Title	代数的手法による、可換な部分構造論理の研究
Author(s)	木原，均
Citation	
Issue Date	2006-03
Type	Thesis or Dissertation
Text version	author
URL	http://hdl.handle.net/10119/971
Rights	
Description	Supervisor:小野　寛晰，情報科学研究科，博士

Commutative Substructural Logics
– an algebraic study

by

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submitted to
Japan Advanced Institute of Science and Technology
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

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March, 2006

Abstract

Substructural logics which are obtained from classical logic or intuitionistic logic by deleting some or all of structural rules have been studied actively in recent years. In proof-theoretical studies of substructural logics, many of important results like the disjunction property and the decidability come out as consequences of the cut elimination theorem. But, when we try to study general properties of substructural logics as a whole, we need to use semantical methods. Recently, algebraic structures called residuated lattices are paid much attention to as a semantics of substructural logics and has been developed much. In this thesis, we investigated logics over $\mathbf{FL_e}$, which is obtained from the sequent system **LJ** for intuitionistic logic by eliminating both the weakening and the contraction rules, and showed the following results by using FL_e-algebras.

When we regard a logic as a set of formulas, the class of all logics over $\mathbf{FL_e}$ forms a bounded lattice whose lattice order is the set inclusion, which has the set of all formulas $\mathcal{F}m$ and $\mathbf{FL_e}$ as the greatest and the smallest element, respectively. One of our purposes in this thesis is to clarify what structure such a lattice does. Our result says that there exist continuum many maximal elements in the lattice. It means that there exist continumm maximal logics over $\mathbf{FL_e}$.

It is an important problem how various logical properties of a given logic are characterized as algebraic properties of corresponding variety. Up to now, algebraic characterizations of several logical properties had been given only for modal logics and intermediate logics. So, how these results can be extended to substructural logics has not been clarified yet. Moreover, there were only a few studies which show relations among algebraic properties from a unified standpoint. In this thesis, we gave algebraic characterizations of several logical properties for logics over $\mathbf{FL_e}$. By using these characterizations, we clarified not only logical relations but also algebraic relations both from a logical and an algebraic standpoint.

Acknowledgments

I would like to express my sincere gratitude to my principal advisor Professor Hiroakira Ono for his constant encouragement and kind guidance during this work. It has been a pleasure to be a student of him.

I also would like to express my deepest gratitude to Dr. Nikolaos Galatos for his helpful discussions and suggestions. In fact, he gave me invaluable comments to early version of my draft.

I wish to express my thanks for Dr. Tomasz Kowalski that gave hints of the result in Chapter 4.

I want to express my gratitude to Toshimasa Matsumoto and Shunichi Amano. They have been helping anything about the computer.

I am grateful to Associate Professor Hajime Ishihara and Dr. Masahiro Hamano for their suggestions and comments. Finally, I acknowledge the members of Ono laboratory for their encouragements.

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Chapter 1

Introduction

Classical logic is usually regarded as standard logic. However, it is often mentioned that there are several differences between logics in human thinking and classical logic. One example is a formula $\neg\neg\phi \to \phi$ (the law of double negation) which is always true in classical logic. Let the formula ϕ denote the following meaning;

ϕ: Jim loves Mary.

Then the formula $\neg\neg\phi \to \phi$ means that if it is not the case that Jim doesn't love Mary then he loves her. But this argument sounds weird since we don't reason in such way. You might think that the formula $\neg\neg\phi$ expresses 'Jim loves Mary a little', so it may be an exaggeration to conclude that the formula $\neg\neg\phi \to \phi$ is true. Intuitionistic logic is a well-known example of nonclassical logics which don't accept the law of double negation.

Another example is the truth of a formula $\phi \to \psi$ when ψ is true. Let ϕ and ψ denote the following meaning;

ϕ: Fishes are plants.

ψ: Beethoven composed nine symphonies.

Now, we know that ψ is true. Then, it follows that the formula $\phi \to \psi$ is also true in classical logic. When we say that ϕ *implies* ψ, we usually suppose that there are some relations between ϕ and ψ. So, if $\phi \to \psi$ is true then you might think that the condition 'Fishes are plants' is necessary to lead 'Beethoven composed nine symphonies'. However, as you know, such a condition is not necessary for him. So, to ignore the relevance of implication causes this strangeness. The motivation of relevant logic is to exclude such strangeness.

Until now, various nonclassical logics has been introduced by their own motivations. Therefore, the following questions will come naturally to mind;

- Are there something common among these logics?
- If so, then is it possible to discuss them within a uniform framework?

The purpose of the study of substructural logics is to find common features among such nonclassical logics, and to introduce a uniform framework for them.

Roughly speaking, substructural logics are logics lacking some or all of the structural rules when they are formalized in sequent systems. They include quite different kinds of nonclassical logics, e.g., Lambek calculus, relevance logic, linear logic, and $\mathbf{FL_{ew}}$ (lacks the contraction rule). For these logics, we can show important results like the disjunction property, interpolation property and decidability in a syntactic way, using the cut elimination theorem for their sequent systems. But, syntactical methods cannot work well for substructural logics in general, and hence we need to use semantical methods. Recently, *residuated lattices* and *pointed residuated lattices* have been actively studied as algebraic semantics for substructural logics, and many important results have been shown by using algebraic approach.

In this paper, we focus on *commutative substructural logics*, i.e., we assume always the exchange rule from the beginning. Hence, our basic sequent system is $\mathbf{FL_e}$ which is obtained from the sequent system **LJ** for intuitionistic logic by eliminating both the weakening and the contraction rules. Then, by a commutative substructural logic, we mean *a logic over* $\mathbf{FL_e}$. Until now, there are not so many studies in algebraic semantics of logics over $\mathbf{FL_e}$ because of their complicated structures. The purpose of this paper is to develop an algebraic study of the class of logics over $\mathbf{FL_e}$, and to clarify relations between logical properties and algebraic properties.

This paper consists of 7 chapters. First, we give a survey of results on logics over $\mathbf{FL_e}$ and FL_e-algebras in Chapter 2 and 3.

In Chapter 2, we discuss logics over $\mathbf{FL_e}$, their *deducibility relation* and a *local deduction theorem* for them, following results by N. Galatos and H. Ono [13]. At first, we will show what will happen if we remove some of structural rules, and why we have to introduce an additional logical connective $\cdot$ (*fusion* or *multiplicative conjunction*) and propositional constants 1 and 0. Then, we show that the class of all logics over $\mathbf{FL_e}$ forms a complete lattice. Especially, intermediate logics form an important subclass of these logics.

In Chapter 3, we introduce *FL_e-algebras*. They are essentially bounded *commutative residuated lattices* with a designated element 0, and provide algebraic semantics for logics over $\mathbf{FL_e}$. We show some of basic results on FL_e-algebras from a view of universal algebra [5]. Then we discuss relations between logics over $\mathbf{FL_e}$ and FL_e-algebras. In fact, it is shown that the lattice of all logics over $\mathbf{FL_e}$ is dually isomorphic to the lattice of varieties of FL_e-algebras.

The remaining chapters consist of results obtained by our study.

In Chapter 4, we investigate the lattice of all logics over $\mathbf{FL_e}$. The following will be a natural question when we discuss the structure of this lattice.

- How many and what kind of maximal logics over $\mathbf{FL_e}$ are there?

To answer this, we prove that there exist continuum minimal varieties of FL_e-algebras. Dually, this means that there exist continuum maximal logics over $\mathbf{FL_e}$. This result makes a remarkable contrast to logics over $\mathbf{FL_{ew}}$.

In Chapters 5 and 6, we discuss algebraic characterizations of several logical properties, e.g., the pseudo-relevance property, Halldén completeness, the principle of variable separation and so on, and relations among them. These logical properties have been studied actively for intermediate logics and modal logics. However, there are not so many such studies for substructural logics except some case studies. So, it is desirable to study them from a unified standpoint.

In Chapter 5, we discuss relations between two types of *interpolation properties* and two types of *pseudo-relevance properties* for logics over $\mathbf{FL_e}$. Then we give an algebraic characterization of the *deductive pseudo-relevance property.* This is a generalization of a result for normal modal logics given by Maksimova [28]. Moreover, we show that the pseudo-relevance property holds for all logics over $\mathbf{FL_{ew}}$ for which Glivenko's theorem holds.

In Chapter 6, we discuss algebraic characterizations of *Halldén completeness* and the *deductive principle of variable separation* for logics over $\mathbf{FL_e}$. We will see that we can extend most of the results for intermediate logics or normal modal logics given by Lemmon [6], Wroński [43] and Maksimova [28] to logics over $\mathbf{FL_{ew}}$, but some modifications on the definitions become necessary in order to make similar results hold for logics over $\mathbf{FL_e}$. But, by these characterizations it is not so clear why the deductive principle of variable separation implies Halldén completeness semantically though this is clear syntactically. Thus, we will introduce a new algebraic notion called *well-connected pair.* By using this, we have succeeded to give alternative characterizations of Halldén completeness and the deductive principle of variable separation in the original forms for logics over $\mathbf{FL_e}$, and clarify their semantical relations with the disjunction property.

Chapter 2

Commutative Substructural Logics

In this chapter, we introduce commutative substructural logics and their deduciblity relations. Then we will show a local deduction theorem for commutative substructural logics.

In this paper, we assume that our language consists of logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\cdot$ (fusion or multiplicative conjunction), $\rightarrow$ (implication), and propositional constants $\top$ (top), $\bot$ (bottom), 1 and 0. Thus, we have four constants from the beginning. These constants are used in the standard formulation of linear logics and relevant logics. *Formulas* are defined inductively as follows;

1. all propositional variables and propositional constants $\{\top, \bot, 1, 0\}$ are formulas,
2. if ϕ and ψ are formuals then $\phi \wedge \psi, \phi \vee \psi, \phi \cdot \psi, \phi \rightarrow \psi$ are formulas.

$\neg\phi$ and $\phi \leftrightarrow \psi$ are abbreviations of formulas $\phi \rightarrow 0$ and $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, respectively.

We identify *formulas* and *terms*, and denote them by letters like ϕ, ψ, δ or t, s, u depending on whether they are used in logical or algebraic context.

The set of all formulas is denoted by $\mathcal{F}m$. If $\mathbf{p}$ is a set of propositional variables, we often write a formula ϕ as $\phi(\mathbf{p})$ to indicate that the variables occuring in ϕ are in $\mathbf{p}$. In this case, we denote the set of all such formulas by $\mathcal{F}m(\mathbf{p})$.

2.1 Gentzen's sequent systems and structural rules

To explain why we introduce the logical connective $\cdot$ and propositional constants 1 and 0, we will discuss some relationships between commas and structural rules.

The sequent system **LJ** for intuitionistic logic was introduced by Gentzen in the middle of the 1930s. A *sequent* of **LJ** is an expression of the form $\phi_1, \ldots, \phi_n \Rightarrow \psi$, where $n \geq 0$ and ψ may be empty. An intuitive meaning of the sequent is that " ψ *follows from assumptions* $\phi_1, \ldots, \phi_n$". In this sequent, $\phi_1, \ldots \phi_n$ and ψ are called the *antecedents* and the *succedent*, respectively. In the following, Greek capital letters Γ, Δ, Π stand for arbitrary (finite, possibly empty) sequences of formulas. There are three structural rules in **LJ**, i.e., the exchange, weakening and contraction rules. For the sake of convenience, we separate the cut rule from the other structural rules. Then, the following holds;

PROPOSITION 2.1 *A sequent $\phi_1, \ldots, \phi_n \Rightarrow \psi$ is provable in* **LJ** *if and only if the sequent $\phi_1 \wedge \ldots \wedge \phi_n \Rightarrow \psi$ is provable in it.*

(proof) We will show this in the case of $n = 2$. Suppose that $\phi_1, \phi_2 \Rightarrow \psi$ is provable in **LJ**. Then,

$$\dfrac{\dfrac{\dfrac{\phi_1, \phi_2 \Rightarrow \psi}{\phi_1 \wedge \phi_2, \phi_2 \Rightarrow \psi}}{\phi_1 \wedge \phi_2, \phi_1 \wedge \phi_2 \Rightarrow \psi}}{\phi_1 \wedge \phi_2 \Rightarrow \psi}\ (c \Rightarrow).$$

Conversely, suppose that $\phi_1 \wedge \phi_2 \Rightarrow \psi$ is provable in **LJ**. Then,

$$\dfrac{\dfrac{\dfrac{\phi_1 \Rightarrow \phi_1}{\phi_1, \phi_2 \Rightarrow \phi_1}\ (w \Rightarrow) \quad \dfrac{\phi_2 \Rightarrow \phi_2}{\phi_1, \phi_2 \Rightarrow \phi_2}\ (w \Rightarrow)}{\phi_1, \phi_2 \Rightarrow \phi_1 \wedge \phi_2} \qquad \phi_1 \wedge \phi_2 \Rightarrow \psi}{\phi_1, \phi_2 \Rightarrow \psi}\ (cut).$$

□

The above result says that in **LJ**, *commas in the left-hand side of a sequent mean conjunctions.* But to prove this, we need to both of the weakening and contraction rules. So, the similar result doesn't hold in a sequent system which lacks either the weakening or contraction. Thus, the following question will come naturally to mind:

- What do commas mean in a sequent system lacking either or both of the weakening and contraction rules?

To answer this, we examine the roles of each of stuructural rules.

Exchange rule:

$$\dfrac{\Gamma, \phi, \psi, \Delta \Rightarrow \delta}{\Gamma, \psi, \phi, \Delta \Rightarrow \delta}\ (e \Rightarrow)$$

The exchange rule allows us to use assumptions in an arbitrary order.

Weakening rule:

$$\dfrac{\Gamma, \Delta \Rightarrow \delta}{\Gamma, \phi, \Delta \Rightarrow \delta}\ (w \Rightarrow)$$

The weakening rule allows us to add any redundant assumption. In other word, when a sequent $\Gamma \Rightarrow \psi$ is provable in a sequent system which has no weakening rule, every assumption, i.e., every formula in Γ, must be used *at least once* in a proof of $\Gamma \Rightarrow \psi$.

Contraction rule:

$$\dfrac{\Gamma, \phi, \phi, \Delta \Rightarrow \delta}{\Gamma, \phi, \Delta \Rightarrow \delta}\ (c \Rightarrow).$$

The contarction rule allows us to use each assumption more than once. In other words, when a sequent is provable in a sequent system that lacks the contarction rule, every assumption must be used *at most once* in its proof.

Hence, in a sequent system with all structural rules, if a sequent $\phi_1, \ldots \phi_n \Rightarrow \psi$ is provable then it means that ψ can be derived from $\phi_1, \ldots \phi_n$ by using them in arbitrary order and number of times.

2.1.1 Comma and fusion

By the above argument, if a sequent $\phi_1, \ldots, \phi_n \Rightarrow \psi$ is provable in a sequent system which has neither the weakening nor contraction then every assumption, namely ϕ_i $(1 \leq i \leq n)$, must be used *exactly once* to derive ψ. Moreover, commas in the left-hand side of sequents don't behave like conjunctions. Then, what does each comma mean in such systems? To see this in a more explicit way, we introduce a new logical connective $\cdot$ which represents a comma in such systems. This connective $\cdot$ is called the *fusion* or the *multiplicative conjunction.* We assume the following rules for $\cdot$.

$$\frac{\Gamma, \phi, \psi, \Delta \Rightarrow \delta}{\Gamma, \phi \cdot \psi, \Delta \Rightarrow \delta} (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi \quad \Delta \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \phi \cdot \psi} (\Rightarrow \cdot)$$

Then, the following holds.

PROPOSITION 2.2 *In a sequent system which lacks both the weakening and contraction rules, a sequent $\phi_1, \ldots, \phi_n \Rightarrow \psi$ is provable if and only if $\phi_1 \cdot \ldots \cdot \phi_n \Rightarrow \psi$ is provable.*

(proof) We will show this in the case of $n = 2$. Suppose that $\phi_1, \phi_2 \Rightarrow \psi$ is provable in such a system. Clearly, by definition of $(\cdot \Rightarrow)$, the sequent $\phi_1 \cdot \phi_2 \Rightarrow \psi$ is also provable.

Conversely, suppose that $\phi_1 \cdot \phi_2 \Rightarrow \psi$ is provable. Then,

$$\frac{\dfrac{\phi_1 \Rightarrow \phi_1 \quad \phi_2 \Rightarrow \phi_2}{\phi_1, \phi_2 \Rightarrow \phi_1 \cdot \phi_2} (\Rightarrow \cdot) \qquad \phi_1 \cdot \phi_2 \Rightarrow \psi}{\phi_1, \phi_2 \Rightarrow \psi} (cut).$$

□

Moreover, we show an important relationship between fusion and implication.

PROPOSITION 2.3 *A sequent $\phi_1 \cdot \phi_2 \Rightarrow \psi$ is provable if and only if $\phi_1 \Rightarrow \phi_2 \rightarrow \psi$ is provable.*

(proof) Suppose that $\phi_1 \cdot \phi_2 \Rightarrow \psi$ is provable. Then, by Proposition 2.2, the sequent $\phi_1, \phi_2 \Rightarrow \psi$ is also provable. Using $(\Rightarrow\rightarrow)$ we have that $\phi_1 \Rightarrow \phi_2 \rightarrow \psi$ is provable.

Conversely, suppose that $\phi_1 \Rightarrow \phi_2 \rightarrow \psi$ is provable. Then,

$$\frac{\dfrac{\phi_1 \Rightarrow \phi_2 \rightarrow \psi \qquad \dfrac{\phi_2 \Rightarrow \phi_2 \quad \psi \Rightarrow \psi}{\phi_2 \rightarrow \psi, \phi_2 \Rightarrow \psi}}{\phi_1, \phi_2 \Rightarrow \psi} (cut)}{\phi_1 \cdot \phi_2 \Rightarrow \psi} (\cdot \Rightarrow) .$$

□

The above results say that deletion of structural rules has a significant effect on the meaning of commas and implication.

To see an intuitive meaning of the fusion, we will give some examples. Let formulas ϕ, ψ, δ represent the following meanings;

ϕ: One pays 100 yen.

ψ: One can get a rice ball.

δ: One can get a cup of coffee.

We assume that one rice ball costs 100 yen and one cup of coffee costs also 100 yen, i.e., both $\phi \Rightarrow \psi$ and $\phi \Rightarrow \delta$ are provable. Then we can show that

1. $\phi \cdot \phi \Rightarrow \psi \cdot \delta$ is provable,
2. $\phi \Rightarrow \psi \cdot \delta$ is not provable,
3. $\phi \Rightarrow \psi \wedge \delta$ is provable.

These sequents denote the following meanings;

1. If one pays 100 plus 100 yen, i.e., 200 yen then one can get both of a rice ball and a cup of coffee, namely, 200 yen is enough to have a lunch.
2. 100 yen is not enough to get both of them.
3. If one pays 100 yen then one can get a rice ball and also one can get a cup of coffee, *but not both*.

One might be confused about that;

- What is a difference between $\psi \cdot \delta$ in 1 and $\psi \wedge \delta$ in 3?
- The behavior of conjunction in 3 is similar to disjunction.

An answer of the former is that in 1, you can eat a rice ball while drinking a cup of coffee. On the other hand, in 3, being able to get is only one though you can choose a favorite thing either of them.

To answer the latter, let

ψ': one can get a sandwich,

and we assume that one sandwich costs 200 yen. Then, $\phi \Rightarrow \psi' \vee \delta$ is provable but $\phi \Rightarrow \psi' \wedge \delta$ is not provable. This means that if you pays 100 yen then you can get a sandwich or you can get a cup of coffee, though you have no choice but to take a cup of coffee.

2.1.2 Propositional constants

Sometimes it is convenient to add propositional constants in our language. For example, in the case of **LJ** we use propositional constants $\top$ and $\bot$ to denote the constantly *true* and *false* propositions, respectively. For these constants, **LJ** has the following initial sequents:

1. $\Gamma \Rightarrow \top$,
2. $\Gamma, \bot, \Delta \Rightarrow \phi$.

The following result says that in **LJ**, a formula ϕ is provable if and only if it is logically equivalent to $\top$, and a formula $\neg\phi$ is an abbreviation of the formula $\phi \rightarrow \bot$.

PROPOSITION 2.4 *In* **LJ**, *the following hold;*

1. $\Rightarrow \phi$ *is provable if and only if* $\Rightarrow \phi \leftrightarrow \top$ *is provable.*

2. $\Rightarrow \neg\phi \leftrightarrow (\phi \to \bot)$ *is provable.*

(proof) *1.* Suppose that $\Rightarrow \phi$ is provable. Then,

$$\dfrac{\dfrac{\phi \Rightarrow \top}{\Rightarrow \phi \to \top} \qquad \dfrac{\dfrac{\Rightarrow \phi}{\top \Rightarrow \phi}\,(w \Rightarrow)}{\Rightarrow \top \to \phi}}{\Rightarrow (\phi \to \top) \wedge (\top \to \phi)} \quad .$$

Conversely, suppose that $\Rightarrow \phi \leftrightarrow \top$ is provable. Then,

$$\dfrac{\dfrac{\Rightarrow (\phi \to \top) \wedge (\top \to \phi) \qquad \dfrac{\top \to \phi \Rightarrow \top \to \phi}{(\phi \to \top) \wedge (\top \to \phi) \Rightarrow \top \to \phi}}{\Rightarrow \top \to \phi}\,(cut) \qquad \dfrac{\Rightarrow \top \quad \phi \Rightarrow \phi}{\top \to \phi \Rightarrow \phi}}{\Rightarrow \phi}\,(cut) \; .$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi, \neg\phi \Rightarrow}}{\phi, \neg\phi \Rightarrow \bot}\,(\Rightarrow w)}{\neg\phi \Rightarrow \phi \to \bot}}{\Rightarrow \neg\phi \to (\phi \to \bot)} \qquad \dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \quad \bot \Rightarrow}{\phi, \phi \to \bot \Rightarrow}}{\phi \to \bot \Rightarrow \neg\phi}}{\Rightarrow (\phi \to \bot) \to \neg\phi}}{\Rightarrow (\neg\phi \to (\phi \to \bot)) \wedge ((\phi \to \bot) \to \neg\phi)} \quad .$$

Therefore $\Rightarrow \neg\phi \leftrightarrow (\phi \to \bot)$ is provable. □

By the weakening rule of **LJ**, it is easy to see that they can be replaced by weaker initial sequents $\Rightarrow \top$ and $\bot \Rightarrow$, respectively. On the other hand, if a sequent system doesn't have the weakening rule then constants defined by these weaker initial sequents behave in a different way, i.e., a similar result of Proposition 2.4 doesn't hold in such a system. Hence, we introduce additional new propositional constants, denoted by 1 and 0, in our language, and assume the following initial sequents and rules of inference for them:

3. $\Rightarrow 1$,
4. $0 \Rightarrow$,

$$\dfrac{\Gamma, \Delta \Rightarrow \delta}{\Gamma, 1, \Delta \Rightarrow \delta}\,(1 \Rightarrow) \qquad \dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\,(\Rightarrow 0) .$$

Intuitively, constants 1 (0) is the *weakest* (*strongest*) proposition among provable formulas (contradictory formulas, respectively). Here, by a *contradictory* formula, we mean a formula ϕ such that $\phi \Rightarrow$ is provable. Note that in a sequent system with the weakening rule, $\top$ ($\bot$) is logically equivalent to 1 (0, respectively). Conversely, if $\top$ is equal to 1 then by using the initial sequent 1, the rule $(1 \Rightarrow)$ and the cut rule, we can derive the weakening rule $(w \Rightarrow)$.

PROPOSITION 2.5 *In a sequent system with no weakening rule, the following hold;*

1. $\Rightarrow \phi$ *is provable if and only if* $\Rightarrow (\phi \wedge 1) \leftrightarrow 1$ *is provable.*

2. $\Rightarrow \neg\phi \leftrightarrow (\phi \to 0)$ *is provable.*

(proof) *1.* Suppose that $\Rightarrow \phi$ is provable. Then,

$$\dfrac{\dfrac{\dfrac{1 \Rightarrow 1}{\phi \wedge 1 \Rightarrow 1}}{\Rightarrow (\phi \wedge 1) \to 1} \qquad \dfrac{\dfrac{\dfrac{\Rightarrow \phi \quad \Rightarrow 1}{\Rightarrow \phi \wedge 1}}{1 \Rightarrow \phi \wedge 1}\,(1 \Rightarrow)}{\Rightarrow 1 \to (\phi \wedge 1)}}{\Rightarrow ((\phi \wedge 1) \to 1) \wedge (1 \to (\phi \wedge 1))} \quad .$$

Conversely, suppose that $\Rightarrow (\phi \wedge 1) \leftrightarrow 1$ is provable. By the following proof, $\Rightarrow 1 \to (\phi \wedge 1)$ is provable;

$$\dfrac{\Rightarrow ((\phi \wedge 1) \to 1) \wedge (1 \to (\phi \wedge 1)) \qquad \dfrac{1 \to (\phi \wedge 1) \Rightarrow 1 \to (\phi \wedge 1)}{((\phi \wedge 1) \to 1) \wedge (1 \to (\phi \wedge 1)) \Rightarrow 1 \to (\phi \wedge 1)}}{\Rightarrow 1 \to (\phi \wedge 1)}\,(cut) .$$

Hence,

$$\dfrac{\dfrac{\Rightarrow 1 \to (\phi \wedge 1) \qquad \dfrac{\Rightarrow 1 \quad \phi \wedge 1 \Rightarrow \phi \wedge 1}{1 \to (\phi \wedge 1) \Rightarrow \phi \wedge 1}}{\Rightarrow \phi \wedge 1}\,(cut) \qquad \dfrac{\phi \Rightarrow \phi}{\phi \wedge 1 \Rightarrow \phi}}{\Rightarrow \phi}\,(cut) .$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi, \neg\phi \Rightarrow}}{\phi, \neg\phi \Rightarrow 0}\,(\Rightarrow 0)}{\neg\phi \Rightarrow \phi \to 0}}{\Rightarrow \neg\phi \to (\phi \to 0)} \qquad \dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \quad 0 \Rightarrow}{\phi, \phi \to 0 \Rightarrow}}{\phi \to 0 \Rightarrow \neg\phi}}{\Rightarrow (\phi \to 0) \to \neg\phi}}{\Rightarrow (\neg\phi \to (\phi \to 0)) \wedge ((\phi \to 0) \to \neg\phi)} \quad .$$

Therefore $\Rightarrow \neg\phi \leftrightarrow (\phi \to 0)$ is provable. □

The above result says that in a sequent system without the weakening rule, a formula ϕ may not be equivalent to $\top$ even if it is provable. Moreover, a formula $\neg\phi$ can be regarded as an abbreviation of the formula $\phi \to 0$, not $\phi \to \bot$. Note that $\top$ (0) is logically equivalent to $\neg\bot$ ($\neg 1$, respectively).

2.2 Sequent System $\mathbf{FL_e}$

Roughly speaking, *substructural logics* are logics lacking some or all of structural rules when they are formulated as sequent systems. As we have discussed in the previous section, they are *sensitive* to the number and order of occurrences of assumptions. By this reason, they are sometimes called *resource-sensitive logics.*

In this paper, we assume always commutativity of assumptions, i.e., the exchange rules. Hence, the basic logic in this paper is $\mathbf{FL_e}$. By $\mathbf{FL_e}$, we mean the sequent system which is obtained from intuitionistic logic **LJ** by eliminating both the contraction and weakening rules. More precisely, $\mathbf{FL_e}$ has the following initial sequents and rules;

Initial Sequents:

1. $\phi \Rightarrow \phi$
2. $\Gamma \Rightarrow \top$
3. $\Gamma, \bot, \Delta \Rightarrow \phi$
4. $\Rightarrow 1$
5. $0 \Rightarrow$

Exchange rule:

$$\frac{\Gamma, \phi, \psi, \Delta \Rightarrow \delta}{\Gamma, \psi, \phi, \Delta \Rightarrow \delta}\ (e \Rightarrow)$$

Cut rule:

$$\frac{\Gamma \Rightarrow \phi \quad \Delta, \phi, \Pi \Rightarrow \delta}{\Delta, \Gamma, \Pi \Rightarrow \delta}\ (cut)$$

Rules for constants:

$$\frac{\Gamma, \Delta \Rightarrow \delta}{\Gamma, 1, \Delta \Rightarrow \delta}\ (1 \Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\Rightarrow 0)$$

Rules for logical connectives:

$$\frac{\Gamma, \phi, \Delta \Rightarrow \delta}{\Gamma, \phi \wedge \psi, \Delta \Rightarrow \delta}\ (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \psi, \Delta \Rightarrow \delta}{\Gamma, \phi \wedge \psi, \Delta \Rightarrow \delta}\ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \phi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \wedge \psi}\ (\Rightarrow \wedge)$$

$$\frac{\Gamma, \phi, \psi, \Delta \Rightarrow \delta}{\Gamma, \phi \cdot \psi, \Delta \Rightarrow \delta}\ (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi \quad \Delta \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \phi \cdot \psi}\ (\Rightarrow \cdot)$$

$$\frac{\Gamma, \phi, \Delta \Rightarrow \delta \quad \Gamma, \psi, \Delta \Rightarrow \delta}{\Gamma, \phi \vee \psi, \Delta \Rightarrow \delta}\ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi}\ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi}\ (\Rightarrow \vee 2)$$

$$\frac{\Gamma \Rightarrow \phi \quad \Pi, \psi, \Delta \Rightarrow \delta}{\Pi, \Gamma, \phi \to \psi, \Delta \Rightarrow \delta}\ (\to \Rightarrow) \qquad \frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi}\ (\Rightarrow \to)$$

$$\frac{\Gamma \Rightarrow \phi}{\neg\phi, \Gamma \Rightarrow}\ (\neg \Rightarrow) \qquad \frac{\Gamma, \phi \Rightarrow}{\Gamma \Rightarrow \neg\phi}\ (\Rightarrow \neg)$$

A *proof* and *provability* of a sequent in $\mathbf{FL_e}$ are defined as usual. If there is a proof of the sequent $\Rightarrow \phi$ in $\mathbf{FL_e}$ then we say that the formula ϕ is *provable* in $\mathbf{FL_e}$. Note that there are four propositional constants in the standard formulation of $\mathbf{FL_e}$.

We denote sequent systems obtained from $\mathbf{FL_e}$ by adding the weakening rule and the contraction rule, by $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ec}}$, respectively.

We often identify a sequent system with the set of all formulas provable in it.

2.3 Deducibility Relations

We say that a formula ψ is *deducible* from a set of formulas Φ in $\mathbf{FL_e}$, $\Phi \vdash_{\mathbf{FL_e}} \psi$, if there is a proof of $\Rightarrow \psi$ in $\mathbf{FL_e}$ by adding $\Rightarrow \phi_i$ for $\phi_i \in \Phi$ as initial sequents. Clearly, $\emptyset \vdash_{\mathbf{FL_e}} \psi$ if and only if ψ is provable in $\mathbf{FL_e}$.

We say that any subset T of $\mathcal{F}m(\mathbf{p})$ is an $\mathbf{FL_e}$-*theory* of the language $\mathcal{F}m(\mathbf{p})$, if it is closed under $\vdash_{\mathbf{FL_e}}$, i.e., $T \vdash_{\mathbf{FL_e}} \psi(\mathbf{p})$ implies $\psi(\mathbf{p}) \in T$.

In [13], Galatos and Ono showed the following;

PROPOSITION 2.6 *A subset T of $\mathcal{F}m$ is an $\mathbf{FL_e}$-theory if and only if*

1. $\mathbf{FL_e} \subseteq T$.
2. *If* $\phi, \phi \to \psi \in T$ *then* $\psi \in T$.
3. *If* $\phi \in T$ *then* $\phi \wedge 1 \in T$.

(proof) ($\Rightarrow$) *1.* Suppose that a formula ϕ is provable in $\mathbf{FL_e}$. Then,

$$\begin{array}{ll} & \vdash_{\mathbf{FL_e}} \phi \\ \Rightarrow & T \vdash_{\mathbf{FL_e}} \phi \\ \Rightarrow & \phi \in T. \end{array}$$

2. If $\phi, \phi \to \psi \in T$ then

$$\dfrac{\Rightarrow \phi \to \psi \qquad \dfrac{\Rightarrow \phi \qquad \dfrac{\phi \Rightarrow \phi \quad \psi \Rightarrow \psi}{\phi, \phi \to \psi \Rightarrow \psi}}{\phi \to \psi \Rightarrow \psi}\,(cut)}{\Rightarrow \psi}\,(cut) \; .$$

Hence, the formula ψ is provable in $\mathbf{FL_e}$ by adding $\Rightarrow \phi$ and $\Rightarrow \phi \to \psi$ as initial sequents. Thus, $T \vdash_{\mathbf{FL_e}} \psi$, i.e., $\psi \in T$.

3. If $\phi \in T$ then

$$\dfrac{\Rightarrow \phi \quad \Rightarrow 1}{\Rightarrow \phi \wedge 1} \; .$$

Thus, $\phi \wedge 1 \in T$.

($\Leftarrow$) Let T be a subset of $\mathcal{F}m$ satisfying the above three conditions. It is sufficient to show that if $\phi_1, \ldots, \phi_n \Rightarrow \psi$ is provable in $\mathbf{FL_e}$ by adding $\Rightarrow \delta_i$ for $\delta_i \in T$ as initial sequents then $(\phi_1 \cdots \phi_n) \to \psi \in T$. For, when $\{\phi_1, \ldots, \phi_n\}$ is empty, this means that $T \vdash_{\mathbf{FL_e}} \psi$ implies $\psi \in T$. We will prove this by induction on the length of a given proof of $\phi_1, \ldots, \phi_n \Rightarrow \psi$. The base case is obvious since the sequent is of the form $\Rightarrow \psi$ for $\psi \in T$ or an initial sequent of $\mathbf{FL_e}$. In the latter case, by Proposition 2.2 and 2.3, the formulas $\psi \to \psi$, $(\phi_1 \cdots \phi_n) \to \top$, $(\phi_1 \cdots \bot \cdots \phi_n) \to \psi$, 1 and $\neg 0$ are provable in $\mathbf{FL_e}$. Thus, by our assumption *1*, they belong to T. Here, we will show the cases of ($\Rightarrow\to$), ($\Rightarrow \wedge$) and ($\vee \Rightarrow$). In the following, to eliminate nonessential complications, we assume that the number of assumptions of the last sequent of given proof is at most one.

($\Rightarrow\to$) Suppose that the last inference of the proof is ($\Rightarrow\to$);

$$\dfrac{\phi_1, \phi_2 \Rightarrow \psi}{\phi_2 \Rightarrow \phi_1 \to \psi} \; .$$

Then, by induction hypothesis, the formula $(\phi_1 \cdot \phi_2) \to \psi$ belongs to T. Note that

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi_1 \Rightarrow \phi_1 \quad \phi_2 \Rightarrow \phi_2}{\phi_1, \phi_2 \Rightarrow \phi_1 \cdot \phi_2} \qquad \psi \Rightarrow \psi}{\phi_1, \phi_2, \phi_1 \cdot \phi_2 \to \psi \Rightarrow \psi}}{\phi_2, \phi_1 \cdot \phi_2 \to \psi \Rightarrow \phi_1 \to \psi}}{\phi_1 \cdot \phi_2 \to \psi \Rightarrow \phi_2 \to (\phi_1 \to \psi)}}{\Rightarrow (\phi_1 \cdot \phi_2 \to \psi) \to (\phi_2 \to (\phi_1 \to \psi))}\ ,$$

hence the formula $(\phi_1 \cdot \phi_2 \to \psi) \to (\phi_2 \to (\phi_1 \to \psi))$ is provable in $\mathbf{FL_e}$, so by our assumption *1*, it belongs to T. Thus, by our assumption *2*, we have $\phi_2 \to (\phi_1 \to \psi) \in T$.

$(\Rightarrow \wedge)$ Suppose that the last inference of the proof is

$$\dfrac{\phi \Rightarrow \psi \quad \phi \Rightarrow \delta}{\phi \Rightarrow \psi \wedge \delta}\ (\Rightarrow \wedge).$$

Then, by induction hypothesis, the formulas $\phi \to \psi, \phi \to \delta$ belong to T. By our assumption *3*, the formulas $(\phi \to \psi) \wedge 1$ and $(\phi \to \delta) \wedge 1$ also belong to T. Note that

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \quad \psi \Rightarrow \psi}{\phi, \phi \to \psi \Rightarrow \psi}}{\phi, (\phi \to \psi) \wedge 1 \Rightarrow \psi}}{\phi, (\phi \to \psi) \wedge 1, 1 \Rightarrow \psi}}{\phi, (\phi \to \psi) \wedge 1, (\phi \to \delta) \wedge 1 \Rightarrow \psi} \quad \dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \quad \delta \Rightarrow \delta}{\phi, \phi \to \delta \Rightarrow \delta}}{\phi, (\phi \to \delta) \wedge 1 \Rightarrow \delta}}{\phi, 1, (\phi \to \delta) \wedge 1 \Rightarrow \delta}}{\phi, (\phi \to \psi) \wedge 1, (\phi \to \delta) \wedge 1 \Rightarrow \delta}}{\phi, (\phi \to \psi) \wedge 1, (\phi \to \delta) \wedge 1 \Rightarrow \psi \wedge \delta}}{\Rightarrow ((\phi \to \delta) \wedge 1) \to (((\phi \to \psi) \wedge 1) \to (\phi \to \psi \wedge \delta))}\ ,$$

hence $((\phi \to \delta) \wedge 1) \to (((\phi \to \psi) \wedge 1) \to (\phi \to \psi \wedge \delta)) \in T$. Thus, by our assumption *2*, we have $\phi \to \psi \wedge \delta \in T$.

$(\vee \Rightarrow)$ Suppose that the last inference of the proof is

$$\dfrac{\phi \Rightarrow \delta \quad \psi \Rightarrow \delta}{\phi \vee \psi \Rightarrow \delta}\ (\vee \Rightarrow).$$

Then, by induction hypothesis, the formulas $\phi \to \delta, \psi \to \delta$ belong to T. By our assumption *3*, the formulas $(\phi \to \delta) \wedge 1$ and $(\psi \to \delta) \wedge 1$ also belong to T. Note that

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \quad \delta \Rightarrow \delta}{\phi, \phi \to \delta \Rightarrow \delta}}{\phi, (\phi \to \delta) \wedge 1 \Rightarrow \delta}}{\phi, (\phi \to \delta) \wedge 1, 1 \Rightarrow \delta}}{\phi, (\phi \to \delta) \wedge 1, (\psi \to \delta) \wedge 1 \Rightarrow \delta} \quad \dfrac{\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi \quad \delta \Rightarrow \delta}{\psi, \psi \to \delta \Rightarrow \delta}}{\psi, (\psi \to \delta) \wedge 1 \Rightarrow \delta}}{\psi, 1, (\psi \to \delta) \wedge 1 \Rightarrow \delta}}{\psi, (\phi \to \delta) \wedge 1, (\psi \to \delta) \wedge 1 \Rightarrow \delta}}{\phi \vee \psi, (\phi \to \delta) \wedge 1, (\psi \to \delta) \wedge 1 \Rightarrow \delta}}{\Rightarrow ((\psi \to \delta) \wedge 1) \to (((\phi \to \delta) \wedge 1) \to ((\phi \vee \psi) \to \delta))}\ ,$$

hence $((\psi \to \delta) \wedge 1) \to (((\phi \to \delta) \wedge 1) \to ((\phi \vee \psi) \to \delta)) \in T$. Thus, by our assumption *2*, we have $\phi \vee \psi \to \delta \in T$. □

If an $\mathbf{FL_e}$-theory is closed under substitution then it is called a *logic over* $\mathbf{FL_e}$ or a *commutative substructural logic* . Clearly, $\mathcal{F}m$ and $\mathbf{FL_e}$ are the largest and smallest logics over $\mathbf{FL_e}$, respectively. Let CSL be the set of all logics over $\mathbf{FL_e}$. Note that if $\mathcal{L}_i \in CSL$ $(i \in I)$ then $\bigcap_{i\in I} \mathcal{L}_i \in CSL$. For any $\mathcal{L}_1, \mathcal{L}_2 \in CSL$, we define $\mathcal{L}_1 \vee \mathcal{L}_2$ by the smallest commuative substructural logic which includes $\mathcal{L}_1 \cup \mathcal{L}_2$. Then, it is easy to see that $\mathbf{CSL} = \langle CSL, \cap, \vee, \mathbf{FL_e}, \mathcal{F}m \rangle$ forms a complete lattice.

In the following figure, **Cl** and **Int** denote classical logic and intuitionistic logic, respectively;

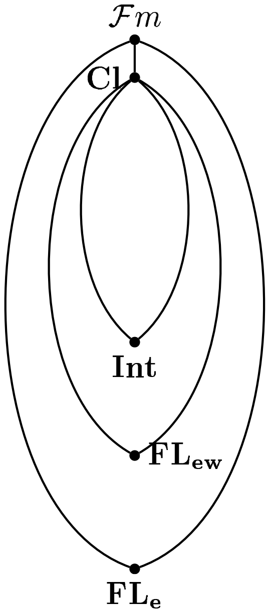

Figure 2.1: the lattice of **CSL**

For any logic $\mathcal{L}$ over $\mathbf{FL_e}$ and a set of formulas $\Phi \cup \{\psi\}$, we write $\Phi \vdash_{\mathcal{L}} \psi$ if $\Phi \cup \mathcal{L} \vdash_{\mathbf{FL_e}} \psi$. The relation $\vdash_{\mathcal{L}}$ on $\mathcal{F}m$ is called the *deducibility relation* of $\mathcal{L}$. As with $\mathbf{FL_e}$-theories, we can define theories with respect to $\vdash_{\mathcal{L}}$ as follows; a subset T of $\mathcal{F}m(\mathbf{p})$ is called an $\mathcal{L}$-*theory* of the language $\mathcal{F}m(\mathbf{p})$, if it is closed under $\vdash_{\mathcal{L}}$, i.e., $T \vdash_{\mathcal{L}} \psi(\mathbf{p})$ implies $\psi(\mathbf{p}) \in T$. Note that every $\mathcal{L}$-theory is an $\mathbf{FL_e}$-theory, since

$$\begin{array}{rc} & T \vdash_{\mathbf{FL_e}} \psi \\ \Rightarrow & T \cup \mathcal{L} \vdash_{\mathbf{FL_e}} \psi \\ \Rightarrow & T \vdash_{\mathcal{L}} \psi \\ \Rightarrow & \psi \in T. \end{array}$$

Moreover, if T is an $\mathcal{L}$-theory then it must include $\mathcal{L}$, since

$$\begin{array}{rc} & \psi \in \mathcal{L} \\ \Rightarrow & \mathcal{L} \vdash_{\mathbf{FL_e}} \psi \\ \Rightarrow & T \cup \mathcal{L} \vdash_{\mathbf{FL_e}} \psi \\ \Rightarrow & T \vdash_{\mathcal{L}} \psi \\ \Rightarrow & \psi \in T. \end{array}$$

In the same way as Proposition 2.6, we can show that for any logic $\mathcal{L}$ over $\mathbf{FL_e}$, a subset T of $\mathcal{F}m$ is an $\mathcal{L}$-theory if and only if it satisfies the following;

1. $\mathcal{L} \subseteq T$,

2. $\phi, \phi \to \psi \in T$ implies $\psi \in T$,

3. $\phi \in T$ implies $\phi \wedge 1 \in T$.

It is well-known that the *deduction theorem* holds for any intermediate logic $\mathcal{L}$, i.e., for subset $\Gamma \cup \Delta \cup \{\phi\}$ of $\mathcal{F}m$,

$$\Gamma, \Delta \vdash_{\mathcal{L}} \phi \iff \Gamma \vdash_{\mathcal{L}} \bigwedge_{i=1}^{n} \psi_i \to \phi \quad \text{for } \exists n \in \mathbb{N}, \psi_i \in \Delta (i \leq n),$$

where $\mathbb{N}$ is the set of natural numbers and $\bigwedge_{i=1}^{n} \psi_i$ denotes the conjunction of formulas ψ_i $(i \leq n)$. But this doesn't hold for substructural logics in general.

For logics over **FL** which is obtained from intuitionistic logic **LJ** by eliminating all of structural rules, Galatos and Ono showed a *parametrized local deduction theorem* in [13].

Here, we see a *local deduction theorem*, which is a simpler version of a parametrized local deduction theorem, for logics over $\mathbf{FL_e}$.

PROPOSITION 2.7 *Let* $\Gamma \cup \Delta \cup \{\phi\}$ *be a subset of* $\mathcal{F}m$ *and* $\mathcal{L}$ *a logic over* $\mathbf{FL_e}$. *Then,*

$$\Gamma, \Delta \vdash_{\mathcal{L}} \phi \iff \Gamma \vdash_{\mathcal{L}} (\prod_{i=1}^{n} (\psi_i \wedge 1)) \to \phi \quad \textit{for } \exists n \in \mathbb{N}, \psi_i \in \Delta (i \leq n),$$

where $\prod_{i=1}^{n} (\psi_i \wedge 1)$ *denotes the fusion of formulas* $\psi_i \wedge 1$ $(i \leq n)$.

In particular, if $\mathcal{L}$ *is a logic over* $\mathbf{FL_{ew}}$ *then we obtain more simpler form;*

$$\Gamma, \Delta \vdash_{\mathcal{L}} \phi \iff \Gamma \vdash_{\mathcal{L}} \prod_{i=1}^{n} \psi_i \to \phi \quad \textit{for } \exists n \in \mathbb{N}, \psi_i \in \Delta (i \leq n).$$

Chapter 3

FL_e-algebras

In this chapter, we will introduce FL_e-algebras, whose reducts are commutative residuated lattices. We will show that FL_e-algebras provide algebraic semantics for logics over $\mathbf{FL_e}$ which are introduced in the previous chapter. We discuss not only basic results on FL_e-algebras from universal algebra [5] but also some relations between logics over $\mathbf{FL_e}$ and FL_e-algebras.

A *commutative residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, 1 \rangle$ such that $\langle A, \wedge, \vee \rangle$ is a lattice, $\langle A, \cdot, 1 \rangle$ is a commutative monoid, and multiplication is residuated with respect to the order by the division operation $\rightarrow$; i.e., for all $a, b, c \in A$,

$$a \cdot b \leq c \iff b \leq a \rightarrow c.$$

An FL_e*-algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, \top, \bot, 0, 1 \rangle$ such that $\langle A, \wedge, \vee, \cdot, \rightarrow, 1 \rangle$ is a commutative residuated lattice, $\top$ and $\bot$ are the greatest and the smallest elements of $\langle A, \wedge, \vee \rangle$, respectively, and 0 is an arbitrary element of A. Note that the language of FL_e-algebras has four constants from the beginning but this doesn't mean that there are at least four elements in FL_e-algebra even if it is non-degenerate. For example, consider the two element Boolean algebra. In this case, $\top = 1, \bot = 0$. $\{\wedge, \vee, \cdot, \rightarrow, \top, \bot, 0, 1\}$ are called *fundamental operations.*

An FL_e-algebra is *integral* if the unit 1 of the commutative monoid is equal to $\top$. In any integral FL_e-algebra, $a \cdot b \leq a$ holds since $a \cdot b \leq a \cdot \top = a \cdot 1 = a$. An FL_e-algebra $\mathbf{A}$ is *increasing idempotent* if $a \leq a \cdot a$ for any $a \in A$. It is easy to see that an FL_e-algebra $\mathbf{A}$ is both integral and increasing idempotent if and only if $a \cdot b = a \wedge b$ holds for all $a, b \in A$.

Well-known examples of FL_e-algebras are Heyting algebras and Boolean algebras. A Heyting algebra is a integral, increasing idempotent FL_e-algebra in which 0 is equal to $\bot$. A Boolean algebra is a Heyting algebra in which $a \vee a' = 1$ holds always, where a' is an abbreviation of $a \rightarrow 0$.

In this chapter, though we discuss the following results in the case of FL_e-algebars, most of those hold for more general algebras.

3.1 Basic Concepts of FL_e-algebras

Let $\mathbf{A}$ and $\mathbf{B}$ be FL_e-algebras. In the following, we will introduce some of basic algebraic concepts of FL_e-algebras.

A map $\alpha : A \to B$ is called a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ if it satisfies the following;

1. $\alpha(\top_{\mathbf{A}}) = \top_{\mathbf{B}},\ \alpha(\bot_{\mathbf{A}}) = \bot_{\mathbf{B}},\ \alpha(1_{\mathbf{A}}) = 1_{\mathbf{B}},\ \alpha(0_{\mathbf{A}}) = 0_{\mathbf{B}}$,

2. for all $x, y \in A$ and for every operation $\oplus \in \{\wedge, \vee, \cdot, \to\}$,

$$\alpha(x \oplus_{\mathbf{A}} y) = \alpha(x) \oplus_{\mathbf{B}} \alpha(y).$$

In the following, we allow symbol $\oplus$ to range over operations in $\{\wedge, \vee, \cdot, \to\}$. The *kernel* of a homomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ is defined to be the set $Ker(\alpha) = \{(a, b) \in A^2 : \alpha(a) = \alpha(b)\}$.

Let B be a subset of A. $\mathbf{B} = \langle B, \wedge_{\mathbf{B}}, \vee_{\mathbf{B}}, \cdot_{\mathbf{B}}, \to_{\mathbf{B}}, \top_{\mathbf{B}}, \bot_{\mathbf{B}}, 0_{\mathbf{B}}, 1_{\mathbf{B}}\rangle$ is called a *subalgebra* of $\mathbf{A}$ if

1. $\top_{\mathbf{B}} = \top_{\mathbf{A}}, \bot_{\mathbf{B}} = \bot_{\mathbf{A}}, 0_{\mathbf{B}} = 0_{\mathbf{A}}, 1_{\mathbf{B}} = 1_{\mathbf{A}}$,

2. B is closed under all binary fundamental operations, i.e., for all $x, y \in B$,

$$x \oplus_{\mathbf{B}} y = x \oplus_{\mathbf{A}} y \quad (\in B).$$

A *congruence relation* on $\mathbf{A}$ is an equivalence relation θ on A that is *compatible* with all binary fundamental operations of $\mathbf{A}$, i.e., for any $x_1, x_2, y_1, y_2 \in A$, if $x_1\theta y_1, x_2\theta y_2$ then $(x_1 \oplus x_2)\theta(y_1 \oplus y_2)$. The collection of all congruences on $\mathbf{A}$ forms a complete lattice denoted by $\mathbf{Con}(\mathbf{A})$. For, if $\{\theta_i | i \in I\}$ is a family of congruences then $\bigcap_{i \in I} \theta_i$ is also a congruence. Every non-degenerate FL_e-algebra $\mathbf{A}$ has at least two congruences, i.e., the *universal congruence* $\nabla := A^2$ and the *diagonal congruence* $\Delta := \{(a, a) | a \in A\}$. When $\mathbf{A}$ has exactly two congruences then it is called *simple*. The *congruence generated* by a set X of pairs of elements in $\mathbf{A}$ is the smallest congruence $Cg(X)$ containing X. The congruence generated by a singleton is called *principal*.

PROPOSITION 3.1 *Let* $\mathbf{A}, \mathbf{B}$ *be* FL_e*-algebras and* α *a homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$. *Then* $Ker(\alpha)$ *is a congruence on* $\mathbf{A}$.

(proof) Clearly, $Ker(\alpha)$ is a equivalent relation on $\mathbf{A}$. For any $(a_1, b_1), (a_2, b_2) \in Ker(\alpha)$,

$$\begin{aligned} \alpha(a_1 \oplus_{\mathbf{A}} a_2) &= \alpha(a_1) \oplus_{\mathbf{B}} \alpha(a_2) \\ &= \alpha(b_1) \oplus_{\mathbf{B}} \alpha(b_2) \\ &= \alpha(b_1 \oplus_{\mathbf{A}} b_2). \end{aligned}$$

Thus, $Ker(\alpha)$ is a congruence on $\mathbf{A}$. □

Let θ be a congruence on $\mathbf{A}$. For $a \in A$, the *θ-congruence block* of a is the set $a/\theta = \{b \in A | (b, a) \in \theta\}$. The set $\{a/\theta | a \in A\}$ is denoted by A/θ. The *quotient algebra* $\mathbf{A}/\theta$ of $\mathbf{A}$ by θ is an FL_e-algebra such that its underlying set is A/θ, and fundamental operations $\oplus_{\mathbf{A}/\theta}$ are defined by $a/\theta \oplus_{\mathbf{A}/\theta} b/\theta = (a \oplus_{\mathbf{A}} b)/\theta$. Note that since θ is a congruence, $\oplus_{\mathbf{A}/\theta}$ is well-defined.

Let $\mathbf{A}$ be an FL_e-algebra and $\mathcal{F}$ a subset of A. Then $\mathcal{F}$ is called a *filter* of $\mathbf{A}$ if it satisfies the following conditions;

1. $1 \leq a$ implies $a \in \mathcal{F}$,

2. $a, a \to b \in \mathcal{F}$ implies $b \in \mathcal{F}$,

3. $a \in \mathcal{F}$ implies $a \wedge 1 \in \mathcal{F}$.

PROPOSITION 3.2 *Let* $\mathbf{A}$ *be an* FL_e*-algebra and* $\mathcal{F}$ *a filter of* $\mathbf{A}$*. Then for any* $a, b \in A$,

1. $a \in \mathcal{F}$ *and* $a \leq b$ *impliy* $b \in \mathcal{F}$,

2. $a, b \in \mathcal{F}$ *implies* $a \cdot b \in \mathcal{F}$,

3. $a, b \in \mathcal{F}$ *implies* $a \wedge b \in \mathcal{F}$.

(proof) *1.* If $a \leq b$ then $1 \leq a \to b$. Hence, by conditions 1 and 2 of a filter, we have $b \in \mathcal{F}$.

2. Note that $a \cdot b \leq a \cdot b \Leftrightarrow a \leq b \to a \cdot b$. Thus, by *1* and condition 2 of a filter, we have $a \cdot b \in \mathcal{F}$.

3. If $a, b \in \mathcal{F}$ then $a \wedge 1, b \wedge 1 \in \mathcal{F}$ by condition 3 of a filter. So, by *2*, $(a \wedge 1) \cdot (b \wedge 1) \in \mathcal{F}$. Since

$$\begin{array}{rl} & \begin{cases} (a \wedge 1) \cdot (b \wedge 1) \leq a \cdot 1 \leq a \\ (a \wedge 1) \cdot (b \wedge 1) \leq b \cdot 1 \leq b \end{cases} \\ \Rightarrow & (a \wedge 1) \cdot (b \wedge 1) \leq a \wedge b, \end{array}$$

by *1*, we have $a \wedge b \in \mathcal{F}$. □

By above proposition, a filter $\mathcal{F}$ of an FL_e-algebra $\mathbf{A}$ is a lattice filter of $\langle A, \wedge, \vee \rangle$.

The collection of all filters of $\mathbf{A}$ forms a complete lattice denoted by $\mathbf{Fil}(\mathbf{A})$. For, if $\{\mathcal{F}_i | i \in I\}$ is a family of filters then $\bigcap_{i \in I} \mathcal{F}_i$ is also a filter. A filter $\mathcal{F}$ of $\mathbf{A}$ is called *proper* if $\mathcal{F} \neq A$. We say that a proper filter $\mathcal{F}$ of $\mathbf{A}$ is *prime* if for any $a, b \in A$, $a \vee b \in \mathcal{F}$ implies $a \in \mathcal{F}$ or $b \in \mathcal{F}$.

PROPOSITION 3.3 *Let* $\mathcal{F}$ *be a filter of an* FL_e*-algebra* $\mathbf{A}$*, and define*

$$\theta_{\mathcal{F}} = \{(a, b) \mid (a \to b) \wedge (b \to a) \in \mathcal{F}\}.$$

Then $\theta_{\mathcal{F}}$ *is a congruence of* $\mathbf{A}$ *and* $\mathcal{F} = \{a \in A | (a \wedge 1, 1) \in \theta_{\mathcal{F}}\}$.

(proof) Clearly, $\theta_{\mathcal{F}}$ is symmetric by the definition, and it is reflexive since for every $a \in A$, $1 \leq (a \to a) \wedge (a \to a)$. If $(a, b), (b, c) \in \theta_{\mathcal{F}}$ then by the definition of $\theta_{\mathcal{F}}$, $(a \to b) \wedge (b \to a), (b \to c) \wedge (c \to b) \in \mathcal{F}$. Since $(a \to b) \wedge (b \to a) \leq a \to b$ and $(b \to c) \wedge (c \to b) \leq b \to c$, we have $a \to b \in \mathcal{F}$ and $b \to c \in \mathcal{F}$. Note that for any $x, y, z \in A$, $(x \to y) \to ((y \to z) \to (x \to z)) \in \mathcal{F}$ since

$$\begin{array}{rl} & x \to y \leq x \to y \\ \Leftrightarrow & x \cdot (x \to y) \leq y \\ \Rightarrow & x \cdot (x \to y) \cdot (y \to z) \leq y \cdot (y \to z) \leq z \\ \Rightarrow & x \cdot (x \to y) \cdot (y \to z) \leq z \\ \Leftrightarrow & 1 \leq (x \to y) \to ((y \to z) \to (x \to z)). \end{array}$$

Hence, by condition 2 of a filter, $a \to c \in \mathcal{F}$. Similarly, from $b \to a \in \mathcal{F}$ and $c \to b \in \mathcal{F}$, it follows that $c \to a \in \mathcal{F}$. By Proposition 3.2, we have $(a \to c) \wedge (c \to a) \in \mathcal{F}$, namely, $(a, c) \in \theta_{\mathcal{F}}$. Therefore $\theta_{\mathcal{F}}$ is an equivalence relation on A. We will show that $\theta_{\mathcal{F}}$ is a congruence. It is sufficient to show that for any fundamental operation $\oplus \in \{\wedge, \vee, \cdot, \to\}$, $(x, y) \in \theta_{\mathcal{F}}$ and $z \in A$ imply $(x \oplus z, y \oplus z), (z \oplus x, z \oplus y) \in \theta_{\mathcal{F}}$. Here we will prove the cases of $\vee$ and $\to$.

($\vee$) Suppose that $(x, y) \in \theta_{\mathcal{F}}$. Then $x \to y, y \to x \in \mathcal{F}$. By condition 3 of a filter, $(x \to y) \wedge 1, (y \to x) \wedge 1 \in \mathcal{F}$. Now,

$$
\begin{aligned}
& \begin{cases} x \cdot ((x \to y) \wedge 1) \leq x \cdot (x \to y) \leq y \\ z \cdot ((x \to y) \wedge 1) \leq z \cdot 1 \leq z \end{cases} \\
\Rightarrow \quad & (x \cdot ((x \to y) \wedge 1)) \vee (z \cdot ((x \to y) \wedge 1)) \leq y \vee z \\
\Leftrightarrow \quad & (x \vee z) \cdot ((x \to y) \wedge 1) \leq y \vee z \\
\Leftrightarrow \quad & 1 \leq ((x \to y) \wedge 1) \to ((x \vee z) \to (y \vee z)),
\end{aligned}
$$

hence $((x \to y) \wedge 1) \to ((x \vee z) \to (y \vee z)) \in \mathcal{F}$. Thus, by condition 2 of a filter, $(x \vee z) \to (y \vee z) \in \mathcal{F}$. Similarly $(y \vee z) \to (x \vee z) \in \mathcal{F}$.

($\to$) Suppose that $(x, y) \in \theta_{\mathcal{F}}$. Then $x \to y, y \to x \in \mathcal{F}$. Since

$$
\begin{aligned}
& y \cdot (y \to x) \cdot (x \to z) \leq z \\
\Leftrightarrow \quad & 1 \leq (y \to x) \to ((x \to z) \to (y \to z)),
\end{aligned}
$$

by condition 2 of a filter, $(x \to z) \to (y \to z) \in \mathcal{F}$. Similarly $(y \to z) \to (x \to z) \in \mathcal{F}$, so $(x \to z, y \to z) \in \theta_{\mathcal{F}}$. On the other hand, since

$$
\begin{aligned}
& z \cdot (z \to x) \cdot (x \to y) \leq y \\
\Leftrightarrow \quad & 1 \leq (x \to y) \to ((z \to x) \to (z \to y)),
\end{aligned}
$$

$(z \to x) \to (z \to y) \in \mathcal{F}$. Similarly $(z \to y) \to (z \to x) \in \mathcal{F}$, thus, $(z \to x, z \to y) \in \theta_{\mathcal{F}}$.

Next, we will show that $\mathcal{F} = \{a \in A | (a \wedge 1, 1) \in \theta_{\mathcal{F}}\}$. Let $a \in \mathcal{F}$. Then, clearly, $(a \wedge 1) \to 1 \in \mathcal{F}$ since $a \wedge 1 \leq 1 \Leftrightarrow 1 \leq (a \wedge 1) \to 1$. By condition 3 of a filter, $a \wedge 1 \in \mathcal{F}$. Now

$$
\begin{aligned}
& (a \wedge 1) \cdot 1 \leq a \wedge 1 \\
\Leftrightarrow \quad & 1 \leq (a \wedge 1) \to (1 \to (a \wedge 1)),
\end{aligned}
$$

hence $1 \to (a \wedge 1) \in \mathcal{F}$, therefore, $(a \wedge 1, 1) \in \theta_{\mathcal{F}}$. Conversely, let $(a \wedge 1, 1) \in \theta_{\mathcal{F}}$. Then $((a \wedge 1) \to 1) \wedge (1 \to (a \wedge 1)) \in \mathcal{F}$, so, $1 \to (a \wedge 1) \in \mathcal{F}$. Since $1 \in \mathcal{F}$ and $a \wedge 1 \leq a$, $a \in \mathcal{F}$. □

PROPOSITION 3.4 *For any FL_e-algebra* **A**, **Fil(A)** *is isomorphic to* **Con(A)**, *via the mutually inverse maps $\mathcal{F} \mapsto \theta_{\mathcal{F}}$ and $\theta \mapsto \mathcal{F}_{\theta} = \{a \in A | (a \wedge 1, 1) \in \theta\}$.*

(proof) By Proposition 3.3, $\theta_{\mathcal{F}}$ is a congruence. Let θ be a congruence. Then $\mathcal{F}_{\theta}$ satisfies the conditions 1 and 3 of a filter. Suppose that $a, a \to b \in \mathcal{F}_{\theta}$. Then, by definition of $\mathcal{F}_{\theta}$, $(a \wedge 1, 1), ((a \to b) \wedge 1, 1) \in \theta$. Since $(a \wedge 1) \cdot ((a \to b) \wedge 1) \leq a \cdot (a \to b) \leq b$,

$$1/\theta = ((a \wedge 1) \cdot ((a \to b) \wedge 1))/\theta \leq (a \cdot (a \to b))/\theta \leq b/\theta.$$

Thus, $1/\theta = 1/\theta \wedge b/\theta = (1 \wedge b)/\theta$, i.e., $(b \wedge 1, 1) \in \theta$, so, $b \in \mathcal{F}_{\theta}$.

We will show that $\mathbf{Fil}(\mathbf{A})$ is isomorphic to $\mathbf{Con}(\mathbf{A})$. Note that for any lattices $\mathbf{L}$ and $\mathbf{M}$, $\mathbf{L}$ is isomorphic to $\mathbf{M}$ if and only if $\mathbf{L}$ is order-isomorphic to $\mathbf{M}$. Now the given maps are clearly order-preserving, so, it is sufficient to show that they are inverses of each other. Proposition 3.3 already proved $\mathcal{F} = \mathcal{F}_{\theta_{\mathcal{F}}}$. In order to show $\theta = \theta_{\mathcal{F}_\theta}$, we will prove that $(x, y) \in \theta$ if and only if $((x \to y) \wedge (y \to x) \wedge 1, 1) \in \theta$. If $x\theta y$ then

$$1 = ((x \to x) \wedge (y \to y) \wedge 1)\theta((x \to y) \wedge (y \to x) \wedge 1).$$

Conversely, suppose that $((x \to y) \wedge (y \to x) \wedge 1, 1) \in \theta$. Since $x \cdot ((x \to y) \wedge (y \to x) \wedge 1) \leq x \cdot x \to y \leq y$,

$$x/\theta = (x \cdot ((x \to y) \wedge (y \to x) \wedge 1))/\theta \leq (x \cdot x \to y)/\theta \leq y/\theta.$$

Similarly $y/\theta \leq x/\theta$. Hence $x/\theta = y/\theta$, i.e., $(x, y) \in \theta$. We will show that $\theta = \theta_{\mathcal{F}_\theta}$. If $(a, b) \in \theta$ then, by above mention, $((a \to b) \wedge (b \to a) \wedge 1, 1) \in \theta$. Thus, $(a \to b) \wedge (b \to a) \in \mathcal{F}_\theta$, so, $(a, b) \in \theta_{\mathcal{F}_\theta}$. Conversely, $(a, b) \in \theta_{\mathcal{F}_\theta}$ implies $((a \to b) \wedge (b \to a) \wedge 1, 1) \in \theta$. Therefore $(a, b) \in \theta$. □

If $\mathcal{A} = \{\mathbf{A}_i | i \in I\}$ is an indexed set of FL_e-algebras then the *product* of $\mathcal{A}$ is the algebra $\mathbf{P} = \Pi_{i \in I}\mathbf{A}_i$ whose underlying set is the Cartesian product of the underlying sets of each algebras in $\mathcal{A}$, and in which all binary fundamental operations $\oplus_{\mathbf{P}}$ are defined by

$$(a_i)_{i \in I} \oplus_{\mathbf{P}} (b_i)_{i \in I} = (a_i \oplus_{\mathbf{A}_i} b_i)_{i \in I},$$

where $a_i, b_i \in A_i$.

A *subdirect product* of an indexed set $\mathcal{A} = \{\mathbf{A}_i | i \in I\}$ of FL_e-algebras is a subalgebra $\mathbf{B}$ of the product of $\mathcal{A}$, such that for every $i \in I$ and for every $a_i \in A_i$, there exists an element of B, whose i-th coordinate is a_i. In other words, the projection to the i-th coodinate map from B to A_i is onto. A homomorphism $\alpha : \mathbf{A} \to \Pi_{i \in I}\mathbf{A}_i$ is called a *subdirect embedding* if it is a one-to-one homomorphism *(embedding)* and $\alpha(\mathbf{A})$ is a subdirect product of $\Pi_{i \in I}\mathbf{A}_i$.

PROPOSITION 3.5 *Let $\theta_i \in Con\mathbf{A}$ $(i \in I)$ and $\bigcap_{i \in I} \theta_i = \Delta$. Then, the homomorphism $\nu : \mathbf{A} \to \Pi_{i \in I}\mathbf{A}/\theta_i$ defined by*

$$\nu(a)(i) = a/\theta_i,$$

is a subdirect embedding.

(proof) Let ν_i be the natural homomorphism from $\mathbf{A}$ to $\mathbf{A}/\theta_i$ for $i \in I$. Then, each ν_i is an onto (*surjective*) homomorphism. We will show that ν is an embedding. For any distinct elements $a, b \in A$, $(a, b) \notin \bigcap_{i \in I} \theta_i$ since $\bigcap_{i \in I} \theta_i = \Delta$. Hence, there exists some $j \in I$ such that $(a, b) \notin \theta_j$. Thus, $\nu_j(a) \neq \nu_j(b)$, namely, $\nu(a) \neq \nu(b)$. □

An FL_e-algebra $\mathbf{A}$ is *subdirectly irreducible*, if for every subdirect embedding $\alpha : \mathbf{A} \to \Pi_{i \in I}\mathbf{B}_i$ there exists some $i \in I$ such that

$$\pi_i \circ \alpha : \mathbf{A} \to \mathbf{B}_i$$

is an isomorphism, where π_i is the i-th coodinate map from $\alpha(\mathbf{A})$ to $\mathbf{B}_i$.

PROPOSITION 3.6 *An FL_e-algebra* $\mathbf{A}$ *is subdirectly irreducible iff either* $\mathbf{A}$ *is trivial or there is a minimum congruence in* $Con\mathbf{A} - \{\Delta\}$. *In the latter case the minimum element is* $\bigcap(Con\mathbf{A} - \{\Delta\})$, *called the monolith.*

(proof) ($\Rightarrow$) If $\mathbf{A}$ is non-degenerate and $Con\mathbf{A} - \{\Delta\}$ has no minimum element then $\bigcap(Con\mathbf{A} - \{\Delta\}) = \Delta$. Let $I = Con\mathbf{A} - \{\Delta\}$. Then the natural map $\alpha : \mathbf{A} \to \Pi_{\theta \in I}\mathbf{A}/\theta$ is a subdirect embedding by Proposition 3.5, and as the natural map $\mathbf{A} \to \mathbf{A}/\theta$ is not an embedding for $\theta \in I$, it follows that $\mathbf{A}$ is not subdirectly irreducible.

($\Leftarrow$) If $\mathbf{A}$ is trivial and $\alpha : \mathbf{A} \to \Pi_{i \in I}\mathbf{A}_i$ is a subdirect embedding then each $\mathbf{A}_i$ is trivial; hence each $\pi_i \circ \alpha$ is an isomorphism. So suppose $\mathbf{A}$ is non-degenerate, and let $\theta = \bigcap(Con\mathbf{A} - \{\Delta\}) \neq \Delta$. Choose $(a, b) \in \theta$ such that $a \neq b$. If $\alpha : \mathbf{A} \to \Pi_{i \in I}\mathbf{A}_i$ is a subdirect embedding then for some i, $(\alpha a)(i) \neq (\alpha b)(i)$; hence $(\pi_i \circ \alpha)(a) \neq (\pi_i \circ \alpha)(b)$. Thus $(a, b) \notin ker(\pi_i \circ \alpha)$ and hence $\theta \not\subseteq ker(\pi_i \circ \alpha)$. But this implies $ker(\pi_i \circ \alpha) = \Delta$, so $\pi_i \circ \alpha : \mathbf{A} \to \mathbf{A}_i$ is an isomorphism. Consequently $\mathbf{A}$ is subdirectly irreducible. □

We say that an element $a(\not\geq 1)$ of FL_e-algebra $\mathbf{A}$ is an *opremum*, if for each $x(< 1)$ there exists some $n \in \omega$ such that $x^n \leq a$.

PROPOSITION 3.7 *Let* $\mathbf{A}$ *be an FL_e-algebra. Then* $\mathbf{A}$ *is subdirectly irreducible if and only if there is an opermum a in it.*

(proof) ($\Rightarrow$) Suppose that $\mathbf{A}$ is subdirectly irreducible. Then, by Proposition 3.6 and 3.4, $\mathbf{A}$ has the second smallest filter $\mathcal{F}$. Thus, there is some $a \in \mathcal{F}$ such that $a \not\geq 1$. For each $x < 1$, let

$$\mathcal{F}_x = \{y \in A | x^n \leq y, \exists n \in \omega\}.$$

It is easy to see that $\mathcal{F}_x$ is a filter generated by x. Since $\mathcal{F} \subseteq \mathcal{F}_x$, $a \in \mathcal{F}_x$. Hence there is some $n \in \omega$ such that $x^n \leq a$, i.e., a is an opremum of $\mathbf{A}$.

($\Leftarrow$) Let a be an opremum of $\mathbf{A}$ and $\mathcal{F}_a = \{y \in A | (a \wedge 1)^n \leq y, \exists n \in \omega\}$. It is sufficient to show that $\mathcal{F}_a$ is the second smallest filter of $\mathbf{A}$. Let $\mathcal{F}$ be a filter other than the smallest filter. Then for any $x \in \mathcal{F}$, there is some $m \in \omega$ such that

$$(x \wedge 1)^m \leq a.$$

Thus for any $y \in \mathcal{F}_a$, there is some $n, m \in \omega$ such that

$$(x \wedge 1)^{mn} \leq (a \wedge 1)^n \leq y.$$

Hence $y \in \mathcal{F}$. □

PROPOSITION 3.8 (Birkhoff) *Every FL_e-algebra* $\mathbf{A}$ *is isomorphic to a subdirect product of subdirectly irreducible FL_e-algebras (which are homomorphic images of* $\mathbf{A}$*).*

(proof) Since trivial algebras are subdirectly irreducible, we suppose that $\mathbf{A}$ is non-degenerate. Then there exists some $a, b \in A$ such that $a \neq b$. Let $\Sigma = \{\theta \in Con\mathbf{A} | (a, b) \notin \theta\}$. By Zorn's lemma, there is maximal element $\theta_{a,b}$ in Σ. Then clearly $Cg(a, b) \vee \theta_{a,b}$ is the smallest congruence in $[\theta_{a,b}, \nabla] - \{\theta_{a,b}\}$, so by Proposition 3.6 we see that $\mathbf{A}/\theta_{a,b}$ is subdirecty irreducible. As $\bigcap\{\theta_{a,b} | a, b \in A, a \neq b\} = \Delta$ we can show that $\mathbf{A}$ is subdirectly embeddable in the product of the indexed family of subdirectly irreducible algebras

$(\mathbf{A}/\theta_{a,b})_{a\neq b}$. □

Let $\mathbf{B} = \langle B, \wedge, \vee, ', 0, 1\rangle$ be a Boolean algebra. Note that since 1 is the greatest element of a Boolean algebra $\mathbf{B}$, a filter $\mathcal{F}$ of $\mathbf{B}$ is defined more simply as follows;

1. $1 \in \mathcal{F}$,
2. $a, a \to b \in \mathcal{F}$ implies $b \in \mathcal{F}$.

It is easy to see that the above condition is equivalent to the following;

1. $1 \in \mathcal{F}$,
2. $a, b \in \mathcal{F}$ implies $a \wedge b \in \mathcal{F}$,
3. $a \in \mathcal{F}$ and $a \leq b$ implies $b \in \mathcal{F}$.

A subset $\mathcal{I}$ of B is called an *ideal* of $\mathbf{B}$ if it satisfies the following;

1. $0 \in \mathcal{I}$,
2. $a, b \in \mathcal{I}$ implies $a \vee b \in \mathcal{I}$,
3. $a \in \mathcal{I}$ and $a \geq b$ implies $b \in \mathcal{I}$.

Let X be a set and $\mathbf{B}(\mathcal{P}(X))$ the Boolean algebra defined by the power set $\mathcal{P}(X)$ of X. A subset $\mathcal{U}$ of $\mathcal{P}(X)$ is called an *ultrafilter* over a set X or an *ultrafilter* of $\mathbf{B}(\mathcal{P}(X))$, if it is a filter of $\mathbf{B}(\mathcal{P}(X))$ which is maximal with respect to the property that $\emptyset \notin \mathcal{U}$.

PROPOSITION 3.9 *Let $\mathcal{F}$ be a filter of a Boolean algebra $\mathbf{B}$. Then the following are equivalent:*

1. *$\mathcal{F}$ is an ultrafilter of $\mathbf{B}$,*
2. *for any a in B, exactly one of a, a' belongs to $\mathcal{F}$,*
3. *for any $a, b \in B$, $a \vee b \in \mathcal{F}$ implise $a \in \mathcal{F}$ or $b \in \mathcal{F}$.*

(proof) *1* ⇒ *2.* If $\mathcal{F}$ is an ultrafilter then $\mathbf{B}/\theta_{\mathcal{F}} \cong \mathbf{2}$ since $\mathbf{B}/\theta_{\mathcal{F}}$ is simple, where $\mathbf{2}$ is the two element Boolean algebra. Let $\nu : \mathbf{B} \to \mathbf{B}/\theta_{\mathcal{F}}$ be the natural homomorphism. For $a \in B$, $\nu(a') = \nu(a)'$ so

$$\nu(a) = 1/\theta_{\mathcal{F}} \quad \text{or} \quad \nu(a') = 1/\theta_{\mathcal{F}},$$

as $\mathbf{B}/\theta_{\mathcal{F}} \cong \mathbf{2}$; hence

$$a \in \mathcal{F} \quad \text{or} \quad a' \in \mathcal{F}.$$

If there exists $a \in B$ such that $a \in \mathcal{F}$ and $a' \in \mathcal{F}$ then $0 = a \wedge a' \in \mathcal{F}$, so this is a contradiction.

2 ⇒ *3.* Suppose $\mathcal{F}$ is a filter with $a \vee b \in \mathcal{F}$. By *2*, $(a \vee b)' = (a' \wedge b') \notin \mathcal{F}$, so $a' \notin \mathcal{F}$ or $b' \notin \mathcal{F}$. Thus, either $a \in \mathcal{F}$ or $b \in \mathcal{F}$.

3 ⇒ *1.* Suppose that $\mathcal{F}'$ is a filter of $\mathbf{B}$ such that $\mathcal{F} \subset \mathcal{F}'$. If $a \in \mathcal{F}' - \mathcal{F}$ then $a' \in \mathcal{F}$, since $1 = a \vee a' \in \mathcal{F}$ and $a \notin \mathcal{F}$, by *3*. Hence, $a' \in \mathcal{F} \subset \mathcal{F}'$, so $0 = a \wedge a' \in \mathcal{F}'$. Thus, $\mathcal{F}' = B$. □

PROPOSITION 3.10 (Jónsson) *Let W be a family of subsets of I $(\neq \emptyset)$ such that*

1. *$I \in W$,*
2. *if $J \in W$ and $J \subseteq K \subseteq I$ then $K \in W$,*
3. *if $J_1 \cup J_2 \in W$ then $J_1 \in W$ or $J_2 \in W$.*

Then there is an ultrafilter U over I such that $U \subseteq W$.

(proof) If $\emptyset \in W$ then $W = \mathcal{P}(I)$, so every ultrafilter is in W. Otherwise, it is easy to see that $\mathcal{P}(I) - W$ is a proper ideal. Hence we can extend it to a maximal ideal and obtain an ultrafilter by taking the complementary ultrafilter. □

If $\mathcal{A} = \{\mathbf{A}_i | i \in I\}$ is an indexed set of FL_e-algebras and $\mathcal{U}$ is an ultrafilter over index set I, then the binary relation $\theta_{\mathcal{U}}$ on the product $\mathbf{P}$ of $\mathcal{A}$, defined by $(a_i)_{i\in I}\theta_{\mathcal{U}}(b_i)_{i\in I}$ iff $\{i \in I | a_i = b_i\} \in \mathcal{U}$, is a congruence on $\mathbf{P}$. The quotient algebra $\mathbf{P}/\theta_{\mathcal{U}}$ is called the *ultraproduct* of $\mathcal{A}$ over the ultrafilter $\mathcal{U}$.

PROPOSITION 3.11 *If $\{\mathbf{A}_i | i \in I\}$ is a finite set of finite FL_e-algebras, say $\{\mathbf{B}_1, \ldots, \mathbf{B}_m\}$, ($I$ can be infinite), and $\mathcal{U}$ is an ultrafilter over I, then $\Pi_{i\in I}\mathbf{A}_i/\mathcal{U}$ is isomorphic to one of the algebras $\mathbf{B}_1, \ldots, \mathbf{B}_m$, namely to the $\mathbf{B}_j$ such that*

$$\{i \in I | \mathbf{A}_i = \mathbf{B}_j\} \in \mathcal{U}.$$

(proof) Let

$$S_j = \{i \in I | \mathbf{A}_i = \mathbf{B}_j\}.$$

Then $I = S_1 \cup \cdots \cup S_m$, so by Proposition 3.9, there is some j $(1 \leq j \leq m)$ such that $S_j \in \mathcal{U}$. Let $B_j = \{b_1, \ldots, b_k\}$, where the b's are all distinct, and choose $a_1, \ldots, a_k \in \Pi_{i\in I}A_i$ such that

$$a_1(i) = b_1, \ldots, a_k(i) = b_k$$

if $i \in S_j$. Then, for every element $a \in \Pi_{i\in I}A_i$,

$$\{i \in I | a(i) = a_1(i)\} \cup \cdots \cup \{i \in I | a(i) = a_k(i)\} \supseteq S_j.$$

Since $S_j \in \mathcal{U}$, $\{i \in I | a(i) = a_1(i)\} \cup \cdots \cup \{i \in I | a(i) = a_k(i)\} \in \mathcal{U}$, this follows

$$\{i \in I | a(i) = a_1(i)\} \in \mathcal{U} \text{ or } \ldots \text{ or } \{i \in I | a(i) = a_k(i)\} \in \mathcal{U};$$

hence

$$a/\theta_{\mathcal{U}} = a_1/\theta_{\mathcal{U}} \text{ or } \ldots \text{ or } a/\theta_{\mathcal{U}} = a_k/\theta_{\mathcal{U}}.$$

Also it is evident that $a_1/\theta_{\mathcal{U}}, \ldots, a_k/\theta_{\mathcal{U}}$ are all distinct. Thus $\Pi_{i\in I}\mathbf{A}_i/\theta_{\mathcal{U}}$ has exactly k elements, $a_1/\theta_{\mathcal{U}}, \ldots, a_k/\theta_{\mathcal{U}}$. Let α be the map from $\Pi_{i\in I}A_i/\theta_{\mathcal{U}}$ to B_j defined by

$$\alpha(a_t/\theta_{\mathcal{U}}) = b_t, \quad 1 \leq t \leq k.$$

Then it is easy to see that α is an isomorphism. □

3.2 Basic Concepts of Classes of FL_e-algebras

In this section, we will discuss some basic concepts of classes of FL_e-algebras.

For a class of FL_e-algebras $\mathcal{K}$, we denote by $S(\mathcal{K})$, $H(\mathcal{K})$, $P(\mathcal{K})$ and $P_S(\mathcal{K})$ the classes of all FL_e-algebras that are isomorphic to a subalgebra, a homomorphic image, a product of algebras of $\mathcal{K}$ and a subdirect product of algebras of $\mathcal{K}$, respectively. $I(\mathcal{K})$ denotes the class of all FL_e-algebras which are isomorphic to some member of $\mathcal{K}$. If O_1 and O_2 are two operators on classes of algebras we write O_1O_2 for the composition of the two operators, and $\leq$ denotes the usual prtial ordering, i.e., $O_1 \leq O_2$ if $O_1(\mathcal{K}) \subseteq O_2(\mathcal{K})$ for all classes of algebra $\mathcal{K}$. An operator O is *idempotent* if $O^2 = O$.

PROPOSITION 3.12 *The following inequalities hold:* $SH \leq HS, PS \leq SP$, *and* $PH \leq HP$. *Also the operators,* H, S, *and* IP *are idempotent.*

(proof) Suppose $\mathbf{A} = SH(\mathcal{K})$. Then for some $\mathbf{B} \in \mathcal{K}$ and onto homomorphism $\alpha : \mathbf{B} \to \mathbf{C}$, we have $\mathbf{A} \leq \mathbf{C}$. Thus, $\alpha^{-1}(\mathbf{A}) \leq \mathbf{B}$, and as $\alpha(\alpha^{-1}(\mathbf{A})) = \mathbf{A}$, so $\mathbf{A} \in HS(\mathcal{K})$. In the same way, we can show that $PS \leq SP$ and $PH \leq HP$.

It is easy to show that H, S and IP are idempotent. □

A nonempty class $\mathcal{K}$ of FL_e-algebras is called a *variety* if it is closed under the three operators S, H and P. If $\mathcal{K}$ is a class of FL_e-algebras then $V(\mathcal{K})$ denote the smallest variety containing $\mathcal{K}$. We say that $V(\mathcal{K})$ is the *variety generated by* $\mathcal{K}$. If $\mathcal{K}$ has a single member $\mathbf{A}$ we write simply $V(\mathbf{A})$.

PROPOSITION 3.13 (Tarski) $V = HSP$.

(proof) Since $HV = SV = IPV = V$ and $I \leq V$, it follows that $HSP \leq HSPV = V$. By the previous proposition, we see that $H(HSP) = HSP$, $S(HSP) \leq HSSP = HSP$, and $P(HSP) \leq HPSP \leq HSPP \leq HSIPIP = HSIP \leq HSHP \leq HHSP = HSP$; hence for any $\mathcal{K}$, $HSP(\mathcal{K})$ is closed under H, S and P. As $V(\mathcal{K})$ is the smallest variety containing $\mathcal{K}$, it must be $V = HSP$. □

Let X be a set of (distinct) objects called *variables.* The set $T(X)$ of *terms over* X is the smallest set such that

1. $X \cup \{\top, \bot, 0, 1\} \subseteq T(X)$,
2. if $t, s \in T(X)$ then the "string" $t \oplus s \in T(X)$, where $\oplus$ is a fundamental operation.

For $t \in T(X)$ we often write t as $t(x_1, \ldots, x_n)$ to indicate that the variables occurring in t are among $x_1, \ldots, x_n$. The *term algebra* or *absolutely free algebra over* X is the algebra $\mathbf{T}(X) = \langle T(X), \wedge, \vee, \cdot, \to, \top, \bot, 0, 1\rangle$.

Let $\mathcal{K}$ be a class of FL_e-algebras and $\mathbf{U}(X)$ an FL_e-algebra generated by X. Then, we say that $\mathbf{U}(X)$ has the *universal mapping property for* $\mathcal{K}$ *over* X, if for every $\mathbf{A} \in \mathcal{K}$ and for every map $\alpha : X \to A$ there is a homomorphism $\beta : \mathbf{U}(X) \to \mathbf{A}$ such that $\beta(x) = \alpha(x)$ for any $x \in X$. Note that for any set X of variables and class $\mathcal{K}$ of FL_e-algebras, the term algebra $\mathbf{T}(X)$ has the universal mapping property for $\mathcal{K}$ over X.

Let $\mathcal{K}$ be a class of FL_e-algebras and $\mathbf{T}(X)$ the term algebra over X. Define a congruence $\theta_{\mathcal{K}}(X)$ on $\mathbf{T}(X)$ by $\theta_{\mathcal{K}}(X) = \bigcap \Phi_{\mathcal{K}}(X)$, where

$$\Phi_{\mathcal{K}}(X) = \{\phi \in Con\mathbf{T}(X) \mid \mathbf{T}(X)/\phi \in IS(K)\}.$$

Then we say that $\mathbf{F}_{\mathcal{K}}(\bar{X})$ is *the $\mathcal{K}$-free algebra over $\bar{X}$*, if it is the quotient algebra of $\mathbf{T}(X)$ by $\theta_{\mathcal{K}}(X)$, i.e., $\mathbf{F}_{\mathcal{K}}(\bar{X}) = \mathbf{T}(X)/\theta_{\mathcal{K}}(X)$, where $\bar{X} = X/\theta_{\mathcal{K}}(X)$. For $x \in X$ we write $\bar{x}$ for $x/\theta_{\mathcal{K}}(X)$.

PROPOSITION 3.14 (Birkhoff) *Let $\mathcal{K}$ be a class of FL_e-algebras and $\mathbf{F}_{\mathcal{K}}(\bar{X})$ the $\mathcal{K}$-free algebra over $\bar{X}$. Then $\mathbf{F}_{\mathcal{K}}(\bar{X})$ has the universal mapping property for $\mathcal{K}$ over $\bar{X}$.*

(proof) Given $\mathbf{A} \in \mathcal{K}$, let α be a map from $\bar{X}$ to A. Let $\nu : \mathbf{T}(X) \to \mathbf{F}_{\mathcal{K}}(\bar{X})$ be the natural homomorphism. Then $\alpha \circ \nu$ maps X into A, so by the universal mapping property of $\mathbf{T}(X)$ there is a homomorphism $\mu : \mathbf{T}(X) \to \mathbf{A}$ such that $\mu(x) = \alpha \circ \nu(x)$ for any $x \in X$. From the definition of $\theta_{\mathcal{K}}(X)$, it is clear that $\theta_{\mathcal{K}}(X) \subseteq Ker(\mu)$ since $Ker(\mu) \in \Phi_{\mathcal{K}}(X)$. Thus there is a homomorphism $\beta : \mathbf{F}_{\mathcal{K}}(\bar{X}) \to \mathbf{A}$ such that $\mu = \beta \circ \nu$. Then, for any $x \in X$,

$$\begin{aligned} \beta(\bar{x}) &= \beta \circ \nu(x) \\ &= \mu(x) \\ &= \alpha \circ \nu(x) \\ &= \alpha(\bar{x}), \end{aligned}$$

so $\beta(x) = \alpha(x)$. Hence $\mathbf{F}_{\mathcal{K}}(\bar{X})$ has the universal mapping property for $\mathcal{K}$ over $\bar{X}$. □

PROPOSITION 3.15 *If $\mathcal{K}$ is a class of FL_e-algebras and $\mathbf{A} \in \mathcal{K}$, then for sufficiently large X, $\mathbf{A} \in H(\mathbf{F}_{\mathcal{K}}(\bar{X}))$.*

(proof) Choose $|X| \geq |A|$ and let $\alpha : \bar{X} \to A$ be a surjection. Then, by Proposition 3.14, there is an onto homomorphism $\beta : \mathbf{F}_{\mathcal{K}}(\bar{X}) \to \mathbf{A}$. □

PROPOSITION 3.16 (Birkhoff) *Let $\mathcal{K}$ be a class of FL_e-algebras. Then $\mathbf{F}_{\mathcal{K}}(\bar{X}) \in ISP(\mathcal{K})$. In particular, if $\mathcal{K}$ is a variety then $\mathbf{F}_{\mathcal{K}}(\bar{X}) \in \mathcal{K}$.*

(proof) Since $\theta_{\mathcal{K}}(X) = \bigcap \Phi_{\mathcal{K}}(X)$,

$$\mathbf{F}_{\mathcal{K}}(\bar{X}) = \mathbf{T}(X)/\theta_{\mathcal{K}}(X) \in IP_S(\{\mathbf{T}(X)/\phi \mid \phi \in \Phi_{\mathcal{K}}(X)\}),$$

so $\mathbf{F}_{\mathcal{K}}(\bar{X}) \in IP_S IS(\mathcal{K})$. Note that $P_S \leq SP$. Thus, by Proposition 3.12, $\mathbf{F}_{\mathcal{K}}(\bar{X}) \in ISP(\mathcal{K})$. □

An *equation* over a variable set X is a pair of terms of $T(X)$. If t, s are terms we write $t \approx s$ for the equation defined by them. We say that an equation $t(x_1, \ldots, x_n) \approx s(x_1, \ldots x_n)$ over X is *valid* in an FL_e-algebra $\mathbf{A}$, or it is *satisfied* by $\mathbf{A}$, in symbols $\mathbf{A} \models t \approx s$, if for any $a_1, \ldots, a_n \in A$, $t(a_1, \ldots, a_n) = s(a_1, \ldots, a_n)$. The notion of validity is extended to classes of algebras and sets of equations. A set $\mathcal{E}$ of equations is said to be valid in a class $\mathcal{K}$ of FL_e-algebras, in symbols $\mathcal{K} \models \mathcal{E}$, if every equation of $\mathcal{E}$ is valid in every algebra of $\mathcal{K}$. For a class $\mathcal{K}$ of FL_e-algebras and a set X of variables, let $E_{\mathcal{K}}(X) = \{t \approx s \mid \mathcal{K} \models t \approx s \text{ for } t, s \in T(X)\}$.

PROPOSITION 3.17 *Let $\mathcal{K}$ be a class of FL_e-algebras and $t \approx s$ an equation over X. Then $\mathcal{K} \models t \approx s$ if and only if for every $\mathbf{A} \in \mathcal{K}$ and for every homomorphism $\alpha : \mathbf{T}(X) \to \mathbf{A}$, $\alpha t = \alpha s$.*

(proof) ($\Rightarrow$) Let $t = t(x_1, \ldots, x_n), s = s(x_1, \ldots, x_n)$, $\mathbf{A} \in \mathcal{K}$ and α a homomorphism from $\mathbf{T}(X)$ to $\mathbf{A}$. By our assumption, $t(a_1, \ldots, a_n) = s(a_1, \ldots, a_n)$ for any $a_1, \ldots, a_n \in A$. Thus,

$$\begin{array}{rc} & t(\alpha(x_1), \ldots, \alpha(x_n)) = s(\alpha(x_1), \ldots, \alpha(x_n)) \\ \Rightarrow & \alpha t(x_1, \ldots, x_n) = \alpha s(x_1, \ldots, x_n) \\ \Rightarrow & \alpha t = \alpha s. \end{array}$$

($\Leftarrow$) Choose $\mathbf{A} \in \mathcal{K}$ and $a_1, \ldots, a_n \in A$. By the universal mapping property of $\mathbf{T}(X)$, there is a homomorphism $\alpha : \mathbf{T}(X) \to \mathbf{A}$ such that $\alpha(x_i) = a_i$ $(1 \leq i \leq n)$. Then,

$$\begin{aligned} t(a_1, \ldots, a_n) &= t(\alpha(x_1), \ldots, \alpha(x_n)) \\ &= \alpha t \\ &= \alpha s \\ &= s(\alpha(x_1), \ldots, \alpha(x_n)) \\ &= s(a_1, \ldots, a_n), \end{aligned}$$

thus $\mathcal{K} \models t \approx s$. □

PROPOSITION 3.18 *For any class $\mathcal{K}$ of FL_e-algebras, all of the classes $\mathcal{K}$, $I(\mathcal{K})$, $S(\mathcal{K})$, $H(\mathcal{K})$, $P(\mathcal{K})$ and $V(\mathcal{K})$ satisfy the same equations over any set of variables X.*

(proof) Clearly $\mathcal{K}$ and $I(\mathcal{K})$ satisfy the same equations. Since $I \leq IS, I \leq H$, and $I \leq IP$, it must be $E_{\mathcal{K}}(X) \supseteq E_{S(\mathcal{K})}(X)$, $E_{\mathcal{K}}(X) \supseteq E_{H(\mathcal{K})}(X)$ and $E_{\mathcal{K}}(X) \supseteq E_{P(\mathcal{K})}(X)$.

Suppose that $\mathcal{K} \models t(x_1, \ldots, x_n) \approx s(x_1, \ldots, x_n)$. If $\mathbf{B} \leq \mathbf{A} \in \mathcal{K}$ and $b_1, \ldots, b_n \in B$ then $t(b_1, \ldots, b_n) = s(b_1, \ldots, b_n)$ since $b_1, \ldots, b_n \in A$ and $\mathbf{A} \models t \approx s$. Thus $\mathbf{B} \models t \approx s$, so $E_{\mathcal{K}}(X) = E_{S(\mathcal{K})}(X)$. Suppose that $\mathbf{A} \in \mathcal{K}$ and $\alpha : \mathbf{A} \to \mathbf{B}$ is an onto homomorphism. Then for any $b_1, \ldots, b_n \in B$ there exist $a_1, \ldots, a_n \in A$ such that $\alpha(a_i) = b_i$ $(1 \leq i \leq n)$. Since $t(a_1, \ldots, a_n) = s(a_1, \ldots, a_n)$, $t(\alpha(a_1), \ldots, \alpha(a_n)) = s(\alpha(a_1), \ldots, \alpha(a_n))$. Hence $t(b_1, \ldots, b_n) = s(b_1, \ldots, b_n)$. Thus $E_{\mathcal{K}}(X) = E_{H(\mathcal{K})}(X)$. Suppose $\mathbf{A}_i \in \mathcal{K}$ for $i \in I$. Then for $a_1, \ldots a_n \in \Pi_{i \in I} \mathbf{A}_i$, $t(a_1(i), \ldots, a_n(i)) = s(a_1(i), \ldots, a_n(i))$. Hence for $i \in I$, $t(a_1, \ldots, a_n)(i) = s(a_1, \ldots, a_n)(i)$, so $\Pi_{i \in I} \mathbf{A}_i \models t \approx s$. Thus $E_{\mathcal{K}}(X) = E_{P(\mathcal{K})}(X)$. Since $V = HSP$, $E_{\mathcal{K}}(X) = E_{V(\mathcal{K})}(X)$. □

PROPOSITION 3.19 *Let $\mathcal{K}$ be a class of FL_e-algebras and t, s terms over X. Then,*

$$\begin{array}{rl} & \mathcal{K} \models t \approx s \\ \Leftrightarrow & \mathbf{F}_{\mathcal{K}}(\bar{X}) \models t \approx s \\ \Leftrightarrow & (t, s) \in \theta_{\mathcal{K}}(X). \end{array}$$

(proof) By Proposition 3.16 and 3.18, it is easy to see that $\mathcal{K} \models t \approx s$ implies $\mathbf{F}_{\mathcal{K}}(\bar{X}) \models t \approx s$. Let $t = t(x_1, \ldots, x_n), s = s(x_1, \ldots, x_n)$ and $\nu : \mathbf{T}(X) \to \mathbf{F}_{\mathcal{K}}(\bar{X})$ the natural homomorphism. Suppose that $\mathbf{F}_{\mathcal{K}}(\bar{X}) \models t \approx s$. Then, clearly, $t(\bar{x}_1, \ldots, \bar{x}_n) = s(\bar{x}_1, \ldots, \bar{x}_n)$. Hence,

$$\nu(t) = t(\bar{x}_1, \ldots, \bar{x}_n) = s(\bar{x}_1, \ldots, \bar{x}_n) = \nu(s),$$

so $(t, s) \in Ker(\nu) = \theta_{\mathcal{K}}(X)$. Finally, suppose $(t, s) \in \theta_{\mathcal{K}}(X)$. For $\mathbf{A} \in \mathcal{K}$ and $a_1, \ldots, a_n \in A$, let $\alpha : \mathbf{T}(X) \to \mathbf{A}$ be a homomorphism such that $\alpha(x_i) = a_i$ $(1 \leq i \leq n)$. Since

$\theta_{\mathcal{K}}(X) = Ker(\nu) \subseteq Ker(\alpha)$, there is a homomorphsim $\beta : \mathbf{F}_{\mathcal{K}}(\bar{X}) \to \mathbf{A}$ such that $\alpha = \beta \circ \nu$. Then,

$$\alpha(t) = \beta \circ \nu(t) = \beta \circ \nu(s) = \alpha(s).$$

Thus, by Proposition 3.17, $\mathcal{K} \models t \approx s$. □

PROPOSITION 3.20 *Let $\mathcal{K}$ be a class of FL_e-algebras and $t, s \in T(X)$. Then for any set of variables Y with $|Y| \geq |X|$,*

$$\mathcal{K} \models t \approx s \Leftrightarrow \mathbf{F}_{\mathcal{K}}(\bar{Y}) \models t \approx s.$$

(proof) ($\Rightarrow$) It is obvious since $\mathbf{F}_{\mathcal{K}}(\bar{Y}) \in ISP(\mathcal{K})$.

($\Leftarrow$) Let X_0 be a set of variables which satisfies $X_0 \supseteq X$ and $|X_0| = |Y|$. Then $\mathbf{F}_{\mathcal{K}}(\bar{X}_0) \simeq \mathbf{F}_{\mathcal{K}}(\bar{Y})$. By Proposition 3.19,

$$\mathcal{K} \models t \approx s \Leftrightarrow \mathbf{F}_{\mathcal{K}}(\bar{X}_0) \models t \approx s,$$

hence

$$\mathcal{K} \models t \approx s \Leftrightarrow \mathbf{F}_{\mathcal{K}}(\bar{Y}) \models t \approx s.$$

□

PROPOSITION 3.21 *Let $\mathcal{K}$ be a class of FL_e-algebras and X a set of variables. Then for any infinite set of variables Y,*

$$E_{\mathcal{K}}(X) = E_{\mathbf{F}_{\mathcal{K}}(\bar{Y})}(X).$$

(proof) For any $t \approx s \in E_{\mathcal{K}}(X)$, there exist a subset $\{x_1, \ldots, x_n\}$ of X such that $t = t(x_1, \ldots, x_n), s = s(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n), s(x_1, \ldots, x_n) \in T(\{x_1, \ldots, x_n\})$. Since $|\{x_1, \ldots, x_n\}| < |Y|$, by Proposition 3.20, $\mathcal{K} \models t \approx s \Leftrightarrow \mathbf{F}_{\mathcal{K}}(\bar{Y}) \models t \approx s$. □

Let $\mathcal{E}$ be a set of equations, and define $Mod(\mathcal{E})$ to be the class of all FL_e-algebras satisfying $\mathcal{E}$. A class $\mathcal{K}$ of FL_e-algebras is an *equational class* if there is a set $\mathcal{E}$ of equations such that $\mathcal{K} = Mod(\mathcal{E})$. In this case, we say that $\mathcal{K}$ is *axiomatized* by $\mathcal{E}$.

PROPOSITION 3.22 *If $\mathcal{V}$ is a variety and X is an infinite set of variables then $\mathcal{V} = Mod(E_{\mathcal{V}}(X))$.*

(proof) Let $\mathcal{V}' = Mod(E_{\mathcal{V}}(X))$. Clearly $\mathcal{V}' \supseteq \mathcal{V}$ and $\mathcal{V}'$ is a variety by Proposition 3.18. Moreover, $E_{\mathcal{V}'}(X) = E_{\mathcal{V}}(X)$ since for any $t \approx s \in E_{\mathcal{V}'}(X)$, $\mathcal{V}' \models t \approx s$ implies $\mathcal{V} \models t \approx s$. Thus for any equation $t \approx s$,

$$\mathcal{V}' \models t \approx s \Leftrightarrow \mathcal{V} \models t \approx s,$$

so by Proposition 3.19, $\theta_{\mathcal{V}'}(X) = \theta_{\mathcal{V}}(X)$, namely, $\mathbf{F}_{\mathcal{V}'}(\bar{X}) = \mathbf{F}_{\mathcal{V}}(\bar{X})$. Let Y be an infinite set of variables. Then by Proposition 3.21,

$$E_{\mathcal{V}'}(Y) = E_{\mathbf{F}_{\mathcal{V}'}(\bar{X})}(Y) = E_{\mathbf{F}_{\mathcal{V}}(\bar{X})}(Y) = E_{\mathcal{V}}(Y)$$

since X is also infinite. Thus again by Proposition 3.19, $\theta_{\mathcal{V}'}(Y) = \theta_{\mathcal{V}}(Y)$, so $\mathbf{F}_{\mathcal{V}'}(\bar{Y}) = \mathbf{F}_{\mathcal{V}}(\bar{Y})$. Let $\mathbf{A} \in \mathcal{V}'$. Then by Proposition 3.15, for suitable infinite Y,

$$\mathbf{A} \in H(\mathbf{F}_{\mathcal{V}'}(\bar{Y})) = H(\mathbf{F}_{\mathcal{V}}(\bar{Y})).$$

Therefore $\mathcal{V}' = \mathcal{V}$. □

PROPOSITION 3.23 (Birkhoff) *$\mathcal{K}$ is an equational class if and only if $\mathcal{K}$ is a variety.*

(proof) ($\Rightarrow$) Suppose that $\mathcal{K} = Mod(\mathcal{E})$. Then, by Proposition 3.18, $V(\mathcal{K}) \models \mathcal{E}$. Hence,

$$V(\mathcal{K}) \subseteq Mod(\mathcal{E}) = \mathcal{K} \subseteq V(\mathcal{K}),$$

so $\mathcal{K}$ is a variety.

($\Leftarrow$) Let $\mathcal{K}$ be a variety and X an infinite set of variables. Then by Proposition 3.22, $\mathcal{K} = Mod(E_{\mathcal{K}}(X))$. □

An FL_e-algebra $\mathbf{A}$ is *congruence-distributive* if $\mathbf{Con}(\mathbf{A})$ is a distributive lattice. A class $\mathbf{K}$ of FL_e-algebras is *congruence-distributive* if every algebra in $\mathcal{K}$ is congruence-distributive. It is well-known that every variety of FL_e-algebras is congruence-distributive. We denote the class of ultraproducts of members of $\mathcal{K}$ by $P_U(\mathcal{K})$. The following is a celebrated result due to B. Jónsson, known as *Jónsson's Lemma.*

PROPOSITION 3.24 (Jónsson's Lemma) *Let $V(\mathcal{K})$ be a congruence-distributive variety. If $\mathbf{A}$ is a subdirectly irreducible algebra in $V(\mathcal{K})$ then*

$$\mathbf{A} \in HSP_U(\mathcal{K}).$$

(proof) Suppose that $\mathbf{A}$ is a non-degenerate subdirectly irreducible algebra in $V(\mathcal{K})$. Then for some $\mathbf{A}_i \in \mathcal{K}$ ($i \in I$), and for some $\mathbf{B} \leq \Pi_{i\in I}\mathbf{A}_i$ there is an onto homomorphism $\alpha : \mathbf{B} \to \mathbf{A}$. Let $\theta = Ker(\alpha)$ and for $J \subseteq I$,

$$\theta_J = \{(a, b) \in (\Pi_{i\in I}\mathbf{A}_i)^2 \mid J \subseteq \{i \in I | a(i) = b(i)\}\}.$$

It is easy to see that for any $J(\subseteq I)$, θ_J is a congruence on $\Pi_{i\in I}\mathbf{A}_i$. Let $\theta_J\restriction_B = \theta_J \cap B^2$ be the restriction of θ_J to $\mathbf{B}$ and

$$W = \{J \subseteq I | \theta_J\restriction_B \subseteq \theta\}.$$

Clearly $I \in W$ and $\emptyset \notin W$. If $J \in W$ and $J \subseteq K \subseteq I$ then $\theta_K\restriction_B \subseteq \theta$ since $\theta_K\restriction_B \subseteq \theta_J\restriction_B$. So $K \in W$. Suppose that $J_1 \cup J_2 \in W$, i.e., $\theta_{J_1\cup J_2}\restriction_B \subseteq \theta$. As $\theta_{J_1\cup J_2} = \theta_{J_1} \cap \theta_{J_2}$, it follows $\theta_{J_1\cup J_2}\restriction_B = \theta_{J_1}\restriction_B \cap \theta_{J_2}\restriction_B$. Since $\theta = \theta \vee (\theta_{J_1}\restriction_B \cap \theta_{J_2}\restriction_B)$, by distributitity of congruences,

$$\theta = (\theta \vee \theta_{J_1}\restriction_B) \cap (\theta \vee \theta_{J_2}\restriction_B).$$

Thus $\theta = \theta \vee \theta_{J_i}\restriction_B$ for $i = 1$ or 2 since $\mathbf{B}/\theta \simeq \mathbf{A}$ is subdirectly irreducible. Hence $\theta_{J_i}\restriction_B \subseteq \theta$ for $i = 1$ or 2, so either J_1 or J_2 is in W. By Proposition 3.10, there is an ultrafilter $\mathcal{U}$ in W. From the definition of W,

$$\theta_{\mathcal{U}}\restriction_B \subseteq \theta,$$

where $\theta_{\mathcal{U}} = \bigvee\{\theta_J | J \in \mathcal{U}\}$. Let ν be the natural homomorphism from $\Pi_{i\in I}\mathbf{A}_i$ to $\Pi_{i\in I}\mathbf{A}_i/\mathcal{U}$. Then let

$$\beta : \mathbf{B} \to \nu(\mathbf{B})$$

be the restriction of ν to $\mathbf{B}$. Since $Ker(\beta) = \theta_{\mathcal{U}}\restriction_B \subseteq \theta$, $\mathbf{A} \simeq \mathbf{B}/\theta \simeq (\mathbf{B}/Ker(\beta))/(\theta/Ker(\beta))$. Now

$$\mathbf{B}/Ker(\beta) \simeq \nu(\mathbf{B}) \leq \Pi_{i\in I}\mathbf{A}_i/\mathcal{U},$$

so

$$\mathbf{B}/Ker(\beta) \in ISP_U(\mathcal{K}),$$

hence $\mathbf{A} \in HSP_U(\mathcal{K})$. □

Let $\mathcal{FL}_e$ be the variety of all FL_e-algebras and $\mathcal{V}_{tri}$ the variety which contains only trivial algebras. We define $sub\mathcal{V} = \{\mathcal{V} \mid \mathcal{V}$ is a subvariety of $\mathcal{FL}_e\}$. Note that if $\mathcal{V}_i \in sub\mathcal{V}$ $(i \in I)$ then $\bigcap_{i\in I} \mathcal{V}_i \in sub\mathcal{V}$. For any $\mathcal{V}_1, \mathcal{V}_2 \in sub\mathcal{V}$, let $\mathcal{V}_1 \vee \mathcal{V}_2$ be the smallest variety including the class $\mathcal{V}_1 \cup \mathcal{V}_2$. Then, it is not hard to see that $\langle sub\mathcal{V}, \cap, \vee, \mathcal{V}_{tri}, \mathcal{FL}_e\rangle$ forms a complete lattice in which $\mathcal{FL}_e$ and $\mathcal{V}_{tri}$ are the largest and smallest elements, respectively. We call it the *subvariety lattice* of $\mathcal{FL}_e$ or the *lattice of varieties of FL_e-algebras.*

In the following figure, $\mathcal{HA}$ and $\mathcal{BA}$ denote the class of all Heyting algebras and Boolean algebras, respectively.

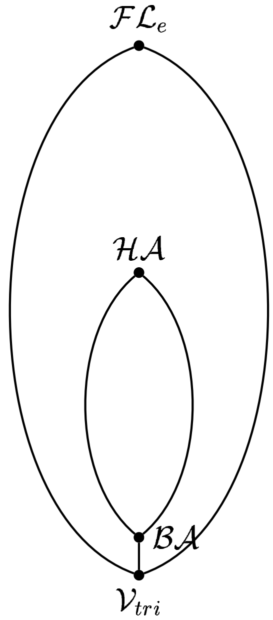

Figure 3.1: the subvariety lattice of $\mathcal{FL}_e$

3.3 Logics over $\mathbf{FL_e}$ and Varieties of FL_e-algebras

Let $\mathbf{A}$ be an FL_e-algebra and X the set of all variables. A *valuation* v on $\mathbf{A}$ is a map from X to A. The notion of valuation is extend to a map from $\mathcal{F}m(X)$ to A as usual. A formula

$\phi(X)$ is *valid* in $\mathbf{A}$, or it is *satisfied* by $\mathbf{A}$, if for every valuation v on $\mathbf{A}$, $v(\phi(X)) \geq 1_{\mathbf{A}}$. Note that a formula ϕ is valid in $\mathbf{A}$ if and only if the equation $\phi \wedge 1 \approx 1$ is valid in $\mathbf{A}$, i.e., $\mathbf{A} \models \phi \wedge 1 \approx 1$. Let $L(\mathbf{A})$ be the set of all formulas which are valid in $\mathbf{A}$.

PROPOSITION 3.25 *For any FL_e-algebra $\mathbf{A}$, $L(\mathbf{A})$ is a logic over $\mathbf{FL_e}$.*

(proof) It is easy to see that $L(\mathbf{A})$ is closed under substitution and $\mathbf{FL_e} \subseteq L(\mathbf{A})$. Suppose that $\phi, \phi \to \psi \in L(\mathbf{A})$. For any valuation v on $\mathbf{A}$,

$$\begin{aligned} & 1_{\mathbf{A}} \leq v(\phi \to \psi) = v(\phi) \to v(\psi) \\ \Rightarrow\ & 1_{\mathbf{A}} \leq v(\phi) \leq v(\psi) \\ \Rightarrow\ & 1_{\mathbf{A}} \leq v(\psi). \end{aligned}$$

Hence $\psi \in L(\mathbf{A})$. Clearly, if $\phi \in L(\mathbf{A})$ then $\phi \wedge 1 \in L(\mathbf{A})$. □

We call $L(\mathbf{A})$ the logic *characterized* by $\mathbf{A}$.

PROPOSITION 3.26 *Let $\mathbf{A}$ be an FL_e-algebra. If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ then $L(\mathbf{A}) \subseteq L(\mathbf{B})$.*

(proof) Suppose that $\phi \notin L(\mathbf{B})$. Then, there is some valuation v on $\mathbf{B}$ such that $v(\phi) \not\geq 1_{\mathbf{B}}$. Since $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, the valuation v is considered as a valuation on $\mathbf{A}$. Thus, $v(\phi) \not\geq 1_{\mathbf{A}}$. □

PROPOSITION 3.27 *Let $\mathbf{A}$ be an FL_e-algebra and θ a congruence on $\mathbf{A}$. Then, $L(\mathbf{A}) \subseteq L(\mathbf{A}/\theta)$.*

(proof) Suppose that there is some formula $\phi(p_1, \ldots, p_n)$ such that $\phi \notin L(\mathbf{A}/\theta)$. Then there exists some valuation v on $\mathbf{A}/\theta$ defined by $v(p_i) = x_i/\theta$ $(1 \leq i \leq n)$ such that

$$v(\phi(p_1, \ldots, p_n)) = \phi(x_1/\theta, \ldots, x_n/\theta) \not\geq 1_{\mathbf{A}/\theta}.$$

Hence $\phi(x_1, \ldots, x_n)/\theta \not\geq 1_{\mathbf{A}/\theta}$, so $\phi(x_1, \ldots, x_n) \not\geq 1_{\mathbf{A}}$. Thus, by the valuation w on $\mathbf{A}$ defined by $w(p_i) = x_1$ for $1 \leq i \leq n$, we have $w(\phi) \not\geq 1_{\mathbf{A}}$. □

The above two results say that for any homomorphism $\alpha : \mathbf{A} \to \mathbf{B}$, if α is an embedding (onto homomorphism) then $L(\mathbf{B}) \subseteq L(\mathbf{A})$ ($L(\mathbf{A}) \subseteq L(\mathbf{B})$, respectively). Clearly, when α is an isomorphism, $L(\mathbf{A}) = L(\mathbf{B})$.

PROPOSITION 3.28 *Let $\mathbf{A}_i$ $(i \in I)$ be an FL_e-algebra. Then,*

$$L(\prod_{i \in I} \mathbf{A}_i) = \bigcap_{i \in I} L(\mathbf{A}_i).$$

(proof) Let ν_i be the i-th coodinate projection. Then ν_i are onto homomorphisms, so $L(\Pi_{i \in I} \mathbf{A}_i) \subseteq L(\mathbf{A}_i)$ for any $i \in I$ by Proposition 3.27. Hence $L(\Pi_{i \in I} \mathbf{A}_i) \subseteq \bigcap_{i \in I} L(\mathbf{A}_i)$. Suppose that $\phi \notin L(\Pi_{i \in I} \mathbf{A}_i)$. Then there is a valuation v on $\Pi_{i \in I} \mathbf{A}_i$ such that $v(\phi) \not\geq 1_{\Pi_{i \in I} \mathbf{A}_i}$, so there is some $j \in I$ such that $v(\phi)(j) \not\geq 1_{\Pi_{i \in I} \mathbf{A}_i}(j)$. Let $w = \nu_j \circ v$. It is easy to see that w is a valuation on $\mathbf{A}_j$ and $w(\phi) \not\geq 1_{\mathbf{A}_j}$. Thus, $\phi \notin L(\mathbf{A}_j)$, therefore

$\phi \notin \bigcap_{i \in I} L(\mathbf{A}_i)$. □

Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$ and T an $\mathcal{L}$-theory of a language $\mathcal{F}m(\mathbf{p})$. We define an equivalence relation $\sim_T$ on the set $\mathcal{F}m(\mathbf{p})$ by

$$\phi \sim_T \psi \;\rightleftharpoons\; \phi \leftrightarrow \psi \in T.$$

It is easy to see that $\sim_T$ is a congruence relation on $\mathcal{F}m(\mathbf{p})$. It allows to define the *Lindenbaum-Tarski* algebra

$$\mathbf{A}(\mathbf{p}, T) = \mathcal{F}m(\mathbf{p})/\sim_T$$

as a quotient algebra of $\mathcal{F}m(\mathbf{p})$. Denote

$$||\phi|| = \{\psi | \phi \sim_T \psi\}$$

for any formula $\phi = \phi(\mathbf{p})$.

PROPOSITION 3.29 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$. Then there is some FL_e-algebra $\mathbf{A}$ such that*

$$\mathcal{L} = L(\mathbf{A}).$$

(proof) Let $\mathbf{A} = \mathcal{F}m/\sim_{\mathcal{L}}$ be the Lindenbaum-Tarski algebra determined by $\mathcal{L}$. We will show that $\mathcal{L} = L(\mathbf{A})$. Note that $\phi \in \mathcal{L}$ if and only if $\phi \wedge 1 \leftrightarrow 1 \in \mathcal{L}$. Let $\phi(p_1, \ldots, p_n) \in \mathcal{L}$ and v a valuation on $\mathbf{A}$ defined by $v(p_i) = ||\psi_i||$ $(1 \leq i \leq n)$. Since $\mathcal{L}$ is a logic, the substitution instance $\phi(\psi_1, \ldots, \psi_n)$ of ϕ is in $\mathcal{L}$. Thus,

$$\begin{array}{ll} & \phi(\psi_1, \ldots, \psi_n) \wedge 1 \leftrightarrow 1 \in \mathcal{L} \\ \Rightarrow & \phi(\psi_1, \ldots, \psi_n) \wedge 1 \sim_{\mathcal{L}} 1 \\ \Rightarrow & ||\phi(\psi_1, \ldots, \psi_n) \wedge 1|| = ||1|| \\ \Rightarrow & \phi(||\psi_1||, \ldots, ||\psi_n||) \wedge 1_{\mathbf{A}} = 1_{\mathbf{A}} \\ \Rightarrow & v(\phi(p_1, \ldots, p_n)) = \phi(v(p_1), \ldots, v(p_n)) \geq 1_{\mathbf{A}}. \end{array}$$

Hence $\phi \in L(\mathbf{A})$. Conversely, let $\phi(p_1, \ldots, p_n) \in L(\mathbf{A})$. Then, for a valuation v on $\mathbf{A}$ determined by $v(p_i) = ||p_i||$ $(1 \leq i \leq n)$,

$$v(\phi(p_1, \ldots, p_n)) = \phi(||p_1||, \ldots, ||p_n||) \geq 1_{\mathbf{A}}.$$

Hence,

$$\begin{array}{ll} & \phi(||p_1||, \ldots, ||p_n||) \wedge 1_{\mathbf{A}} = 1_{\mathbf{A}} \\ \Rightarrow & ||\phi(p_1, \ldots, p_n) \wedge 1|| = ||1|| \\ \Rightarrow & \phi(p_1, \ldots, p_n) \wedge 1 \sim_{\mathcal{L}} 1 \\ \Rightarrow & \phi(p_1, \ldots, p_n) \wedge 1 \leftrightarrow 1 \in \mathcal{L} \\ \Rightarrow & \phi(p_1, \ldots, p_n) \in \mathcal{L}. \end{array}$$

□

For any logic $\mathcal{L}$ over $\mathbf{FL_e}$, we define the class of FL_e-algebras $V(\mathcal{L})$ as follows;

$$V(\mathcal{L}) = Mod(\{\phi \wedge 1 \approx 1 \mid \phi \in \mathcal{L}\}).$$

PROPOSITION 3.30 *For any logic $\mathcal{L}$ over $\mathbf{FL_e}$, the class $V(\mathcal{L})$ is a variety.*

(proof) By definition, $V(\mathcal{L})$ is an equational class. Thus, by Proposition 3.23, $V(\mathcal{L})$ is a variety. □

Conversely, for any variety $\mathcal{V}$ of FL_e-algebras, we define the set of formulas $L(\mathcal{V})$ as follows;

$$L(\mathcal{V}) = \{\phi \in \mathcal{F}m | \mathcal{V} \models \phi \wedge 1 \approx 1\}.$$

PROPOSITION 3.31 *For every variety $\mathcal{V}$ of FL_e-algebras, $L(\mathcal{V})$ is a logic over $\mathbf{FL_e}$.*

(proof) At first, we will show that $L(\mathcal{V})$ is closed under substitution. Let $\phi(p_1, \ldots, p_n) \in L(\mathcal{V})$ and $\phi(\psi_1, \ldots, \psi_n)$ a substitution instance of ϕ. Suppose that $\phi(\psi_1, \ldots, \psi_n) \notin L(\mathcal{V})$. Then there exist some $\mathbf{A} \in \mathcal{V}$ and valuation v on $\mathbf{A}$ such that

$$v(\phi(\psi_1, \ldots, \psi_n)) = \phi(v(\psi_1), \ldots, v(\psi_n)) \ngeq 1_{\mathbf{A}}.$$

Let w be a valuation on $\mathbf{A}$ defined by $w(p_i) = v(\psi_i)$ for $i \in I$. Then,

$$\begin{array}{rcl} w(\phi(p_1, \ldots, p_n)) & = & \phi(w(p_1), \ldots, w(p_n)) \\ & = & \phi(v(\psi_1), \ldots, v(\psi_n)) \\ & \ngeq & 1_{\mathbf{A}}. \end{array}$$

Thus $\phi \notin L(\mathcal{V})$, but this is a contradiction. It is easy to see that if $\phi \in \mathbf{FL_e}$ then for any FL_e-algebra $\mathbf{A}$ and valuation v on $\mathbf{A}$, $v(\phi) \geq 1_{\mathbf{A}}$. Hence $\phi \in L(\mathcal{V})$. Let $\phi, \phi \to \psi \in L(\mathcal{V})$. Then for any $\mathbf{A} \in \mathcal{V}$ and valuation v on $\mathbf{A}$, $1_{\mathbf{A}} \leq v(\phi)$ and $1_{\mathbf{A}} \leq v(\phi \to \psi)$. Thus,

$$\begin{array}{rcl} 1_{\mathbf{A}} \leq v(\phi \to \psi) & \iff & 1_{\mathbf{A}} \leq v(\phi) \to v(\psi) \\ & \iff & v(\phi) \leq v(\psi) \\ & \iff & 1_{\mathbf{A}} \leq v(\psi). \end{array}$$

Therefore $\phi \in L(\mathcal{V})$. Clearly, $\phi \in L(\mathcal{V})$ implies $\phi \wedge 1 \in L(\mathcal{V})$. □

PROPOSITION 3.32 *$\langle CSL, \cap, \vee, \mathbf{FL_e}, \mathcal{F}m\rangle$ and $\langle sub\mathcal{V}, \cap, \vee, \mathcal{V}_{tri}, \mathcal{FL}_e\rangle$ are dually isomorphic, via the mutually inverse dual maps $\mathcal{L} \mapsto V(\mathcal{L})$ and $\mathcal{V} \mapsto L(\mathcal{V})$.*

(proof) First, we will show that the given maps are antitone. Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$. Then $V(\mathcal{L}_2) \models \phi \wedge 1 \approx 1$ for $\phi \in \mathcal{L}_2$. Thus, $V(\mathcal{L}_2) \models \psi \wedge 1 \approx 1$ for every $\psi \in \mathcal{L}_1$, so, $V(\mathcal{L}_2) \subseteq V(\mathcal{L}_1)$. Let $\mathcal{V}_1 \subseteq \mathcal{V}_2$ and $\phi \in L(\mathcal{V}_2)$. Then, $\mathcal{V}_2 \models \phi \wedge 1 \approx 1$, so clearly, $\mathcal{V}_1 \models \phi \wedge 1 \approx 1$. Thus they are antitone. We will show that $\mathcal{L} = L(V(\mathcal{L}))$ and $\mathcal{V} = V(L(\mathcal{V}))$. If $\phi \in \mathcal{L}$ then $V(\mathcal{L}) \models \phi \wedge 1 \approx 1$. Hence, $\phi \in L(V(\mathcal{L}))$. If $\phi \notin \mathcal{L}$ then, by Proposition 3.29, there is some FL_e-algebra $\mathbf{A}$ such that $\mathbf{A} \not\models \phi \wedge 1 \approx 1$ and $\mathbf{A} \in V(\mathcal{L})$. Hence, $V(\mathcal{L}) \not\models \phi \wedge 1 \approx 1$, so $\phi \notin L(V(\mathcal{L}))$. Therefore $\mathcal{L} = L(V(\mathcal{L}))$. Let $\mathbf{A} \in \mathcal{V}$. Then for any $\phi \wedge 1 \approx 1$, if $\mathcal{V} \models \phi \wedge 1 \approx 1$ then $\mathbf{A} \models \phi \wedge 1 \approx 1$. Thus,

$$\begin{array}{rl} \Rightarrow & \mathbf{A} \in Mod(\{\phi \wedge 1 \approx 1 \mid \mathcal{V} \models \phi \wedge 1 \approx 1\}) \\ \Rightarrow & \mathbf{A} \in V(L(\mathcal{V})). \end{array}$$

Conversely, suppose that $\mathbf{A} \in V(L(\mathcal{V}))$. Then, for any $\phi \wedge 1 \approx 1$, $\mathcal{V} \models \phi \wedge 1 \approx 1$ impliess $\mathbf{A} \models \phi \wedge 1 \approx 1$ by definitions of V and L. Note that for any equation $t \approx s$ and FL_e-algebra $\mathbf{B}$,

$$\mathbf{B} \models t \approx s \Leftrightarrow \mathbf{B} \models ((t \to s) \wedge (t \to s)) \wedge 1 \approx 1.$$

Thus,

$$\begin{array}{rl} & \mathcal{V} \models t \approx s \\ \Leftrightarrow & \mathcal{V} \models ((t \to s) \wedge (s \to t)) \wedge 1 \approx 1 \\ \Rightarrow & \mathbf{A} \models ((t \to s) \wedge (s \to t)) \wedge 1 \approx 1 \\ \Leftrightarrow & \mathbf{A} \models t \approx s \\ \Rightarrow & \mathbf{A} \in Mod(\{t \approx s | \mathcal{V} \models t \approx s\}). \end{array}$$

By Proposition 3.22, $\mathcal{V} = Mod(\{t \approx s | \mathcal{V} \models t \approx s\})$, therefore $\mathbf{A} \in \mathcal{V}$. □

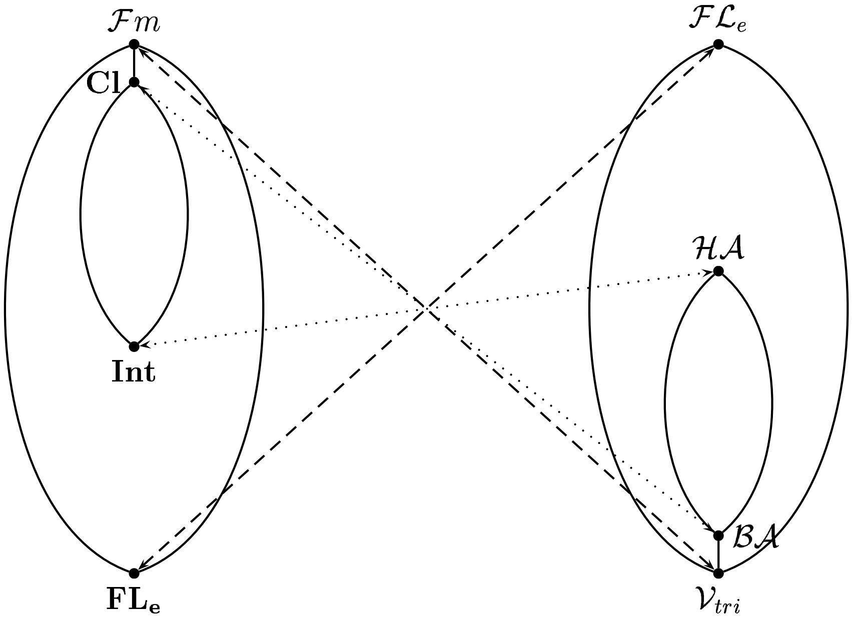

Figure 3.2:

PROPOSITION 3.33 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$. For any $\mathcal{L}$-theory T of the language $\mathcal{F}m(\mathbf{p})$, the Lindenbaum-Tarski algebra $\mathbf{A}(\mathbf{p}, T)$ is in $V(\mathcal{L})$; the canonical mapping α : $\mathcal{F}m \to \mathbf{A}$, i.e., for each formula $\phi = \phi(\mathbf{p})$*

$$\alpha(\phi) = ||\phi|| = \phi / \sim_T,$$

is a homomorphism and, moreover,

$$\phi \in T \Longleftrightarrow ||\phi|| \geq 1_{\mathbf{A}}.$$

(proof) First, we will show that $\mathbf{A}(\mathbf{p}, T) \in V(\mathcal{L})$. Note that for any $\phi(p_1, p_2, \cdots, p_n) \in \mathcal{L}$ and $\psi_i \in \mathcal{F}m(\mathbf{p})$ $(i = 1, 2, \cdots, n)$, $\phi(\psi_1, \psi_2, \cdots, \psi_n)$ is also in $\mathcal{L}$ since $\mathcal{L}$ is closed under substitution. Then

$$\begin{array}{rcl} \phi(\psi_1, \psi_2, \cdots, \psi_n) \in \mathcal{L} & \Rightarrow & \phi(\psi_1, \psi_2, \cdots, \psi_n) \wedge 1 \leftrightarrow 1 \in \mathcal{L} \\ & \Rightarrow & \phi(\psi_1, \psi_2, \cdots, \psi_n) \wedge 1 \leftrightarrow 1 \in T \\ & \Rightarrow & 1_{\mathbf{A}} = ||1|| = ||\phi(\psi_1, \psi_2, \cdots, \psi_n) \wedge 1|| \\ & \Rightarrow & ||\phi(\psi_1, \psi_2, \cdots, \psi_n)|| \geq 1_{\mathbf{A}} \end{array}$$

Let v be any valuation on $\mathbf{A}$. We assume that $v(p_i) = ||\psi_i||$. Then,

$$\begin{aligned} v(\phi(p_1, p_2, \cdots, p_n)) &= \phi(v(p_1), v(p_2), \cdots, v(p_n)) \\ &= \phi(||\psi_1||, ||\psi_2||, \cdots, ||\psi_n||) \\ &= ||\phi(\psi_1, \psi_2, \cdots, \psi_n)||. \end{aligned}$$

Hence, $v(\phi(p_1, p_2, \cdots, p_n)) \geq 1_{\mathbf{A}}$, so $\mathbf{A} \in V(\mathcal{L})$. It is clear that α is a homomorphism since $\sim_T$ is a congruence relation on $\mathcal{F}m(\mathbf{p})$. Next, we will show that $\phi \in T \iff ||\phi|| \geq 1_{\mathbf{A}}$. Note that for any formula ψ, $\psi \in T$ if and only if $\psi \wedge 1 \leftrightarrow 1 \in T$. Thus,

$$\begin{aligned} \phi \in T &\iff \phi \wedge 1 \leftrightarrow 1 \in T \\ &\iff \phi \wedge 1 \sim_T 1 \\ &\iff ||\phi \wedge 1|| = 1_{\mathbf{A}} \\ &\iff ||\phi|| \geq 1_{\mathbf{A}}. \end{aligned}$$

□

PROPOSITION 3.34 *Let T be an $\mathcal{L}$-theory of the language $\mathcal{F}m(\mathbf{p})$, $\phi \in \mathcal{F}m(\mathbf{p})$ and $\phi \notin T$. Then there exists a maximal $\mathcal{L}$-theory T_0 of the language $\mathcal{F}m(\mathbf{p})$ such that $T \subseteq T_0$ and $\phi \notin T_0$. Moreover, $\mathbf{A}(\mathbf{p}, T_0)$ is a subdirectly irreducible algebra and $||\phi||$ is its opremum element.*

(proof) Let

$$\Sigma = \{G : \mathcal{L}\text{-theory} \mid T \subseteq G \text{ and } \phi \notin G\},$$

since $T \in \Sigma$, Σ is not empty. By Zorn's lemma, Σ has a maximal element. So let T_0 be a maximal element of Σ. We will show that $\mathbf{A}(\mathbf{p}, T_0)$ is subdirectly irreducible. It is sufficient to show that $||\phi||$ is an opremum. Note that for any $||\psi|| \in A$, $||\psi|| < 1_{\mathbf{A}}$ implies $\psi \notin T_0$ by Proposition 3.33. So for any $||\psi|| < 1_{\mathbf{A}}$, let

$$T_\psi = \{\sigma \in \mathcal{F}m(\mathbf{p}) \mid T_0, \psi \vdash_{\mathcal{L}} \sigma\}.$$

Then T_ψ is an $\mathcal{L}$-theory of the language $\mathcal{F}m(\mathbf{p})$ which includes T_0. Moreover, since $\psi \notin T_0$ and $\psi \in T_\psi$, T_ψ is strictly bigger than T_0. Since T_0 is maximal with respect to ϕ, T_ψ must contain ϕ, i.e., $T_0, \psi \vdash_{\mathcal{L}} \phi$. Hence, by local deduction theorem, there exists some $n \in \omega$ such that

$$\begin{aligned} T_0 \vdash_{\mathcal{L}} (\psi \wedge 1)^n \to \phi &\Rightarrow (\psi \wedge 1)^n \to \phi \in T_0 \\ &\Rightarrow 1_{\mathbf{A}} \leq ||(\psi \wedge 1)^n \to \phi|| \\ &\Rightarrow 1_{\mathbf{A}} \leq ||(\psi \wedge 1)||^n \to ||\phi|| \\ &\Rightarrow ||(\psi \wedge 1)||^n \leq ||\phi|| \\ &\Rightarrow ||\psi||^n \leq ||\phi||. \end{aligned}$$

□

Chapter 4

Maximal Commutative Logics

In this chapter, we will discuss *maximal* logics over $\mathbf{FL_e}$. By a maximal logic, we mean a logic $\mathcal{L}$ such that $\mathcal{L} \subset \mathcal{F}m$, and $\mathcal{L} \subseteq \mathcal{L}' \subset \mathcal{F}m$ implies $\mathcal{L}' = \mathcal{L}$ for any logic $\mathcal{L}'$ over $\mathbf{FL_e}$. For logics over $\mathbf{FL_{ew}}$ which is obtained from intuitionistic logic $\mathbf{LJ}$ by eliminating the contraction rule, it is well-known that there is only one maximal logic, which is classical logic $\mathbf{Cl}$. But there are many maximal logics over $\mathbf{FL_e}$.

The goal of this chapter is to show that there exist in fact continuum maximal logics over $\mathbf{FL_e}$. In order to show this, we will investigate the subvariety lattice $\langle subV, \cap, \vee, \mathcal{V}_{tri}, \mathcal{FL}_e \rangle$ and show that there are continuum *minimal* varieties of FL_e-algebras. Here, a non-trivial variety $\mathcal{V}$ of FL_e-algebras is mimimal if for any variety $\mathcal{W}$ of FL_e-algebars, $\mathcal{W} \subset \mathcal{V}$ implies $\mathcal{W} = \mathcal{V}_{tri}$, i.e., the subvarieties of $\mathcal{V}$ are exactly $\mathcal{V}_{tri}$ and $\mathcal{V}$ itself.

Before showing our result, we discuss several results related to maximal logics for substructural logics and to minimal varieties for residuated lattices.

As mentioned above, for logics over $\mathbf{FL_{ew}}$ there is exactly one maximal logic, i.e., the classical logic $\mathbf{Cl}$ is the only maximal logic. Hence, for logics over $\mathbf{FL_{ew}}$, *almost maximal logics* have been studied. By an almost maximal logic, we mean a logic $\mathcal{L}$ such that $\mathcal{L} \subset \mathbf{Cl}$, and $\mathcal{L} \subseteq \mathcal{L}' \subset \mathbf{Cl}$ implies $\mathcal{L}' = \mathcal{L}$ for any logic $\mathcal{L}'$ over $\mathbf{FL_{ew}}$. In [41], M. Ueda showed that there exist countably many almost maximal logics over $\mathbf{FL_{ew}}$ (see also [31]). Then, T. Kowalski and M. Ueda extended the result and proved in [21] that there exist *continuum* many almost maximal logics over $\mathbf{FL_{ew}}$.

On the other hand, P. Jipsen and C. Tsinakis showed in [15] that there exist continuum many minimal varieties of non-commutative residuated lattices that satisfy the equation $p^4 \approx p^3$, but not the equation $p^3 \approx p^2$. In [10], N. Galatos improved this result by constructing continuum many minimal varieties of non-commutative residuated lattices that satisfy the idempotent law, i.e., $p^2 \approx p$ and distributivity. Moreover, he showed that there are only two minimal varieties of commutative residuated lattices that satisfy the idempotent law. These results imply that there exist continuum many maximal logics over $\mathbf{FL}$ which is obtained from $\mathbf{LJ}$ by eliminating all of the structural rules.

The following figure represents the lattice of all logics over $\mathbf{FL}$.

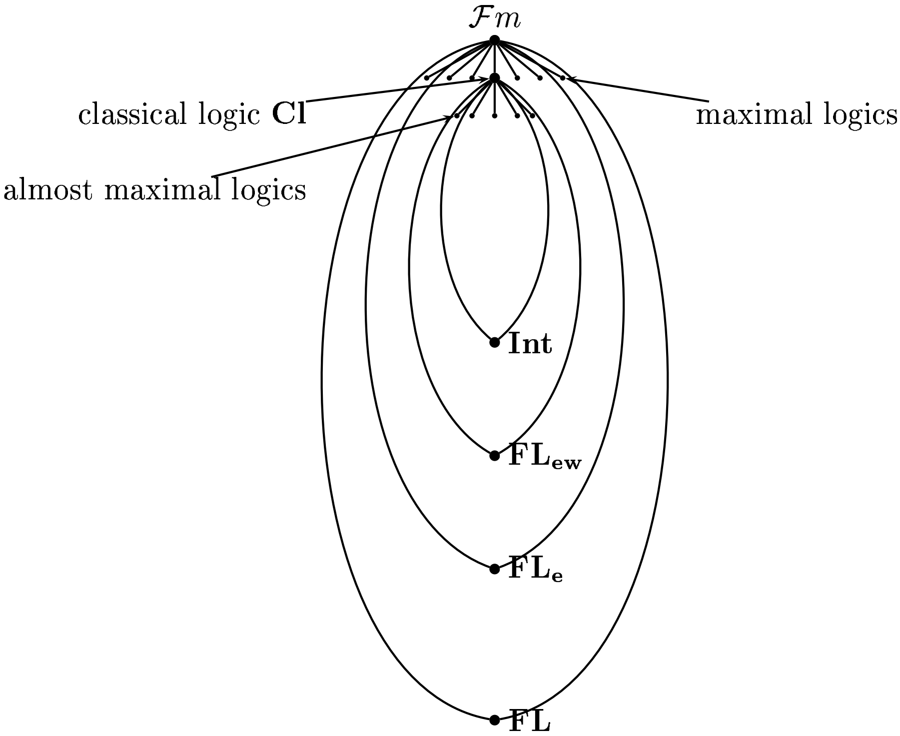

Figure 4.1: the lattice of all logics over **FL**

4.1 Algebras $\mathbf{B}_k$

To explain our basic ideas developed in the next section, we introduce algebras $\mathbf{B}_k = \langle B_k, \wedge, \vee, \cdot, \rightarrow, \top, \bot, 0, 1\rangle$ for $k \in \mathbb{N}$ defined as follows;

(underlying set)
For any $k \in \mathbb{N}$,

$$B_k = \{\top, \bot\} \cup \{0^i \mid 0 \leq i \leq 2k+1\},$$

where 0^0 and 0^1 denote constants 1 and 0, respectively.
(lattice order)

$$\bot < 1 < 0^2 < 0^4 < \cdots < 0^{2k} < \top,$$
$$\bot < 0 < 0^3 < 0^5 \cdots < 0^{2k+1} < \top.$$

(monoid operation)

$$\bot \cdot x = x \cdot \bot = \bot \quad \forall x \in B_k.$$

$$\top \cdot x = x \cdot \top = \top \quad \forall x \in B_k \text{ with } x \neq \bot.$$

$$0^m \cdot 0^n = \begin{cases} 0^{m+n} & \text{if} \quad m+n \leq 2k+1, \\ 0^{2k} & \text{if} \quad m+n > 2k+1 \text{ and } m+n \text{ is even}, \\ 0^{2k+1} & \text{if} \quad m+n > 2k+1 \text{ and } m+n \text{ is odd}. \end{cases}$$

(residuation)

$$x \to \top = \top \qquad\qquad \bot \to x = \top$$

$$\top \to x = \begin{cases} \top & \text{if } x = \top \\ \bot & \text{otherwise} \end{cases} \qquad x \to \bot = \begin{cases} \top & \text{if } \ x = \bot \\ \bot & \text{otherwise} \end{cases}$$

$$1 \to x = x$$

$$x \to 1 = \begin{cases} \top & \text{if } \ x = \bot \\ 1 & \text{if } \ x = 1 \\ \bot & \text{otherwise} \end{cases}$$

$$0^{2i} \to 0^{2l} = \begin{cases} 0^{2k} & \text{if } \ l = k \\ 0^{2(l-i)} & \text{if } \ i \leq l < k \\ \bot & \text{if } \ l < i \end{cases}$$

$$0^{2i} \to 0^{2l+1} = \begin{cases} 0^{2k+1} & \text{if } \ l = k \\ 0^{2(l-i)+1} & \text{if } \ i \leq l < k \\ \bot & \text{if } \ l < i \end{cases}$$

$$0^{2i+1} \to 0^{2l+1} = \begin{cases} 0^{2k} & \text{if } \ l = k \\ 0^{2(l-i)} & \text{if } \ i \leq l < k \\ \bot & \text{if } \ l < i \end{cases}$$

$$0^{2i+1} \to 0^{2l} = \begin{cases} 0^{2k+1} & \text{if } \ l = k \\ 0^{2(l-i)-1} & \text{if } \ i < l < k \\ \bot & \text{if } \ l \leq i \end{cases}$$

The following are some examples of algebras $\mathbf{B}_k$.

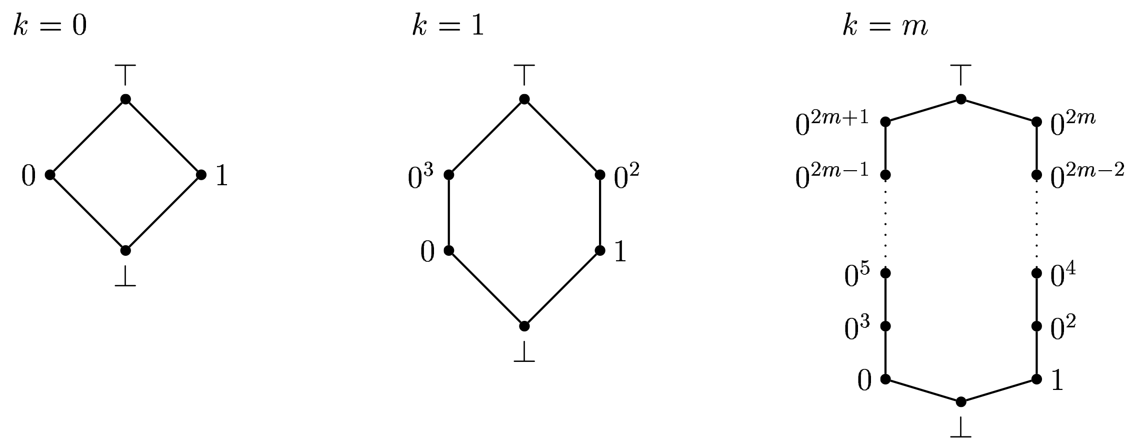

Figure 4.2: algebras $\mathbf{B}_k$

LEMMA 4.1 *For any $k \in \mathbb{N}$, $\mathbf{B}_k$ is a simple FL_e-algebra.*

(proof) It is a routine work to check that $\mathbf{B}_k$ is an FL_e-algebra. We will show that it is simple. Suppose that there exist some elements $x, y \in B_k$ and congruence θ on $\mathbf{B}_k$ such that $x \neq y$ and $(x, y) \in \theta$. Then, without loss of generality, we can assume that either $x < y$, or both $x \not< y$ and $y \not< x$.

($x < y$) Then $y \to x = \bot$ and $y \to y$ is $\top, 1$ or 0^{2k}. So,

$$\begin{aligned} (x,y) \in \theta &\Rightarrow (y \to x, y \to y) \in \theta \\ &\Rightarrow ((y \to x) \wedge 1, (y \to y) \wedge 1) \in \theta \\ &\Rightarrow (\bot, 1) \in \theta. \end{aligned}$$

($x \not< y$ and $y \not< x$) Let $x \in \{0^{2i} | 0 \leq i \leq k\}$ and $y \in \{0^{2j+1} | 0 \leq j \leq k\}$. Then,

$$(x,y) \in \theta \Rightarrow (x \wedge 1, y \wedge 1) \in \theta \Rightarrow (1, \bot) \in \theta.$$

Hence, in each cases, we obtain $(\bot, 1) \in \theta$. For any elements $z_1, z_2 \in B_k$,

$$\begin{aligned} (\bot \cdot z_1, 1 \cdot z_1), (\bot \cdot z_2, 1 \cdot z_2) \in \theta &\Rightarrow (\bot, z_1), (\bot, z_2) \in \theta \\ &\Rightarrow (z_1, z_2) \in \theta. \end{aligned}$$

Thus, $\theta = \nabla$. □

LEMMA 4.2 *For any $k \in \mathbb{N}$, $V(\mathbf{B}_k)$ is a minimal variety.*

(proof) Suppose that $\mathcal{V}$ is a nontrivial variety such that $\mathcal{V} \subseteq V(\mathbf{B}_k)$ and $\mathbf{A}$ is a non-degenerate subdirectly irreducible algebra in $\mathcal{V}$. Then $\mathbf{A} \in V(\mathbf{B}_k)$, hence, by Jónsson's Lemma, $\mathbf{A} \in HSPu(\mathbf{B}_k)$. Since $\mathbf{B}_k$ is a finite algebra, by Proposition 3.11 , every algebra in $Pu(\mathbf{B}_k)$ is isomorphic to $\mathbf{B}_k$. Hence,

$$\mathbf{A} \in HS(\mathbf{B}_k).$$

Note that $\mathbf{B}_k$ has no subalgebras other than $\mathbf{B}_k$ itself since every element in B_k is generated by the constants $\{\top, \bot, 0, 1\}$. Moreover, by Lemma 4.1, $\mathbf{B}_k$ is simple, so $\mathbf{A}$ must be isomorphic to $\mathbf{B}_k$. Thus, $\mathbf{B}_k \cong \mathbf{A} \in \mathcal{V}$. Hence $\mathcal{V} = V(\mathbf{B}_k)$. □

THEOREM 4.3 *There exist at least countably many minimal varieties of FL_e-algebras.*

(proof) By Lemma 4.1 and 4.2, $\mathbf{B}_k$ is simple, so it is subdirectly irreducible and $V(\mathbf{B}_k)$ is minimal variety. Moreover, as shown in proof of Lemma 4.2, if $V(\mathbf{B}_{k_1}) = V(\mathbf{B}_{k_2})$ then $\mathbf{B}_{k_1} \cong \mathbf{B}_{k_2}$, i.e., $k_1 = k_2$. Thus, for each $k_1, k_2 \in \mathbb{N}$ with $k_1 \neq k_2$, $V(\mathbf{B}_{k_1}) \neq V(\mathbf{B}_{k_2})$. □

4.2 Algebras $\mathbf{B}_k^S$ and $\mathbf{B}_S$

By extending the idea of the previous section, we will show in this section that there exist continuum minimal varieties of FL_e-algebras. This implies that there exist continuum maximal logics over $\mathbf{FL_e}$. At first, we will introduce algebras $\mathbf{B}_k^S$ which are obtained from $\mathbf{B}_k$ by adding new elements $\star_i$ ($i \in S$) for any subset S of natural number $\mathbb{N}$. More precisely, see below;

For any subset S of $\mathbb{N}$ and $k \in \mathbb{N}$, we define an algebra $\mathbf{B}_k^S = \langle B_k^S, \wedge, \vee, \cdot, \to, \top, \bot, 0, 1\rangle$ by the following;

(underlying set)

$$B_k^S = \{\top, \bot\} \cup \{0^i \mid 0 \le i \le 2k+1\} \cup \{\star_i \mid 0 < i < k, i \in S\},$$

where 0^0 and 0^1 denote constants 1 and 0, respectively.

(lattice order)

$$\bot < 1 < 0^2 < 0^4 < \cdots < 0^{2k} < \top$$
$$\bot < 0 < \star_1 < 0^3 < \star_2 < \cdots < 0^{2i-1} < \star_i < 0^{2i+1} < \cdots < 0^{2k+1} < \top.$$

(monoid operation)

$$\bot \cdot x = x \cdot \bot = \bot \quad \forall x \in B_k.$$

$$\top \cdot x = x \cdot \top = \top \quad \forall x \in B_k \text{ with } x \neq \bot.$$

$$0^m \cdot 0^n = \begin{cases} 0^{m+n} & \text{if } m+n \le 2k+1, \\ 0^{2k} & \text{if } m+n > 2k+1 \text{ and } m+n \text{ is even}, \\ 0^{2k+1} & \text{if } m+n > 2k+1 \text{ and } m+n \text{ is odd.} \end{cases}$$

$$\star_l \cdot 0^m = 0^m \cdot \star_l = \begin{cases} \star_l & \text{if } m = 0, \\ 0^{2l-1+m} & \text{if } 2l-1+m \le 2k+1, \\ 0^{2k} & \text{if } 2l-1+m > 2k+1 \text{ and } 2l-1+m \text{ is even}, \\ 0^{2k+1} & \text{if } 2l-1+m > 2k+1 \text{ and } 2l-1+m \text{ is odd.} \end{cases}$$

$$\star_l \cdot \star_m = \begin{cases} 0^{2l-1+2m-1} & \text{if } 2l-1+2m-1 \le 2k+1, \\ 0^{2k} & \text{if } 2l-1+2m-1 > 2k+1 \text{ and } 2l-1+2m-1 \text{ is even}, \\ 0^{2k+1} & \text{if } 2l-1+2m-1 > 2k+1 \text{ and } 2l-1+2m-1 \text{ is odd.} \end{cases}$$

(residuation)

$$x \to \top = \top \qquad \bot \to x = \top$$

$$\top \to x = \begin{cases} \top & \text{if } x = \top \\ \bot & \text{otherwise} \end{cases} \qquad x \to \bot = \begin{cases} \top & \text{if } x = \bot \\ \bot & \text{otherwise} \end{cases}$$

$$1 \to x = x$$

$$x \to 1 = \begin{cases} \top & \text{if } x = \bot \\ 1 & \text{if } x = 1 \\ \bot & \text{otherwise} \end{cases}$$

$$0^{2i} \to 0^{2l} = \begin{cases} 0^{2k} & \text{if } l = k \\ 0^{2(l-i)} & \text{if } i \leq l < k \\ \bot & \text{if } l < i \end{cases}$$

$$0^{2i} \to \star_l = \begin{cases} 0^{2(l-i)-1} & \text{if } i < l \text{ and } l - i \notin S \\ \star_{l-i} & \text{if } i < l \text{ and } l - i \in S \\ \bot & \text{if } l \leq i \end{cases}$$

$$0^{2i} \to 0^{2l+1} = \begin{cases} 0^{2k+1} & \text{if } l = k \\ 0^{2(l-1)+1} & \text{if } i \leq l < k \text{ and } l - i + 1 \notin S \\ \star_{l-i+1} & \text{if } i \leq l < k \text{ and } l - i + 1 \in S \\ \bot & \text{if } l < i \end{cases}$$

$$0^{2i+1} \to \star_l = \begin{cases} 0^{2(l-i-1)} & \text{if } i < l \\ \bot & \text{if } l \leq i \end{cases}$$

$$0^{2i+1} \to 0^{2l+1} = \begin{cases} 0^{2k} & \text{if } l = k \\ 0^{2(l-i)} & \text{if } i \leq l < k \\ \bot & \text{if } l < i \end{cases}$$

$$0^{2i+1} \to 0^{2l} = \begin{cases} 0^{2k+1} & \text{if } l = k \\ 0^{2(l-i)-1} & \text{if } i < l < k \text{ and } l - i \notin S \\ \star_{l-i} & \text{if } i < l < k \text{ and } l - i \in S \\ \bot & \text{if } l \leq i \end{cases}$$

$$\star_i \to \star_l = \begin{cases} 0^{2(l-i)} & \text{if } i \leq l \\ \bot & \text{if } l < i \end{cases}$$

$$\star_i \to 0^{2l+1} = \begin{cases} 0^{2k} & \text{if } l = k \\ 0^{2(l-i+1)} & \text{if } i \leq l < k \\ \bot & \text{if } l < i \end{cases}$$

$$\star_i \to 0^{2l} = \begin{cases} 0^{2k+1} & \text{if } l = k \\ 0^{2(l-i)+1} & \text{if } i \leq l < k \text{ and } l - i + 1 \notin S \\ \star_{l-i+1} & \text{if } i \leq l < k \text{ and } l - i + 1 \in S \\ \bot & \text{if } l < i. \end{cases}$$

It is a routine work to check that $\mathbf{B}_k^S$ is an FL_e-algebra for any $S \subseteq \mathbb{N}$ and $k \in \mathbb{N}$.

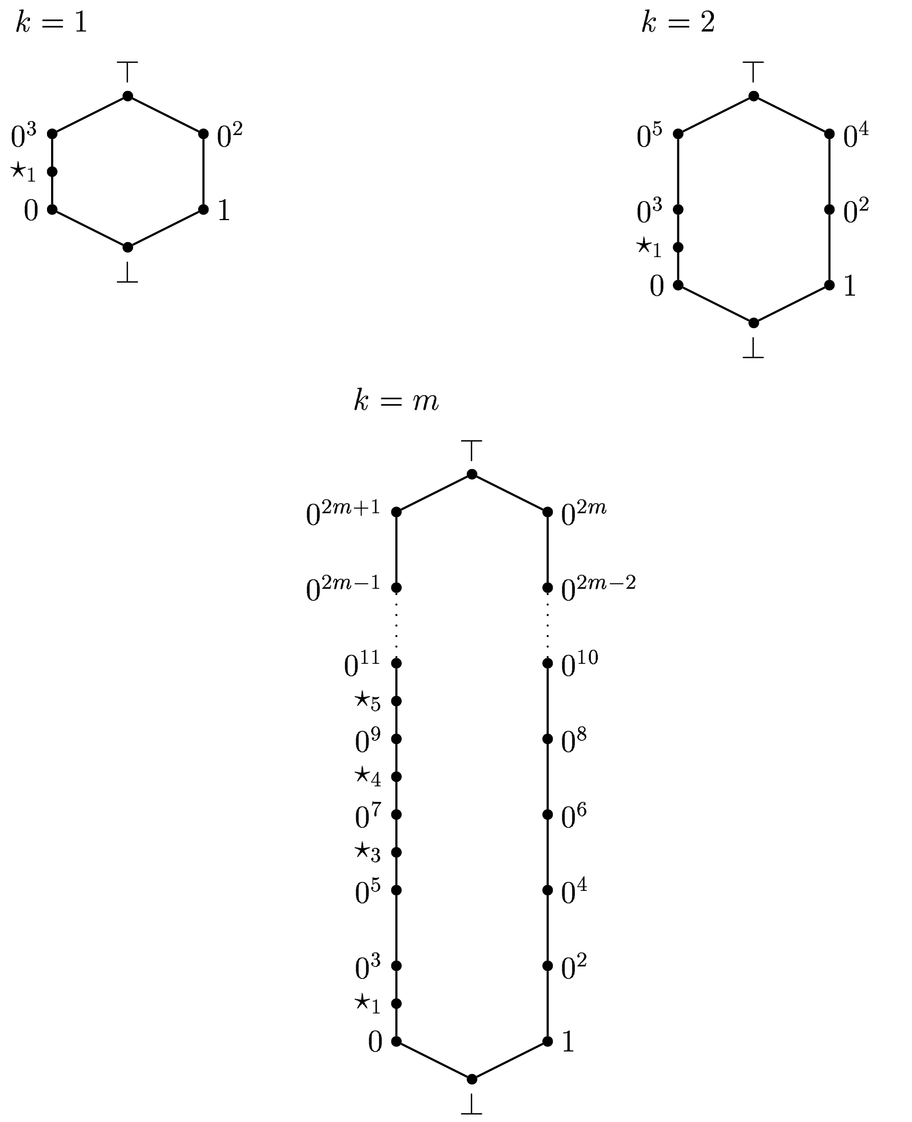

Figure 4.3: algebras $\mathbf{B}_k^S$ for $S = \{1, 3, 4, 5\}$

Let S be a subset of $\mathbb{N}$, $\mathcal{U}$ an ultrafilter over $\mathbb{N}$ and $\prod_{k\in\mathbb{N}} \mathbf{B}_k^S/\mathcal{U}$ an ultraproduct of the family $\{\mathbf{B}_k^S | k \in \mathbb{N}\}$. Then, let $\mathbf{B}_S$ be the zero-generated subalgebra of $\prod_{k\in\mathbb{N}} \mathbf{B}_k^S/\mathcal{U}$, i.e., it is the subalgebra generated by only constants $\top, \bot, 1, 0$ of $\prod_{k\in\mathbb{N}} \mathbf{B}_k^S/\mathcal{U}$.

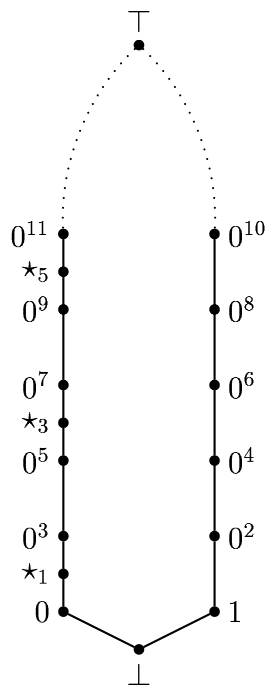

Figure 4.4: $\mathbf{B}_S$ for $S = \{n | n \text{ is odd}\}$

LEMMA 4.4 *For any $S \subseteq \mathbb{N}$, $\mathbf{B}_S$ is simple.*

(proof) In the same way as Lemma 4.1, we can show that for any congruence θ on $\mathbf{B}_S$, $\theta \neq \Delta$ implies $\theta = \nabla$. □

LEMMA 4.5 *For every $S \subseteq \mathbb{N}$, $V(\mathbf{B}_S)$ is a minimal variety.*

(proof) Let $\mathbf{A}$ be a non-degenerate subdirectly irreducible algebra in $V(\mathbf{B}_S)$. First, we will show that $\mathbf{B}_S$ is isomorphic to some subalgebra of $\mathbf{A}$. By Jónsson's Lemma, $\mathbf{A} \in HSPu(\mathbf{B}_S)$, i.e., there is some subalgeba $\mathbf{C}$ of an ulterproduct $\mathbf{B}_S{}^I/\mathcal{U}$ and an onto homomorphism α from $\mathbf{C}$ to $\mathbf{A}$, where I is an arbitrary set and $\mathcal{U}$ is an ultrafilter over I. Let β be the map from $\mathbf{B}_S$ to the direct product $\mathbf{B}_S{}^I$ defined by

$$\beta(x_{\mathbf{B}_S}) = (\cdots, x, x, x, \cdots)_{\mathbf{B}_S{}^I},$$

namely, every i-th coordinate is equal to x. Then it is easy to see that β is an embedding. Moreover $\beta(\mathbf{B}_S)$ is the smallest subalgebra of $\mathbf{B}_S{}^I$ since $\mathbf{B}_S$ is a zero-generated algebra. Let γ be the natural map from $\mathbf{B}_S{}^I$ to $\mathbf{B}_S{}^I/\mathcal{U}$. For every $x, y \in \beta(B_S)$, $x \neq y$ implies $\gamma(x) \neq \gamma(y)$. Since $x \neq y$ implies $\{i \in I | x(i) = y(i)\} = \emptyset$, $\{i \in I | x(i) = y(i)\} \notin \mathcal{U}$.

Thus, $\mathbf{B}_S$ is isomorphic to the zero-generated subalgebra of ${\mathbf{B}_S}^I/\mathcal{U}$. If $\mathbf{C}$ is a subalgebra of ${\mathbf{B}_S}^I/\mathcal{U}$ then

$$\mathbf{B}_S \leq \mathbf{C}.$$

We will show that for every onto homomorphism α from $\mathbf{C}$ to non-degenerate $\mathbf{A}$, $\alpha(\mathbf{B}_S) \cong \mathbf{B}_S$. If not, for every $a, b \in B_S$, $\alpha(a) = \alpha(b)$ since $\mathbf{B}_S$ is simple. Hence,

$$\top_{\mathbf{A}} = \alpha(\top_{\mathbf{C}}) = \alpha(\top_{\mathbf{B}_S}) = \alpha(\bot_{\mathbf{B}_S}) = \alpha(\bot_{\mathbf{C}}) = \bot_{\mathbf{A}}.$$

Thus $\mathbf{A}$ is a trivial algebra. But this is a contradiction. Therefore $\mathbf{B}_S$ is isomorphic to some subalgebra of $\mathbf{A}$.

Suppose that $\mathcal{V}$ is a nontrivial subvariety of $V(\mathbf{B}_S)$ and $\mathbf{A}$ is a non-degenerate subdirectly irreducible algebra in $\mathcal{V}$. Then $\mathbf{B}_S$ is isomorphic to some subalgebra of $\mathbf{A}$, hence $\mathbf{B}_S \in \mathcal{V}$. Thus $\mathcal{V} = V(\mathbf{B}_S)$, so $V(\mathbf{B}_S)$ is a minimal variety. □

THEOREM 4.6 *There exist continuum minimal varieties of FL_e-algebras.*

(proof) Suppose that there exist some subsets S_1, S_2 of $\mathbb{N}$ such that $S_1 \neq S_2$ and $V(\mathbf{B}_{S_1}) = V(\mathbf{B}_{S_2})$. Without loss of generality, we can assume $S_2 \not\subseteq S_1$. Then,

$$\begin{array}{rl} & V(\mathbf{B}_{S_1}) = V(\mathbf{B}_{S_2}) \\ \Rightarrow & \mathbf{B}_{S_2} \in V(\mathbf{B}_{S_1}) \text{ and } \mathbf{B}_{S_1} \in V(\mathbf{B}_{S_2}). \end{array}$$

Since $\mathbf{B}_{S_1}$ and $\mathbf{B}_{S_2}$ are non-degenerate subdirectly irreducible algebras, by the proof of Lemma 4.5, $\mathbf{B}_{S_1}$ and $\mathbf{B}_{S_2}$ are isomorphic to some subalgebras of $\mathbf{B}_{S_2}$ and $\mathbf{B}_{S_1}$, respectively. Thus $\mathbf{B}_{S_1} \cong \mathbf{B}_{S_2}$ because $\mathbf{B}_S$ does not have any proper subalgebras for all $S \subseteq \mathbb{N}$. Let α be an isomorphism from $\mathbf{B}_{S_1}$ to $\mathbf{B}_{S_2}$. Now there is some $l \in S_2 - S_1$ such that $\star_l \in B_{S_2}$ and $\star_l \notin B_{S_1}$. Then,

$$\begin{array}{rrcl} & \alpha(0_{\mathbf{B_{S_1}}} \rightarrow_{\mathbf{B_{S_1}}} {0^{2l}}_{\mathbf{B_{S_1}}}) & = & \alpha(0_{\mathbf{B_{S_1}}}) \rightarrow_{\mathbf{B_{S_2}}} \alpha({0^{2l}}_{\mathbf{B_{S_1}}}) \\ \Rightarrow & \alpha({0^{2l-1}}_{\mathbf{B_{S_1}}}) & = & 0_{\mathbf{B_{S_2}}} \rightarrow_{\mathbf{B_{S_2}}} {0^{2l}}_{\mathbf{B_{S_2}}} \\ \Rightarrow & {0^{2l-1}}_{\mathbf{B_{S_2}}} & = & {\star_l}_{\mathbf{B_{S_2}}}. \end{array}$$

But this is a contradiction. Therefore $V(\mathbf{B}_{S_1}) = V(\mathbf{B}_{S_2})$ implies $S_1 = S_2$. □

In the previous chapter, we have already seen that the subvariety lattice $\langle subV, \cap, \vee, \mathcal{V}_{tri}, \mathcal{FL}_e\rangle$ is dually isomorphic to the lattice $\langle CSL, \cap, \vee, \mathbf{FL_e}, \mathcal{F}m\rangle$. Then by the above result, we can show that there exist continuum maximal logics over $\mathbf{FL_e}$.

Chapter 5

Interpolation Property and Pseudo-relevance Property

In this chapter, we will show that an algebraic characterization of the *deductive pseudo-relevance property* given by L. Maksimova [28] works well also for commutative substructural logics, i.e., extensions of $\mathbf{FL_e}$. More pricisely, a commutative substructural logic $\mathcal{L}$ has the deductive pseudo-relevance property if and only if the joint embedding property holds in the corresponding variety $V(\mathcal{L})$. We will discuss also how the weakening rules have an effect on relations between the deductive pseudo-relevance property and the deductive interpolation property, or equivalently, between the joint embedding property and the amalgamation property.

5.1 Interpolation Property for Commutative Substructural Logics

We say that a logic $\mathcal{L}$ has *Craig's interpolation property* (IP) if for all formulas ϕ and ψ the condition $\vdash_{\mathcal{L}} \phi \to \psi$ implies there exists some formula σ such that

1. $\vdash_{\mathcal{L}} \phi \to \sigma$ and $\vdash_{\mathcal{L}} \sigma \to \psi$,
2. $Var(\sigma) \subseteq Var(\phi) \cap Var(\psi)$,

where $Var(\phi)$ denotes the set of all propositional variables in ϕ. A formula σ that satisfies the above condition 1 and 2 is called an *interpolant* of $\phi \to \psi$.

We say that a logic $\mathcal{L}$ has the *deductive interpolation property* (DIP) if for all formulas ϕ and ψ the condition $\phi \vdash_{\mathcal{L}} \psi$ implies there exists some formula σ such that

1. $\phi \vdash_{\mathcal{L}} \sigma$ and $\sigma \vdash_{\mathcal{L}} \psi$,
2. $Var(\sigma) \subseteq Var(\phi) \cap Var(\psi)$.

PROPOSITION 5.1 *For any logic $\mathcal{L}$ over $\mathbf{FL_e}$, if $\mathcal{L}$ has IP then it has also DIP.*

(proof) If $\phi \vdash_{\mathcal{L}} \psi$ then, by the local deduction theorem, there exists a non-negative integer n such that $\vdash_{\mathcal{L}} (\phi \wedge 1)^n \to \psi$. So, by IP, there is an interpolant σ of $(\phi \wedge 1)^n \to \psi$. Hence,

$\phi \vdash_{\mathcal{L}} \sigma, \sigma \vdash_{\mathcal{L}} \psi$ and $Var(\sigma) \subseteq Var(\phi) \cap Var(\psi)$ hold. □

In the case of intermediate logics, the converse direction of the above result holds since the deduction theorem holds for them, but this doesn't hold in general.

A class of $\mathcal{K}$ of algebras is said to have the *amalgamation property* (AP) if for all $\mathbf{D}, \mathbf{A}, \mathbf{B}$ in $\mathcal{K}$ with monomorphisms $f : \mathbf{D} \to \mathbf{A}, g : \mathbf{D} \to \mathbf{B}$, there exist $\mathbf{C} \in \mathcal{K}$ and monomorphisms $f' : \mathbf{A} \to \mathbf{C}, g' : \mathbf{B} \to \mathbf{C}$ such that $f' \circ f = g' \circ g$.

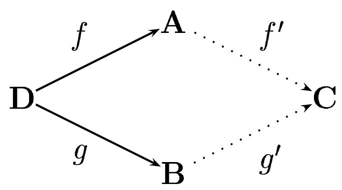

Figure 5.1: amalgamation property

In [13], Galatos and Ono showed the following.

PROPOSITION 5.2 *For any logic $\mathcal{L}$ over $\mathbf{FL_e}$, $\mathcal{L}$ has DIP if and only if the associated variety $V(\mathcal{L})$ of FL_e-algebras has AP.*

5.2 Pseudo-Relevance Property for Commutative Substructural Logics

We say that a logic $\mathcal{L}$ possesses the *pseudo-relevance property* (PRP), if for all formulas ϕ and ψ without common variables the condition $\vdash_{\mathcal{L}} \phi \to \psi$ implies $\vdash_{\mathcal{L}} \phi \to \bot$ or $\vdash_{\mathcal{L}} \psi$. The pseudo-relevance property was introduced and was studied for intermediate predicate logics by N.-Y. Suzuki [38] (see also [39]).

We say that a logic $\mathcal{L}$ possesses the *deductive pseudo-relevance property* (DPRP), if for all formulas ϕ and ψ without common variables the condition $\phi \vdash_{\mathcal{L}} \psi$ implies $\phi \vdash_{\mathcal{L}} \bot$ or $\vdash_{\mathcal{L}} \psi$. Note that in the case of commutative substructural logics, by compactness theorem, we can replace ϕ by an arbitrary nonempty set Γ of formulas whenever Γ and ψ have no variables in common.

It is easy to see that for each logic $\mathcal{L}$ over $\mathbf{FL_e}$, PRP implies DPRP. For, if $\phi \vdash_{\mathcal{L}} \psi$ holds for ϕ and ψ without common variables then, by the local deduction theorem, $\vdash_{\mathcal{L}} (\phi \wedge 1)^n \to \psi$ holds for some $n \in \mathbb{N}$. Hence, by PRP, either $\vdash_{\mathcal{L}} (\phi \wedge 1)^n \to \bot$ or $\vdash_{\mathcal{L}} \psi$ holds, i.e., $\phi \vdash_{\mathcal{L}} \bot$ or $\vdash_{\mathcal{L}} \psi$ hold. In the case of intermediate logics, the converse direction also holds, but this doesn't hold in general.

PRP and DPRP can be regarded as a special case of IP and DIP, respectively. Here, we will see relations between DIP and DPRP.

For each logic $\mathcal{L}$ over $\mathbf{FL_{ew}}$, DIP implies DPRP. For, if $\phi \vdash_{\mathcal{L}} \psi$ holds for ϕ and ψ without common variables, there exists σ without any variables such that it is an interpolant of $\phi \vdash_{\mathcal{L}} \psi$. Such a formula σ must be constructed by constants $\bot$ and $\top$. It is easy to see that for any logic over $\mathbf{FL_{ew}}$, every formula constructed by only constants $\bot$ and $\top$ is equivalent either to $\bot$ or to $\top$. Hence σ is equivalent either to $\bot$ or to $\top$. In the former case, $\phi \vdash_{\mathcal{L}} \bot$ holds. Otherwise $\vdash_{\mathcal{L}} \psi$ holds.

On the other hand, DIP doesn't always implies DPRP for logics over $\mathbf{FL_e}$. For, interpolants of $\phi \vdash_{\mathcal{L}} \psi$ are formulas consisting only of constants $0, 1, \top$ and $\bot$, and it might be neither $\top$ nor $\bot$. In fact, Ono [32] showed that IP holds for $\mathbf{FL_e}$ by using cut elimination of $\mathbf{FL_e}$ and Maehara's method. Hence, DIP also holds for $\mathbf{FL_e}$. On the other hand, since $p \cdot \neg p \vdash_{\mathbf{FL_e}} (0 \to q) \to q$ holds but neither $p \cdot \neg p \vdash_{\mathbf{FL_e}} \bot$ nor $\vdash_{\mathbf{FL_e}} (0 \to q) \to q$ hold. Therefore DPRP doesn't hold.

In the following figure, dotted arrows hold for logics over $\mathbf{FL_{ew}}$.

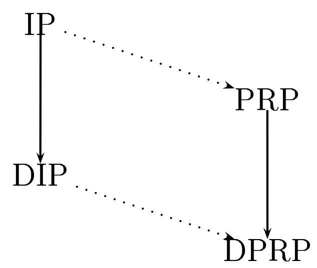

Figure 5.2: relations among several properties

A class $\mathcal{K}$ of algebras is said to have the *joint embedding property* (JEP) if for all algebras $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$ there exist $\mathbf{C} \in \mathcal{K}$ and monomorphisms $\alpha : \mathbf{A} \to \mathbf{C}$ and $\beta : \mathbf{B} \to \mathbf{C}$.

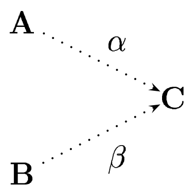

Figure 5.3: joint embedding property

We can extend a result on a characterization of DPRP for normal modal logics given by Maksimova [28] to logics over $\mathbf{FL_e}$ as follows.

THEOREM 5.3 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$, and $V(\mathcal{L})$ the associated variety of FL_e-algebras. Then the following are equivalent:*

1. *$\mathcal{L}$ has the DPRP,*

2. *the class of all non-degenerate FL_e-algebras of $V(\mathcal{L})$ has the JEP,*

3. *every two subdirectly irreducible FL_e-algebras of $V(\mathcal{L})$ are jointly embeddable into a suitable algebra in $V(\mathcal{L})$.*

(proof) *1* $\Rightarrow$ *2.* Assume that $\mathcal{L}$ has the DPRP, $\mathbf{A}$ and $\mathbf{B}$ are non-degenerate algebras in $V(\mathcal{L})$. Let

$$\mathbf{p}' = \{p_a \mid a \in \mathbf{A}\}, \quad \mathbf{p}'' = \{q_b \mid b \in \mathbf{B}\}, \quad \mathbf{p} = \mathbf{p}' \cup \mathbf{p}'',$$

i.e., $\mathbf{p}'$ and $\mathbf{p}''$ are disjoint sets of variables constructed by $\mathbf{A}$ and $\mathbf{B}$, respectively. Denote $\mathcal{F}m(\mathbf{p})$, $\mathcal{F}m(\mathbf{p}')$, and $\mathcal{F}m(\mathbf{p}'')$ the sets of all formulas of variables from $\mathbf{p}$, $\mathbf{p}'$, and $\mathbf{p}''$, respectively. Let us take valuations $v' : \mathbf{p}' \to \mathbf{A}$ and $v'' : \mathbf{p}'' \to \mathbf{B}$, where $v'(p_a) = a$ for $a \in \mathbf{A}$ and $v''(q_b) = b$ for $b \in \mathbf{B}$. Take sets T' and T'' of formulas as follows,

$$T' = \{\phi \in \mathcal{F}m(\mathbf{p}') \mid v'(\phi) \geq 1_{\mathbf{A}}\}, \quad T'' = \{\psi \in \mathcal{F}m(\mathbf{p}'') \mid v''(\psi) \geq 1_{\mathbf{B}}\}.$$

Then T' and T'' are $\mathcal{L}$-theories of the languages $\mathcal{F}m(\mathbf{p}')$ and $\mathcal{F}m(\mathbf{p}'')$, respectively. Let us define

$$T = \{\sigma \in \mathcal{F}m(\mathbf{p}) \mid T' \cup T'' \vdash_{\mathcal{L}} \sigma\}.$$

Then T is an $\mathcal{L}$-theory of the language $\mathcal{F}m(\mathbf{p})$. By Proposition 3.33, one can define the Lindenbaum-Tarski algebra $\mathbf{C} = \mathbf{C}(\mathbf{p}, T) = \mathcal{F}m(\mathbf{p})/\sim_T$. Let us define the mappings $\alpha : \mathbf{A} \to \mathbf{C}$ and $\beta : \mathbf{B} \to \mathbf{C}$ by $\alpha(a) = ||p_a||$ for $a \in \mathbf{A}$, $\beta(b) = ||q_b||$ for $b \in \mathbf{B}$. First, we will show that α and β are homomorphisms. For all $a_1, a_2 \in A$ and every operation $\oplus$,

$$\alpha(a_1 \oplus_{\mathbf{A}} a_2) = ||p_{a_1 \oplus_{\mathbf{A}} a_2}||$$

and

$$\begin{aligned} \alpha(a_1) \oplus_{\mathbf{C}} \alpha(a_2) &= ||p_{a_1}|| \oplus_{\mathbf{C}} ||p_{a_2}|| \\ &= ||p_{a_1} \oplus_{\mathcal{F}m(\mathbf{p})} p_{a_2}|| \\ &= ||p_{a_1} \oplus_{\mathcal{F}m(\mathbf{p}')} p_{a_2}||. \end{aligned}$$

It is easy to see that $p_{a_1 \oplus_{\mathbf{A}} a_2} \leftrightarrow p_{a_1} \oplus_{\mathcal{F}m(\mathbf{p}')} p_{a_2} \in T'$. Since $T' \subseteq T$, we have $p_{a_1 \oplus_{\mathbf{A}} a_2} \leftrightarrow p_{a_1} \oplus_{\mathcal{F}m(\mathbf{p}')} p_{a_2} \in T$, i.e., $p_{a_1 \oplus_{\mathbf{A}} a_2} \sim_T p_{a_1} \oplus_{\mathcal{F}m(\mathbf{p}')} p_{a_2}$. Thus,

$$\alpha(a_1 \oplus_{\mathbf{A}} a_2) = ||p_{a_1 \oplus_{\mathbf{A}} a_2}|| = ||p_{a_1} \oplus_{\mathcal{F}m(\mathbf{p}')} p_{a_2}|| = \alpha(a_1) \oplus_{\mathbf{C}} \alpha(a_2).$$

Similarly, since $T'' \subseteq T$, β is a homomorphism. Let us show that they are monomorphisms. It is sufficient to show that $\alpha(a) \not\geq 1_{\mathbf{C}}$ for $a \not\geq 1_{\mathbf{A}}$ and $\beta(b) \not\geq 1_{\mathbf{C}}$ for $b \not\geq 1_{\mathbf{B}}$. Let $a \in \mathbf{A}$ and $\alpha(a) \geq 1_{\mathbf{C}}$. Then $\alpha(a) = ||p_a|| \geq 1_{\mathbf{C}}$, so $p_a \in T$ by Proposition 3.33. Thus, $T' \cup T'' \vdash_{\mathcal{L}} p_a$. By the local deduction theorem, there exists $\phi_{p_a} \in T'$ such that

$$T'' \vdash_{\mathcal{L}} \phi_{p_a} \to p_a.$$

By DPRP, we obtain

$$T'' \vdash_{\mathcal{L}} \bot \quad \text{or} \quad \vdash_{\mathcal{L}} \phi_{p_a} \to p_a.$$

If $T'' \vdash_{\mathcal{L}} \bot$, i.e., $\bot \in T''$ then

$$\begin{aligned} & v''(\bot) \geq 1_{\mathbf{B}} \\ \Rightarrow \quad & \bot_{\mathbf{B}} \geq 1_{\mathbf{B}} \\ \Rightarrow \quad & \bot_{\mathbf{B}} = 1_{\mathbf{B}}. \end{aligned}$$

Hence,

$$\top_{\mathbf{B}} = \bot_{\mathbf{B}} \to \bot_{\mathbf{B}} = 1_{\mathbf{B}} \to 1_{\mathbf{B}} = 1_{\mathbf{B}} = \bot_{\mathbf{B}}.$$

Thus $\mathbf{B}$ is degenerate, but this is a contradiction. Thus, $\vdash_{\mathcal{L}} \phi_{p_a} \to p_a$ must hold. This means that $p_a \in T'$ and hence $v'(p_a) = a \geq 1_{\mathbf{A}}$.

2 $\Rightarrow$ *3.* Obvious.

3 $\Rightarrow$ *1.* Assume that there exist some $\phi(\mathbf{p}')$ and $\psi(\mathbf{p}'')$, where $\mathbf{p}'$ and $\mathbf{p}''$ are disjoint lists of variables, such that neither $\phi \vdash_{\mathcal{L}} \perp$ nor $\vdash_{\mathcal{L}} \psi$ holds. Let $\mathbf{p} = \mathbf{p}' \cup \mathbf{p}''$. By Proposition 3.34 , there exist maximal $\mathcal{L}$-theories $T_\perp, T_\psi$ of the language $\mathcal{F}m(\mathbf{p})$ such that $\phi \in T_\perp$ and $\perp \notin T_\perp$, and $\psi \notin T_\psi$. So, $\mathbf{A}(\mathbf{p}, T_\perp)$ and $\mathbf{B}(\mathbf{p}, T_\psi)$ are subdirectly irreducible and $||\perp||$ and $||\psi||$ are oprema of $\mathbf{A}$ and $\mathbf{B}$, respectively. Take the canonical mappings $v_1 : \mathcal{F}m \to \mathbf{A}$ and $v_2 : \mathcal{F}m \to \mathbf{B}$ as valuations on $\mathbf{A}$ and $\mathbf{B}$, respectively, i.e.,

$$v_1(\sigma) = ||\sigma|| = \sigma/\sim_{T_\perp} \quad \text{and} \quad v_2(\sigma) = ||\sigma|| = \sigma/\sim_{T_\psi} .$$

Then,

$$v_1(\phi) = ||\phi|| \geq 1_{\mathbf{A}} \quad \text{and} \quad v_2(\psi) = ||\psi|| \not\geq 1_{\mathbf{B}}.$$

Now, by Proposition 3.33 , $\mathbf{A}, \mathbf{B} \in V(\mathcal{L})$. Thus, by our assumption *3*, there exists some $\mathbf{C}$ in $V(\mathcal{L})$ into which both $\mathbf{A}$ and $\mathbf{B}$ are jointly embeddable by monomorphisms $\alpha : \mathbf{A} \to \mathbf{C}$ and $\beta : \mathbf{B} \to \mathbf{C}$. Since ϕ and ψ have no variables in common, we can construct a valuation v on $\mathbf{C}$ defined by $v = \alpha v_1$ for variables in $\mathbf{p}'$ and $v = \beta v_2$ for variables in $\mathbf{p}''$. Then,

$$v(\phi) = \alpha v_1(\phi) = \alpha(||\phi||) \geq 1_{\mathbf{C}}$$
and
$$v(\psi) = \beta v_2(\psi) = \beta(||\psi||) \not\geq 1_{\mathbf{C}}.$$

Hence for any $n \in \omega$,

$$\begin{aligned} v((\phi \wedge 1)^n \to \psi) &= v(\phi \wedge 1)^n \to_{\mathbf{C}} v(\psi) \\ &= 1_{\mathbf{C}} \to_{\mathbf{C}} v(\psi) \\ &= v(\psi) \\ &\not\geq 1_{\mathbf{C}}. \end{aligned}$$

Then by local deduction theorem, $\phi \not\vdash_{\mathcal{L}} \psi$. Thus $\mathcal{L}$ has DPRP. □

From Theorem 5.3, for any logic $\mathcal{L}$ over $\mathbf{FL_{ew}}$ AP of $V(\mathcal{L})$ implies JEP of the class of all non-degenerate algebras of $V(\mathcal{L})$. The following result is a direct proof of it.

COROLLARY 5.4 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_{ew}}$. Then, if $V(\mathcal{L})$ has AP then the class of all non-degenerate FL_{ew}-algebras of $V(\mathcal{L})$ has JEP.*

(proof) Let $\mathbf{A}, \mathbf{B}$ be non-degenerate FL_{ew}-algebras in $V(\mathcal{L})$ and $\mathbf{2}$ the two-elements Boolean algebra. Then, it is easy to see that maps $\alpha : \mathbf{2} \to \mathbf{A}$ and $\beta : \mathbf{2} \to \mathbf{B}$ defined by

$$\alpha(1_{\mathbf{2}}) = 1_{\mathbf{A}} \quad \text{and} \quad \alpha(0_{\mathbf{2}}) = 0_{\mathbf{A}},$$
$$\beta(1_{\mathbf{2}}) = 1_{\mathbf{B}} \quad \text{and} \quad \beta(0_{\mathbf{2}}) = 0_{\mathbf{B}}$$

are monomorphisms. Thus, by AP of $V(\mathcal{L})$, there exist some $\mathbf{C}$ in $V(\mathcal{L})$ and monomorphisms $\alpha' : \mathbf{A} \to \mathbf{C}$ and $\beta' : \mathbf{B} \to \mathbf{C}$. □

In [19], Komori showed that PRP holds always for all intermediate logic by using Glivenko's theorem. We can extend this result to any logic over $\mathbf{FL_{ew}}$ for which Glivenko's theorem holds. Here, we will give a proof of this by proof-theoretical method suggested by Ono. (see e.g. [12])

THEOREM 5.5 *PRP holds always for any logic $\mathcal{L}$ which includes $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$.*

(proof) Note first that the following logics are equivalent;

WEM: $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$,

$WCon1$: $\mathbf{FL_{ew}} + \neg\delta^2 \to \neg\delta$,

$WCon2$: $\mathbf{FL_{ew}} + \dfrac{\delta, \delta, \Gamma \Rightarrow}{\delta, \Gamma \Rightarrow}$.

First, we will show that Glivenko's theorem holds for any logic which includes $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$. In fact, $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$ is the smallest logic among logics over $\mathbf{FL_{ew}}$ for which Glivenko's theorem holds. For, if Glivenko's theorem holds for a logic $\mathcal{L}$ over $\mathbf{FL_{ew}}$ then $\vdash_{\mathcal{L}} \neg(\delta \wedge \neg\delta)$ since $\vdash_{\mathbf{LK}} \neg(\delta \wedge \neg\delta)$. Thus, $\mathcal{L} \supseteq \mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$. In order to prove that Glivenko's theorem holds for $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$, it is sufficient to show that if a sequent $\Gamma \Rightarrow \Delta$ is provable in **LK** then $\neg\Delta, \Gamma \Rightarrow$ is provable in our third system, where $\neg\Delta$ means $\neg\psi_1, \cdots, \neg\psi_n$ for all $\psi_i \in \Delta$. In fact, if this is the case then for any formula ϕ provable in **LK** the sequent $\neg\phi \Rightarrow$ is provable in $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$ and hence $\neg\neg\phi$ is also provable in it. We will prove it by induction on the length of a given proof of $\Gamma \Rightarrow \Delta$ in **LK**. The base case is obvious. Here, we will show the cases of $(\to\Rightarrow)$, $(\Rightarrow\wedge)$ and $(\Rightarrow\to)$.

$(\to\Rightarrow)$ By induction hypothesis, $\neg\phi, \neg\Delta, \Gamma \Rightarrow$ and $\neg\Sigma, \psi, \Pi \Rightarrow$ are provable in $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$. Then,

$$\dfrac{\dfrac{\dfrac{\neg\phi, \neg\Delta, \Gamma \Rightarrow}{\neg\Delta, \Gamma \Rightarrow \neg\neg\phi} \quad \dfrac{\neg\Sigma, \psi, \Pi \Rightarrow}{\neg\Sigma, \Pi \Rightarrow \neg\psi}}{\neg\Delta, \Gamma\neg\Sigma, \Pi \Rightarrow \neg\neg\phi \cdot \neg\psi} \quad \dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \quad \dfrac{\psi \Rightarrow \psi}{\neg\psi, \psi \Rightarrow}}{\phi, \neg\psi, \phi \to \psi \Rightarrow}}{\neg\neg\phi, \neg\psi, \phi \to \psi \Rightarrow}}{\neg\neg\phi \cdot \neg\psi, \phi \to \psi \Rightarrow}}{\neg\Delta, \neg\Sigma, \phi \to \psi, \Gamma, \Pi \Rightarrow}\ (cut).$$

$(\Rightarrow\wedge)$ By induction hypothesis, $\neg\phi, \neg\Delta, \Gamma \Rightarrow$ and $\neg\psi, \neg\Delta, \Gamma \Rightarrow$ are provable in $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$. Then,

$$\dfrac{\dfrac{\dfrac{\neg\phi, \neg\Delta, \Gamma \Rightarrow}{\neg\Delta, \Gamma \Rightarrow \neg\neg\phi} \quad \dfrac{\neg\psi, \neg\Delta, \Gamma \Rightarrow}{\neg\Delta, \Gamma \Rightarrow \neg\neg\psi}}{\neg\Delta, \Gamma \Rightarrow \neg\neg\phi \wedge \neg\neg\psi} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi, \psi \Rightarrow \phi} \quad \dfrac{\psi \Rightarrow \psi}{\phi, \psi \Rightarrow \psi}}{\phi, \psi \Rightarrow \phi \wedge \psi}}{\phi, \psi, \neg(\phi \wedge \psi) \Rightarrow}}{\neg\neg\phi, \neg\neg\psi, \neg(\phi \wedge \psi) \Rightarrow}}{\neg\neg\phi \wedge \neg\neg\psi, \neg\neg\phi \wedge \neg\neg\psi, \neg(\phi \wedge \psi) \Rightarrow}}{\neg\neg\phi \wedge \neg\neg\psi, \neg(\phi \wedge \psi) \Rightarrow}\ (WCon2)}{\neg(\phi \wedge \psi), \neg\Delta, \Gamma \Rightarrow}\ (cut).$$

($\Rightarrow\rightarrow$) By induction hypothesis, $\neg\psi, \neg\Delta, \phi, \Gamma \Rightarrow$ is provable in $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$. Then,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi}{\phi, \neg\phi \Rightarrow}}{\phi, \neg\phi \Rightarrow \psi}}{\neg\phi \Rightarrow \phi \rightarrow \psi}}{\neg\phi, \neg(\phi \rightarrow \psi) \Rightarrow}}{\neg(\phi \rightarrow \psi) \Rightarrow \neg\neg\phi} \quad \dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\phi, \psi \Rightarrow \psi}}{\psi \Rightarrow \phi \rightarrow \psi}}{\neg(\phi \rightarrow \psi) \Rightarrow \neg\psi}}{\neg(\phi \rightarrow \psi) \Rightarrow \neg\neg\phi \wedge \neg\psi} \quad \dfrac{\dfrac{\dfrac{\dfrac{\phi, \neg\psi, \neg\Delta, \Gamma \Rightarrow}{\neg\psi, \neg\Delta, \Gamma \Rightarrow \neg\phi}}{\neg\neg\phi, \neg\psi, \neg\Delta, \Gamma \Rightarrow}}{\neg\neg\phi \wedge \neg\psi, \neg\neg\phi \wedge \neg\psi, \neg\Delta, \Gamma \Rightarrow}}{\neg\neg\phi \wedge \neg\psi, \neg\Delta, \Gamma \Rightarrow} {\scriptstyle (WCon2)}}{\neg(\phi \rightarrow \psi), \neg\Delta, \Gamma \Rightarrow} {\scriptstyle (cut)}$$

Let $\mathcal{L}$ be a logic including $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$. Suppose that there exist some formulas ϕ and ψ, which have no variables in common, such that neither $\vdash_{\mathcal{L}} \phi \rightarrow \bot$ nor $\vdash_{\mathcal{L}} \psi$ hold. By $\nvdash_{\mathcal{L}} \psi$, there is an algebra $\mathbf{A} \in V(\mathcal{L})$ and a valuation v on $\mathbf{A}$ such that $v(\psi) \neq 1_{\mathbf{A}}$. Since $\mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta) \subseteq \mathcal{L}$, $\neg\phi = \phi \rightarrow \bot \notin \mathbf{FL_{ew}} + \neg(\delta \wedge \neg\delta)$. By Glivenko's theorem, $\neg\phi \notin \mathbf{LK}$. Hence, there exists a valuation u on $\mathbf{A}$ defined by $u(p) = 1_{\mathbf{A}}$ or $u(p) = 0_{\mathbf{A}}$ for any propositional variable p such that $u(\neg\phi) = 0_{\mathbf{A}}$, i.e., $u(\phi) = 1_{\mathbf{A}}$. Since ϕ and ψ have no varialbes in common, there is some valuation w on $\mathbf{A}$ determined by

$$\begin{array}{ll} w(p) = v(p) & \text{if } p \in Var(\psi), \\ w(q) = u(q) & \text{if } q \in Var(\phi). \end{array}$$

Hence,

$$w(\phi \rightarrow \psi) = u(\phi) \rightarrow v(\psi) = 1_{\mathbf{A}} \rightarrow v(\psi) = v(\psi) \neq 1_{\mathbf{A}}.$$

Therefore, $\nvdash_{\mathcal{L}} \phi \rightarrow \psi$. □

Since PRP always implies DPRP, DPRP also holds for any logic which includes $\mathbf{FL_{ew}}+ \neg(\delta \wedge \neg\delta)$.

If we don't suppose the variable's condition of DPRP then we obtain the following result. It is interesting to compare the following result with the above.

THEOREM 5.6 *A logic $\mathcal{L}$ satisfies the following condition, for all formulas ϕ and ψ,*

$$\phi \vdash_{\mathcal{L}} \psi \Rightarrow \phi \vdash_{\mathcal{L}} \bot \ \textit{or} \ \vdash_{\mathcal{L}} \psi$$

if and only if $\mathcal{L}$ is equal to $\mathcal{F}m$.

(proof) Cleary, $\mathcal{F}m$ satisfies the above condition. Assume that a logic $\mathcal{L}$ is not $\mathcal{F}m$. Then, there is a non-degenerate algebra $\mathbf{A} \in V(\mathcal{L})$. Note that for any propositional variable p, $p \notin \mathcal{L}$, since there is a valuation v on $\mathbf{A}$ defined by $v(p) = \bot_{\mathbf{A}} \not\geq 1_{\mathbf{A}}$. Hence $\nvdash_{\mathcal{L}} p$. Moreover, since $\vdash_{\mathcal{L}} p \rightarrow p$ always holds, $p \vdash_{\mathcal{L}} p$. If $p \vdash_{\mathcal{L}} \bot$ then there is some $n \in \omega$ such that $\vdash_{\mathcal{L}} (p \wedge 1)^n \rightarrow \bot$. Define a valuation w on $\mathbf{A}$ by $w(p) = 1_{\mathbf{A}}$. Then,

$$w((p \wedge 1)^n \rightarrow \bot) = 1_{\mathbf{A}}^n \rightarrow \bot_{\mathbf{A}} = \bot_{\mathbf{A}} \not\geq 1_{\mathbf{A}}.$$

This is a contradiction, so $p \nvdash_{\mathcal{L}} \bot$. Hence $\mathcal{L}$ does not satisfy the above condition. □

Chapter 6

Halldén Completeness and Principle of Variable Separation

The disjunction property, Halldén completeness, and the principle of variable separation have been studied actively in modal logics and intermediate logics. Some relationships among them were shown in [8].

In this chapter, we discuss algebraic characterizations of Halldén completeness and the deductive principle of variable separation for commutative substructural logics. Though to logics over $\mathbf{FL_{ew}}$ we can extend most of results on Halldén completeness and the deductive principle of variable separation obtained by Lemmon [6], Wroński [43] and Maksimova [28], the lack of the weakening rule will cause some difficulties in extending them to logics over $\mathbf{FL_e}$. So, some modifications of definitions of them become necessary to make similar results hold for logics over $\mathbf{FL_e}$. We will give not only partial results on modified Halldén completeness and the deductive principle of variable separation, but also algebraic characterizations of them in the original form for logics over $\mathbf{FL_e}$.

6.1 Halldén Completeness

It is well-known that a logic $\mathcal{L}$ has the *disjunction property* (DP) if for all formulas ϕ and ψ, $\vdash_{\mathcal{L}} \phi \vee \psi$ implies $\vdash_{\mathcal{L}} \phi$ or $\vdash_{\mathcal{L}} \psi$.

We say that a logic $\mathcal{L}$ is *Halldén complete* (HC) if for all formulas ϕ and ψ which have no variables in common, $\vdash_{\mathcal{L}} \phi \vee \psi$ implies $\vdash_{\mathcal{L}} \phi$ or $\vdash_{\mathcal{L}} \psi$. It is easy to see that Halldén completeness is a special case of the disjunction property. An example of a logic which is Halldén complete but doesn't have the disjunction property is classical logic **Cl**. Wroński [42] showed that there are a continuum of intermediate logics for which the disjunction property holds, and Galanter [9] showed that there are a continuum of intermediate logics which are Halldén complete but don't have the disjunction property.

LEMMA 6.1 *Let $\mathcal{G}$ be a proper filter of an FL_{ew}-algebra* $\mathbf{A}$ *and $a \notin \mathcal{G}$. Then there exists a prime filter $\mathcal{F}_a$ such that it is maximal in the set*

$$\Sigma = \{\mathcal{F} : \textit{filter} | \mathcal{G} \subseteq \mathcal{F}, a \notin \mathcal{F}\}.$$

(proof) By Zorn's lemma, Σ has a maximal element. So let $\mathcal{F}_a$ be a maximal element of Σ. We will show that $\mathcal{F}_a$ is prime. Assume $x \notin \mathcal{F}_a$ and $y \notin \mathcal{F}_a$, and define $\mathcal{H}_x$ as follows:

$$\mathcal{H}_x = \{z \in A | \ x^k \cdot u \leq z, \ \exists k \in \omega, \exists u \in \mathcal{F}_a\}.$$

Then $\mathcal{H}_x$ is the filter generated by $\mathcal{F}_a \cup \{x\}$. Since $\mathcal{F}_a$ is maximal in Σ and $x \notin \mathcal{F}_a$, $a \in \mathcal{H}_x$. So there exist some $l \in \omega$ and $u \in \mathcal{F}_a$ such that

$$x^l \cdot u \leq a.$$

Similarly there exists some $m \in \omega$ and $v \in \mathcal{F}_a$ such that

$$y^m \cdot v \leq a.$$

Let $t = l + m - 1$. Then, by the distributivity of $\cdot$ with $\vee$

$$\begin{aligned} & (x \vee y)^t \cdot u \cdot v \\ = & \bigvee_{i=0}^{t} x^i \cdot y^{t-i} \cdot u \cdot v. \end{aligned}$$

Since $i \geq l$ or $t - i \geq m$, either of the following holds:

$$\begin{aligned} (1) & \quad x^i \cdot y^{t-i} \cdot u \cdot v \\ \leq & \quad x^l \cdot u \\ \leq & \quad a \end{aligned}$$

$$\begin{aligned} (2) & \quad x^i \cdot y^{t-i} \cdot u \cdot v \\ \leq & \quad y^m \cdot v \\ \leq & \quad a. \end{aligned}$$

So if $x \vee y \in \mathcal{F}_a$ then $a \in \mathcal{F}_a$. But this is a contradiction. Hence, $x \vee y \notin \mathcal{F}_a$. □

An algebra **A** is said to be *well-connected* if for every $x, y \in A$, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$. We can extend the results for intermediate logics given by Lemmon (e.g. [6] Theorem 15.22) and Wroński [43] to logics over $\mathbf{FL_{ew}}$ as follows.

THEOREM 6.2 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_{ew}}$. Then the following are equivalent:*

(i) *$\mathcal{L}$ is Halldén complete,*

(ii) *$\mathcal{L}$ is meet-irreducible in the lattice of all logics over $\mathbf{FL_{ew}}$, i.e., for any logics $\mathcal{L}_1, \mathcal{L}_2$,*

$$\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2 \text{ implies } \mathcal{L}_1 = \mathcal{L} \text{ or } \mathcal{L}_2 = \mathcal{L},$$

(iii) *$\mathcal{L} = L(\mathbf{A})$ for some well-connected FL_{ew}-algebra $\mathbf{A}$, where $L(\mathbf{A})$ denotes the set of all formulas which are valid in $\mathbf{A}$.*

(proof) (iii) ⇒(ii) Let **A** be a well-connected FL_{ew}-algebra and $\mathcal{L} = L(\mathbf{A})$, and $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$. Then, clearly $\mathcal{L} \subseteq \mathcal{L}_1$ and $\mathcal{L} \subseteq \mathcal{L}_2$. Suppose that neither $\mathcal{L} = \mathcal{L}_1$ nor $\mathcal{L} = \mathcal{L}_2$. Then there exist some formulas $\phi \in \mathcal{L}_1 - \mathcal{L}$ and $\psi \in \mathcal{L}_2 - \mathcal{L}$ such that they have no variable in common, since every logic is closed under substitution. So, there is some valuation u on **A** such that

$$u(\phi) < 1 \qquad \text{and} \qquad u(\psi) < 1.$$

Since **A** is well-connected, $u(\phi \vee \psi) = u(\phi) \vee u(\psi) < 1$. Thus $\phi \vee \psi \notin \mathcal{L}$. But, by our assumption,

$$\begin{aligned} \phi \in \mathcal{L}_1, \psi \in \mathcal{L}_2 \quad & \Rightarrow \quad \phi \vee \psi \in \mathcal{L}_i \ (i = 1, 2) \\ & \Rightarrow \quad \phi \vee \psi \in \mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}. \end{aligned}$$

This is a contradiction.

(ii) $\Rightarrow$ (i) Suppose that there exist some ϕ_1, ϕ_2 which have no variables in common, $\vdash_{\mathcal{L}} \phi_1 \vee \phi_2$ but $\not\vdash_{\mathcal{L}} \phi_1$ and $\not\vdash_{\mathcal{L}} \phi_2$. Let $\mathcal{L}_i = \mathcal{L} + \phi_i$ $(i = 1, 2)$. Clearly, $\mathcal{L} \subseteq \mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_1 \neq \mathcal{L}$ and $\mathcal{L}_2 \neq \mathcal{L}$. It remains to show that $\mathcal{L}_1 \cap \mathcal{L}_2 \subseteq \mathcal{L}$. Suppose that $\psi \in \mathcal{L}_1 \cap \mathcal{L}_2$. Then, by local deduction theorem for $\mathbf{FL_{ew}}$ there are substitution instances δ_i $(i = 1, \cdots, n)$ and σ_j $(j = 1, \cdots, m)$ of ϕ_1 and ϕ_2, respectively, such that

$$\vdash_{\mathcal{L}} \prod_{i=1}^{n} \delta_i \to \psi, \quad \vdash_{\mathcal{L}} \prod_{j=1}^{m} \sigma_j \to \psi.$$

Thus,

$$\vdash_{\mathcal{L}} \Big(\prod_{i=1}^{n} \delta_i \vee \prod_{j=1}^{m} \sigma_j\Big) \to \psi.$$

Since for any formulas A_i $(i = 1, \cdots k)$ and B_j $(j = 1, \cdots l)$,

$$\prod_{i=1}^{k} \prod_{j=1}^{l} (A_i \vee B_j) \to \Big(\prod_{i=1}^{k} A_i \vee \prod_{j=1}^{l} B_j\Big)$$

is provable in $\mathbf{FL_{ew}}$,

$$\vdash_{\mathcal{L}} \prod_{i=1}^{n} \prod_{j=1}^{m} (\delta_i \vee \sigma_j) \to \psi.$$

Now each $\delta_i \vee \sigma_j$ is a substitution instance of $\phi_1 \vee \phi_2$, so it is provable in $\mathcal{L}$. Hence, $\psi \in \mathcal{L}$.

(i) $\Rightarrow$ (iii) Let $\mathbf{A}$ be the Lindenbaum-Tarski algebra determined by $\mathcal{L}$ and for any $a \in A - \{1\}$, $\mathcal{F}_a$ be a prime filter obtained by Lemma 6.1. For every formula ϕ define a subset $ref(\phi)$ of $A - \{1\}$ as follows:

$$ref(\phi) = \{a \in A - \{1\} \mid \phi \notin L(\mathbf{A}/\mathcal{F}_a)\},$$

where $\mathbf{A}/\mathcal{F}_a$ denotes the quotient algebra determined by $\mathcal{F}_a$.

We will show that the family $\{ref(\phi) | \phi \notin \mathcal{L}\}$ has the finite intersection property. Suppose $\phi, \psi \notin \mathcal{L}$. If ϕ and ψ have variables in common then take a formula ϕ' which is a renaming of ϕ, i.e., ϕ' and ψ have no variables in common and there exists a bijection from the set of all variables of ϕ to the counterpart of ϕ'. Since $\phi \notin \mathcal{L}$, $\phi' \notin \mathcal{L}$. By Halldén completeness of $\mathcal{L}$, $\phi' \vee \psi \notin \mathcal{L}$. Thus, there is some valuation u on $\mathbf{A}$ such that $u(\phi' \vee \psi) < 1_{\mathbf{A}}$. Let $x = u(\phi' \vee \psi)$. Then by the definition of $\mathcal{F}_x$,

$$\begin{aligned} & u(\phi' \vee \psi) \notin \mathcal{F}_x \\ \Rightarrow\ & u(\phi') \vee_{\mathbf{A}} u(\psi) \notin \mathcal{F}_x \\ \Rightarrow\ & u(\phi') \notin \mathcal{F}_x \text{ and } u(\psi) \notin \mathcal{F}_x. \end{aligned}$$

Define a valuation w by $w(\phi) = u(\phi')$. Then,

$$\begin{aligned} & w(\phi) \notin \mathcal{F}_x \text{ and } u(\psi) \notin \mathcal{F}_x \\ \Rightarrow\ & \phi \notin L(\mathbf{A}/\mathcal{F}_x) \text{ and } \psi \notin L(\mathbf{A}/\mathcal{F}_x) \\ \Rightarrow\ & x \in ref(\phi) \text{ and } x \in ref(\psi) \\ \Rightarrow\ & x \in ref(\phi) \cap ref(\psi). \end{aligned}$$

In a same way, we can show that for finite many $\phi_i \notin \mathcal{L}$ $(0 \leq i \leq n)$, we have $\bigcap_{0\leq i\leq n} ref(\phi_i) \neq \emptyset$, and hence the family $\{ref(\phi)|\phi \notin \mathcal{L}\}$ has the finite intersection property. Thus, there exists some ultrafilter Δ over $A - \{1\}$ such that

$$\{ref(\phi)|\phi \notin \mathcal{L}\} \subseteq \Delta.$$

We will show that the ultraproduct

$$\mathbf{A}^* = \Big(\prod_{a\in A-\{1\}} (\mathbf{A}/\mathcal{F}_a)\Big)\Big/\Delta$$

is a well-connected FL_{ew}-algebra. In the following, the notation $x_{\mathbf{A}}, x_{\mathbf{A}^*}$ and $x_{\mathbf{A}/\mathcal{F}_a}$ stand for elements of $\mathbf{A}, \mathbf{A}^*$ and $\mathbf{A}/\mathcal{F}_a$, respectively. Suppose $x_{\mathbf{A}^*} \vee_{\mathbf{A}^*} y_{\mathbf{A}^*} = 1_{\mathbf{A}^*}$. Then,

$$\begin{aligned}
& \{a \in A - \{1\} \mid (x_{\mathbf{A}^*} \vee_{\mathbf{A}^*} y_{\mathbf{A}^*})(a) = (1_{\mathbf{A}^*})(a)\} \\
= \; & \{a \in A - \{1\} \mid x_{\mathbf{A}/\mathcal{F}_a} \vee_{\mathbf{A}/\mathcal{F}_a} y_{\mathbf{A}/\mathcal{F}_a} = 1_{\mathbf{A}/\mathcal{F}_a}\} \\
= \; & \{a \in A - \{1\} \mid x_{\mathbf{A}} \vee_{\mathbf{A}} y_{\mathbf{A}} \in \mathcal{F}_a\} \\
= \; & \{a \in A - \{1\} \mid x_{\mathbf{A}} \in \mathcal{F}_a \text{ or } y_{\mathbf{A}} \in \mathcal{F}_a\} \qquad \text{(since } \mathcal{F}_a \text{ is prime.)} \\
= \; & \{a \in A - \{1\} \mid x_{\mathbf{A}} \in \mathcal{F}_a\} \cup \{a \in A - \{1\} \mid y_{\mathbf{A}} \in \mathcal{F}_a\} \\
= \; & \{a \in A - \{1\} \mid x_{\mathbf{A}/\mathcal{F}_a} = 1_{\mathbf{A}/\mathcal{F}_a}\} \cup \{a \in A - \{1\} \mid y_{\mathbf{A}/\mathcal{F}_a} = 1_{\mathbf{A}/\mathcal{F}_a}\}.
\end{aligned}$$

By definition of the ultraproduct $\mathbf{A}^*$, $\{a \in A - \{1\} \mid (x_{\mathbf{A}^*} \vee_{\mathbf{A}^*} y_{\mathbf{A}^*})(a) = (1_{\mathbf{A}^*})(a)\} \in \Delta$, so $\{a \in A - \{1\} \mid x_{\mathbf{A}/\mathcal{F}_a} = 1_{\mathbf{A}/\mathcal{F}_a}\} \cup \{a \in A - \{1\} \mid y_{\mathbf{A}/\mathcal{F}_a} = 1_{\mathbf{A}/\mathcal{F}_a}\} \in \Delta$. Since Δ is a ultrafilter,

$$\{a \in A - \{1\} \mid x_{\mathbf{A}/\mathcal{F}_a} = 1_{\mathbf{A}/\mathcal{F}_a}\} \in \Delta \text{ or } \{a \in A - \{1\} \mid y_{\mathbf{A}/\mathcal{F}_a} = 1_{\mathbf{A}/\mathcal{F}_a}\} \in \Delta.$$

Thus, $x_{\mathbf{A}^*} = 1_{\mathbf{A}^*}$ or $y_{\mathbf{A}^*} = 1_{\mathbf{A}^*}$, hence $\mathbf{A}^*$ is well-connected. Moreover,

$$\begin{aligned}
\phi \in \mathcal{L} \;&\Rightarrow\; \phi \in L(\mathbf{A}^*) \quad \text{(since } \mathbf{A}^* \in V(\mathcal{L})) \\
\phi \notin \mathcal{L} \;&\Rightarrow\; \phi \notin L(\mathbf{A}^*) \quad \text{(since } \{ref(\phi) \mid \phi \notin \mathcal{L}\} \subseteq \Delta).
\end{aligned}$$

Therefore, $\mathcal{L} = L(\mathbf{A}^*)$. □

In [43], Wroński showed that for every intermediate logic $\mathcal{L}$, $\mathcal{L}$ is Halldén complete if and only if $\mathcal{L} = L(\mathbf{A})$ for some *subdirectly irreducible* Heyting algebra $\mathbf{A}$. On the other hand, by Proposition 3.7, an FL_{ew}-algebra $\mathbf{A}$ is subdirectly irreducible if and only if

$$\exists a \ngeq 1 \; \forall x < 1 \; \exists n \in \omega, \; s.t, \quad x^n \leq a.$$

Since this is not a first-order sentence, it is not necessarily preserved under ultraproducts. Thus, we can't extend it in the same way as in Wroński's proof. It is an open problem whether Halldén completeness of a logic over $\mathbf{FL_{ew}}$ is characterized by some subdirectly irreducible FL_{ew}-algebra. But if we assume the n-potency, i.e., $\phi^n \to \phi^{n+1}$, to a logic $\mathcal{L}$ over $\mathbf{FL_{ew}}$, then we can show that $\mathcal{L}$ is Halldén complete if and only if $\mathcal{L} = L(\mathbf{A})$ for some subdirectly irreducible FL_{ew}-algebra.

The previous theorem doesn't hold always if we replace $\mathbf{FL_{ew}}$ by $\mathbf{FL_e}$, and FL_{ew}-algebras by FL_e-algebras, since 1 is not the greatest element on FL_e-algebras. In other words, we need to modify definitions of Halldén completeness and well-connectedness so as to make it true.

LEMMA 6.3 *Let $\mathcal{G}$ be a proper filter of FL_e-algebra $\mathbf{A}$ and $a \notin \mathcal{G}$. Then there exists a filter $\mathcal{F}_a$ which is maximal in the set*

$$\Sigma = \{\mathcal{F} : filter | \mathcal{G} \subseteq \mathcal{F}, a \notin \mathcal{F}\}.$$

Moreover, $\mathcal{F}_a$ satisfies the following condition:

$$if \ (x \wedge 1) \vee (y \wedge 1) \in \mathcal{F}_a \ then \ x \in \mathcal{F}_a \ or \ y \in \mathcal{F}_a.$$

(proof) By analogy with the proof of Lemma 6.1. □

Note that the above condition is equal to the following condition:

$$\text{if } (x \wedge 1) \vee (y \wedge 1) \in \mathcal{F}_a \text{ then } x \wedge 1 \in \mathcal{F}_a \text{ or } y \wedge 1 \in \mathcal{F}_a.$$

Therefore, when $\mathbf{A}$ is a FL_{ew}-algebra, i.e., 1 is the greatest element of $\mathbf{A}$, the above condition is equal to primeness.

LEMMA 6.4 *For every logic $\mathcal{L}$ over $\mathbf{FL_e}$, if $\delta_i \to 1$ and $\gamma_j \to 1$ $(i = 1, \cdots, m, j = 1, \cdots, n)$ are provable in $\mathcal{L}$ then*

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(\delta_i \vee \gamma_j) \to (\prod_{i=1}^{m}\delta_i \vee \prod_{j=1}^{n}\gamma_j)$$

is also provable in $\mathcal{L}$.

(proof) By induction on $k = (m, n)$. It is trivial if $k = (1, 1)$. Suppose that the above holds for $k = (1, l)$. Then,

$$\dfrac{\dfrac{\Pi_{j=1}^{l}(\delta_1 \vee \gamma_j) \Rightarrow \delta_1 \vee \Pi_{j=1}^{l}\gamma_j \quad \delta_1 \vee \gamma_{l+1} \Rightarrow \delta_1 \vee \gamma_{l+1}}{(\Pi_{j=1}^{l}(\delta_1 \vee \gamma_j)), (\delta_1 \vee \gamma_{l+1}) \Rightarrow (\delta_1 \vee \Pi_{j=1}^{l}\gamma_j) \cdot (\delta_1 \vee \gamma_{l+1})}\ (\cdot \Rightarrow)}{(\Pi_{j=1}^{l}(\delta_1 \vee \gamma_j)) \cdot (\delta_1 \vee \gamma_{l+1}) \Rightarrow (\delta_1 \vee \Pi_{j=1}^{l}\gamma_j) \cdot (\delta_1 \vee \gamma_{l+1})}\ (\Rightarrow \cdot),$$

so $(\Pi_{j=1}^{l}(\delta_1 \vee \gamma_j)) \cdot (\delta_1 \vee \gamma_{l+1}) \to (\delta_1 \vee \Pi_{j=1}^{l}\gamma_j) \cdot (\delta_1 \vee \gamma_{l+1})$ is provable in $\mathcal{L}$. Now, by distributivity of $\cdot$ with $\vee$, and our asomptions of δ_i and γ_j, the formula $(\delta_1 \vee \prod_{j=1}^{l}\gamma_j) \cdot (\delta_1 \vee \gamma_{l+1}) \to (\delta_1 \vee \prod_{j=1}^{l+1}\gamma_j)$ is also provable in $\mathcal{L}$. Thus, $\prod_{j=1}^{l+1}(\delta_1 \vee \gamma_i) \to (\delta_1 \vee \prod_{j=1}^{l+1}\gamma_j)$ is provable in $\mathcal{L}$. One can show in the same way for m. □

As the above lemmas show, it seems to be necessary to modify the notion of Halldén completeness and well-connectedness. The following conditions **(i)** and **(*)** seem to be strictly weaker than Halldén completeness and well-connectedness, respectively. Nevertheless, they characterize the meet-irreducibility of a given logic.

COROLLARY 6.5 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$. Then the following are equivalent:*

(i) *for every formulas ϕ and ψ which have no variables in common,*

$$\vdash_{\mathcal{L}} (\phi \wedge 1) \vee (\psi \wedge 1) \ implies \ \vdash_{\mathcal{L}} \phi \ or \ \vdash_{\mathcal{L}} \psi,$$

(ii) *$\mathcal{L}$ is meet-irreducible in the lattice of all logics over* $\mathbf{FL_e}$*,*

(iii) *$\mathcal{L} = L(\mathbf{A})$ for some FL_e-algebra $\mathbf{A}$ satisfying the following.*

(*) *for any $x, y \in A^- = \{a \in A | a \leq 1\}$,*

$$if\ x \vee y = 1\ then\ x = 1\ or\ y = 1.$$

(proof) (iii) $\Rightarrow$ (ii) Note that for any logic $\mathcal{L}$ over $\mathbf{FL_e}$, $\phi \in \mathcal{L}$ implies $\phi \wedge 1 \in \mathcal{L}$. Then, by analogy with the proof of Theorem 6.2, one can show this direction.

(ii) $\Rightarrow$ (i) In the same way as the proof of Theorem 6.2, we will prove this by taking a contraposition. Suppose that there exist some ϕ_1, ϕ_2 which have no variables in common, $\vdash_{\mathcal{L}} (\phi_1 \wedge 1) \vee (\phi_2 \wedge 1)$ but $\not\vdash_{\mathcal{L}} \phi_1$ and $\not\vdash_{\mathcal{L}} \phi_2$. Let $\mathcal{L}_i = \mathcal{L} + \phi_i$ $(i = 1, 2)$. It is sufficient to show that $\mathcal{L}_1 \cap \mathcal{L}_2 \subseteq \mathcal{L}$. For any $\psi \in \mathcal{L}_1 \cap \mathcal{L}_2$, by local deduction theorem for $\mathbf{FL_e}$ there are substitution instances δ_i $(i = 1, \cdots, n)$ and σ_j $(j = 1, \cdots, m)$ of ϕ_1 and ϕ_2, respectively, such that

$$\vdash_{\mathcal{L}} \prod_{i=1}^{n} (\delta_i \wedge 1) \to \psi, \quad \vdash_{\mathcal{L}} \prod_{j=1}^{m} (\sigma_j \wedge 1) \to \psi.$$

Thus,

$$\vdash_{\mathcal{L}} \Big(\prod_{i=1}^{n} (\delta_i \wedge 1) \vee \prod_{j=1}^{m} (\sigma_j \wedge 1)\Big) \to \psi.$$

Note that $(\delta_i \wedge 1) \to 1$ $(i = 1, \ldots, n)$ and $(\sigma_j \wedge 1) \to 1$ $(j = 1, \ldots, m)$ are provable in $\mathcal{L}$, so by Lemma 6.4,

$$\prod_{i=1}^{n} \prod_{j=1}^{m} ((\delta_i \wedge 1) \vee (\sigma_j \wedge 1)) \to \Big(\prod_{i=1}^{n} (\delta_i \wedge 1) \vee \prod_{j=1}^{m} (\sigma_j \wedge 1)\Big)$$

is provable in $\mathcal{L}$. Hence

$$\vdash_{\mathcal{L}} \prod_{i=1}^{n} \prod_{j=1}^{m} ((\delta_i \wedge 1) \vee (\sigma_j \wedge 1)) \to \psi.$$

Therefore $\psi \in \mathcal{L}$.

(i) $\Rightarrow$ (iii) Let $\mathbf{A}$ be a Lindenbaum algebra of $\mathcal{L}$ and for any $a \in \bar{A} = \{a \in A | a \not\geq 1\}$, $\mathcal{F}_a$ a filter obtained by Lemma 6.3. For every formula ϕ, define a subset $ref(\phi)$ of $\bar{A}$ as follows:

$$ref(\phi) = \{a \in \bar{A} \mid \phi \notin L(\mathbf{A}/\mathcal{F}_a)\}.$$

In the same way as Theorem 6.2, one can show that the family $\{ref(\phi) | \phi \notin \mathcal{L}\}$ has the finite intersection property. Thus, there exists some ultrafilter Δ over $\bar{A}$ such that

$$\{ref(\phi) | \phi \notin \mathcal{L}\} \subseteq \Delta.$$

We will show that the ultraproduct

$$\mathbf{A}^* = \Big(\prod_{a \in \bar{A}} (\mathbf{A}/\mathcal{F}_a)\Big) \Big/ \Delta$$

satisfies the condition (*). Note that the condition (*) is equivalent to the following:

$$\forall x, y \in A, \quad x \not\geq 1 \text{ and } y \not\geq 1 \text{ implies } (x \wedge 1) \vee (y \wedge 1) \neq 1.$$

If $x_{\mathbf{A}^*} \not\geq 1_{\mathbf{A}^*}$ then $x_{\mathbf{A}^*} \wedge_{\mathbf{A}^*} 1_{\mathbf{A}^*} \neq 1_{\mathbf{A}^*}$, so,

$$\begin{array}{rl} & \{a \in \bar{A} \mid (x_{\mathbf{A}^*} \wedge_{\mathbf{A}^*} 1_{\mathbf{A}^*})(a) \neq 1_{\mathbf{A}^*}(a)\} \in \Delta \\ \Rightarrow & \{a \in \bar{A} \mid x_{\mathbf{A}/\mathcal{F}_a} \wedge_{\mathbf{A}/\mathcal{F}_a} 1_{\mathbf{A}/\mathcal{F}_a} \neq 1_{\mathbf{A}/\mathcal{F}_a}\} \in \Delta \\ \Rightarrow & \{a \in \bar{A} \mid x_{\mathbf{A}} \wedge_{\mathbf{A}} 1_{\mathbf{A}} \notin \mathcal{F}_a\} \in \Delta \\ \Rightarrow & \{a \in \bar{A} \mid x_{\mathbf{A}} \notin \mathcal{F}_a\} \in \Delta. \end{array}$$

Hence for any $x_{\mathbf{A}^*} \not\geq 1_{\mathbf{A}^*}$ and $y_{\mathbf{A}^*} \not\geq 1_{\mathbf{A}^*}$,

$$\begin{array}{rl} & \{a \in \bar{A} \mid x_{\mathbf{A}} \notin \mathcal{F}_a\} \in \Delta \text{ and } \{a \in \bar{A} \mid y_{\mathbf{A}} \notin \mathcal{F}_a\} \in \Delta \\ \Rightarrow & \{a \in \bar{A} \mid x_{\mathbf{A}} \notin \mathcal{F}_a\} \cap \{a \in \bar{A} \mid y_{\mathbf{A}} \notin \mathcal{F}_a\} \in \Delta \\ \Rightarrow & \{a \in \bar{A} \mid x_{\mathbf{A}} \notin \mathcal{F}_a \text{ and } y_{\mathbf{A}} \notin \mathcal{F}_a\} \in \Delta. \end{array}$$

By the condition of the filter $\mathcal{F}_a$,

$$\{a \in \bar{A} \mid x_{\mathbf{A}} \notin \mathcal{F}_a \text{ and } y_{\mathbf{A}} \notin \mathcal{F}_a\} = \{a \in \bar{A} \mid (x_{\mathbf{A}} \wedge_{\mathbf{A}} 1_{\mathbf{A}}) \vee_{\mathbf{A}} (y_{\mathbf{A}} \wedge_{\mathbf{A}} 1_{\mathbf{A}}) \notin \mathcal{F}_a\}.$$

Thus, $\{a \in \bar{A} \mid (x_{\mathbf{A}} \wedge_{\mathbf{A}} 1_{\mathbf{A}}) \vee_{\mathbf{A}} (y_{\mathbf{A}} \wedge_{\mathbf{A}} 1_{\mathbf{A}}) \notin \mathcal{F}_a\} \in \Delta$. So,

$$\begin{array}{rl} & \{a \in \bar{A} \mid (((x_{\mathbf{A}^*} \wedge_{\mathbf{A}^*} 1_{\mathbf{A}^*}) \vee_{\mathbf{A}^*} (y_{\mathbf{A}^*} \wedge_{\mathbf{A}^*} 1_{\mathbf{A}^*})) \wedge 1_{\mathbf{A}^*})(a) \neq 1_{\mathbf{A}^*}(a)\} \in \Delta \\ \Rightarrow & ((x_{\mathbf{A}^*} \wedge_{\mathbf{A}^*} 1_{\mathbf{A}^*}) \vee_{\mathbf{A}^*} (y_{\mathbf{A}^*} \wedge_{\mathbf{A}^*} 1_{\mathbf{A}^*})) \wedge 1_{\mathbf{A}^*} \neq 1_{\mathbf{A}^*} \\ \Rightarrow & (x_{\mathbf{A}^*} \wedge_{\mathbf{A}^*} 1_{\mathbf{A}^*}) \vee_{\mathbf{A}^*} (y_{\mathbf{A}^*} \wedge_{\mathbf{A}^*} 1_{\mathbf{A}^*}) \neq 1_{\mathbf{A}^*}. \end{array}$$

Moreover,

$$\begin{array}{lcll} \phi \in \mathcal{L} & \Rightarrow & \phi \in L(\mathbf{A}^*) & (\text{since } \mathbf{A}^* \in V(\mathcal{L})) \\ \phi \notin \mathcal{L} & \Rightarrow & \phi \notin L(\mathbf{A}^*) & (\text{since } \{ref(\phi) \mid \phi \notin \mathcal{L}\} \subseteq \Delta). \end{array}$$

Therefore, $\mathcal{L} = L(\mathbf{A}^*)$. □

Obviously, each of **(i)** and **(iii)** is equivalent to the counterpart of Theorem 6.2, when the weakening rule or the distributive law holds in a given logic $\mathcal{L}$. Moreover, whenever the axiom of n-potency, i.e., $\phi^n \to \phi^{n+1}$, holds in $\mathcal{L}$ over $\mathbf{FL_e}$, they are equivalent to the following (the same holds for logics over $\mathbf{FL_{ew}}$):

(iv) $\mathcal{L} = L(\mathbf{A})$ for some subdirectly irreducible FL_e-algebra $\mathbf{A}$ satisfing $x^n \leq x^{n+1}$.

6.2 An Alternative Characterization of Halldén Completeness

In the previous section, we have already seen that it is necessarly to modify the definition of Halldén completeness in order to give the similar algebraic characterizaiton of it for logics over $\mathbf{FL_e}$. In this section, we will give another algebraic characterization of Halldén completeness in the original form and see an algebraic relation between the disjunction property and Halldén completeness.

We say that the subalgebras $\mathbf{B}, \mathbf{C}$ of $\mathbf{A}$ are a *well-connected pair* if for any elements $x \in B$ and $y \in C$, $x \vee_{\mathbf{A}} y \geq 1_{\mathbf{A}}$ implies $x \geq 1_{\mathbf{A}}$ or $y \geq 1_{\mathbf{A}}$. It is easy to see that the following conditions are equivalent;

1. $\mathbf{A}$ is a well-connected algebra,

2. every two subalgebras $\mathbf{B}, \mathbf{C}$ of $\mathbf{A}$ form a well-connected pair,

3. $\mathbf{A}, \mathbf{A}$ are a well-connected pair of $\mathbf{A}$.

LEMMA 6.6 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$ and $\mathbf{F}_{V(\mathcal{L})}(\bar{X})$ a $V(\mathcal{L})$-free algebra. Then the following are equivalent;*

1. *$\mathcal{L}$ is Halldén complete,*

2. *for each disjoint pair of subsets $\bar{X}_1, \bar{X}_2$ of $\bar{X}$, the subalgebras $\mathbf{B}(\bar{X}_1)$ and $\mathbf{C}(\bar{X}_2)$ of $\mathbf{F}_{V(\mathcal{L})}(\bar{X})$ generated by $\bar{X}_1$ and $\bar{X}_2$, respectively, are a well-connected pair.*

(proof) *1* $\Rightarrow$ *2.* Let $\bar{X}_1$ and $\bar{X}_2$ be disjoint subsets of $\bar{X}$, and $\mathbf{B}(\bar{X}_1)$ and $\mathbf{C}(\bar{X}_2)$ be the subalgebras of $\mathbf{F}_{V(\mathcal{L})}(\bar{X})$ generated by $\bar{X}_1$ and $\bar{X}_2$, respectively. Then, for any $b \in B(\bar{X}_1), c \in C(\bar{X}_2)$ there are some terms p, q such that

$$b = p(\bar{X}_1) \quad \text{and} \quad c = q(\bar{X}_2).$$

Suppose that $b = p(\bar{X}_1) \not\geq 1_{\mathbf{F}_{V(\mathcal{L})}(\bar{X})}$ and $c = q(\bar{X}_2) \not\geq 1_{\mathbf{F}_{V(\mathcal{L})}(\bar{X})}$. Then neither $\vdash_{\mathcal{L}} p(X_1)$ nor $\vdash_{\mathcal{L}} q(X_2)$ hold. Now $p(X_1)$ and $q(X_2)$ have no variables in common, by our assumption *1*, $\vdash_{\mathcal{L}} p(X_1) \vee q(X_2)$ doesn't hold. Hence $p(\bar{X}_1) \vee q(\bar{X}_2) \not\geq 1_{\mathbf{F}_{V(\mathcal{L})}(\bar{X})}$.

2 $\Rightarrow$ *1.* Suppose that there exist some formulas $\phi(\mathbf{p})$ and $\psi(\mathbf{q})$, where $\mathbf{p}$ and $\mathbf{q}$ are disjoint lists of variables, such that neither $\vdash_{\mathcal{L}} \phi(\mathbf{p})$ nor $\vdash_{\mathcal{L}} \psi(\mathbf{q})$ hold. Let $X = \mathbf{p} \cup \mathbf{q}$ and $\mathbf{F}_{V(\mathcal{L})}(\bar{X})$ the $V(\mathcal{L})$-free algebra. Since $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$ are disjoint subsets of $\bar{X}$, by our assumption *2*, the subalgebras $\mathbf{B}(\bar{\mathbf{p}})$ and $\mathbf{C}(\bar{\mathbf{q}})$ generated by $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$, respectively, are well-connected pair of $\mathbf{F}_{V(\mathcal{L})}(\bar{X})$. Let v be the valuation on $\mathbf{F}_{V(\mathcal{L})}(\bar{X})$ defined by the canonical mapping, i.e.,

$$v(p) = \bar{p} \quad \text{for} \quad p \in X = \mathbf{p} \cup \mathbf{q}.$$

Then,

$$v(\phi(\mathbf{p})) = \phi(\bar{\mathbf{p}}) \not\geq 1_{\mathbf{F}_{V(\mathcal{L})}(\bar{X})} \quad \text{and} \quad \phi(\bar{\mathbf{p}}) \in \mathbf{B}(\bar{\mathbf{p}}),$$
$$v(\psi(\mathbf{q})) = \psi(\bar{\mathbf{q}}) \not\geq 1_{\mathbf{F}_{V(\mathcal{L})}(\bar{X})} \quad \text{and} \quad \psi(\bar{\mathbf{q}}) \in \mathbf{C}(\bar{\mathbf{q}}).$$

Hence $\phi(\bar{\mathbf{p}}) \vee \psi(\bar{\mathbf{q}}) \not\geq 1_{\mathbf{F}_{V(\mathcal{L})}(\bar{X})}$, so $\vdash_{\mathcal{L}} \phi(\mathbf{p}) \vee \psi(\mathbf{q})$ doesn't hold. □

THEOREM 6.7 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$. Then the following are equivalent;*

1. *$\mathcal{L}$ is Halldén complete,*

2. *for every two non-degenerate FL_e-algebras $\mathbf{A}, \mathbf{B}$ in $V(\mathcal{L})$, there is a well-connected pair $\mathbf{C}_1, \mathbf{C}_2$ of some algebra $\mathbf{C}$ in $V(\mathcal{L})$ such that $\mathbf{A}$ and $\mathbf{B}$ are quotient algebras of $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively.*

(proof) *1* $\Rightarrow$ *2.* For any non-degenerate algebras $\mathbf{A}, \mathbf{B}$ in $V(\mathcal{L})$, let

$$\mathbf{p} = \{p_a | a \in A\}, \quad \mathbf{q} = \{q_b | b \in B\}, \quad \mathbf{r} = \mathbf{p} \cup \mathbf{q}.$$

Denote $\mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{r}}), \mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{p}})$ and $\mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{q}})$ the $V(\mathcal{L})$-free algebras generated by $\bar{\mathbf{r}}, \bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$, respectively. Clearly, $\mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{r}}) \in V(\mathcal{L})$. Now $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$ are disjoint sets of $\bar{\mathbf{r}}$, by our assumption *1* and Lemma 6.6, the subalgebras $\mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{p}}), \mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{q}})$ of $\mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{r}})$ are a well-connected pair. Since $|\bar{\mathbf{p}}| = |\mathbf{A}|$ and $|\bar{\mathbf{q}}| = |\mathbf{B}|$, there are onto homomorphisms

$$\alpha : \mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{p}}) \to \mathbf{A} \quad \text{and} \quad \beta : \mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{q}}) \to \mathbf{B}.$$

Thus, $\mathbf{A}$ and $\mathbf{B}$ are quotient algebras of $\mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{p}})$ and $\mathbf{F}_{V(\mathcal{L})}(\bar{\mathbf{q}})$, respectively.

2 $\Rightarrow$ *1*. Assume that there are some formulas $\phi(\mathbf{p})$ and $\psi(\mathbf{q})$, where $\mathbf{p}$ and $\mathbf{q}$ are disjoint lists of variables, such that neither $\vdash_{\mathcal{L}} \phi(\mathbf{p})$ nor $\vdash_{\mathcal{L}} \psi(\mathbf{q})$ hold. Then there are some FL_e-algebras $\mathbf{A}, \mathbf{B}$ in $V(\mathcal{L})$ and valuations v_1 on $\mathbf{A}$ and v_2 on $\mathbf{B}$ such that

$$v_1(\phi) \not\geq 1_{\mathbf{A}} \quad \text{and} \quad v_2(\psi) \not\geq 1_{\mathbf{B}}.$$

By our assumption *2*, there are a well-connected pair $\mathbf{C}_1, \mathbf{C}_2$ of some $\mathbf{C}$ in $V(\mathcal{L})$ and onto homomorphisms $\alpha : \mathbf{C}_1 \to \mathbf{A}$ and $\beta : \mathbf{C}_2 \to \mathbf{B}$. Since $\mathbf{p}$ and $\mathbf{q}$ are disjoint, we can define a valuation w on $\mathcal{C}$ by

$$\begin{array}{ll} w(p) = c & \text{for } p \in \mathbf{p}, \\ w(q) = d & \text{for } q \in \mathbf{q}, \end{array}$$

where c and d are arbitrary elements in $\alpha^{-1}v_1(p)$ and $\beta^{-1}v_2(q)$, respectively. Then,

$$\begin{array}{l} \alpha w(\phi) = v_1(\phi) \not\geq 1_{\mathbf{A}}, \\ \beta w(\psi) = v_2(\psi) \not\geq 1_{\mathbf{B}}, \end{array}$$

hence $w(\phi) \not\geq 1_{\mathbf{C}}, w(\phi) \in C_1$ and $w(\psi) \not\geq 1_{\mathbf{C}}, w(\psi) \in C_2$. Thus

$$w(\phi \vee \psi) = w(\phi) \vee w(\psi) \not\geq 1_{\mathbf{C}},$$

so $\vdash_{\mathcal{L}} \phi \vee \psi$ doesn't hold. □

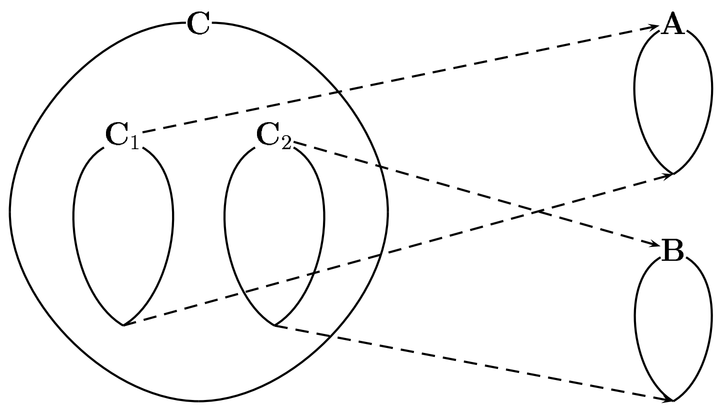

Figure 6.1: characterization of HC

D. Souma [37] has shown that a characterization of the disjunction property for intermediate logics given by L. Maksimova [27] holds also for logics over $\mathbf{FL_e}$. More precisely, the following holds.

PROPOSITION 6.8 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$. Then the following are equivalent;*

1. *$\mathcal{L}$ has the disjunction property,*
2. *for any FL_e-algebras $\mathbf{A}, \mathbf{B}$ in $V(\mathcal{L})$ there exists a well-connected $\mathbf{C}$ in $V(\mathcal{L})$ such that $\mathbf{A} \times \mathbf{B}$ is a quotient algebra of $\mathbf{C}$.*

Syntactically, it is easy to see that the disjunction property implies Halldén completeness and the condition (i) of Corollary 6.5. But semantically, it has not been so clear why the above condition *2* implies the condition (iii) of Corollary 6.5. Now, by using the characterization of Theorem 6.7, we can give a direct proof of it, since if $\mathbf{C}$ is well-connected then $\mathbf{C}$ and $\mathbf{C}$ are a well-connected pair of it.

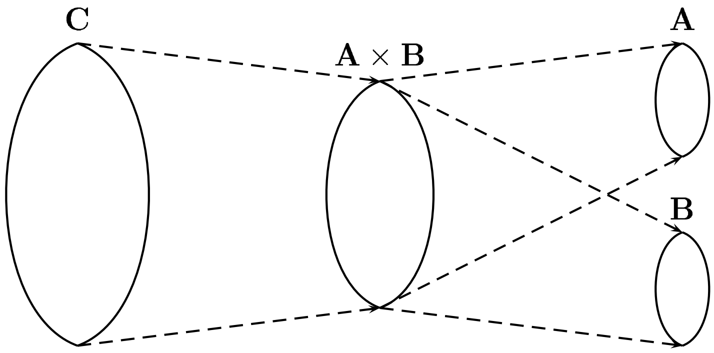

Figure 6.2: DP implies HC

6.3 Principle of Variable Separation

We say that a logic $\mathcal{L}$ has the *principle of variable separation* (PVS) if for every formulas $\phi_1, \phi_2, \psi_1, \psi_2$, where $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ have no variables in common, the condition $\vdash_{\mathcal{L}} \phi_1 \wedge \psi_1 \rightarrow \phi_2 \vee \psi_2$ implies $\vdash_{\mathcal{L}} \phi_1 \rightarrow \phi_2$ or $\vdash_{\mathcal{L}} \psi_1 \rightarrow \psi_2$. Clearly, both Halldén completeness and PRP are special cases of PVS.

We say that a logic $\mathcal{L}$ has the *deductive principle of variable separation* (DPVS) if for every formulas $\phi_1, \phi_2, \psi_1, \psi_2$, where $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ have no variables in common, the condition $\phi_1 \wedge \psi_1 \vdash_{\mathcal{L}} \phi_2 \vee \psi_2$ implies $\phi_1 \vdash_{\mathcal{L}} \phi_2$ or $\psi_1 \vdash_{\mathcal{L}} \psi_2$. It is easy to see that both Halldén completeness and DPRP are special cases of DPVS. Note that for any logic $\mathcal{L}$ over $\mathbf{FL_e}$,

$$\phi \wedge \psi \vdash_{\mathcal{L}} \delta \iff \phi, \psi \vdash_{\mathcal{L}} \delta.$$

Hence, as with DPRP, we can replace formulas ϕ_1 and ψ_1 by sets of formulas Γ and Σ, respectively, i.e., whenever $\Gamma \cup \{\phi_2\}$ and $\Sigma \cup \{\psi_2\}$ have no variables in common,

$$\Gamma, \Sigma \vdash_{\mathcal{L}} \phi_2 \vee \psi_2 \text{ implies } \Gamma \vdash_{\mathcal{L}} \phi_2 \text{ or } \Sigma \vdash_{\mathcal{L}} \psi_2.$$

In the case of intermediate logics, by the deduction theorem, PVS and DPVS are equivalent logical properties. But this is not true in general.

In [8], Chagrov and Zakharyaschev showed that there are a continuum of intermediate logics which are Halldén complete but don't have the PVS. Moreover, they showed that there are a continuum of intermediate logics which have both the PVS and the disjunction property, and as many intermediate logics which have the PVS but don't have the disjunction property. Thus, the relation between the disjunction property, Halldén completeness, and PVS for intermediate logics can be represented as following figure in which the cardinality of each set of logics are continuum.

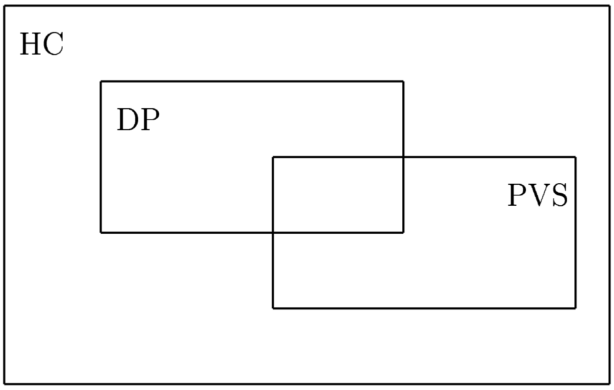

Figure 6.3:

Note that since every intermediate logics is a logic over $\mathbf{FL_e}$, the above relation holds also for logics over $\mathbf{FL_e}$.

By using a syntactic method, Naruse, Surarso and Ono [30] showed that if our language does not contain any propositional constant then PVS holds for $\mathbf{FL_e}$. On the other hand, since DPRP doesn't hold for $\mathbf{FL_e}$, neither PVS nor DPVS holds for $\mathbf{FL_e}$ when our language contains propositional constants.

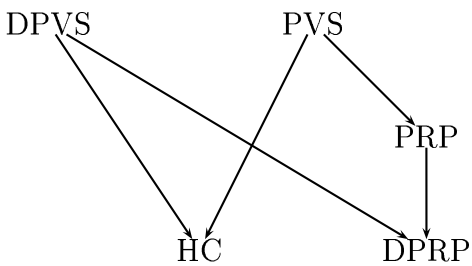

Figure 6.4: relations among several properties

As described before, PVS and DPVS are equivalent logical properties for intermediate logics. Moreover, for logics over $\mathbf{FL_{ec}}$, PVS implies DPVS. For, if $\phi_1 \wedge \psi_1 \vdash_{\mathcal{L}} \phi_1 \vee \psi_2$ holds for formulas $\phi_1, \phi_2, \psi_1, \psi_2$, where $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ have no variables in common, then $\vdash_{\mathcal{L}} (\phi_1 \wedge \psi_1 \wedge 1)^n \to \phi_2 \vee \psi_2$ holds for some $n \in \mathbb{N}$. Note that a formula $\sigma_1 \wedge \sigma_2 \wedge 1$ is equivalent to the formula $(\sigma_1 \wedge 1) \wedge (\sigma_2 \wedge 1)$, and a formula $\delta \to \delta^n$ is provable in $\mathcal{L}$ by using the contraction rule, so, we have $\vdash_{\mathcal{L}} ((\phi_1 \wedge 1) \wedge (\psi_1 \wedge 1)) \to \phi_2 \vee \psi_2$. By PVS, $\vdash_{\mathcal{L}} (\phi_1 \wedge 1) \to \phi_2$ and $\vdash_{\mathcal{L}} (\psi_1 \wedge 1) \to \psi_2$ hold. Thus, we have $\phi_1 \vdash_{\mathcal{L}} \phi_2$ and $\psi_1 \vdash_{\mathcal{L}} \psi_2$, i.e., $\mathcal{L}$ has DPVS.

But, for logics over $\mathbf{FL_e}$, we don't know a general relations between PVS and DPVS.

In this and next sections, we will discuss algebraic characterizations of DPVS and semantical relations between Halldén completeness and DPVS.

The following result is a generalization of the result for intermediate logics given by Maksimova [28].

THEOREM 6.9 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_{ew}}$. Then the following are equivalent:*

1. *DPVS holds in $\mathcal{L}$,*
2. *the class of all subdirectly irreducible FL_{ew}-algebras of $V(\mathcal{L})$ has the JEP,*
3. *every two subdirectly irreducible FL_{ew}-algebras of $V(\mathcal{L})$ are jointly embeddable into a well-connected algebra in $V(\mathcal{L})$.*

(proof) *1* $\Rightarrow$ *2*. Assume that DPVS holds in $\mathcal{L}$ and $\mathbf{A}$ and $\mathbf{B}$ in $V(\mathcal{L})$ are subdirectly irreducible with its opremum c and d, respectively. Similarly with Theorem 5.3, let

$$\mathbf{p}' = \{p_a \mid a \in \mathbf{A}\}, \quad \mathbf{p}'' = \{q_b \mid b \in \mathbf{B}\}, \quad \mathbf{p} = \mathbf{p}' \cup \mathbf{p}'',$$

denote by $\mathcal{F}m(\mathbf{p}), \mathcal{F}m(\mathbf{p}')$ and $\mathcal{F}m(\mathbf{p}'')$ the sets of all formulas with variables in $\mathbf{p}, \mathbf{p}'$ and $\mathbf{p}''$, respectively. Let us take the valuations $v' : \mathbf{p}' \to \mathbf{A}$ and $v'' : \mathbf{p}'' \to \mathbf{B}$ determined by

$$v'(p_a) = a \text{ for } a \in \mathbf{A}, \qquad v''(q_b) = b \text{ for } b \in \mathbf{B}.$$

Let

$$T' = \{\phi \in \mathcal{F}m(\mathbf{p}') \mid v'(\phi) = 1_{\mathbf{A}}\}, \quad T'' = \{\psi \in \mathcal{F}m(\mathbf{p}'') \mid v''(\psi) = 1_{\mathbf{B}}\}.$$

Obviously $p_c \notin T'$ and $q_d \notin T''$. Denote

$$T^* = \{\sigma \in \mathcal{F}m(\mathbf{p}) \mid T' \cup T'' \vdash_{\mathcal{L}} \sigma\},$$

then T^* is an $\mathcal{L}$-theory of the language $\mathcal{F}m(\mathbf{p})$. Suppose that the formula $p_c \vee q_d$ belongs to T^*. Then $T' \cup T'' \vdash_{\mathcal{L}} p_c \vee q_d$. Since $T' \cup \{p_c\}$ and $T'' \cup \{q_d\}$ have no variables in common, by DPVS, it follows

$$T' \vdash_{\mathcal{L}} p_c \quad \text{or} \quad T'' \vdash_{\mathcal{L}} q_d.$$

But the former case is refutable on $\mathbf{A}$ and the latter is also refutable on $\mathbf{B}$. Thus, T^* doesn't contain $p_c \vee q_d$. By Proposition 3.34, there exists some maximal $\mathcal{L}$-theory T of the language $\mathcal{F}m(\mathbf{p})$ such that $T^* \subseteq T$ and $p_c \vee q_d \notin T$. Then the Lindenbaum-Tarski algebra $\mathbf{C} = \mathbf{C}(\mathbf{p}, T) = \mathcal{F}m(\mathbf{p})/\sim_T$ is subdirectly irreducible, and $||p_c \vee q_d||$ is its opremum. Let us define mappings $\delta : \mathbf{A} \to \mathbf{C}$ and $\epsilon : \mathbf{B} \to \mathbf{C}$ as follows;

$$\begin{aligned} \delta(a) &= ||p_a|| \quad \text{for} \quad a \in \mathbf{A}, \\ \epsilon(b) &= ||q_b|| \quad \text{for} \quad b \in \mathbf{B}. \end{aligned}$$

Since $T', T'' \subseteq T$, δ and ϵ are homomorphisms. Now we will show that δ and ϵ are monomorphisms. It is sufficient to show that $\delta(a) \neq 1_{\mathbf{C}}$ for $a \neq 1_{\mathbf{A}}$ and $\epsilon(b) \neq 1_{\mathbf{C}}$ for

$b \neq 1_{\mathbf{B}}$. As c is an opremum of $\mathbf{A}$, for every $a \neq 1_{\mathbf{A}}$ there exists some $n \in \omega$ such that $a^n \leq c$. So,

$$\delta(a^n) \leq \delta(c) = ||p_c|| \leq ||p_c \vee q_d|| < 1_{\mathbf{C}}.$$

Hence, $\delta(a) \neq 1_{\mathbf{C}}$. Analogously, $\epsilon(b) \neq 1_{\mathbf{C}}$ for $b \neq 1_{\mathbf{B}}$.

2 $\Rightarrow$ *3.* It is obvious, since if FL_{ew}-algebra is subdirectly irreducible then it is well-connected (see e.g. Theorem 4.2 in [31]).

3 $\Rightarrow$ *1.* Let us suppose that $\mathbf{p}$ and $\mathbf{q}$ are disjoint lists of variables and neither $\phi_1(\mathbf{p}) \vdash_{\mathcal{L}} \phi_2(\mathbf{p})$ nor $\psi_1(\mathbf{q}) \vdash_{\mathcal{L}} \psi_2(\mathbf{q})$ hold. By Zorn's lemma, we can get an $\mathcal{L}$-theory T_{ϕ_1} of the language $\mathcal{F}m(\mathbf{p})$ such that it is maximal among $\mathcal{L}$-theories T which contain ϕ_1 but don't contain ϕ_2. Similarly, let T_{ψ_1} be a maximal $\mathcal{L}$-theory of the language $\mathcal{F}m(\mathbf{q})$ with respect to ψ_2. Let $\mathbf{A} = \mathcal{F}m(\mathbf{p})/\sim_{T_{\phi_1}}$ and $\mathbf{B} = \mathcal{F}m(\mathbf{q})/\sim_{T_{\psi_1}}$. Then they are subdirectly irreducible algebras in $V(\mathcal{L})$ such that $||\phi_2||$ and $||\psi_2||$ are oprema of $\mathbf{A}$ and $\mathbf{B}$, respectively, and the canonical mappings $v : \mathcal{F}m(\mathbf{p}) \to \mathbf{A}$ and $u : \mathcal{F}m(\mathbf{q}) \to \mathbf{B}$ satisfies

$$\begin{array}{l} v(\phi_1) = ||\phi_1|| = 1_{\mathbf{A}} \\ v(\phi_2) = ||\phi_2|| < 1_{\mathbf{A}} \end{array} \quad \text{and} \quad \begin{array}{l} u(\psi_1) = ||\psi_1|| = 1_{\mathbf{B}} \\ u(\psi_2) = ||\psi_2|| < 1_{\mathbf{B}}. \end{array}$$

By our assumption *3*, there exist a well-connected algebra $\mathbf{C} \in V(\mathcal{L})$ and monomorphisms α from $\mathbf{A}$ into $\mathbf{C}$ and β from $\mathbf{B}$ into $\mathbf{C}$. Since

$$\alpha(v(\phi_2)) = \alpha(||\phi_2||) < 1_{\mathbf{C}} \qquad \text{and} \qquad \beta(u(\psi_2)) = \beta(||\psi_2||) < 1_{\mathbf{C}},$$

$$\alpha(v(\phi_2)) \vee \beta(u(\psi_2)) < 1_{\mathbf{C}}.$$

Since $\mathbf{p}$ and $\mathbf{q}$ are disjoint lists of variables, we can take a valuation w on $\mathbf{C}$ defined by the following;

$$\begin{array}{lll} w(p) = \alpha v(p) & \text{for} & p \in \mathbf{p} \\ w(q) = \beta u(q) & \text{for} & q \in \mathbf{q}. \end{array}$$

Then $w(\phi_1) = \alpha v(\phi_1) = 1_{\mathbf{C}}, w(\psi_1) = \beta v(\psi_1) = 1_{\mathbf{C}}$ and $w(\phi_2 \vee \psi_2) = \alpha(v(\phi_2)) \vee \beta(u(\psi_2)) < 1_{\mathbf{C}}$, thus $\phi_1(\mathbf{p}), \psi_1(\mathbf{q}) \vdash_{\mathcal{L}} \phi_2(\mathbf{p}) \vee \psi_2(\mathbf{q})$ doesn't hold on $\mathbf{C}$. □

As well as in the case of Halldén completeness, the previous theorem doesn't hold always, if we replace $\mathbf{FL_{ew}}$ by $\mathbf{FL_e}$, and FL_{ew}-algebra by FL_e-algebra. So, we need to modify the definitions of DPVS and well-connectedness so as to make it true.

COROLLARY 6.10 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$. Then the following are equivalent:*

1. *for every formulas $\phi_1, \phi_2, \psi_1, \psi_2$, where $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ have no variables in common,*

   $$\phi_1 \wedge \psi_1 \vdash_{\mathcal{L}} (\phi_2 \wedge 1) \vee (\psi_2 \wedge 1) \quad \textit{implies} \quad \phi_1 \vdash_{\mathcal{L}} \phi_2 \textit{ or } \psi_1 \vdash_{\mathcal{L}} \psi_2,$$

2. *the class of all subdirectly irreducible FL_e-algebras of $V(\mathcal{L})$ has the JEP,*

3. *every two subdirectly irreducible FL_e-algebras of $V(\mathcal{L})$ are jointly embeddable into an algebra $\mathbf{C}$ in $V(\mathcal{L})$ which satisfies the following,*

   for any $x, y \in C^- = \{c \in C \mid c \leq 1\}$,

   $$x \vee y = 1 \textit{ implies } x = 1 \textit{ or } y = 1.$$

(proof) Similar with the proof of Theorem 6.9. □

In the following figure, HC* and DPVS* denote the logical properties introduced in Corollary 6.5 and 6.10, i.e., the modified versions of HC and DPVS, respectively.

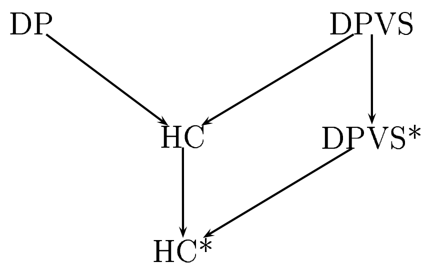

Figure 6.5: relations among several properties

As well as in the case of HC, the algebraic condition of DPVS for intermediate logics characterizes the modified version of DPVS for logics over $\mathbf{FL_e}$.

6.4 An Alternative Characterization of Principle of Variable Separation

In this section, by taking the idea in section 6.2, we will give another algebraic characterization of DPVS in the original form and see algebraic relations among DPVS, HC and DPRP.

We say that a class $\mathcal{K}$ of algebras has the *joint super-embedding property* (JSEP) if for every two algebras $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$ there exist some $\mathbf{C}$ in $\mathcal{K}$ and monomorphisms $\alpha : \mathbf{A} \to \mathbf{C}$ and $\beta : \mathbf{B} \to \mathbf{C}$ such that for every $a \in A - \{\bot\}$ and $b \in B - \{\top\}$ the inequality $\alpha(a) \leq \beta(b)$ does not hold. The following result is given by L. Maksimova [28].

PROPOSITION 6.11 *Let $\mathcal{L}$ be a normal modal logic, $V(\mathcal{L})$ the associated variety of modal algebras. Then the following are equivalent;*

1. *$\mathcal{L}$ is Halldén complete,*
2. *the class of all non-degenerate algebras of $V(\mathcal{L})$ has the JSEP.*

Since for any normal modal logic, a formula $\phi \to \psi$ is an abbreviation of the formula $\neg\phi \vee \psi$, so Halldén completeness, PVS and DPVS are equivalent. Thus the above condition *2* is also an algebraic characterization of PVS and DPVS for normal modal logics. Though it doesn't hold for logics over $\mathbf{FL_e}$, we can give a similar result of it. To do this, we need to introduce a new property.

We say that a class $\mathcal{K}$ of algebras has the *well-connected joint embedding property* (WCJEP) if for each $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$ there exist $\mathbf{C}$ in $\mathcal{K}$ and monomorphisms $\alpha : \mathbf{A} \to \mathbf{C}$ and $\beta : \mathbf{B} \to \mathbf{C}$ such that for every $a \in A$ and $b \in B$, $\alpha(a) \vee \beta(b) \geq 1_{\mathbf{C}}$ implies $a \geq 1_{\mathbf{A}}$ or $b \geq 1_{\mathbf{B}}$. Note that JSEP and WCJEP are equivalent for any class of modal algebras.

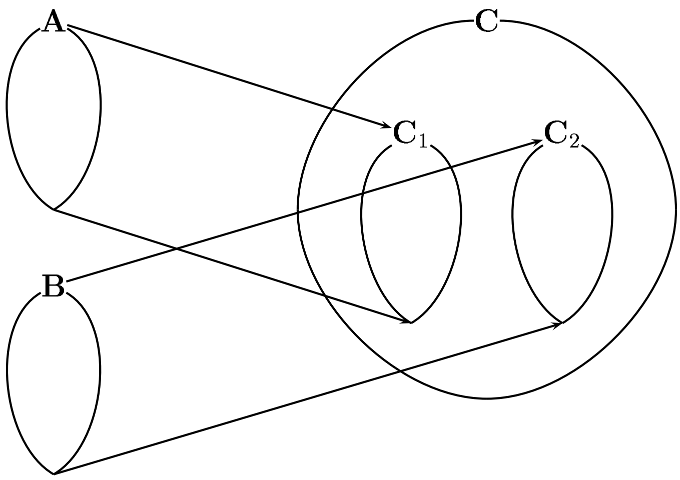

Figure 6.6: well-connected joint embedding property

THEOREM 6.12 *Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$. Then the following are equivalent;*

1. *DPVS holds in $\mathcal{L}$,*
2. *the class of all non-degenerate FL_e-algebras of $V(\mathcal{L})$ has the WCJEP.*

(proof) *1* $\Rightarrow$ *2.* Suppose that DPVS holds in $\mathcal{L}$ and $\mathbf{A}$ and $\mathbf{B}$ are non-degenerate FL_e-algebras in $V(\mathcal{L})$. Similarly with Theorem 6.9, let

$$\mathbf{p}' = \{p_a \mid a \in \mathbf{A}\}, \quad \mathbf{p}'' = \{q_b \mid b \in \mathbf{B}\}, \quad \mathbf{p} = \mathbf{p}' \cup \mathbf{p}'',$$

denote by $\mathcal{F}m(\mathbf{p})$, $\mathcal{F}m(\mathbf{p}')$ and $\mathcal{F}m(\mathbf{p}'')$ the sets of all formulas with variables in $\mathbf{p}, \mathbf{p}'$ and $\mathbf{p}''$, respectively. Let us take the valuations $v' : \mathbf{p}' \to \mathbf{A}$ and $v'' : \mathbf{p}'' \to \mathbf{B}$ determined by

$$v'(p_a) = a \text{ for } a \in \mathbf{A}, \qquad v''(q_b) = b \text{ for } b \in \mathbf{B}.$$

Let

$$T' = \{\phi \in \mathcal{F}m(\mathbf{p}') \mid v'(\phi) \geq 1_{\mathbf{A}}\}, \quad T'' = \{\psi \in \mathcal{F}m(\mathbf{p}'') \mid v''(\psi) \geq 1_{\mathbf{B}}\}.$$

Denote

$$T = \{\sigma \in \mathcal{F}m(\mathbf{p}) \mid T' \cup T'' \vdash_{\mathcal{L}} \sigma\},$$

then T is an $\mathcal{L}$-theory of the language $\mathcal{F}m(\mathbf{p})$. Take the Lindenbaum-Tarski algebra $\mathbf{C} = \mathbf{C}(\mathbf{p}, T) = \mathcal{F}m(\mathbf{p})/\sim_T$ and mappings $\alpha : \mathbf{A} \to \mathbf{C}$ and $\beta : \mathbf{B} \to \mathbf{C}$ determined by

$$\begin{aligned} \alpha(a) &= ||p_a||_{\sim_T} \quad \text{for } a \in A \\ \beta(b) &= ||q_b||_{\sim_T} \quad \text{for } b \in B. \end{aligned}$$

Since $T', T'' \subseteq T$, the mappings α, β are homomorphisms. We will show that they are monomorphisms, and for every $a \in A, b \in B$, $\alpha(a) \vee \beta(b) \geq 1_{\mathbf{C}}$ implies $a \geq 1_{\mathbf{A}}$ or $b \geq 1_{\mathbf{B}}$. Assume that $\alpha(a) \geq 1_{\mathbf{C}}$. Then $||p_a||_{\sim_T} \geq 1_{\mathbf{C}}$, so $p_a \in T$. By definition of T, $T' \cup T'' \vdash_{\mathcal{L}} p_a$ holds. Hence, by local deduction theorem, there exists some $\phi \in T'$ such that

$$T'' \vdash_{\mathcal{L}} \phi \to p_a.$$

By our assumption *1*, $\vdash_{\mathcal{L}} \phi \to p_a$ or $T'' \vdash_{\mathcal{L}} \bot$ hold. But $T'' \vdash_{\mathcal{L}} \bot$ is refutable on $\mathbf{B}$, so $\vdash_{\mathcal{L}} \phi \to p_a$, i.e., $T' \vdash_{\mathcal{L}} p_a$ holds. Therefore $p_a \in T'$, namely, $a \geq 1_{\mathbf{A}}$. Thus α is monomorphism. Similarly, β is monomorphism. Suppose that for $a \in A, b \in B$,

$$\alpha(a) \vee \beta(b) = ||p_a \vee q_b||_{\sim_T} \geq 1_{\mathbf{C}}.$$

Then $p_a \vee q_b \in T$, so $T' \cup T'' \vdash_{\mathcal{L}} p_a \vee q_b$. By our assumption *1*, $T' \vdash_{\mathcal{L}} p_a$ or $T'' \vdash_{\mathcal{L}} q_b$ hold. Hence $a \geq 1_{\mathbf{A}}$ or $b \geq 1_{\mathbf{B}}$.

2 $\Rightarrow$ *1.* Suppose that there exist some formulas $\phi_1(\mathbf{p}), \phi_2(\mathbf{p}), \psi_1(\mathbf{q}), \psi_2(\mathbf{q})$, where $\mathbf{p}$ and $\mathbf{q}$ are disjoint lists of variables, neither $\phi_1 \vdash_{\mathcal{L}} \phi_2$ nor $\psi_1 \vdash_{\mathcal{L}} \psi_2$ hold. Let T_{ϕ_1} and T_{ψ_1} be $\mathcal{L}$-theories generated by ϕ_1 and ψ_1, respectively. Clearly, $\phi_2 \notin T_{\phi_1}$ and $\psi_2 \notin T_{\psi_1}$. Then, the Lindenbaum-Tarski algebras $\mathbf{A} = \mathbf{A}(\mathbf{p}, T_{\phi_1}) = \mathcal{F}m(\mathbf{p})/ \sim_{T_{\phi_1}}$ and $\mathbf{B} = \mathbf{B}(\mathbf{q}, T_{\psi_1}) = \mathcal{F}m(\mathbf{q})/ \sim_{T_{\psi_1}}$ are non-degenerate algebras such that

$$||\phi_1||_{\sim_{T_{\phi_1}}} \geq 1_{\mathbf{A}} \text{ and } ||\phi_2||_{\sim_{T_{\phi_1}}} \not\geq 1_{\mathbf{A}},$$
$$||\psi_1||_{\sim_{T_{\psi_1}}} \geq 1_{\mathbf{B}} \text{ and } ||\psi_2||_{\sim_{T_{\psi_1}}} \not\geq 1_{\mathbf{B}}.$$

By our assumption *2*, there exist $\mathbf{C} \in V(\mathcal{L})$ and monomorphisms $\alpha : \mathbf{A} \to \mathbf{C}$ and $\beta : \mathbf{B} \to \mathbf{C}$ such that for every $a \in A$ and $b \in B$, $\alpha(a) \vee \beta(b) \geq 1_{\mathbf{C}}$ implies $a \geq 1_{\mathbf{A}}$ or $b \geq 1_{\mathbf{B}}$. Take a valuation w on $\mathbf{C}$ defined by

$$w(p) = \alpha(||p||_{\sim_{T_{\phi_1}}}) \text{ for } p \in \mathbf{p},$$
$$w(q) = \beta(||q||_{\sim_{T_{\psi_1}}}) \text{ for } q \in \mathbf{q}.$$

If $\phi_1, \psi_1 \vdash_{\mathcal{L}} \phi_2 \vee \psi_2$ holds then, by local deduction theorem, there exist some $n, m \in \omega$ such that

$$\vdash_{\mathcal{L}} (\phi_1 \wedge 1)^n \cdot (\psi_1 \wedge 1)^m \to (\phi_2 \vee \psi_2),$$

so,

$$\begin{aligned} 1_{\mathbf{C}} \leq\ & w((\phi_1 \wedge 1)^n \cdot (\psi_1 \wedge 1)^m \to (\phi_2 \vee \psi_2)) \\ =\ & (w(\phi_1) \wedge 1_{\mathbf{C}})^n \cdot (w(\psi_1) \wedge 1_{\mathbf{C}})^m \to (w(\phi_2) \vee w(\psi_2)) \\ =\ & 1_{\mathbf{C}}^n \cdot 1_{\mathbf{C}}^m \to \alpha(||\phi_2||_{\sim_{T_{\phi_1}}}) \vee \beta(||\psi_2||_{\sim_{T_{\psi_1}}}) \\ =\ & \alpha(||\phi_2||_{\sim_{T_{\phi_1}}}) \vee \beta(||\psi_2||_{\sim_{T_{\psi_1}}}). \end{aligned}$$

Hence $||\phi_2||_{\sim_{T_{\phi_1}}} \geq 1_{\mathbf{A}}$ or $||\psi_2||_{\sim_{T_{\psi_1}}} \geq 1_{\mathbf{B}}$. But this is a contradiction, thus $\phi_1, \psi_1 \vdash_{\mathcal{L}} \phi_2 \vee \psi_2$ does not hold. □

Obviously, the WCJEP is stronger than the JEP. This is why DPVS implies DPRP semantically.

Note that the following condition is equivalent to the WCJEP of the class of all non-degenerate algebras of $V(\mathcal{L})$;

($\star$) for every two non-degenerate FL_e-algebras $\mathbf{A}, \mathbf{B}$ in $V(\mathcal{L})$, there is a well-connected pair $\mathbf{C}_1, \mathbf{C}_2$ of some algebra $\mathbf{C}$ in $V(\mathcal{L})$ such that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic to $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively.

It is easy to see that ($\star$) implies the algebraic characterization of Halldén completeness given in Theorem 6.7. This is why DPVS implies Halldén completeness semantically.

Chapter 7

Conclusions and Open Problems

In this thesis we investigated logics over $\mathbf{FL_e}$ by using algebraic methods. In Chapter 4, we showed that there exist continuum maximal logics over $\mathbf{FL_e}$. In Chapter 5, we showed that an algebraic characterization of the deductive pseudo-relevance property for normal modal logics worked well also for logics over $\mathbf{FL_e}$. In Chapter 6, we introduced some modifications of definitions of Halldén completenss and the deductive principle of variable separation in order to extend results for intermediate logics and modal logics to those for logics over $\mathbf{FL_e}$. So, by using a new algebraic notion, namely well-connected pair, we were able to give alternative characterizations of them, and clarified their semantical relations with the disjunction property. But still there remain many open problems. The following is a list of them that have come up from our study. The author believes that these problems are worth while to study in the future.

- Do our algebraic characterizations of Halldén completeness, the deductive pseudo-relevance property and the deductive principle of variable separation work well for logics over $\mathbf{FL}$, i.e., non-commutative case?
- What are algebraic characterizations of original (i.e., non-deductive) form of both the pseudo-relevance property and the principle of variable separation for logics over $\mathbf{FL_e}$, respectively?
- Are there some relations between the principle of variable separation and the deductive principle of variable separation?
- What kind of logical property does our algebraic condition of Halldén completeness characterize for normal modal logics? (cf. Propositon 6.11 and Theorem 6.12)
- For any logic $\mathcal{L}$ over $\mathbf{FL_e}$, $\mathcal{L}$ is Halldén complete and has the deductive pseudo-relevance property if and only if it satisfies the following;

  for all formulas ϕ, ψ, δ which have no variables in common,

  $$\phi \vdash_{\mathcal{L}} \psi \vee \delta \text{ implies } \phi \vdash_{\mathcal{L}} \bot \text{ or } \vdash_{\mathcal{L}} \psi \text{ or } \vdash_{\mathcal{L}} \delta.$$

  - What is an algebraic characterization of the above logical propety?
  - Is there a logic which has the above property but doesn't have the deductive principle of variable separation?

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