 \Rightarrow [P]\Lambda_2, [P]\Theta_2 \\ \hline
[F]\Sigma_2, \Sigma_2 \Rightarrow [P]\Lambda_2, [P]\Theta_2, \neg\chi \\ \hline
[F]\Sigma_2 \Rightarrow [P]\Lambda_2, \Theta_2, [F]\neg\chi \\ \hline
\neg[F]\neg\chi, [F]\Sigma_2 \Rightarrow [P]\Lambda_2, \Theta_2
\end{array} \quad .
$$

Hence $\neg[F]\neg\chi$ serves as an interpolant of the present partition of the lower sequent.

2.2. The partition is of the form $\langle\{[F]\Sigma_1; [P]\Lambda_1, \Theta_1\}, \{[F]\Sigma_2; [P]\Lambda_2, \Theta_2, [F]\varphi\}\rangle$. By taking the partition $\langle\{[F]\Sigma_1, \Sigma_1; [P]\Lambda_1, [P]\Theta_1\}, \{[F]\Sigma_2, \Sigma_2; [P]\Lambda_2, [P]\Theta_2, \varphi\}\rangle$ of the upper sequent and applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant χ such that both $[F]\Sigma_1, \Sigma_1 \Rightarrow [P]\Lambda_1, [P]\Theta_1, \chi$ and $\chi, [F]\Sigma_2 \Rightarrow [P]\Lambda_2, [P]\Theta_2, \varphi$ are provable in $\mathcal{S}(K_t \otimes K_B)^*$. Then we can obtain following two proofs:

$$
\begin{array}{c}
\ddots\vdots\iddots \\ \hline
[F]\Sigma_1, \Sigma_1 \Rightarrow [P]\Lambda_1, [P]\Theta_1, \chi \\ \hline
[F]\Sigma_1 \Rightarrow [P]\Lambda_1, \Theta_1, [F]\chi
\end{array}
\qquad
\begin{array}{c}
\ddots\vdots\iddots \\ \hline
\chi, [F]\Sigma_2, \Sigma_2 \Rightarrow [P]\Lambda_2, [P]\Theta_2, \varphi \\ \hline
[F]\chi, \chi, [F]\Sigma_2, \Sigma_2 \Rightarrow [P]\Lambda_2, [P]\Theta_2, \varphi \\ \hline
[F]\chi, [F]\Sigma_2 \Rightarrow [P]\Lambda_2, \Theta_2, [F]\varphi
\end{array} \quad .
$$

Hence $[F]\chi$ serves as an interpolant of the present partition of the lower sequent.

Case 3. The last inference is

$$
\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\textit{acceptable cut}) \quad ,
$$

where $\varphi \in \mathit{Sub}(\Gamma, \Pi, \Delta, \Sigma)$. Then the partition is of the form $\langle\{\Gamma_1, \Pi_1; \Delta_1, \Sigma_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Sigma_2\}\rangle$. This is only the case which never happens when the cut-elimination theorem holds.

3.1. If $\varphi \in \mathit{Sub}(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1)$, by taking partitions $\langle\{\Gamma_1; \Delta_1, \varphi\}, \{\Gamma_2; \Delta_2\}\rangle$ and $\langle\{\varphi, \Pi_1; \Sigma_1\}, \{\Pi_2; \Sigma_2\}\rangle$ of upper sequents and applying the induction hypothesis to the proof of the upper sequent, there exist interpolants χ_1 and χ_2 such that all of (1) $\Gamma_1 \Rightarrow \Delta_1, \varphi, \chi_1$, (2) $\chi_1, \Gamma_2 \Rightarrow \Delta_2$, (3) $\varphi, \Pi_1 \Rightarrow \Sigma_1, \chi_2$ and (4) $\chi_2, \Pi_2 \Rightarrow \Sigma_2$ are provable in $\mathcal{S}(K_t \otimes K_B)^*$. Then we can obtain following two proofs:

$$
\begin{array}{c}
\begin{array}{c} \vdots \\ \hline \Gamma_1 \Rightarrow \Delta_1, \varphi, \chi_1 \end{array}
\quad
\begin{array}{c} \vdots \\ \hline \varphi, \Pi_1 \Rightarrow \Sigma_1, \chi_2 \end{array} \\ \hline
\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1, \chi_1, \chi_2 \\ \hline\hline
\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1, \chi_1 \vee \chi_2, \chi_1 \vee \chi_2 \\ \hline
\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1, \chi_1 \vee \chi_2
\end{array}
\qquad
\begin{array}{c}
\begin{array}{c} \vdots \\ \hline \chi_1, \Gamma_2 \Rightarrow \Delta_2 \\ \hline\hline \chi_1, \Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2 \end{array}
\quad
\begin{array}{c} \vdots \\ \hline \chi_2, \Pi_2 \Rightarrow \Sigma_2 \\ \hline\hline \chi_2, \Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2 \end{array} \\ \hline
\chi_1 \vee \chi_2, \Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2
\end{array} \quad .
$$

Note that the cut of the proof in the left hand is a acceptable by our assumption. Hence $\chi_1 \vee \chi_2$ serves as interpolants of the present partition the lower sequent.

3.2. If $\varphi \in Sub(\Gamma_2, \Pi_2, \Delta_2, \Sigma_2)$, by taking partitions $\langle\{\Gamma_1; \Delta_1\}, \{\Gamma_2; Delta_2, \varphi\}\rangle$ and $\langle\{\Pi_1; \Sigma_1\}, \{\varphi, \Pi_2; \Sigma_2\}\rangle$ of upper sequents and applying the induction hypothesis to the proof of the upper sequent, there exist interpolants χ_1 and χ_2 such that all of (1) $\Gamma_1 \Rightarrow \Delta_1, \chi_1$, (2) $\chi_1, \Gamma_2 \Rightarrow \Delta_2, \varphi$, (3) $\Pi_1 \Rightarrow \Sigma_1, \chi_2$ and (4) $\chi_2, \varphi, \Pi_2 \Rightarrow \Sigma_2$ are provable in $\mathcal{S}(K_t \otimes K_B)^*$. Then we can obtain following two proofs:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \Rightarrow \Delta_1, \chi_1}}{\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1, \chi_1} \quad \dfrac{\dfrac{\vdots}{\Pi_1 \Rightarrow \Sigma_1, \chi_2}}{\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1, \chi_2}}{\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1, \chi_1 \wedge \chi_2} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\chi_1, \Gamma_2 \Rightarrow \Delta_2, \varphi} \quad \dfrac{\vdots}{\chi_2, \varphi, \Pi_2 \Rightarrow \Sigma_2}}{\chi_1, \Gamma_2, \chi_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2}}{\chi_1 \wedge \chi_2, \Gamma_2, \chi_1 \wedge \chi_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2}}{\chi_1 \wedge \chi_2, \Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2} .$$

Note that the cut of the proof in the right hand is acceptable by our assumption. Hence $\chi_1 \wedge \chi_2$ serves as interpolants of the present partition of the lower sequent. ∎

Theorem 5.16 (Craig's interpolation theorem)
If $\varphi \to \psi$ is provable in $\mathcal{S}(K_t \otimes K_B)^*$, then there exists a formula χ such that

1) $\varphi \to \chi$ and $\chi \to \psi$ are both provable in $\mathcal{S}(K_t \otimes K_B)^*$,

2) $V(\chi) \subseteq V(\varphi) \cap V(\psi)$.

Proof. Assume that $\varphi \supset \psi$ is provable in $\mathcal{S}(K_t \otimes K_B)^*$. Clearly, the sequent $\varphi \Rightarrow \psi$ is provable in it. Then by Lemma 5.15, taking φ as Γ_1 and ψ as Δ_2, there exists a formula χ satisfying *1)* and *2)* of Theorem 5.16. ∎

5.5 Proof-search procedure for temporal epistemic logics

By our main theorem (Theorem 5.4) of this chapter, it follows that both systems $\mathcal{S}(L)$ and $\mathcal{S}(L)^-$ determin the logic L. Different from $\mathcal{S}(L)$, the sequent system $\mathcal{S}(L)^-$ will be suitable for implementing a theorem prover for L. In the following, we will give a decision procedure, using the following subformula property. The next proposition can be ascertained easily by checking each inference rules.

Proposition 5.17 *Sequent systems $\mathcal{S}(K_t \otimes K_B)^-$ and $\mathcal{S}(K_t \otimes K_K)^-$ have the subformula property, i.e. for $L \in \{K_t \otimes K_B, K_t \otimes K_K\}$, every sequent $\Gamma \Rightarrow \Delta$ provable in $\mathcal{S}(L)^-$ has such a proof P that every formula appearing in P is a subformula of a formula in $\Gamma \Rightarrow \Delta$.*

In rules (AC), (T1)$'$ and (T2)$'$, all the formulas in their upper sequents consists of a subformula of a formula in their lower sequents by conditions for applying their rules.

Though, for the sake of brevity, we have defined antecedents and succedents of sequents as finite sets of formulas in the present chapter, we can also define them as finite sequences of formulas by introducing both exchange rule and contraction rule. A *decision procedure* for L is a concrete finite procedure which decides whether a given formula is provable or not in a logic L. A sequent $\Gamma \Rightarrow \Delta$ is *reduced*, if each formula occurs at most three times in both Γ and Δ. We say that a sequent $\Gamma \Rightarrow \Delta$ is *1-reduced* if in the antecedent and also once in the succedent, every formula in it occurs exactly once. We show the decidability by giving a concrete proof-search procedure.

Suppose $\Gamma \Rightarrow \Delta$ is not (1-)reduced. Then a (1-)reduced sequent $\Gamma' \Rightarrow \Delta'$ can be obtain from $\Gamma \Rightarrow \Delta$ by using contraction and exchange rules repeatedly. Conversely, $\Gamma \Rightarrow \Delta$ can be obtained from $\Gamma' \Rightarrow \Delta'$ by means of weakening and exchange rules. So, for any sequent $\Gamma \Rightarrow \Delta$, there exists a (1-)reduced sequent $\Gamma' \Rightarrow \Delta'$ such that $\Gamma' \Rightarrow \Delta'$ is provable in $\mathcal{S}(L)^-$ if and only if $\Gamma \Rightarrow \Delta$ is provable in $\mathcal{S}(L)^-$. So, it is enough to give a proof-search procedure for 1-reduced sequents.

Proposition 5.18 *Let $L \in \{K_t \otimes K_B, K_t \otimes K_K\}$. Suppose that $\Gamma \Rightarrow \Delta$ is a sequent which is provable in $\mathcal{S}(L)^-$ and that $\Gamma' \Rightarrow \Delta'$ is any 1-reduced sequent obtained from $\Gamma \Rightarrow \Delta$. Then, there exists a $\mathcal{S}(L)^-$-proof of $\Gamma' \Rightarrow \Delta'$ such that every sequent appearing in it is reduced.*

PROOF. Let P be a $\mathcal{S}(L)^-$-proof of $\Gamma \Rightarrow \Delta$. We prove this proposition by induction on the length of P. The case that $\Gamma \Rightarrow \Delta$ is an initial sequent is trivial. Suppose that $\Gamma \Rightarrow \Delta$ is the lower sequent of an application of a rule I. When I has two upper sequents, it must be of the following form:

$$\frac{\Sigma_1 \Rightarrow \Lambda_1 \quad \Sigma_2 \Rightarrow \Lambda_2}{\Gamma \Rightarrow \Delta}\ (I)$$

Let $\Sigma_1' \Rightarrow \Lambda_1'$ and $\Sigma_2' \Rightarrow \Lambda_2'$ be any 1-reduced sequents obtained from $\Sigma_1 \Rightarrow \Lambda_1$ and $\Sigma_2 \Rightarrow \Lambda_2$, respectively, by means of contraction and exchange rules. By the induction hypothesis, each of $\Sigma_1' \Rightarrow \Lambda_1'$ and $\Sigma_2' \Rightarrow \Lambda_2'$ has a proof consisting only of reduced sequents. Let $\Gamma^* \Rightarrow \Delta^*$ be the sequent obtained from $\Sigma_1' \Rightarrow \Lambda_1'$ and $\Sigma_2' \Rightarrow \Lambda_2'$ by applying the rule I. If we can show that $\Gamma^* \Rightarrow \Delta^*$ is reduced, by using contraction and exchange rules, we can obtain a 1-reduced sequent for $\Gamma \Rightarrow \Delta$. Clearly, the whole proof consists only of reduced sequents. Similarly, we can see that the lower sequent is reduced when I consists of a single upper sequent.

So, it remains to show that $\Gamma^* \Rightarrow \Delta^*$ is always reduced. For each inference of **LK**, the claim holds (see e.g. [34]); i.e. if upper sequents are 1-reduced, then the lower sequent is reduced. We need to check this for (T1)$'$ and (T2)$'$, since this is clear for other cases. Consider the case where the inference is (T1)$'$:

$$\frac{[F]\Sigma, \Sigma \Rightarrow [P]\Lambda, [P]\Theta, \varphi}{[F]\Sigma \Rightarrow [P]\Lambda, \Theta, [F]\varphi}$$

Suppose that the upper sequent is 1-reduced, i.e. every formula in the antecedent of the upper sequent and every formula in the succedent occurs once. If one of formulas in sequence Θ is $[F]\varphi$, then the succedent of the lower sequent contains exactly two occurrences of $[F]\varphi$. If the sequence Θ contains $[P]\psi$ for a formula $\psi \in \Lambda$, then the succedent of the lower sequent contains exactly two occurrences of $[P]\psi$. Therefore in either case the lower sequent is reduced. Similarly to this, we can ascertain it for the case of rule (T2)$'$. ∎

Theorem 5.19 *The temporal epistemic logics $K_t \otimes K_B$ and $K_t \otimes K_K$ are decidable. In fact, a proof-search procedure exists for these temporal epistemic logics.*

PROOF. This theorem is shown by giving a concrete proof-search procedure of the restricted system $\mathcal{S}(L)^-$. Suppose that a given sequent $\Gamma \Rightarrow \Delta$ is 1-reduced. Here, a reduced sequent which consists of formulas in $Sub(\Gamma \cup \Delta)$ is called an *suitable sequent*. In searching a proof of the sequent $\Gamma \Rightarrow \Delta$ in $\mathcal{S}(L)^-$, it is enough to search a proof which consists only of suitable sequents. For each sequent $\Theta \Rightarrow \Sigma$ in a proof, if the same sequent appear above $\Theta \Rightarrow \Sigma$, then the proof contains redundancies. Therefore every proof can be transformed into the proof without any repetition of sequents. Here we say "partially constructed proofs", *inference figures*. In inference figure, each rule must be applied in a correct way,, but uppermost sequents are not necessarily initial sequents. For each i, let $\mathcal{G}_i$ be the the set of all inference figures in which inference rules are applied at most $i-1$ times. Giving attention to these things, we can obtained the following procedure.

1. $\mathcal{G}_1$ is the singleton set of only $\Gamma \Rightarrow \Delta$. The figure of only that sequent is a inference figure.

2. Suppose that $\mathcal{G}_i$ is already defined. Then, first we put every inference figure in $\mathcal{G}_i$ into $\mathcal{G}_{i+1}$. Next, take any inference figure $\mathcal{F}$ in $\mathcal{G}_i$, and take any uppermost sequent $\Sigma \Rightarrow \Pi$ of $\mathcal{F}$ which is not an initial sequent, if any. Find an inference rule I whose lower sequent is of the form $\Sigma \Rightarrow \Pi$, and form an inference figure $\mathcal{F}'$ by putting upper sequents of I on this $\Sigma \Rightarrow \Pi$. If this $\mathcal{F}'$ has no repetitions of sequents in any branch of it, then put $\mathcal{F}'$ also into $\mathcal{G}_{i+1}$.

3. If $\mathcal{G}_{i+1} = \mathcal{G}_i$, that is, no fresh inference figure can be constructed from inference figures in $\mathcal{G}_i$, then this procedure give the output "$\Gamma \Rightarrow \Delta$ is not provable", and terminates.

4. Otherwise, check whether or not there exists an inference figure in $\mathcal{G}_{i+1} - \mathcal{G}_i$ such that every uppermost sequent of it is an initial sequent. If exists, then this procedure gives the output "$\Gamma \Rightarrow \Delta$ is provable" and terminates. If not, the we add 1 to i and return back the step 2.

Since the set of all the non-repetition inference figures which consist only of acceptable sequents is finite, and so there must exist a natural number j such that $\mathcal{G}_{j+1} = \mathcal{G}_j$. Therefore, the above procedure eventually terminates. ∎

Example 1. Here, we will consider the following formula as an example.

$$\neg([\alpha]\Box p \wedge \Box\neg p) \qquad ([\alpha] \in \{\mathsf{K}_\alpha, \mathsf{B}_\alpha\})$$

This is an example taken from [49]. We can represent $\Box\varphi$ as $[F]\varphi \wedge \varphi$ by our language. So the above formula can be represented by

$$\neg([\alpha]([F]p \wedge p) \wedge ([F]\neg p \wedge \neg p))$$

First, in the case that $[\alpha]$ is K_α, we can obtain a $\mathcal{S}(K_t \otimes K_K)^-$-proof of this formula as follows by a finite procedure. (See the proof of Theorem 5.19.)

Suppose that If P_1 is $\Rightarrow \neg(\mathsf{K}_\alpha([F]p \wedge p) \wedge [F]\neg p \wedge \neg p)$, then $\mathsf{P}_1 \in \mathcal{G}_1$.

If P_2 is $\dfrac{\mathsf{K}_\alpha([F]p \wedge p) \wedge [F]\neg p \wedge \neg p \Rightarrow}{\mathsf{P}_1}$, then $\mathsf{P}_2 \in \mathcal{G}_2$.

$\vdots$

If P_{11} is $\dfrac{\mathsf{K}_\alpha([F]p \wedge p), [F]\neg p, \neg p \Rightarrow}{\mathsf{P}_{10}}$, then $\mathsf{P}_{11} \in \mathcal{G}_{11}$.

If P_{12} is $\dfrac{[F]p \wedge p, [F]\neg p, \neg p \Rightarrow}{\mathsf{P}_{11}}$, then $\mathsf{P}_{12} \in \mathcal{G}_{12}$.

If P_{13} is $\dfrac{[F]p \wedge p, \neg p \Rightarrow}{\mathsf{P}_{12}}$, then $\mathsf{P}_{13} \in \mathcal{G}_{13}$.

If P_{14} is $\dfrac{p, \neg p \Rightarrow}{\mathsf{P}_{13}}$, then $\mathsf{P}_{14} \in \mathcal{G}_{14}$.

If P_{15} is $\dfrac{p \Rightarrow p}{\mathsf{P}_{14}}$, then $\mathsf{P}_{15} \in \mathcal{G}_{15}$.

Thus, we have the following proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p, \neg p \Rightarrow}}{[F]p \wedge p, \neg p \Rightarrow}}{[F]p \wedge p, [F]\neg p, \neg p \Rightarrow}}{\mathsf{K}_\alpha([F]p \wedge p), [F]\neg p, \neg p \Rightarrow}}{}}{\mathsf{K}_\alpha([F]p \wedge p) \wedge ([F]\neg p \wedge \neg p) \Rightarrow}}{\Rightarrow \neg(\mathsf{K}_\alpha([F]p \wedge p) \wedge ([F]\neg p \wedge \neg p))}$$

The double line in the above represents that applications of $\wedge\rightarrow$ -rule are omitted.

When $[\alpha]$ is B_α, our decision procedure stops in finite steps and tell us that the above formula is not provable in $\mathcal{S}(K_t \otimes K_B)^-$.

Example 2. Next, we will consider the following formula as an example.

$$[P]\neg[F]\mathsf{K}_\alpha p \vee p$$

Suppose that If P_1 is $\Rightarrow [P]\neg[F]\mathsf{K}_\alpha p \vee p$, then $\mathsf{P}_1 \in \mathcal{G}_1$.

$\vdots$

If P_4 is $\dfrac{\Rightarrow [P]\neg[F]\mathsf{K}_\alpha p, p}{\mathsf{P}_3}$, then $\mathsf{P}_4 \in \mathcal{G}_4$.

If P_5 is $\dfrac{\Rightarrow \mathsf{K}_\alpha p, [P]\neg[F]\mathsf{K}_\alpha p \qquad \mathsf{K}_\alpha p \Rightarrow p}{\mathsf{P}_4}$, then $\mathsf{P}_5 \in \mathcal{G}_5$.

If $\mathsf{P}_{5.1}$ and $\mathsf{P}_{5.2}$ are $\Rightarrow \mathsf{K}_\alpha p, [P]\neg[F]\mathsf{K}_\alpha p$ and $\mathsf{K}_\alpha p \Rightarrow p$, respectively,

then $\dfrac{\dfrac{\Rightarrow [F]\mathsf{K}_\alpha p, \neg[F]\mathsf{K}_\alpha p}{\mathsf{P}_{5.1}} \qquad \dfrac{p \Rightarrow p}{\mathsf{P}_{5.2}}}{\mathsf{P}_5} \in \mathcal{G}_6$.

If P_6 is $\Rightarrow [F]\mathsf{K}_\alpha p, \neg[F]\mathsf{K}_\alpha p$, then $\dfrac{\dfrac{\dfrac{[F]\mathsf{K}_\alpha p \Rightarrow [F]\mathsf{K}_\alpha p}{\mathsf{P}_6}}{\mathsf{P}_{5.1}} \qquad \dfrac{p \Rightarrow p}{\mathsf{P}_{5.2}}}{\mathsf{P}_5} \in \mathcal{G}_6$.

Thus we have the following proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[F]\mathsf{K}_\alpha p \Rightarrow [F]\mathsf{K}_\alpha p}{\Rightarrow [F]\mathsf{K}_\alpha p, \neg[F]\mathsf{K}_\alpha p}}{\Rightarrow \mathsf{K}_\alpha p, [P]\neg[F]\mathsf{K}_\alpha p} \qquad \dfrac{p \Rightarrow p}{\mathsf{K}_\alpha p \Rightarrow p}}{\Rightarrow [P]\neg[F]\mathsf{K}_\alpha p, p}}{}}{\Rightarrow [P]\neg[F]\mathsf{K}_\alpha p \vee p}$$

5.6 Notes and further researches

Temporal logics and epistemic logics are very useful for formalizing various notions which appear in computer science. Moreover, the development of temporal epistemic logics has been paid much attention recently. For instance, in [49] M. Wooldridge, C. Dixon and M. Fisher introduced tableau systems for temporal epistemic logics with the unary "next" operator $\bigcirc$ and the binary "until" operator $\mathcal{U}$ as temporal operators, and adopted *product* of temporal and knowledge/belief as Kripke style possible world semantics. In the paper, they gave a *model-search procedure* for these logics; i.e. the procedure which constructs a model for a given formula whenever it is satisfiable. In contrast with this, our decision procedure presented here is a *proof procedure*; i.e. the procedure which gives us a proof of of a given formula whenever it is provable. The following table shows differences between [49] and ours.

	[49]	Ours
Operators	$\bigcirc, \mathcal{U}$	$[F]$, $[P]$
Semantics	(t, w)	w
	product	fusion
Output	satisfiable	provable
	$\Downarrow$	$\Downarrow$
	model	proof

The decidability of both temporal epistemic logics $K_t \otimes K_B$ and $K_t \otimes K_K$ is shown by giving a proof-search procedure in this chapter. This result provides us not only a proof-search for temporal epistemic logics but also a relatively feasible procedure. Our idea is to introduce restrictions of applications of inference rules so that the subformula property holds in sequent systems with these restrictions. In fact, the subformula property, not the cut elimination theorem itself, is essential in getting a proof-search procedure. In [29], Mouri implemented a proof-search procedure for modal logic **S5**, based on Takano's result. Indeed, he constructed a proof assistant system xpe (X window system Proof Editor), and implement a proof-search procedure for **S5** system with acceptable cut rules on xpe. This is enough feasible. The procedure we introduced here still needs to check the repetition of sequents, which is necessary to make sure the termination of the procedure. This, of course, causes the inefficiency of the procedure, as we need to check whether each sequent obtained now appears already or not in the inference figure. So, it is one of most important problems to find a system for these temporal epistemic logics without checking the repetition.

It will be also interesting to find an efficient procedure for these logics equipped with both proof-search (for provable sequents) and model-search (for unprovable sequents, or for satisfiable sequents). In this respect, the approach taken by Mouri [29, 30] for modal logics **S4** and **K4** seems to be suggestive.

We have discussed two basic temporal epistemic logics in our thesis. In either case, we assume no interdependency between epistemic operators and temporal ones. But in practice, it will be more natural to assume certain dependency between them. While this will cause some complications and difficulties from a logical point of view, this will be an interesting and important research subject in future.

Chapter 6

Concluding remarks

In this chapter, we summarize the results of this thesis and mention further studies.

1. pseudo-Euclidean logics (Chapter 3)

For non-negative integers m and n, we gave a complete answer to inclusion relationship between pseudo-Euclidean logics $\mathbf{K} \oplus \{\Diamond^k \varphi \to \Box^m \Diamond^n \varphi\}$, denoted by E_k, where $k \leq 0$ by using Kripke type semantics; viz. we showed when $E_k \supseteq E_{k'}$ holds. As generalization of our results, it is interested in what happen if we allow both m and n to change. More precisely, let $E_k^{m,n}$ be the logic which is obtained from the smallest normal logic $\mathbf{K}$ by adding the axiom $\Diamond^k \phi \to \Box^m \Diamond^n \phi$, where $k, m, n \geq 0$. Then it is to see when $E_k^{m,n} \supseteq E_{k'}^{m',n'}$ holds.

2. fusions of modal logics (Chapter 4)

We discussed fusions of well-known modal logics. As for these, we attempt to derive some of both semantical and proof theoretic properties.

- semantical property
  If both modal logics L_1 and L_2 are any of **S4**and **S5**, then the finite model property of fusions with weak interdependent axioms which are of the form $\tau_i \varphi \to \sigma_i \varphi$ where both τ_i and σ_i are sequences of modalities holds.

- proof theoretic property for fusions
  Let L_1 and L_2 be of the form $\mathbf{K} \oplus Q$ where $Q \subseteq \{T, D, 4, 5, B\}$. If sequent systems $\mathcal{S}(L_1)$ and $\mathcal{S}(L_2)$ have a property $\mathcal{P}$, then $\mathcal{S}(L_1 \otimes L_2)$ has the property, where $\mathcal{P}$ is any of the cut elimination property, the cut restriction property and the extended cut restriction property.

- proof theoretic property for fusions with interdependent axioms
  Let L_1 and L_2 be of the form $\mathbf{K} \oplus Q$ where $Q \subseteq \{T, 4\}$. Section 4.4 discussed the cut

elimination theorem for each of $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\varphi\}$ and $L_{1\Box} \otimes L_{2\blacksquare} \oplus \{\Box\varphi \to \blacksquare\Box\varphi\}$. We can not construct sequent systems with subformula property, for $L_{1\Box} \otimes L_{2\blacksquare} \oplus Q$ where $L_1 \in \{\mathbf{K}, \mathbf{KT}\}$, $L_2 \in \{\mathbf{K4}, \mathbf{S4}\}$ and $Q \in \{\Box\varphi \to \blacksquare\varphi, \Box\varphi \to \blacksquare\Box\varphi\}$.

From the point of view of applications of multimodal logics, fusions with more general interdependent axioms will be desired.

3. temporal epistemic logics (Chapter 5)

The decidability of temporal epistemic logics $K_t \otimes K_B$ and $K_t \otimes K_K$ were shown by giving a proof search procedure. In order to do this, we restricted the formulas occurring in the upper sequents of rules cut rule and rules for temporal operators in Section 5.2 to certain subformulas so that the sequent system has subformula property. Then we showed the decidability in the bottom-up manner. However the procedure given in this thesis demands the backtracking for the sake of the loop-checking. we would like to give more efficient algorithm by proof search for temporal epistemic logics.

We discussed logics with axioms constructed independently between epistemic notion and temporal one. But finding out the logical property for some dependently temporal epistemic logics will be expected in many applications, which are logics with interdependent axioms among temporal notions and epistemic notions.

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