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Description	

Combination of Uncertain Class Membership Degrees with Probabilistic, Belief, and Possibilistic Measures

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Abstract One important issue of uncertain or fuzzy object-oriented models is that uncertain membership degrees of an object to the classes in a class hierarchy may be obtained from different sources while they are actually constrained by the subclass relation. In this paper we present the notion of admissible combination functions and an algorithm to propagate and combine prior uncertain membership degrees on a class hierarchy, which are possibly conflicting, in order to produce a tightly consistent uncertain membership assignment. We assume uncertain membership degrees to be measured by support pairs represented by sub-intervals of $[0,1]$. The usual probabilistic interval intersection, Dempster-Shafer, and possibilistic combination rules are examined and proved to be admissible ones.

1 Introduction

Object-oriented models have been shown to be useful for designing and implementing information and intelligent systems. The uncertain and fuzzy nature of real world problems has motivated significant research effort in extension of the classical object-oriented framework to a more powerful one involving uncertain and fuzzy values [4, 9].

Uncertain and imprecise attribute values lead to partial membership of an object to a class. Representing, computing, and reasoning with partial class membership have been one of the key issues in development of uncertain and fuzzy object-

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oriented systems. There were different measures proposed for uncertain class membership degrees. For instance, [12] defined for each class a membership function on a set of objects. In [3] linguistic labels were used to express the strength of the link of an object to a class. In [7] class membership was defined as similarity degrees between objects and classes. Meanwhile, [2] mentioned different measures, including probabilistic one, to be used for membership degrees.

However, most of the literature about uncertain and fuzzy object-oriented systems does not address and deal with the fact that membership degrees of an object can be obtained from different sources and to different classes in a class hierarchy, which can also be conflicting to each other. Meanwhile, a membership degree of an object to a class imposes constraints on membership degrees of the object to the subclasses and super-classes of that class. Therefore, a posterior membership degree of an object to a class should be a combination of a prior assigned one and those constrained and propagated from the subclasses and super-classes of that class.

In this paper we introduce the notion of admissible combination functions for uncertain membership degrees represented by sub-intervals of $[0,1]$, called support pairs. The lower and upper bounds of such a support pair can be interpreted as those of a probability interval, belief and plausibility degrees as in Dempster-Shafer theory [11], or necessity and possibility degrees as in possibility theory [8]. We then present an algorithm to propagate and combine membership support pairs, in order to produce a tightly consistent membership assignment for an object on a whole class hierarchy. These are refinement and extension of the early proposal in [5].

Section 2 defines the properties of an admissible uncertain class membership combination function and presents the propagation and combination algorithm. Sections 3, 4, and 5 particularly examine and prove the admissibility of the usual probabilistic interval intersection, Dempster-Shafer, and possibilistic combination rules. Finally, Section 6 concludes the paper with some remarks.

2 Combination Functions and Algorithm

Definition 1. Class Hierarchy
A *class hierarchy* is defined as a pair $(\mathbf{C}, \subseteq)$ where $\mathbf{C}$ is a set of classes and $\subseteq$ is the subclass partial order. Given $c_1, c_2 \in \mathbf{C}$, $c_1 \subseteq c_2$ denotes that c_1 is a subclass of c_2.

From now on, $\mathscr{I}([0,1])$ denotes the set of all sub-intervals of $[0,1]$.

Definition 2. Uncertain Membership Assignment
Let $(\mathbf{C}, \subseteq)$ be a class hierarchy and $\mathbf{O}$ be a set of objects. An *uncertain membership assignment* is a function $m : \mathbf{C} \times \mathbf{O} \rightarrow \mathscr{I}([0,1])$. For every $c \in \mathbf{C}$, $o \in \mathbf{O}$, $m(c,o)$ denotes the uncertain membership degree of o to c; $m(c,o) = []$ means that there is inconsistency about the membership of o to c.

The subclass relation imposes a constraint on membership degrees of an object to classes as stated in the following assumption, which was first proposed in [6].

Assumption 1

1. *If an object is a member of a class with some* positive characteristic *degree, then it is a member of any super-class of that class with the same degree.*
2. *If an object is a member of a class with some* negative characteristic *degree, then it is a member of any subclass of that class with the same degree.*

For fuzzy truth degrees, for instance, the positive and negative characteristics could be defined to be *true* and *false* characteristics, respectively. For examples, "(Object #1 is an EAGLE) is *very true*" entails "(Object #1 is a BIRD) is *very true*", and "(Object #1 is a BIRD) is *very false*" entails "(Object #1 is an EAGLE) is *very false*", provided that EAGLE⊆BIRD. The assumption here is that, if one can assign a class to an object with a TRUE-characteristic degree, then one can assign a super-class of this class to the object with at least the same truth degree (i.e., it is possibly truer), which is actually the least specific statement subsuming all other possible statements of the case. Dually, if one can assign a class to an object with a FALSE-characteristic degree, then one can assign a subclass of this class to the object with at least the same falsity degree (i.e., it is possibly falser).

Here, uncertainty lower bounds are considered as positive characteristic degrees, while uncertainty upper bounds are considered as negative characteristic ones. Therefore, if an object is a member of a class with a support pair $[l,u]$, then it is a member of any super-class of that class with the support pair $[l,1]$, and a member of any subclass of that class with the support pair $[0,u]$. This is in agreement with [10], for instance, which states that the membership degree of an object to a class is at least equal to its membership degree to a subclass of that class.

In this paper, given two support pairs $[x_1,x_2]$ and $[y_1,y_2]$, we write $[x_1,x_2] \leq_\mu [y_1,y_2]$ to denote that $x_1 \leq y_2$, and $[x_1,x_2] \leq_\tau [y_1,y_2]$ to denote that $x_1 \leq y_1$ and $x_2 \leq y_2$.

Definition 3. Consistent Uncertain Membership Assignment
An uncertain membership assignment m on $(\mathbf{C},\subseteq)$ and $\mathbf{O}$ is said to be *consistent* wrt (with respect to) $(\mathbf{C},\subseteq)$ iff (if and only if):

1. $m(c,o) \neq []$, for every $c \in \mathbf{C}$ and $o \in \mathbf{O}$, and
2. $m(c_i,o) \leq_\mu m(c_j,o)$, for every $c_i \subseteq c_j \in \mathbf{C}$.

It is called *tightly consistent* when $m(c_i,o) \leq_\tau m(c_j,o)$.

This notion of consistency of support pair assignment wrt the subclass relation constraint on a class hierarchy was first proposed in [6]. Its rational is that, if $m(c_i,o) \leq_\mu m(c_j,o)$ then there exist $u \in m(c_i,o)$ and $v \in m(c_j,o)$ such that $u \leq v$. The notion of tight consistency added here requires further that both the lower and upper bounds of $m(c_i,o)$ are respectively smaller than those of $m(c_j,o)$. One can observe that $\leq_\tau$ is a partial order, while $\leq_\mu$ is not, and $\leq_\tau$ is stronger than $\leq_\mu$ in the sense that $[x_1,x_2] \leq_\tau [y_1,y_2]$ implies $[x_1,x_2] \leq_\mu [y_1,y_2]$.

Due to Assumption 1 above, given a prior uncertain class membership assignment on a class hierarchy, the posterior membership degree of an object to a class is determined not only by a prior one of the object to that class alone, but also by the constrained membership degrees of the object to the super-classes and subclasses of that class. That is, if $(\{c_1, c_2, \ldots, c_n\}, \subseteq)$ is the class hierarchy and $[u_i, v_i]$ is the prior membership support pair of the object to the class c_i, for every i from 1 to n, then the posterior support pair for the object belonging to the class c_k is a combination of the support pairs in the set $\{[u_k, v_k]\} \cup \{[0, v_i] | c_k \subseteq c_i, i \neq k\} \cup \{[u_j, 1] | c_j \subseteq c_k, j \neq k\}$.

An important issue here is that a used combination function should maintain the consistency of membership degrees of every object on a whole class hierarchy as expressed in Definition 3. For this, we introduce the notion of admissible functions as defined below (cf. [5]).

Definition 4. Admissible Combination Function
An *uncertain membership combination function* $\otimes : \mathscr{I}([0,1]) \times \mathscr{I}([0,1]) \to \mathscr{I}([0,1])$ is said to be *admissible* if satisfying the following properties as long as not resulting in the empty interval []:

1. $\otimes$ is commutative and associative,
2. $\otimes$ is monotonic: $[x_1, x_2] \leq_\tau [y_1, y_2] \Rightarrow [x_1, x_2] \otimes [u, v] \leq_\tau [y_1, y_2] \otimes [u, v]$
3. $[x_1, x_2] \otimes [0, z] \leq_\tau [x_1, x_2]$
4. $[y_1, y_2] \leq_\tau [y_1, y_2] \otimes [z, 1]$.

The first two properties are desirable for any combination function. Meanwhile, properties 3 and 4 show that $[0, z]$, as a negative constraint, and $[z, 1]$, as a positive constraint, respectively decreases and increases the support pairs they are combined with.

Moreover, one has the following derived properties for an admissible combination function:

5. $[x_1, x_2] \otimes [0, 1] = [x_1, x_2]$
6. $[x_1, x_2] \otimes [0, y_2] \leq_\tau [x_1, 1] \otimes [y_1, y_2]$

Property 5 is a direct consequence of properties 3 and 4, due to $[x_1, x_2] \otimes [0, 1] \leq_\tau [x_1, x_2]$ and $[x_1, x_2] \leq_\tau [x_1, x_2] \otimes [0, 1]$. Intuitively, since $[0, 1]$ denotes an absolutely uninformative support pair, it should be neutral when combined with another support pair. Property 6 is a consequence of properties 2, 3, and 4, because $[x_1, x_2] \leq_\tau [x_1, 1]$ and $[0, y_2] \leq_\tau [y_1, y_2]$ and $\otimes$ is monotonic. Here $[x_1, 1]$ means "at least x_1" and $[0, y_2]$ means "at most y_2", which self-explain the property.

Algorithm 1 below exploits the subclass relation constraint on uncertain membership to combine and resolve possibly inconsistent prior support pairs of an object on a class hierarchy. Suppose a class hierarchy is $(\{c_1, c_2, \ldots, c_n\}, \subseteq)$ and the support pair of an object o to each class c_i is $[u_i, v_i]$. The idea of the algorithm is that, for every i and j from 1 to n, if c_i is a subclass of c_j, then pass $[u_i, 1]$ to c_j and $[0, v_j]$ to c_i, on the basis that the membership degree of o to c_i is smaller than to c_j as assumed above. The resulting support pair of o to each class is then obtained as a conjunction of $[u_i, v_i]$ and those passed from c_i's subclasses and super-classes. As such, the computational complexity of this algorithm is $O(n^2)$.

The algorithm was first proposed in [6], but for only the interval intersection function and without any proof for its correctness. We present a proof for it here.

Algorithm 1 The propagation and combination algorithm

Input: A prior uncertain membership assignment m for an object o wrt a class hierarchy $(\{c_1, c_2, \ldots, c_n\}, \subseteq)$ and an admissible membership combination function $\otimes$.

Output: A posterior uncertain membership assignment m' for an object o wrt $(\{c_1, c_2, \ldots, c_n\}, \subseteq)$ such that, for every $c_i \subseteq c_j$, $m'(c_i, o) \leq_\tau m'(c_j, o)$, as long as $m'(c_i, o) \neq []$ and $m'(c_j, o) \neq []$.

for every i from 1 to n **do**
    $S_i = \{[u_i, v_i] = m(c_i, o)\}$
end for
for every i from 1 to $n-1$ **do**
    for every j from $i+1$ to n **do**
        if $c_i \subseteq c_j$ **then**
            $S_i = S_i \cup \{[0, v_j]\}, S_j = S_j \cup \{[u_i, 1]\}$
        else
            if $c_j \subseteq c_i$ **then**
                $S_i = S_i \cup \{[u_j, 1]\}, S_j = S_j \cup \{[0, v_i]\}$
            end if
        end if
    end for
end for
return $m'(c_i, o) = \bigotimes_{[u,v] \in S_i} [u, v] (1 \leq i \leq n)$.

Proposition 1. *Algorithm 1 is correct wrt its input-output specification.*

Proof. For simplicity, but without loss of generality, suppose that c_1 and c_2 are two arbitrary classes such that $c_1 \subseteq c_2$. One has:

1. $S_1 = \{[u_1, v_1], [0, v_2]\} \cup \{[0, v_j] | c_1 \subseteq c_j, j \neq 1, 2\} \cup \{[u_i, 1] | c_i \subseteq c_1, i \neq 1\}$.
   $S_2 = \{[u_2, v_2], [u_1, 1]\} \cup \{[u_i, 1] | c_i \subseteq c_2, i \neq 1, 2\} \cup \{[0, v_j] | c_2 \subseteq c_j, j \neq 2\}$.
2. $[u_1, v_1] \otimes [0, v_2] \leq_\tau [u_2, v_2] \otimes [u_1, 1]$ (due to Property 6 presented above).
3. $\{[0, v_j] | c_2 \subseteq c_j, j \neq 2\} \subseteq \{[0, v_j] | c_1 \subseteq c_j, j \neq 1, 2\}$, because of $c_1 \subseteq c_2$. Since a combination with $[0, z]$ decreases a membership support, due to Property 3 in Definition 4, the following holds:
   $$[u_1, v_1] \otimes [0, v_2] \otimes_{\{j | c_1 \subseteq c_j, j \neq 1,2\}} [0, v_j] \leq_\tau [u_2, v_2] \otimes [u_1, 1] \otimes_{\{j | c_2 \subseteq c_j, j \neq 2\}} [0, v_j]$$
4. $\{[u_i, 1] | c_i \subseteq c_1, i \neq 1\} \subseteq [u_i, 1] | c_i \subseteq c_2, i \neq 1, 2\}$, because of $c_1 \subseteq c_2$. Since a combination with $[z, 1]$ increases a membership support, due to Property 4 in Definition 4, the following holds:
   $$\begin{aligned} &[u_1, v_1] \otimes [0, v_2] \otimes_{\{j | c_1 \subseteq c_j, j \neq 1,2\}} [0, v_j] \otimes_{\{i | c_i \subseteq c_1, i \neq 1\}} [u_i, 1] \\ &\leq_\tau [u_2, v_2] \otimes [u_1, 1] \otimes_{\{j | c_2 \subseteq c_j, j \neq 2\}} [0, v_j] \otimes_{\{i | c_i \subseteq c_2, i \neq 1,2\}} [u_i, 1] \end{aligned}$$

Therefore $m'(c_1, o) \leq_\tau m'(c_2, o)$, as long as $m'(c_i, o) \neq []$ and $m'(c_j, o) \neq []$.

3 Interval Intersection

In this section we examine the common and simple combination function that intersects involved support pairs, which could be interpreted as probability lower and upper bounds.

Definition 5. Interval Intersection Function
Let $\otimes_i : \mathscr{I}([0,1]) \times \mathscr{I}([0,1]) \to \mathscr{I}([0,1])$ be defined by

$$[x_1,x_2] \otimes_i [y_1,y_2] = [x_1,x_2] \cap [y_1,y_2] = [\max(x_1,y_1), \min(x_2,y_2)].$$

Proposition 2. *$\otimes_i$ is an admissible uncertain membership combination function.*

Proof.

1. It is obvious that $\otimes_i$ is commutative and associative, because the min and max functions are so.
2. $[x_1,x_2] \otimes_i [u,v] = [\max(x_1,u), \min(x_2,v)]$
   $[y_1,y_2] \otimes_i [u,v] = [\max(y_1,u), \min(y_2,v)]$
   Since $[x_1,x_2] \leq_\tau [y_1,y_2]$, i.e., $x_1 \leq y_1$ and $x_2 \leq y_2$, one has $\max(x_1,u) \leq \max(y_1,u)$ and $\min(x_2,v) \leq \min(y_2,v)$, and thus $[x_1,x_2] \otimes_i [u,v] \leq_\tau [y_1,y_2] \otimes_i [u,v]$.
3. $[x_1,x_2] \otimes_i [0,z] = [x_1, \min(x_2,z)] \leq_\tau [x_1,x_2]$.
4. $[y_1,y_2] \leq_\tau [\max(y_1,z),y_2] = [y_1,y_2] \otimes_i [z,1]$.

Example 1. Suppose the uncertain membership assignment μ for an object wrt the class hierarchy illustrated in Figure 1. It expresses that it is certain to a degree between 0.7 and 1 that the object belongs to the class BIRD, and between 0.8 and 1 to the class PENGUIN. Meanwhile, there is inconsistency as the object does not belong to the class ADULT-BIRD, i.e. with the membership support $[0,0]$, but to its subclass ADULT-PENGUIN with the membership support $[.5,.5]$. Also, the membership support pairs assigned to the classes BIRD and PENGUIN are not tightly consistent.

Applying Algorithm 1 using the interval intersection function, one obtains the membership support pair for each class as follows:

BIRD: $[.7,1] \otimes_i [0,1] \otimes_i [.8,1] \otimes_i [.5,1] = [.8,1]$
ADULT-BIRD: $[0,0] \otimes_i [.5,1] \otimes_i [0,1] = []$
PENGUIN: $[.8,1] \otimes_i [.5,1] \otimes_i [0,1] = [.8,1]$
ADULT-PENGUIN: $[.5,.5] \otimes_i [0,0] \otimes_i [0,1] \otimes_i [0,1] = []$

The empty membership support pairs for ADULT-BIRD and ADULT-PENGUIN are due to the inconsistency of the given membership assignment as noted above. Except for that, the posterior membership support pairs computed for the classes BIRD and PENGUIN become tightly consistent.

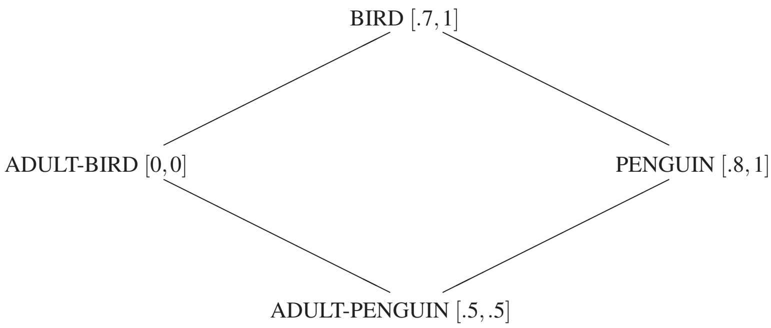

Fig. 1 A class hierarchy with an uncertain membership assignment

Proposition 3. *Given a prior consistent uncertain membership assignment for an object wrt a class hierarchy, Algorithm 1 using $\otimes_i$ produces a posterior tightly consistent membership assignment for the object wrt the class hierarchy.*

Proof. What is to be proved is only that no combination in Algorithm 1 results in []. Indeed, for every $c_i \subseteq c_j$ and the current membership support pairs to c_i and c_j being respectively $[u_i, v_i]$ and $[u_j, v_j]$, the combinations are only $[u_i, v_i] \otimes_i [0, v_j]$ and $[u_j, v_j] \otimes_i [u_i, 1]$. Meanwhile, $[u_i, v_i] \leq_\mu [u_j, v_j]$, i.e., $u_i \leq v_j$, because the given membership assignment is consistent. So, for $\otimes_i$, one has:
$[u_i, v_i] \otimes_i [0, v_j] = [u_i, \min(v_i, v_j)] \neq []$
$[u_j, v_j] \otimes_i [u_i, 1] = [\max(u_j, u_i), v_j] \neq []$
because $u_i \leq \min(v_i, v_j)$ and $\max(u_j, u_i) \leq v_j$.

4 Dempster-Shafer Combination

As shown in Example 1, the interval intersection function may result in empty membership support pairs. Dempster-Shafer combination rule [11] can resolve join of conflicting support pairs, whose intersection is empty.

We recall that, in Dempster-Shafer theory, a basic probability mass is assigned to each non-empty subset A of the set of all hypotheses, and denoted by $m(A)$. The joint mass assignment of two mass assignments $m_1(A)$ and $m_2(A)$ is defined as follows:

$$m(A) = \frac{\sum_{X \cap Y = A} m_1(X).m_2(Y)}{\sum_{X \cap Y \neq \emptyset} m_1(X).m_2(Y)}$$

This combination function is thus commutative and associative.

In [1], a support pair $[x_1, x_2]$ for a proposition p is interpreted as the following mass assignment on the power set of $\{p, \neg p\}$:

$$\{p\} : x_1, \{\neg p\} : 1 - x_2, \{p, \neg p\} : x_2 - x_1$$

Dempster-Shafer combination of two support pairs $[x_1, x_2]$ and $[y_1, y_2]$ for p can be first performed as the combination of their corresponding mass assignments, yielding the following one:

$\{p\}$: $K(x_1y_2 + x_2y_1 - x_1y_1)$
$\{\neg p\}$: $1 - Kx_2y_2$
$\{p, \neg p\}$: $K(x_2y_2 + x_1y_1 - x_1y_2 - x_2y_1)$
where $K = 1/(1 + x_1y_2 + x_2y_1 - x_1 - y_1)$. Then the combined support pair for p can be derived as $[K(x_1y_2 + x_2y_1 - x_1y_1), Kx_2y_2]$. We note that it is always a valid support pair, i.e., $0 \leq K(x_1y_2 + x_2y_1 - x_1y_1) \leq Kx_2y_2 \leq 1$.

Definition 6. Dempster-Shafer Combination Function
Let $\otimes_{ds} : \mathscr{I}([0,1]) \times \mathscr{I}([0,1]) \to \mathscr{I}([0,1])$ be defined by

$$[x_1, x_2] \otimes_{ds} [y_1, y_2] = [K(x_1y_2 + x_2y_1 - x_1y_1), Kx_2y_2]$$

where $K = 1/(1 + x_1y_2 + x_2y_1 - x_1 - y_1)$.

Proposition 4. *$\otimes_{ds}$ is an admissible uncertain membership combination function.*

Proof.

1. Since Dempster-Shafer rule of combining probability masses is commutative and associative, so is $\otimes_{ds}$.
2. $[z_1, z_2] \otimes_{ds} [u, v] = [K(z_1v + z_2u - z_1u), Kz_2v]$
   where $K = 1/(1 + z_1v + z_2u - z_1 - u)$.
   Consider the function $f(z_1, z_2) = K(z_1v + z_2u - z_1u)$. One has

$$\partial f(z_1, z_2)/\partial z_1 = K^2[(v - u)(1 + z_2u - u) + (1 - v)z_2u] \geq 0,$$
$$\partial f(z_1, z_2)/\partial z_2 = K^2u(1 - u)(1 - z_1)] \geq 0.$$

   So $f(z_1, z_2)$ is increasing wrt both z_1 and z_2.
   Similarly, consider the function $g(z_1, z_2) = Kz_2v$. One has

$$\begin{aligned} \partial g(z_1, z_2)/\partial z_1 &= K^2v(1 - v)z_2 \geq 0, \text{ and} \\ \partial g(z_1, z_2)/\partial z_2 &= K^2v(1 + z_1v - z_1 - u) \\ &\geq K^2v(1 + z_1v - z_1 - v) = K^2v(1 - v)(1 - z_1) \geq 0. \end{aligned}$$

   So $g(z_1, z_2)$ is also increasing wrt both z_1 and z_2.
   Hence, $[x_1, x_2] \otimes_{ds} [u, v] \leq_\tau [y_1, y_2] \otimes_{ds} [u, v]$ if $[x_1, x_2] \leq_\tau [y_1, y_2]$.

3. $[x_1,x_2] \otimes_{ds} [0,z] = [Kx_1z, Kx_2z]$, where $K = 1/(1+x_1z-x_1)$.
   It is easy to check that $Kz \leq 1$, and thus $[Kx_1z, Kx_2z] \leq_\tau [x_1,x_2]$.
4. $[y_1,y_2] \otimes_{ds} [z,1] = [K(zy_2+y_1-zy_1), Ky_2]$, where $K = 1/(1+zy_2-z)$.
   It is easy to check that $K(zy_2+y_1-zy_1) \geq y_1$ and $Ky_2 \geq y_2$, and thus $[y_1,y_2] \leq_\tau [y_1,y_2] \otimes_{ds} [z,1]$.

Example 2. Applying Algorithm 1 using Dempster-Shafer combination function on the class hierarchy and membership assignment as in Example 1, one obtains the membership support pair for each class as follows:
BIRD: $[.7,1] \otimes_{ds} [0,1] \otimes_{ds} [.8,1] \otimes_{ds} [.5,1] = [.97,1]$
ADULT-BIRD: $[0,0] \otimes_{ds} [.5,1] \otimes_{ds} [0,1] = [0,0]$
PENGUIN: $[.8,1] \otimes_{ds} [.5,1] \otimes_{ds} [0,1] = [.9,1]$
ADULT-PENGUIN: $[.5,.5] \otimes_{ds} [0,0] \otimes_{ds} [0,1] \otimes_{ds} [0,1] = [0,0]$
One can observe that the posterior membership support pairs computed for all the classes become tightly consistent.

Proposition 5. *Using $\otimes_{ds}$, Algorithm 1 always produces a tightly consistent membership assignment.*

Proof. This is due to a property of Dempster-Shafer combination function that it never results in the empty interval [] as noted above.

5 Possibilistic Combination

In possibility theory, uncertainty of a proposition p is expressed by a pair $[N(p), \Pi(p)]$, where $N(p)$ and $\Pi(p)$ are respectively called the necessity and possibility degrees and satisfy the condition $\max(1-N(p), \Pi(p)) = 1$. Different combination rules were proposed for necessity and possibility degrees obtained from various sources [8]. Here we apply a multiplicative and associative one for combining membership support pairs as defined below.

Definition 7. Possibilistic Combination Function
Let $\otimes_p : \mathscr{I}([0,1]) \times \mathscr{I}([0,1]) \rightarrow \mathscr{I}([0,1])$ be defined by

$$[x_1,x_2] \otimes_p [y_1,y_2] = [1-D(1-x_1)(1-y_1), Dx_2y_2]$$

where $D = 1/\max((1-x_1)(1-y_1), x_2y_2)$.

Proposition 6. *$\otimes_p$ is an admissible uncertain membership combination function.*

Proof.

1. $\otimes_p$ is clearly commutative. The associativity of the function was proved in [8].
2. For the monotonic property, we have to prove that $[x_1,x_2] \leq_\tau [y_1,y_2] \Rightarrow [x_1,x_2] \otimes_p [u,v] \leq_\tau [y_1,y_2] \otimes_p [u,v]$. According to the above-mentioned condition of a necessity-possibility pair, either u is 0 or v is 1. So we prove this property in these two cases.

(a) $[x_1,x_2] \otimes_p [0,v] \leq_\tau [y_1,y_2] \otimes_p [0,v]$
Indeed, one has:
$[x_1,x_2] \otimes_p [0,v] = [1 - \frac{(1-x_1)}{\max(1-x_1,x_2v)}, \frac{x_2v}{\max(1-x_1,x_2v)}]$ and
$[y_1,y_2] \otimes_p [0,v] = [1 - \frac{(1-y_1)}{\max(1-y_1,vy_2)}, \frac{vy_2}{\max(1-y_1,vy_2)}]$

- $x_1 = 0$ and $y_1 = 0$: $[x_1,x_2] \otimes_p [0,v] = [0,x_2v] \leq_\tau [y_1,y_2] \otimes_p [0,v] = [0,vy_2]$, because $x_2 \leq y_2$.
- $x_1 = 0$ and $y_2 = 1$: $[x_1,x_2] \otimes_p [0,v] = [0,x_2v] \leq_\tau [y_1,y_2] \otimes_p [0,v] = [1 - \frac{(1-y_1)}{\max(1-y_1,v)}, \frac{v}{\max(1-y_1,v)}]$, because $x_2v \leq v \leq v/\max(1-y_1,v)$.
- $x_2 = 1 \Rightarrow y_2 = 1$, $1-x_1 \leq v \Rightarrow 1-y_1 \leq v$: $[x_1,x_2] \otimes_p [0,v] = [1 - \frac{(1-x_1)}{v}, 1] \leq_\tau [y_1,y_2] \otimes_p [0,v] = [1 - \frac{(1-y_1)}{v}, 1]$, because $x_1 \leq y_1$.
- $x_2 = 1 \Rightarrow y_2 = 1$, $v \leq 1-x_1$: $[x_1,x_2] \otimes_p [0,v] = [0, v/(1-x_1)] \leq_\tau [y_1,y_2] \otimes_p [0,v] = [1 - \frac{(1-y_1)}{\max(1-y_1,v)}, \frac{v}{\max(1-y_1,v)}]$, because $\max(1-y_1,v) \leq (1-x_1)$.

(b) $[x_1,x_2] \otimes_p [u,1] \leq_\tau [y_1,y_2] \otimes_p [u,1]$
In this case, one has:
$[x_1,x_2] \otimes_p [u,1] = [1 - \frac{(1-x_1)(1-u)}{\max((1-x_1)(1-u),x_2)}, \frac{x_2}{\max((1-x_1)(1-u),x_2)}]$ and
$[y_1,y_2] \otimes_p [u,1] = [1 - \frac{(1-u)(1-y_1)}{\max((1-u)(1-y_1),y_2)}, \frac{y_2}{\max((1-u)(1-y_1),y_2)}]$

- $x_1 = 0$ and $y_1 = 0$, $y_2 \leq 1-u \Rightarrow x_2 \leq 1-u$:
$[x_1,x_2] \otimes_p [u,1] = [0, x_2/(1-u)] \leq_\tau [y_1,y_2] \otimes_p [u,1] = [0, y_2/(1-u)]$, because $x_2 \leq y_2$.
- $x_1 = 0$ and $y_1 = 0$, $1-u \leq y_2$:
$[x_1,x_2] \otimes_p [u,1] = [1 - \frac{(1-u)}{\max(1-u,x_2)}, \frac{x_2}{\max(1-u,x_2)}] \leq_\tau [y_1,y_2] \otimes_p [u,1] = [1 - (1-u)/y_2, 1]$, because $\max(1-u,x_2) \leq y_2$.
- $x_1 = 0$ and $y_2 = 1$:
$[x_1,x_2] \otimes_p [u,1] = [1 - \frac{(1-u)}{\max(1-u,x_2)}, \frac{x_2}{\max(1-u,x_2)}] \leq_\tau [y_1,y_2] \otimes_p [u,1] = [1 - (1-u)(1-y_1), 1]$, because $(1-y_1) \leq 1/\max(1-u,x_2)$.
- $x_2 = 1 \Rightarrow y_2 = 1$:
$[x_1,x_2] \otimes_p [tu,1] = [1-(1-x_1)(1-u), 1] \leq_\tau [y_1,y_2] \otimes_p [u,1] = [1-(1-u)(1-y_1), 1]$, because $x_1 \leq y_1$.

3. $[x_1,x_2] \otimes_p [0,z] = [1 - \frac{(1-x_1)}{\max(1-x_1,x_2z)}, \frac{x_2z}{\max(1-x_1,x_2z)}]$

- $x_1 = 0$: $[x_1,x_2] \otimes_p [0,z] = [0,x_2z)] \leq_\tau [x_1,x_2]$.
- $x_2 = 1$: $[x_1,x_2] \otimes_p [0,z] = [1 - \frac{(1-x_1)}{\max(1-x_1,z)}, \frac{z}{\max(1-x_1,z)}] \leq_\tau [x_1,x_2]$, because $1 - x_1 \leq (1-x_1)/\max(1-x_1,z)$.

4. $[y_1,y_2] \otimes_p [u,1] = [1 - \frac{(1-u)(1-y1)}{\max((1-u)(1-y_1),y_2)}, \frac{y_2}{\max((1-u)(1-y_1),y_2)}]$

- $y_1 = 0$: $[y_1,y_2] \leq_\tau [y_1,y_2] \otimes_p [u,1] = [1 - \frac{(1-u)}{\max(1-u,y_2)}, \frac{y_2}{\max(1-u,y_2)}]$, because $y_2 \leq y_2/\max(1-u,y_2)$.
- $y_2 = 1$: $[y_1,y_2] \leq_\tau [y_1,y_2] \otimes_p [u,1] = [1-(1-u)(1-y_1), 1]$, because $(1-u)(1-y_1) \leq 1-y_1$.

Example 3. In possibility theory, the assigned membership support pairs to the classes ADULT-BIRD and ADULT-PENGUIN in Example 1 are not valid ones. Applying Algorithm 1 using the defined possibilistic combination function on only the classes BIRD and PENGUIN, one obtains the membership support pair for each class as follows:

BIRD: $[.7, 1] \otimes_p [.8, 1] = [.94, 1]$

PENGUIN: $[.8, 1] \otimes_p [0, 1] = [.8, 1]$

As such, the posterior membership support pairs computed for these two classes become tightly consistent.

Proposition 7. *Using $\otimes_p$, Algorithm 1 always produces a tightly consistent membership assignment.*

Proof. The normalization factor D in Definition 7 assures that $\max(D(1-x_1)(1-y_1), Dx_2y_2) = 1$. So $\otimes_p$ never results in the empty interval [].

6 Conclusion

We have presented an algorithm to propagate and combine uncertain membership support pairs on a class hierarchy. As proved, given a prior membership assignment from various sources for an object to the classes in the hierarchy, the algorithm produces a tightly consistent posterior membership assignment for that object to the classes. As such, it also resolves possibly conflicting prior membership support pairs. The algorithm is based on an admissible combination function whose properties have been defined and membership constraints due to the subclass relation.

Three specific combination functions, namely, the interval intersection, Dempster-Shafer, and possibilistic ones have been examined and proved to be admissible. We have also proved that interval intersection produces a tightly consistent posterior membership assignment if the prior assignment is consistent. Meanwhile, Dempster-Shafer and possibilistic combination functions always produce a tightly consistent one.

The results can be applied for computation and reasoning in object-oriented or ontology-based systems involving uncertainty, in particular one of class membership. Moreover, the framework of uncertain membership combination presented here could be adapted for other belief or uncertainty measures as well. These are among the topics we are investigating.

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