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Semisimplicity, Amalgamation Property and Finite Embeddability Property of Residuated Lattices

by

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Abstract

In this thesis, we study semisimplicity, amalgamation property and finite embeddability property of residuated lattices. We prove semisimplicity and amalgamation property of residuated lattices which are of purely algebraic character, by using proof-theoretic methods and results of substructural logics. On the other hand, we show the finite model property (FMP) for various substructural logics, including fuzzy logics as a consequence of the finite embeddability property (FEP) of corresponding classes of residuated lattices. Thus all of our studies are attempts at bridging gaps between algebras and logics.

The first topics of our thesis is finite embeddability property (FEP) of various classes of integral residuated lattices. A class of algebras has the FEP if every finite partial subalgebra of a member of the class can be embedded into a finite member of the same class. W. Blok and C. J. van Alten showed that the class of all partially ordered biresiduated integral groupoids has the FEP. This implies that the variety of all integral residuated lattices ($\mathcal{IRL}$) has the FEP. The FEP of a given variety of $\mathcal{IRL}$ implies the finite model property (FMP) for the corresponding logic. We prove the FEP for various classes of the variety $\mathcal{IRL}$. From this the FMP follows for various substructural logics including fuzzy logics.

Next, we study the semisimplicity of free $\mathbf{FL}_w$-algebras. An algebra is semisimple if it has a subdirect representation with simple factors. V. N. Grišin proved that every free $\mathbf{CFL}_{ew}$-algebra is semisimple. To show this Grišin introduced a new sequent system which is equivalent to $\mathbf{CFL}_{ew}$ and showed that the cut elimination theorem holds for the sequent system. Later, T. Kowalski and H. Ono proved that every free $\mathbf{FL}_{ew}$-algebras is also semisimple using Grišin's idea. By using this, they proved that the variety of all $\mathbf{FL}_{ew}$-algebras is generated by it finite simple members. By using the similar technique, we show that every free $\mathbf{FL}_w$-algebras is semisimple. We will introduce a new sequent system $\mathbf{FL}_w^+$ which is equivalent to $\mathbf{FL}_w$ and for which cut elimination theorem holds. Using proof-theoretic properties of $\mathbf{FL}_w^+$, we show the semisimplicity of free $\mathbf{FL}_w$-algebras.

Lastly, we discuss the amalgamation property (AP) of commutative residuated lattices. Kowalski showed the AP for the variety $\mathcal{FL}_{ew}$ of all $\mathbf{FL}_e$-algebras. The result is obtained by the fact that (1) the logical system $\mathbf{FL}_{ew}$ has the Craig's interpolation property (CIP), and (2) the variety of $\mathcal{FL}_{ew}$ has the equational interpolation property (EIP). A. Wroński proved that the EIP of a variety implies the AP. Therefore the variety $\mathcal{FL}_{ew}$ has the AP. We show that Kowalski's proof of the AP works well also for the variety $\mathcal{CRL}$ of all commutative residuated lattices. To show this result, we introduce a sequent for commutative residuated lattices and show the CIP, and using them we prove that the variety $\mathcal{CRL}$ has the EIP. By considering filters on residuated lattices, we can show that many important subclasses of $\mathcal{CRL}$ has the AP. Moreover, we can show that if $\mathbf{L}$ is a logic which is an extension of $\mathbf{FL}_e$ with the CIP and $\mathcal{K}$ is the variety which is corresponding to $\mathbf{L}$, then $\mathcal{K}$ has the EIP, from which the AP of $\mathcal{K}$ follows.

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Contents

Chapter 1

Introduction

In this thesis, we study semisimplicity, amalgamation property and finite embeddability property of residuated lattices. We prove semisimplicity and amalgamation property of residuated lattices which are of purely algebraic character, by using proof-theoretic methods and results of substructural logics. On the other hand, we show the finite model property for various substructural logics, including fuzzy logics as a consequence of the finite embeddability property of corresponding classes of residuated lattices. Thus all of our studies are attempts at bridging gaps between algebras and logics.

In this chapter, we will give a short review of residuated lattices and substructural logics in Section 1.1. Also we will give a survey of contents of this thesis in Section 1.2.

1.1 Residuated lattices and substructural logics

Residuated Lattices

A residuated lattice is an algebraic structure which consists both lattice and monoid structures, and has binary operations called residuations. Residuation is a fundamental concept of ordered structures and categories.

In 1930s, residuated lattices were studied by M. Ward and R. P. Dilworth[12, 46, 47]. They investigated that the general properties of ideals of rings. The structure theory of residuated lattices were studied by K. Blount and C. Tsinakis [9]. There are many studies of residuated lattices. For examples, the word problems of various classes of residuated lattices and researches about cancellative residuated lattice were investigated by P. Jipsen and C. Tsinakis[22], and N. galatos[16].

We give a precise definition of a residuated lattice. An algebraic structure $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ is called a residuated lattice if it satisfies the following conditions:

> (1) $\langle A, \wedge, \vee \rangle$ is a lattice, (2) $\langle A, \cdot, 1 \rangle$ is a monoid, (3) $\backslash$ and $/$ are binary operations which satisfy that $a \cdot b \leq c \Leftrightarrow a \leq c/b \Leftrightarrow b \leq a\backslash c$ hold for all a, b, c in A.

Operations $\backslash$ and $/$ are called *left* and *right* residuations, respectively. Note that terms in a residuated lattice, multiplication $\cdot$ has priority over the residuations $\backslash, /$, which have priority over the lattice operations $\wedge, \vee$.

We introduce some of the important subclasses of residuated lattices.

(1) Commutativity: $xy = yx$.

(2) Integrality: $x \leq 1$.

(3) Increasing-idempotency: $x \leq x^2$.

(4) Cancellativity: $xz = yz \Longrightarrow x = y$, and $zx = zy \Longrightarrow x = y$.

(5) Distributivity: $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Substructural logics

Basic substructural logics are defined as sequent systems obtained from either **LK** for classical logic or **LJ** for intuitionistic logic by deleting some or all substructural rules. We say that substructural logics are extensions of these basic substructural logics when they are formalized in sequent system.

The basic **FL**-systems are extensions of the **FL** which are obtained **FL** by adding with some or all structural rules defined by the following.

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \theta}{\Gamma, \beta, \alpha, \Delta \Rightarrow \theta}\ (e \Rightarrow) : \textit{exchange}$$

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \theta}{\Gamma, \alpha, \Delta \Rightarrow \theta}\ (c \Rightarrow) : \textit{contraction}$$

$$\frac{\Gamma, \Delta \Rightarrow \theta}{\Gamma, \alpha, \Delta \Rightarrow \theta}\ (w \Rightarrow) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \theta}\ (\Rightarrow w) \quad : \textit{weakening}$$

Many non-classical logics including linear logic, many-valued logics, Hájeck's basic logic (BL), relevant logics are regarded as substructural logics which are originated from different motivations. The study of substructural logics will give us a uniform framework for investigating various kinds of non-classical logics. In the following we introduce various systems of basic substructural logics. We will express those by adding the corresponding letters e, c and w to **FL** as subscripts. Here, we define $\neg\phi$ by $\phi \to 0$.

$\mathbf{FL}_e = \mathbf{FL}$ + exchange = $\mathbf{LJ} - \{w, c\}$.
$\mathbf{FL}_w = \mathbf{FL}$ + weakening = $\mathbf{LJ} - \{e, c\}$.
$\mathbf{FL}_c = \mathbf{FL}$ + contraction = $\mathbf{LJ} - \{e, w\}$.
$\mathbf{FL}_{ew} = \mathbf{FL}$ + exchange + weakening = $\mathbf{LJ} - \{c\}$.
$\mathbf{CFL}_e = \mathbf{LK} - \{w, c\} = \mathbf{FL}_e + [\neg\neg\phi \to \phi]$
$\mathbf{CFL}_{ew} = \mathbf{LK} - \{c\} = \mathbf{CFL}_e$ + weakening (Grišin's logic)

A logic **L** is one of the **FL**-systems (**CFL**-systems) if **L** is defined as sequent systems obtained from **LJ** (**LK**) for intuitionistic logic (classical logic) by deleting some or all substructural rules.

Residuated lattices and substructural logics

It is well known that for each substructural logic, there exists a class of residuated lattices as an algebraic semantics for the logic. we introduce various classes of residuated lattices which are algebraic semantics for substructural logics.

In the usual way, we can give an interpretation of formulas in an **FL**-algebra. An **FL**-algebra **A** is a residuated lattice with fixed, but arbitrary element 0 in A. An $\mathbf{FL}_\perp$-algebra is an **FL**-algebra with bottom element $\perp$. In this case, an $\mathbf{FL}_\perp$-algebra has the four constants, $1, 0, \perp, \top$.

Let **A** be an **FL**-algebra. A valuation v on **A** is any mapping from the set of all propositional variable to the set A. We can extend each valuation v to a mapping from the set of all formulas to A inductively as follows. We use the same symbols for logical connectives and constants as those for corresponding algebraic operations and constants, respectively.

$v(1) = 1$ and $v(0) = 0$,
$v(\top) = \top$ and $v(\perp) = \perp$, when the language has $\top$ and $\perp$ and **A** is bounded.
$v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$,
$v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$,
$v(\alpha * \beta) = v(\alpha) \cdot v(\beta)$,
$v(\alpha \backslash \beta) = v(\alpha) \backslash v(\beta)$,
$v(\alpha / \beta) = v(\alpha) / v(\beta)$.

A formula α is *valid* in **A** if $v(\alpha) \geq 1$ for any valuation v on **A**. Also a given sequent $\alpha_1, \cdots, \alpha_m \Rightarrow \beta$ is said to be valid in **A** iff the formula $(\alpha_1 * \cdots * \alpha_m) \rightarrow \beta$ is valid in **A** or $v(\alpha_1) \cdots v(\alpha_m) \leq v(\beta)$ holds for any valuation v on **A**.

We can show the following completeness theorem for basic substructural logics, by using the standard argument on Lindenbaum algebras.

Completeness theorem For any provable sequent S in **FL** iff it is valid in all $\mathbf{FL}_\perp$-algebras. This holds also for other basic substructural logics and corresponding classes of **FL**-algebras.

The following we give algebras for the logics $\mathbf{FL}_e$, $\mathbf{FL}_w$, $\mathbf{FL}_c$, $\mathbf{FL}_{ew}$, and $\mathbf{FL}_{ec}$.

$\mathbf{FL}_e$-algebra : $\mathbf{FL}_\perp$-algebra + Commutativity.
$\mathbf{FL}_w$-algebra : $\mathbf{FL}_\perp$-algebra + Integrality + $[\perp = 0]$.
$\mathbf{FL}_c$-algebra : $\mathbf{FL}_\perp$-algebra + Increasing-idempotency.
$\mathbf{FL}_{ew}$-algebra : $\mathbf{FL}_e$-algebra + Integrality + $[\perp = 0]$.
$\mathbf{FL}_{ec}$-algebra : $\mathbf{FL}_e$-algebra + Increasing-idempotency.

Algebraic approaches are quite useful to obtain general results on substrucrural logics. Therefore we study residuated lattices to investigate the structure of substructural logics. In this thesis, we will show that the finite model property for various substructural logics to prove the finite embeddability property for corresponding classes of residuated lattices.

On the other hand, proof-theoretic methods are also powerful to analyze the structure of residuated lattices. In this thesis, we will show the semisimplicity of every free $\mathbf{FL}_w$-algebras using proof-theoretic methods, the amalgamation property for $\mathcal{CRL}$ using proof-theoretic results.

1.2 Contents of this thesis

In this thesis, we study finite embeddability property, semisimplicity and amalgamation property of residuated lattices.

We will show the finite embeddability property of various classes of integral residuated lattices which implies that the finite model property for various substructural logics. Moreover, we will show semisimplicity and amalgamation property of residuated lattices which are purely algebraic properties, using proof-theoretic methods and results.

Our main results are as follows:

- Let $\mathbf{A}$ be an algebra which is $\mathbf{FL}_w$-algebras with adding some or all of the following axiom schemes: commutativity, Wcon$_l$ (Wcon$_r$), classic, representation. Then $\mathbf{A}$ has the FEP.
- Every free $\mathbf{FL}_w$-algebras is semisimple.
- The variety $\mathcal{CRL}$ has the AP.
- Let $\mathbf{L}$ be a logic which is an extension of $\mathbf{FL}_e$ with the CIP and $\mathcal{K}$ be the variety of all $\mathbf{L}$-algebras. Then $\mathcal{K}$ has the EIP. Therefore, $\mathcal{K}$ has the AP.

All of our researches can be considered as the bridges between algebras and logics.

Finite embeddability property

In Chapter 4 and 5, we will consider the *finite embeddability property* (FEP) for various classes of residuated lattices and the *finite model property* (FMP) for various substructural logics including fuzzy logics.

A class of algebras has the FEP if every finite partial subalgebra of an algebra in the class can be embedded into a finite algebra in the same class. This notion was first introduced and studied by T. Evans. He also investigated the relationship between the FEP and the word problem of a given algebra.

It is known that the FMP is quite powerful method to show the decidability in the study of modal logics. It is well-known argument by Harrop that the decidability follows from the FMP and the finite axiomatizability. However, it is also known that it is hard to show the FMP of substructural logics and we do not have any powerful method like the *filtration method* in modal logic yet.

Studies of the FMP of substructural logics were made by R.K. Meyer in 1972, R.K. Meyer and H. Ono in 1994, W. Buszkowski in 1996 and also C.J. van Alten and J.G. Raftery in 1999. All of these studies (except W. Buszkowski's work) are implicational fragment of substructural logics. Studies of other fragments of substructural logics were made by Y. Lafont in 1997. He proved that each of the **CFL**-systems except $\mathbf{CFL}_c$ has the FMP. and also M. Okada and K. Terui in 1999. They proved that each of the **FL**-systems except $\mathbf{FL}_c$ has the FMP. Therefore, they showed the FMP of most basic substructural logics, but they used cut elimination theorem to show the FMP. That means, to show the FMP is much harder than to show decidability.

In Chapter 4, we will consider the FEP for integral residuated lattices. The idea of this chapter is mainly due to W. Blok and C. J. van Alten's papers [6, 7]. First, we introduce the FEP and consider the relationships among the FEP, the FMP, and the *strong finite model property* (SFMP). We can show that if a class of algebras $\mathcal{K}$ has the FEP then $\mathcal{K}$ has the FMP. Hence, the FEP is one of the algebraic methods to show the FMP. Next, we introduce the Blok-Alten's construction. The original construction introduced by W. Blok and C. J. van Alten is for the structure, called partially ordered biresiduated groupoids, however our interests are not so general setting, we only focus the residuated lattices. Thus, we modify their construction to the residuated lattices. Lastly, we will show that the FEP for integral residuated lattices.

In Chapter 5, we will consider the FEP for various classes of residuated lattices and the FMP for various substructural logics including fuzzy logics. First, we consider the *full* left (right) integral residuated lattices which is obtained by deleting right (left) residuation from residuated lattices. Second, we will prove that the FEP for various classes of integral residuated lattices. Next, we summarize the FMP for various substructural logics including fuzzy logics. Lastly, we consider some classes of residuated lattices that are failure of the FEP.

Semisimplicity

In Chapter 6, we will investigate subdirect irreducibity, simplicity and semisimplicity of residuated lattices. Moreover we will show that every *free* $\mathbf{FL}_w$-algebra is semisimple.

We say that an algebra is semisimple if it is isomorphic to a subdirect product of simple algebras. We can characterize that an algebra $\mathbf{A}$ is semisimple iff the intersection of a set of all maximal members in $\mathrm{Con}(\mathbf{A})$ is equal to the least congruence of $\mathbf{A}$. Using the fact that the relationship between filters and congruences in residuated lattices, we can say that a residuated lattice $\mathbf{A}$ is semisimple iff the intersection of a set of all maximal filters of $\mathbf{A}$ is equal to the smallest filter of $\mathbf{A}$.

In[18], V. N. Grišin proved that every free $\mathbf{CFL}_{ew}$-algebra is semisimple. V. N. Grišin introduced a new sequent system which is equivalent to $\mathbf{CFL}_{ew}$ and using the fact that the cut elimination theorem holds for the sequent system to show the semisimplicity of every free $\mathbf{CFL}_{ew}$-algebras by using proof-theoretic methods.

In [26], T. Kowalski and H. Ono show that variety of $\mathbf{FL}_{ew}$-algebras is generated by its finite simple members. The result is obtained by first showing that every free $\mathbf{FL}_{ew}$-algebra is semisimple and then showing that every variety generated by a simple $\mathbf{FL}_{ew}$-algebra is generated by a set of finite simple $\mathbf{FL}_{ew}$-algebras. They used Grišin's idea in [18] to show the semisimplicity of every free $\mathbf{FL}_{ew}$-algebras. They introduced a sequent system SFL^{+}_{ew} such that

1. algebras for SFL^{+}_{ew} are exactly equal to $\mathbf{FL}_{ew}$-algebras,
2. cut elimination theorem holds for SFL^{+}_{ew}.

Then, using proof-theoretic properties of SFL^{+}_{ew}, the semisimplicity of every free $\mathbf{FL}_{ew}$-algebras is obtained.

Our result is also based on Grišin's idea and Kowalski-Ono's technique. We will introduce a new sequent system, $\mathbf{FL}^{+}_{w}$ which is equivalent to $\mathbf{FL}_{w}$. Using the fact that cut elimination theorem holds for $\mathbf{FL}^{+}_{w}$ and using proof-theoretic properties of $\mathbf{FL}^{+}_{w}$, the proof of the semisimplicity works also for free $\mathbf{FL}_{w}$-algebras. It is very interesting to see how nicely proof-theoretic methods work to bring about purely algebraic consequence.

Those results make an interesting contrast with the case of Heyting algebras. In the case of Heyting algebras, it is easy to see that every simple Heyting algebra is a two valued Boolean algebra. Hence any semisimple Heyting algebra is a Boolean algebra. Thus, any free Heyting algebra can never be semisimple.

Amalgamation property

In Chapter 7, we will study the amalgamation property of various classes of commutative residuated lattices. In particular, we will prove that the *amalgamation property* (AP) for the variety $\mathcal{CRL}$ of all commutative residuated lattices and the variety $\mathcal{FL}_e$ of all $\mathbf{FL}_e$-algebras. Moreover, we will show that the following.

Let $\mathbf{L}$ be a logic which is an extension of $\mathbf{FL}_e$ with the CIP and $\mathcal{K}$ the variety of all $\mathbf{L}$-algebras. Then $\mathcal{K}$ has the EIP. Therefore, $\mathcal{K}$ has the AP.

In[24], T. Kowalski showed that the amalgamation property (AP) for the variety $\mathcal{FL}_{ew}$ of all $\mathbf{FL}_e$-algebras, The result is obtained by showing that

1. the logical system $\mathbf{FL}_{ew}$ has the Craig's interpolation property (CIP),
2. the variety of $\mathcal{FL}_{ew}$ has the equational interpolation property (EIP).

We will show that Kowalski's proof of the AP works well also the variety of $\mathcal{CRL}$ and $\mathcal{FL}_e$ We use the CIP of the logic $\mathbf{FL}_e$ and $\mathbf{FL}_e$ with only constant 1, denoted by $\mathbf{FL}^{-}_{e}$ to show that the AP for $\mathcal{FL}_e$ and $\mathcal{CRL}$. Therefore our result is obtained by using the proof theoretical results. It is very interesting to see how nicely proof-theoretic results (CIP) work to bring about algebraic consequences (AP).

For a class of algebra $\mathcal{K}$, there is a natural problem that how member of $\mathcal{K}$ can be glued together to obtain a larger member of $\mathcal{K}$. One of the answers to this problem is called the amalgamation property (AP). The AP is the following property.

If $\mathbf{A}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are in $\mathcal{K}$, f_1 is an embedding of $\mathbf{A}$ into $\mathbf{B}_1$ and f_2 is an embedding of $\mathbf{A}$ into $\mathbf{B}_2$, then there exists $\mathbf{C} \in \mathcal{K}$ and embeddings g_1 of $\mathbf{B}_1$ into $\mathbf{C}$ and g_2 of $\mathbf{B}_2$ into $\mathbf{C}$ such that $g_1 f_1 = g_2 f_2$ holds.

The AP is studied in model theory but no satisfactory criterion is known. Some well known structures satisfy the AP, for example, the class of all groups, commutative groups, field, partially ordered sets, lattices and Boolean algebras has the AP. On the other hand, it is known that the class of rings and semigroups do not have the AP.

In 1957, W. Craig proved that the following theorem.

If $A \to B$ is provable in classical logic then there exists a formula C such that both $A \to C$ and $C \to B$ are provable, and every propositional variables in C appears both A and B.

Now, we call the above theorem Craig's interpolation theorem. We say that such a formula C is an *interpolant* of $A \to B$. Craig's interpolation theorem is one of the important theorems in mathematical logic. The same result holds also for the intuitionistic logic. We say that a logic $\mathbf{L}$ has the *Craig's interpolation property* (CIP) if the statement holds for $\mathbf{L}$. The CIP was proved in many important logics. For example, L. Maksimova proved the CIP of modal logics, H. Ono and Y. Komori proved the CIP of substructural logics. In particular, L. Maksimova proved the striking result that the CIP holds for only 7 logics between the intuitionistic logic and the classical logic.

The connection between the AP and the CIP was studied by B. Jónsson and A. Daigneault independently. This connection was further studied by D. Pigozzi. It is known that th AP is equivalent to the CIP in many classes of algebras. For example, L. Maksimova proved that a normal modal logic with a single unary modality has the Craig's interpolation property iff the corresponding class of algebras has the super-amalgamation property, and also that the intuitionictic logic has the CIP iff the variety of Heyting algebras has the super-amalgamation property.

In Chapter 7, we will use proof-theoretic results, the CIP holds for both $\mathbf{FL}_e$ and $\mathbf{FL}_e^-$ to prove the AP for the variety of $\mathcal{FL}_e$ and $\mathcal{CRL}$. It is an interesting connection to see how nicely proof-theoretic results work to show the algebraic property. Lastly, we will show that if $\mathbf{L}$ is a logic which is an extension of $\mathbf{FL}_e$ with the CIP and $\mathcal{K}$ is the variety of all $\mathbf{L}$-algebras then $\mathcal{K}$ has the EIP. Therefore, $\mathcal{K}$ has the AP. Thus, we can show most of important classes of commutative residuated lattices have the AP.

Organization of this thesis

In Chapter 2, We will introduce algebraic preliminaries of this thesis. First, we will summarize the basic facts of ordered structures, lattices and basic concepts of algebras. Next, we will introduce residuated lattices and its basic results.

In Chapter 3, we will introduce substructural logics its basic results.

In Chapter 4,5, we will study the finite embeddability property for various classes of integral residuated lattices. Moreover we will study the finite model property for various substructural logics.
In Chapter 6, we will study subdirect irreducibility, simplicity and semisimplicity of residuated lattices. Lastly, we will show the semisimplicity of free $\mathbf{FL}_w$-algebras.

In Chapter 7, We will study the amalgamation property (AP) of various classes of commutative residuated lattices. First, we will intoroduce the AP and the CIP. Next we will show that the CIP holds for the logic $\mathbf{FL}_e^-$ which is the logic for commutative residuated lattices. Lastly, we will prove that the variety $\mathcal{CRL}$ has the AP. Moreover we will prove that the following. Let $\mathbf{L}$ be a logic which is an extension of $\mathbf{FL}_e$ with the CIP and $\mathcal{K}$ be the variety of all $\mathbf{L}$-algebras. Then $\mathcal{K}$ has the EIP. Therefore, $\mathcal{K}$ has the AP.

Lastly, in Chapter 8, we will summarize this thesis and state further studies.

Chapter 2

Algebraic Preliminaries

In this chapter, we will introduce the preliminaries of this thesis. First, we will summarize the basic facts of universal algebra. Next, we will introduce residuated lattices and their basic results.

2.1 Basics of algebra

Order and lattice theory

A structure $\langle X, \leq \rangle$ is a *poset* if X is a set and $\leq$ is a reflexive, transitive and antisymmetric relation (order relation) on X. An order relation $\leq$ on X gives rise to a relation $<$ of a strict order defined as follows: $x < y$ iff $x \leq y$ and $x \neq y$. A poset $\langle X, \leq \rangle$ is a *chain* if for all $x, y \in X$, either $x \leq y$ or $y \leq x$, i.e., any two elements of X are *comparable*. A poset $\langle X, \leq \rangle$ is an antichain if $x \leq y$ in X only if $x = y$, i.e., any two elements of X are *incomparable*.

Let $\langle X, \leq \rangle$ be a poset and A a nonempty subset of X. An element $a \in A$ is said to be *minimal* (*maximal*) in A if for any $x \in X$, $x < a(x > a)$ implies $x \notin A$. An element $a \in A$ is the *least* (*greatest*) element in A if $a \leq x(x \leq a)$ for all $x \in A$. An element $x \in X$ is an *upper bound* (a *lower bound*) for A in X if $a \leq x(a \geq x)$ for all $a \in A$. The least upper bound, which is also called the *supremum* of A and denoted by $\sup A$. Similarly, the greatest lower bound is also called the *infimum* of A and denoted by $\inf A$.

Let X and Y be posets. A mapping f from X to Y is said to be *monotone* (or order-preserving), if for all $x, y \in X$ such that $x \leq y$ implies $f(x) \leq f(y)$.

Let X be a poset. A map $f : X \to X$ is called *residuated* if there exists a map $f^* : X \to X$ such that $f(x) \leq y \iff x \leq f^*(y)$ for all $x, y \in X$. We say that f^* is the *residual* of f. Let X be a poset. A map $c : X \to X$ is called a *closure operator* on X if it satisfies that (1) $x \leq c(x)$, (2) $x \leq y$ implies $c(x) \leq c(y)$, (3) $c(c(x)) = c(x)$ for all $x, y \in X$. An element $x \in X$ is called closed if $c(x) = x$. A map $i : X \to X$ is called an *interior operator* on X if it satisfies that (1) $i(x) \leq x$, (2) $x \leq y$ implies $i(x) \leq i(y)$, (3) $i(i(x)) = i(x)$ for all $x, y \in X$.

Note that for a residual pair f, f^*, the composition f^*f is a closure operator and ff^* is an interior operator.

A poset L is called a *lattice* iff for every $a, b \in L$ both $\sup\{a, b\}$ (denoted by $a \vee b$) and $\inf\{a, b\}$ (denoted by $a \wedge b$) exist in L. A lattice is *bounded* if it has both the greatest element 1 and the least element 0. Obviously these two elements satisfy that $BL1$: $x \wedge 0 = 0$ and $x \vee 1 = 1$. A lattice L is *distributive* if it satisfies a *distributive law*, $D1$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. It is known that a lattice is distributive iff it satisfies $D2$: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. A poset X is *complete* is for every subset A of X both sup A and inf A exist in X denoted $\bigvee A$ and $\bigwedge A$. A lattice is a *complete lattice* if it is complete as a poset.

Universal algebra

A finitary operation on a set A is an n-ary operation which is a function from A^n into A for some natural number n. A type is a set $\mathcal{F}$ of function symbols. A non-negative integer n which is called *arity* is assigned to each member f of $\mathcal{F}$. Then we say f is an n-ary function symbol. $\mathcal{F}_n$ denotes the set of all n-ary function symbols in $\mathcal{F}$.

Definition 2.1.1 (Algebra) An *algebra* $\mathbf{A}$ of type $\mathcal{F}$ is a pair $\langle A, F\rangle$ where A is a nonempty set and F is a family of finitary operations on A indexed by the type $\mathcal{F}$

For each n-ary function symbol f in $\mathcal{F}$ there is an n-ary operation $f^{\mathbf{A}}$ on A. The set A is called the *universe* of $\mathbf{A}$ and $f^{\mathbf{A}}$'s are called the fundamental operations of A.

A *subuniverse* of $\mathbf{A}$ is a subset B of A which is closed under the fundamental operations of $\mathbf{A}$, i.e., if f is a fundamental n-ary operation of $\mathbf{A}$ and $a_1, \cdots, a_n \in B$ then $f(a_1, \cdots, a_n) \in B$. Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same type. $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$ and every fundamental operations of $\mathbf{B}$ is the restriction of the corresponding operation of $\mathbf{A}$.

Let $\mathcal{F}$ be a type of algebra and G a subset of F. $\langle A, G\rangle$ is called *reduct* of an algebra $\mathbf{A}$ and a subalgebra of $\langle A, G\rangle$ is called *subreduct* of an algebra $\mathbf{A}$.

Examples of Algebras

(1) A *group* $\mathbf{G}$ is an algebra $\langle G, \cdot, ^{-1}, 1\rangle$ which satisfies that $G1 : x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $G2 : x \cdot 1 = x = 1 \cdot x$ and $G3 : x \cdot x^{-1} = 1 = x^{-1}x$. A group $\mathbf{G}$ is called *commutative* if it satisfied also $G4 : x \cdot y = y \cdot x$.

(2) A *semigroup* is a groupoid $\langle G, \cdot\rangle$ which satisfies that $G1$.

(3) A *monoid* $\mathbf{M}$ is an algebra $\langle M, \cdot, 1\rangle$ which satisfies that both $G1$ and $G2$.

(4) A *Boolean algebra* is an algebra $\langle B, \vee, \wedge, ^{\prime}, 0, 1\rangle$ which satisfies that $B1 : \langle B.\vee, \wedge\rangle$ is a distributive lattice, $B2 : x \vee 1 = 1$ and $x \wedge 0 = 0$, $B3 : x \vee x^{'} = 1$ and $x \wedge x^{'} = 0$.

(5) An algebra $\langle H, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a *Heyting algebra* if it satisfies that $\langle H, \vee, \wedge\rangle$ is a distributive lattice, 0 and 1 are the least and greatest elements of this lattice, and the following condition holds: $z \leq x \rightarrow y \Longleftrightarrow z \wedge x \leq y$ for all $x, y, z \in H$.

Definition 2.1.2 (Congruence) Let $\mathbf{A}$ be an algebras of type $\mathcal{F}$ and θ a equivalence relation on $\mathbf{A}$. Then θ is a *congruence* on $\mathbf{A}$ if θ satisfies the following condition:
For each $f \in \mathcal{F}_n$ and $a_i, b_i \in A$, if $a_i \theta b_i$ holds for $1 \leq i \leq n$ then $f^{\mathbf{A}}(a_1, \cdots, a_n) \theta f^{\mathbf{A}}(b_1, \cdots, b_n)$ holds.

We denote $[a]_\theta = \{b \in A : a\theta b\}$ and $A/\theta = \{[a]_\theta : a \in A\}$.

Definition 2.1.3 (Quotient algebra) Let θ be a congruence on an algebra $\mathbf{A}$. Then the *quotient algebra* of $\mathbf{A}$ by θ, written $\mathbf{A}/\theta$, is the algebra whose universe is A/θ and whose fundamental operations satisfy $f^{\mathbf{A}/\theta}([a_1]_\theta, \cdots, [a_n]_\theta) = [f^{\mathbf{A}}(a_1, \cdots, a_n)]_\theta$ for every $a_1, \cdots, a_n \in A$ and for every n-ary function symbol f in $\mathcal{F}_n$.

Note that quotient algebras of $\mathbf{A}$ are of the same type as $\mathbf{A}$. We denote $\mathrm{Con}(\mathbf{A})$ the set of all congruence on an algebra $\mathbf{A}$. $\langle \mathrm{Con}(\mathbf{A}), \subseteq\rangle$ forms a complete lattice. We say that $\langle \mathrm{Con}(\mathbf{A}), \subseteq\rangle$ is congruence lattice of $\mathbf{A}$.

Definition 2.1.4 (Homomorphism) Suppose $\mathbf{A}$ and $\mathbf{B}$ are two algebras of the same type $\mathcal{F}$. A mapping α is called a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ if $\alpha f^{\mathbf{A}}(a_1, \cdots, a_n) = f^{\mathbf{B}}(\alpha(a_1), \cdots, \alpha(a_n))$ for each n-ary f in $\mathcal{F}$ and $a_1, \cdots, a_n \in A$.

If a homomorphism α is surjective then B is said to be a *homomorphic image* of A and α is called an *epimorphism*. If a homomorphism α is injective (bijective) then α is called a *monomorphism* or an *embedding* (*isomorphism*).

Definition 2.1.5 (Kernel of a homomorphism) Let $\alpha : \mathbf{A} \rightarrow \mathbf{B}$ be a homomorphism. Then the *kernel* of α, written $\ker(\alpha)$, is defined by $\ker(\alpha) = \{\langle a, b\rangle \in A^2 : \alpha(a) = \alpha(b)\}$.

Let $\alpha : \mathbf{A} \rightarrow \mathbf{B}$ be a homomorphism. Then it is easy to see that the kernel of α, $\ker(\alpha)$ is a congruence on $\mathbf{A}$.

Definition 2.1.6 (Direct product) Let $(\mathbf{A_i})_{i\in I}$ be an indexed family of algebras of the same type $\mathcal{F}$. The *direct product* $\mathbf{A} = \Pi_{i\in I}\mathbf{A}_i$ is an algebra with universe $\Pi_{i\in I}A_i$ and such that $f \in \mathcal{F}$ and $a_1, \cdots, a_n \in \Pi_{i\in I}A_i$, $f^{\mathbf{A}}(a_1, \cdots, a_n)(i) = f^{\mathbf{A}}(a_1(i), \cdots, a_n(i))$ for $i \in I$, i.e., $f^{\mathbf{A}}$ is defined coordinatewise.

Note that the empty product $\Pi\emptyset$ is the trivial algebra $\{\emptyset\}$. For each $j \in I$ the projection map $\pi_j : \Pi_{i\in I}A_i \rightarrow A_j$ is defined by $\pi_j(a) = a(j)$, which gives a surjective homomorphism from $\Pi_{i\in I}\mathbf{A}_i$ onto $\mathbf{A}_j$.

Definition 2.1.7 (Subdirect product) An algebra $\mathbf{A}$ is a *subdirect product* of an indexed family $(\mathbf{A}_i)_{i\in I}$ of algebras if it satisfies (i) $\mathbf{A}$ is a subalgebra of $\Pi_{i\in I}\mathbf{A}_i$ and (ii) $\pi_i(\mathbf{A}) = \mathbf{A}_i$.

An embedding $\alpha : \mathbf{A} \to \Pi_{i\in I}\mathbf{A}_i$ is a *subdirect embedding* if $\alpha(\mathbf{A})$ is a subdirect product of $(\mathbf{A}_i)_{i\in I}$.

Definition 2.1.8 (Subdirect irreducible algebra) An algebra $\mathbf{A}$ is *subdirectly irreducible* if for every subdirect embedding $\alpha : \mathbf{A} \to \Pi_{i\in I}\mathbf{A}_i$ there is an $i \in I$ such that $\pi_i\alpha : \mathbf{A} \to \mathbf{A}_i$ is an isomorphism.

Definition 2.1.9 (Simple algebra) An algebra $\mathbf{A}$ is *simple* if $\mathrm{Con}(\mathbf{A}) = \{\Delta_{\mathbf{A}}, \nabla_{\mathbf{A}}\}$ where $\Delta_{\mathbf{A}}$ is the diagonal relation, i.e., the least congruence on $\mathbf{A}$ and $\nabla_{\mathbf{A}}$ is the greatest congruence on $\mathbf{A}$.

Definition 2.1.10 (Semisimple algebra) An algebra is *semisimple* if it is isomorphic to a subdirect product of simple algebras.

Let $\mathcal{K}$ be a class of algebras of the same type. Define the classes of algebras $I(\mathcal{K})$, $S(\mathcal{K})$, $H(\mathcal{K})$, $P(\mathcal{K})$ and $P_S(\mathcal{K})$ as follows:

$I(\mathcal{K})$ is the class of all isomorphic images of members of $\mathcal{K}$.

$S(\mathcal{K})$ is the class of all isomorphic images of subalgebras of members of $\mathcal{K}$.

$H(\mathcal{K})$ is the class of all homomorphic images of members of $\mathcal{K}$.

$P(\mathcal{K})$ is the class of all isomorphic images of direct products of a nonempty family of algebras in $\mathcal{K}$.

$P_S(\mathcal{K})$ is the class of all isomorphic images of subdirect products of a nonempty family of algebras in $\mathcal{K}$.

Each of I, S, H, P and P_S is called a class operator.

Definition 2.1.11 (Variety) A non-empty class of algebra of type $\mathcal{F}$ is called *variety* if it is closed under subalgebras, homomorphic images and direct products.

Let $\mathcal{K}$ be a class of algebras with same type. $V(\mathcal{K})$ denotes the smallest variety containing $\mathcal{K}$.

Theorem 2.1.1 (Tarski) *If $\mathcal{K}$ is a class of algebras then $HSP(\mathcal{K})$ is the variety generated by $\mathcal{K}$, i.e., V=HSP.*

Let X be a set of variables and $\mathcal{F}$ a type of algebras. The set $T(X)$ of terms of type $\mathcal{F}$ over X is the smallest set such that (i) $X \bigcup \mathcal{F}_0 \subseteq T(X)$. (ii) If $p_1, \cdots, p_n \in T(X)$ and $f \in \mathcal{F}_n$ then $f(p_1, \cdots, p_n) \in T(X)$.

Let $p(x_1, \cdots, x_n)$ be a term of type $\mathcal{F}$ over some set X and $\mathbf{A}$ an algebra of type $\mathcal{F}$. We define a mapping $p^{\mathbf{A}} : A^n \to A$ as follows: (1) if p is a variable x_i then $p^{\mathbf{A}}(a_1, \cdots, a_n) = a_i$ for $a_1, \cdots, a_n \in A$, i.e., $p^{\mathbf{A}}$ is the i-th projection. (2) If p is of the form $f(p_1(x_1, \cdots, x_n), \cdots, p_k(x_1, \cdots, p_n))$, where $f \in \mathcal{F}_n$ then $p^{\mathbf{A}}(a_1, \cdots, a_n) = f^{\mathbf{A}}(p_1^{\mathbf{A}}(x_1, \cdots, x_n), \cdots, p_k^{\mathbf{A}}(x_1, \cdots, x_n))$. In particular, if $p = f \in \mathcal{F}$ then $p^{\mathbf{A}} = f^{\mathbf{A}} \cdot p^{\mathbf{A}}$ is the term function on $\mathbf{A}$ corresponding to the term p.

We can transform the set $T(X)$ into an algebra in a natural way.

Definition 2.1.12 (Term algebra) Let $\mathcal{F}$ be a type of algebras and X a set. If $T(X) \neq \emptyset$ then the *term algebra* of type $\mathcal{F}$ over X, written $\mathcal{T}(X)$, is of the form $\langle T(X), \mathcal{F} \rangle$ and the fundamental operations satisfy $f^{\mathcal{T}(X)} : \langle p_1, \cdots, p_n \rangle \mapsto f(p_1, \cdots, p_n)$ for $f \in \mathcal{F}_n$, $p_i \in T(X)$ $1 \leq i \leq n$.

We note that $\mathcal{T}(\emptyset)$ exists iff $\mathcal{F}_0 \neq \emptyset$.

Definition 2.1.13 (Universal mapping property) Let $\mathcal{K}$ be a class of algebra of type $\mathcal{F}$ and $\mathbf{U}(X)$ an algebra of type $\mathcal{F}$ generated by a set X. If for every $\mathbf{A} \in \mathcal{K}$ and for every map $f : X \to A$ there is a unique homomorphism $\alpha : \mathbf{U}(X) \to \mathbf{A}$ that extends f ($f(x) = \alpha(x)$ for all $x \in X$) then $\mathbf{U}(X)$ has the *universal mapping property* for $\mathcal{K}$ over X, X is called a set of *free generators* of $\mathbf{U}(X)$ and $\mathbf{U}(X)$ said to be *freely generated* by X.

Let $\mathcal{K}$ be a class of algebra of type $\mathcal{F}$ and X a set of variables. Define the congruence $\theta_{\mathcal{K}}(X)$ on $\mathcal{T}(X)$ by $\theta_{\mathcal{K}}(X) = \bigcap \Phi_{\mathcal{K}}(X)$, where $\Phi_{\mathcal{K}}(X) = \{\phi \in \mathrm{Con}(\mathcal{T}(X)) : \mathcal{T}(X)/\phi \in IS(\mathcal{K})\}$ and define $\mathcal{F}_{\mathcal{K}}(\overline{X})$ the *$\mathcal{K}$-free algebra over* $\overline{X}$ by $\mathcal{F}_{\mathcal{K}}(\overline{X}) = \mathcal{T}(X)/\theta_{\mathcal{K}}(X)$, where $\overline{X} = X/\theta_{\mathcal{K}}(X)$. For $x \in X$ we write $\overline{x}$ for $x/\theta_{\mathcal{K}}(X)$, for $p = p(x_1, \cdots, x_n) \in T(X)$ we write $\overline{p}$ for $p^{\mathcal{F}_{\mathcal{K}}(\overline{X})}(\overline{x}_1, \cdots .\overline{x}_n)$.

Theorem 2.1.2 (Birkhoff) *Suppose $\mathcal{T}(X)$ exists. Then $\mathcal{F}_{\mathcal{K}}(\overline{X})$ has the universal mapping property for $\mathcal{K}$ over X. Moreover, for $\mathcal{K} \neq \emptyset$, $\mathcal{F}_{\mathcal{K}}(\overline{X}) \in ISP(\mathcal{K})$, in particular if $\mathcal{K}$ is a variety then $\mathcal{F}_{\mathcal{K}}(\overline{X}) \in \mathcal{K}$.*

Let Σ be a set of identities of type $\mathcal{F}$ and define $\mathcal{M}(\Sigma)$ to be the class of algebras $\mathbf{A}$ satisfying Σ. A class $\mathcal{K}$ of algebras is an equational class if there is a set of identities Σ such that $\mathcal{K} = \mathcal{M}(\Sigma)$. In this case, we say that $\mathcal{K}$ is axiomatized by Σ. It is easy to see that for any class $\mathcal{K}$ of algebras with the same type, all the classes $\mathcal{K}$, $P(\mathcal{K})$, $S(\mathcal{K})$, $H(\mathcal{K})$, and $P_S(\mathcal{K})$ have precisely the same valid equations.

The following theorem is called Birkhoff's variety theorem which says that a variety can be characterize the set of equations.

Theorem 2.1.3 (Birkhoff) *$\mathcal{K}$ is an equational class iff K is a variety.*

Let I be a set. Consider the power set $\wp(I)$ of I. $\wp(I)$ forms a Boolean algebra with set-theoretical operations. An ultrafilter of $\wp(I)$ is said to ultrafilter over I. Let $\{\mathbf{A}_i\}_{i\in I}$ be a set of algebra with same type and U be an ultrafilter over I. Then we define the binary relation θ_U on the product $\Pi\mathbf{A}_i$ by $[a_i]_{i\in I}\theta_U[b_i]_{i\in I}$ iff $\{i \in I : a_i = b_i\} \in U$. We can prove that the binary relation θ_U is a congruence on $\Pi\mathbf{A}_i$. Then we define a ultra-product $\Pi\mathbf{A}_i/U$ to be $\Pi\mathbf{A}_i/\theta_U$.

$P_U(\mathcal{K})$ denotes the class of all ultraproducts of collections of algebras from a class $\mathcal{K}$. P_U is a class operator of ultraproducts.

Definition 2.1.14 (Quasivariety) A *quasi-identity* is an identity or a formula of the form $(p_1 = q_2 \wedge \cdots \wedge p_n = q_n) \Rightarrow p = q$. A *quasivariety* is a class of algebras closed under I, S, P, and P_U.

Theorem 2.1.4 *Let $\mathcal{K}$ be a class of algebras. Then the following are equivalent.*
(1) $\mathcal{K}$ *is a quasivariety,*
(2) $\mathcal{K}$ *is closed under $ISPP_U$ and contains a trivial algebra.*
(3) $\mathcal{K}$ *can be axiomatized by quasi-identities.*

2.2 Residuated lattices

A residuated lattice is an algebraic structure which consists both lattice and monoid structures, and has binary operations called residuations. Residuation is a fundamental concept of ordered structures and categories. In this section, we introduce residuated lattices and investigate their basic results.

Residuated lattices

Definition 2.2.1 (Residuated lattice) A *residuated lattice* is an algebraic structure $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, /\rangle$ such that

> (1) $\langle A, \wedge, \vee\rangle$ is a lattice, (2) $\langle A, \cdot, 1\rangle$ is a monoid, (3) $\backslash$ and $/$ are binary operations which satisfy that $a \cdot b \leq c \Leftrightarrow a \leq c/b \Leftrightarrow b \leq a\backslash c$ hold for all a, b, c in A. ($\leq$ is a lattice order).

Operations $\backslash$ and $/$ are called *left* and *right* residuation, respectively. If we assume the commutativity of the monoid operation $\cdot$, then these two residuations become identical and the algebra $\mathbf{A}$ is called a commutative residuated lattices. In this case, $x\backslash y = y/x$ which is sometimes written as $x \to y$. Note that the notation of terms in a residuated lattice, multiplication $\cdot$ has priority over the residuations $\backslash, /$, which have priority over the lattice operations $\wedge, \vee$. In the rest of the paper, we write xy instead of $x \cdot y$. For example, we write $xy/z \wedge v\backslash wu = [(xy)/z] \wedge [v\backslash(wu)]$.

The following proposition is useful in algebraic properties of residuated lattices. The proof can be found in [9].

Proposition 2.2.1 *Let* **A** *be a residuated lattice. For any a, b, c in A and any subset Y of A, following conditions satisfy:*

(1) $a(b \vee c) = ab \vee ac$, $(b \vee c)a = ba \vee ca$
(2) *If* $\bigvee Y$ *exists then* $a(\bigvee Y) = \bigvee\{ay : y \in Y\}$, $(\bigvee Y)a = \bigvee\{ya : y \in Y\}$
(3) $(a \wedge b)/c = (a/c) \wedge (b/c)$, $c\backslash(a \wedge b) = (c\backslash a) \wedge (c\backslash b)$
(4) *If* $\bigwedge Y$ *exists then* $(\bigwedge Y)/c = \bigwedge\{y/c : y \in Y\}$, $c\backslash(\bigwedge Y) = \bigwedge\{c\backslash y : y \in Y\}$
(5) $a/(b \vee c) = (a/b) \wedge (a/c)$, $(b \vee c)\backslash a = (b\backslash a) \wedge (c\backslash a)$
(6) *If* $\bigvee Y$ *exists then* $a/(\bigwedge Y) = \bigwedge\{a/y : y \in Y\}$, $(\bigwedge Y)\backslash a = \bigwedge\{y\backslash a : y \in Y\}$
(7) $(a/c)c \leq a$, $c(c\backslash a) \leq a$
(8) $a(c/b) \leq ac/b$, $(a\backslash c)b \leq a\backslash cb$
(9) $(c/b)(b/a) \leq c/a$, $(a\backslash b)(b\backslash c) \leq a\backslash c$
(10) $c/b \leq (c/a)/(b/a)$, $b\backslash c \leq (a\backslash b)\backslash(a\backslash c)$
(11) $b/a \leq (c/a)\backslash(c/a)$, $a\backslash b \leq (a\backslash c)/(b\backslash c)$
(12) $c/b \leq ca/ba$, $a\backslash c \leq ba\backslash bc$
(13) $(c/a)/b = c/ba$, $b\backslash(a\backslash c) = ab\backslash c$
(14) $a\backslash(c/b) = (a\backslash c)/b$
(15) $c \leq (a/c)\backslash a$, $c \leq a/(c\backslash a$
(16) $a/1 = a$, $1\backslash a = a$
(17) $a/a \geq 1$, $a\backslash a \geq 1$
(18) $(a/b)(1/c) \leq a/cb$, $(c\backslash 1)(b\backslash a) \leq bc\backslash a$
(19) $(a/a)a = a$, $a(a\backslash a) = a$
(20) $(a/a)^2 = (a/a)$, $(a\backslash a)^2 = (a\backslash a)$

A *has the least element,* $\bot$*, then* **A** *has also the greatest element,* $\top$*. In this case, the following holds.* (i) $a\bot = \bot a = \bot$ *for all* $a \in A$, (ii) $a/\bot = \bot\backslash a = \top$ *for all* $a \in A$, (iii) $\top/a = a\backslash\top = \top$ *for all* $a \in A$.

A residuated lattice is *bounded* if it has the least (also greatest) element.

It is easy to see that a residuated lattice can be also defined by equations for lattices and monoids, with the following six identities: $a = a \wedge ((ab \vee c)/b)$, $b = b \wedge (a\backslash(ab \vee c))$, $a(b \vee c) = ab \vee ac$, $(b \vee c)a = ba \vee ca$, $(a/b)b \vee a = a$, $b(b\backslash a) \vee a = a$,. Therefore, we can show that the following.

Theorem 2.2.2 *The class of all residuated lattices forms a variety.*

Subclasses of residuated lattices

We introduce some of the important subclasses of residuated lattices. Some of them are also important subclasses which have a close relationship to basic substructural logics.

(1) Commutativity: $xy = yx$.
(2) Integrality: $x \leq 1$.
(3) Increasing-idempotency: $x \leq x^2$.
(4) Cancellativity: $xz = yz \Longrightarrow x = y$, and $zx = zy \Longrightarrow x = y$.

(5) Distributivity: $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

$\mathcal{RL}$, $\mathcal{CRL}$, $\mathcal{IRL}$, $\mathcal{RL}^2$, $Can\mathcal{RL}$, and $\mathcal{DRL}$ denote the class of commutative, integral, increasing-idempotent cancellative, and distributive residuated lattices.

$\mathcal{RL}$: the class of all residuated lattices,
$\mathcal{CRL}$: the class of all commutative residuated lattices.
$\mathcal{IRL}$: the class of all integral residuated lattices.
$\mathcal{RL}^2$: the class of all increasing-idempotent residuated lattices.
$\mathcal{CIRL}$: the class of all commutative integral residuated lattices, i.e., $\mathcal{CRL} \bigcap \mathcal{IRL}$.
$Can\mathcal{RL}$: the class of all cancellative residuated lattices.
$\mathcal{DRL}$: the class of all distributive residuated lattices.

It is clear that all of these classes are varieties except $Can\mathcal{RL}$. To see that the class of $Can\mathcal{RL}$ is also a variety, it is necessary to show that the cencellativity can be defined by identities in a residuated lattice. This can be confirmed as follows [22].

Lemma 2.2.3 *A residuated lattice is cancellative iff it satisfies the identities* $x\backslash xy = y$ *and* $yx/x = y$.

Examples of Residuated Lattices

There are many examples of residuated lattices[16]. Historically, residuated lattices were first studied by M. Dilworth and R. P. Ward in 1930s. They investigated properties of ideals of rings and got a concept of residuated lattices. In the following we introduce some interesting examples of residuated lattices.

ℓ-groups

An ℓ-groups or lattice ordered group is an algebra $\mathbf{G} = \langle G, \wedge, \vee, \cdot, ^{-1}, 1\rangle$, if it satisfies that (1) $\langle G, \wedge, \vee\rangle$ is a lattice, (2) $\langle G, \cdot, ^{-1}, 1\rangle$ is a group and (3) $x \leq y$ implies both $xz \leq yz$ and $zx \leq zy$. We define the residuations as follows. $z/x = z \cdot x^{-1}$ and $x\backslash z = x^{-1} \cdot z$. Thus $\mathbf{G}$ forms a residuated lattice.

Relational algebras

A relational algebra is an algebra $\mathbf{R} = \langle R, \wedge, \vee, ^{-}, 0, 1, \cdot, e, ^{\cup}\rangle$ such that (1) $\langle R, \wedge, \vee, ^{-}, 0, 1\rangle$ is a Boolean algebra, (2) $\langle R, \cdot, e\rangle$ is a monoid and (3) $(a^{\cup})^{\cup} = a$, $(ab)^{\cup} = b^{\cup}a^{\cup}$, $a(b \vee c) = ab \vee ac$, $(b \vee c)a = ba \vee ca$, $(a \vee b)^{\cup} = a^{\cup} \vee b^{\cup}$ and $a^{\cup}(ab)^{-} \leq b^{-}$ for all $a, b, c, \in R$. Now we define residuations by $a\backslash b = (a^{\cup}b^{-})^{-}$ and $b/a = (b^{-}a^{\cup})^{-}$. Then the reduct of relational algebra $\langle R, \wedge, \vee, \cdot, \backslash, /e\rangle$ forms a residuated lattice.

Quantales

An algebra $\mathbf{Q} = \langle Q, \wedge, \vee, \cdot\rangle$ is a quantale if it satisfies that (1) $\langle Q, \wedge, \vee\rangle$ is a complete lattice, (2) $\langle Q, \cdot\rangle$ is a semigroup satisfying (i) $(\bigvee x_i) \cdot y = \bigvee(x_i \cdot y)$, (ii) $y \cdot (\bigvee x_i) = \bigvee(y \cdot x_i)$.

If $\langle Q, \cdot \rangle$ is a monoid $\langle Q, \cdot, 1\rangle$ then $\mathbf{Q}$ forms a residuated lattice. We can define that residuations as follows: $x\backslash y = \max\{z : x \cdot z \leq y\}$, $y/x = \max\{z : z \cdot x \leq y\}$. Then the structure $\langle Q, , \wedge, \vee, \cdot, 1, \backslash, / \rangle$ forms a residuated lattice.

Triangular norms in fuzzy logic

The next example comes from fuzzy logic. This structure forms a totally ordered residuated lattice. The interval $[0, 1]$ of real number with max and min forms a complete lattice. A map T from $[0, 1]^2$ to $[0, 1]$ is a *t-norm* if it satisfies that the following conditions: (1) $\langle [0, 1], \cdot, 1\rangle$ is a commutative monoid, (2) $x \leq y$ implies $x \cdot z \leq y \cdot z$. Here we use $x \cdot y$ instead of $T(x, y)$. A t-norm T is said to *left-continuous*, if it satisfies that $(\sup x_i) \cdot y = y \cdot (\sup x_i) = \sup(x_i \cdot y)$. The complete lattice $[0, 1]$ with a left-continuous t-norm forms a quantale and so it can be regarded as a residuated lattice.

Power set of a monoid

The next example is one of the important examples. This tells us how to construct residuated lattices.

Let $\mathbf{M} = \langle M, \cdot, 1\rangle$ be a monoid. For any subsets X and Y of M, we define $X \cdot Y = \{x \cdot y : x \in X, y \in Y\}$, $X/Y = \{z : \{z\} \cdot Y \subseteq X\}$ and $X\backslash Y = \{z : Y \cdot \{z\} \subseteq X\}$. Then $\wp(\mathbf{M}) = \langle \wp(M), \cap, \cup, \cdot, \backslash, /, \{1\}\rangle$ is a residuated lattice.

A closure operator C in a residuated lattice $\mathbf{A}$ is said to be a nucleus if it satisfies $C(x)C(y) \leq C(xy)$. Let C be a nucleus in a residuated lattice $\mathbf{A}$. An element x in $\mathbf{A}$ is C-closed if $C(x) = x$ holds. We denote $C(A)$ by the set of all C-closed elements of $\mathbf{A}$. Now consider the structure $C(\mathbf{A}) = \langle C(A), \cap, \cup_C, \cdot_C, \backslash, /, C(1)\rangle$, where $\cup_C$ and $*_C$ are defined by $x \cup_C y = C(x \cup y)$, $x \cdot_C y = C(x \cdot y)$. Then we can prove that $C(\mathbf{A})$ is also a residuated lattice.

Combining this with Power set construction of residuated lattice. The structure $C(\wp(\mathbf{M})) = \langle C(\wp(M)), \cap, \cup_C, \cdot_C, \backslash, /, C(\{1\})\rangle$ is a residuated lattice. For more informations, see [41].

Filters on residuated lattices

Definition 2.2.2 (Filter) A subset F of A is a *filter* of $\mathbf{A}$ if it satisfies that
(1) If $1 \leq x$ then $x \in F$,
(2) If $x, x\backslash y \in F$ then $y \in F$,
(3) If $x, y \in F$ then $x \wedge y \in F$,
(4) If $x \in F$ and $z \in A$ then $z\backslash xz, zx/z \in F$.

Note that a filter F is a upward closed set, i.e., if $x \in F$ and $x \leq y$ then $y \in F$. Indeed, suppose $x \leq y$, we have $1 \leq x\backslash y$. Thus $x\backslash y \in F$ by (1). By (2), we have $y \in F$. We can show that $x\backslash y \in F$ iff $y/x \in F$. Indeed, suppose $x\backslash y \in F$. Use (4), we have $x(x\backslash y)/x \in F$ and we can prove that $x(x\backslash y)/x \leq y/x$. Hence $y/x \in F$. Similarly, we can show the converse direction. In the case of commutative residuated lattices, the condition (4) is redundant. Indeed, suppose $x \in F$. Then we have $x \leq z\backslash zx = z\backslash xz$. Thus we have $z\backslash xz \in F$ since F is upward closed. We can also show that if $x, y \in F$ then $xy \in F$. Indeed, suppose $x, y \in F$. By (4), we have $y\backslash xy \in F$. Use (2), we have $xy \in F$. In other case, we can prove similarly.

We prepare the following conditions.

(A) $1 \in F$,
(B) If $x \in F$ and $x \leq y$ then $y \in F$,
(C) If $x \in F$ and $z \in A$ then $z\backslash xz \wedge 1, zx/z \wedge 1 \in F$,
(D) If $x, y \in F$ then $xy \in F$,
(E) If $x \in F$ then $x \wedge 1 \in F$,
(F) If $x \in F$ and $z \in A$ then $(x\backslash z)\backslash z, z(/z/x) \in F$,
(G) If $x \in F$ and $z \in A$ then $(x\backslash z)\backslash z \wedge 1, z(/z/x) \wedge 1 \in F$,
(H) If $x \in F$ and $z, w \in A$ then $w(z\backslash xz)/w \in F$,
(I) If $x \in F$ and $z, w \in A$ then $w(z\backslash xz)/w \wedge 1 \in F$,
(J) If $x \in F$ and $z, w \in A$ then $w/(w/(x\backslash z)\backslash z)) \in F$,
(K) If $x \in F$ and $z, w \in A$ then $w/(w/(x\backslash z)\backslash z)) \wedge 1 \in F$.

Then we can show that the following proposition which is a useful characterization of filters in a residuated lattice.

Proposition 2.2.4 *Let* $\mathbf{A}$ *be a residuated lattice and* F *a nonempty subset of* A*. Then the following conditions are equivalent.*

1. F *is a filter of* $\mathbf{A}$,
2. F *satisfies* (2), (A), (B) *and* (C),
3. F *satisfies* (A), (B), (C) *and* (D),
4. F *satisfies* (4), (A), (B), (D) *and* (E),
5. F *satisfies* (2), (3), (4), (A) *and* (B),
6. F *satisfies* (1), (2) *and* (C),
7. F *satisfies* (1), (2) *and* (G),
8. F *satisfies* (1), (2) *and* (I),
9. F *satisfies* (1), (2) *and* (K),
10. F *satisfies* (1), (2), (3) *and* (F),
11. F *satisfies* (1), (2), (3) *and* (H),
12. F *satisfies* (1), (2), (3) *and* (J).

For each a in A we define $\lambda_a(x) = (ax/a) \wedge 1$ and $\rho_a(x) = (a\backslash xa) \wedge 1$. We say that λ_a, ρ_a are right, left conjugation. Using the proposition above we can have the representation of the filter generated by given nonempty subset of A. Let S be a nonempty subset of A. Then the filter generated by S is of the form $\langle S \rangle = \{x \in A : z \leq x \text{ for some } z \in \Pi\Gamma\bar{\Delta}(S)\}$, where $\bar{\Delta}$, Γ and Π are defined as follows.

$\bar{\Delta}(S) = \{s \wedge 1 : s \in S\}$
$\Gamma(S) = \{\mu_{u_1} \cdots \mu_{u_n}(s) : n \in \mathbb{N}, u_i \in A, s \in S\}$ where $\mu_{u_i} \in \{\lambda_{u_i}, \rho_{u_i}\}$.
$\Pi(S) = \{s_1 \cdots s_n : n \in \mathbb{N}, s_i \in S\} \bigcup \{1\}$.

We first remark that $\lambda_u(x)\lambda_u(y) \leq \lambda_u(xy)$. Indeed, $\lambda_u(x)\lambda_u(y) \leq (u\backslash xu)(u\backslash yu) \wedge 1 \leq u\backslash xu(u\backslash yu) \wedge 1 \leq u\backslash xyu \wedge 1 \leq \lambda_u(xy)$. The argument for ρ_u is similar.

We will show that $\langle S \rangle$ is the filter of a residuated lattice $\mathbf{A}$. By Proposition 2.2.4, it is enough to show that $\langle S \rangle$ contains 1, $\langle S \rangle$ is an upward closed, closed with respect to the monoid operation and satisfies that $x \in \langle S \rangle$ implies both $x \wedge 1$ and $z\backslash xz, zx/z \in \langle S \rangle$ for all $x \in \langle S \rangle$. It is clear that $\langle S \rangle$ contains 1 and $\langle S \rangle$ is a upward closed set. By definition of $\langle S \rangle$, $\langle S \rangle$ is closed with respect to the monoid operation and satisfies that $x \in \langle S \rangle$ implies $x \wedge 1$.

Finally, we need to show that $\langle S \rangle$ satisfies that $x \in \langle S \rangle$ implies $z\backslash xz, zx/z \in \langle S \rangle$ for all $x \in \langle S \rangle$. Let x be an arbitrary element of $\langle S \rangle$. Then x can be expressed by $x = y_1 \cdots y_k$ for some $y_i \in \Gamma\bar{\Delta}(S)$. Recall that if $z \in \Gamma\bar{\Delta}(S)$ then $\mu_u(z)$ is also in $\Gamma\bar{\Delta}(S)$, where μ_u is λ_u or ρ_u. Consider $\mu_u(x)$. Using the inequation considered above, we have $\mu_u(y_1) \cdots \mu_u(y_k) \leq \mu_u(y_1 \cdots y_k) = \mu_u(x)$. Hence $\mu_u(x) \in \Gamma\bar{\Delta}(S)$ and so $\Gamma\bar{\Delta}(S)$ is the filter generated by a set S. It is trivial that $\Gamma\bar{\Delta}(S)$ is the minimum filter, since any filters containing S must also contain $\Pi\Gamma\bar{\Delta}(S)$.

$\mathcal{CF}(\mathbf{A})$ denotes the lattice of all congruence filters on $\mathbf{A}$. In the following we introduce the relationship between filters and congruences on residuated lattices. For more information, see [41].

Lemma 2.2.5 *Let θ be a congruence on $\mathbf{A}$. Put $F_\theta = \{a \in A : 1\theta(a \wedge 1)\}$. Then the set F_θ is a congruence filter on $\mathbf{A}$.*

Lemma 2.2.6 *Let F be a filter on $\mathbf{A}$. Define a binary relation θ_F on $\mathbf{A}$ by $x\theta_F y \iff x\backslash y, y\backslash x \in F$. Then the set θ_F forms a congruence on $\mathbf{A}$.*

Theorem 2.2.7 *The lattice of $\mathcal{CF}(\mathbf{A})$ of all congruence filter of $\mathbf{A}$ is isomorphic to its congruence lattice* Con($\mathbf{A}$). *The isomorphism is given by the mutually inverse maps: $\theta \to F_\theta$ and $F \to \theta_F$.*

The relationship between convex normal subalgebras and congruences are studied by K. Blount and C. Tsinakis. They showed that there exists a lattice isomorphism between the lattice of all convex normal subalgebras of $\mathbf{A}$ and its congruence lattice Con($\mathbf{A}$). For more information, see [9].

Chapter 3

Substructural Logics

Basic substructural logics are defined as sequent systems obtained from either **LK** for classical logic or **LJ** for intuitionistic logic by deleting some or all substructural rules. We say that substructural logics are extensions of these basic substructural logics when they are formalized in sequent systems.

3.1 Substructural logics

Sequent system for substructural logic FL

A sequent of **FL** is an expression of the form $\Gamma \Rightarrow \theta$, where Γ is a finite sequence of formulas and θ is a formula. Both Γ and θ may be empty. We use capital Greek letters will denote finite (possibly empty) sequences of formulas.

Initial sequents of **FL** are of the following forms:

(1) $\alpha \Rightarrow \alpha$,
(2) $\Rightarrow 1$,
(3) $0 \Rightarrow$,
(4) $\Gamma \Rightarrow \top$,
(5) $\Gamma, \bot, \Delta \Rightarrow \theta$.

Cut rule of **FL** is the following:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \theta}{\Delta, \Gamma, \Sigma \Rightarrow \theta}\ (cut)$$

Rules for constants are the following:

$$\frac{\Gamma, \Delta \Rightarrow \theta}{\Gamma, 1, \Delta \Rightarrow \theta}\ (1w) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0w)$$

Rules for logical connectives are the following:

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \theta \quad \Gamma, \beta, \Delta \Rightarrow \theta}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \theta} \; (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \; (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \; (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \theta}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \theta} \; (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \theta}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \theta} \; (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \; (\Rightarrow \wedge)$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \theta}{\Gamma, \alpha * \beta, \Delta \Rightarrow \theta} \; (* \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha * \beta} \; (\Rightarrow *)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \theta}{\Delta, \Gamma, \alpha \backslash \beta, \Sigma \Rightarrow \theta} \; (\backslash \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \backslash \beta} \; (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \theta}{\Delta, \beta / \alpha, \Gamma, \Sigma \Rightarrow \theta} \; (/ \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta / \alpha} \; (\Rightarrow /)$$

The basic **FL**-systems are extension of the **FL** which are obtained **FL** adding with some or all structural rules defined by the following.

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \theta}{\Gamma, \beta, \alpha, \Delta \Rightarrow \theta} \; (e \Rightarrow) : \textit{exchange}$$

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \theta}{\Gamma, \alpha, \Delta \Rightarrow \theta} \; (c \Rightarrow) : \textit{contraction}$$

$$\frac{\Gamma, \Delta \Rightarrow \theta}{\Gamma, \alpha, \Delta \Rightarrow \theta} \; (w \Rightarrow) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \theta} \; (\Rightarrow w) \qquad : \textit{weakening}$$

Basic substructural logics

Many non-classical logics including linear logic, many-valued logics, Hájek's basic logic, relevant logics are regarded as substructural logics those are originated from different motivations. To study substructural logics will give us a uniform framework for investigating various kinds of non-classical logics. In the following we introduce various systems of basic substructural logics. We will express various basic substructural logics by adding the corresponding letters e, c and w to **FL** as subscripts. Here, we define $\neg\phi$ by $\phi \to 0$.

$\mathbf{FL}_e = \mathbf{FL}$ + exchange $= \mathbf{LJ} - \{w, c\}$ (Intuitionistic linear logic).

$\mathbf{FL}_w = \mathbf{FL}$ + weakening $= \mathbf{LJ} - \{e, c\}$.

$\mathbf{FL}_c = \mathbf{FL}$ + contraction $= \mathbf{LJ} - \{e, w\}$.

$\mathbf{FL}_{ew} = \mathbf{FL}$ + exchange + weakening $= \mathbf{LJ} - \{c\}$.

$\mathbf{CFL}_e = \mathbf{LK} - \{w, c\} = \mathbf{FL}_e + [\neg\neg\phi \to \phi]$.

$\mathbf{CFL}_{ew} = \mathbf{LK} - \{c\} = \mathbf{CFL}_e$ + weakening (Grišin's logic).

We say that a logic **L** is one of the **FL**-systems (**CFL**-systems) if **L** is defined as sequent systems obtained from **LJ** (**LK**) for classical logic by deleting some or all substructural rules.

Examples of Substructural Logics

Lambek calculus

In 1958, T. Lambek introduced that the formal system which is later called Lambek calculus for analyzing the mathematical property of English sentences (categorial grammar). Later, J. van Benthem and W. Buszkowski further studied. This formal system is equivalent to the logic without substructural rules.

Logics without contraction rule

The logics without contraction rule were studied by V. Grišin in 1970's. He pointed out that the native set theory based this logics never causes the Russell paradox. In 1980's H. Ono and Y. Komori further studied these logics and introduced both Kripke type semantics and algebraic semantics. Now these logics can be regarded as a fundamental substructural logics.

Relevant logics

Relevant logics were studied from philosophical motivations. The relevant logics are roughly the logics without weakening rules. One of the motivations of relevant logic is to exclude paradoxes of classical implication. It is known that there are two classes of paradoxes (paradox of relevance and paradox of consistency).

Linear logic

Linear logic is a substructural logic only with exchange rule which was introduced by J. Y. Girard. This logic gives great influence not only logic but also to theoretical computer science. It is known that the classical (intuitionistic) logic can be embedded into the full linear logic[17].

Many valued logic and Fuzzy logics

There are many logics for fuzzy inference and fuzzy reasoning. These logics can be considered the family of many-valued logics which was introduced by Łukasiewicz and algebraic studied by C. C. Chang. Monoidal logic (ML) in fuzzy logic is exactly equal to $\mathbf{FL}_{ew}$. The fundamental logic in fuzzy logic is called Hájek's basic logic (BL) which is an extension of $\mathbf{FL}_{ew}$. Later, we will give precisely definition of fuzzy logics around BL.

Algebras for basic substructural logics

It is well known that for each substructural logic, there exists a class of residuated lattices as an algebraic semantics for the logic. we introduce various classes of residuated lattices which are algebraic semantics for substructural logics.

In the usual way, we can give an interpretation of formulas in an **FL**-algebra. An **FL**-algebra **A** is a residuated lattice with fixed, but arbitrary element 0 in A. An $\mathbf{FL}_{\perp}$-algebra is an **FL**-algebra with bottom element $\perp$. In this case, an $\mathbf{FL}_{\perp}$-algebra has the four constants, $1, 0, \perp, \top$.

Let **A** be an **FL**-algebra. A valuation v on **A** is any mapping from the set of all propositional variable to the set A. We can extend each valuation v to a mapping from the set of all formulas to A inductively as follows. We use the same symbols for logical connectives and constants as those for corresponding algebraic operations and constants, respectively.

$v(1) = 1$ and $v(0) = 0$,

$v(\top) = \top$ and $v(\perp) = \perp$, when the language has $\top$ and $\perp$ and **A** is bounded.

$v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$,

$v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$,

$v(\alpha * \beta) = v(\alpha) \cdot v(\beta)$,

$v(\alpha \backslash \beta) = v(\alpha) \backslash v(\beta)$,

$v(\alpha / \beta) = v(\alpha) / v(\beta)$.

We say that a formula α is *valid* in **A** if $v(\alpha) \geq 1$ for any valuation v on **A**. Also a given sequent $\alpha_1, \cdots, \alpha_m \Rightarrow \beta$ is said to be valid in **A** iff the formula $(\alpha_1 * \cdots * \alpha_m) \rightarrow \beta$ is valid in **A** or $v(\alpha_1) \cdots v(\alpha_m) \leq v(\beta)$ holds for any valuation v on **A**.

We can show the following completeness theorem for basic substructural logics, by using the standard argument on Lindenbaum algebras.

Theorem 3.1.1 (Completeness theorem) *For any provable sequent S in* **FL** *if and only if it is valid in all* $\mathbf{FL}_{\perp}$*-algebras. This holds also for other basic substructural logics and corresponding classes of* **FL***-algebras.*

The following we give algebras for the logics $\mathbf{FL}_e$, $\mathbf{FL}_w$, $\mathbf{FL}_c$, $\mathbf{FL}_{ew}$, and $\mathbf{FL}_{ec}$.

$\mathbf{FL}_e$-algebra : $\mathbf{FL}_\perp$-algebra + Commutativity.

$\mathbf{FL}_w$-algebra : $\mathbf{FL}_\perp$-algebra + Integrality + $[\perp = 0]$.

$\mathbf{FL}_c$-algebra : $\mathbf{FL}_\perp$-algebra + Increasing-idempotency.

$\mathbf{FL}_{ew}$-algebra : $\mathbf{FL}_e$-algebra + Integrality + $[\perp = 0]$.

$\mathbf{FL}_{ec}$-algebra : $\mathbf{FL}_e$-algebra + Increasing-idempotency.

3.2 Cut elimination theorem

Cut elimination theorem

Gentzen proved the theorem which is called *cut elimination theorem* for sequent systems both classical logic (**LK**) and intuitionistic logic (**LJ**). The cut elimination theorem is one of the most important theorems in proof theory formalized by sequent calculus.

Theorem 3.2.1 (Gentzen) *Let S be a provable sequent in* **LK**. *Then S has a proof without using the cut rule. This holds also for* **LJ**.

A proof is called a *cut-free* proof if it has no applications of the cut rule. We say that a sequent system **L** is a *cut-free* system if the cut elimination theorem holds for **L**. By modifying the proof of the cut elimination theorem for **LK** and **LJ**, we can have the cut elimination theorem for some basic substructural logics.

Note that the cut elimination theorem for substructural logics without contraction rule is easier than for the logic which has contraction rule. Gentzen proved the cut elimination theorem for **LK** and **LJ** using Mix-elimination. We can prove that the cut elimination for contraction-free substructural logics by using double induction of the *grade* (the complexity for the cut formula which is equal to the number of occurrences of logical connectives) and the *length* of a proof without using Mix-elimination.

Theorem 3.2.2 (Cut elimination) *Cut elimination theorem holds for* $\mathbf{FL}$, $\mathbf{FL}_e$, $\mathbf{FL}_w$, $\mathbf{FL}_{ew}$, $\mathbf{FL}_{ec}$, *and* $\mathbf{FL}_{ecw}(= INT)$. *It holds also for* $\mathbf{CFL}_e$, $\mathbf{CFL}_{ew}$, $\mathbf{CFL}_{ec}$, *and* $\mathbf{CFL}_{ecw}(= CL)$.

On the other hand, it is known that the logic $\mathbf{FL}_c$ and $\mathbf{FL}_{cw}$ fail to the cut elimination theorem proved by Bayu Surarso and Ono[5]. They proved the following

Theorem 3.2.3 *Let* **L** *be a logic* $\mathbf{FL}_c$ ($\mathbf{FL}_{cw}$) *or its implicational fragment. Then there exists a sequent which is provable in* **L** *but there is no cut-free proof of the sequent.*

Note that the logic $\mathbf{FL}_{cw}$ is equivalent to the intuitionistic logic (INT). But the cut elimination theorem does not hold for $\mathbf{FL}_{cw}$. We can show that the cut elimination theorem holds for $\mathbf{FL}_{ec}$ to modify the contraction rule, called global contraction.

Some consequences of cut elimination theorem

Subformula property

A proof in a sequent system **L** has the *subformula property* if it contains only formulas which are subformulas of some formulas in its end-sequent. A sequent system **L** has the subformula property if for any provable sequent S in **L** has a proof of S with the subformula property.

In the case of basic substructural logics, we can show that the following theorem.

Theorem 3.2.4 *Subformula property holds for* $\mathbf{FL}$, $\mathbf{FL}_e$, $\mathbf{FL}_w$, $\mathbf{FL}_{ew}$, $\mathbf{FL}_{ec}$, *and* $\mathbf{FL}_{ecw}(= INT)$. *It holds also for* $\mathbf{CFL}_e$, $\mathbf{CFL}_{ew}$, $\mathbf{CFL}_{ec}$, *and* $\mathbf{CFL}_{ecw}(= CL)$.

Disjunction property

A logic **L** has the *disjunction property* when for any formulas α and β, if $\alpha \vee \beta$ is provable in **L** then either α or β is provable in **L** We note that the classical logic does not have the disjunction property, but the intuitionistic logic has this property.

In the case of basic substructural logics, we can show that the following theorem.

Theorem 3.2.5 *The sequent systems* $\mathbf{FL}$, $\mathbf{FL}_e$, $\mathbf{FL}_w$, $\mathbf{FL}_{ew}$, $\mathbf{FL}_{ec}$, $\mathbf{FL}_{ecw}(= INT)$, $\mathbf{CFL}_e$, *and* $\mathbf{CFL}_{ew}$ *have the disjunction property*

Craig's interpolation theorem

In 1957, W. Craig proved that the following theorem.

Theorem 3.2.6 (Craig) *If* $A \to B$ *is provable in classical logic then there exists of a formula* C *such that both* $A \to C$ *and* $C \to B$ *are provable, and every propositional variables in* C *appears both* A *and* B.

Now, the above theorem called Craig's interpolation theorem. A formula C in the theorem is an *interpolant* of $A \to B$. The same result holds also for the intuitionistic logic. A logic **L** has the *Craig's interpolation property* (CIP) if the statement holds for **L**.

It is known that the relationship among *Beth's definability theorem*, *Robinson's consistency theorem* and Craig's interpolation theorem. It is known that the all of these three properties are equivalent in classical logic and intuitionistic logic [11].

In 1977, after 20 years Craig's interpolation theorem, L. Maksimova proved the following amazing result[30].

Theorem 3.2.7 (Maksimova) *There are only 7 logics which have the CIP among the intermediate logics between the intuitionistic logic and the classical logic.*

Maksimova's theorem is striking, because it is shown that there are uncountable many intermediate logics between the intuitionistic logic and the classical logic. But the CIP holds for only 7 logics among them.

There are many way to prove the CIP. Original proof of Craig is obtained by using a semantical method. In 1960's, S. Maehara succeeded to show the CIP for classical logic follows from the cut elimination theorem [29]. This technique is called now Maehara's method.

In the following we will explain that what Maehara proved using the intuitionictic logic (INT). Let $\mathbf{LJ}^+$ be the sequent system obtained from $\mathbf{LJ}$ by adding propositional constants $\top$ and $\bot$ and initial sequents for these constants as follows: $\rightarrow \top$ and $\rightarrow \bot$. It is easy to see that the cut elimination theorem holds for $\mathbf{LJ}^+$ and $\mathbf{LJ}^+$ is a conservative extension of $\mathbf{LJ}$.

For any sequent Γ of formulas, $\langle \Gamma_1, \Gamma_2 \rangle$ of (possibly empty) sequences of formulas Γ_1 and Γ_2 is said to partition of Γ if the multiset of Γ_1 and Γ_2 is equal to Γ. The CIP for $\mathbf{LJ}^+$ is the following

Theorem 3.2.8 (Maehara) *Let $\Gamma \Rightarrow \alpha$ be a provable sequent in $\mathbf{LJ}^+$. and $\langle \Gamma_1, \Gamma_2 \rangle$ be any partition of Γ. Then there exists a formula β such that both $\Gamma_1 \Rightarrow \beta$ and $\beta, \Gamma_2 \Rightarrow \alpha$ are provable in $\mathbf{LJ}^+$, and moreover that $V(\beta) \subseteq V(\Gamma_1) \cap V(\Gamma, \alpha)$, where $V(A)$ denotes the variables occurring in a formula A.*

Maehara's method can be applied also to both **FL**-systems and **CFL**-systems except $\mathbf{FL}_c$ and $\mathbf{CFL}_c$. In the case of **CFL**-systems, it is necessary to modify the definition of partition. A sequent in **CFL**-systems is of the form, $\Gamma \rightarrow \Delta$, where both Γ and Δ are arbitrary finite sequence of formulas. We say that $\langle (\Gamma_1, \Delta_1); (\Gamma_2, \Delta_2) \rangle$ is a partition of $\Gamma \rightarrow \Delta$ where multiset union of Γ_1 and Γ_2 (Δ_1 and Δ_2, respectively) is equal to Γ (Δ, respectively) as a multiset of formulas.

Most of basic substractural logics enjoy cut elimination theorem. Thus, by using Maehara's method, we can show the CIP for basic substractural logics both **FL**-systems and **CFL**-systems except $\mathbf{FL}_c$ and $\mathbf{CFL}_c$.

Theorem 3.2.9 *Craig's interpolation theorem holds for substructural logics* $\mathbf{FL}$, $\mathbf{FL}_w$, $\mathbf{FL}_e$, $\mathbf{FL}_{ew}$, $\mathbf{FL}_{ec}$, $\mathbf{CFL}_e$, $\mathbf{CFL}_{ew}$ *and* $\mathbf{CFL}_{ec}$.

It is necessary to modify the definition of partitions to show the CIP for substructural logics without exchange rule. Suppose $\Gamma \rightarrow D$ is any sequent (of **FL**, or $\mathbf{FL_w}$). Then $\langle \Gamma_1, \Gamma_2, \Gamma_3 \rangle$ is a partition of Γ, if the sequence $\Gamma_1, \Gamma_2, \Gamma_3$ is equal to Γ without exchanging the order of formulas.

There are other consequences of the cut elimination. It is known that Maksimova's principle of variable separation and variable sharing property are important consequences of the cut elimination. For more information, see [37].

Some extensions of $\mathbf{FL}_e$ and fuzzy logics

Some axioms for extensions

The following axioms are for typical extensions of $\mathbf{FL}_e$.

C (contraction) : $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$,
W (weakening) : $\alpha \to (\beta \to \alpha)$,
EM (exclusive middle) : $\alpha \vee \neg\alpha$,
DN (double negation) : $\neg\neg\alpha \to \alpha$,
Wcon (weak contraction) : $(\alpha \to \neg\alpha) \to \neg\alpha$ equivalently, $\neg\alpha^2 \to \neg\alpha$,
P (Peirce's law) : $((\alpha \to \beta) \to \alpha) \to \alpha$,
WP (weak Peirce's law) : $(\neg\alpha \to \alpha) \to \alpha$,
Lin (linearity) : $(\alpha \to \beta) \vee (\beta \to \alpha)$,
Dis (distributive) : $(\alpha \wedge (\beta \vee \delta)) \to ((\alpha \wedge \beta) \vee (\alpha \wedge \delta))$.

Note that Wcon and WP are obtained from Con and P by replacing β by 0.

We write $\mathbf{FL}_e[X]$ for the logic an extension of $\mathbf{FL}_e$ with a set of axioms X. In usual convention, we use $\mathbf{FL}_{ew}$ $\mathbf{FL}_{ec}$ and $\mathbf{FL}_{ewc} = INT$ instead of $\mathbf{FL}_e[W]$, $\mathbf{FL}_e[C]$ and $\mathbf{FL}_e[W, C]$.

Fuzzy logics and many valued logic

The fundamental logic in fuzzy logic is called Hájek's basic logic (BL) which is defined by the following axioms and *modus ponens* which is only deduction rule for BL.

BL1: $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \beta))$
BL2: $(\alpha * \beta) \to \alpha$
BL3: $(\alpha * \beta) \to (\beta * \alpha)$
BL4: $(\alpha * (\alpha \to \beta)) \to (\beta * (\beta \to \alpha))$
BL5a: $(\alpha \to (\beta \to \gamma)) \to ((\alpha * \beta) \to \gamma)$
BL5b: $((\alpha * \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$
BL6: $((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma)$
BL7: $0 \to \alpha$

The family of fuzzy logics, called BL (Hájek's basic fuzzy logic), MTL (monoidal t-norm logic), Ł (Łukasiwicz's logic), PL (product logic), and other logics related fuzzy logic which can be defined by the following as extensions of substructural logics. Note that ML (monoidal logic) is precisely the same $\mathbf{FL}_{ew}$. We can also show that the many valued logic and Gödel's logic can be consider the extensions of substructural logics.

MTL : $\mathbf{FL}_{ew}[Lin]$.

$IMTL$: $\mathbf{FL}_{ew}[Lin, DN] = MTL + DN$.

$SMTL$: $\mathbf{FL}_{ew}[Lin, Wcon] = MTL + [\alpha \wedge \neg\alpha \to 0]$.

$WMTL$: $MTL + Wcon$ $(= \mathbf{FL}_{ew}[Lin, Wcon] = SMTL)$.

$\Pi MTL:\ MTL + [\alpha \wedge \neg\alpha \to 0] + [\neg\neg\beta \to [((\alpha * \beta) \to (\gamma * \beta)) \to (\alpha \to \gamma)]]$.

$BL:\ \mathbf{FL}_{ew}[Lin] + [\alpha \wedge \beta \to \alpha(\alpha \to \beta)]$.

$SBL:\ BL + [\alpha \wedge \neg\alpha \to 0]$.

$PL:\ BL + [\alpha \wedge \neg\alpha \to 0] + [\neg\neg\beta \to [((\alpha * \beta) \to (\gamma * \beta)) \to (\alpha \to \gamma)]]$.

Moreover, we can characterize that many valued logic (Ł) which is introduced by Łukasiewicz and Gödel's logic (G) as extensions of basic substructural logics and also can be regarded as extensions of fuzzy logics.

Ł : $\mathbf{FL}_{ew}[Lin] + [\alpha \wedge \beta \to \alpha(\alpha \to \beta)] + DN = BL + DN$.

$G:\ \mathbf{FL}_{ecw} + Lin = INT + Lin$.

Chapter 4

Finite Embeddability Property I

In this chapter, we will consider the *finite embeddability property* (FEP) for integral residuated lattices. The idea of this chapter is mainly due to W. Blok and C. J. van Alten's papers [6, 7]. First, we introduce the FEP and consider the relationships among the FEP, the *strong finite model property* (SFMP) and the *finite model property* (FMP). We can show that if an algebra has the FEP then it has the FMP. Hence, the FEP is one of the algebraic methods to show the FMP. Next, we introduce the Blok - van Alten's construction. The original construction introduced by W. Blok and C. J. van Alten is for the structure, partially ordered biresiduated groupoids, but our interests are not so general setting, we only focus the residuated lattices. We modefy their argument and proofs to the residuated lattices. Next, we shall prove that the FEP for integral residuated lattices. Lastly, we consider some classes of residuated lattices that are failure of the FEP.

4.1 Finite embeddability property

Let $\mathbf{A}$ be an algebra of the form $\langle A, \langle f_i^{\mathbf{A}} : i \in I \rangle\rangle$ of finite type and any nonempty subset $B \subseteq A$, the partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ with domain B is the partial algebra $\langle B, \langle f_i^{\mathbf{B}} : i \in I \rangle\rangle$, where for $i \in I$, f_i is n-ary function symbol, and $b_1, \cdots, b_n \in B$

$$f_i^{\mathbf{B}}(b_1, \cdots, b_n) = \begin{cases} f_i^{\mathbf{A}}(b_1, \cdots, b_n) & f_i^{\mathbf{A}}(b_1, \cdots, b_n) \in B \\ undefined & f_i^{\mathbf{A}}(b_1, \cdots, b_n) \notin B \end{cases}$$

Definition 4.1.1 (Finite embeddability property) A class of algebras $\mathcal{K}$ has the *finite embeddability property* (FEP) if every finite partial subalgebra of a member of $\mathcal{K}$ can be embedded into a finite member of $\mathcal{K}$.

It is known that if also a class of algebras $\mathcal{K}$ is *finitely axiomatizable* then its universal theory is decidable.

This notion, FEP was introduced and studied for varieties by T. Evans [15] which is closely related the word problem. Indeed T. Evans proved that if finitely presented algebra $\mathbf{A}$ in a variety $\mathcal{V}$ has the FEP, then the word problem is solvable for $\mathbf{A}$.

Historically, J. C. C. McKinsey and A. Tarski studied the same concepts before T. Evans and proved that the variety $\mathcal{HA}$ of all Heyting algebra has the FEP.

Next, we define the algebraic version of the *finite model property* (FMP) and the *strong finite model property* (SFMP). In the following, $\mathcal{K}$ denotes a class of algebras and $\mathcal{K}_F$ denotes the class of finite algebras in $\mathcal{K}$.

Definition 4.1.2 (Finite model property) A class of algebras $\mathcal{K}$ has the *finite model property* (FMP) if every identity that fail to hold in $\mathcal{K}$ can be refuted in a finite member of $\mathcal{K}$.

In other word, the FMP is the same as the following: A class $\mathcal{K}$ has the finite model property (FMP) if $\mathcal{K}_F \models s = t$ implies $\mathcal{K} \models s = t$ for all identity $s = t$. It is also equivalent to the condition $\mathcal{K} \subseteq HSP(\mathcal{K}_F)$.

Definition 4.1.3 (Strong finite model property) A class of algebras $\mathcal{K}$ has the *strong finite model property* (SFMP) if every quasi-identity that fail to hold in $\mathcal{K}$ can be refuted in a finite member of $\mathcal{K}$.

In other word, the SFMP is the same as the following A class $\mathcal{K}$ has the strong finite model property (SFMP) if $\mathcal{K}_F \models \sigma$ implies $\mathcal{K} \models \sigma$ for all quasi-identities σ. It is also equivalent to the condition $\mathcal{K} \subseteq ISPP_U(\mathcal{K}_F)$.

A class of algebra $\mathcal{K}$ is said to *locally finite* if every finitely generated subalgebra is finite. It is easy to see that a class of algebra $\mathcal{K}$ is *locally finite* then $\mathcal{K}$ has the FEP. Indeed, for any $\mathbf{A} \in \mathcal{K}$ and any finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$, the algebra gen($\mathbf{B}$) generated by B is finite algebra in $\mathcal{K}$. Hence $\mathbf{B}$ can be embedded into gen($\mathbf{B}$).

It is also easy to see that if a quasivariety $\mathcal{K}$ has the FEP then $\mathcal{K}$ has the SFMP. Indeed, let $\mathcal{K}$ be a quasivariety. Then $\mathcal{K}$ satisfies that $\mathcal{K} = ISPP_U(\mathcal{K})$. Let $\mathcal{K}$ satisfy the FEP and $\mathcal{K}_F$ be the class of finite algebra in $\mathcal{K}$. Then for any quasi-identity σ, $\mathcal{K}$ satisfies that $\mathcal{K}_F \models \sigma \Longrightarrow \mathcal{K} \models \sigma$, Hence, $\mathcal{K} \subseteq ISPP_U(\mathcal{K}_F)$, and so $\mathcal{K}$ has the SFMP.

To the end of this section, we remark the following.

(1) If a class of algebras $\mathcal{K}$ has the locally finite then $\mathcal{K}$ has the FEP.

(2) If a class of algebras $\mathcal{K}$ has the FEP then $\mathcal{K}$ has the SFMP.

(3) If a class of algebras $\mathcal{K}$ has the SFMP then $\mathcal{K}$ has the FMP.

4.2 The Blok - van Alten's construction

In this section we introduce the construction by Blok - van Alten [6]. Their original construction is for the structure called *partially ordered biresiduated groupoids* which is much general setting for our attention. Therefore we introduce the construction for residuated lattices.

Let $\mathbf{A}$ be an integral residuated lattice and $\mathbf{B}$ be a partial subalgebra of $\mathbf{A}$. We shall sometimes omit the multiplicative symbol $\cdot$. Let $\mathbf{M}$ be the submonoid of $\mathbf{A}$ generated by $\mathbf{B}$. If $X, Y \subseteq M$ and $a \in M$ then XY denotes $\{ab : a \in X, b \in Y\}$, Xa denotes $X\{a\}$ and aX denotes $\{a\}X$. For each $x, y \in M$ and $b \in B$, define $(x : y \rightsquigarrow b] = \{c \in M : xcy \leq b\}$. This set is a downward closed subset of M. Note that $(\emptyset \rightsquigarrow b] = \{a \in M : a \leq b\}$. Define $\overline{D} = \{(x : y \rightsquigarrow b] : x, y \in M, b \in B\}$. $D = \{\bigcap \Xi : \Xi \subseteq \overline{D}\}$. There is a closure operator C on subset M associated with D. For $X \subseteq M$, $\mathsf{C}(X) = \bigcap\{(x : y \rightsquigarrow b] \in \overline{D} : X \subseteq (x : y \rightsquigarrow b]\}$. Define for all $X, Y \subseteq M$,

$$X \cdot^D Y = \mathsf{C}(XY)$$
$$X \backslash^D Y = \{a \in M : Xa \subseteq Y\}$$
$$X /^D Y = \{a \in M : aX \subseteq Y\}$$
$$\bigwedge_{i \in I}^{D} X_i = \bigcap_{i \in I} X_i$$
$$\bigvee_{i \in I}^{D} X_i = \mathsf{C}(\bigcup_{i \in I} X_i)$$
$$1^D = M$$
$$0^D = \bigcap \overline{D}$$

.

Theorem 4.2.1 *The structure* $\mathbf{D}(A, B) = \langle D, \cdot^D, \backslash^D, /^D, \wedge^D, \vee^D, 1^D \rangle$ *is an integral residuated lattice.*

To prove the theorem, we prepare some lemmas,

Lemma 4.2.2 *If* $X \subseteq M$ *and* $Y_i \subseteq M$ *for* $i \in I$ *then* $X \backslash^D \bigcap_{i \in I} Y_i = \bigcap_{i \in I}(X \backslash^D Y_i)$ *and* $X /^D \bigcap_{i \in I} Y_i = \bigcap_{i \in I}(X /^D Y_i)$.

Proof. For any $a \in M$, $Xa \subseteq \bigcap_{i \in I} Y_i \Leftrightarrow Xa \subseteq Y_i$ for each $i \in I$. This follows the first statement. The second case is similar to prove.

Lemma 4.2.3 *If* $X \subseteq M$*and* $Y \subseteq D$ *then* $X \backslash^D Y, X /^D Y \in D$.

Proof. We first show that for $X \subseteq M, x, y, c \in M$ and $b \in B$, $X \backslash^D (x : y \rightsquigarrow b] = \bigcap\{(xc : y \rightsquigarrow b] : c \in X\}$, $(x : y \rightsquigarrow b] /^D X = \bigcap\{(x : cy \rightsquigarrow b] : c \in X\}$. Note that is $c \in X$ then $(xc : y \rightsquigarrow b] \in \overline{D}$ since $X \subseteq M$. Hence, the first equation imply that $X \backslash^D (xc : y \rightsquigarrow b] \in D$. To see the equation is satisfied,

$$\begin{aligned} d \in X \backslash^D (x : y \rightsquigarrow b] &\Leftrightarrow cd \in (x : y \rightsquigarrow b] \\ &\Leftrightarrow xcdy \leq b \\ &\Leftrightarrow d \in (x : cy \rightsquigarrow b] \end{aligned}$$

for each $c \in X$. Since $Y \in D$, Y can be expressed of the form $Y = \bigcap\{(x_i : y_i \rightsquigarrow b_i] : i \in I\}$. By the Lemma 1, we can get $X\backslash^D Y = X\backslash^D \bigcap((x_i : y_i \rightsquigarrow b_i]) = \bigcap(X\backslash^D (x_i : y_i \rightsquigarrow b_i])$. Thus, $X\backslash^D Y \in D$. The proof that $X/^D Y \in D$ is similar to prove.

Lemma 4.2.4 *The operator* C *satisfies that the following additional condition which is related monoid operation* $\cdot$. $\mathsf{C}(X)\mathsf{C}(Y) \subseteq \mathsf{C}(XY)$, *for all* $X, Y \subseteq M$.

Proof. Let $X, Y, Z \subseteq M$.

$$\begin{aligned} XY \subseteq \mathsf{C}(XY) &\Leftrightarrow Y \subseteq X\backslash^D \mathsf{C}(XY) \\ &\Rightarrow \mathsf{C}(Y) \subseteq X\backslash^D \mathsf{C}(XY) \\ &\Leftrightarrow X\mathsf{C}(Y) \subseteq \mathsf{C}(XY) \\ &\Leftrightarrow X \subseteq \mathsf{C}(XY)/^D \mathsf{C}(Y) \\ &\Rightarrow \mathsf{C}(X) \subseteq \mathsf{C}(XY)/^D \mathsf{C}(Y) \\ &\Leftrightarrow \mathsf{C}(X)\mathsf{C}(Y) \subseteq \mathsf{C}(XY) \end{aligned}$$

Note that 1^D is an identity element of $\mathbf{D}(A, B)$. Indeed, let $X \in D$. Since X is downward closed, $X \supseteq XM(MX)$, and since $1 \in M$ we also have $X \subseteq XM(MX)$, so $X = XM = MX$. We claim that the operations $\backslash^D$, $/^D$ and $\bigwedge^D$ are closed under the operation C. Indeed, to see the operation $/^D$ is closed under C. We need to show that $\mathsf{C}(X/^D Y) \subseteq X/^D Y$, since the converse direction is always hold. Let X, Y be in D. Then $X \cdot^D \mathsf{C}(X/^D Y) = \mathsf{C}(X) \cdot^D \mathsf{C}(X/^D Y) \subseteq \mathsf{C}(X \cdot (X/^D Y) \subseteq \mathsf{C}(Y) = Y$. This implies that $\mathsf{C}(X/^D Y) \subseteq X/^D Y$. In the case of the operation $\backslash^D$ can be similar to prove. Next, to show that $\bigwedge^D$ is closed under C. Let $X_i (i \in I)$ be in D. We need to show that $\mathsf{C}(\bigwedge^D X_i) \subseteq \bigwedge^D X_i$, since the converse direction is always hold. For each $i \in I$, $\bigwedge^D X_i = \bigcap^D X_i \subseteq X_i \Rightarrow \mathsf{C}(\bigcap^D X_i) \subseteq \mathsf{C}(X_i) = X_i$. Thus $\mathsf{C}(\bigwedge^D X_i) \subseteq \bigwedge^D X_i$.

Proof of Theorem 4.2.1
The operation $\cdot^D$ is monoid operation. It is easy to see the associativity of $\cdot^D$. Indeed, let $X, Y, Z \in D$. Then $(X \cdot^D Y) \cdot^D Z = \mathsf{C}(\mathsf{C}(XY)Z)$. By Lemma 21, $\mathsf{C}(\mathsf{C}(XY)Z) = \mathsf{C}(\mathsf{C}(XY)\mathsf{C}(Z)) \subseteq \mathsf{C}(\mathsf{C}((XY)Z)) = \mathsf{C}((XY)Z)$. Since C is a closure operator, we have $\mathsf{C}((XY)Z) \subseteq \mathsf{C}(\mathsf{C}((XY)Z))$. Hence $(X \cdot^D Y) \cdot^D Z = \mathsf{C}((XY)Z)$. Similarly, we have $X \cdot^D (Y \cdot^D Z) = \mathsf{C}(X(YZ))$. Since $\mathbf{M}$ is a monoid, we have $(XY)Z = X(YZ))$, hence $\cdot^D$ is associative on elements of D. We have shown that D is closed under the operations $/^D$ and $\backslash^D$. The lattice operations $\wedge^D$ and $\vee^D$ can be defined by set theoretic inclusion $\subseteq$. Thus, the things we need to prove is the residuations. For any $X, Y, Z \in D$, we can show that $XZ \subseteq Y \Leftrightarrow Z \subseteq X\backslash^D Y$ by the definition of $\backslash^D$, and also Y is closed with respect to the operation C. Hence, $X \cdot^D Z \subseteq Y \Leftrightarrow XZ \subseteq Y$. Therefore $\backslash^D$ satisfies the condition of the left residuation. The operation $/^D$ is similar to prove.

Theorem 4.2.5 *If* $\mathbf{A}$ *is an integral residuated lattice and* $\mathbf{B}$ *is a partial subalgebra of* $\mathbf{A}$, *then* $\mathbf{B}$ *can be embedded, as a partial subalgebra, into the integral residuated lattice* $\mathbf{D}(A, B)$.

Proof. We define an embedding from $\mathbf{B}$ to $\mathbf{D}(A,B)$ by $b \to (\emptyset \rightsquigarrow b]$. To show the theorem we need to prove that all operations $\backslash, /, \wedge, \vee, \cdot$ and constants $1, 0$ in $\mathbf{B}$ preserve into $\mathbf{D}(A,B)$ by this embedding.

(1) Suppose $a \backslash^{\mathbf{A}} b \in B$. To see $(a \backslash^{\mathbf{A}} b] = (a] \backslash^{D} (b]$. We need to show that $\{c \in M : ac \leq b\} = \{c \in M : (a]c \subseteq (b]\}$. From left to right direction, let $c \in M$ such that $ac \leq b$ and $d \leq a$. Then $dc \leq ac \leq b$. Thus, $dc \in (b]$. The converse direction, suppose $(a]c \subseteq (b]$. Then $ac \leq b$. The operation $/$ is similar to prove. (2) Suppose $ab \in B$. We need to show that $(ab] = (a] \cdot^{D} (b]$. From left to right direction, $ab \in (a](b] \subseteq (a] \cdot^{D} (b]$. The converse direction, take $c \leq a, d \leq b$. Then $cd \leq ac$ and so $(a](b] \subseteq (ab]$. Thus, $\mathsf{C}((a](b]) = (a] \cdot^{D} (b] \subseteq (ab]$. (3) Suppose $a_i \in B$ and $\bigwedge a_i$ exists in A and also $\bigwedge a_i \in B$. It is clear that $(\bigwedge a_i] = \bigcap (a_i] = \bigwedge^{D} (a_i]$. (4) Suppose $a_i \in B$ and $\bigvee a_i$ exists in A and also $\bigvee a_i \in B$. We need to show that $(\bigvee a_i] = \bigcap (a_i] = \bigvee^{D} (a_i]$. From left to right direction, $(a_i] \subseteq (\bigvee_{i \in I} a_i] \Rightarrow \bigcup_{i \in I} (a_i] \subseteq (\bigvee_{i \in I} a_i] \Rightarrow \mathsf{C}(\bigcup_{i \in I} (a_i]) = \bigvee^{D} (a_i] \subseteq (\bigwedge_{i \in I} a_i]$. The converse direction, let $(x : y \rightsquigarrow d] \in \overline{D}$ such that $\bigcup_i (a_i] \subseteq (x : y \rightsquigarrow d]$. Then $a_i \in (x : y \rightsquigarrow d] \Rightarrow x a_i y \leq d \Rightarrow \bigvee x a_i y \leq d \Rightarrow x \bigvee a_i y \leq d$. Thus, $\bigvee a_i \in (x : y \rightsquigarrow d]$. Therefore $\bigvee_i a_i \in \bigcap \{(x : y \rightsquigarrow d] \in \overline{D} : \bigcup_i (a_i] \subseteq (a : y \rightsquigarrow d]\} = \mathsf{C}(\bigcup_i (a_i]) = \bigvee_i (a_i]$. (5) Suppose $1^A \in B$. Then $(1^A] = M = 1^D$. (6) Suppose $0^A \in B$. Then $(x : y \rightsquigarrow b]$ is downward closed set. Thus $0^A \in (x : y \rightsquigarrow b]$ and so $0^A \in \bigcap \overline{D}$. $(0^A] = \{0^A\} \subseteq \bigcap \overline{D} \subseteq (0^A]$. So $(0^A] = 0^D$. This completes the theorem

Well-quasi order and Higman's theorem

In this section we discuss well-quasi order and Higman's theorem. For more information, see [48].

Well-quasi order

Let $\leq$ be a quasi order on a set X. An infinite sequence $(x_n : n \in \mathbb{N})$ in X is called *good* if there are indices i and j such that $i < j$ and $x_i \leq x_j$, otherwise it is called a bad sequence. We write $x < y$ to abbreviate $x \leq y \wedge y \not\leq x$.

We say that $\leq$ is *well-behaved* if every infinite sequence is good.

Definition 4.2.1 (Well-quasi order) A well-behaved quasi order is called a *well-quasi* order.

For the sake of convenience, we will say that a quasi order $\leq$ is well-founded if it has no infinite strictly descending $>$ sequence.

Lemma 4.2.6 *Every well-quasi order $\leq$ is well-founded.*

Proof. Suppose $\leq$ is a well-quasi order on a set A. Take an infinite sequence $(x_n : n \in \mathbb{N})$ in A such that $x_n > x_{n+1}$ for all $n \in \mathbb{N}$. By the transitivity of $\leq$, $x_n \geq x_{n+m}$ holds. Moreover we have $x_n \not\leq x_{n+m}$. Indeed, suppose $x_n \leq x_{n+m}$. We also have $x_{n+1} \geq x_{n+m}$.

Thus, $x_n \leq x_{n+1}$. But it is a contradiction. Since $\leq$ is a well-behaved by the assumption, there exists some indices $i, j > 0$ such that $i < j$ and $x_i \leq x_j$. But this is contradiction.

Let $(x_n : n \in \mathbb{N})$ in a set X be an infinite sequence. Then we say that $(y_n : n \in \mathbb{N})$ is an infinite subsequence of $(x_n : n \in \mathbb{N})$ if there exists an injective monotone mapping f from $\mathbb{N}$ to $\mathbb{N}$ such that $y_n = x_{f(n)}$ for all $n \in \mathbb{N}$. We also say that an element x_i in an sequence $(x_n : n \in \mathbb{N})$ is *terminal* if there is no index j with $j > i$ such that $x_i \leq x_j$.

Theorem 4.2.7 *Let $\leq$ be a quasi order on a set X. Then the following conditions are equivalent.*
(1) *$\leq$ is a well-quasi order.*
(2) *$\leq$ is a well-founded and every antichain in X is finite.*
(3) *Every infinite sequence $(x_n : n \in \mathbb{N})$ in X contains an infinite subsequence $(x_{f(n)} : n \in \mathbb{N})$ such that $x_{f(n)} \leq x_{f(n+1)}$ for all $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2). Suppose $\leq$ is a well-quasi order on X. Then $\leq$ is a well-founded by Lemma. We need to show that every antichain in X is finite. Assume that $(x_n : n \in \mathbb{N})$ is an infinite antichain. This means that any pair of distinct elements in $(x_n : n \in \mathbb{N})$ are incomparable. But we assume that $\leq$ is a well-quasi order, i.e., well-behaved. This is contradiction. Therefore every antichain in X is finite.
(2) $\Rightarrow$ (1). Suppose $\leq$ is a well-founded and there exists a bad sequence $(x_n : n \in \mathbb{N})$ in X. By well-foundedness of $\langle X, \leq \rangle$, there exists m_1, for any $n \geq m_1$ such that $x_{m_1} \not\geq x_n$. Since $(x_n : n \in \mathbb{N})$ is bad, $x_{m_1} \not\leq x_n$ for all $n > m_1$. Consider $\{x_n\}_{n > m_1}$. By well-foundedness of $\langle X, \leq \rangle$, we can take $m_2 > m_1$ such that $x_{m_2} \not\geq x_n$ for any $n \geq m_2$. Now, $x_{m_1} \not\geq x_{m_2}$ and $x_{m_2} \not\leq x_{m_2}$, so both x_{m_1} and x_{m_2} are imcpmparable. Next, consider $\{x_n\}_{n > m_2}$ and iterate the same argument. Finally, we can get the set $\{x_{m_i}\}_{i \in I}$ which is a infinite antichain in X.

(1) $\Rightarrow$ (3). Suppose $\leq$ is a well-quasi order on X. Let $(x_n : n \in \mathbb{N})$ be an infinite sequence in X. It is easy to see that the number of the terminal element in $(x_n : n \in \mathbb{N})$ is finite. Otherwise, the infinite sequence of terminal elements in it is a bad sequence (because if the sequence of terminal elements are good, then we have $x_i \leq x_j$ which contardicts the fact that x_i is terminal), and this contradicts the fact that $\leq$ is a well-quasi order. Hence there exists some $m > 0$ such that x_i is not terminal for every $i \geq m$. Define an injective monotone mapping f as follows. $f(0) = m$, and for any $n \geq 0$ $f(n+1)$ is the least integer such that $x_{f(n)} \leq x_{f(n+1)}$ and $f(n+1) > f(n)$. Then we can construct the sequence $(x_{f(n)} : n \in \mathbb{N})$ as required.
(3) $\Rightarrow$ (1) is trivial by the definition of a well-quasi order.

It is known that the direct proof of (2) $\Rightarrow$ (3) is given by using the infinite version of Ramsey's theorem. In the following is due to I. Hodkinson [20]. Let S be a set and κ a cardinal. Then $[S]^\kappa$ is the set of subset of S of size κ.

Theorem 4.2.8 (Ramsey's theorem) *If $f : [\mathbb{N}]^n \to k$, where $n, k \in \mathbb{N}$, then there is infinite $I \subseteq \mathbb{N}$ such that $f|_{[I]^n}$ is constant.*

We give a sketch of the direct proof of (2) $\Rightarrow$ (3). Assume (2). Let $f : [\mathbb{N}]^2 \to \{\leq, >, \perp\}$ be a function such that $x_i f(i,j) x_j$ for all $i < j$. Here $a \perp b$ means a and b are incomparable. By Ramsey's theorem, there exists an infinite set $X \subseteq \mathbb{N}$ such that $f|_{[X]^2}$ is constant. Supposing (2) the constant value must be $\leq$ and so (3) holds.

Next, we will show that the products of natural numbers $\mathbb{N}^k$ is well-quasi order. To see that we first show the case $k = 2$. Let $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$ be well-quasi order. Consider the structure $\langle A \times B, \leq \rangle$, where $\leq$ is a product order on $A \times B$. Take an arbitrary infinite sequence $\langle (a_n, b_n) : n \in \mathbb{N} \rangle$ of $A \times B$. Then $\langle a_n : n\mathbb{N} \rangle$ is an infinite sequence of A. By Theorem 4.2.7, there exists an infinite subsequence $\langle a_{g(n)} : n\mathbb{N} \rangle$ such that $a_{g(n)} \leq_A a_{g(n+1)}$ for all n. Then $\langle b_{g(n)} : n\mathbb{N} \rangle$ is also an infinite sequence of B. By Theorem 4.2.7, there exists m_1 and m_2 such that $g(m_1) < g(m_2)$ and $b_{g(m_1)} \leq_B b_{g(m_2)}$. Now g is a monotone, thus $m_1 < m_2$. Hence $a_{g(m_1)} \leq_A a_{g(m_2)}$. Therefore we have $g(m_1) < g(m_2)$ and $(a_{g(m_1)}, b_{g(m_1)}) \leq (a_{g(m_2)}, b_{g(m_2)})$. To repeat this argument, we have the following result.

Proposition 4.2.9 $(\mathbb{N}^k, \leq)$ *is well-quasi order.*

Higman's theorem

Let $\mathbf{A}$ be an algebra of type Σ. A given quasi order $\leq$ on Σ is called a precedence ordering of operation symbols.

Definition 4.2.2 (Divisibility order) Let $\mathbf{A}$ be an algebra of type Σ and $\leq$ is a precedence ordering on Σ. A quasi order $\ll$ on $\mathbf{A}$ is called a *divisibility order* based on $\leq$ if for all operation symbols σ, τ and all elements $a, a_1, \cdots, a_m, b_1, \cdots, b_n$ in A, the following conditions are satisfied
(1) If $a \ll a_i$ for some $1 \leq i \leq m$, then $a \ll \sigma^{\mathbf{A}}(a_1, \cdots, a_m)$,
(2) If $\sigma \leq \tau$ and $a_i \ll b_{j_i}$ for $1 \leq i \leq m$ and some $j_1, \cdots, j_m$ with $1 \leq j_1 < j_2 < \cdots < j_m \leq n$, then $\sigma^{\mathbf{A}}(a_1, \cdots, a_m) \ll \tau^{\mathbf{A}}(b_1, \cdots, b_n)$.

Divisibility orders have a very important property. G. Higman proved that the following theorem.

Theorem 4.2.10 (Higman's Theorem) *Let* $\mathbf{A}$ *be an algebra of type* Σ*. Assume that* A *is equipped with a divisibility order which is based on a well-quasi order on* Σ*. If the divisibility order restricted to any generating set of* A *is a well-quasi order, then the divisibility order on* $\mathbf{A}$ *is already a well-quasi order.*

4.3 Finite embeddability property for integral residuated lattices

Let $\mathbf{F}(k)$ be the free monoid with identity element 1 on k generators $x_1, \cdots, x_k$. Each element of $F(k)$ is assumed to be reduced, i.e., each element contains no 1's.

Define a relation $\leq$ on $F(k)$ by $s \leq t$ iff some instances of variables in s can be replaced by 1's in such a way that it reduced to t. It is obvious that the relation $\leq$ is partially order. For example, $x_1x_2x_1x_3 \leq {x_1}^2x_3$. Indeed, we replace x_2 by 1. Then the left hand side is equal to the right hand side.

Lemma 4.3.1 *On the structure $\langle \mathbf{F}(k), \leq \rangle$, we can define the left and right residuations.*

Proof. It is enough to consider the case of left residuation $\backslash$. Define the left residuation by the following. Let s, t be elements in $F(k)$. Then (1) If $s \leq t$ then $s \backslash t = 1$. (2) If $s \not\leq t$ but $s \leq t_1$ where t_1 is maximal decomposition of t, i.e., $t = t_1t_2$ then $s \backslash t = t_2$. (3) If $s \not\leq t$ and $s \not\leq t_1$ for any decomposition of t then $s \backslash t = t$. Then, it is easy to see that $\backslash$ is the left residuation on $F(k)$.

A quasi ordered set has the *finite basis property* if every downward closed set is the downward closure of a finite set which is equivalent to a quasi order is well-quasi ordered. For convenience in our setting, we consider dual quasi order. Moreover, we can consider the condition of *divisibility order* the following, since type Σ of algebra in our case has only one type, monoid operation.

Give an algebra, a quasi order $\leq$ on a set A is called *divisibility order* if it satisfies the following conditions.
(1) Each operation of A is order-preserving in each its argument.
(2) If f^A is an n-ary operation of A and $a_1, \cdots a_n \in A$ then $f^A(a_1, \cdots, a_n) \leq a_i$ for each $i = 1, \cdots, n$.

Then we can prove the following theorem using the Higman's theorem.

Theorem 4.3.2 *An algebra of a finite type Σ has the finite basis property in a divisibility order if any generating set has.*

Theorem 4.3.3 *The order $\leq$ on the algebra $\mathbf{F}(k)$ has the finite basis property.*

Proof. First, to see that $\leq$ is divisibility order on $\mathbf{F}(k)$. The algebra $\mathbf{F}(k)$ has only one operation (without constant) which is monoid operation $\cdot$. It is easy to see that monoid operation satisfies the conditions (1) and (2). Next, consider that a generating set of $\mathbf{F}(k)$ has the finite basis property. This is trivial since a quasi order on a finite set $\{x_1, \cdots, x_k\}$ always satisfies the finite basis property.

As an important consequence of the theorem, let X be a subset of $\mathbf{F}(k)$ and we write $\mathrm{Max}(X)$ the maximal elements of X. Then it is easy to see that $(X]$ is equal to $(\mathrm{Max}(X)]$ by the definition of $\leq$ on $\mathbf{F}(k)$. Indeed, each larger element has fewer number of symbols. By the finite basis property, $(\mathrm{Max}(X)]$ is the downward closure of a finite set, hence $\mathrm{Max}(X)$ must be finite. This is also obtained that $\leq$ is well-quasi order since the set $\mathrm{Max}(X)$ is clearly an antichain.

Lemma 4.3.4 *If $\mathbf{A}$ is an integral residuated lattice and $\mathbf{B}$ is a finite partial subalgebra of $\mathbf{A}$ then $\mathbf{D}(A, B)$ is finite.*

Proof. Let $B = \{b_1, \cdots, b_k\}$. Recall that $F(k)$ be the free monoid on k generators $\{x_1, \cdots, x_k\}$. The map that takes x_i to b_i for each $i \in \{1, \cdots, k\}$ extends naturally to a map h from $F(k)$ to A that preserves the monoid operation. Consider the $h^{-1}((b])$ which is a subset of $F(k)$. We denote $\mathrm{Crit}(b)$ the set of all maximal elements in $h^{-1}((b])$. Then $\mathrm{Crit}(b)$ is a finite set.

Put $Z = \bigcup_{b \in B} \bigcup_{z \in \mathrm{Crit}(b)} [z)$, where $[z)$ is the upward closure of $\{z\}$. It is obvious that $[z)$ is finite since each larger element has fewer number of symbols. Thus B, each $\mathrm{Crit}(b)$ and each $[z)$ is finite. Hence Z is also finite.

Let $b \in B, a, c \in M$. We show that $h^{-1}((a : c \rightsquigarrow b]) = (Y]$ for some $Y \subseteq Z$. Let $x, z \in F(k)$ such that $h(x) = a, h(z) = c$. Then

$$\begin{aligned}
y \in h^{-1}((a : c \rightsquigarrow b]) &\Leftrightarrow h(y) \in (a : c \rightsquigarrow b] \\
&\Leftrightarrow ah(y)c \leq b \\
&\Leftrightarrow h(x)h(y)h(z) \leq b \\
&\Leftrightarrow h(xyz) \leq b \\
&\Leftrightarrow xyz \in h^{-1}((b]) \\
&\Leftrightarrow xyz \leq w, (\exists w \in \mathrm{Crit}(b)) \\
&\Leftrightarrow y \leq x\backslash(w/z), (\exists w \in \mathrm{Crit}(b))
\end{aligned}$$

Therefore $y \in h^{-1}((a : c \rightsquigarrow b]) \Leftrightarrow y \leq x\backslash(w/z)$ for some $w \in \mathrm{Crit}(b)$. Put $Y = \{x\backslash(w/z) : w \in \mathrm{Crit}(b)\}$ which is a subset of Z. Then $h^{-1}((a : c \rightsquigarrow b]) = (Y]$. Since h is surjective on M $((a : c \rightsquigarrow b]) = h(Y]$. Since Z is finite, therefore there is only finitely many distinct $(a : c \rightsquigarrow b]$. Hence $\mathbf{D}(A, B)$ is finite.

Then we have the following results.

Theorem 4.3.5 *The variety $\mathcal{IRL}$ of all integral residuated lattices has the FEP.*

These arguments above work also well in the case of commutative integral residuated lattices. Then we have the following corollary.

Corollary 4.3.6 *The variety $\mathcal{CIRL}$, $\mathcal{FL}_w$ and $\mathcal{FL}_{ew}$ have the FEP.*

Next, considering the corresponding logical systems, we have the following.

Theorem 4.3.7 *The logical systems $\mathbf{FL}_w$ and $\mathbf{FL}_{ew}$ have the FMP, and hence both are decidable.*

Of course the decidability of both $\mathbf{FL}_w$ and $\mathbf{FL}_{ew}$ can be proved without showing the FEP for algebras related those logics, since cut elimination theorem holds for both of them. And both $\mathbf{FL}_w$ and $\mathbf{FL}_{ew}$ have the FMP were proved by M. Okada and K. Terui. However they proved the FMP by using cut elimination theorem.

Failure of finite embeddability property

Residuated lattices and $\mathbf{FL_e}$-algebras

M. Okada and K. Terui proved that the logical system **FL**-systems except $\mathbf{FL}_c$ has the FMP, hence equational theory of these corresponding algebras are decidable. Otherwise, we can show that universal theory of **FL**-algebras and $\mathbf{FL}_e$-algebras does not have the FEP, moreover both are undecidable. In [22], P. Jipsen and C. Tsinakis proved that the quasi-equational theory of residuated lattices is undecidable.

Thus, we can show the $\mathbf{FL}_e$-algebras does not have the FEP, moreover its universal theory is undecidable. P. Lincoln, J. Mitchell, A. Scedrov and N. Shankar studied decision problems for propositional linear logic. They showed that the undecidability of the full system of linear logic. By analyzing their proof carefully, H. Ono pointed out that the undecidability of the deducibility relation of $\mathbf{FL}_e$ follows from it. The same argument was mentioned by W. Blok and C. J. van Alten. It follows that the quasi-equational theory of $\mathbf{FL}_e$-algebras is undecidable, and hence the SFMP doesn't hold in $\mathbf{FL}_e$-algebras. They also give a simple direct proof of this fact[6].

Cancellative residuated lattices

Next, example is cancellative residuated lattices. We can show that a class of cancellative residuated lattices does not have the FEP. It is easy to see that if a cancellative residuated lattice has the bottom element 0 then $0x = x0 = 0$. Hence a cancellative residuated lattice with 0 is only trivial algebra. Furthermore, we can show that a variety of cancellative residuated lattices does not have the FEP. Indeed, without loss of generality, we only consider the 0-free cancellative residuated lattices Suppose 0-free cancellative residuated lattices **A** has the FEP. By assumption, for any finite partial subalgebra **B** can be embedded into a finite algebra **F**. Take $x \in \mathbf{B}$ which is not equal to 1. By the finiteness of **F**, there exists a natural number k such that $x^k = x^{k+1}$. Using the cancellative law iteratively, we can get $x = 1$. This contradicts the assumption. Therefore we conclude that the variety $Can\mathcal{RL}$ does not have the FEP. Hence any non-trivial cancellative residuated lattice is infinite. For more information about cancellative residuated lattices, see [22].

To the end of this chapter, we summarize that the relationship between the FMP and the FEP in **FL**-systems and **CFL**-systems as follows. Note that in 2004, van Alten and Raftery proved the FEP for the variety of $\mathbf{FL}_{ec}$-algebras [2].

	FL	$\mathbf{FL}_e$	$\mathbf{FL}_w$	$\mathbf{FL}_c$	$\mathbf{FL}_{ec}$	$\mathbf{FL}_{ew}$	INT
FMP	◯	◯	◯	?	◯	◯	◯
FEP	×	×	◯	?	◯	◯	◯

	CFL	$\mathbf{CFL}_e$	$\mathbf{CFL}_w$	$\mathbf{CFL}_c$	$\mathbf{CFL}_{ec}$	$\mathbf{CFL}_{ew}$	CL
FMP	◯	◯	◯	?	◯	◯	◯
FEP	×	×	◯	?	◯	◯	◯

Chapter 5

Finite Embeddability Property II

In this chapter, we will consider the *finite embeddability property* (FEP) for various classes of residuated lattices and *finite model property* (FMP) for various substructural logics including fuzzy logics.

Using the idea of previous chapter, we will show that the FEP for various classes of integral residuated lattices which are closely related to the substructural logics. Next, we summarize the FMP for various substructural logics including fuzzy logics. Lastly, we consider some classes of residuated lattices that are failure of the FEP.

The FMP is quite powerful method to show the decidability in the study of modal logics, since the decidability follows from the FMP and the finite axiomatizability. It is well known argument by Harrop. But It is known that it is hard to show the FMP of substructural logics and we do not have any powerful method like the *filtration method* in modal logic yet.

Studies of the FMP of substructural logics were made by R.K. Meyer in 1972, R.K. Meyer and H. Ono in 1994, W. Buszkowski in 1996 and also C.J. van Alten and J.G. Raftery in 1999. All of these studies (except W. Buszkowski's work) are implicational fragment of substructural logics. Studies of other fragments of substructural logics were made by Y. Lafont in 1997. He proved that each of the **CFL**-systems except $\mathbf{CFL}_c$ has the FMP. and also M. Okada and K. Terui in 1999. They proved that each of the **FL**-systems except $\mathbf{FL}_c$ has the FMP. Therefore, they showed the FMP of most basic substructural logics, but they used cut elimination theorem to show the FMP. That means, to show the FMP is much harder than to show decidability.

First, we consider *full* left (right) integral residuated lattices which is obtained by deleting right (left) residuation from integral residuated lattices. Second, we will prove that the FEP for various classes of integral residuated lattices. Lastly, we summarize the FMP for various substructural logics including fuzzy logics.

5.1 Finite embeddability property for full left (right) integral residuated lattices

We say that a reduct of residuated lattice which is obtained by deleting right (left) residuation, *full left residuated lattice*. In this case we can show that the variety of full left (right) integral residuated lattices has the FEP using the argument of previous section stating with **A** is a left residuated lattice.

Theorem 5.1.1 *The variety of full left (right) integral residuated lattices has the FEP.*

We note that a full left residuated lattice is also obtained by left residuated lattice with the following conditions.

(1) $x \leq y$ implies both $xz \leq yz$ and $zx \leq zy$,
(2) $(y \vee z)x = yx \vee zx$.

Both conditions holds in a residuated lattice. C. J. van Alten and J. G. Raftery pointed out that if it drops the condition (2), there exists a counter example that a left integral residuated lattice cannot be embedded into an integral residuated lattice and so Blok-van Alten's construction does not work in this case.

C. J. van Alten and J. G. Raftery show that the following.

Let **A** be a left integral residuated lattice with additional conditions (1) and (2) mentioned above and $\mathcal{S}$ be the set of subset of **A** that are downward closed and closed under joins. Define operations on $\mathcal{S}$ as follows. Let X, Y be elements of $\mathcal{S}$.

$$X \cdot^{\mathcal{S}} Y = (\{xy : x \in X, y \in Y\},]$$
$$X \backslash^{\mathcal{S}} Y = \{a \in A : Xa \subseteq Y\},$$
$$X /^{\mathcal{S}} Y = \{a \in A : aX \subseteq Y\},$$
$$X \bigwedge^{\mathcal{S}} Y = X \bigcap Y,$$
$$X \bigvee^{\mathcal{S}} Y = (\{x \vee y : x \in X, y \in Y\}],$$
$$1^{\mathcal{S}} = (1^{\mathbf{A}}].$$

Then $\mathcal{S}$ with these operations form the integral residuated lattice, and also **A** can be embedded into $\mathcal{S}$ by the map $a \rightarrow (a]$.

C. J. van Alten and J. G. Raftery also pointed out that the condition (2) of the identities of a full left integral residuated lattice is crucial. If we drop this condition then there is no guarantee that $Y/^{\mathcal{S}}X$ is closed under join. They showed also the counter example that the structure left integral residuated lattice without the condition (2) can not be embedded into an integral residuated lattice.

5.2 Finite embeddability property for various classes of $\mathbf{FL}_w$-algebras

$\mathbf{FL}_w$-algebras with Com

We first note that $\mathbf{FL}_w$-algebras with commutativity (Com) has the FEP. This algebraic structure is exactly equivalent to $\mathbf{FL}_{ew}$-algebras.

Theorem 5.2.1 *The variety of* $\mathbf{FL}_{ew}$*-algebras has the FEP.*

$\mathbf{FL}_w$-algebras with Wcon

In $\mathbf{FL}_{ew}$, the weak-contraction (Wcon) is of the form $(\alpha \to \neg\alpha) \to \neg\alpha$ which is a special case of contraction (Con): $\alpha \to (\alpha \to \beta) \to (\alpha \to \beta)$, which is equivalent to $\alpha \to \alpha^2$. We can get Wcon from Con to replace β by 0. The condition Wcon is equivalent to the form $\neg\alpha^2 \to \alpha$ which can be considered as a contrapositive of contraction rule. In FL_w, we can consider two types of Wcon, $\alpha^2\backslash 0 = \alpha\backslash 0$ (Wcon$_l$) and $0/\alpha^2 = 0/\alpha$ (Wcon$_r$).

An $\mathbf{FL}_w$-algebra $\mathbf{A}$ is called $\mathbf{FL}_w$-algebra with Wcon$_l$ (Wcon$_r$, respectively) if it satisfies that $x^2\backslash 0 = x\backslash 0$ $(0/x^2 = 0/x)$ for all $x \in \mathbf{A}$. When $\mathbf{FL}_w$-algebra is commutative, i.e., $\mathbf{FL}_{ew}$-algebra, both Wcon$_l$ and Wcon$_r$ coincide with Wcon: $\neq x^2 = \neq x$.

Theorem 5.2.2 *The variety of* $\mathbf{FL}_w$*-algebra with Wcon*$_x$*,* $x \in \{l, r\}$ *has the FEP.*

Proof. First we claim that Wcon$_l$ is equivalent to the condition $(x\backslash 0) \wedge x = 0$. Let $\mathbf{A}$ be an $\mathbf{FL}_w$-algebra with Wcon$_l$ and $\mathbf{B}$ is a finite partial subalgebra of $\mathbf{A}$. By Lemma 4.3.4, $\mathbf{D}(A, B)$ is a finite $\mathbf{FL}_w$-algebra, thus it is enough to show that $\mathbf{D}(A, B)$ also satisfies $X\backslash 0^D \wedge^D X = \{0\}$ for any $X \in \mathbf{D}(A, B)$. Take $x \in X\backslash 0^D \wedge^D X$. By the definition of $X\backslash 0$, $x^2 = 0$. Thus $x^2\backslash 0 = x\backslash 0 = 1$. Therefore $x = x \wedge 1 = x \wedge x\backslash 0 = 0$. We conclude that $X\backslash 0^D \wedge^D X = \{0\}$. The proof that Wcon$_r$ is similar to prove.

Classical $\mathbf{FL}_w$-algebras

An $\mathbf{FL}_w$-algebra $\mathbf{A}$ is *classical* if it satisfies the equation $(0/x)\backslash 0 = x = 0/(x\backslash 0)$. When $\mathbf{FL}_w$-algebra is commutative, i.e., $\mathbf{FL}_{ew}$-algebra, there is only one negation, thus we call this condition DN: $(x \to 0) \to 0$.

Theorem 5.2.3 *The variety of classical* $\mathbf{FL}_w$*-algebra has the FEP.*

Proof. In this case, small modification is necessary to construct the new structure. Let $\mathbf{A}$ be a classical $\mathbf{FL}_w$-algebra and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$. Let $\mathbf{B}^+ = \mathbf{B} \bigcup \mathbf{B}' \bigcup \mathbf{B}''$, where $\mathbf{B}' = \{b\backslash 0 : b \in \mathbf{B}\}$, $\mathbf{B}'' = \{0/b : b \in \mathbf{B}\}$. It is trivial to see that $\mathbf{B}^+$ is a finite partial subalgebra of $\mathbf{A}$. Let $\mathbf{D}(A, B^+)$ be an algebra, constructed by $\mathbf{B}^+$, instead of $\mathbf{B}$. By Lemma 4.3.4, $\mathbf{D}(A, B^+)$ is a finite $\mathbf{FL}_w$-algebra, thus it is enough to show that

$\mathbf{D}(A, B^+)$ is also a classical $\mathbf{FL}_w$-algebra. Take any element $X \in \mathbf{D}(A, B^+)$, X is of the form $\bigcap_i (x_i : y_i \rightsquigarrow b_i]$, where $x_i, y_i \in M, b_i \in \mathbf{B}^+$. Suppose $x \in (0^D/X)\backslash 0^D$, i.e., for any element $w \in \mathbf{M}(wX = \{0\} \Longrightarrow wx = 0) \cdots (1)$. Take any element $z \in X$, by the definition of X then $x_i z y_i \leq b_i$, for each i, $(0/b_i)x_i z y_i \leq (0/b_i)b_i \leq 0$. We can get $z \leq [(0/b_i)x_i]\backslash(0/y_i)$. Hence $0/([(0/b_i)x_i]\backslash(0/y_i)) \cdot z \leq 0$ and so $0/([(0/b_i)x_i]\backslash(0/y_i)) \cdot X \leq 0$. By the assumption (1), $0/([(0/b_i)x_i]\backslash(0/y_i)) \cdot x \leq 0$. Using the classical in $\mathbf{A}$, we have proved that $(0^D/X)\backslash 0^D \subseteq X$. Note that the converse direction always holds in $\mathbf{D}(A, B^+)$. Thus, we conclude $(0^D/X)\backslash 0^D = X$. The equation $0^D/(X\backslash 0^D) = X$ is similar to prove.

Representable $\mathbf{FL}_w$-algebras

A residuated lattice is *representable* if its subdirectly irreducible members are totally ordered. A representable residuated lattice is characterized by adding the following identity[1].

$$(x\backslash y) \vee ([w(z\backslash((y\backslash x)z))]/w) = 1$$

In the case of commutative residuated lattices, the condition *representable* is called linearity (Lin) which is of the form, $(x \rightarrow y) \vee (y \rightarrow x) = 1$. The condition Lin is also called prelinearity. H. Ono proved that for any subdirectly irreducible $\mathbf{FL}_{ew}$-algebra $\mathbf{A}$, Lin is valid in $\mathbf{A}$ if and only if $\mathbf{A}$ is totally ordered.

To prove the FEP of the representable $\mathbf{FL}_w$-algebras, we use the following algebraic property [8].

Lemma 5.2.4 *A variety $\mathcal{V}$ has the FEP if and only if $\mathcal{V}_{si}$ has the FEP, where $\mathcal{V}_{si}$ is the class of subdirectly irreducible members of $\mathcal{V}$.*

Proof. Note that the only-if part is trivial, so we prove the if part. Let $\mathbf{B}$ be a partial subalgebra of $\mathbf{A}$ and $\Pi_{i\in I}\mathbf{C}_i$ be a subdirect decomposition of $\mathbf{A}$. Then each i-th projection of $\mathbf{B}$ is a subset of $\mathbf{C}_i$. By the assumption $\mathcal{V}_{si}$ has the FEP, i-th projection of $\mathbf{B}$ can be embedded in some finite algebra $\mathbf{D}_i$. Therefore $\mathbf{B}$ can be embedded in $\Pi_{i\in I}\mathbf{D}_i$. We write $b(i)$ for the i-th component of the image of b by this embedding, for each $b \in \mathbf{B}$. If a and b are distinct in $\mathbf{B}$ then there exists i such that $a(i) \neq b(i)$. Let i_{ab} be the one of these i and define the subset I_o of I by $I_o = \{i_{ab} : a, b \in B, a \neq b\}$. Then I_o is finite and $\mathbf{B}$ can be embedded into finite algebra $\Pi_{i\in I_o}\mathbf{D}_i$.

Now, turn to the FEP of representable $\mathbf{FL}_w$-algebra.

Theorem 5.2.5 *The variety of representable* $\mathbf{FL}_w$*-algebras has the FEP.*

Proof. By Lemma 5.2.4, it is enough to consider subdirectly irreducible algebras in the variety of $\mathbf{FL}_w$-algebras. Let $\mathbf{A}$ be a subdirectly irreducible $\mathbf{FL}_w$-algebra and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$. Then $\mathbf{A}$ is totally ordered. Since each element of $\overline{D}$ is a downward closed subset of $\mathbf{A}$, hence each element of $\overline{D}$ is totally ordered and in this case $\overline{D}$ and $\mathbf{D}$ coincide. Therefore it is obvious that $\mathbf{D}$ is also totally ordered.

In the case of $\mathbf{FL}_{ew}$-algebra with some or all conditions mentioned above are also proved by H. Ono[39].

Simple $\mathbf{FL}_{ew}$-algebras

Recall, we say an algebra is *simple* if it has only two congruences. We will discuss simplicity of residuated lattices in Chapter 6. In the case of $\mathbf{FL}_{ew}$-algebras, simple algebra is characterized as follows.

An $\mathbf{FL}_{ew}$-algebra $\mathbf{A}$ is simple iff for any $x(< 1)$ in A there exists a positive integer m such that $x^m = 0$.

Using this characterization, we can prove that the FEP for simple $\mathbf{FL}_{ew}$-algebras.

Theorem 5.2.6 *The variety of simple* $\mathbf{FL}_{ew}$*-algebras has the FEP.*

Proof. Let $\mathbf{A}$ be a simple $\mathbf{FL}_{ew}$-algebra and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$. Construct the structure $\mathbf{D}(A, B)$. By Lemma 4.3.4, $\mathbf{D}(A, B)$ is a finite. Thus, it is enough to show that $\mathbf{D}(A, B)$ is also a simple $\mathbf{FL}_{ew}$-algebra. By the assumption for each $x \in A$, there exists $n \in \mathbb{N}$ such that $x^n = 0$. Let $\mathbf{M}$ be the submonoid of $\mathbf{A}$ generated by $\mathbf{B}$. Recall that for any element of $\mathbf{D}(A, B)$ is a downward closed subset of $\mathbf{M}$. Then for any $X \in D$, the set $\mathrm{Max}(X)$ of is finite, since $\mathbf{D}(A, B)$ is well-quasi order. Now we write $k = |\mathrm{Max}(X)|$. For all $x_i \in \mathrm{Max}(X)$ there exists $n_i \in \mathbb{N}$ such that $x_i^{n_i} = 0$. Since $\mathrm{Max}(X)$ is finite there exists $max\{n_i\}$. It is obvious that $X^{k \cdot max\{n_i\}} = 0^D$ for any X. We conclude that $\mathbf{D}(A, B)$ is also simple.

T. Kowalski and H. Ono proved that the following.

Theorem 5.2.7 *The variety of* $\mathbf{FL}_{ew}$*-algebras is generated by its finite simple members.*

To show the theorem above, they first proved the semisimplicity of free $\mathbf{FL}_{ew}$-algebras, we will discuss in chapter 6, and next showed that the variety of all simple $\mathbf{FL}_{ew}$-algebras has the FEP. They proved the FEP for the variety of all simple $\mathbf{FL}_{ew}$-algebras in different way. For more information, see [25]

Finite model property for substructural logics

In this section, by the considerations of the FEP for various extensions of residuated lattices, we can show that the FMP for various substructural logics.

There are many extensions of basic substructural logic $\mathbf{FL}$. It is shown that $\mathbf{FL}$-algebra and $\mathbf{FL}_e$-algebra do not have the FEP. Thus, it is natural to start with $\mathbf{FL}_w$, whose corresponding algebras are $\mathbf{FL}_w$-algebras. We have already showed that the variety of all $\mathbf{FL}_w$-algebras has the FEP. In Chapter 4.1, we show that if the given subvariety of $\mathcal{IRL}$ has the FEP then the corrsponding logic has the FMP.

First we summarize our results of the FMP. The following result is the FMP for extensions of $\mathbf{FL}_w$-algebra.

Theorem 5.2.8 *Let* $\mathbf{L}$ *be a logic which is an extension of* $\mathbf{FL}_w$ *by adding some or all of axiom schemes, commutativity* (Com)*,* $Wcon_l$ ($Wcon_r$)*, classical, representable. Then* $\mathbf{L}$ *has the FMP, and hence* $\mathbf{L}$ *is decidable.*

Recall that the FEP for a given variety of $\mathcal{IRL}$ implies the FMP for the corresponding logic. The following axioms are typical extensions of $\mathbf{FL}_{ew}$.

(1) EM (exclusive middle) : $\alpha \vee \neg\alpha$,

(2) DN (double negation) : $\neg\neg\alpha \to \alpha$,

(3) Con (contraction) : $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$,

(4) Wcon (weak contraction) : $(\alpha \to \neg\alpha) \to \neg\alpha = \neg\alpha^2 \to \neg\alpha$,

(5) P (Peirce's law) : $((\alpha \to \beta) \to \alpha) \to \alpha$,

(6) WP (weak Peirce's law) : $(\neg\alpha \to \alpha) \to \alpha$,

(7) Lin (linearity) : $(\alpha \to \beta) \vee (\beta \to \alpha)$.

Corollary 5.2.9 *Let* $\mathbf{L}$ *be a logic which is an extension of* $\mathbf{FL}_{ew}$ *by adding some or all of axiom schemes,* (1) $\sim$ (7)*. Then* $\mathbf{L}$ *has the FMP, and hence* $\mathbf{L}$ *is decidable.*

Note that the following is the FMP for fuzzy logics.

Corollary 5.2.10 *Fuzzy logics* MTL*,* $IMTL$*, and* $SMTL$ ($= WMTL$) *have the FMP, and hence decidable.*

Chapter 6

Semisimplicity

In this chapter, we investigate subdirect irreducibility, simplicity and semisimplicity of residuated lattices. In the study of algebras, Birkhoff's subdirect representation theorem is one of the fundamental theorems which says that every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras. We characterize subdirect irreducibility, simplicity and semisimplicity of residuated lattices by using Birkhoff's theorem. Lastly, we show that every *free* $\mathbf{FL}_w$-algebra is semisimple.

In[18] V. N. Grišin proved that every free $\mathbf{CFL}_{ew}$-algebra is semisimple. To show that Grišin introduced new sequent system which is equivalent to $\mathbf{CFL}_{ew}$ and using the fact that the cut elimination theorem holds for the sequent system.

In[26], T. Kowalski and H. Ono show that variety $\mathcal{FL}_{ew}$ of all $\mathbf{FL}_{ew}$-algebras is generated by its finite simple members. The result is obtained by first showing that every free $\mathbf{FL}_{ew}$-algebras is semisimple and then showing that every variety generated by a $\mathbf{FL}_{ew}$-algebra is generated by a set of finite simple $\mathbf{FL}_{ew}$-algebras. To show the former, based on Grišin's idea in [18]. They introduced a sequent system SFL_{ew}^{+} such that

1. algebras for SFL_{ew}^{+} are exactly equal to $\mathbf{FL}_e$- algebras,
2. cut elimination theorem holds for SFL_{ew}^{+}.

Then, using proof-theoretic properties of SFL_{ew}^{+}, the semisimplicity of every free $\mathbf{FL}_{ew}$-algebra is obtained.

Our result is also based on Grišin's idea and and Kowalski-Ono's technique. We will introduce a new sequent system, $\mathbf{FL}_w^{+}$ which is equivalent to $\mathbf{FL}_w$. Using the fact that cut elimination theorem holds for $\mathbf{FL}_w^{+}$ and using proof-theoretic properties of $\mathbf{FL}_w^{+}$, we show that the proof of the semisimplicity works also for every free $\mathbf{FL}_w$-algebra. It is very interesting to see how nicely proof-theoretic methods work to bring about purely algebraic consequence.

Those results are interesting contrast in the case of Heyting algebras. In the case of Heyting algebras, it is easy to see that there is the only simple Heyting algebra which is exactly the two valued Boolean algebra. Hence any semisimple Heyting algebra is a Boolean algebra.

6.1 Subdirect irreducibility, simplicity and semisimplicity of residuated lattices

In this section, we study subdirect irreducibility, simplicity, and semisimplicity of residuated lattices.

Subdirect irreducibility of residuated lattices

Suppose $\mathbf{A}$ and $\mathbf{B}_i$ for each $i \in I$ are residuated lattices. We say that a *subdirect representation* of $\mathbf{A}$ with factors $\mathbf{B}_i$ if there is an embedding $\imath : \mathbf{A} \to \Pi_{i \in I} \mathbf{B}_i$ such that each f_i defined by $f_i = \pi_i \circ \imath$ is onto $\mathbf{B}_i$ for each $i \in I$, where π_i is the i-th projection. A residuated lattice is *subdirectly irreducible* if it is non-degenerate and for any subdirect representation $f : \mathbf{A} \to \Pi_{i \in I} \mathbf{B}_i$, there exists a $\jmath$ such that $f_\jmath$ is an isomorphism of $\mathbf{A}$ onto $\mathbf{B}_\jmath$.

The following theorem gives us a useful characterization of subdirectly irreducible algebras [10].

Theorem 6.1.1 *A non trivial algebra* $\mathbf{A}$ *is a subdirectly irreducible iff* $\mathbf{A}$ *has the second smallest congruence relation* $\mathrm{Con}(\mathbf{A})$.

The next theorem is called Birkhoff's subdirect representation theorem it follows that every algebra has a subdirect representation with subdirectly irreducible algebras with same type [10].

Theorem 6.1.2 (Birkhoff) *Every algebra* $\mathbf{A}$ *is isomorphic to a subdirect product of subdirectly irreducible algebras each of which is a homomorphic image of* $\mathbf{A}$.

From Birkhoff's subdirect representation theorem, every residuated lattice has a subdirect representation with subdirectly irreducible residuated lattices. By Theorem 2.2.7 and 6.6.1, we can see that a residuated lattice $\mathbf{A}$ is subdirectly irreducible iff it has the second smallest filter.

Proposition 6.1.3 *A residuated lattice* $\mathbf{A}$ *is subdirectly irreducible iff there exists an element* $c (\not\geq 1)$ *such that for any* $x \not\geq 1$ *there exists an element* $z \in \Pi\Gamma\bar{\Delta}(x)$ *for which* $z \leq c$.

Proof. Suppose that $\mathbf{A}$ is subdirectly irreducible. By Theorem 6.1.1, there exists the second smallest filter F_0 which contains the smallest filter $\{a \in A : 1 \leq a\}$ properly. Then, we can take an element $c(\not\geq 1)$ in F_0. Let G_x be a filter generated by $x \not\geq 1$. Then $G_x = \{u \in A : z \leq u \text{ for some } z \in \Pi\Gamma\bar{\Delta}(x)\}$. Since F_0 is the minimal filter containing $\{a \in A : 1 \leq a\}$. Thus, $F_0 \subseteq G_x$. Therefore $z \leq c$ holds. Conversely, suppose that $c(\not\geq 1)$ is the element which satisfies the condition. Consider the filter F_c generated by c which is not the smallest filter. Then $F_c = \{x \in A : z \leq x$ for some $z \in \Pi\Gamma\bar{\Delta}(c)\}$. Let F be a arbitrary filter but not the smallest one. Then for any $w \in F$ $\Pi\Gamma\bar{\Delta}(w) \subseteq F$ holds. By assumption, there exists an element $z \in \Pi\Gamma\bar{\Delta}(x)$ for which $z \leq c$. Since F is a filter, $c \in F$. Hence, $F_c \subseteq F$ and F_c is the second smallest filter. We have **A** has the second smallest filter. Therefore **A** is subdirectly irreducible by Theorem 6.1.1.

In the following results are due to T. Kowalski and H. Ono [26].

Lemma 6.1.4 *Let* **A** *be a subdirectly irreducible* $\mathbf{FL}_{ew}$*-algebra. If* $x \vee y = 1$ *then either* $x = 1$ *or* $y = 1$ *holds.*

Proof. We will prove that taking the contraposition. It is sufficient to show that $x, y < 1$ implies $x \vee y < 1$. Since **A** is subdirectly irreducible, there exists an element $a < 1$ such that for any $z < 1$ there exists a positive integer k satisfying $z^k \leq a$. We can take some positive integers m, n both $x^m \leq a$ and $y^n \leq a$ hold. Put $s = \max\{m, n\}$ and $t = 2s - 1$. Then $x^s \leq a$ and $y^s \leq a$ hold. By the distributivity of $\cdot$ and $\vee$, we have the following

$$(x \vee y)^t = \bigvee_{i=0}^{t} x^i \cdot y^{t-i}.$$

It is easy to see that either $i \geq s$ or $t - i \geq s$. In the former case,

$$x^i \cdot y^{t-i} \leq x^i \leq x^s \leq a$$

and the latter case,

$$x^i \cdot y^{t-i} \leq y^{t-i} \leq y^s \leq a.$$

Hence, in either case, we have $(x \vee y)^t \leq a$. Therefore, $x \vee y < 1$.

An element a in an $\mathbf{FL}_{ew}$-algebra **A** is called *coatom* if it is maximal among elements in $\mathbf{A}\backslash\{1\}$. Then we have the following result from the above lemma.

Corollary 6.1.5 *Every subdirectly irreducible* $\mathbf{FL}_{ew}$*-algebra has either the single coatom or no coatom.*

The following result is essentially due to T. Kowalski[23]. That is a characterization an $\mathbf{FL}_{ew}$-algebra has the unique coatom which is interesting contrast with Lemma 6.1.4.

Lemma 6.1.6 *An* $\mathbf{FL}_{ew}$*-algebra* **A** *has the unique coatom iff there exists an element* $a(< 1)$ *and a positive integer* m *such that* $x^m \leq a$ *holds for any* $x < 1$.

Proof. To show the only-if part, it is clear to take the coatom for a and 1 for m. To show conversely, suppose that there exists an element $a < 1$ and $m \geq 1$ such that $x^m \leq a$ for any $x < 1$. Then **A** is clearly subdirectly irreducible. Take any such a and also smallest number k among such ms for a. If $k = 1$ then a is the unique coatom of **A**. Suppose $k > 1$. By assumption, there exists an element b such that $b^{k-1} \not\leq a$ and $b^k \leq a$. Put d by $d = b^{k-1} \to a$. Note that $d < 1$. Take any $y(< 1)$ and $z = d \vee y$. Then $d, y \leq z$ and $z < 1$ by Lemma. Since $b^k \leq a$, we have $b \leq b^{k-1} \to a = d \leq z$. By $z < 1$, we have $z^k \leq a$. Hence, $y \leq z \leq z^{k-1} \to a \leq b^{k-1} \to a = d$. Therefore $y \leq d$ and d is the coatom.

Simplicity and semisimplicity of residuated lattices

Recall that an algebra $\mathbf{A}$ is *simple* if it has only two congruences on $\mathbf{A}$. Let $\mathbf{A}$ be a residuated lattice. Then $\mathbf{A}$ is simple iff it has only two filters. It is easy to see that for any filter F of a residuated lattice $\mathbf{A}$ the quotient algebra $\mathbf{A}/F$ is simple iff F is a maximal filter.

We have the following characterization that a bounded residuated lattice is simple.

Lemma 6.1.7 *A bounded residuated lattice* $\mathbf{A}$ *is simple iff for any* $x(\not\geq 1)$ *in* $\mathbf{A}$, $\bot \in \Pi\Gamma\bar{\Delta}(\{x\})$.

In the case of commutative bounded residuated lattices, there is a nice characterization that $\mathbf{A}$ is simple. The following result is essentially due to T. Kowalski and H. Ono who proved that the case of $\mathbf{FL}_{ew}$-algebras.

Corollary 6.1.8 *A commutative bounded residuated lattice* $\mathbf{A}$ *is simple iff for any* $x(\not\geq 1)$ *in* $\mathbf{A}$ *there exists a positive integer* m *such that* $(x \wedge 1)^m = \bot$.

Recall that an algebra $\mathbf{A}$ is *semisimple* if it has a subdirect representation with simple factors. It is known that an algebra $\mathbf{A}$ is semisimple iff the intersection of a set $\Omega_{\mathbf{A}}$ of all maximal members in $\mathrm{Con}(\mathbf{A})$ is equal to the least congruence $\Delta_{\mathbf{A}}$ (i.e., the diagonal relation) of $\mathbf{A}$[32]. Indeed, suppose that $\bigcap \Omega_{\mathbf{A}} = \Delta_{\mathbf{A}}$. Consider $\Pi\mathbf{A}/\theta$, where $\theta \in \Omega_{\mathbf{A}}$. Then each $\mathbf{A}/\theta$ is simple algebra. It is easy to see that $\Pi\mathbf{A}/\theta$ is a subdirect representation of $\mathbf{A}$. Conversely, suppose that $\bigcap \Omega_{\mathbf{A}} \neq \Delta_{\mathbf{A}}$. Then $\Pi_{\theta\in\Omega_{\mathbf{A}}}\mathbf{A}/\theta$ is not a subdirect representation of $\mathbf{A}$. On the other hand, by Birkhoff's subdirect representation theorem, there exists a set Φ of congruence on $\mathbf{A}$ such that $\Pi_{\theta\in\Phi}\mathbf{A}/\theta$ is a subdirect representation of $\mathbf{A}$. Hence, there exists a congruence $\vartheta \in \Phi$ such that $\mathbf{A}/\vartheta$ is not simple. Therefore $\mathbf{A}$ is not semisimple.

Then we have the following.

Theorem 6.1.9 *An algebra* $\mathbf{A}$ *is semisimple iff the intersection of the set of all maximal members in* $\mathrm{Con}(\mathbf{A})$ *is equal to the least congruence i.e., the diagonal relation of* $\mathbf{A}$.

Let Φ be the set of all maximal filters of a residuated lattice $\mathbf{A}$. Define the *radical* $\mathrm{Rad}_{\mathbf{A}}$ of $\mathbf{A}$ by $\mathrm{Rad}_{\mathbf{A}} = \bigcap_{F\in\Phi} F$. Then, the following can be easily shown by Theorem 6.1.9 and the connections between congruences and filters by Theorem 2.2.7.

Lemma 6.1.10 *For any residuated lattices,* $\mathbf{A}$ *is semisimple iff* $\mathrm{Rad}_{\mathbf{A}} = \{a \in A | 1 \leq a\}$.

Corollary 6.1.11 *For any integral residuated lattices,* $\mathbf{A}$ *is semisimple iff* $\mathrm{Rad}_{\mathbf{A}} = \{1\}$.

Proposition 6.1.12 *Let* $\mathbf{A}$ *be a commutative bounded residuated lattice. For any* $x(\not\geq 1)$ *in* $\mathbf{A}$, $x \in \mathrm{Rad}_{\mathbf{A}}$ *iff for any* $n \geq 1$ *there exists* $m \geq 1$ *such that* $\sim(\sim x^n \wedge 1)^m = \top$, *where* $\sim x = x \rightarrow \bot$.

Proof. ($\Leftarrow$) Assume that for any $n \geq 1$ there exists $m \geq 1$ such that $\sim (\sim x^n \wedge 1) = \top$. Suppose $x \notin \mathrm{Rad}_{\mathbf{A}}$. Then there exists a maximal filter F such that $x \notin F$. Since F is maximal, there exists $k \geq 1$ such that $\sim x^k \in F$ and so $\sim x^k \wedge 1 \in F$. Now we take n for k and m for l. Then $\sim (\sim x^k \wedge 1)^l = \top$ by assumption. Hence $(\sim x^k \wedge 1)^l = \bot \in F$. It contradicts F is maximal filter. Therefore $x \in \mathrm{Rad}_{\mathbf{A}}$.

($\Rightarrow$) To show the other direction, we take the contraposition. Suppose there exist $n \geq 1$ such that $\sim (\sim x^n \wedge 1)^m \neq \top$ for any $m \geq 1$. If $(\sim x^n \wedge 1)^m = \bot$ then $\sim (\sim x^n \wedge 1)^m = \top$. Thus, $(\sim (x^n))^m > \bot$ for any $m \geq 1$. Let $z = \sim (x^n)$ and H be the filter generated by z. Clearly H is proper as $z^m > \bot$ for any $m \geq 1$. By Zorn's lemma, there exists maximal filter G such that $F \subseteq G$. Suppose $x \in G$. Then $x^n \in G$ for any $n \geq 1$. But this contradicts $z = \sim (x^n) \in G$ Hence $x \notin G$, Therefore $x \notin G$.

In[25], T. Kowalski and H. Ono proved that the necessary and sufficient condition for $x \in \mathbf{A}$ to be a member of $\mathrm{Rad}_{\mathbf{A}}$ for $\mathbf{FL}_{ew}$-algebra as follows.

Proposition 6.1.13 *Let* $\mathbf{A}$ *be an* $\mathbf{FL}_{ew}$*-algebra. For any* x *in* $\mathbf{A}$*,* $x \in \mathrm{Rad}_{\mathbf{A}}$ *iff for any* $n \geq 1$ *there exists* $m \geq 1$ *such that* $\tilde{m}(x^n) = 1$*, where* $\tilde{m}(x) = \neg(\neg x)^m$*.*

Corollary 6.1.14 *An* $\mathbf{FL}_{ew}$*-algebra* $\mathbf{A}$ *is semisimple iff for every* $a \in A \backslash \{1\}$ *there exists an* $N \geq 1$ *such that for any* $m \geq 1$*, we have* $\tilde{m}(a^N) < 1$*.*

Corresponding to Proposition 6.1.12, we can show the following result, which, however gives only a necessary condition for an element x in a bounded residuated lattice $\mathbf{A}$ to be a member of $\mathrm{Rad}_{\mathbf{A}}$.

Proposition 6.1.15 *Let* $\mathbf{A}$ *be a bounded residuated lattice. If an element* $x \ngeq 1$ *is in* $\mathrm{Rad}_{\mathbf{A}}$ *then for any* $n \geq 1$ *there exist* $d_1, \cdots, d_t \in \Gamma\overline{\Delta}(\sim x^n)$ *such that* $d_1 \cdots d_t = \bot$*, where* $\sim x$ *stands for either* $x \backslash \bot$ *or* $\bot / x$*.*

Proof. It is easy to see that it is enough to show that if $x \ngeq 1$ is in $\mathrm{Rad}_{\mathbf{A}}$ then $\bot \in \Pi\Gamma\overline{\Delta}(\neg x^n)$ for any $n \geq 1$. Taking the contraposition, suppose that there exists $n \geq 1$ such that $\bot \notin \Pi\Gamma\overline{\Delta}(\neg x^n)$. This means that the filter H generated by $\{\neg x^n\}$ is proper. By Zorn's lemma, there exists a maximal filter G including H. If $x \in G$ then $x^n \in G$. On the other hand, $\neg x^n \in G$ since G includes H. This contradicts the fact that G is proper. Thus $x \notin G$. Hence $x \notin \mathrm{Rad}_{\mathbf{A}}$.

6.2 Semisimplicity for free $\mathbf{FL}_w$-algebras

Sequent system $\mathbf{FL}_w^+$

In this section we show that every free $\mathbf{FL}_w$-algebra is semisimple, using the sequent system FL_w^+ introduced below. Our proof proceeds similarly to Grišin[18] and [25].

Similarly to the sequent system SFL_{ew} introduced in [25], we introduce a sequent system, which we call FL_w^+ as follows. A sequent is of the form $\Gamma \to \alpha$ where Γ is a finite sequence of formulas:

1. initial sequents

   (1) $\Gamma, p, \Delta \Rightarrow p$ where p is a propositional variable,
   (2) $\Gamma, 0, \Delta \Rightarrow \alpha$.

2. rules of inference

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \theta}{\Delta, \Gamma, \Sigma \Rightarrow \theta} \ (cut)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \theta \quad \Gamma, \beta, \Delta \Rightarrow \theta}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \theta} \ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 2)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \ (\Rightarrow \wedge)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \theta}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \theta} \ (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \theta}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \theta} \ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \theta}{\Delta, \Gamma, \alpha\backslash\beta, \Sigma \Rightarrow \theta} \ (\backslash \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta} \ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \theta}{\Delta, \beta/\alpha, \Gamma, \Sigma \Rightarrow \theta} \ (/ \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha} \ (\Rightarrow /)$$

$$\frac{\Gamma_1 \Rightarrow \alpha_1, \cdots, \Gamma_m \Rightarrow \alpha_m}{\Gamma_1, \cdots, \Gamma_m \Rightarrow \alpha_1 * \cdots * \alpha_m} \ (\Rightarrow *) \qquad \frac{\Gamma, \alpha_1, \cdots, \alpha_m, \Delta \Rightarrow \theta}{\Gamma, \alpha_1 * \cdots * \alpha_m, \Delta \Rightarrow \theta} \ (* \Rightarrow)$$

Here, we assume that in each application of rules $(\Rightarrow *)$ and $(* \Rightarrow)$, none of α_i must be fusion formulas, i,e, formulas whose outermost logical connective is the fusion $*$.

Basic Results

The reason why we need the system FL_w^+ will be explained by the following Lemma, which can be shown in the same way as [25].

Lemma 6.2.1 *Cut elimination holds for FL_w^+.*

It is easy to see that our sequent system FL_w^+ is equivalent to $\mathbf{FL}_w$. It suffices to show that any application of ($\Rightarrow *$) and ($* \Rightarrow$) not satisfying the condition mentioned above in a given cut-free proof can be replaced by one with the condition. We have the following lemma.

Lemma 6.2.2 *A sequent $\Gamma \Rightarrow C$ is provable in $\mathbf{FL}_w$ if and only if it is provable in FL_w^+.*

Using the equivalence between $\mathbf{FL}_w^+$ and FL_w. Thus, we have the following.

Lemma 6.2.3 *Free $\mathbf{FL}_w$-algebras are precisely Lindenbaum algebras of FL_w^+.*

Semisimplicity for free $\mathbf{FL}_w$-algebras

To show the semisimplicity for free FL_w-algebras, we prepare the notations as follows.

Let a formula α be given. In the following, $\neg\alpha$ denotes either $\alpha\backslash 0$ or $0/\alpha$. Also $\Gamma(\alpha)$ and $\Pi(\alpha)$ denote sets of formulas which are defined in the same way as those defined in Section 2, though in the present case, $*, \backslash$ and $/$ denote logical connectives. For each formula α, let $\ell(\alpha)$ denote the length of α as a sequence of symbols. For a sequence Γ of formulas $\alpha_1, \cdots, \alpha_m$, the lengh $\ell(\Gamma)$ is defined by $\ell(\Gamma) = \ell(\alpha_1) + \cdots + \ell(\alpha_m)$.

Also we need to introduce some notations for our main lemma. The expression $\{\alpha^N\}^m$ stands for the sequence $\alpha^N, \cdots, \alpha^N$ with m times, where α^N is of the form $\alpha * \cdots * \alpha$ (N times). Let β be any member of $\Pi\Gamma(\alpha)$. Then β is of the form $\gamma_1 * \cdots * \gamma_n$ where each γ_i is of the form $\mu_{\delta_{1i}} \cdots \mu_{\delta_{m_i i}}(\alpha)$ for some formulas $\delta_{1i}, \cdots, \delta_{m_i i}$. Define the rank $r(\beta)$ by $r(\beta) = \sum_{i=1}^{n} m_i$. For each nonempty multiset X of $\Pi\Gamma(\alpha)$, where X is $\beta_1, \cdots, \beta_k$, define $r(X)$ and $|X|$ by $r(X) = \sum_{j=1}^{k} r(\beta_j)$ and $|X| = k$, respectively.

To show that any free $\mathbf{FL}_w$-algebra A is semisimple, Corollary 6.1.11 says that it is enough to show that the radical $\mathrm{Rad}_{\mathbf{A}}$ of any Lindenbaum algebra of FL_w^+ is equal to $\{1\}$, where 1 is the greatest element of a given Lindenbaum algebra. Since the element 1 consists of all provable formulas. By Lemma 6.2.2 and 6.2.3, this follow from the following lemma.

Lemma 6.2.4 (Main Lemma) *Suppose that a formula α is not provable in FL_w^+ and that $N > \ell(\alpha)$. For any sequent $\Gamma_1, \cdots, \Gamma_{K+1} \Rightarrow \sigma$ such that $\ell(\Gamma, \sigma) \leq \ell(\alpha)$, where Γ is equal to $\Gamma_1, \cdots, \Gamma_{K+1}$ and any nonempty multisets $X_1, \cdots, X_K$ of $\Gamma(\neg\alpha^N)$, if $\Gamma_1, X_1, \cdots, X_k, \Gamma_{K+1} \Rightarrow \sigma$ is provable in FL_w^+ then $\Gamma_1, \cdots . \Gamma_{K+1} \Rightarrow \sigma$ is provable in FL_w^+.*

The following is the contrapositon of Proposition 6.1.15 in the case of $\mathbf{FL}_w$-algebras.

Corollary 6.2.5 *Let* $\mathbf{A}$ *be an* $\mathbf{FL}_w$*-algebra and* $x \in \mathbf{A}$*. If there exists* $n \geq 1$ *such that for any* $d_1, \cdots, d_t \in \Gamma(\neg x^n)$*,* $d_1 \cdots d_t \neq \bot$ *then* $x \notin \mathrm{Rad}_{\mathbf{A}}$*, where* $\neg x$ *stands for either* $x\backslash 0$ *or* $0/x$*.*

Clearly, the sequent system FL_w^+ is consistent, i.e., the sequent $\Rightarrow 0$ is not provable. Let $\delta_1, \cdots, \delta_n$ be arbitrary formulas in $\Gamma(\neg\alpha^N)$. Then the sequent $\delta_1, \cdots, \delta_n \Rightarrow 0$ is not provable in FL_w^+. For, otherwise, $\Rightarrow 0$ is provable in FL_w^+ by Lemma 6, which is a contradiction. Recall here that any formula β in $\Pi\Gamma(\neg\alpha^N)$ is of the form $\delta_1 * \cdots * \delta_n$ for some $\delta_1, \cdots, \delta_n \in \Gamma(\neg\alpha^N)$. Thus, we have the following.

Proposition 6.2.6 *Let* α *be any formula which is not provable in* FL_w^+*. Then there exists* $N \geq 1$ *such that for any* $\delta_1, \cdots, \delta_n \in \Gamma(\neg\alpha^N)$*,* $\delta_1, \cdots, \delta_n \Rightarrow 0$ *is not provable in* FL_w^+*. Thus, for any* β *in* $\Pi\Gamma(\neg\alpha^N)$*,* $\beta \Rightarrow 0$ *is neither provable in* FL_w^+*.*

In the term of Lindenbaum algebra $\mathbf{A}$ of FL_w^+ , the above proposition says that if $[\alpha] \neq [1]$ in $\mathbf{A}$ then $[0] \notin \Pi\Gamma([\neg\alpha^N])$ for some $N \geq 1$, where $[\gamma]$ denotes the equivalence class, to which a given formula γ belongs. Thus, using Proposition 6.2.6, we have the following theorem.

Theorem 6.2.7 *Every free* $\mathbf{FL}_w$*-algebra is semisimple.*

The proof of our main lemma also work well in the case of $\mathbf{FL}_{ew}$-algebras. Thus, we have the following corollary which is first proved by T. Kowalski and H. Ono [25].

Corollary 6.2.8 *Every* $\mathbf{FL}_{ew}$*-algebra is semisimple.*

Proof of Main Lemma

Proof. The proof will be given by using double induction on the total rank of multisets X_i's and $\ell(\Gamma, \sigma)$, where Γ is $\Gamma_1, \Gamma_2, \cdots \Gamma_{k+1}$. We note first that without loss of generality, it is enough to consider the case for $\neg\alpha$ is $\alpha\backslash 0$. Therefore our proof will be given by using double induction on $\langle \sum_{i=1}^{K} r(X_i), \ell(\Gamma, \sigma)\rangle$. When $\sum_{i=1}^{K} r(X_i) = 0$, i.e., every X_i consists only of the formula $\alpha^N\backslash 0$, the proof goes essentially the same as one in [25], as shown in the following and therefore our lemma can be regarded as an extension of the one for commutative case.

(1) Suppose that the given sequent $\Gamma_1, X_1, \cdots, X_K, \Gamma_{K+1} \Rightarrow \sigma$ is an initial sequent. In this case, either σ is a propositional variable which occurs also in some Γ_i, or 0 occurs in some Γ_i. It is clear that $\Gamma_1, \cdots . \Gamma_{K+1} \Rightarrow \sigma$ is provable in either case.

(2) Suppose next that the given sequent $\Gamma_1, X_1, \cdots, X_k, \Gamma_{K+1} \Rightarrow \sigma$ is the lower sequent of an inference rule I. By Lemma 6.2.1, this sequent has a cut-free proof P. We need to consider two possibilities. The first case is that the principal formula of I is either in some Γ_i or in σ, and the second case is that the principal formula of I is one of an element in X_i.

Consider the first case. Since the proof P is a cut-free proof, it is easily seen that the length of each of the upper sequent of a given inference is smaller than that of the lower sequent by subformula property. Thus, we can use the hypotheses of induction and apply the same inference rule I, we conclude that $\Gamma_1, \cdots, \Gamma_{K+1} \Rightarrow \sigma$ is provable.

Consider the second case. The principal formula of the inference rule I is one of the element γ in X_i. In this case, there two possibilities that (i) $r(\gamma) = 0$ or (ii) $r(\gamma) > 0$.

(i) In this case the given sequent of the form that $\Gamma_1, X_1, \cdots, X_i^l, \gamma, X_i^r, \cdots \Gamma_{K+1}$, where X_i^l, γ, X_i^r is equal to X_i. So the inference rule I is of the form that;

$$\frac{\Pi_2, X_i^l \Rightarrow \alpha^N \quad \Pi_1, 0, X_i^r, \Pi_3 \Rightarrow \sigma}{\Pi_1, \Pi_2, X_i^l, \alpha^N \backslash 0, X_i^r, \Pi_3 \Rightarrow \sigma}$$

or

$$\frac{X_i^{l_1} \Rightarrow \alpha^N \quad \Pi_1, \Pi_2, X_i^{l_2}, 0, X_i^r, \Pi_3 \Rightarrow \sigma}{\Pi_1, \Pi_2, X_i^l, \alpha^N \backslash 0, X_i^r, \Pi_3 \Rightarrow \sigma}$$

where Π_1, Π_2 is equal to $\Gamma_1, X_i, \cdots, \Gamma_i$, and Π_3 is equal to $\Gamma_{i+1}, X_{i+1}, \cdots, \Gamma_{K+1}$. and also X_i^l is equal to $X_i^{l_1}, X_i^{l_2}$.

Since the proof goes essentially in the same way, we consider only the first case. Consider the proof R of the left upper sequent $\Pi_2, X_i^l \Rightarrow \alpha^N$. We will trace back branches of R, which consists of sequents having α^N in the conclusion, to the places where this α^N is introduced. Note that α^N is introduced at one place in each branch of R. It is easy to see that each α^N is introduced either as an initial sequent, or by $(\Rightarrow *)$ rule. We will show that any α^N is introduced only as an initial sequent. Suppose that at least one place , α^N is introduced by $(\Rightarrow *)$, whose lower sequent is of the form $\Delta \Rightarrow \alpha^N$. We assume here that α is of the from $D_1 * \cdots * D_w$ and none of D_j are fusion-formulas. Then, I must have $N \cdot w$ upper sequents, each of which is of the form $\Delta_i \Rightarrow D_{n_i}$, where $1 \leq n_i \leq w$ and the list $\Delta_1, \cdots, \Delta_{N \cdot w}$ is equal to Δ. For each j such that $1 \leq j \leq w$, there exists exactly N sequents with the conclusion D_j among those sequents. We enumerate them as $S_1^j, \cdots, S_N^j$. Next, for each h such that $1 \leq h \leq N$, take $S_h^1, \cdots, S_h^w$ for upper sequent and apply $(\Rightarrow *)$ rule to them. Then we can have a sequent of the form $\Sigma_h \Rightarrow \alpha$ for $1 \leq h \leq N$ and the list of $\Sigma_1, \cdots, \Sigma_N$ is equal to Δ. Now $\ell(\Delta) \leq \ell(\Pi_2) \leq \ell(\Gamma, \sigma) \leq \ell(\alpha) < N$. If we assume that $\Sigma_i > 0$ for any i such that $1 \leq i \leq N$ then $\ell(\Delta) \geq N$, which is a contradiction. Therefore, Σ_i must be empty for some i. But this means that $\Rightarrow \alpha$ is provable. This contradicts the assumption that α is unprovable. Hence, we conclude that at any place α^N is introduced by initial sequent of the form $\Pi, 0, \Lambda \Rightarrow \alpha^N$.

We will modify the proof R of $\Pi_2, X_i^l \Rightarrow \alpha^N$ as follows. We replace every sequent $\Theta \Rightarrow \alpha^N$ in a branch which we have traced in R, including initial sequent of the form $\Pi, 0, \Lambda \Rightarrow \alpha^N$ mentioned above, by the sequent $\Pi_1, \Theta, \Pi_3 \Rightarrow \sigma$, which is equal to $\Gamma_1, X_1, \cdots, X_i^l, \Gamma_{i+1}, \cdots, \Gamma_{K+1} \Rightarrow \sigma$. Then we will have the proof whose end sequent is $\Pi_1, \Pi_2, X_i^l, \Pi_3 \Rightarrow \sigma$. This sequent has the smaller length than the original sequent $\Gamma_1, X_1, \cdots, , \Gamma_{K+1} \Rightarrow \sigma$. Hence, by hypothesis of induction, we conclude that $\Gamma_1, \cdots, \Gamma_{K+1} \Rightarrow \sigma$ is provable. We note that the above proof (i) works well also for the case when $\sum_{i=1}^{K} r(X_i) = 0$, i.e., each X_i consists only of the formula $\alpha^N \backslash 0$. Hence, the above proof assures us that the base step of the induction of our proof holds.

(ii) The rank $r(\gamma)$ is greater than 0. In this case the principal formula γ is of the form $\mu_{\beta_1} \cdots \mu_{\beta_m}(\neg\alpha^N)$ with $m > 0$, where μ is either ρ or λ operator and $\beta_1, \cdots, \beta_m$ are formulas. There are two possible cases. The first case is that μ_{β_1} is $\rho_{\beta 1}$, and the second case is that μ_{β_1} is λ_{β_1}. We give here only a proof of the first case. In the following, we let χ denote $\mu_{\beta_2} \cdots \mu_{\beta_m}(\neg\alpha^N)$. Thus, γ is $\beta_1 \backslash \chi * \beta_1$. In this case, the inference rule I is either of the following form;

$$\frac{\Pi_2, X_i^l \Rightarrow \beta_1 \quad \Pi_1, \chi * \beta_1, X_i^r, \Pi_3 \Rightarrow \sigma}{\Pi_1, \Pi_2, X_i^l, \beta_1 \backslash \chi * \beta_1, X_i^r, \Pi_3 \Rightarrow \sigma}$$

or

$$\frac{X_i^{l_1} \Rightarrow \beta_1 \quad \Pi_1, \Pi_2, X_i^l, \chi * \beta_1, X_i^r, \Pi_3 \Rightarrow \sigma}{\Pi_1, \Pi_2, X_i^l, \beta_1 \backslash \chi * \beta_1, X_i^r, \Pi_3 \Rightarrow \sigma}$$

where (1) Π_1, Π_2 is equal to $\Gamma_1, X_1, \cdots, \Gamma_i$, (2) Π_3 is equal to $\Gamma_{i+1}, X_{i+1}, \cdots, \Gamma_{K+1}$, (3) X_i is equal to $X_i^l, \beta_1 \backslash \chi * \beta_1, X_i^r$ and (4) X_i^l is equal to $X_i^{l_1}, X_i^{l_2}$.

We consider the first case. In this case, the right upper sequent of the inference rule I implies the provability of $\Pi_1, \chi, \beta_1, X_i^r, \Pi_3 \Rightarrow \sigma$. Taking the left upper sequent and $\Pi_1, \chi, \beta_1, X_i^r, \Pi_3 \Rightarrow \sigma$, and applying the cut rule, as shown below,

$$\frac{\Pi_2, X_i^l \Rightarrow \beta_1 \quad \Pi_1, \chi, \beta_1, X_i^r, \Pi_3 \Rightarrow \sigma}{\Pi_1, \chi, \Pi_2, X_i^l, X_i^r, \Pi_3 \Rightarrow \sigma}$$

We can see the sequent

$$\Pi_1, \chi, \Pi_2, X_i^l, X_i^r, \Pi_3 \Rightarrow \sigma$$

is provable. In this sequent $\sum_{j \neq i} r(X_j) + r(\chi) < \sum_{i=1} r(X_i)$. Hence, by hypothesis of induction, we conclude that $\Gamma_1, \cdots, \Gamma_{K+1} \Rightarrow \sigma$ is provable. Similarly, we can show this in the second case. This completes the proof of our lemma.

Chapter 7

Amalgamation Property

In this chapter, we study the amalgamation property (AP) of various classes of commutative residuated lattices.

In[24], T. Kowalski showed that the AP for the variety $\mathcal{FL}_{ew}$ of all $\mathbf{FL}_{ew}$-algebras, The result is obtained by showing that

1. the logical system $\mathbf{FL}_{ew}$ has the Craig's interpolation property (CIP),
2. the variety of $\mathcal{FL}_{ew}$ has the equational interpolation property (EIP).

We will show that his proof of the AP works well also the variety of $\mathcal{FL}_e$ and $\mathcal{CRL}$.

In this section, first we introduce that the AP and related algebraic properties. In particular, we study the relationship between the EIP and the AP. Next, we introduce the CIP and prove that the CIP for $\mathbf{FL}_e$ with only constant 1 which is exactly equal to commutative residuated lattices. Lastly, we prove that the variety of all commutative residuated lattices $\mathcal{CRL}$ has the EIP, hence it has the AP. By considering filters on residuated lattices, we show that many important subvarieties of commutative residuated lattices has the AP. Moreover, we show that if $\mathbf{L}$ is a logic which is an extension of $\mathbf{FL}_e$ with the CIP and $\mathcal{K}$ be the variety which is corresponding to $\mathbf{L}$, then $\mathcal{K}$ has the EIP. Therefore, $\mathcal{K}$ has the AP.

7.1 Amalgamation property and Craig's interpolation property

Amalgamation property

First, we introduce the *amalgamation property* (AP) and other related properties.

The AP was first considered by O. Schreier. He researched the AP for groups. In a general form of the AP was formulated by R. Fraisse in connection with certain embedding properties. The *strong amalgamation property* was introduced by B. Jónsson who developed general investigation of the AP

Definition 7.1.1 (Amalgamation property) A class of algebra $\mathcal{K}$ has the *amalgamation property* iff if $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$, $f_1 : \mathbf{A} \to \mathbf{B}_1, f_2 : \mathbf{A} \to \mathbf{B}_2$ are embeddings then there exist an algebra $\mathbf{C} \in \mathcal{K}$ and embeddings $g_1 : \mathbf{B}_1 \to \mathbf{C}, g_2 : \mathbf{B}_2 \to \mathbf{C}$ such that $g_1 f_1 = g_2 f_2$.

We say that a class of algebras $\mathcal{K}$ has the *strong amalgamation property* if it satisfies that the condition of the AP and, moreover, $g_1(\mathbf{B}_1) \cap g_2(\mathbf{B}_2) = g_i f_i(\mathbf{A})$ for $i = 1, 2$. A class of algebras $\mathcal{K}$ has the *super-amalgamation property* if it satisfies that the condition of th AP and, moreover, satisfies the following additional condition: For any $b \in B_i$ and $c \in B_i$ $(i \in \{1, 2\})$, if $g_i(b) \leq g_j(c)$ in $\mathbf{C}$, then there exists $d \in A$ such that $b \leq f_i(d)$ in $\mathbf{B}_i$ and $f_j(d) \leq c$ in $\mathbf{B}_j$ hold.

The AP is studied in model theory but no satisfactory criterion is known. Some of well known structures satisfy the AP. For example, the class of all groups, commutative groups, fields, partially ordered sets, lattices and Boolean algebras have the AP. Moreover the class of lattices has the strong amalgamation property. On the other hand, it is known that neither the class of rings nor semigroups have the AP.

The AP of algebraic structures related to some fragments of substructural logics were studied by many researchers. For instance, K. Iseki proved the AP for the class of all BCK-algebras [21]. A. Wroński proved that an algebraic version of interpolation theorem holds for the class of all BCK-algebras and also proved that the strong amalgamation property for the class of all BCK-algebras [51]. The fact that the variety of MV-algebras which is equivalent to BL-algebras +DN has the AP was proved by D. Mundici [34]. However it is not even known whether the variety of all commutative BCK-algebras enjoys the AP, though it fails to have the strong amalgamation property.

The concept of the connection between the AP and Craig's interpolation property was first studied by B. Jónsson and A. Daigneault independently. This connection was further studied by D. Pigozzi[44] and many other researchers. It is shown by L. Maksimova that a normal modal logic with a single unary modality has the Craig's interpolation property iff the corresponding class of algebras has the super-amalgamation property [31], and also that intuitionictic logic has the CIP iff the variety of Heyting algebras has the super-amalgamation property [30].

Next, we define an algebraic version of the interpolation property which is called *equational interpolation property.*

Definition 7.1.2 (Equational interpolation property) A variety $\mathcal{V}$ has the equational interpolation property (EIP) iff for all finite sets $\Sigma, \Gamma \cup \{\delta\}$ of identities in the language of $\mathcal{V}$ the following holds: if $\mathcal{V}$ satisfies the quasi-identity $\bigwedge(\Sigma \cup \Gamma) \Rightarrow \delta$ and the set of terms over $V(\Sigma) \cap V(\Gamma \cup \{\delta\})$ is non-empty, then there exists a finite set Δ of identities over $V(\Sigma) \cap V(\Gamma \cup \{\delta\})$ such that:

(1) $\mathcal{V} \models \bigwedge \Sigma \Rightarrow \bigwedge \Delta$, and

(2) $\mathcal{V} \models \bigwedge(\Delta \cup \Gamma) \Rightarrow \delta$.

The EIP was introduced by A. Wroński and the similar concepts were investigated by B. Jónsson, D. Pigozzi and P. D. Bacsich.

The following theorem is crucial to prove the AP in our argument. The proof is essentially due to A. Wroński[50]. We will give a proof below for the sake of completeness.

Theorem 7.1.1 *If a variety $\mathcal{V}$ has the equational interpolation property (EIP) then $\mathcal{V}$ has the amalgamation property (AP).*

Proof. Let $\alpha : \mathbf{A} \to \mathbf{B}_1$ and $\beta : \mathbf{A} \to \mathbf{B}_2$ be embeddings. Take a set X of variables as enough as we can define an epimorphism $\phi_A : \mathcal{T}(X) \to \mathbf{A}$, where $\mathcal{T}(X)$ is a term algebra over X in $\mathcal{V}$ (the existence of such X is assured by considering to take X as for instance, a universe A of $\mathbf{A}$). Next, we take sets Y and Z of variables such that $X \subseteq Y, Z$, $X = Y \cap Z$. Define an epimorphism $\phi_{B_1} : \mathcal{T}(Y) \to \mathbf{B}_1$ such that $\phi_{B_1}(p) = \alpha(\phi_A(p))$ for any $p \in \mathcal{T}(X)$, and for any $p \notin \mathcal{T}(X)$, $\phi_{B_1}(p)$ is a suitable element of $\mathbf{B}_1$. Also define an $\phi_{B_2} : \mathcal{T}(Z) \to \mathbf{B}_2$ such that $\phi_{B_2}(p) = \alpha(\phi_A(p))$ for any $p \in \mathcal{T}(X)$, and for any $p \notin \mathcal{T}(X)$, $\phi_{B_1}(p)$ is a suitable element of $\mathbf{B}_2$.

Put $\Phi_{B_1} = \{p = q : \langle p, q\rangle \in \ker(\phi_{B_1})\}$ and $\Phi_{B_2} = \{p = q : \langle p, q\rangle \in \ker(\phi_{B_2})\}$. Consider the term algebra $\mathcal{T}(Y \cup Z)$ and define a binary relation θ on $\mathcal{T}(Y \cup Z)$ by

$$\langle p, q\rangle \in \theta \Longleftrightarrow \mathcal{V} \models \bigwedge(\Phi_{B_1} \bigcup \Phi_{B_2}) \Rightarrow p = q$$

Then, we can show that θ is a congruence. Define the algebra $\mathcal{C}$ by $\mathcal{T}(Y \cup Z)/\theta$.

Define $\gamma : \mathbf{B}_1 \to \mathcal{C}$ and $\delta : \mathbf{B}_2 \to \mathcal{C}$ with $\gamma(b_1) = [p]_\theta$ and $\delta(b_2) = [q]_\theta$ for any $b_1 \in B_1$, $b_2 \in B_2$, where $p \in \phi_{B_1}^{-1}(b_1)$, $q \in \phi_{B_2}^{-1}(b_2)$ and $[a]_\theta$ is the equivalence class by θ. Then, it is easy to see that both γ and δ are homomorphisms.

Next, we will see $\gamma\alpha = \delta\beta$. Take $a \in A$, $p \in \phi_{B_1}^{-1}(\alpha(a))$ and $q \in \phi_{B_2}^{-1}(\beta(a))$. It is enough to show that $\langle p, q\rangle \in \theta$. Since $\phi_{B_1}(p) = \alpha(a)$ there exists $p' \in \mathcal{T}(X)$ such that $\phi_{B_1}(p) = \phi_{B_1}(p')$, i.e., $\langle p, p'\rangle \in ker(\phi_{B_1})$. Since $\phi_A(p') = \alpha^{-1}(\phi_{B_1}(p)) = \alpha^{-1}(\phi_{B_1}(p')) = a$, we have $\phi_{B_2}(q) = \beta(a) = \beta(\phi_A(p')) = \phi_{B_2}(p')$. Hence $\langle p', q\rangle \in ker(\phi_{B_2})$. Therefore, $\langle p, q\rangle \in ker(\phi_{B_1}) \cup ker(\phi_{B_2}) \subseteq \theta$.

Finally, we will see both γ and δ are embeddings. Assume $\delta(c) = \delta(c')$ for some $c, c' \in B_2$. Take $q, q' \in \mathcal{T}(Z)$ such that $q \in \phi_{B_2}^{-1}(c)$ and $q' \in \phi_{B_2}^{-1}(c')$ and $\langle q, q'\rangle \in \theta$. Then $\mathcal{V} \models \bigwedge(\Phi_{B_1} \bigcup \Phi_{B_2}) \Rightarrow q = q'$. By compactness theorem, there exist finites sets $\Phi'_{B_1}(\subseteq \Phi_{B_1})$ and $\Phi'_{B_2}(\subseteq \Phi_{B_2})$ such that $\mathcal{V} \models \bigwedge(\Phi'_{B_1} \bigcup \Phi'_{B_2}) \Rightarrow q = q'$. By the EIP, there exists a finite set Δ of identities over X such that (1) $\mathcal{V} \models \bigwedge(\Phi'_{B_1}) \Rightarrow \bigwedge\Delta$, and (2) $\mathcal{V} \models \bigwedge(\Delta \cup \Phi'_{B_2}) \Rightarrow q = q'$. Thus, $\mathcal{V} \models \bigwedge(\Phi_{B_1}) \Rightarrow \bigwedge\Delta$, and $\mathcal{V} \models \bigwedge(\Delta \cup \Phi_{B_2}) \Rightarrow q = q'$. We can take Δ is a subset of Φ_{B_1} and we will show that $\Delta \subseteq \Phi_{B_2}$. Indeed $\ker(\phi_{B_1}|_{\mathcal{T}(X)}) \subseteq \ker(\phi_A) \subseteq \ker(\phi_{B_2})$. Hence, (2) can be reduced that $\mathcal{V} \models \bigwedge(\Phi_{B_2}) \Rightarrow q = q'$. Therefore $\langle q, q'\rangle \in \ker(\phi_{B_2})$ and so $c = \phi_{B_2}(q) = \phi_{B_2}(q') = c'$. We can show that γ is an embedding similarly. This completes the theorem.

Next, we glimpse the relationship the EIP and other related topics.

A class of algebras has the *congruence extension property* iff for every congruence Φ on a subalgebra $\mathbf{B}$ of $\mathbf{A}$ in the class there exists a congruence Θ on $\mathbf{A}$ such that $\Theta|_{\mathbf{B}} = \Phi$.

It is known that both the class of abelian groups and the class of distributive lattices have the CEP, but the class of groups and the class of lattices does not have the CEP. In particular, it is easy to see that the variety $\mathcal{CRL}$ of all commutative residuated lattices has the CEP.

Definition 7.1.3 (Transferable injections) A variety $\mathcal{V}$ has transferable injections iff for every embedding $\alpha : \mathbf{A} \to \mathbf{B}_1$ and homomorphism $\beta : \mathbf{A} \to \mathbf{B}_2$ where $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{V}$, there exist $\mathbf{C} \in \mathcal{V}$, a homomorphism $\gamma : \mathbf{B}_1 \to \mathbf{C}$ and an embedding $\delta : \mathbf{B}_2 \to \mathbf{C}$ such that $\gamma\alpha = \delta\beta$.

P. D. Bacsich was studied the relationship among the AP, the CEP and the TI [3, 4].

Theorem 7.1.2 (Bacsich) *A variety $\mathcal{V}$ has the TI iff $\mathcal{V}$ has both the AP and the CEP.*

A. Wroński also investigated the relationship among the AP, the CEP and the EIP. He proved the following theorem[50]. Precisely, Wroński proved the equivalence the EIP and the TI.

Theorem 7.1.3 (Wroński) *A variety $\mathcal{V}$ has the EIP iff $\mathcal{V}$ has both congruence extension property (CEP) and amalgamation property (AP).*

Craig's interpolation property

In chapter 3, we discuss Craig's interpolation property (CIP) for basic substructural logics. Recall that A logic $\mathbf{L}$ has the CIP if the following statement holds for $\mathbf{L}$.

If $A \to B$ is provable in $\mathbf{L}$ then there exists of a formula C such that both $A \to C$ and $C \to B$ are provable, and every propositional variables in C appears both A and B.

Now, we prove that the CIP holds for $\mathbf{FL}_e$ with only constant 1, denoted by $\mathbf{FL}_e^-$. Note that algebras for $\mathbf{FL}_e^-$ are precisely commutative residuated lattices.

For any formula A, $V(A)$ denotes the set of all propositional variables appearing in A. When Γ is a sequent of $A_1, \cdots, A_n$, denotes $V(\Gamma) = V(A_1) \cup \cdots \cup V(A_n)$.

Theorem 7.1.4 *Craig's interpolation theorem holds for $\mathbf{FL}_e^-$.*

To see the theorem above, we prepare the lemma.

Lemma 7.1.5 *Let $\Gamma \to C$ is provable in $\mathbf{FL}_e^-$ and $\langle \Gamma_1, \Gamma_2 \rangle$ is any partition of Γ. Then there exists a formula D such that both $\Gamma_1 \to D$ and $D, \Gamma_2 \to C$ are provable in $\mathbf{FL}_e^-$ and moreover $V(D) \subseteq V(\Gamma) \cap V(C)$.*

Proof. The proof is shown by the induction on the length n of a cut-free proof P of $\Gamma \to C$. In this proof we omit exchange rule for simplicity's sake, when no confusions will occur. When n is equal to 1 then the sequent $\Gamma \to C$ is an initial sequent. In this case we must check the two cases.

(i) $\Gamma \to C$ is of the form $A \to A$.

Case 1. $\Gamma_1 = A$. Then we can take A as an interpolant. Indeed, we take A as the formula C, then we have $A \to A$ and $A \to A$. Both are initial sequents, which are trivially provable.

Case 2. $\Gamma_2 = A$. Then we can take 1 as an interpolant. Indeed, we take 1 as the formula C, then we have $\to 1$ and $1, A \to A$. $\to 1$ is an initial sequent, trivially provable and $1, A \to A$ can be proved by the following inference rule,

$$\frac{A \to A}{1, A \to A}$$

(ii) $\Gamma \to C$ is of the form $\to 1$.
Then we can take 1 as an interpolant. Indeed, we take 1 as the formula C, then we have $1 \to 1$ and $\to 1$. Both are trivially provable.

Next, we consider the case $n > 1$. Then that the cut-free proof P of $\Gamma \to C$ has at least one inference rule. We denote the last inference rule in P by I.

(1) I is of the form,

$$\frac{A, \Gamma \to D}{A \wedge B, \Gamma \to D}$$

Case 1. Consider the partition $\langle A, \Gamma_1; \Gamma_2 \rangle$ of A, Γ. Then by the induction of hypothesis, there exists a formula C such that both $A, \Gamma_1 \to C$ and $C, \Gamma_2 \to D$ are provable. Then,

$$\frac{A, \Gamma_1 \to C}{A \wedge B, \Gamma_1 \to C}$$

Hence, both $A \wedge B, \Gamma_1 \to C$ and $C, \Gamma_2 \to D$ are provable. Moreover, the formula C satisfies the condition of variables, $V_\perp(C_1) \subseteq V_\perp(\{A \wedge B\} \cup \Gamma_1) \cap V(\Gamma_2 \cup \{D\})$. Therefore we can take the formula C as an interpolant.

Case 2. Consider the partition $\langle \Gamma_1; A, \Gamma_2 \rangle$ of A, Γ. We can prove similarly.

(2) I is of the form,

$$\frac{B, \Gamma \to D}{A \wedge B, \Gamma \to D}$$

We can prove similarly in the case of (1).

(3) I is of the form,

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}$$

Then by the induction of hypothesis, there exist formulas C_1 and C_2 such that all $\Gamma_1 \to C_1$, $C_1, \Gamma_2 \to A$, $\Gamma_1 \to C_2$ and $C_2, \Gamma_2 \to B$ are provable. Then,

$$\frac{\Gamma_1 \to C_1 \quad \Gamma_1 \to C_2}{\Gamma_1 \to C_1 \wedge C_2}$$

and

$$\frac{\dfrac{C_1, \Gamma_2 \to A}{C_1 \wedge C_2, \Gamma_2 \to A} \quad \dfrac{C_1, \Gamma_2 \to B}{C_1 \wedge C_2, \Gamma_2 \to B}}{C_1 \wedge C_2, \Gamma_2 \to A \wedge B}$$

Thus, both $\Gamma_1 \to C_1 \wedge C_2$ and $C_1 \wedge C_2, \Gamma_2 \to A \wedge B$ are provable. Moreover, all formulas $C_1, C_2, C_1 \wedge C_2$ satisfy the condition of variables. Therefore we can take the formula $C_1 \wedge C_2$ as an interpolant.

(4) I is of the form,

$$\frac{A, \Gamma \to D \quad B, \Gamma \to D}{A \vee B, \Gamma \to D}$$

Case 1. Consider the partition $\langle \Gamma_1; \Gamma_2, A \vee B \rangle$. Then by the induction of hypothesis, there exist formulas C_1 and C_2 such that all $\Gamma_1 \to C_1$, $C_2, \Gamma_2, A \to D$, $\Gamma_1 \to C_2$ and $C_2, \Gamma_2, B \to D$ are provable. Thus, both $\Gamma_1 \to C_1 \wedge C_2$ and $C_1 \wedge C_2, \Gamma_2, A \vee B \to D$ are provable. Moreover, all formulas $C_1, C_2, C_1 \wedge C_2$ satisfy the condition of variable. Therefore we can take the formula $C_1 \wedge C_2$ as an interpolant.

Case 2. Consider the partition $\langle \Gamma_1, A \vee B; \Gamma_2 \rangle$. We can prove similarly.

(5) I is of the form,

$$\frac{\Gamma \to A}{\Gamma \to A \vee B}$$

Then by the induction of hypothesis, there exists a formula C such that both $\Gamma_1 \to C$ and $C, \Gamma \to A$ are provable. Thus, both $\Gamma_1 \to C$ and $C, \Gamma \to A \vee B$ are provable. The formula C satisfies the condition of variables. We can take C as an interpolant.

(6) I is of the form,

$$\frac{\Gamma \to B}{\Gamma \to A \vee B}$$

We can prove similarly in the case of (5).

(7) I is of the form,

$$\frac{\Gamma_1 \to A \quad B, \Gamma_2 \to D}{A \supset B, \Gamma \to D}$$

Consider the partition $\langle \Gamma_1, A \supset B; \Gamma_2 \rangle$. Then by the induction of hypothesis, there exist formulas C_1 and C_2 such that all $\Gamma_{11} \to C_1$, $C_1, \Gamma_{12} \to A$, $B, \Gamma_{21} \to C_2$ and $C_2, \Gamma_{22} \to D$ are provable, where $\Gamma_1 = \Gamma_{11}, \Gamma_{12}$ and $\Gamma_2 = \Gamma_{21}, \Gamma_{22}$. Thus, $A \supset B, \Gamma_1, \Delta_1 \to C_1 \supset C_2$ and

$C_1 \supset C_2, \Gamma_2, \Delta_2 \to D$ are provable. Moreover, all formulas C_1, C_2 and $C_1 \supset C_2$ satisfy the condition of variables. Therefore we can take the formula $C_1 \supset C_2$ as an interpolant.

Consider the partition $\langle \Gamma_1; A \supset B, \Gamma_2 \rangle$.

(8) I is of the form,

$$\frac{A, \Gamma \to B}{\Gamma \to A \supset B}$$

Then by the induction of hypothesis, there exists a formula C such that both $\Gamma_1 \to C$ and $C, \Gamma_2, A \to B$ are provable. Thus, both $\Gamma_1 \to C$ and $C, \Gamma_2 \to A \supset B$ are provable. Moreover, C satisfies the condition. We can take a formula C as an interpolant.

(9) I is of the form,

$$\frac{A, B, \Gamma \to D}{A * B, \Gamma \to D}$$

Case 1. Consider the partition $\langle \Gamma_1, A * B; \Gamma_2 \rangle$. Then by the induction of hypothesis, there exists a formula C_1 such that both $\Gamma_1, A, B, \Delta_1 \to C_1$ and $C_1, \Gamma_2, \Delta_2 \to D$ are provable. Thus, $\Gamma_1, A * B, \Delta_1 \to C_1$ and $C_1, \Gamma_2, \Delta_2 \to D$ are provable. Moreover, C_1 satisfies the condition of an interpolant.

Case 2. Consider the partition $\langle \Gamma_1; \Gamma_2, A * B \rangle$. We can prove similarly.

(10) I is of the form,

$$\frac{\Gamma_1 \to A \quad \Gamma_2 \to B}{\Gamma \to A * B}$$

Then by the induction of hypothesis, there exist formulas C_1 and C_2 such that $\Gamma_{11} \to C_1$, $C_1, \Gamma_{12} \to A$, $\Gamma_{21} \to C_1$ and $C_2, \Gamma_{22} \to D$ are all provable, where $\Gamma_1 = \Gamma_{11}, \Gamma_{12}$ and $\Gamma_2 = \Gamma_{21}, \Gamma_{22}$. Thus, $\Gamma_1, \Delta_1 \to C_1 * C_2$ and $C_1 * C_2, \Gamma_2, \Delta_2 \to A * B$ are provable. Moreover all variables C_1, C_2 and $C_1 * C_2$ are satisfy the condition of variables. Therefore we can take a formula $C_1 * C_2$ as an interpolant.

(11) I is of the form,

$$\frac{A, B, \Gamma \to D}{B, A, \Gamma \to D}$$

Then by the induction of hypothesis, the CIP holds for any partition of Γ, A, B. Then it is also a partition of Γ, B, A, hence the CIP holds for $B, A, \Gamma \to D$.

(12) I is of the form,

$$\frac{\Gamma \to D}{\Gamma, 1 \to D}$$

Then by the induction of hypothesis, there exists a formula C such that both $\Gamma_1 \to C$ and $C, \Gamma_2 \to D$ are provable. Hence $\Gamma_1 \to C$ and $C, \Gamma_2, 1 \to D$ are provable. Moreover C satisfies the condition of variables. Therefore we can take a formula C_1 as an interpolant. This completes the lemma.

The above lemma is crucial that a proof of the AP for the variety $\mathcal{CRL}$ of all commutative residuated lattices. The same argument can be considered some other restricted constants in substructural logics.

7.2 Amalgamation property for $\mathcal{CRL}$ and $\mathcal{FL}_e$

In this section, we prove the AP for the variety $\mathcal{CRL}$ of all commutative residuated lattices and the variety $\mathcal{FL}_e$ of all $\mathbf{FL}_e$-algebras.

In[24], T. Kowalski showed that the AP for the variety $\mathcal{FL}_{ew}$ of all $\mathbf{FL}_{ew}$-algebras, The result is obtained by showing that

(1) the logical system $\mathbf{FL}_{ew}$ has the CIP, and

(2) showing that the variety of $\mathcal{FL}_{ew}$ has the equational interpolation property (EIP).

To show the EIP for the variety of all $\mathbf{FL}_{ew}$-algebras, T. Kowalski proved the following lemma and theorem.

Lemma 7.2.1 (Kowalski) *The variety $\mathcal{FL}_{ew}$ of all $\mathbf{FL}_{ew}$-algebras satisfies a quasi-identity $\tau = 1 \Rightarrow \sigma = 1$ iff there exists a positive integer N such that $\mathcal{FL}_{ew}$ satisfies a identity $\tau^N \to \sigma = 1$.*

Theorem 7.2.2 (Kowalski) *The variety $\mathcal{FL}_{ew}$ has the EIP, hence it has the AP.*

We will show that his proof of the AP also works well the variety of $\mathcal{FL}_e$ and $\mathcal{CRL}$.

It is easy to see that for any identity $s = t$ can be translated $\tau \wedge 1 = 1$ for some term τ, and also a finite set of identities into a single identity. In the following we use an inequality $\tau \geq 1$ instead of an identity $\tau \wedge 1 = 1$ for simplicity.

Lemma 7.2.3 *The variety $\mathcal{CRL}$ of all commutative residuated lattices satisfies the quasi-identity $\tau \geq 1 \& \sigma \geq 1 \Rightarrow \delta \geq 1$ iff there exist natural numbers n and m with $n + m \geq 1$ such that $\mathcal{CRL}$ satisfies the identity $(\tau \wedge 1)^n(\sigma \wedge 1)^m \to \delta \geq 1$.*

Proof. It is enough to show that the following are equivalent.

(1) $\mathcal{CRL} \models \tau \geq 1 \& \sigma \geq 1 \Rightarrow \delta \geq 1$

(2) $\mathcal{CRL} \models (\tau \wedge 1)^n(\sigma \wedge 1)^m \to \delta \geq 1$ for some n, m.

(1) $\Rightarrow$ (2): Suppose $\mathcal{CRL} \not\models (\tau \wedge 1)^n(\sigma \wedge 1)^m \to \delta \geq 1$ for any n, m. Thus for each (n, m) there exists a commutative residuated lattice $\mathbf{A}_{nm}$ such that $\mathbf{A}_{nm} \not\models (\tau \wedge 1)^n(\sigma \wedge 1)^m \to \delta \geq 1$. Then, there is a vector $\vec{a}_{nm} = \langle a_{nm,0}, \cdots, a_{nm,k-1} \rangle$ of elements from A_{nm} such that $(\tau \wedge 1)^n[\vec{a}_{nm}](\sigma \wedge 1)^m[\vec{a}_{nm}] \not\leq \delta[\vec{a}_{nm}]$. Take $\mathbf{A} = \Pi_{0<n,m<\omega}\mathbf{A}_{nm}$ and put $\vec{a} = \langle\langle a_{nm,i} : 0 < n, m < \omega\rangle : 0 \leq i \leq k-1\rangle$. It is obvious that $\mathbf{A}$ is a commutative residuated lattice and also we have $(\tau \wedge 1)^n[\vec{a}](\sigma \wedge 1)^m[\vec{a}] \not\leq \delta[\vec{a}]$ for every n, m in $\mathbf{A}$. Consider the filter F of $\mathbf{A}$

generated by $\tau[\vec{a}]$ and $\sigma[\vec{a}]$. It is clear that $\delta[\vec{a}] \notin F$. Suppose θ is a congruence associated with F. Then, in the algebra $\mathbf{A}/\theta$ which is trivially a commutative residuated lattice, we have $\tau(\vec{a}/\theta), \sigma(\vec{a}/\theta) \geq 1$ but $\delta(\vec{a}/\theta) \not\geq 1$. Therefore, $\mathbf{A}/\theta$ does not satisfies (1).

(2) $\Rightarrow$ (1): Suppose $\mathcal{FL}_e \models (\tau \wedge 1)^n(\sigma \wedge 1)^m \rightarrow \delta \geq 1$ for some n, m. Take any algebra $\mathbf{A}$ in $\mathcal{FL}_e$ and any vector $\vec{a} \in \mathbf{A}$ such that $(\tau)[\vec{a}], (\sigma)[\vec{a}] \geq 1$. Since (2) holds in $\mathbf{A}$, we can have $1^{n+m} \rightarrow \delta[\vec{a}] \geq 1$ and so $1 \leq \delta(\vec{a})$.

Theorem 7.2.4 *The variety $\mathcal{CRL}$ of all commutative residuated lattices has the EIP. Therefore, $\mathcal{CRL}$ has the AP.*

Proof. Suppose $\mathcal{CRL}$ satisfies the quasi-identity $\bigwedge(\Sigma \cup \Gamma) \Rightarrow \epsilon$. Since we have a constant 1, the set of terms over $V(\Sigma) \cap V(\Gamma \cup \{\epsilon\})$ is non-empty. It is necessary to show that there exists a finite set of identities Δ which satisfies (1) and (2) of the definition of the EIP.

Recall that any identity can be represented of the form $\tau \wedge 1 = 1$ for some term τ. Moreover, any finite set of identities can be also represented $\tau \wedge 1 = 1$ for some term τ. We write τ_Σ, τ_Γ for the set of identities Σ and Γ. In particular, we assume that ϵ is of the form $\sigma \wedge 1 = 1$.

Then, $\mathcal{CRL}$ satisfies the quasi-identity $(\tau_\Sigma \wedge 1 = 1) \& (\tau_\Gamma \wedge 1 = 1) \Rightarrow (\sigma \wedge 1 = 1)$. By Lemma 7.3.3, we have $\mathcal{CRL} \models (\tau_\Sigma \wedge 1)^n(\tau_\Gamma \wedge 1)^m \rightarrow \sigma \geq 1$ for some $n + m \geq 1$. It is also equivalent that $\mathcal{CRL} \models (\tau_\Sigma \wedge 1)^n \rightarrow [(\tau_\Gamma \wedge 1)^m \rightarrow \sigma] \geq 1$.

Since the logic $\mathbf{FL}_e^-$ has the CIP. And algebras for $\mathbf{FL}_e^-$ are exactly equal to commutative residuated lattices, there exists a term δ over $V((\tau_\Sigma \wedge 1)^n) \cap V((\tau_\Gamma \wedge 1)^m \rightarrow \sigma)$ such that

(*i*) $\mathcal{CRL} \models (\tau_\Sigma \wedge 1)^n \rightarrow \delta \geq 1$,

(*ii*) $\mathcal{CRL} \models \delta \rightarrow ((\tau_\Gamma \wedge 1)^m \rightarrow \sigma) \geq 1$.

By Lemma 7.3.3, in the case $m = 0$, (*i*) is equivalent to $\mathcal{CRL} \models \tau_\Sigma \geq 1 \Rightarrow \delta \geq 1$. This is also equivalent to $\mathcal{CRL} \models \bigwedge \Sigma \Rightarrow \delta \geq 1$. This proves (1) of the definition of the EIP with Δ being $\{\delta \wedge 1 = 1\}$.

Next, we will show (2) of the EIP. Let $\mathbf{A}$ be any algebra from $\mathcal{CRL}$ and $\vec{a}$ a vector from $\mathbf{A}$ such that $\mathbf{A} \models \bigwedge(\Delta \cup \Gamma)[\vec{a}]$. This means that $(\delta)[\vec{a}], (\tau_\Gamma)[\vec{a}] \geq 1$ by considering the representation for Δ and Γ. By (*ii*), we have $(\delta \wedge 1) \rightarrow ((\tau_\Gamma \wedge 1)^m \rightarrow \sigma) \geq 1$. Thus, $1 \rightarrow (1 \rightarrow \sigma) \geq 1$ and so $\sigma \wedge 1 = 1$. Hence we have $\mathbf{A} \models (\sigma \wedge 1)[\vec{a}]$. This shows that $\mathbf{A} \models \bigwedge(\Delta \cup \Gamma) \Rightarrow \sigma \geq 1$. This completes the proof.

Corollary 7.2.5 *The variety $\mathcal{FL}_e$ of all $\mathbf{FL}_e$-algebras has the EIP. Therefore, $\mathcal{FL}_e$ has the AP.*

Moreover, by considering filters of residuated lattices, we can show that the following result.

Theorem 7.2.6 *Let* $\mathbf{L}$ *be a logic which is an extension of* $\mathbf{FL}_{ew}$ *with the CIP and* $\mathcal{K}$ *the variety which is corresponding to* $\mathbf{L}$*. Then* $\mathcal{K}$ *has the EIP. Therefore,* $\mathcal{K}$ *has the AP.*

Therefore, we can show the following important subclasses of commutative residuated lattices have the EIP, and hence these have the AP.

(1) The variety $\mathcal{FL}_{ew}$ of all $\mathbf{FL}_{ew}$-algebras,

(2) The variety $\mathcal{FL}_{ec}$ of all $\mathbf{FL}_{ec}$-algebras,

(3) The variety $\mathcal{CFL}_{e}$ of all $\mathbf{FL}_{e}$-algebras,

(4) The variety $\mathcal{CFL}_{ew}$ of all $\mathbf{CFL}_{ew}$-algebras,

(5) The variety $\mathcal{CFL}_{ec}$ of all $\mathbf{CFL}_{ec}$-algebras,

(6) The variety $\mathcal{CRL}^2$ of all increasing-idempotent commutative residuated lattices,

(7) The variety $\mathcal{CCRL}$ of all classical commutative residuated lattices,

(8) The variety $\mathcal{CIRL}$ of all commutative integral residuated lattices,

(9) The variety $\mathcal{CCIRL}$ of all classical commutative integral residuated lattices,

(10) The variety $\mathcal{CCRL}^2$ of all increasing-idempotent classical commutative residuated lattices.

Some remarks

We consider the AP for class of all noncommutative residuated lattices. To show the AP for $\mathcal{CRL}$, we use the CIP and prove the EIP. A. Wroński showed that the EIP is equivalent to both the AP and the CEP. It is known that the variety $\mathcal{CRL}$ of all commutative residuated lattices has the CEP [16]. It is also known that there exists an noncommutative residuated lattice which does not have the CEP. The following is an example that does not have the CEP.

Let $\mathbf{A}$ be a totally ordered integral residuated lattice whose universe is $A = \{1, a, b, c, d\}$ with $1 > a > b > c > d$. Put $a = a^2$, $b = b^2 = ab = ba$, $c = ca = cb$ and other multiplications $= d$ and the residuations in A are easy to define. Moreover, it is easy to see that $\mathbf{A}$ is simple. Now, we can show that $\mathbf{B} = \{1, a, b\}$ forms a subalgebra of $\mathbf{A}$ and $\mathbf{B}$ has a non trivial filter $\{1, a\}$. Thus $\mathbf{A}$ does not hold for the CEP. In the term of congruence, $\{\{1, a\}, \{b\}\}$ is a congruence on $\mathbf{B}$, however, there exists no congruence on $\mathbf{A}$ such that its restriction is equal to $\{\{1, a\}, \{b\}\}$, since $\mathbf{A}$ is simple.

Next counter example comes from $\mathbf{FL}_w$-algebras. Let $\mathbf{A}$ be an $\mathbf{FL}_w$-algebra whose universe is $A = \{1, x, y, z, w, 0\}$, and $1 > x > y > z > w > 0$. Define the multiplication on $\mathbf{A}$ by $x = x^2$, $y = y^2 = xy = yx$, $z = zx = zy$ and $w = xz = yz = z^2 = wz = zw = wx = xw$ and $w^2 = 0$. Residuations in A are easy to define. Moreover, we can show that

A is simple. Now, we can show that $\mathbf{B} = \{1, x, y, 0\}$ is a subalgebra of **A**. It is easy to see that $\{1, x\}$ is a non trivial filter of **B**. Thus **A** does not hold for the CEP.

Hence, our technique works only well in the class of residuated lattices which has the CEP. On the other hand, the CIP work well also for substructural logics without exchange rule, like **FL** and $\mathbf{FL}_w$.

Chapter 8

Conclusion and Further Works

In this thesis, we study semisimplicity, amalgamation property and finite embeddability property of residuated lattices. In Chapter 5, we discuss the finite embeddability property for various classes of integral residuated lattices. As a consequence of these result, we show that the finite model property for various substructural logics including fuzzy logics. In Chapter 6, we study subdirect irreducibility, simplicity and semisimplicity of residuated lattices. Lastly, we prove the semisimplicity for free $\mathbf{FL}_w$-algebras by using proof-theoretical methods. In Chapter 7, we prove the amalgamation property of the variety $\mathcal{CRL}$ and $\mathcal{FL}_e$. Moreover we show that if $\mathbf{L}$ is a logic which is an extension of $\mathbf{FL}_e$ with the CIP and $\mathcal{K}$ is the variety which is corresponding to $\mathbf{L}$, then $\mathcal{K}$ has the EIP. Therefore, $\mathcal{K}$ has the AP.

We mention below a number of problems that are for further studies.

(1) Which other varieties of residuated lattices have the FEP or the AP ?

(2) Which varieties of residuated lattices have the joint embeddability property (JEP) ?

(3) Which varieties of residuated lattices have the super (strong) amalgamation property ?

(4) Are there cut-free sequent systems for distributive or cancellative residuated lattices in the sense of the corresponding logics ?

(5) To investigate further the relationship between residuated lattices and substructural logics by studying algebraizable logics in the sense of Blok and Pigozzi.

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