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An Algebraic Approach to Incompleteness in Modal Logic

by

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Contents

Chapter 1

Introduction

The thesis is written in a rather abstract manner. Nevertheless, the motivation and goals of our research are far simpler and more natural than it may seem.

In this work, we study semantics for *normal propositional modal logics*. The field has a long and complicated history; we refer the reader to Goldblatt [28]. From the semantic point of view, modal logic can be investigated in at least two ways:

- either as a logical counterpart of the theory of *boolean algebras with operators* (BAOs);
- or, more in the spirit of the Amsterdam school, as a simple yet expressive formalism for talking about relational structures from an internal, local perspective (cf. e.g., Blackburn et al. [3]).

Why those two seemingly distinct approaches are possible? The mathematical reason is that BAOs have very nice *duality theory*. It was discovered as early in 1950's by Jónsson and Tarski [31]. This paper, whose importance cannot be overestimated, has shown that every BAO can be embedded in its *canonical extension*: a complete, atomic and completely additive algebra. From our perspective, we may say that complete, atomic and completely additive algebras ($\mathcal{CAV}$-BAOs) are precisely the duals of relational structures or *Kripke frames*: they are graphs in disguise.

In the 1960's, logicians began to study the notion of *Kripke completeness*. For some time, it was believed that every logic is Kripke complete: that every logic is determined by class of graphs, relational structures or Kripke frames, whichever name the reader prefers. From an algebraic point of view, that would mean that every variety of BAOs is HSP-generated by the class of its complete, atomic and completely additive algebras ($\mathcal{CAV}$-BAOs). For some small lattices of varieties of BAOs, this is indeed true, as was shown by Bull [7] or Fine [19]. But in general, it is false. In 1972, Thomason [50] has shown that there exists a consistent logic without any Kripke frames, i.e., a non-trivial variety without $\mathcal{CAV}$-BAOs. Other incompleteness results were soon found by e.g., Fine [18] or van Benthem [53]. Perhaps the most surprising result was found by Blok [4]. He has shown that in most cases, a variety of BAOs shares its class of $\mathcal{CAV}$-BAOs with continually many varieties. Speaking in terms of logic and frames, it means that in general for a given logic Λ, there exists a continuum of other logics which cannot be distinguished from Λ by means of Kripke frames.

But why should we restrict our attention to $\mathcal{CAV}$-BAOs? Of course, they have a number of nice properties; some of them are discussed below. But in what way is the combination of completeness, atomicity and complete additivity crucial for their nice behaviour? Maybe we can have our cake and eat it: weaken the definition of $\mathcal{CAV}$-BAOs in such a way that, e.g., the class thus obtained still allows for nice representation theorems — but, as opposed to $\mathcal{CAV}$-BAOs, it also allows for general completeness results? To put things simply: does it make sense to study notions of completeness weaker than Kripke completeness? This question was the starting point of the present research.

Our findings can be described as follows.

- There are many non-equivalent notions of completeness weaker than Kripke completeness (Theorem 4.1). None of the notions of completeness with respect to a class of algebras from the following list is equivalent to any of remaining ones: completeness with respect to atomic BAOs, ω-complete BAOs, complete BAOs, BAOs which admit residuals ($\mathcal{T}$-BAOs), atomic and completely additive BAOs ($\mathcal{AV}$-BAOs), complete and atomic BAOs ($\mathcal{CA}$-BAOs, duals of *normal neighbourhood frames*; cf. Došen [15]), complete BAOs which admit residuals, $\mathcal{CAV}$-BAOs. This list of non-equivalences is not exhaustive. One of few questions we were not able to solve is whether there are logics which are not complete with respect to completely additive BAOs.

- For almost all of the above completeness notions, Blok's result can be generalized (Theorem 5.9). In most cases, for a variety V of BAOs, there is a continuum of other varieties which share with V its class of $\mathcal{AV}$-BAOs, ω-complete BAOs, BAOs which admit residuals ... and so on. The class of atomic BAOs is an exception here: we don't know whether The Blok Alternative holds for it or not. Nevertheless, as we show below, the robustness of the Blok Alternative does not mean than weakening of the notion of Kripke completeness does not allow to obtain more general completeness results.

- Some of the above completeness notions turn out to coincide with conservativity of certain natural *minimal extensions* (Section 3.3). For example, completeness with respect to $\mathcal{AV}$-BAOs coincides with conservativity of *minimal hybrid extensions*, completeness with respect to BAOs which admit residuals — with conservativity of *minimal tense extensions*, completeness with respect to ω-complete BAOs — with conservativity of *minimal infinitary extensions.*

- The second-order character of consequence over Kripke frames is caused by the lattice-completeness of their dual algebras. Consequence over complete BAOs, just as consequence over Kripke frames, is non-compact (Theorem 2.15). There are logics complete with respect to κ-complete BAOs which do not have any complete BAO in the associated variety (Theorem 4.6). Thomason's reduction of second-order consequence to consequence over Kripke frames is actually a reduction to consequence over subdirectly irreducible complete BAOs (Theorem 7.10); we hope that the additional assumption of subdirect irreducibility can be removed.

- Most of the "nice" properties of $\mathcal{CAV}$-BAOs are inherited by the class of $\mathcal{AV}$-BAOs:

  (1) Thomason's [51] duality results can be lifted to duality between $\mathcal{AV}$-BAOs and *discrete frames* (Section 3.2);

  (2) just as in the case of $\mathcal{CAV}$-BAOs, every $\mathcal{AV}$-BAO is a subdirect product of its complete morphic images (Theorem 6.2). Thus, the class of $\mathcal{AV}$-BAOs in a given variety is determined by its subdirectly irreducible members;

  (3) the fact that for Kripke frames, strong local completeness, strong global completeness and $\mathcal{CAV}$-complexity coincide holds for $\mathcal{AV}$-BAOs (or dually, discrete frames) as well (Theorem 7.2).

- All those regularities should not be taken for granted. For example, for normal neighbourhood frames ($\mathcal{CA}$-BAOs) or ω-complete BAOs (2) and (3) above do not hold (Theorem 6.1 and Corollary 7.6).

- In addition, the class of $\mathcal{AV}$-BAOs has some other nice properties $\mathcal{CAV}$-BAOs do not have:

  (1) $\mathcal{AV}$-completeness coincides with strong $\mathcal{AV}$-completeness, something which has no Kripke equivalent at all (Corollary 7.4);

  (2) the Goldblatt-Thomason Theorem 7.9 for discrete frames seems actually nicer than its original form for Kripke frames;

  (3) general $\mathcal{AV}$-completeness results are available even for some lattices of logics where incompleteness with respect to ω-complete BAOs — and thus with respect to Kripke frames — is a common phenomenon. An example is provided by the lattice of tense logics of linear time flows (Corollary 8.7).

- Moreover, the existence of general $\mathcal{AV}$-completeness results for a given lattice of logics can indicate that one may prove, e.g., general low computational results. An example is provided again by the lattice of tense logics of linear time flows: all finitely axiomatizable logics and all $\bigcap$-prime logics in this lattice are coNP-complete (Section 8.4), even though many of them are Kripke-incomplete in a nasty way (Section 8.3).

The most important conclusion of our research seems to be that completeness with respect to $\mathcal{AV}$-BAOs — or, dually, discrete frames — deserves particular attention. Indeed, if anything deserves to be called *the* thesis of the present thesis, it is the claim that the notion of $\mathcal{AV}$-completeness is *the* proper extension of the notion of Kripke completeness. In other words, discrete frames provide the most attractive first-order alternative to Kripke frames; more attractive than $\mathcal{CA}$-BAOs (neighbourhood frames), for example. We disagree here with, e.g., Shehtman [45] who claims that *in neighbourhood semantics, modal logics enjoy the compactness property in numerous cases and thus they acquire features of classical first-order theories.* What he had in mind, is the *local* consequence over neighbourhood frames; the global neighbourhood consequence is non-compact, as was mentioned above. In contrast with the Kripke case, strong local $\mathcal{CA}$-completeness and strong global $\mathcal{CA}$-completeness are distinct notions. And this is the problem with neighbourhood frames: they don't really allow to overcome any problems arising with Kripke semantics. Instead, they add some problems of their own. This contrasts with behaviour of discrete frames, i.e., $\mathcal{AV}$-BAOs.

Even if the reader does not agree with such a strong conclusion, we hope that at least we succeed in convincing him of a weaker thesis: that it makes sense to study algebraically notions of completeness weaker than Kripke completeness. Moreover, this study can deepen our understanding of the notion of Kripke completeness itself. In a similar way, research on "resource-conscious" logics, like linear, relevant or substructural logics, refined our insight in the structure of intuitionistic or classical proofs. It is just a metaphor, but hopefully it makes the motivations behind the present research clearer.

Chapter 2

Basic notions

2.1 Preliminaries

In our notation, we carefully distinguish between connectives in metalanguage — &, OR, $\Rightarrow$, $\Leftrightarrow$ — and algebraic operations — $\wedge$, $\vee$, $\to$, $\leftrightarrow$. For standard mathematical entities, the set of natural numbers is denoted by $\mathbb{N}$ and identified with ω (cf. appendices), the set of integers by $\mathbb{Z}$, the set of rationals by $\mathbb{Q}$, the set of reals by $\mathbb{R}$. The absolute value of r is denoted by $Abs(r)$. The set of all ultrafilters of BAO $\mathfrak{A}$ is denoted by $Uf\mathfrak{A}$; the set of all atoms of BAO $\mathfrak{A}$ is denoted by $At\mathfrak{A}$.

For a given cardinal κ, set $\textsc{var}(\kappa) = \{p_\iota\}_{\iota\in\kappa}$; it is the set of κ many *modal propositional variables*. If not stated otherwise, we tacitly assume that the underlying set of variables is $\textsc{var}(\omega)$; we also often use notation $p, q, r, p_1, q_1, r_1 \ldots$ for variables. A *modal similarity type* is a finite set TYPE of arbitrary symbols; *the basic modal similarity type* is an arbitrary singleton. A language is a pair $\langle \textsc{type}, \textsc{var}(\kappa)\rangle$. Formulas of a given language are defined as

$$\varphi ::= \bot \mid \top \mid p \mid \neg\varphi \mid \psi \wedge \varphi \mid \psi \vee \varphi \mid \psi \to \varphi \mid \Diamond_\pi\varphi$$

where $\pi \in \textsc{type}$, $p \in \textsc{var}(\kappa)$. In addition, for every $\pi \in \textsc{type}$, define the *dual operator* $\Box_\pi\varphi \leftrightharpoons \neg\Diamond_\pi\neg\varphi$. For the basic modal similarity type, we often drop the subscript. The absolutely free algebra of all formulas in $\langle \textsc{type}, \textsc{var}(\kappa)\rangle$ is denoted as $\textsc{form}_\kappa$. *The modal degree* of φ is defined as follows: $md(p) \leftrightharpoons 0$, $md(\neg\varphi) \leftrightharpoons md(\varphi)$, $md(\psi\wedge\varphi) \leftrightharpoons md(\psi\vee\varphi) \leftrightharpoons md(\psi \to \varphi) \leftrightharpoons max(md(\psi), md(\varphi))$, $md(\Diamond_\pi\varphi) = md(\varphi) + 1$. $sub(\varphi)$ is the closure of the set of subformulas of φ under single negation and $l(\varphi)$ is the cardinality of $sub(\varphi)$.

There is a standard axiom

$\mathbf{K}_\pi$ $\Box_\pi(p \to q) \to (\Box_\pi p \to \Box_\pi q)$

and three standard rules:

M $\dfrac{\psi, \psi \to \varphi}{\varphi}$ (Modus Ponens);

N_π $\dfrac{\varphi}{\Box_\pi \varphi}$ (π-necessitation);

S $\dfrac{\varphi}{s(\varphi)}$, where s is any uniform substitution.

Assume the underlying language $\langle \text{TYPE}, \text{VAR}(\kappa)\rangle$ is fixed. For arbitrary set of modal formulas Λ, let $\mathsf{Log}_\kappa(\Lambda)$ be the minimal closure of Λ in FORM_κ under the axioms of classical propositional logic, $\mathbf{K}_\pi$ for all $\pi \in \text{TYPE}$, M, S and N_π for all $\pi \in \text{TYPE}$. If the cardinality of the set of variables is fixed (usually it is ω), we may drop the subscript κ. $\mathsf{Log}(\Lambda)$ is usually called *the (normal modal) logic axiomatized by* Λ. In addition, for arbitrary sets of modal formulas Λ and Γ in $\langle \text{TYPE}, \text{VAR}(\kappa)\rangle$, let $\mathsf{M}_\kappa(\Lambda, \Gamma)$ be the minimal closure of $\mathsf{Log}_\kappa(\Lambda) \cup \Gamma$ under M and let $\mathsf{MN}_\kappa(\Lambda, \Gamma)$ be the minimal closure of $\mathsf{Log}_\kappa(\Lambda) \cup \Gamma$ under M and N_π for all $\pi \in \text{TYPE}$. Convention about dropping the subscript if κ is kept fixed applies here again. It is a straightforward observation that $\mathsf{Log}(\Lambda) = \mathsf{M}(\Lambda, \emptyset) = \mathsf{MN}(\Lambda, \emptyset)$. A set of formulas Γ is M-*consistent* (MN-*consistent*) with $\mathsf{Log}(\Lambda)$ iff $\bot \notin \mathsf{M}(\Lambda, \Gamma)$ ($\bot \notin \mathsf{MN}(\Lambda, \Gamma)$). A formula φ is M-*consistent* (MN-*consistent*) with $\mathsf{Log}(\Lambda)$ if $\{\varphi\}$ is M-consistent (MN-consistent) with $\mathsf{Log}(\Lambda)$.

We introduce a useful notational convention. For arbitrary $\Delta \in \text{FORM}_\kappa$, define

$$\begin{aligned}
\blacklozenge^0 \Delta &\leftrightharpoons \Delta \\
\blacklozenge \Delta &\leftrightharpoons \{\Diamond_\pi \delta | \delta \in \Delta, \pi \in \text{TYPE}\} \\
\blacklozenge^{n+1} \Delta &\leftrightharpoons \blacklozenge \blacklozenge^n \Delta \\
\blacklozenge^{\leq \kappa} \Delta &\leftrightharpoons \bigcup_{n \in \kappa} \blacklozenge^\kappa \Delta,
\end{aligned}$$

where $\kappa \in \omega'$. Definitions of $\blacksquare$, $\blacksquare^{\leq n}$ and $\blacksquare^{\leq \omega}$ are dual. For $\Delta = \{\varphi\}$, we write $\blacklozenge \varphi$ instead of $\blacklozenge\{\varphi\}$ and similarly for other operators.

The following syntactic observations are useful in what follows:

Fact 2.1 (Compactness of derivations) *For every* Λ, Γ, κ *and* φ, $\varphi \in \mathsf{M}_\kappa(\Lambda, \Gamma)$ *(*$\varphi \in \mathsf{MN}_\kappa(\Lambda, \Gamma)$*) iff there are finite subsets* $\lambda \subseteq_{fin} \Lambda$ *and* $\gamma \subseteq_{fin} \Gamma$ *s.t.* $\varphi \in \mathsf{M}_\kappa(\lambda, \gamma)$ *(*$\varphi \in \mathsf{MN}_\kappa(\lambda, \gamma)$*).*

Fact 2.2 (cf. [37], Theorem 2.1.5) *(1)* $\mathsf{MN}_\kappa(\Lambda, \Gamma) = \mathsf{M}_\kappa(\Lambda, \blacksquare^{\leq\omega}\Gamma)$
(2) $\mathsf{Log}_\kappa(\Lambda \cup \Gamma) = \mathsf{M}_\kappa(\Lambda, \mathsf{S}(\blacksquare^{\leq\omega}\Gamma))$

Fact 2.3 (cf. [57], 1.4.1) $\bot \in \mathsf{M}_\kappa(\Lambda, \Gamma)$ *iff* $\neg p \in \mathsf{MN}_\kappa(\Lambda, p \to \gamma | \gamma \in \Gamma)$ *for arbitrary* p *which does not appear in* Γ*.*

Fact 2.4 (cf. Wolter [57], 1.4.1) $\bot \in \mathsf{M}_\kappa(\Lambda, \Gamma)$ *iff* $\neg p \in \mathsf{MN}_\kappa(\Lambda, p \to \gamma | \gamma \in \Gamma)$ *for arbitrary* p *which does not appear in* Γ*.*

There are some logics of particular historical and/or mathematical importance; they appear often in this thesis. We list their axioms in Table 2.1. The following convention is useful: for any $i \leq n$, let $\bar{p}^i_n \leftrightharpoons \bigwedge\limits_{j<i} p_j \wedge \neg p_i \wedge \bigwedge\limits_{i<j\leq n} p_j$. Also, let $\Diamond^+_\pi p \leftrightharpoons p \vee \Diamond_\pi p$. $\Box^+_\pi$ is defined dually.

Let us recall the following straightforward consequence of The Zorn Lemma and compactness of derivations:

Lemma 2.5 *Every logic is contained in a maximal consistent logic.*

A boolean algebra with operators (BAO) is a structure

$$\langle \mathfrak{A}, \to, \wedge, \vee, \neg, \top, \bot, \{\Diamond_\pi\}_{\pi \in \text{TYPE}} \rangle$$

s.t. $\langle \mathfrak{A}, \to, \wedge, \vee, \neg, \top, \bot \rangle$ is a boolean algebra and $\Diamond_\pi \bot = \bot$, $\Diamond_\pi(x \vee y) = \Diamond_\pi x \vee \Diamond_\pi y$ hold for every $\pi \in \text{TYPE}$ and every $x, y \in \mathfrak{A}$. It follows from the definition that the class of all boolean algebras in a given similarity type forms a variety. *A valuation* for $\text{VAR}(\kappa)$ in $\mathfrak{A}$ is a mapping $\mathfrak{V} : \text{VAR}(\kappa) \mapsto \mathfrak{A}$. It is readily extended to a valuation of all formulas in $\langle \text{TYPE}, \text{VAR}(\kappa) \rangle$, i.e., homomorphism $\bar{\mathfrak{V}}$ from FORM_κ to $\mathfrak{A}$. Thus, if not stated otherwise, we make systematic confusion between (1) $\mathfrak{V}$ and $\bar{\mathfrak{V}}$, (2) an algebra and its carrier set, (3) syntactic connectives (or derived terms) and corresponding algebraic operations; at least, if the underlying algebra is clear from the context.

A *model* is a pair $\langle \mathfrak{A}, \mathfrak{V} \rangle$ consisting of a BAO $\mathfrak{A}$ and a valuation in $\mathfrak{A}$. A formula φ *is true in* (*holds in*) $\langle \mathfrak{A}, \mathfrak{V} \rangle$, in symbols $\langle \mathfrak{A}, \mathfrak{V} \rangle \vDash \varphi$, if $\mathfrak{V}(\varphi) = \top$. A formula φ *is true in* (*holds in*) a BAO $\mathfrak{A}$, in symbols $\mathfrak{A} \vDash \varphi$, if it is true in every $\langle \mathfrak{A}, \mathfrak{V} \rangle$ based on $\mathfrak{A}$. A formula φ *is true in* (*holds in*) of a class of

$\mathbf{4}_\pi$	$\Diamond_\pi^2 p \to \Diamond_\pi p$
$\mathbf{.3}_\pi$	$\Box_\pi(\Box_\pi^+ p \to q) \vee \Box_\pi(\Box_\pi^+ q \to p)$
$\mathbf{T}_\pi$	$\Box_\pi p \to p$
$\mathbf{S4}_\pi$	$\mathbf{4}_\pi \wedge \mathbf{T}_\pi$
$\mathbf{GL}_\pi$	$\Diamond_\pi p \to \Diamond_\pi(p \wedge \Box_\pi \neg p)$
$\mathbf{Func}_\pi$	$\Diamond_\pi p \leftrightarrow \Box_\pi p$
$\mathbf{D}_\pi$	$\Diamond_\pi \top$
$\mathbf{Ver}_\pi$	$\Box_\pi \bot$
$\mathbf{Triv}_\pi$	$p \leftrightarrow \Box_\pi p$
$\mathbf{McK}_\pi$	$\Box_\pi \Diamond_\pi p \to \Diamond_\pi \Box_\pi p$
$\mathbf{Conj}_{\pi'}^{\pi}$	$p \to \Box_\pi \Diamond_{\pi'} p$
$\mathbf{S5}_\pi$	$\mathbf{4}_\pi \wedge \mathbf{T}_\pi \wedge \mathbf{Conj}_\pi^\pi$
$\mathbf{Lin}_{\pi'}^{\pi}$	$\mathbf{Conj}_{\pi'}^{\pi} \wedge \mathbf{Conj}_{\pi}^{\pi'} \wedge \mathbf{4}_\pi \wedge (\Diamond_{\pi'} \Diamond_\pi p \vee \Diamond_{\pi'} \Diamond_\pi p \to \Diamond_\pi p \vee p \vee \Diamond_{\pi'} p)$
$\mathbf{wgrz}_\pi$	$\neg\left(p \wedge \Box_\pi^+ \left(p \to \Diamond_\pi \left(\neg p \wedge \Diamond_\pi p\right)\right)\right)$
$\mathbf{nocycle}_\pi^n$	$\neg(\bar{p}_n^0 \wedge \bigwedge_{i<n} \Box_\pi^+ (\bar{p}_n^i \to \Diamond_\pi \bar{p}_n^{i+1\, mod n}))$
$\mathbf{nochain}_n$	$\neg\left(\bar{p}_n^0 \wedge \bigvee \blacklozenge \left(\bar{p}_n^1 \wedge \ldots \wedge \bigvee \blacklozenge \bar{p}_n^n\right)\right)$
$\mathbf{nobranch}_n$	$\neg \bigvee \blacklozenge^{\leq m} \bigwedge_{i \in n+1} \bigvee \blacklozenge \bar{p}_n^i$
$\mathbf{Fin}_n$	$\mathbf{nochain}_n \wedge \mathbf{nobranch}_n$

Table 2.1: Some important axioms. If no confusion arises, we use the same symbol to denote both the logic axiomatized by a given formula and the axiom itself

algebras $\mathcal{X}$, in symbols $\mathcal{X} \vDash \varphi$ if it is true in every algebra from $\mathcal{X}$. For a set of formulas Γ, write $\langle \mathfrak{A}, \mathfrak{V} \rangle \vDash \Gamma$ ($\mathfrak{A} \vDash \Gamma$, $\mathcal{X} \vDash \Gamma$) if every formula from Γ is true. It is readily checked that the set of formulas true in a given model is MN-closed and the set of formulas true in an algebra or a class of algebras is a logic. Conversely, for arbitrary Λ, $\mathbb{V}(\Lambda) \leftrightharpoons \{\mathfrak{A} | \mathfrak{A} \vDash \Lambda\}$ is a variety.

2.2 Properties

We are interested in algebras with certain *properties.* A property, according to the naïve notion, is any formula or a set of formulas in the natural language — or the highly standardized fragment of natural language used in mathematics. And on most occasion this somewhat imprecise definition is good enough. However, sometimes there may arise a need for more precision. We should define formally what *a class* of algebras is.

One of problems with the notion of class in most standard set-theoretical frameworks is that one is not allowed to form collections of classes; and yet we want to be able them on some occasions. There are two possible solutions to this problem. One is to define a class as an extension of *a property. A property,* in turn, would be a second-order sentence (or, say, a countable set of second-order sentences) in the similarity type of boolean algebras with operators. Thus, instead of dealing with collections of classes themselves, we could identify such a collection with the collection of corresponding properties — i.e., linguistic objects. It was the road taken in the first version of this work.

We finally opted for another solution. Namely, we decided to adopt a quite strong system of set theory, where all "unproblematic" classes of classes are legitimate. The system we have in mind is the one denoted as ST_2 in Fraenkel et al. [20, Chapter II,§7.2]. A more detailed introduction and a motivation of this choice is contained in Appendix B.

The reason why we initially found the first option appealing is that the syntactic form a sentence defining a property can reveal some useful information about the corresponding class. For example, if the sentence is equivalent to a formula without second-order variables or a Σ^1_1-sentence, we know that the class of algebras satisfying the sentence is closed under ultraproducts. For this reason, we first define some important classes of algebras in a standard way and then write down a formalized second-order definition (or first-order definition, if possible). Most properties introduced below appear in Appendix

A.

symbol	class of algebras
$\mathcal{ALG}$	all algebras
$\kappa\mathcal{C}$	κ-complete BAOs:
	every family of no more than κ-elements has supremum
$\mathcal{C}$	complete BAOs: all families of elements have supremum
$\mathcal{A}$	atomic BAOs: every element is above some atom
$\mathcal{V}$	completely additive BAOs:
	operators distribute over all existing joins
$\mathcal{T}$	BAOs *admitting residuals*: for every $\pi \in$ TYPE,
	there exists a mapping $\mathbf{h}_\pi$ s.t. $\Diamond_\pi x \leq y$ iff $x \leq \mathbf{h}_\pi y$
$\mathcal{F}$	finite BAOs
$\mathcal{I}_\pi$	π-cycle free BAOs: for no $x \neq \bot$, $x \leq \Diamond_\pi x$
$\mathcal{I}$	cycle-free BAOs: for no $x \neq \bot$, $x \leq \bigvee \blacklozenge x$
$\mathcal{S}$	subdirectly irreducible BAOs

Table 2.2: List of some important properties.

In order to reformulate definitions from Table 2.2, we introduce some abbreviations.

$$
\begin{aligned}
Atom(x) &\leftrightharpoons (x \neq \bot)\&\forall y(y \leq x \implies y = \bot \,\mathrm{OR}\, y = x)\\
|X| \leq \kappa &\leftrightharpoons \text{cardinality of}\, X \text{ is no greater than}\, \kappa\\
&\qquad \text{for a second-order definable}\, \kappa\\
|X| = \kappa &\leftrightharpoons \text{cardinality of}\, X \text{ is equal to}\, \kappa\\
&\qquad \text{for a second-order definable}\, \kappa\\
Inf(y, X) &\leftrightharpoons \forall z(X(z) \Rightarrow y \leq z)\&\forall x(\forall z(X(z) \Rightarrow x \leq z) \Rightarrow x \leq y).\\
Inf(y, \Box_\pi X) &\leftrightharpoons \forall z(X(z) \Rightarrow y \leq \Box_\pi z)\&\forall x(\forall z(X(z) \Rightarrow x \leq \Box_\pi z) \Rightarrow x \leq y)\\
OpFilter(X) &\leftrightharpoons \forall x, y(X(x \wedge y) \Leftrightarrow X(x)\&X(y))\&X(\top)\&\\
&\qquad \neg X(\bot)\&X(x) \Rightarrow \bigwedge_{\pi\in \mathrm{TYPE}} X(\Box_\pi x)
\end{aligned}
$$

As a rule, we prefer the standard notation $y = \bigwedge X$ and $y = \bigwedge \Box_\pi X$ to $Inf(y, X)$, $Inf(y, \Box_\pi X)$ respectively. Note here the restriction to *second-order definable* cardinals; cf. Shapiro [44, Chapter 5.1] or Garland [23] for a discussion which cardinals are second-order definable. Difficulties of this kind made us finally choose the other approach to handling collections of classes.

symbol	defining sentence(s)
$\mathcal{ALG}$	$\forall x\, x = x$
$\kappa\mathcal{C}$	$\forall X(\lvert X\rvert \leq \kappa \Rightarrow \exists y\, y = \bigwedge X)$
$\mathcal{C}$	$\forall X \exists y\, y = \bigwedge X$
$\mathcal{A}$	$\forall x \exists y (x \neq \bot \Rightarrow Atom(y) \& y \leq x)$
$\mathcal{V}$	$\forall X(\exists y\, y = \bigwedge X) \Rightarrow \exists z\, (z = \bigwedge \Box_\pi X \& \exists y\, y = \bigwedge X \& z = \Box_\pi y)$ for every $\pi \in$ TYPE
$\mathcal{T}$	$\exists h \forall x, y\, (\Diamond_\pi x \leq y \Leftrightarrow x \leq h(y))$ for every $\pi \in$ TYPE
$\mathcal{F}$	$\forall X(\lvert X\rvert \leq \omega \& \neg(\lvert X\rvert = \omega))$
$\mathcal{I}_\pi$	$\forall x (x \leq \Diamond_\pi x \Rightarrow x = \bot)$
$\mathcal{I}$	$\forall x (x \leq \bigvee_{\pi \in \mathrm{TYPE}} \Diamond_\pi x \Rightarrow x = \bot)$
$\mathcal{S}$	$\exists X(OpFilter(X) \& \forall Y(OpFilter(Y) \Rightarrow \exists x((x \neq \bot) \& Y(x)) \Leftrightarrow$ $\Leftrightarrow \forall x(X(x) \Rightarrow Y(x))))$

Table 2.3: Second-order definitions of classes from 2.2.

Some of those classes have alternative nice characterizations.

Lemma 2.6 (Rautenberg [43]) *$\mathfrak{A} \in \mathcal{S}$ iff there exists $* \neq \top$ s.t. for any $x \neq \top$ there exists $n \in \omega$ s.t. $\bigwedge \blacksquare^{\leq n} x \leq *$. Any such element $*$ is called* an opremum *of $\mathfrak{A}$.*

Lemma 2.7 (cf., e.g., Chagrov et al. [11], section 10.5) *$\mathfrak{A} \in \mathcal{FI}$ iff $\mathfrak{A}$ is finite and there exists $n \in \omega$ s.t. $\mathfrak{A} \models \bigwedge \blacksquare^n \bot$.*

Proof: ($\Rightarrow$). Assume $\mathfrak{A} \not\models \bigwedge \blacksquare^n \bot$ for all $n \in \omega$, i.e., for all $n \in \omega$, $\bigvee \blacklozenge^n \top \neq \bot$. As $\bigvee \blacklozenge^{n+1} \top \rightarrow \bigvee \blacklozenge^n \top \in \mathbf{K}_{\mathrm{TYPE}}$ and a finite algebra cannot contain an infinite descending chain, $a \fallingdotseq \bigvee \blacklozenge^{n+1} \top = \bigvee \blacklozenge^n \top \neq \bot$ for some n. But then $a \leq \bigvee_{\pi \in \mathrm{TYPE}} \Diamond_\pi a$ and $\mathfrak{A} \notin \mathcal{I}$.

($\Leftarrow$). Assume there exists $a \neq \bot$ s.t. $a \leq \bigvee_{\pi \in \mathrm{TYPE}} \Diamond_\pi a$. But then $a \leq \bigvee \blacklozenge^n a \leq \bigvee \blacklozenge^n \top$ for every $n \in \omega$. ⊣

It is worth noting that the name *cycle-free* may be misleading: duals of cycle-free Kripke frames (the notions of Kripke frame and its dual algebra are defined later) are cycle-free BAOs in the finite case, but not necessarily in the infinite case. Another thing we have to make explicit is that the class $\mathcal{T}$ of BAOs admitting residuals does not coincide with the class of BAOs where residuals are term-definable, i.e., with the class of *residuated* BAOs of Jipsen [30]. Let us also note here well-known

Fact 2.8 *(1)* $\mathcal{T} \subseteq \mathcal{V}$,

(2) $\mathcal{CT} = \mathcal{CV}$

(3) class $\mathcal{T}$ *is the same as the class* $\mathcal{T}'$ *of* conjugated BAO*s: i.e., algebras s.t. for every* $\pi \in$ TYPE *there is a mapping* $\mathbf{p}_\pi$ *s.t. for all* x *and* y, $\Diamond_\pi x \wedge y = \bot$ *iff* $x \wedge \mathbf{p}_\pi y = \bot$*). Conjugates and residuals are interdefinable:* $\mathbf{p}_\pi x = \neg \mathbf{h}_\pi \neg x$.

By convention, we use notation $\mathcal{XY}$ for intersection of classes $\mathcal{X}$, $\mathcal{Y}$ or conjunction of their definining sentences (if there are only finitely many of them). Thus, for example, $\mathcal{AV}$ is the class of atomic and completely additive BAOs. Note that this class is first-order definable by sentence(s)

$$\mathcal{A}\&\forall x, y\,(x \neq \bot \& x \leq \Diamond_\pi y \Rightarrow \exists z(Atom(z) \& z \leq y \& x \wedge \Diamond_\pi z \neq \bot)),$$

even though the class $\mathcal{V}$ was defined using a second-order sentence.

We set PROPERTIES $\leftrightharpoons \{\mathcal{ALG}, \omega\mathcal{C}, \mathcal{C}, \mathcal{A}, \mathcal{V}, \mathcal{T}, \mathcal{CA}, \mathcal{CV}, \mathcal{AV}, \mathcal{AT}, \mathcal{CAV}\}$. This is our *standard class of properties.*

In the present thesis, the concept of *the root filter of* x plays an important role. Take any $x \in \mathfrak{A}$. $\mathsf{Root}(x) \leftrightharpoons \{y | \forall n \in \omega x \leq \bigwedge \blacksquare^{\leq n} y\}$. It is straightforward to see that every root filter is open. The root filter of x should not be confused with the open filter generated by x; in fact, $x \in \mathsf{Root}(x)$ iff x is open. Recall that x is *an open element* if $x \leq \bigwedge \blacksquare x$.

Fact 2.9 *If* $\mathfrak{A} \in \mathcal{V}$*, then every root filter is complete.*

Recall from the appendix that complete filters correspond to complete homomorphisms. And complete homomorphisms are a very important tool from our point of view.

Fact 2.10 *All classes of algebras from* PROPERTIES*, with the possible exception of* $\mathcal{V}$*, are closed under complete homomorphisms.*

2.3 Completeness

Let $\mathcal{X}$ be an arbitrary property of BAOs. Let Λ, Γ be arbitrary sets of modal formulas in language $\langle \text{TYPE}, \text{VAR}(\kappa)\rangle$. Define

$$\mathcal{X}_\kappa(\Lambda, \Gamma) \leftrightharpoons \{\varphi \in \text{FORM}_\kappa \mid \forall \mathfrak{A} \in \mathcal{X} \cap \mathbb{V}(\Lambda)\, \forall \langle \mathfrak{A}, \mathfrak{V}\rangle (\langle \mathfrak{A}, \mathfrak{V}\rangle \vDash \Gamma \Rightarrow \langle \mathfrak{A}, \mathfrak{V}\rangle \vDash \varphi)\}.$$

As a rule, we drop the subscript when $\kappa = \omega$. Again, it is immediate to check that for arbitrary Λ, Γ, $\mathcal{X}(\Lambda, \Gamma)$ is MN-closed and $\mathcal{X}(\Lambda, \emptyset)$ is a logic. Thus, write $\mathsf{Log}\mathcal{X}(\Lambda) \leftrightharpoons \mathcal{X}(\Lambda, \emptyset)$. We also use notation $\Lambda \models_{\mathcal{X}} \alpha$ for $\alpha \in \mathsf{Log}\mathcal{X}(\Lambda)$ and $\Gamma \vDash_{\mathcal{X},\Lambda} \alpha$ for $\alpha \in \mathcal{X}(\Lambda, \Gamma)$.

We say that $\mathsf{Log}(\Lambda)$, the logic axiomatized by Λ, is *(weakly) $\mathcal{X}$-complete* if $\mathsf{Log}(\Lambda) = \mathsf{Log}\mathcal{X}(\Lambda)$. $\mathsf{Log}(\Lambda)$ is *strongly (globally) $\mathcal{X}_\kappa$-complete* if for every Δ in κ variables, $\mathsf{MN}(\Lambda, \Delta) = \mathcal{X}_\kappa(\Lambda, \Delta)$. The logic is *strongly $\mathcal{X}$-complete* if it is strongly $\mathcal{X}_\kappa$-complete for every κ. Obviously, strong completeness implies weak completeness and for any $\mathcal{X} \subseteq \mathcal{Y}$, weak (strong) $\mathcal{X}$-completeness implies weak (strong) $\mathcal{Y}$-completeness.

Remark 2.11 *The reader should be careful to distinguish this notion from, e.g,* strong finite model property *as defined in, e.g., Blok et al. [6] It does not exactly correspond to strong $\mathcal{F}$-completeness in the sense of the present work. According to the definition from Blok et al. [6], a logic has strong fmp is* $\mathsf{MN}(\Lambda, \Delta) = \mathcal{F}(\Lambda, \Delta)$ *for every finite Δ. In other words, it means that the corresponding variety has fmp as a quasivariety. The present notion of strong completeness with respect to a given class of algebras is much stronger, to the extent of being of hardly any use for reformulations of fmp.*

Theorem 2.12 *If the logic axiomatized by Λ is not* tabular*, i.e., it does not correspond to a finite variety, it is not strongly $\mathcal{F}_\omega$-complete.*

Proof: According to, e.g., Theorem 12.1 and Exercise 12.3 in [11], $\mathsf{Log}(\Lambda)$ is tabular iff $\mathbf{Fin}_n \in \mathsf{Log}(\Lambda)$ for some $n \in \mathbb{N}$. Thus, $p \in \mathcal{F}_\omega(\Lambda, \{\mathbf{Fin}_n \to p\}_{n\in\omega})$ for arbitrary Λ. Assume then $p \in \mathsf{MN}(\Lambda, \{\mathbf{Fin}_n \to p\}_{n\in\omega})$. By compactness of the MN-consequence, there exists n such that $p \in \mathsf{MN}(\Lambda, \{\mathbf{Fin}_i \to p\}_{i\leq n})$ and using the fact that $\mathbf{Fin}_i \to \mathbf{Fin}_n \in \mathsf{Log}(\mathbf{K}_\pi)$ for any $i \leq n$ and any π, we may reduce it further to $p \in \mathsf{MN}(\Lambda, \mathbf{Fin}_n \to p)$. But, as $\mathsf{Log}(\Lambda)$ is not tabular, there exists $\langle \mathfrak{A}, \mathfrak{V}\rangle$ s.t. $\mathfrak{A} \vDash \Lambda$ and $\langle \mathfrak{A}, \mathfrak{V}\rangle \nvDash \mathbf{Fin}_n$. Extend $\mathfrak{V}$ to $\mathfrak{V}'$ by stipulating $\mathfrak{V}'(p) = \mathfrak{V}(\mathbf{Fin}_n)$. $\langle \mathfrak{A}, \mathfrak{V}'\rangle \vDash \mathbf{Fin}_n \to p$ and $\langle \mathfrak{A}, \mathfrak{V}'\rangle \nvDash p$, which is a contradiction.

Observe that from a formal point of view, we should first rename variables to obtain a variable p which does not occur in any $\mathbf{Fin}_n$ to avoid extending language to $\langle \textsc{type}, \textsc{var}(\omega+1)\rangle$. From now on, we use this trick without explanation any time we need a supply of no more than κ 'new' variables for any infinite cardinal κ. $\dashv$

Nevertheless, the present notion has not been chosen arbitrarily. This is justified by the following theorem, which is basically a generalization of Theorem 1.4.1 in Wolter [57].

Theorem 2.13 *Let Λ be any set of formulas, κ be any cardinal and $\mathcal{X}$ be any class closed under products. The following are equivalent*

(1) $\mathsf{Log}(\Lambda)$ is strongly $\mathcal{X}_\kappa$-complete.

(2) Every κ-generated algebra in $\mathbb{V}(\Lambda)$ is a subalgebra of an algebra from $\mathcal{X}$; i.e., $\mathsf{Log}(\Lambda)$ is $\mathcal{X}_\kappa$-complex.

Proof: (1) $\Rightarrow$ (2). Take any κ-generated $\mathfrak{B} \in \mathbb{V}(\Lambda)$. Choose any epimorphism α from the absolutely free algebra $\textsc{form}_\kappa$ onto $\mathfrak{B}$. Define $\Gamma = \{\varphi \in \textsc{form}_\kappa | \alpha(\varphi) = \top\}$. Obviously for any $\psi \notin \Gamma$, $\psi \notin \mathsf{MN}_\kappa(\Lambda, \Gamma)$ as Γ is MN-closed set. But then by strong $\mathcal{X}_\kappa$-completeness, there must exist $\langle \mathfrak{A}_\psi, \mathfrak{V}_\psi \rangle$ s.t. $\mathfrak{A}_\psi \in \mathbb{V}(\Lambda) \cap \mathcal{X}$, $\langle \mathfrak{A}_\psi, \mathfrak{V}_\psi \rangle \vDash \Gamma$ and $\langle \mathfrak{A}_\psi, \mathfrak{V}_\psi \rangle \nvDash \psi$. Define $\langle \mathfrak{A}, \mathfrak{V} \rangle \leftrightharpoons \prod_{\psi \notin \Gamma} \langle \mathfrak{A}_\psi, \mathfrak{V}_\psi \rangle$; by the assumption that $\mathcal{X}$ is closed under products, $\mathfrak{A} \in \mathcal{X}$. Let $\mathfrak{C} \leftrightharpoons \{\mathfrak{V}(\varphi) | \varphi \in \textsc{form}_\kappa\}$. It is obviously a subalgebra of $\mathfrak{A}$. To show it is isomorphic to $\mathfrak{B}$, it is enough to show that $\mathfrak{V}(\varphi) = \top$ iff $\alpha(\varphi) = \top$. If $\alpha(\varphi) = \top$, then $\varphi \in \Gamma$ and thus for every ψ, $\langle \mathfrak{A}_\psi, \mathfrak{V}_\psi \rangle \vDash \varphi$. Otherwise, $\langle \mathfrak{A}_\varphi, \mathfrak{V}_\varphi \rangle \nvDash \varphi$.

(2) $\Rightarrow$ (1). Let Λ axiomatize a $\mathcal{X}_\kappa$-complex logic, Γ be an arbitrary set of formulas in $\langle \textsc{type}, \textsc{var}(\kappa)\rangle$. Assume $\alpha \notin \mathsf{MN}(\Lambda, \Gamma)$. It means that α does not belong to the open filter $F_\Box(\Gamma)$ generated by Γ in the Lindenbaum algebra $\textsc{form}_\kappa(\Lambda)/_\Gamma$. But then we can divide $\textsc{form}_\kappa(\Lambda)/_\Gamma$ by $F_\Box(\Gamma)$ and obtain an $\mathfrak{A} \in \mathbb{V}(\Lambda)$ and $\langle \mathfrak{A}, \mathfrak{V} \rangle$ s.t. $\langle \mathfrak{A}, \mathfrak{V} \rangle \vDash \Gamma$ and $\langle \mathfrak{A}, \mathfrak{V} \rangle \nvDash \alpha$. By $\mathcal{X}_\kappa$-complexity, $\mathfrak{A}$ is a subalgebra of some algebra from $\mathbb{V}(\Lambda) \cap \mathcal{X}$.

$\dashv$

Weak completeness does not, in general, imply strong completeness even for classes closed under products. Here is a telling example in the basic similarity type.

Theorem 2.14 **GL.3** *is weakly* $\mathcal{FI}$*-complete.*

Proof: See, e.g., Corollary 5.47 in Chagrov et al. [11] $\dashv$

On the other hand, we have the following

Theorem 2.15 **GL.3** *is not strongly* $\omega\mathcal{C}$*-complete.*

Proof: Let $\Gamma = \{p_i \rightarrow \Diamond p_{i+1} \mid i \in \omega\}$. We show that $\neg p_0 \notin \mathsf{MN}(\mathbf{GL.3}, \Gamma)$. To this end it is enough to take the algebra of finite and cofinite subsets of the integers with the operator $\Diamond X \leftrightharpoons \{n \in \omega | \exists x \in X 0 \leq x < n\} \cup \{z \in \mathbb{Z} - \mathbb{N} | \exists x \in X (x \geq 0 \,\text{or}\, Abs(x) > Abs(z))\}$. This algebra verifies **GL.3**. Let $\mathfrak{V}(p_i) \leftrightharpoons \{i\}$. A straightforward calculation verifies that $\langle \mathfrak{A}, \mathfrak{V} \rangle \models \Gamma$ and $\langle \mathfrak{A}, \mathfrak{V} \rangle \not\models p_0$.

Assume $\mathfrak{A} \in \mathbb{V}(\mathbf{GL.3}) \cap \omega\mathcal{C}$ and there exists a valuation $\mathfrak{V}$ in $\mathfrak{A}$ s.t. $\langle \mathfrak{A}, \mathfrak{V} \rangle \models \Gamma$ and yet $\langle \mathfrak{A}, \mathfrak{V} \rangle \not\models \neg p_0$. Let $a \leftrightharpoons \bigvee\limits_{i \in \omega} \mathfrak{V}(p_i)$. As $\mathfrak{V}(p_0) \neq \bot$, $a \neq \bot$. As for each $i \in \omega$, $\mathfrak{V}(p_i) \leq \Diamond \mathfrak{V}(p_{i+1}) \leq \Diamond a$, $a \leq \Diamond a$. It means, however, that $\Box \neg a \leq \neg a$ and for any $\mathfrak{V}$ s.t. $\mathfrak{V}(p) = \neg a$, $\langle \mathfrak{A}, \mathfrak{V} \rangle \not\models \Box(\Box p \rightarrow p) \rightarrow \Box p$, contrary to the assumption that $\mathfrak{A} \in \mathbb{V}(\mathbf{GL})$. $\dashv$

Corollary 2.16 *For any* $\mathcal{X}$ *s.t.* $\mathcal{FI} \subseteq \mathcal{X} \subseteq \omega\mathcal{C}$*, weak* $\mathcal{X}$*-completeness does not imply strong* $\mathcal{X}$*-completeness.*

There is, however, one more notion of completeness investigated by modal logicians: it is *strong local completeness*, completeness for M-consequence. To define this notion, we introduce first the *local* $\mathcal{X}_\kappa$*-closure* over Λ of Γ in no more than κ variables. $\mathcal{X}^l_\kappa(\Lambda, \Gamma)$ is the set of all formulas α in less than κ variables s.t for every $\mathfrak{A} \in \mathcal{X} \cap \mathbb{V}(\Lambda)$, for every $x \neq \bot$ and for every $\mathfrak{V}$ — a valuation in $\mathfrak{A}$, if $x \leq \mathfrak{V}(\gamma)$ for every $\gamma \in \mathfrak{A}$ (sometimes, we write it as $x \leq \mathcal{V}[\Gamma]$), then $x \leq \mathfrak{V}(\alpha)$. A logic is *strongly locally* $\mathcal{X}_\kappa$*-complete* if for every Γ in κ variables, $\mathsf{M}_\kappa(\Lambda, \Delta) = \mathcal{X}^l_\kappa(\Lambda, \Gamma)$. For any algebra $\mathfrak{A}$ and a set of formulas Γ, we say Γ is *consistent* in $\mathfrak{A}$ if there is a valuation $\mathcal{V}$ in $\mathfrak{A}$ and $x \neq \bot$ s.t. $x \leq \mathcal{V}[\Gamma]$. Thus, for arbitrary class of BAOs $\mathcal{X}$, arbitrary sets of formulas Λ and Γ, $\bot \notin \mathcal{X}^l(\Lambda, \Gamma)$ iff Γ is consistent in a $\mathcal{X}$-BAO from $\mathbb{V}(\Lambda)$. For a finite Γ, Γ is consistent in $\mathfrak{A}$ iff $\mathfrak{A} \not\models \neg \bigwedge \Gamma$. Also, we can note the following

Fact 2.17 (Wolter [57], Proposition 2.4.9) $\blacksquare^{\leq \omega}\Gamma \cup \{\varphi\}$ *is consistent in* $\mathfrak{A}$ *iff for every class of algebras* $\mathcal{X}$ *closed under homomorphic images,* $\neg\varphi \notin \mathcal{X}(Th(\mathfrak{A}), \Gamma)$.

In most of the present thesis, we are concerned mainly with the notion of weak completeness. An in-depth discussion of strong completeness notions is postponed until Chapter 7.

One of the most interesting questions from our perspective is the existence of non-trivial completeness results with respect to members of PROPERTIES other than the smallest and the greatest one (i.e., other than $\mathcal{CAV}$ and $\mathcal{ALG}$). One of preciously few related results in the existing literature is the following

Theorem 2.18 (Buszkowski [9]) *Assume that for every $\alpha \in \Lambda$ and every propositional variable p appearing in α, p is in the scope of $\Diamond_\pi$, for some $\pi \in$ TYPE. Then $\mathsf{Log}(\Lambda)$ is $\mathcal{A}$-complete.*

Proof: See Buszkowski [9] or Venema [56], Theorem 6.6. $\dashv$

In forthcoming chapters, we are going to see why it may be important to investigate such notions of completeness. We are also going to discover unavoidable limitations to general completeness theorems similar to Theorem 2.18.

2.4 Wolter's Splitting Theorem

The material in this section is based on Wolter [57, Chapter 2]. The techniques introduced in that work are a very convenient tool to deal with the abstract concept of *splittings* of lattices of logic. In addition, they allow us to prove the Jónsson Lemma for the particular case of BAOs instead of just *using* it.

Assume $\mathfrak{A} \in \mathcal{S}$ is a κ-generated algebra and the set of generators is $\{*\} \cup \{a_{i+1}\}_{i \in \kappa}$, where $a_0 = *$ is an opremum, let $\mathfrak{V}^{\mathfrak{A}}(p_\iota) \leftrightharpoons a_\iota$ for any $\iota \leq \kappa$ and define $\Gamma^{\mathfrak{A}}$ in the same way as in the proof of Theorem 2.13, i.e., $\Gamma^{\mathfrak{A}} \leftrightharpoons \{\varphi \in \text{FORM}_\kappa | \langle \mathfrak{A}, \mathfrak{V}^{\mathfrak{A}} \rangle \vDash \varphi\}$. For any set of formulas Λ s.t. $\mathfrak{A} \vDash \Lambda$ and any set of formulas Δ in $\langle \text{TYPE}, \text{VAR}(\kappa) \rangle$, we say Δ is *a Λ-diagram* of $\mathfrak{A}$ if $\mathsf{MN}_\kappa(\Lambda, \Delta) = \Gamma^{\mathfrak{A}}$. $\mathfrak{A}$ is *finitely presentable* above Λ if it has a finite Λ-diagram.

For a finite algebra $\mathfrak{A}$, a particular case of a diagram is provided by *the Jankov formula*. For any $a \in \mathfrak{A}$, let p_a be a distinct variable.

$$\delta(\mathfrak{A}) \leftrightharpoons \bigwedge\{p_a \wedge p_b \leftrightarrow p_{a \wedge b}\} \wedge \bigwedge\{\neg p_a \leftrightarrow p_{\neg a}\} \wedge \bigwedge\{\Diamond_\pi p_a \leftrightarrow p_{\Diamond_\pi a}\}$$

Lemma 2.19 *For any $\mathfrak{A}$ and any Λ s.t. $\mathfrak{A} \models \Lambda$, $\delta(\mathfrak{A})$ is a Λ-diagram of $\mathfrak{A}$.*

Proof: See Proposition 7.3.6 in Kracht [37]. $\dashv$

Lemma 2.20 (Wolter [57], Proposition 2.4.11) *Assume $\mathfrak{A} \in \mathbb{V}(\Lambda) \cap \mathcal{S}$, Δ is a Λ-diagram of $\mathfrak{A}$. For any $\mathfrak{B} \in \mathbb{V}(\Lambda)$ the following are equivalent:*

(1) $\blacksquare^{\leq\omega}\Delta \cup \{\neg p_*\}$ is consistent in $\mathfrak{B}$;

(2) $\mathfrak{A} \in ISH(\mathfrak{B})$;

(3) $\mathfrak{A} \in HS(\mathfrak{B})$.

Proof: (1) $\Rightarrow$ (2). By Fact 2.17, there is $\mathfrak{C} \in H(\mathfrak{B})$ and a valuation $\mathfrak{V}$ in $\mathfrak{C}$ s.t. $\mathfrak{V}[\Delta] = \top$ and $\mathfrak{V}(p_*) \neq \bot$. Then

$$\mathsf{MN}_\kappa(\Lambda, \Delta) \subseteq \{\varphi \in \text{FORM}_\kappa | \mathfrak{V}(\varphi) = \top\}. \tag{2.1}$$

Define a mapping $f : \mathfrak{A} \mapsto \mathfrak{C}$ by $f(a) \leftrightharpoons \mathfrak{V}(\psi_a)$, where ψ_a is an arbitrarily chosen formula s.t. $\mathfrak{V}^{\mathfrak{A}}(\psi) = a$. This mapping is well-defined: if $\mathfrak{V}^{\mathfrak{A}}(\varphi) = \mathfrak{V}^{\mathfrak{A}}(\psi)$ then $\varphi \leftrightarrow \psi \in \Gamma^{\mathfrak{A}} = \mathsf{MN}_\kappa(\Lambda, \Delta)$ and by 2.1, $\mathfrak{V}(\varphi) = \mathfrak{V}(\psi)$. In the same way, one can prove f is a homomorphism. For every $a \in \mathfrak{A}$, $a \neq \top$ iff $\bigwedge \blacksquare^{\leq n} a \leq *$ for some $n \in \omega$, this implies $\bigwedge \blacksquare^{\leq n} f(a) \leq f(*) = \mathfrak{V}(p_*) \neq \top$, hence $f(a) \neq \top$, thus f is an embedding, hence $\mathfrak{A} \in ISH(\mathfrak{B})$

(2) $\Rightarrow$ (3) Follows by $ISH(\mathfrak{B}) \subseteq HS(\mathfrak{B})$.

(3) ($\Rightarrow$) (1). $\blacksquare^{\leq\omega}\Delta \cup \{\neg p_*\}$ is consistent in $\mathfrak{A}$, the rest follows by standard techniques. $\dashv$

Wolter [57] used this theorem to prove the following crucial

Theorem 2.21 (The Jónsson Lemma for BAOs) *Assume $\mathfrak{A} \in \mathbb{V}(\Lambda) \cap \mathcal{S}$, Δ is a Λ-diagram of $\mathfrak{A}$, $K \subseteq \mathbb{V}(\Lambda)$. The following are equivalent:*

(1) $\mathfrak{A} \in HSP_U(K)$;

(2) $\mathfrak{A} \in HSP(K)$;

(3) Every finite subset of $\blacksquare^{\leq\omega}\Delta \cup \{\neg p_*\}$ is consistent in some algebra from K.

Proof: Only (3) $\Rightarrow$ (1) requires proof. But this is almost immediate by now. By standard model-theoretic techniques (Corollary 4.1.11 in Chang and Keisler [13]), we obtain an algebra $\mathfrak{C}$ in $P_U(K)$ s.t. $\blacksquare^{\leq\omega}\Delta \cup \{\neg p_*\}$ is consistent in $\mathfrak{C}$. By Lemma 2.20, $\mathfrak{A} \in HS(\mathfrak{C})$. $\dashv$

This theorem has far-reaching consequences. Perhaps the most striking one is

Theorem 2.22 (Wolter's splitting theorem) *Let* $\theta \fallingdotseq Th(\mathfrak{A})$ *for a finitely generated* $\mathfrak{A} \in \mathbb{V}(\Lambda) \cap \mathcal{S}$ *and let* Δ *be a* Λ*-diagram of* $\mathfrak{A}$*. The following are equivalent:*

(1) θ *splits* $Ext\Lambda$*.*

(2) $\{\mathfrak{B} \in \mathbb{V}(\Lambda) | \mathfrak{A} \notin HSP_U(\mathfrak{B})\}$ *is a variety.*

(3) $\{\mathfrak{B} \in \mathbb{V}(\Lambda) | \mathfrak{A} \notin HSP_U(\mathfrak{B})\}$ *is closed under ultraproducts.*

(4) There is $m \in \omega$ *and a finite* $\delta \subseteq \Delta$ *s.t. for every* $\mathfrak{B}$*, if* $\mathfrak{B} \not\models \bigwedge \blacksquare^{\leq m}\delta \to p_*$*, then every finite subset of* $\blacksquare^{\leq\omega}\Delta \cup \{p_*\}$ *is consistent in* $\mathfrak{B}$*.* $\bigwedge \blacksquare^{\leq m}\delta \to p_*$ *is called* the splitting formula *of* $\mathfrak{A}$*.*

Proof: See Wolter [57]. ⊣

Some more well-known consequences of the Jónsson Lemma are:

Corollary 2.23

- *If* $\Lambda \supseteq Th(\mathfrak{A})$ *for a finite* $\mathfrak{A}$*, then* $\Lambda = Th(\mathfrak{B})$ *for a finite* $\mathfrak{B} \in HS(\mathfrak{A})$*.*
- *If* Λ *has the finite model property (i.e., is* $\mathcal{F}$*-complete) and* θ *splits* $Ext\Lambda$*, then* $\theta = Th(\mathfrak{A})$ *for a* finite $\mathfrak{A} \in \mathcal{S}$*.*
- *If* Λ *has the fmp and* $Ext\Lambda$ *is lower-continuous, then every extension of* Λ *has the finite model property.*

Chapter 3

Duality theory and minimal extensions

This chapter explains the motivations behind the decision to study completenes with respect to classes of algebras with particular properties. One reason is that some classes of BAOs allow for nice representation theorems. Moreover, some classes in PROPERTIES with appropriately defined morphisms can be characterized by strong category-theoretical duality results. Another reason is that the notion of completeness with respect to a class of BAOs can coincide with a syntactic notion of conservativity of some natural minimal extension.

3.1 Basics of duality theory, general case

In this section, we sketch duality theory for BAO-s via canonical extensions. This theory seems well-known. One of the most readable and clear expositions may be found in Chapter 5 of Blackburn et al. [3]. Therefore, our introduction is going to be brief.

As usual, define *a Kripke frame* (or *a relational structure* as a pair $\mathfrak{F} = \langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$, where $R_\pi \in W \times W$ for every $\pi \in$ TYPE. Let $\mathfrak{F} \leftrightharpoons \langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$ and $\mathfrak{G} \leftrightharpoons \langle V, \{S_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$. A mapping f from W to V is called *a bounded morphism* from $\mathfrak{F}$ to $\mathfrak{G}$ iff the following two conditions are satisfied:

forth for every $\pi \in$ TYPE and every $x, y \in W$, $xR_\pi y$ implies $f(x)S_\pi f(y)$;

back for every $\pi \in$ TYPE, every $x \in W$ and $v \in V$, $f(x)S_\pi v$ implies the existence of $y \in W$ s.t. $xR_\pi y$ and $f(y) = v$.

A *generated subframe* of $\mathfrak{F}$ is any frame $\mathfrak{G}$ s.t. the identity mapping on $\mathfrak{G}$ is an injective bounded morphism into $\mathfrak{F}$. A *disjoint union* of a family of Kripke frames $\{\mathfrak{F}_i\}_{i \in J}$ is a frame whose universe is defined as a (set-theoretical) disjoint union of universes of $\mathfrak{F}_i$'s and the accessibility relation is defined componentwise. There are a few other notions related to Kripke frames which prove useful in what follows. *A path* in $\langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$ from x to y is a finite sequence of points $\{x_i\}_{i \leq n}$ s.t. $x = x_0$, $y = x_n$ and for every $i < n$ there is $\pi \in$ TYPE s.t. $x_i R_\pi x_{i+1}$. A *cycle* is a path from x to x. For any x and any $\mathfrak{F} = \langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$, the frame $\mathfrak{F}_x$ consisting of all points y s.t. there is a path in $\mathfrak{F}$ from x to y is called *the subframe of* $\mathfrak{F}$ generated by x; it is straightforward to see that this is indeed a generated subframe of $\mathfrak{F}$. If there is x s.t. $\mathfrak{F} = \mathfrak{F}_x$, then $\mathfrak{F}$ is called *a rooted frame* and x is its *root*.

The dual of a Kripke frame $\mathfrak{F} = \langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$ is a BAO $\mathfrak{F}^\#$ s.t. the boolean reduct of $\mathfrak{A}$ is the powerset algebra of W and for every $\pi \in$ TYPE and every $X \subseteq W$, $\Diamond^\#_\pi X = \{y \in W | \exists x \in X y R_\pi x\}$ for every π. This definition allows us to define satisfaction and truth in Kripke frames. A valuation in $\mathfrak{F} = \langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$ is a valuation in $\mathfrak{F}^\#$ (i.e., a mapping of propositional variables into subsets of W, extended to all formulas) and $\langle \mathfrak{F}, \mathfrak{V} \rangle \vDash \alpha$ iff $\langle \mathfrak{F}^\#, \mathfrak{V} \rangle \vDash \alpha$. The notion of (weak and strong) completeness may be now defined for Kripke frames in the standard way. Nevertheless, as we are going to see soon, Kripke completeness is nontrivial issue and many logics do fail to be complete in this sense. For now, let us just record the following well-known fact, implicit already in classical papers of Jónsson and Tarski [31], [32].

Theorem 3.1 *The dual of a Kripke frame is a* $\mathcal{CAV}$-BAO.

Proof: (sketch) Suprema are set-theoretical sums, atoms are singletons. Verification of complete additivity is straightforward. $\dashv$

Corollary 3.2 *Kripke completeness implies* $\mathcal{CAV}$*-completeness.*

The question whether converses of Theorem 3.1 and Corollary 3.2 hold, i.e., whether $\mathcal{CAV}$-BAOs are Kripke frames in disguise, is answered in the next section.

Given a BAO $\mathfrak{A}$, define its *canonical frame* $\mathfrak{A}_\# \leftrightharpoons \langle Uf\mathfrak{A}, \{R_\pi^{\mathfrak{A}}\}_{\pi\in\text{TYPE}}\rangle$ s.t. for every $\pi \in$ TYPE and $U, V \in Uf\mathfrak{A}$, $UR_\pi^{\mathfrak{A}}V$ iff for every $v \in V$, $\Diamond_\pi v \in U$. Let us recall another standard result.

Theorem 3.3 *Every $\mathfrak{A}$ is embeddable into $(\mathfrak{A}_\#)^\#$.*

Proof: (sketch) The mapping $f_{\mathfrak{A}}(a) \leftrightharpoons \{U \in Uf\mathfrak{A} | a \in U\}$ is the desired *canonical embedding.* ⊣

Given a BAO-morphism $f : \mathfrak{A} \mapsto \mathfrak{B}$, define $f_\# : \mathfrak{B}_\# \mapsto \mathfrak{A}_\#$ by $f_\#(U) \leftrightharpoons \{a \in \mathfrak{A} | f(a) \in U\}$ (one may easily verify that $f_\#(U)$ is indeed an ultrafilter and thus $f_\#$ is well-defined). Conversely, given a bounded morphism $g : \mathfrak{F} \mapsto \mathfrak{G}$, define $g^\# : \mathfrak{G}^\# \mapsto \mathfrak{A}^\#$ by $f^\#(X) \leftrightharpoons \{x | f(x) \in X\}$.

Theorem 3.4

- *If f is a BAO-morphism, then $f_\#$ is a bounded morphism.*
- *If g is a bounded morphism, then $g^\#$ is a BAO-morphism.*
- *If f is onto (injective), then $f_\#$ is injective (onto).*
- *If g is onto (injective), then $f^\#$ is injective (onto).*

Proof: See, e.g., Theorem 5.47 in Blackburn et al. [3]. ⊣

Thus, to use the language of category theory, $(\,)_\#$ and $(\,)^\#$ form a pair of *contravariant functors*. But it does not follow that categories of BAOs and Kripke frames with appropriate morphisms are *equivalent*. To this end, we would have to show that $\mathfrak{A}$ is isomorphic to $(\mathfrak{A}_\#)^\#$ and the same for frames. Alas, in the infinite case it *never* holds.

Theorem 3.5 *$\mathfrak{A}$ is isomorphic to $(\mathfrak{A}_\#)^\#$ iff $\mathfrak{A}$ is finite. If $\mathfrak{A}$ is infinite, then the cardinality of $(\mathfrak{A}_\#)^\#$ is strictly greater than that of $\mathfrak{A}$.*

Proof: (sketch) We only prove the second statement of the theorem. If $|\mathfrak{A}| = \kappa$ for some infinite κ, then $\mathfrak{A}$ has at least κ ultrafilters (cf. Theorem 5.31 in Koppelberg [33]). But then the cardinality of $(\mathfrak{A}_\#)^\#$ is not smaller than 2^κ. ⊣

In the same way, one may prove that

Theorem 3.6 *$\mathfrak{F}$ is isomorphic to $(\mathfrak{F}^{\#})_{\#}$ iff $\mathfrak{F}$ is finite. If $\mathfrak{F}$ is infinite, then the cardinality of $(\mathfrak{F}_{\#})^{\#}$ is strictly greater than the cardinality of $\mathfrak{F}$.*

In some cases, we may prove far more than that. There are cases when $(\mathfrak{A}_{\#})^{\#}$ does not even belong to the variety generated by $\mathfrak{A}$; i.e., there are some formulas true in $\mathfrak{A}$ which fail in its canonical extension. Let us call a logic Λ *canonical* if for every $\mathfrak{A} \in \mathbb{V}(\Lambda)$, $(\mathfrak{A}_{\#})^{\#} \in \mathbb{V}(\Lambda)$.

Theorem 3.7 *Any logic between* **GL** *and* **GL.3** *is not canonical.*

Proof: This is a weak form of Theorem 2.15. That theorem together with Theorem 2.13 implies that there are algebras in $\mathbb{V}(\mathbf{GL.3})$ which cannot be embedded in any $\omega\mathcal{C}$-BAO from $\mathbb{V}(\mathbf{GL})$. But if a logic is canonical, then — by Theorems 3.1 and 3.3 — every algebra from the corresponding variety is embeddable into a $\mathcal{CAV}$-BAO from the same variety. ⊣

Can we do any better than this? Yes, we can. To begin with, let us generalize the concept of a frame. *A general frame* is a triple $\mathfrak{F} = \langle W, \{R_\pi\}_{\pi\in\text{TYPE}}, A\rangle$, where $R_\pi \in W \times W$ for every $\pi \in$ TYPE and A is a family of subsets of W which forms the universe of a subalgebra of $\mathfrak{F}^{\#}$; $\langle W, \{R_\pi\}_{\pi\in\text{TYPE}}\rangle$ is called the *the underlying frame* of $\langle W, \{R_\pi\}_{\pi\in\text{TYPE}}, A\rangle$ and A is called its *set of admissible subsets.* Theorem 3.3 suggests a natural way of representing every BAO in this way. For a given $\mathfrak{A}$, let $\mathfrak{A}_+$ be the general frame whose underlying frame is $\mathfrak{A}_{\#}$ and the set of admissible subsets is $f_{\mathfrak{A}}[\mathfrak{A}]$, i.e., the family of all subsets of $Uf\mathfrak{A}$ which are images of some $a \in \mathfrak{A}$ via the canonical embedding defined in the proof of the Theorem 3.3. Every general frame which can be represented as $\mathfrak{A}_+$ for some $\mathfrak{A}$ is called *descriptive.* The following nice topological characterization of *descriptive frames* is well-known.

Theorem 3.8 *$\langle W, \{R_\pi\}_{\pi\in\text{TYPE}}, A\rangle$ is descriptive iff the following three conditions hold:*

differentiatedness *for any $x, y \in W$, if $\forall X \in A, x \in X \Leftrightarrow y \in X$, then $u = v$;*

tightness *for any $x, y \in W$ and any $\pi \in$ TYPE, if $\forall X \in A, y \in X \Rightarrow x \in \Diamond^{\#}_{\pi} X$, then $xR_\pi y$;*

compactness *any family of elements of A with the finite intersection property has non-empty intersection (we do not require that the intersection itself belongs to A).*

Proof: See, e.g., Theorem 8.51 in Chagrov et al. [11]. $\dashv$

Given a general frame $\mathfrak{F} = \langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}}, A \rangle$, let $\mathfrak{F}^+$ be the the subalgebra of $\langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}}, A \rangle^\#$ whose universe is A. The following is straightforward now.

Theorem 3.9 *For every $\mathfrak{A}$, $(\mathfrak{A}_+)^+$ is isomorphic to $\mathfrak{A}$. For a general frame $(\mathfrak{F}^+)_+$ is isomorphic to $\mathfrak{F}$ iff $\mathfrak{F}$ is descriptive.*

Definitions of satisfaction and truth in a general frame $\mathfrak{F}$ may now be given by the use of $\mathfrak{F}^+$ in the same way they were given for Kripke frames. Ditto for definition of completeness of a logic with respect to a class of Kripke frames. We obtain an immediate

Corollary 3.10 *Every logic is complete with respect to descriptive frames.*

In order to extend $(\,)^+$ and $(\,)_+$ to morphisms, we need to define first what a bounded morphism for general frames actually is. Let $\mathfrak{F} \leftrightharpoons \langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}}, A \rangle$ and $\mathfrak{G} \leftrightharpoons \langle V, \{S_\pi\}_{\pi \in \mathrm{TYPE}}, B \rangle$. A mapping from W to V is *a bounded morphism* if it is a bounded morphism between $\langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$ and $\langle V, \{S_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$ and, in addition, for every $b \in B$, $f^{-1}[b] \in A$. Observe that we *don't* require that for every $a \in A$, $f(a) \in B$. A bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$ is *an embedding* if it is injective *and* for every $a \in A$ there is $b \in B$ s.t. $f[a] = f[W] \cap b$. Other constructions on general frames may be defined analogously as on Kripke frames. Thus, $\langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}}, A \rangle$ is a generated subframe of $\mathfrak{G}$ iff the identity mapping on $\langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}}, A \rangle$ is an embedding (in the new sense) into $\mathfrak{G}$ and a disjoint union of a family of general frames is a general frame whose underlying frame is the disjoint union of the underlying frames and the admissible subsets are all sums of admissible subsets of frames from the family.

For an arbitrary morphism f from $\mathfrak{A}$ to $\mathfrak{B}$ we may define now f_+ from $\mathfrak{B}_+$ to $\mathfrak{A}_+$ in the same way we defined $f_\#$; i.e., $f_+(U) \leftrightharpoons \{a \in \mathfrak{A} | f(a) \in U\}$. Analogously, for an arbitrary bounded morphism g from $\mathfrak{F} \leftrightharpoons \langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}}, A \rangle$ to $\mathfrak{G} \leftrightharpoons \langle V, \{S_\pi\}_{\pi \in \mathrm{TYPE}}, B \rangle$ we may define g^+ from $\mathfrak{G}^+$ to $\mathfrak{F}^+$ as a restriction of $g^\#$ to $\mathfrak{G}^+$, i.e., as $g^+(b) \leftrightharpoons \{x \in W | g(x) \in B\}$. Again, we may prove that

Theorem 3.11 • *If f is a* BAO*-morphism, then f_+ is a bounded morphism.*

- *If g is a bounded morphism (of general frames), then g^+ is a* BAO*-morphism.*
- *If f is onto (injective), then f_+ is an embedding (onto).*
- *If g is onto (an embedding), then f^+ is injective (onto).*

Proof: Cf., e.g., Propositions 5.79 and 5.80 in Blackburn et al. [3]. ⊣

Thus, $(\,)^+$ and $(\,)_+$ are indeed morphisms; but actually we showed far more than that. By Theorem 3.9 it follows that the categories of BAOs and descriptive frames with respective morphisms are indeed *equivalent* by $(\,)^+$ and $(\,)_+$. So, is this exactly what we wanted to obtain?

Of course, the answer depends on one's goals. But Kripke frames were introduced because of their simplicity. This was considered to be one of their main advantages (if not *the main* advantage) over BAOs: if the universe of $\mathfrak{F}$ is of cardinality κ, then $\mathfrak{F}^{\#}$ has cardinality 2^{κ}. But assume now κ is infinite and take the underlying frame of $(\mathfrak{F}^{\#})_+$, i.e., $(\mathfrak{F}^{\#})_{\#}$ (this frame is known as *the ultrafilter extension* of $\mathfrak{F}$). The powerset algebra of any set of infinite cardinality κ has $2^{2^{\kappa}}$ ultrafilters. This was discovered first by Tarski; even more general observation may be actually obtained as a corollary of an important and difficult result in the theory of boolean algebras, known as the Balcar-Franěk Theorem.

Theorem 3.12 *Assume $\mathfrak{A} \in \mathcal{C}$. Then $\mathfrak{A}$ has $2^{|\mathfrak{A}|}$ ultrafilters.*

Proof: See Corollary 13.7 in Koppelberg [33]. ⊣

In other words, the underlying frame of the dual of a Kripke frame with a countable universe consists of $2^{\mathfrak{c}}$ points. The dual algebra itself has "only" a continuum of elements. Informally speaking, in the infinite case we have the following "hierarchy of complexity": Kripke frames < BAOs < descriptive frames.

We cannot hope for any better representation theorem for the class of all BAOs: this class is simply too big and contains too many counterexamples of all sorts, as we are going to see in the in forthcoming chapters. This is one of the reasons why it may pay off to restrict attention to classes of algebras with particular properties: they may admit a better dual representation than the class of all BAOs does. Theorem 3.1 suggests that at least for some classes of algebras containing $\mathcal{CAV}$-BAOs we should be able to attain this goal. There should be a duality which takes $\mathfrak{F}^{\#}$ back to $\mathfrak{F}$ itself rather than to something incomparably more complex. The next section shows this is indeed the case.

3.2 Duality theory for $\mathcal{AV}$-BAOs

As stated by Theorem 3.8, there are three defining properties of descriptive frames: differentiatedness, tightness and compactness. General frames with the first two properties are called *refined*. In this section, we focus on another important class of refined frames; the results described here are natural generalizations of those obtained by Thomason [51] for duals of Kripke frames. Say that $\langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}}, A\rangle$ is *discrete* if for all $x \in W$, $\{x\} \in A$, i.e., all singletons are admissible.

Remark 3.13 *In personal communication, M. Zakharyaschev suggested that the name* discrete *is misleading; it is often applied to frames based on discretely ordered sets. He suggested the name* atomic *should be used instead. The author acknowledges this comment, but leaves the terminology unchanged.*

The following is well-known and rather straightforward.

Theorem 3.14 *All discrete frames are refined. A finite general frame $\mathfrak{F}$ is discrete iff it is descriptive. An infinite discrete frame is never descriptive.*

Proof: We prove only the last statement. Assume $\langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}}, A\rangle$ is discrete and $|W| \geq \omega$. The family $\{W - \{x\} | x \in W\}$ has the finite intersction property, but its intersection is empty. $\dashv$

Kripke frames may be treated as a subclass of discrete frames. Say that a general frame is *full* if all sets are admissible; thus, there is no real difference between the general frame itself an its underlying frame. Of course, all full frames are discrete. The notions of bounded morphism, generated subframe and disjoint union for discrete frames are defined in the same way as for general frames. The starting point for the duality theory we are about to develop is the following observation, generalizing Theorem 3.1:

Lemma 3.15 *For any discrete frame $\mathfrak{F}$, $\mathfrak{F}^+$ is a $\mathcal{AV}$-BAO.*

Proof: Assume $\mathfrak{A} = \langle W, \{R_\pi\}_{\pi\in\text{TYPE}}, A\rangle^+$, where $\langle W, \{R_\pi\}_{\pi\in\text{TYPE}}, A\rangle$ is a discrete frame. Atomicity follows from the fact that every $X \neq \emptyset$ is above a singleton. Assume now $X \subseteq \Diamond_\pi Y$. It means for every $x \in X$ there is $y_x \in Y$ s.t. $xR_\pi y_x$. But then $\{x\} \subseteq X \cap \Diamond_\pi\{y_x\}$ and the theorem is proven. ⊣

Moreover, we can prove the following.

Lemma 3.16 *Assume $\mathfrak{F} \leftrightharpoons \langle W, \{R_\pi\}_{\pi\in\text{TYPE}}, A\rangle$ and $\mathfrak{G} \leftrightharpoons \langle V, \{S_\pi\}_{\pi\in\text{TYPE}}, B\rangle$ are discrete frames and $g : \mathfrak{F} \mapsto \mathfrak{G}$ is a bounded morphism. Then g^+ is a complete homomorphism of $\mathcal{AV}$-BAOs.*

Proof: That both $\mathfrak{F}^+$ and $\mathfrak{G}^+$ are $\mathcal{AV}$-BAOs follows from Lemma 3.15, whereas the fact that f^+ is a BAO-morphism follows from Theorem 3.11. We only have to prove that it is a complete homomorphism, i.e., that it preserves arbitrary existing joins. Assume $X \subseteq V$ and X is a supremum of a family of atoms $\{v_i\}_{i\in I}$; i.e., $X = \{v_i | i \in I\}$. Furthermore, assume $Y \subseteq W$, for all $i \in I$, $f^+(\{v_i\}) \subseteq Y$ but $Y \subsetneq f^+(X)$, i.e., there is $x \in f^+(X)$ s.t. for no $i \in I$, $x \in f^+(\{v_i\})$. But this means that for no $v_i \in X$, $f(x) = v_i$, a contradiction with the fact that $f(x) \in X$. ⊣

Thus, we have a natural candidate for the dual of the category of discrete frames with bounded morphisms: it is the category of $\mathcal{AV}$-BAOs with complete morphisms. For arbitrary $\mathfrak{A} \in \mathcal{AV}$, define $\mathfrak{A}_*$ as a frame whose underlying set is $At\mathfrak{A}$, for every $\pi \in \text{TYPE}$, $xR_\pi y$ iff $x \leq \Diamond_\pi y$ and the admissible sets are of exactly those the form $X_a \leftrightharpoons \{x \in At\mathfrak{A} | x \leq a\}$ for some $a \in \mathfrak{A}$. In order to check that this definition is correct, we have to verify that for every $a \in \mathfrak{A}$, $\Diamond_\pi X_a = X_{\Diamond_\pi a}$. $\Diamond_\pi X_a \subseteq X_{\Diamond_\pi a}$ is straightforward. Conversely, if $x \in X_{\Diamond_\pi a}$, then (by complete additivity and x being an atom) there is $y \in X_{\Diamond_\pi a}$ s.t. $x \leq \Diamond_\pi y$ but then $x \in \Diamond_\pi X_a$. For every $x \in At\mathfrak{A}$, $X_x = \{x\}$, thus all singletons are admissible.

Theorem 3.17 *Every $\mathfrak{A} \in \mathcal{AV}$ is isomorphic to $(\mathfrak{A}_*)^+$ and every discrete $\mathfrak{G}$ is isomorphic to $(\mathfrak{G}^+)_*$. The respective isomorphisms are defined as $f(a) \leftrightharpoons X_a$ and $g(x) \leftrightharpoons \{x\}$.*

Proof: Follows from the above. ⊣

Corollary 3.18 *A logic is complete with respect to a class of discrete frames iff it is complete with respect to a class of* $\mathcal{AV}$-BAO*s.*

Thus, we have obtained a nice representation theorem for $\mathcal{AV}$-BAOs. In order to show category-theoretical equivalence, one needs to define duals of complete BAO-morphisms. It can be proven that if $f : \mathfrak{A} \mapsto \mathfrak{B}$, the suitable definition is $f_*(b) = a$ iff $b \leq f(a)$. We can show that the definition is correct, i.e., for every $b \in At\mathfrak{B}$, there exists exactly one $a \in At\mathfrak{A}$ s.t. $b \leq f(a)$. If there exists none, then for all $a \in At\mathfrak{A}$, $b \leq \neg f(a)$. But $\bigvee At\mathfrak{A} = \top$, a contradiction with f being a complete homomorphism. If there are two distinct atoms a, a' s.t. $b \leq f(a) \wedge f(a')$, then by f being a homomorphism, $b = \bot$, a contradiction. Now one may prove an analogue of Theorems 3.4 and 3.11. The functor $(\,)^+$ restricted to discrete frames can be written as $(\,)^*$.

It is worthwile to point out that that the work of Thomason [51] shows a way to obtain a duality for the class of $\mathcal{AV}$-BAOs with *all* morphisms, not only the complete ones. To this aim one needs to replace bounded morphisms with *di-morphisms* (the name introduced in ten Cate, Litak [VII], the concept itself by Thomason [51] for Kripke frames). A di-morphism from $\mathfrak{F} \leftrightharpoons \langle W, \{R_\pi\}_{\pi\in\text{TYPE}}, A\rangle$ to $\mathfrak{G} \leftrightharpoons \langle V, \{S_\pi\}_{\pi\in\text{TYPE}}, B\rangle$ is a function g from W to $Uf\mathfrak{G}^*$ satisfying

reverse image $\{x \in W | b \in g(x)\} \in A$, for every $b \in B$;

forth for every $\pi \in$ TYPE and every $x, y \in W$, $xR_\pi y$ implies that for every $Y \in g(y)$, $\Diamond_\pi Y \in g(x)$;

back for every $\pi \in$ TYPE, every $x \in W$ and $Y \in B$, $\Diamond_\pi Y \in f(x)$ implies the existence of $y \in W$ s.t. $xR_\pi y$ and $Y \in g(y)$.

With a suitably defined identity morphism (namely, the mapping attributing to every element the principal ultrafilter generated by the corresponding singleton) and composition, discrete frames with di-morphisms do form a category, too. A di-morphism is *onto* if its range includes all principal ultrafilters of $\mathfrak{G}^*$ ($\mathfrak{G}$ is then called a *di-morphic image* of $\mathfrak{F}$) and is *an embedding* if for every $a \in A$, there exists $b_a \in \mathfrak{B}$ s.t. $x \in a$ iff $b_a \in g(x)$ (observe that it implies injectivity of g). We may now define functors $(\,)^\circ$, $(\,)_\circ$ which on objects are defined in exactly the same way as $(\,)^*$ and $(\,)_*$. For g a di-morphism from $\langle W, \{R_\pi\}_{\pi\in\text{TYPE}}, A\rangle$ to $\langle V, \{S_\pi\}_{\pi\in\text{TYPE}}, B\rangle$,

$g^\circ(b) \fallingdotseq \{x \in W | b \in g(x)\}$ (it is well-defined by the reverse image condition) and for $f : \mathfrak{A} \mapsto \mathfrak{B}$, $f_\circ(b) \fallingdotseq \{X_a | b \leq f(a)\}$.

Theorem 3.19

- *If f is a* BAO*-morphism, then $f_\circ$ is a di-morphism.*
- *If g is a di-morphism, then g° is a* BAO*-morphism.*
- *If f is onto (injective), then $f_\circ$ is an embedding (onto).*
- *If g is onto (an embedding), then g° is injective (onto).*

Sketch of proof: For the first statement: $f_\circ(b)$ must be an ultrafilter, as b is an atom, $\{v \in V | a \in f_\circ(v)\} = X_{f(a)}$, the forth condition is immediate, the back condition uses complete additivity. For the second statement: that g° is boolean homomorphism, follows from properties of ultrafilters, $g^\circ(\Diamond_\pi b) \leq \Diamond_\pi g^\circ(b)$ by the back condition, reverse inequality by the forth condition. For the third statement: if f is onto, then for every $b \in \mathfrak{B}$ there is $a \in \mathfrak{A}$ s.t. $f(a) = b$ and thus for every $v \in At\mathfrak{B}$, $v \in X_b$ iff $X_a \in f_\circ(v)$; if f is an embedding, then every atom is sent onto an element distinct from the bottom and thus for every principal ultrafilter $\{X_a | a' \leq a\}$ for $a' \in At\mathfrak{A}$, there is b s.t. $b \leq f(a)$ iff $a' \leq a$. For the fourth statement, if g is onto, then every two elements in an atomic algebra can be distinguished by a principal ultrafilter; if g is an embedding, then for every $a \in A$, $a = g^\circ(b_a)$. ⊣

Let us conclude with an observation that we can prove a rather weak representation theorem also for $\mathcal{T}$-BAOs.

Lemma 3.20 *An algebra is a $\mathcal{T}$-*BAO *iff it can be represented as reduct of some $\mathfrak{A} = \langle W, \{R_\pi, R_\pi^{-1}\}_{\pi \in \text{TYPE}}, A\rangle^+$.*

Proof: See the proof of Corollary 3.25 and use duality for descriptive frames. The only thing we have to prove in addition is that if for every $\pi \in$ TYPE there is $\pi' \in$ TYPE s.t. $\mathfrak{A} \vDash \mathbf{Conj}^\pi_{\pi'}$, then for any $\Gamma, \Delta \in Uf\mathfrak{A}$, $\Gamma R^{\mathfrak{A}}_\pi \Delta$ implies $\Delta R^{\mathfrak{A}}_{\pi'} \Gamma$. Assume $a \in \Gamma$ and $\Gamma R^{\mathfrak{A}}_\pi \Delta$. If $\Diamond_{\pi'} a \notin \Delta$, then $\Box_{\pi'} \neg a \in \Delta$. But then $\Diamond_\pi \Box_{\pi'} \neg a \in \Gamma$, a contradiction with the fact that by $\mathbf{Conj}^\pi_{\pi'}$, $\Box_\pi \Diamond_{\pi'} a \in \Gamma$. ⊣

3.3 Conservativity of minimal extensions

Some notions of completeness we are going to study have an independent motivation in terms of conservativity of certain minimal extensions. Informally speaking, a minimal extension of a modal logic Λ is the smallest set of formulas in an extension of modal syntax — i.e., formulas are allowed to contain new types of variables, constants, connectives or operators — which contains Λ and is closed with respect to new axioms and rules governing the behaviour of those syntax new objects. If the intersection of such a minimal extension with the class of formulas in a given similarity type is equal to Λ, we say that the minimal extension is *conservative*. It turns out that conservativity of some natural minimal extensions is equivalent to $\mathcal{X}$-completeness for $\mathcal{X} \in$ PROPERTIES. In this section, we study three such conservativity notions: conservativity of minimal hybrid extensions, minimal conjugate extensions and minimal infinitary extensions.

First, let us consider *hybrid extensions*. Let κ be the maximum of the cardinality of the set of variables and ω. Introduce a new set of variable-like objects NOMINALS $= \{i_n\}_{n\in\kappa}$ called *nominals*. Given a modal similarity type, the set of hybrid formulas is defined by

$$\varphi ::= \bot \mid \top \mid p \mid i \mid \neg\varphi \mid \psi \wedge \varphi \mid \psi \vee \varphi \mid \psi \to \varphi \mid \Diamond_\pi \varphi$$

where $\pi \in$ TYPE, $p \in$ VAR(κ), $i \in$ NOMINALS. The additional axiom scheme is

Nom $\bigwedge \blacksquare^{\leq n}(i \to p) \vee \bigwedge \blacksquare^{\leq n}(i \to \neg p)$

for arbitrary $n \in \omega$, $i \in$ NOMINALS. Inference rules M and N_π remain unchanged. The rule of substitution requires some adjustment; we also need an additional rule to handle nominals.

Shybr $\dfrac{\varphi}{s(\varphi)}$, where s is any substitution which replaces uniformly proposition letters with formulas and nominals with nominals;

COV $\dfrac{L(\neg i), \text{for some } i \notin Sub(L(p))}{L(\bot)}$,

where L is *a necessity form*, introduced by Goldblatt [27]. Necessity form $L(\#)$ is defined inductively:

- $\#$ is a necessity form;
- $L(\#) \wedge \phi$ is a necessity form, for arbitrary necessity form L and φ a formula;
- $\Box_\pi L(\#)$ is a necessity form, for arbitrary $\pi \in$ TYPE and arbitrary necessity form L;
- only an object obtained by means of the above rules in finitely many steps is a necessity form.

The notion of *possibility form* is defined dually, by replacing $\wedge$ with $\vee$ in the second clause and $\Diamond_\pi$ replacing $\Box_\pi$ in the third clause. Define $\mathsf{HybrLogic}(\Lambda)$, the *hybrid logic* axiomatized by Λ, as closure of Λ under under the axioms of classical propositional logic, $\mathbf{K}_\pi$ for all $\pi \in$ TYPE, M, Shybr, $\mathsf{N}\ _\pi$ for all $\pi \in$ TYPE, **Nom** and COV. Similarly, we may define $\mathsf{HybrClosure}(\Lambda, \Gamma)$ as closure of $\mathsf{HybrLogic}(\Lambda) \cup \Gamma$ under M. Again, it is straightforward to oberve that $\mathsf{HybrLogic}(\Lambda) = \mathsf{HybrClosure}(\Lambda, \emptyset)$. Minimal hybrid extension of $\mathsf{Log}(\Lambda)$, where Λ is a set of formulas which do not contain nominals, can be thus defined simply as $\mathsf{HybrLogic}(\Lambda)$; it is straightforward to verify that $\mathsf{Log}(\Lambda) \subseteq \mathsf{HybrLogic}(\Lambda)$.

Remark 3.21 *Let us observe, following Gargov et al. [21] that* COV *can be formulated either as above, i.e., as a finitiary non-structural (or non-orthodox) rule or an infinitary rule with the side condition replaced by phrase* for all $i \in$ NOMINALS. *The main reason why the latter choice seems not satisfying for the present author is not that the resulting rules would be infinitary. Even though there is, in principle, nothing wrong with infinitary rules, the resulting notion of inference becomes dependant on the underlying set of nominals; a valid inference may cease to be proper if the set of nominals is enlarged.*

The intuitive interpretation of the new formalism is that nominals are true in exactly one point in every model, where *model* is understood as a (general) frame with a valuation. Models for the new language are required to satisfy this additional condition. The inductive definition of satisfaction is extended to all formulas in the standard way. An alternative option, more in line with the spirit of the present work, is to define models as BAOs where propositional variables, as before, are allowed to range over arbitrary elements and nominals are mapped to atoms. The justification of this section is that — as the reader learned from the section on duality — singletons correspond to atoms in powerset algebras. Once again, though, let us recall that not every atomic algebra can be depicted as a general frame whose universe is the set of atoms of the algebra; as we learnt, complete additivity is a necessary additional condition here.

Let us then formally define *a hybrid valuation* in an algebra $\mathfrak{A}$ as a mapping which assigns arbitrary elements of $\mathfrak{A}$ to propositional variables and atoms of $\mathfrak{A}$ to nominals. A pair consisting of an algebra and a hybrid valuation is called a *a hybrid model*. Definition of (hybrid) truth of a formula in a model is the same as before: $\langle \mathfrak{A}, \mathfrak{V} \rangle \vDash \varphi$ if $\mathfrak{V}(\varphi) = \top$. (Hybrid) truth of a formula or a set of formulas is defined as truth under all valuations. Let us observe that this definition implies certain peculiarity: if an algebra is atomless, then we can't define any hybrid valuation and all formulas are true in this new sense. It shows clearly that we have to restrict our attention to algebras where atoms play a crucial role.

Theorem 3.22 (with Balder ten Cate) *Every hybrid logic is (strongly) complete with respect to a class of $\mathcal{AV}$-BAOs. In other words, for any set of hybrid formulas Λ and any hybrid formula α, $\alpha \in \mathsf{HybrLogic}(\Lambda)$ iff for all $\mathfrak{A} \in \mathcal{AV}$, if $\mathfrak{A} \vDash \Lambda$ then $\mathfrak{A} \vDash \alpha$. Moreover, for any set of hybrid formulas Γ, $\alpha \in \mathsf{HybrClosure}(\Lambda, \Gamma)$ iff $\alpha \in \mathcal{AV}^l(\Lambda, \Gamma)$.*

Proof: Theorem 1.2 in ten Cate, Litak [VII] imply that every hybrid logic is (strongly) complete with respect to a class of discrete frames. The theorem follows by Corollary 3.18.

Nevertheless, there is a straightforward way of proving the theorem, without using duality results. We only sketch the proof. Soundness is proven by checking that **Nom** is true under all hybrid valuations in all $\mathcal{AV}$-BAOs and that COV is admissible in all $\mathcal{AV}$-BAOs. For completeness, we need to use a technique introduced first by Gargov et al. [21]; cf. also ten Cate [49] and Blackburn et al. [3, Chapter 7.3]. Namely, one may prove that given a hybrid Λ-consistent set of formulas Γ — i.e., Γ s.t. $\bot \notin \mathsf{HybrClosure}(\Lambda, \Gamma)$ — and a supply of new nominals J equinumerous with the set of all possibility forms in the language, Γ may be extended to a *named* Λ-consistent set of formulas Γ'. A named set of formulas Γ' is one such that for every possibility form M, either $\neg M(\top) \in \Gamma'$, or there is a variable $j_M \in J$ s.t. $M(j_M) \in \Gamma'$. In a word, new variables act like Henkin witnesses of a sort. Cf. Goldblatt [28] for a very elegant general framework for this sort of proofs. Indeed, the fact that every Λ-consistent set of formulas can be extended to named Λ-consistent set of formulas is a straightforward consequence of Goldblatt's Abstract Henkin Principle [28, Section 8.1]. Γ', in turn, may be extended to a maximal consistent Γ^*. Let $\Gamma^*_\square \leftrightharpoons \{\varphi | \forall n \in \omega \bigwedge \blacksquare^{\leq n}\varphi \in \Gamma^*\}$. Divide the absolutely free algebra of all formulas in the set of propositional variables extended by NOMINALS and J by the open filter of formulas in $\Gamma^*_\square$. Denote the resulting algebra by $\mathfrak{A}$. It can be proven that

- $\{j_M/\Gamma^*_\square | M \in M_{\Gamma^*}\}$, where M_{Γ^*} is the set of possibility forms s.t. $M(\top) \in \Gamma^*$, are atoms of $\mathfrak{A}$;
- if $\bot \notin \alpha/\Gamma^*_\square$, then for some $M \in M_{\Gamma^*}$, $j_M \to \alpha \in \Gamma^*_\square$. Thus $\mathfrak{A}$ is atomic and $At\mathfrak{A} = \{j_M/\Gamma^*_\square | M \in M_{\Gamma^*}\}$;
- if α, β are formulas s.t. $\bot \notin \alpha/\Gamma^*_\square \cup \beta/\Gamma^*_\square$ and $\alpha \to \Diamond_\pi \beta \in \Gamma^*_\square$, then there is $M \in M_{\Gamma^*}$ s.t. $j_M \to \beta \in \Gamma^*_\square$ and $\bot \notin (\alpha \wedge \Diamond_\pi j_M)/\Gamma^*_\square$. Thus, $\mathfrak{A}$ is not only atomic, but also completely additive;
- there is $M \in M_{\Gamma^*}$ s.t. $j_M \to \gamma \in \Gamma^*_\square$ iff $\gamma \in \Gamma^*$.

The theorem follows. $\dashv$

Corollary 3.23 *The minimal hybrid extension of a logic is conservative iff the logic is $\mathcal{AV}$-complete. In symbols, for any Λ, $\mathsf{HybrLogic}(\Lambda) \cap \text{FORM}_\kappa = \mathsf{Log}\mathcal{AV}(\Lambda)$.*

Proof: Assume $\alpha \notin \mathsf{HybrLogic}(\Lambda)$. By Theorem 3.22, it is equivalent to the existence of $\mathfrak{A} \in \mathcal{AV}$ s.t. $\mathfrak{A} \models \Lambda$ and $\mathfrak{A} \not\models \alpha$. But this is equivalent to $\alpha \notin \mathsf{Log}\mathcal{AV}(\Lambda)$. $\dashv$

Another kind of extensions we are going to consider are *minimal conjugate extensions*, which in the case of unimodal similarity are known as *minimal tense extensions*. Let TYPE be an arbitrary modal similarity type and Λ be a set of formulas in this type. TYPE.$\mathfrak{t}$ consists of TYPE$\cup\{\pi^{>} | \pi \in \text{TYPE}\}$. Minimal conjugate extension $\Lambda.\mathfrak{t}$ is the closure of Λ under the axioms of $\mathbf{K}_{\text{TYPE}.\mathfrak{t}}$, $\mathbf{Conj}_\pi^{\pi^>}$ and $\mathbf{Conj}_{\pi^>}^\pi$ for all $\pi \in$ TYPE. The following is straightforward.

Theorem 3.24 *For every Λ, $\Lambda.\mathfrak{t}$ is (strongly) complete with respect to $\mathcal{T}$-algebras.*

Proof: For arbitrary cardinal κ, arbitrary $\pi \in$ TYPE and arbitrary sets of formulas Λ and Γ in κ variables, we may define conjugates in $\text{FORM}_\kappa(\Lambda)/_\Gamma$ as $\mathbf{p}_\pi x \leftrightharpoons \Diamond_{\pi^>} x$ and $\mathbf{p}_{\pi^>} x \leftrightharpoons \Diamond_\pi x$. $\dashv$

Corollary 3.25 *The minimal conjugate extension of* $\mathsf{Log}(\Lambda)$ *is conservative iff* $\mathsf{Log}(\Lambda)$ *is* $\mathcal{T}$*-complete. In symbols for any* κ, $\Lambda.\mathfrak{t} \cap \text{FORM}_\kappa = \mathsf{Log}\mathcal{T}(\Lambda)$.

Proof: Assume $\alpha \notin \Lambda.\mathfrak{t}$. By Theorem 3.24, it means it is refuted in a $\mathcal{T}$-BAO $\mathfrak{A}$ in the similarity type determined by TYPE.$\mathfrak{t}$. The TYPE-reduct of $\mathfrak{A}$ belongs to $\mathcal{T} \cap \mathbb{V}(\Lambda)$, thus $\alpha \notin \mathsf{Log}\mathcal{T}(\Lambda)$. Conversely, assume $\alpha \notin \mathsf{Log}\mathcal{T}(\Lambda)$. It means there is $\mathfrak{A} \in \mathcal{T} \cap \mathbb{V}(\Lambda)$ s.t. $\mathfrak{A} \not\models \alpha$. But then we may define expansion of $\mathfrak{A}$ to TYPE.$\mathfrak{t}$, setting $\Diamond_{\pi^>} x \leftrightharpoons \mathbf{p}_\pi x$ for arbitrary $\pi \in$ TYPE. It is straghtforward to verify that $\mathfrak{A} \models \Lambda.\mathfrak{t}$. $\dashv$

Lastly, let us focus on *minimal infinitary extensions.* It is the most radical and controvensial way of extending the language: we allow infinitary connectives and thus formulas of infinite size. To keep things under control, we restrict our attention — as it often happens — to countable formulas. Without putting any restriction on cardinality of formulas, we would have to agree that formulas form a proper class.

Thus, given TYPE, the formulas are defined as

$$\varphi ::= \bot \mid \top \mid p \mid \neg\varphi \mid \psi \wedge \varphi \mid \bigwedge_{n\in\omega} \varphi_n \mid \psi \vee \varphi \mid \psi \to \varphi \mid \Diamond_\pi \varphi$$

and $\bigvee_{n\in\omega} \varphi_n \leftrightharpoons \neg \bigwedge_{n\in\omega} \neg\varphi_n$. The additional new axiom scheme is

Inf $\bigwedge_{n\in\omega} \varphi_n \to \varphi_m$, for every $m \in \omega$

and the new inference rule is

$$\mathsf{INF}\quad \frac{\psi \to \varphi_n \text{ for all } n \in \omega}{\psi \to \bigwedge_{n\in\omega} \varphi_n}.$$

Exactly as in the two cases above, define $\mathsf{InfLogic}(\Lambda)$, the *infinitary logic* axiomatized by Λ, as closure of Λ under under the axioms of classical propositional logic, $\mathbf{K}_\pi$ for all $\pi \in$ TYPE, M, S, N_π for all $\pi \in$ TYPE, **Inf** and INF. As before, it is straightforward to oberve that $\mathsf{InfLogic}(\Lambda) = \mathsf{InfClosure}(\Lambda, \emptyset)$.

Minimal infinitary extension of $\mathsf{Log}(\Lambda)$, where Λ is a set of finitary formulas, can be thus defined simply as $\mathsf{InfLogic}(\Lambda)$; it is straightforward to verify that $\mathsf{Log}(\Lambda) \subseteq \mathsf{InfLogic}(\Lambda)$.

The problem we are facing now is that for some valuations in some algebras, suitable values of $\bigwedge\limits_{n\in\omega} \varphi_n$ may fail to exist. Thus, let us say that that a valuation $\mathfrak{V}$ in $\mathfrak{A}$ is *infinitary* if for arbitrary countable family $\{\varphi_i\}_{i\in\omega}$ $\bigwedge\limits_{i\in I} \mathfrak{V}(\varphi_i)$ exists. Infinitary valuations may be readily extended to all infinitary formulas. Thus, truth of a formula in an infinitary model (i.e., algebra with an infinitary valuation) and truth of a set of formulas may be defined in exactly the same way as in the standard case. The (infinitary) truth of a formula in an algebra may be thus defined as truth under all infinitary valuations. Similarly to the hybrid case, it may happen that there are no inifinitary valuations at all in a given algebra and thus all formulas are true in this new sense. Nevertheless, it is immediate that if $\mathfrak{A} \in \omega\mathcal{C}$, then all valuations are infinitary.

Theorem 3.26 *Every infinitary logic is complete with respect to $\omega\mathcal{C}$-BAOs.*

Proof: Soundness follows from the fact that **Inf** is valid under all valuations in ω-complete algebras and INF is admissible.

To prove completeness, let $\Lambda = \mathsf{InfLogic}(\Lambda)$. We just generalize the Lindenbaum algebra construction. The universe of our algebra consists of equivalence classes of formulas; i.e., $[\varphi]_\Lambda = \{\psi | \varphi \leftrightarrow \psi \in \mathsf{InfLogic}(\Lambda)\}$. The operations are defined in the standard way. The only thing we have to prove is that $\bigwedge\limits_{n\in\omega} [\varphi_n]_\Lambda = [\bigwedge\limits_{n\in\omega} \varphi_n]_\Lambda$. This also implies that the Lindenbaum algebra is ω-complete.

First, let us prove that $[\bigwedge\limits_{n\in\omega} \varphi_n]_\Lambda \leq [\varphi_n]_\Lambda$ for any $n \in \omega$, i.e., that $\bigwedge\limits_{n\in\omega} \varphi_n \rightarrow \varphi_n \in \Lambda$. But this follows from **Inf**.

Conversely, assume $[\psi]_\Lambda \leq [\varphi_n]_\Lambda$ for every $n \in \omega$. We want to show that $[\psi]_\Lambda \leq [\bigwedge\limits_{n\in\omega} \varphi_n]_\Lambda$. But Λ is closed under INF. Thus, $\psi \rightarrow \varphi_n \in \Lambda$ for every $n \in \omega$ entails $\psi \rightarrow \bigwedge\limits_{n\in\omega} \varphi_n \in \Lambda$. The theorem is proven. ⊣

Remark 3.27 *The proof above may be strengthened to a proof of strong ωC-completeness. It is enough to observe that for every infinitary logic Λ and every set of formulas Γ which is closed under the axioms of Λ, M, all N_π*

and INF*, the open filter associated with* Γ *in the above Lindenbaum algebra is* ω*-complete.*

Corollary 3.28 *The minimal infinitary extension of* Λ *is conservative iff* Λ *is* ωC*-complete. In symbols, for any* κ *and any set of finitary formuals* Λ*,* $\mathsf{InfLogic}(\Lambda) \cap \text{FORM}_\kappa = \mathsf{Log}\omega\mathcal{C}(\Lambda)$*.*

Proof: Assume $\alpha \notin \mathsf{InfLogic}(\Lambda)$ for a finitary α. Thus, α is refuted in $\mathfrak{A}$ — the infinitary Lindenbaum algebra of $\mathsf{InfLogic}(\Lambda)$ from Theorem 3.26. But $\mathfrak{A} \in \mathbb{V}(\Lambda) \cap \omega\mathcal{C}$.

Conversely, $\mathfrak{A} \in \mathbb{V}(\Lambda) \cap \omega\mathcal{C}$ and $\mathfrak{A} \nvDash \alpha$. But, as $\mathfrak{A}$ is ω-complete, all valuations in $\mathfrak{A}$ are infinitary and $\mathfrak{A} \vDash \mathsf{InfLogic}(\Lambda)$. ⊣

The next chapter shows that there are logics with non-conservative extensions of the above three kinds; moreover, neither of those conservativity notions implies any of the others. As $\mathcal{CAV} \subseteq \mathcal{AV} \cap \mathcal{T} \cap \omega\mathcal{C}$, we have three notions of completeness much weaker than Kripke completeness and yet of some independent motivation and interest.

3.4 Notes

An exhaustive analysis of the history of ideas developed in Sections 3.1 and 3.2 is presented in Goldblatt [26]. We are going to be brief: there is no reason to compete with good existing expositions. We are sorry for any possible inaccuracies.

All results in Section 3.1 have been known by the community for at least 30 years. In fact, most of the ideas can be traced back to Jónsson and Tarski [31] paper. At the time the paper was written, the notion of Kripke frame was not used at all. But Theorems 3.1 and 3.3 are, in fact, proven in that work. Nevertheless, Jónsson and Tarski [31] failed to see the connection with modal logic, for reasons which remain mysterious even today. Relational semantics for modal logic were introduced by Kripke several years later. Hence the name *Kripke frames.* The question whether the name is adequate is totally irrelevant for us here. General frames were introduced by Thomason [50] under the name *first-order semantics.*

In his PhD Thesis (reprinted in Goldblatt [26]), Goldblatt undertook a systematic investigation of the relationship between algebraic and relational semantics for modal logic. The thesis introduced the notions of refined and descriptive frame and proved category-theoretical equivalence between BAOs and descriptive frames.

Section 3.2 takes ideas of Thomason [51] as a starting point. That paper proved that the category of Kripke frames with bounded morphisms is equivalent to the category of $\mathcal{CAV}$-BAOs. He also introduced analogues of di-morphisms for Kripke frames (thus, of course, without the reverse image condition) and proved that the equivalence of category thus obtained with $\mathcal{C}AV$-BAOs with BAO-morphisms.

The generalization of those results to category-theoretical equivalence between $\mathcal{A}V$-BAOs and discrete frames seems new; it has been implicitly used in ten Cate, Litak [VII]. But the observation that $\mathcal{AV}$-BAOs are duals of discrete frames can be derived from Proposition 5.1 in Venema [54]. The author does not know who introduced the name *discrete frames* first, but it is certainly used in papers on modal model theory (cf., e.g., Goranko and Otto [29]).

Hybrid logic has an interesting and long history; cf. Blackburn et al. [3] or ten Cate [49] for some information. The idea of adding a new kind of variables which are true exactly in one point under every valuation seems to have been rediscovered independently several times. Ten Cate [49] proves that every hybrid logic is complete with respect to discrete frames. This gave rise to joint Theorem 3.22 and Corollary 3.23. It has to be observed that Theorem 6.1 of Gargov and Goranko [22] was a mistaken attempt to prove a general hybrid conservativity result. That theorem stated, in effect, that every modal logic has a conservative minimal hybrid extension. It would follow every logic is $\mathcal{AV}$-complete, or, equivalently, complete with respect to discrete frames. We are going to see it is not true.

We are not going to sketch here the whole history of tense logic and conjugates in modal logic. Minimal tense extensions, i.e., minimal conjugate extensions of unimodal logics, have been investigated thoroughly by Wolter; cf. Zakharyaschev et al. [60] for references. As early as in Wolter [57] it was observed that the issue of conservativity of a minimal tense extension is non-trivial, but that minimal tense extensions of $\mathcal{CAV}$-complete logics are conservative. The author was unable to find any result equivalent to Corollary 3.25 explicitly formulated in the existing literature, but it is almost trivial anyhow.

There is something puzzling about minimal infinitary extensions. Infinitary modal logic has been often studied. There are some systematic results concerning infinitary extensions; cf., for example, Tanaka and Ono [48]. And yet, the author has not been able to find any result equivalent to Theorem 3.26 in the literature, despite its simplicity.

Chapter 4

Non-equivalence of completeness notions

The whole chapter is devoted to the proof of the following

Theorem 4.1 *If* $\mathcal{X}, \mathcal{Y} \in$ PROPERTIES *and* $\mathcal{X} \neq \mathcal{Y}$ *then the notion of* $\mathcal{X}$*-completeness is not equivalent to the notion of* $\mathcal{Y}$*-completeness (with a possible exception of* $\{\mathcal{X}, \mathcal{Y}\} = \{\mathcal{ALG}, \mathcal{V}\}$*).*

Observe that to this aim it is enough to show the following

NEq1. $\models_{\mathcal{A}} \;\not\subseteq\; \models_{\mathcal{CT}}$,

NEq2. $\models_{\omega\mathcal{C}} \;\not\subseteq\; \models_{\mathcal{AT}}$,

NEq3. $\models_{\mathcal{C}} \;\not\subseteq\; \models_{\omega\mathcal{C}}$,

NEq4. $\models_{\mathcal{T}} \cap \models_{\mathcal{AV}} \;\not\subseteq\; \models_{\mathcal{CA}}$,

NEq5. $\models_{\mathcal{T}} \;\not\subseteq\; \models_{\mathcal{AV}}$.

The other inequalities follow, as $X' \subseteq X$, $Y \subseteq Y'$ and $X' \not\subseteq Y'$ imply $X \not\subseteq Y$.

4.1 $\mathcal{A}$-inconsistency

This section proves NEq1.

Theorem 4.2 *There exists a nontrivial* $\mathcal{CV}$-BAO $\mathfrak{A}$ *s.t. the variety generated by* $\mathfrak{A}$ *— i.e.,* $HSP(\mathfrak{A})$ *– contains no atomic members. In other words, there exists a logic which is (weakly)* $\mathcal{CV}$*-complete and* $\mathcal{A}$*-inconsistent.*

Proof: Let $\mathbf{I}$ be the unit interval $(0,1)$, $\mathbf{P} = [\mathbf{I} \cup \{0,1\}]^2$, $\mathfrak{P}$ be the boolean algebra whose universe is $2^{\mathbf{P}}$ and the operations coincide with standard set-theoretical ones and let $\mathfrak{O}$ be the boolean algebra of regular open subsets (i.e., interiors of closed sets) of $\mathbf{I}$. Note that $\vee$ and $\neg$ in $\mathfrak{O}$ differ from the set-theoretical operations. It is well-known that both $\mathfrak{P}$ and $\mathfrak{O}$ are complete lattices. For $0 \leq a < b \leq 1$, the open interval determined by a and b is denoted as (a,b), whereas $\langle a,b \rangle$ denotes, as usual, an ordered pair. The proof proceeds via a series of easily verifiable claims.

Claim 1: $B \in \mathfrak{O}$ only if there exists at most countable set J s.t. $B = \bigcup_{j \in J} (a_j, b_j)$, where for each $j \in J$, $0 \leq a_j < b_j \leq 1$ and for each $j \neq k$, either $b_j < a_k$ or $b_k < a_j$. The set $\{\{a_j, b_j\} \mid j \in J\}$ is called *the canonical representation of* B and denoted as Z_B.

For any $z \in \mathbf{P}$, $z_< \leftrightharpoons \min z$, $z_> \leftrightharpoons \max z$ and $\llcorner z \lrcorner \leftrightharpoons (z_<, z_>)$. Our main object of interest is going to be a BAO whose underlying boolean reduct is $\mathfrak{A}_- \leftrightharpoons \mathfrak{P} \times \mathfrak{O}$.

Claim 2: $\mathfrak{A}_-$ is a complete boolean algebra.

Proof of claim: $\mathfrak{A}_-$ is a product of two complete boolean algebras. $\dashv$

Now we are going to define five unary operations on the universe of our algebra. Subscripts are chosen so as to emphasize some analogies with the construction in Venema [55].

First, we are going to define four auxiliary mappings:

$$
\begin{aligned}
\eta_{\lhd} : \mathfrak{O} \ni B &\longrightarrow \{z \in \mathbf{P} \mid \exists y \in B \; y \in \llcorner z \lrcorner\} \in \mathfrak{P},\\
\eta_{\rhd} : \mathfrak{P} \ni A &\longrightarrow \bigvee_{a \in A}{}^{\mathfrak{O}} \llcorner a \lrcorner \in \mathfrak{O},\\
\eta_{>} : \mathfrak{P} \ni A &\longrightarrow \{z \in \mathbf{P} \mid \exists z^* \in A \; \llcorner z^* \lrcorner \supsetneq \llcorner z \lrcorner\} \in \mathfrak{P},\\
\eta_{L} : \mathfrak{P} \ni A &\longrightarrow \{z \in \mathbf{P} \mid \exists z^* \in A \; z_< = z^*_<, z_> = z^*_< + (z^*_> - z^*_<)/2\} \in \mathfrak{P}.
\end{aligned}
$$

Through this section, we set TYPE $\leftrightharpoons \{\lhd, \rhd, >, L, \mathbf{E}\}$.

Definition 4.3 *Define* $\mathfrak{A} \leftrightharpoons \langle \mathfrak{A}_-, \{\Diamond_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$, *where* $\Diamond_\lhd \langle A, B \rangle \leftrightharpoons \langle \eta_\lhd B, \emptyset \rangle$, $\Diamond_\rhd \langle A, B \rangle \leftrightharpoons \langle \emptyset, \eta_\rhd A \rangle$, $\Diamond_> \langle A, B \rangle \leftrightharpoons \langle \eta_> A, \emptyset \rangle$, $\Diamond_L \langle A, B \rangle \leftrightharpoons \langle \eta_L A, \emptyset \rangle$, *and* $\Diamond_{\mathbf{E}}$ *is the unary boolean discriminator function on* $\mathfrak{A}_-$ *(cf., e.g., [30] for a definition).*

Claim 3: $\Diamond_\lhd$, $\Diamond_\rhd$, $\Diamond_>$, $\Diamond_L$ and $\Diamond_{\mathbf{E}}$ are completely additive operators.

Proof of claim: It is obvious for the unary discriminator, $\Diamond_>$ and $\Diamond_L$ as both $\Diamond_>$ and $\Diamond_L$ are easily seen to be operators given by certain relations on a relational frame whose universe is $\mathbf{P}$. Assume now we have a family of pairs $\{\langle A_j, B_j \rangle\}_{j \in J} \subseteq \mathfrak{A}$. Then

$$\bigvee_{j \in J} \Diamond_\lhd \langle A_j, B_j \rangle = \bigvee_{j \in J} \langle \eta_\lhd B_j, \emptyset \rangle = \langle \bigcup_{j \in J} \eta_\lhd B_j, \emptyset \rangle$$

and, similarly,

$$\bigvee_{j \in J} \Diamond_\rhd \langle A_j, B_j \rangle = \bigvee_{j \in J} \langle \emptyset, \eta_\rhd A_j \rangle = \langle \emptyset, \bigvee_{j \in J} \eta_\rhd A_j \rangle.$$

Hence, it is enough to establish

$$\bigcup_{j \in J} \eta_\lhd B_j = \eta_\lhd \bigvee_{j \in J} B_j \tag{4.1}$$

and

$$\bigvee_{j \in J} \eta_\rhd A_j = \eta_\rhd \bigcup_{j \in J} A_j. \tag{4.2}$$

Observe that in both cases, the $\subseteq$-direction is trivial. To establish the converse for 4.1, assume there exists $i \in \mathbf{P}$ s.t.

$$(\exists y \in \bigvee_{j \in J} B_j \ y \in \llcorner i \lrcorner) \ \& \ (\forall j \in J \ \forall y \in B_j \ y \notin \llcorner i \lrcorner).$$

But then for each $j \in J$, $B_j \subseteq \neg \llcorner i \lrcorner$, hence $B_j = Int(B_j) \subseteq Int(\neg \llcorner i \lrcorner) \in \mathfrak{O}$. Thus, we get $\bigvee_{j \in J} B_j \subseteq Int(\neg \llcorner i \lrcorner) \subseteq \neg \llcorner i \lrcorner$, a contradiction.

To establish

$$\bigvee_{a\in \bigcup\limits_{j\in J} A_j} \llcorner a \lrcorner \subseteq \bigvee_{j\in J} \bigvee_{a\in A_j} \llcorner a \lrcorner,$$

take any X s.t. for each $j \in J$, $\bigvee\limits_{a\in A_j} \llcorner a \lrcorner \subseteq X$. Obviously, for each $a \in \bigcup\limits_{j\in J} A_j$ there exists $j_a \in J$ s.t. $a \in A_{j_a}$. Hence for each $a \in \bigcup\limits_{j\in J} A_j$, $\llcorner a \lrcorner \subseteq \bigvee\limits_{a\in A_{j_a}} \llcorner a \lrcorner \subseteq X$ and thus we get $\bigvee\limits_{a\in \bigcup\limits_{j\in J} A_j} \llcorner a \lrcorner \subseteq X$, which gives us 4.2. In fact, this is a sort of law of infinite associativity which holds in every complete lattice; cf., e.g., Kuratowski and Mostowski [41]. $\dashv$

Define $c \leftrightharpoons \Diamond_{\triangleright}\top$ and $Fx \leftrightharpoons \Diamond_{\triangleright}\Box_L(\Box_{\triangleleft}x \wedge \neg\Diamond_{>}\Box_{\triangleleft}x)$. The following claims are easily verifiable.

Claim 4: In $\mathfrak{A}$, c is equal to $\langle \emptyset, \mathbf{I}\rangle$. Hence, an element of $\mathfrak{A}$ is below c iff it is of the form $\langle \emptyset, B\rangle$ for some $B \in \mathfrak{O}$.

Claim 5: For any $B \in \mathfrak{O}$,

$$\Box_{\triangleleft}\langle \emptyset, B\rangle = \langle \{z \in \mathbf{P} \mid \llcorner z \lrcorner \subseteq B\}, \mathbf{I}\rangle,$$

$$\Box_{\triangleleft}\langle \emptyset, B\rangle \wedge \neg\Diamond_{>}\Box_{\triangleleft}\langle \emptyset, B\rangle = \langle Z_B, \mathbf{I}\rangle,$$

$$F\langle \emptyset, B\rangle = \langle \emptyset, \bigcup_{z\in Z_B} (z_<, z_< + (z_> - z_<)/2)\rangle.$$

This gives us the following

Claim 6: There exists a constant term c and an unary term Fx in language determined by TYPE s.t. $\mathfrak{A} \models c > \bot$ and $\mathfrak{A} \models \forall x\ (\bot < x \leq c \rightarrow \bot < Fx < x)$. As $\mathfrak{A}$ is a discriminator algebra, i.e., $\Diamond_{\mathbf{E}}$ behaves like universal modality, those two facts may be reformulated as follows: $\mathfrak{A} \models \Diamond_{\mathbf{E}}c = \top$ and $\mathfrak{A} \models \Diamond_{\mathbf{E}}x \wedge \Box_{\mathbf{E}}(x \rightarrow c) \ \leq\ \Diamond_{\mathbf{E}}Fx \wedge \Diamond_{\mathbf{E}}(x \wedge \neg Fx) \wedge \Box_{\mathbf{E}}(Fx \rightarrow x)$.

The above claim implies Theorem 4.2. For similar arguments cf. Venema [55] or an earlier, undebugged attempt of such a construction by Kracht and Kowalski [38].

$\dashv$

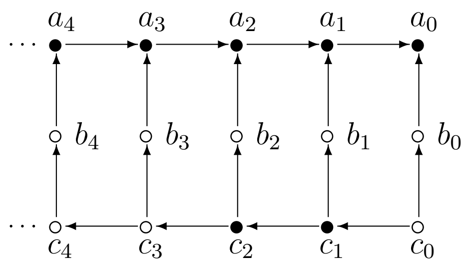

Figure 4.1: Frame $\mathfrak{F}ine_I$ for $I = \{0, 3, 4, \ldots\}$

4.2 $\omega\mathcal{C}$-inconsistency

This section proves NEq2. Through the section, set TYPE $\leftrightharpoons \{>\}$, i.e., we work in the basic modal similarity type.

Theorem 4.4 *There exists a continuum of* $\mathbf{4}_>$*-logics sharing the same class of* $\omega\mathcal{C}$*-BAOs. Their extension with universal modality are* $\omega\mathcal{C}$*-inconsistent.*

We use this example here because it is going to be of importance in the chapters to follow. It is also of some independent interest: as far as the author is aware, it is the first example of a continuum of $\mathbf{4}$-logics with the same class of Kripke frames. We are going to encounter more examples of ωC-incomplete, or even $\omega\mathcal{C}$-inconsistent logics in Section 8.3.

Proof: For arbitrary $I \subseteq \omega$, let $\mathfrak{F}ine_I \leftrightharpoons \langle W, \{R^I\}_{\pi \in \text{TYPE}}, F\rangle$ be the general frame defined as

- $W \leftrightharpoons A \cup B \cup C$, where $A \leftrightharpoons \{a_n\}_{n\in\omega}$, $B \leftrightharpoons \{b_n\}_{n\in\omega}$, $C \leftrightharpoons \{c_n\}_{n\in\omega}$,
- $R^I_> \leftrightharpoons R_> \cup \Delta_{C^I}$, where $R_> \leftrightharpoons \{\langle a_i, a_j\rangle | i > j\} \cup \{\langle b_i, a_j\rangle | i \geq j\} \cup \Delta_B \cup \{\langle c_i, b_j\rangle | i \leq j\} \cup C \times A$, $C^I \leftrightharpoons \{\langle c_i, c_i\rangle | i \in I\}$ and $\Delta_X \leftrightharpoons \{\langle x, x\rangle | x \in X\}$,
- F is the set of finite and cofinite subsets of W.

Claim 1: $\mathfrak{F}ine_I$ is a general frame for any I. In addition, $\mathfrak{F}ine_I^+ \vDash \mathbf{4}_>$.

Define the following three sequences of polynomials in one variable:

$$\begin{array}{rcl} \alpha_n^\pi(x) & \leftrightharpoons & \Diamond_\pi^n x \wedge \Box_\pi^{n+1}\neg x, \\ \beta_n^\pi(x) & \leftrightharpoons & \Diamond_\pi^2 \alpha_n^\pi(x) \wedge \Box_\pi \neg \alpha_{n+1}^\pi(x), \\ \gamma_0^\pi(x) & \leftrightharpoons & \Diamond_\pi \beta_0^\pi(x) \wedge \Diamond_\pi \beta_1^\pi(x), \\ \gamma_i^\pi(x) & \leftrightharpoons & \neg\Diamond_\pi^+ \gamma_{i-1}^\pi(x) \wedge \Diamond_> \beta_n^\pi(x) \wedge \Diamond_\pi \beta_{i+1}^\pi(x) \quad (n \in \omega, i > 0). \end{array}$$

Claim 2: For arbitrary $I \subseteq \omega$, $n \in \omega$ and arbitrary valuation $\mathfrak{V}$ in $\mathfrak{Fine}_I^+$, $\mathfrak{V}(\alpha_n^>(\Box_>\top)) = \{a_n\}$, $\mathfrak{V}(\beta_n^>(\Box_>\top)) = \{b_n\}$, $\mathfrak{V}(\gamma_n^>(\Box_>\top)) = \{c_n\}$.

Proof of claim: A straightforward verification, analogous to the one found in Chagrov and Zakharyaschev [11], section 6.3. $\dashv$

This justifies the following convention: $\underline{a_n} \leftrightharpoons \alpha_n^>(\top)$, $\underline{b_n} \leftrightharpoons \beta_n^>(\top)$, $\underline{c_n} \leftrightharpoons \gamma_n^>(\top)$.

Claim 3: For arbitrary $I \subseteq \omega$ and arbitrary $n \in \omega$, $\mathfrak{Fine}_I^+ \vDash \underline{c_n} \to \Diamond_> \underline{c_{n+1}}$. $\mathfrak{Fine}_I^+ \vDash \underline{c_n} \to \Diamond_> \underline{c_n}$ iff $n \in I$; otherwise, $\mathfrak{Fine}_I^+ \vDash \underline{c_n} \to \Box_> \neg \underline{c_n}$.

Proof of claim: A consequence of Claim 2. $\dashv$

Claim 4: $\mathfrak{Fine}_I^+ \vDash \mathbf{wgrz}_>$.

Proof of claim: Assume $\mathfrak{V}(p) \neq \emptyset$. Then for every $x \in \mathfrak{V}(p)$

- either $x \in A \cup B$, thus $x \in \mathfrak{V}(\Diamond_>^+(p \wedge \Box_>(p \vee \Box_> \neg p)))$;
- or $x \in C$ and then either there is some $y \in A \cap \mathfrak{V}(p)$ s.t. $xR_>y$ or else there exists $c \in \mathfrak{V}(p \wedge \Box_> \neg p)$ (otherwise $\mathfrak{V}(p)$ would be neither finite nor cofinite) and we are done.

$\dashv$

Set

$$\mathbf{LogFine}_> \leftrightharpoons \mathsf{Log}(\{\mathbf{4}_>, \underline{c_i} \to \Diamond_> \underline{c_{i+1}} | i \in \omega\}).$$

Claim 5: For any $\mathfrak{A} \in \mathbb{V}(\mathbf{LogFine}_>) \cap \omega\mathcal{C}$, $\mathfrak{A} \nvDash \neg \underline{c_n}$ implies $\mathfrak{A} \nvDash \mathbf{wgrz}_>$ for arbitrary $n \in \omega$.

Proof of claim: Define $\mathfrak{V}(p) \leftrightharpoons \bigvee_{i \geq n} \underline{c_{2i}}$. It can be easily proven now $\mathfrak{V}(p) \leq \Diamond_{>}\mathfrak{V}(\neg p \wedge \Diamond_{>} p)$; analogous arguments are provided in proofs of Theorems 5.10, 6.1, 8.9 and 8.10. See also Litak [V]. $\dashv$

Now let $\Lambda_I \leftrightharpoons \mathsf{Log}(\mathbf{LogFine}_{>} \cup \{\mathbf{wgrz}_{>}\} \cup \{\underline{c_n} \to \Diamond_{>}\underline{c_n} | n \in I\} \cup \{\underline{c_n} \to \Box_{>}\neg\underline{c_n} | n \notin I\})$.

Claim 6: $\mathfrak{Fine}_I^+ \models \Lambda_J$ iff $I = J$.

Proof of claim: A consequence of Claims 3 and 4. $\dashv$

Claim 7: For arbitrary $\mathfrak{A} \in \omega\mathcal{C}$, and for arbitrary $I, J \subseteq \omega$, $\mathfrak{A} \models \Lambda_I$ iff $\mathfrak{A} \models \Lambda_J$.

Proof of claim: As follows from Claim 5, for every $\mathfrak{A} \in \omega\mathcal{C} \cap \mathbb{V}(\Lambda_I)$ and every $n \in \omega$, $\mathfrak{A} \models \neg\underline{c_n}$. $\dashv$

Now let us add universal modality $\Diamond_{\mathbf{E}}$ to the language. It is clear from Claims 5 and 2 that $\Lambda_I \cup \{\Diamond_{\mathbf{E}}\underline{c_{minI}}\}$ is a consistent logic with no ω-complete BAOs.

$\dashv$

4.3 $\mathcal{C}$-inconsistency

This section proves NEq3. In this section, set TYPE $\leftrightharpoons \{<, >\}$. For simplicity, we write **Lin** instead of $\mathbf{Lin}^{<}_{>}$. The proof is based on an important technique, which reappears in forthcoming chapters. Thus, we single the trick out as a separate

Lemma 4.5 (Ordinal Slicing) *Take any nontrivial $\mathfrak{A} \in \mathbb{V}(\{\mathbf{Lin}, \mathbf{GL}_{>}, \mathbf{D}_{<}\}) \cap \mathcal{C}$. There exist an infinite limit ordinal κ no greater than the cardinality of $\mathfrak{A}$ s.t. we may define a sequence $\{a_\lambda\}_{\lambda \in \kappa}$ of elements of $\mathfrak{A}$ satisfying the following conditions:*

1. *$\forall \lambda \in \kappa$, $a_\lambda \neq \bot$ and $a_\lambda \wedge \Diamond_{>} a_\lambda = \bot$;*
2. *$\forall \lambda \in \kappa$, $n \geq 1$, $a_\lambda \leq \Diamond_{<} a_{\lambda+n}$;*
3. *$\forall \mu < \lambda \in \kappa$, $a_\mu \wedge (a_\lambda \vee \Diamond_{>} a_\lambda) = \bot$;*

4. $\forall \lambda \in \kappa$, $a_\lambda \leq \Box_> \bigvee_{\mu \in \lambda} a_\mu$

5. $\bigvee_{\lambda \in \kappa} a_\lambda = \top$.

Proof: **Claim 1:** Take any $\mathfrak{A} \in \mathbb{V}(\mathbf{GL}_>)$ and arbitrary $x \in \mathfrak{A}$.

$$\Diamond_> x \leq (\Diamond_> x \wedge \neg \Diamond^2_> x) \vee \Diamond_> (\Diamond_> x \wedge \neg \Diamond^2_> x).$$

In addition, $\mathfrak{A} \in \mathcal{I}_>$, i.e., $\mathfrak{A}$ is $>$-cycle free.

Proof of claim: The second statement was already proved and used in the proof of Theorem 2.15. For the first one, observe that by $\mathbf{GL}_>$, we have $\Diamond^2_> x \leq \Diamond_> (\Diamond_> x \wedge \neg \Diamond^2_> x)$ and thus $\Diamond_> x \wedge \Diamond^2_> x \leq \Diamond_> (\Diamond_> x \wedge \neg \Diamond^2_> x)$. Hence

$$\Diamond_> x \wedge (\Diamond^2_> x \vee \neg \Diamond_> x) \wedge \Box_> (\Diamond^2_> x \vee \neg \Diamond_> x) = \bot.$$

$\dashv$

To simplify the notation, define $\Diamond^!_> x \leftrightharpoons \Diamond_> x \wedge \neg \Diamond^2_> x$.

Claim 2: Take any $\mathfrak{A} \in \mathbb{V}(\{\mathbf{Lin}, \mathbf{GL}_>, \mathbf{D}_<\})$ and arbitrary $x \in \mathfrak{A}$ s.t. $x \wedge \Diamond_> x = \bot$. Then $x \leq \Diamond_< \Diamond^!_> x$ and $\Diamond^!_> x \wedge \Diamond_> \Diamond^!_> x = \bot$.

Proof of claim: Observe first that $x \leq \Box_< \Diamond_> x$ by **Lin**, this by $\mathbf{D}_<$ implies $x \leq \Diamond_< \Diamond_> x$, and this in turn implies $x \leq \Diamond_< (\Diamond^!_> x \vee \Diamond_> \Diamond^!_> x)$ by Claim 1. This gives us $x \leq \Diamond_< \Diamond^!_> x \vee \Diamond_< \Diamond^!_> x$ and, by using **Lin**, we get that $x \leq \Diamond_< \Diamond^!_> x \vee \Diamond^!_> x \vee \Diamond_> \Diamond^!_> x$. Note that until now, we have not used the additional assumption about x. The assumption, however, implies that $x \wedge (\Diamond_> \Diamond^!_> x \vee \Diamond^!_> x) = \bot$; it is enough to observe that

$$x \wedge \Diamond_> \Diamond^!_> x \leq x \wedge \Diamond^2_> x \leq x \wedge \Diamond_> x = \bot$$

(we used here the fact $\mathbf{GL}_>$ implies **4**). The first inequality of the claim follows; the second is immediate from the definition. $\dashv$

We may now construct the desired sequence by transfinite induction.

0°. Let $\lambda = 0$. Define $a_0 \leftrightharpoons \Box_> \bot$. Claim 1 ensures the first statement of 1; the second follows from the definition of a_0. 4 follows from the definition. $a_0 \neq \top$; otherwise, $\Box_< \bot = \top$ by **Lin**, but that would contradict $\mathbf{D}_<$. Thus, 5 cannot be satisfied and construction cannot be completed at this stage.

1°. Let $\lambda = \iota'$. $a_\lambda \leftrightharpoons \Diamond^!_> a_\iota$. Claim 2, the induction hypothesis and **4** imply 1 and 2. For any $\mu \leq \iota$,

$$a_\mu \wedge (a_\lambda \vee \Diamond_> a_\lambda) \leq a_\mu \wedge (\Diamond_> a_\iota \vee \Diamond^2_> a_\iota) \leq_{(\mathbf{4})} a_\mu \wedge \Diamond_> a_\iota =_{(IH)} \bot;$$

this is how we get 3. To establish 4, observe that by **Lin**, $\Diamond_> a_\iota \wedge \Diamond_< \neg \Diamond^2_> a_\iota \leq \Box_> \Diamond^2_> a_\iota \wedge \Diamond_< \neg \Diamond^2_> a_\iota = \bot$. At the same time,

$$\Diamond_< \Diamond^!_> a_\iota \wedge \neg a_\iota \wedge \neg \Diamond_< a_\iota \leq \Diamond_< \Diamond_> a_\iota \wedge \neg a_\iota \wedge \neg \Diamond_< a_\iota \leq_{(\mathbf{Lin})} \Diamond_> a_\iota.$$

These two facts together entail that $\Diamond_< \Diamond^!_> a_\iota \wedge \neg a_\iota \wedge \neg \Diamond_< a_\iota = \top$, thus by **Lin**, $\Diamond^!_> a_\iota \leq \Box_> (a_\iota \vee \Diamond_< a_\iota)$. By the induction hypothesis $\Diamond_< a_\iota \leq \bigvee_{\mu \in \iota} a_\mu$ and hence finally we got 4. Lastly, as for any $\mu \leq \lambda$, $a_\mu \wedge \Diamond_> a_\lambda = \bot$, $\bigvee_{\mu \leq \lambda} a_\mu \leq \neg \Diamond_> a_\lambda$ and yet $\neg \Diamond_> a_\lambda \neq \top$ by Claim 2. Thus 5 cannot be satisfied and construction cannot be completed at this stage.

2°. Let λ be a limit ordinal. If $\bigvee_{\mu \in \lambda} a_\mu = \top$, then set $\kappa = \lambda$ and we are done. Otherwise, let $z \leftrightharpoons \bigwedge_{\mu \in \lambda} \neg a_\mu$; by assumption $z \neq \bot$. Thus by Claim 1, $z \wedge \neg \Diamond_> z \neq \bot$ and $a_\lambda \leftrightharpoons z \wedge \neg \Diamond_> z$ must satisfy 1. For arbitrary $\mu < \lambda$, $a_\mu \leq \neg z$ and $a_\mu \leq \Box_> \neg z$ by IH 4. In this way, we get 3. 4 we have by definition. Again, $\bigvee_{\mu \leq \lambda} a_\mu \leq \neg \Diamond_> a_\lambda \neq \top$ and 5 cannot be satisfied.

It is clear that the construction has to stop at some stage, otherwise one would be able to find more pairwise multiplicatively disjoint elements distinct from the bottom than there are elements in $\mathfrak{A}$. And, as follows from the above, the only steps when the construction can possibly stop are those when λ is a limit ordinal. $\dashv$

And here is the main result of the section.

Theorem 4.6 *The logic* **Succ** $\leftrightharpoons \mathsf{Log}(\{\mathbf{Lin}, \mathbf{D}_<, \mathbf{McK}_<, \mathbf{GL}_>, \})$ *is* $\kappa\mathcal{C}$*-complete for every infinite cardinal* κ*, but* $\mathcal{C}$*-inconsistent.*

Proof: We prove only $\mathcal{C}$-inconsistency and $\kappa\mathcal{C}$-soundness of **Succ**, postponing the proof of $\kappa\mathcal{C}$-completeness till Section 8.5.

Claim 1: For any nontrivial $\mathfrak{A} \in \mathbb{V}(\mathbf{Succ}) \cap \mathcal{C}$, $\mathfrak{A} \not\models \mathbf{McK}_<$. Hence, **Succ** is $\mathcal{C}$-inconsistent.

Proof of claim: By Lemma 4.5, we may find a suitable sequence $\{a_\lambda\}_{\lambda\in\kappa}$. As κ is a limit ordinal, $\kappa = \omega*\lambda$ for some $\lambda \leq \kappa$. Define $b \leftrightharpoons \bigvee_{\mu\in\lambda}\bigvee_{n\in\omega} a_{\omega*\mu+2*n}$ and $c \leftrightharpoons \bigvee_{\mu\in\lambda}\bigvee_{n\in\omega} a_{\omega*\mu+2*n+1}$. Lemma 4.5 guarantees that $b\wedge c = \bot$, $b\vee c = \top$ (and thus $c = \neg b$), $\Diamond_< b = \Diamond_< c = \top$ and hence $\Box_<\Diamond_<\top = \Box_<\Diamond_<\top = \top$, $\Box_< b = \Box_< c = \bot$ and hence $\Diamond_<\Box_< b = \Diamond_<\Box_< c = \bot$. Thus, any valuation $\mathfrak{V}$ in $\mathfrak{A}$ satisfying $\mathfrak{V}(p) = b$ refutes $\mathbf{McK}_<$. ⊣

Definition 4.7 *Let κ be any ordinal. Set $\mathfrak{S}ucc_\kappa \leftrightharpoons \langle \aleph_{\kappa+1}, \{R_\pi\}_{\pi\in\mathrm{TYPE}}, A_\kappa\rangle$, where $R_<$ is the standard strict order on $\aleph_{\kappa+1}$ and $R_>$ is its converse, A_κ is the algebra of all sets whose cardinality is no greater than κ and their complements.*

Claim 2: A_κ is closed under all boolean operations, $\Box_<$ and $\Box_>$. In other words, $\mathfrak{F}_\kappa$ is a general frame. Moreover, $\mathfrak{S}ucc^+_\kappa$ is κ-complete.

Proof of claim: For the first statement, the only important thing is to prove closure under modal operators. The fact that every successor cardinal is regular (cf. Appendix) implies that $\Box_< X = \bot$ for every X s.t. $|X| \leq \kappa$, $|\Box_< X| = \kappa + 1$ for every X whose complement is of cardinality no greater than κ and $|\Box_> X| \leq \kappa$ for every $X \neq \top$. $\mathfrak{S}ucc^+_\kappa \in \kappa\mathcal{C}$, because a sum of κ-many sets of cardinality no greater than κ is of cardinality no greater than κ. ⊣

Claim 3: $\mathfrak{S}ucc_\kappa \vDash \mathbf{Succ}$, for every κ.

Proof of claim: The only statement that requires proof is $\mathfrak{S}ucc_\kappa \vDash \mathbf{McK}_<$. Assume then $\mathfrak{V}(\Box_<\Diamond_< p) \neq \bot$ for some $\mathfrak{V}$ in $\mathfrak{S}ucc_\kappa$. It means that for every $\lambda \in \kappa$, there exists $\gamma > \lambda$ s.t. $\gamma \in \mathfrak{V}(p)$. It means that $\mathfrak{V}(p)$ is cofinal in $\aleph_{\kappa+1}$, but this implies that the complement of $\mathfrak{V}(p)$ is of cardinality no greater than κ, as $\mathfrak{V}(p)$ is an admissible set. Thus, $\mathfrak{V}(\Diamond_<\Box_< p) = \top$. ⊣

Thus, we prove $\kappa\mathcal{C}$-soundness and $\mathcal{C}$-inconsistency of the logic in question. The only thing left to show is that **Succ** is actually complete with respect to $\mathfrak{S}ucc_\kappa$, for every infinite cardinal κ. This proof is provided in Section 8.5. Observe, however, that even on the basis of what was proven until now, one may show that for every κ, there is a logic which $\kappa\mathcal{C}$-complete and $\mathcal{C}$-inconsistent; namely, the logic of $\mathfrak{S}ucc_\kappa$. ⊣

4.4 $\mathcal{AV} \cup \mathcal{T}$-inconsistency

This section proves NEq4. In this section TYPE $\leftrightharpoons \{\mathbf{E}, >\}$. Again, as in Section 4.2, we need the universal modality only in order to prove the inconsistency result. The incompleteness result is provable in the basic modal similarity type.

Theorem 4.8 *There exists a $\mathcal{CA}$-BAO $\mathfrak{v}B$ s.t. $HSP(\mathfrak{v}B) \cap (\mathcal{AV} \cup \mathcal{T}) = \emptyset$, i.e., the associated logic is both $\mathcal{AV}$- and $\mathcal{T}$-inconsistent.*

Proof: The algebra we are going to use is almost the same as the one defined in Wolter [57], section 4.6; the only difference is that we add the universal modality. The boolean reduct of $\mathfrak{v}B$ is the powerset algebra of $\omega+1$. $\Diamond_{\mathbf{E}}$ is the unary boolean discriminator. To define $\Diamond_{>}$, fix any non-principal ultrafilter $\nabla \in Uf(\mathfrak{v}B)$.

$$\Diamond_{>} x \leftrightharpoons \begin{cases} \{\omega\} \cup \{m | \exists n < m \, \{n\} \subseteq x\} & : \quad x \in \nabla, \\ \{m | \exists n < m \, \{n\} \subseteq x\} & : \quad x \notin \nabla. \end{cases}$$

Claim 1: $\mathfrak{v}B \models \Diamond_{\mathbf{E}}(\Box^{+}_{>} \Diamond_{>} \top)$.

Proof of claim: It is enough to observe that $\Box^{+}_{>} \Diamond_{>} \top = \{\omega\} \neq \bot$. ⊣

Claim 2: $\mathfrak{v}B \models \Box_{>} \Diamond_{>} \top \to \Box_{>}(\Box_{>}(\Box_{>} p \to p) \to p)$.

Proof of claim: It can be proven by means of Wolter [57] and van Benthem [53]. We sketch the argument here to make our paper more self-contained. $\Box_{>} \Diamond_{>} \top = \{0, \omega\}$. As for any valuation, $\{0\}$ is trivially below the value of any formula starting with $\Box_{>}$, it is enough to prove that $\{\omega\} \leq \mathfrak{V}(\Box_{>}(p \to \Diamond_{>}(p \wedge \Box_{>} \neg p)))$ for arbitrary valuation $\mathfrak{V}$; in other words, that $\mathfrak{V}(\neg p \vee \Diamond_{>}(p \wedge \Box_{>} \neg p)) \in \nabla$. If $\mathfrak{V}(p) \notin \nabla$, then $\mathfrak{V}(\neg p) \in \nabla$ and we are done. If $\mathfrak{V}(p) \in \nabla$, then let $n = \min \mathfrak{V}(p)$. It is clear that $\mathfrak{V}(\Diamond_{>}(p \wedge \Box_{>} \neg p)) = \{m \in \omega | m > n\} \in \nabla$ and we are done. ⊣

Claim 3: For every $\mathfrak{A} \in \mathcal{AV} \cup \mathcal{T}$,

$$\mathfrak{A} \models \Box_{>} \Diamond_{>} \top \to \Box_{>}(\Box_{>}(\Box_{>} p \to p) \to p)$$

only if $\mathfrak{A} \models \Diamond^{+}_{>} \Box_{p} \bot$.

Proof of claim: Assume $\mathfrak{A} \not\models \Diamond^{+}_{>}\Box_{>}\bot$. We show that if it is either $\mathcal{AV}$-BAO or $\mathcal{T}$-BAO, there exist b s.t.

$$\bot \neq \Box_{>}\Diamond_{>}\top \wedge \Diamond_{>}(\Box_{>}(\Diamond_{>}b \vee \neg b) \wedge b). \tag{4.3}$$

Assume first $\mathfrak{A} \in \mathcal{AV}$ and let a be an arbitrary atom below $\Box^{+}_{>}\Diamond_{>}\top$. By complete additivity, there must exist an atom b s.t. $a \leq \Diamond_{>}b$. As $x \leq \Box_{>}(\Diamond_{>}x \vee \neg x)$ holds for any atom x in any BAO, we get the theorem.

If $\mathfrak{A} \in \mathcal{T}$, then there exists $\mathbf{p}_{>} : \mathfrak{A} \mapsto \mathfrak{A}$ — the conjugate of $\Diamond_{>}$. Let $a \leftrightharpoons \Box^{+}_{>}\Diamond_{>}\top$. It is either the case that $\mathbf{p}_{>}a \leq \Box(\Diamond_{>}\mathbf{p}_{>}a \vee \neg\mathbf{p}_{>}a)$ or $\mathbf{p}_{>}a \wedge \Diamond(\Box_{>}\neg\mathbf{p}_{>}a \wedge \mathbf{p}_{>}a) \neq \bot$. In the first case, let $b \leftrightharpoons \mathbf{p}_{>}a$. In the second case, $b \leftrightharpoons \mathbf{p}_{>}a \wedge \mathbf{p}^{2}_{>}a$. ⊣

⊣

4.5 $\mathcal{T}$-inconsistency

This section proves NEq5. Set TYPE $\leftrightharpoons \{<, >\}$.

Theorem 4.9 *There exists a $\mathcal{AV}$-algebra which generates a $\mathcal{T}$-incomplete logic. Its expansion with unary discriminator (univeral modality) has $\mathcal{T}$-inconsistent theory.*

Proof: Consider a frame $\mathfrak{G} \leftrightharpoons \langle W, \{R_{\pi}\}_{\pi\in\mathrm{TYPE}}, A\rangle$. $W \leftrightharpoons \{\infty\} \cup \{a_n\}_{n\in\omega}$, $R_{<} \leftrightharpoons \{\langle a_i, a_j\rangle | i < j\} \cup \{\langle\infty, a_{2i}\rangle | i \in \omega\}$, $R_{>} \leftrightharpoons \{\langle a_i, a_j\rangle | i > j\}$ (i.e., $R_{>} = R^{-1}_{<} \cap (W - \{\infty\})^2$), A is the family of finite and cofinite subsets of W. Observe that — just like in case of frames from Section 4.2 — all points are definable by variable-free formulas. For our purposes, we need only the following

Claim 1: Define $\underline{1} \leftrightharpoons \Diamond_{>}\top \wedge \Box^{2}_{>}\bot$, $\underline{\infty} \leftrightharpoons \Diamond_{<}\Box_{>}\bot$. Then $\underline{\infty} = \{\infty\}$ and $\underline{1} = \{a_1\}$ in $\mathfrak{G}^{+}$.

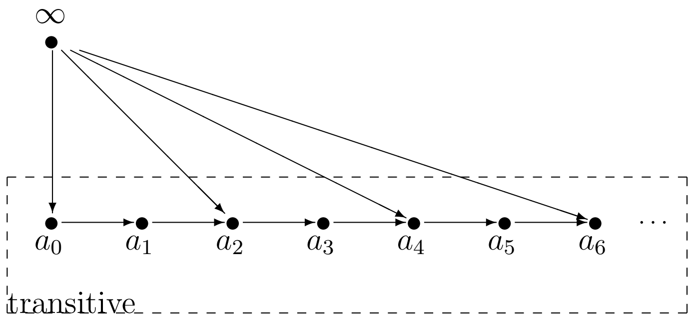

Figure 4.2: Frame $\mathfrak{G}$ with $R_<$ displayed.

Claim 2: Let $\boxtimes\varphi \leftrightharpoons \neg\varphi \wedge \Box_<\varphi$. The following formulas hold in $\mathfrak{G}^+$:

$$\begin{aligned}
\underline{\infty} &\rightarrow \Diamond^2_< \underline{1} \wedge \Box_< \neg \underline{1}, & (4.4)\\
\underline{\infty} \wedge \Diamond_< \Box_< p \wedge \Diamond^2_< \neg p &\rightarrow \Diamond_< \boxtimes p \vee \Diamond^2_< \boxtimes p, & (4.5)\\
\Diamond_< \boxtimes p &\rightarrow \Diamond_<(p \wedge \Box_< p \wedge \Diamond^2_> \boxtimes p \wedge \Box^3_> \neg \boxtimes p), & (4.6)\\
\underline{\infty} \wedge \neg \Diamond_< \boxtimes p &\rightarrow \Box_< \neg(p \wedge \Box_< p \wedge \Diamond^2_> \boxtimes p \wedge \Box^3_> \neg \boxtimes p), & (4.7)\\
\underline{\infty} \wedge \Box_< p &\rightarrow \Box_< \Diamond_< p, & (4.8)\\
\Box_< \Diamond_< p &\rightarrow \Diamond_< \Box_< p. & (4.9)
\end{aligned}$$

Conjunction of these formulas is denoted as Γ.

Proof of claim: Statement 4.4 follows directly from the definition of the frame and Claim 1. For 4.5, assume $\{\infty\} \leq \mathfrak{V}(\Diamond_< \Box_< p \wedge \Diamond^2_< \neg p)$. It means that for some i, $\{a_i\} \leq \mathfrak{V}(\Box p)$ and yet $\mathfrak{V}(\neg p)$ is nonempty. Thus, there must exist a maximal point a_j in $\mathfrak{V}(\neg p)$ and ∞ $R_<$-sees a_j in one or two steps. For statement 4.6, assume xRa_i and $\{a_i\} \leq \mathfrak{V}(\boxtimes p)$. But then $xR_< a_{i+2}$ and a_{i+2} is the one and only point $\mathfrak{V}$-satisfying the formula arising from the successor by erasing the initial $\Diamond_<$. Similar reasoning establishes 4.7. Statement 4.8 is straightforward. For 4.9, assume $\{w\} \leq \mathfrak{V}(\Box_< \Diamond_< p)$. It means that for every i, there exists $j > i$ s.t. $\{a_j\} \leq \mathfrak{V}(p)$, hence $\mathfrak{V}(p)$ is infinite. But then the complement of $\mathfrak{V}(p)$ must be finite and for some i, $a_i \leq \mathfrak{V}(\Box_< p)$. $\dashv$

Claim 3: For any $\mathfrak{A} \in \mathbb{V}(\Gamma) \cap \mathcal{T}$, $\mathfrak{A} \models \neg\underline{\infty}$.

Proof of claim: Assume that for some $\mathcal{T}$-BAO $\mathfrak{A}$ $\mathfrak{A} \not\models \underline{\infty} = \bot$ and there exists the conjugate $\mathbf{p}_<$ of $\Diamond_<$. We show that $\underline{\infty} \leq \Box_<\Diamond_<\mathbf{p}_<\underline{\infty} \wedge \Box_<\Diamond_<\neg\mathbf{p}_<\underline{\infty}$, thus contradicting the fact that $\mathfrak{A}$ validates $\mathbf{McK}_<$ (statement 4.9). That $\underline{\infty} \leq \Box_<\Diamond_<\mathbf{p}_<\underline{\infty}$ follows from 4.8 of the previous claim. Assume now $\underline{\infty} \wedge \Diamond_<\Box_<\mathbf{p}_<\underline{\infty} \neq \bot$. By 4.4, $\underline{\infty} \leq \Diamond^2_<\underline{1} \leq \Diamond^2_<\neg\mathbf{p}_<\underline{\infty}$. Thus, $\underline{\infty} \wedge \Diamond_<\Box_<\mathbf{p}_<\underline{\infty} \wedge \Diamond^2_<\neg\mathbf{p}_<\underline{\infty} \neq \bot$. By 4.5, it means that

$$\Diamond_< \boxtimes \mathbf{p}_<\underline{\infty} \neq \bot. \tag{4.10}$$

Define

$$c \fallingdotseq \mathbf{p}_<\underline{\infty} \wedge \Box_<\mathbf{p}_<\underline{\infty} \wedge \Diamond^2_> \boxtimes \mathbf{p}_<\underline{\infty} \wedge \Box^3_>\neg \boxtimes \mathbf{p}_<\underline{\infty}.$$

By definition, $c \leq \mathbf{p}_<\underline{\infty}$. On the other hand, 4.7 and the fact that

$$\underline{\infty} \leq \Box_<\mathbf{p}_<\underline{\infty} \leq \Box_<(\mathbf{p}_<\underline{\infty} \vee \Diamond_<\neg\mathbf{p}_<\underline{\infty}) = \Box_<\neg \boxtimes \mathbf{p}_<\infty$$

imply $\mathbf{p}_<\underline{\infty} \leq \neg c$. Thus, $c = \bot$ but this contradicts 4.6 and 4.10. ⊣

This proves the incompleteness part of the claim. For the inconsistency result, add universal modality to the signature of $\mathfrak{G}$ and reason as in Sections 4.2 and 4.4. ⊣

4.6 Notes

The first example ever of a Kripke incomplete — i.e., $\mathcal{CAV}$-incomplete — logic was obtained by Thomason [50]. Actually, we can say more: it was also the first example of $\mathcal{CA}$-inconsistent logic. More on that result below. Following Thomason's paper, several authors obtained $\mathcal{CAV}$-incompleteness theorems for restricted similarity types (in particular, the basic modal similarity type), well-behaved lattices of logics (for example, Ext**S4**) and surprisingly simple axioms. We have to mention here Fine [18] and van Benthem [53]; both papers are discussed below. In addition, Gerson [25], [24] has proven that even in the basic similarity type, one may find both examples of logics which are $\mathcal{CA}$-incomplete and examples of logics which are $\mathcal{CA}$-complete but $\mathcal{CAV}$-incomplete. Strangely enough, no systematic investigation of notions of completeness weaker than Kripke completeness followed. Only recently, there has been some revival of interest in related issues; a landmark paper is Venema [55].

This article has proven the existence of an *atomless* variety of BAOs. In other words, it has been shown that there exists a logic Λ s.t. every algebra in $\mathbb{V}(\Lambda)$ is *atomless*. It is even stronger incompleteness result that $\mathcal{A}$-inconsistency. Nevertheless, Venema's algebra was neither complete nor completely additive; and there was no straightforward way of turning it into a BAO from $\mathcal{CV}$. In fact, the question whether it is possible to obtain $\mathcal{A}$-inconsistency result by the use of a $\mathcal{T}$-BAO was posed by Y. Venema himself during the open problems session of a conference in Tbilisi [iv]. The construction from Section 4.1 provides an answer to this question. It has been published first in Litak [V]. Unfortunately, the algebra constructed in Section 4.1 is not atomless. The problem whether there exists a $\mathcal{CV}$-BAO generating an atomless variety seems to be open.

But the construction from Section 4.1 has another advantage: it is a *complete* $\mathcal{T}$-BAO. Hence, it answers negatively an open question, posed to the author by V. Shehtman (p.c.): whether $\mathcal{C}$-completeness and $\mathcal{CA}$-completeness coincide. According to V. Shehtman, this problem was discussed by the Russian modal logicians — L. Maksimova and V. Rybakov, in particular — in 1970's. Apparently, the question was not formulated in print.

It should be also noted that the paper of Kracht and Kowalski [38] was the first attempt to construct $\mathcal{A}$-inconsistent logic by the use of a $\mathcal{T}$-BAO. But, as it was put by Venema [55], *a number of errors makes it difficult to judge whether their approach might succeed or not.*

Litak [V] contained a different example of $\omega\mathcal{C}$-inconsistent logic than the one presented in Section 4.2. The result from that paper is generalized in Section 8.3. The one presented in the present chapter is a generalization of Fine [18]. Fine's construction was simplified by Chagrov and Zakharyaschev [11, Chapter 6]; this provided a good starting point for results in Section 4.2. As was noted, it seems to be the first example of an extension of **4** whose *degree of incompleteness* (cf. Chapter 5 for the definition) with respect to **4** is $\mathfrak{c}$. This also strengthens results from Litak [III] continually many incomplete logics constructed in that work did not even share the same class of adequate Kripke frames.

The logic **Succ** discussed in Section 4.3 is very closely related to the first example of $\mathcal{CAV}$-inconsistent logic in Thomason [50]. In fact, the only difference is that instead of **Lin**, Thomason used a slightly weaker axiom whose meaning is intuitively rendered as *no branching to the right.* It should not be very difficult to adopt our $\mathcal{C}$-inconsistency proof for this system as well. But something more can be said. Chagrov [10] complained that Thomason's proof relies too heavily on the axiom of choice. Our proof, more explicit than that of Thomason, shows that the very nature of his axioms makes resorting to Zermelo's well-ordering theorem necessary. In investigating **Succ**, one cannot use more constructive means than we did.

Section 4.4 strengthens van Benthem [53]; this result was published before in Litak [V]. Van Benthem was interested in completeness with respect to (duals of) *elementary* general frames. The class of duals of such frames is properly contained both in $\mathcal{AV}$ and in $\mathcal{T}$. The algebra used by van Benthem [53] was the subalgebra of $\mathfrak{v}B$ generated by its finite elements; thus, it was not complete. It may be proven, however, that both algebras satisfy exactly the same formulas. $\mathfrak{v}B$ itself was used to prove a $\mathcal{T}$-incompleteness result by Wolter [57] (the present author obtained this result independently, but more than 10 years later). The observation that the logic of $\mathfrak{v}B$ is $\mathcal{AV}$-incomplete appears to be new.

The result in Section 4.5 also seems new. Just as in case of Sections 4.1 and 4.4, it was published first in Litak [V].

Chapter 5

The Generalized Blok Alternative

5.1 Introduction

This section is devoted to various generalizations of theorem proven by Blok [4]. We are going to show that every logic which is not a join-splitting (iterated splitting) of the lattice of all logics in a given modal similarity type shares its class of $\mathcal{X}$-BAO's with continually many other logics for almost all $\mathcal{X} \in$ PROPERTIES, whereas join-splittings have strict finite model property.

First, a definition. $\mathsf{Spec}\mathcal{X}(\Lambda) \coloneqq \{\mathsf{Log}(\Gamma) | \mathsf{Log}\mathcal{X}(\Gamma) = \mathsf{Log}\mathcal{X}(\Lambda)\}$. The following holds:

Lemma 5.1 *Λ is $\mathcal{X}$-complete iff for every $\Delta \in \mathsf{Spec}\mathcal{X}(\Lambda)$, $\Delta \subseteq \mathsf{Log}(\Lambda)$.*

Proof: ($\Rightarrow$). By definition, $\Delta \in \mathsf{Spec}\mathcal{X}(\Lambda)$ implies $\Delta \subseteq \mathsf{Log}\mathcal{X}(\Lambda) = \mathsf{Log}(\Lambda)$, for $\mathcal{X}$-complete Λ.

($\Leftarrow$). $\mathsf{Log}\mathcal{X}(\Lambda) \in \mathsf{Spec}\mathcal{X}(\Lambda)$ and $\mathsf{Log}(\Lambda) \subseteq \mathsf{Log}\mathcal{X}(\Lambda)$. Thus, the assumption implies $\mathsf{Log}(\Lambda) = \mathsf{Log}\mathcal{X}(\Lambda)$, i.e., $\mathcal{X}$-completeness of $\mathsf{Log}(\Lambda)$. ⊣

A special role in our investigations is played by join-splittings of $Ext\mathbf{K}_{\mathrm{TYPE}}$. The reason is given by the following

Lemma 5.2 *Let $\mathcal{X}$ be any property such that $\mathcal{F} \subseteq \mathcal{X}$. If $\mathsf{Log}(\Lambda)$ is a join-splitting of the lattice $Ext\mathbf{K}_{\mathrm{TYPE}}$, it is the smallest element of $\mathsf{Spec}\mathcal{X}(\Lambda)$. Thus, it is $\mathcal{X}$-complete iff $|\mathsf{Spec}\mathcal{X}(\Lambda)| = 1$.*

Proof: Assume $\mathsf{Log}(\Lambda) = Ext\mathbf{K}_{\text{TYPE}}/F$ and $\Delta \in \mathsf{Spec}\mathcal{X}(\Lambda)$. $\mathsf{Log}(\Lambda) \not\subseteq \Delta$ iff $\Delta \subseteq Th(\mathfrak{F})$, for some $\mathfrak{F} \in F$. But $\mathsf{Log}(\Lambda) \subseteq \mathsf{Log}\mathcal{X}(\Lambda)$ and thus $\mathfrak{F} \notin \mathbb{V}(\Lambda) \cap \mathcal{X} = \mathbb{V}(\Delta) \cap \mathcal{X}$, for any $\Delta \in \mathsf{Spec}\mathcal{X}(\Lambda)$. As $\mathfrak{F} \in \mathcal{X}$ (cf. Appendix), the first statement follows. The second statement follows by Lemma 5.1. ⊣

Let us see more exactly how splittings of $Ext\mathbf{K}_{\text{TYPE}}$ look like. We already know that a logic which splits $Ext\mathbf{K}_{\text{TYPE}}$ must be determined by a single subdirectly irreducible finite algebra. We can tell exactly which finite algebras induce splittings:

Theorem 5.3 (Blok) $\mathsf{Log}(\Lambda)$ *splits* $Ext\mathbf{K}_{\text{TYPE}}$ *iff* $\mathsf{Log}(\Lambda) = Th(\mathfrak{A})$ *for some* $\mathfrak{A} \in \mathcal{FIS}$.

Proof: (⇒). Follows from $\mathcal{FIS}$-completeness of $\mathbf{K}_{\text{TYPE}}$ and HS-closure of $\mathcal{FI}$.

(⇐). We use here Wolter's splitting theorem. Assume $\mathfrak{A} \models \bigwedge \blacksquare^n \bot$. We show that $\chi \leftrightharpoons \bigwedge \blacksquare^n \bot \wedge \bigwedge \blacksquare^{\leq n-1} \delta(\mathfrak{A}) \wedge p_*$ (recall that $\delta(\mathfrak{A})$ is the Jankov formula of $\mathfrak{A}$), where p_* is the opremum variable, is a splitting formula for $\mathfrak{A}$. Assume $\mathfrak{V}(\chi) \neq \bot$ for some valuation $\mathfrak{V}$ in an algebra $\mathfrak{B}$. $\bigwedge \blacksquare^n \bot$ belongs to $\mathsf{Root}(\chi)$, the root filter associated with $\mathfrak{V}(\chi)$, and thus $\blacksquare^{\leq m} \mathfrak{V}(\delta(\mathfrak{A}))$ belongs to this root filter for every $m \geq n$. On the other hand, $\mathfrak{V}(\chi) \not\leq \mathfrak{V}(\neg p_*)$. By Theorem 2.22, it follows that $\mathsf{Log}(\Lambda)$ splits $Ext\mathbf{K}_{\text{TYPE}}$. ⊣

Theorem 5.4 (Blok) *If* $\bigwedge \blacksquare^n \bot \in \Lambda$ *for some* $n \in \omega$, *then* $\mathbb{V}(\Lambda)$ *is locally finite.*

Proof: By induction on n.

(1) $i = 1$. For every $\pi \in$ TYPE, $\Box_\pi \bot \in \Lambda$. Thus, there are no non-trivial variable-free formulas and local finiteness of $\mathbb{V}(\Lambda)$ follows from local finiteness of boolean algebras.

(2) Suppose the theorem is true for n and let $m \leftrightharpoons n+1$, $\bigwedge \blacksquare^m \bot \in \Lambda$. Assume that $\mathfrak{A}$ is an k-generated subdirectly irreducible algebra in $\mathbb{V}(\Lambda)$. If $\mathfrak{A} \models \bigwedge \blacksquare^n \bot$, then it is finite by IH. So, assume $\bigwedge \blacksquare^n \neq \top$. We claim this is a coatom of $\mathfrak{A}$. Assume $\bigwedge \blacksquare^n \bot < x < \bigwedge \blacksquare^m \bot = \top$. Then for every $\pi \in$ TYPE, $\top = \bigwedge \blacksquare^m \bot \leq \Box_\pi \bigwedge^n \bot \leq \Box_\pi x$. Thus, the principal filter generated by x is open and so is the principal filter generated by $\bigwedge^n \blacksquare \bot \vee \neg x$. Hence, we have two open filters whose intersection is equal to $\{\top\}$ and $\mathfrak{A}$ cannot be subdirectly irreducible. Denote $* \leftrightharpoons \bigwedge \blacksquare^n \bot$ ($*$ is an opremum). The principal filter $*\uparrow$ generated by $*$ is open. thus, the boolean reduct of $\mathfrak{A}/*\uparrow$

is isomorphic with the principal ideal generated by $*$. $\mathfrak{A}/*\uparrow\models \bigwedge \blacksquare^n\bot$ and is finitely generated (as a homomorphic image of a finitely generated algebra). Thus, it is finite by IH. Finiteness of $\mathfrak{A}$ follows by a standard argument for boolean algebras ($*$ is a coatom). If every finitely generated subdirectly irreducible algebra in the variety is finite, then the variety is locally finite.

$\dashv$

The following generalizes a result of Blok [4]. Nevertheless, we stick here so closely to his techniques that it deserves the name of observation rather than a new result.

Theorem 5.5 (Blok) *Every join-splitting of $Ext\mathbf{K}_{\mathrm{TYPE}}$ has the finite embeddability property.*

Proof: Let Λ be a join-splitting of $Ext\mathbf{K}_{\mathrm{TYPE}}$, i.e., let Λ be of the form $Ext\mathbf{K}_{\mathrm{TYPE}}/\{Th(\mathfrak{S}_n)\}_{n\in\omega}$, $\mathfrak{D} \in \mathbb{V}(\Lambda)$, C be a finite partial subalgebra of $\mathfrak{D}$, $\mathfrak{B}$ — the subalgebra of $\mathfrak{D}$ generated by C and A_0 be the boolean subalgebra of $\mathfrak{B}$ generated by C. Of course, $\mathfrak{B} \in \mathbb{V}(\Lambda)$. As $\{\bigwedge\blacksquare^n\bot\}_{n\in\omega}$ form an ascending chain and A_0 is finite, there is $i \in \omega$ s.t. for every $a \in A_0$, $\exists m \in \omega$ s.t. $a \leq \bigwedge\blacksquare^m\bot$ iff $a \leq \bigwedge\blacksquare^i\bot$. Let $Irr \leftrightharpoons \bigwedge\blacksquare^i\bot$. The principal filter generated by Irr is open, thus the boolean reduct of $\mathfrak{B}/Irr\uparrow$ is isomorphic to the principal ideal generated by Irr. Moreover, as $\mathfrak{B}/Irr\uparrow\models \bigwedge\blacksquare^i\bot$ and is finitely generated (being a homomorphic image of a finitely generated algebra), it is finite by Theorem 5.4. Let A_1 be the finite boolean subalgebra of $\mathfrak{B}$ generated by $A_0 \cup Irr\downarrow$ (A_0 and the principal ideal generated by Irr). $\mathfrak{A}$ is the BAO whose universe is A_1 and operators are defined as

$$\Diamond^{\mathfrak{A}}_\pi x \leftrightharpoons \bigwedge\{a \in A_1 | \Diamond^{\mathfrak{B}}_\pi x \leq a\}.$$

This was called by B. Jónsson *the upper McKinsey closure* of A; it is known (and easy to show) that the algebra thus constructed is always a BAO, that all $\Diamond_\pi$ which existed in A_1 are preserved and that $\Diamond^{\mathfrak{B}}_\pi a \leq \Diamond^{\mathfrak{A}}_\pi a$ in general (hence the name). We only have to show that $\mathfrak{A} \in \mathbb{V}(\Lambda)$.

Claim 1: For every $n \leq i$, $\bigwedge\blacksquare^{n\mathfrak{B}}\bot = \bigwedge\blacksquare^{n\mathfrak{A}}\bot$.

Proof of claim: By induction. (1) $n = 1$. For every $\pi \in$ TYPE, $a \leq \Box^{\mathfrak{A}}_\pi\bot$ iff (by definition of $\Box^{\mathfrak{A}}_\pi$) for every $\pi \in$ TYPE, $a \leq \Box^{\mathfrak{B}}_\pi\bot$ iff $a \leq \bigwedge\blacksquare^{\mathfrak{B}}\bot$. But $\bigwedge\blacksquare^{\mathfrak{B}}\bot \in Irr\downarrow\subseteq \mathfrak{A}$ and thus $\bigwedge\blacksquare^{\mathfrak{B}}\bot = \bigwedge\blacksquare^{\mathfrak{A}}\bot$. (2) Induction step is proven the same way. The only thing which matters is that for $n \leq i$, $\bigwedge\blacksquare^{n\mathfrak{B}}\bot \in Irr\downarrow$. $\dashv$

Claim 2: [1] Let $\mathfrak{Z}'$ be a finite boolean subalgebra of $\mathfrak{B}$, let $b \in \mathfrak{B}$ and $i \leq b$ s.t. $i\downarrow$ is finite and for every $a \in \mathfrak{Z}'$, $a \leq i$ iff $a \leq b$ and let $\mathfrak{Z}$ be the boolean subalgebra of $\mathfrak{B}$ generated by $\mathfrak{Z}' \cup i\downarrow$. Then for every $a \in \mathfrak{Z}$, $a \leq i$ iff $a \leq b$.

Proof of claim: $i\downarrow$ is atomic, being finite. Let atoms of $i\downarrow$ be denoted as $i_1, \ldots i_k$. Thus, $\mathfrak{Z}$ is generated by $\mathfrak{Z}' \cup \{i_1, \ldots, i_k\}$. Clearly, if we can prove the claim for $k = 1$, then we can prove it for arbitrary finite k. Assume then $k = 1$. Every $a \in \mathfrak{Z}$ can be represented as a join of finitely many elements of the form $a_m \wedge j_m \wedge j'_m$, where $a_m \in \mathfrak{Z}'$, $j_m \in \{i_1, \top\}$, $j'_m \in \{\neg i_1, \top\}$. If any j_m is equal to i_1, then $a \leq i$. Thus, we can restrict attention to $a = a' \wedge \neg i_1$, where $a' \in \mathfrak{Z}$ (we make use here of three facts: distributivity of boolean algebras, that $\mathfrak{Z}'$ is closed under boolean operations and that $\neg i_1$ is a coatom). Assume $a' \wedge \neg i_1 \leq b$; this is equivalent to $a' \leq i_1 \vee b$ and this in turn to $a' \leq b$. By assumption, $a' \leq i$ and thus $a \leq i$. ⊣

Claim 3: $\mathfrak{A} \vDash \bigwedge \blacksquare^{i+1}\bot \leftrightarrow \bigwedge \blacksquare^{i}\bot$.

Proof of claim: We want to show that $\bigwedge\limits_{\pi \in \text{TYPE}} \Box^{\mathfrak{A}}_{\pi} \bigwedge \blacksquare^{i\mathfrak{A}}\bot \leq \bigwedge \blacksquare^{i\mathfrak{A}}\bot$. By definition of the upper McKinsey closure, this boils down to showing that for every $a \in \mathfrak{A}$, if for every $\pi \in \text{TYPE}$ $a \leq \Box^{\mathfrak{A}}_{\pi} \bigwedge \blacksquare^{i\mathfrak{A}}\bot$, then $a \leq \bigwedge \blacksquare^{i\mathfrak{A}}\bot$. By definition of $\Diamond^{\mathfrak{A}}_{\pi}$, $a \leq \Box^{\mathfrak{A}}_{\pi} \bigwedge \blacksquare^{i\mathfrak{A}}\bot$ for every $\pi \in \text{TYPE}$ iff $a \leq \Box^{\mathfrak{B}}_{\pi} \bigwedge \blacksquare^{i\mathfrak{A}}\bot$ for every $\pi \in \text{TYPE}$ iff (by Claim 1) $a \leq \Box^{\mathfrak{B}}_{\pi} \bigwedge \blacksquare^{i\mathfrak{B}}\bot$ for every $\pi \in \text{TYPE}$ iff $a \leq \bigwedge \blacksquare^{i+1\mathfrak{B}}\bot$ iff (by Claim 2) $a \leq \bigwedge \blacksquare^{i\mathfrak{B}}\bot$ iff (by Claim 1) $a \leq \bigwedge \blacksquare^{i\mathfrak{A}}\bot$. ⊣

Claim 4: For arbitrary splitting algebra $\mathfrak{S}$ (i.e., for arbitrary $\mathfrak{S} \in \mathcal{FIS}$), if $\mathfrak{S} \in SH(\mathfrak{A})$, then $\mathfrak{S} \in SH(\mathfrak{B})$.

Proof of claim: Assume $\mathfrak{S} \in \mathcal{FIS} \cap SH(\mathfrak{A})$. $\mathfrak{S} \vDash \bigwedge \blacksquare^{n}\bot$ for some $n \in \omega$; thus, by the previous claim, $\mathfrak{S} \vDash \bigwedge \blacksquare^{i}\bot$. Moreover, as $\bigwedge \blacksquare^{i}\bot$ is a variable-free term, every algebra in which $\mathfrak{S}$ is embeddable must verify $\bigwedge \blacksquare^{i}\bot$. Thus, $\mathfrak{S}$ is a subalgebra of $\mathfrak{S}' \in H(\mathfrak{A}/\bigwedge \blacksquare^{i}\bot\uparrow)$. It is thus enough to prove that $\mathfrak{A}/\bigwedge \blacksquare^{i}\bot\uparrow$ is isomorphic to $\mathfrak{B}/\bigwedge \blacksquare^{i}\bot\uparrow$. Irr belongs to both algebras and the principal filter generated by Irr is open in both of them. The boolean part of both algebras is isomorphic to $Irr\downarrow$. $\Box^{\mathfrak{A}/Irr\uparrow}_{\pi} x = \Box^{\mathfrak{A}}_{\pi} x \wedge Irr =_{(\text{by def. of } \mathfrak{A})} \Box^{\mathfrak{B}}_{\pi} x \wedge Irr = \Box^{\mathfrak{B}/Irr\uparrow}_{\pi} x$. ⊣

[1] The proof was kindly supplied by Felix Bou

Thus, by Lemma 2.20 for no $n \in \omega$, $\mathfrak{S}_n \in HS(\mathfrak{A})$. Hence, no $\mathfrak{S}_n \in HSP_U(\mathfrak{A})$ and the result follows by Theorem 2.22.

$\dashv$

Blok's result in the original formulation concerned only the lattice of unimodal logics. Perhaps the main problem was posed by the use of Makinson's theorem, characterizing maximal consistent unimodal logics. However, Makinson's theorem can be disposed of. Let us start with an useful

Theorem 5.6 (Kowalski and Kracht [35]) *For every logic Λ, t.f.a.e.*

(1) $\mathbb{V}(\Lambda)$ is semisimple.

(2) There exists $n \in \omega$ s.t. $\bigvee \blacklozenge^{\leq n+1} p \to \bigvee \blacklozenge^{\leq n} p \in \Lambda$ and for every $\pi \in$ TYPE, $p \to \Box_\pi \bigvee \blacklozenge^{\leq n} p \in \Lambda$.

(3) $\mathbb{V}(\Lambda)$ is a discriminator variety.

Theorem 5.7 *For every Λ, Λ is a maximal consistent logic iff $\mathbb{V}(\Lambda)$ is generated by a simple, zero-generated algebra.*

Proof: For "if" direction see, for example, Kowalski [34]; this direction is of no relevance for us here. For the converse direction, take any non-trivial algebra $\mathfrak{A} \in \mathbb{V}(\Lambda)$ and let $\mathfrak{B}$ be its zero-generated subalgebra. As $\mathbb{V}(\Lambda)$ is minimal, it generated by $\mathfrak{B}$ as well. Assume that $\mathfrak{B}$ has a non-trivial proper homomorphic image $\mathfrak{C}$. $\mathfrak{C}$ is also zero-generated. It means that for some variable-free terms t_1 and t_2, $\mathfrak{B} \not\models t_1 \leftrightarrow t_2$ and $\mathfrak{C} \models t_1 \leftrightarrow t_2$. Thus, the variety generated by $\mathfrak{B}$ is not minimal, a contradiction. $\dashv$

Theorem 5.8 (Maximal Logic) *For every maximal consistent logic Λ, either there exist $n \in \omega$ and $\pi \in$ TYPE s.t. $\bigvee \blacklozenge^{\leq n} \Box_\pi \bot \in \Lambda$ or $\mathbb{V}(\Lambda)$ is generated by the two element algebra satisfying $\Diamond_\pi \top = \top$ for every $\pi \in$ TYPE.*

Proof: By Theorem 5.7, $\Lambda = Th(\mathfrak{A})$ for some zero-generated and simple $\mathfrak{A}$. Assume first $\Box_\pi \bot \neq \bot$ for some $\pi \in$ TYPE and let $n \in \omega$ be such that $\bigvee \blacklozenge^{\leq n}$ is a unary discriminator (universal modality) on $\mathfrak{A}$; such n exists by Theorem 5.6. Then $\bigvee \blacklozenge^{\leq n} \Box_\pi \bot \in \Lambda$.

Assume now $\Diamond_\pi \top = \top$ for every $\pi \in$ TYPE. But then, as $\mathfrak{A}$ is zero-generated, every element is equal either to $\top$ or $\bot$. $\dashv$

We are ready to formulate the main

Theorem 5.9 (The Generalized Blok Alternative) *For arbitrary* TYPE *and arbitrary $\mathcal{X} \in$ PROPERTIES $- \{\mathcal{A}, \mathcal{V}\}$, $|\mathsf{Spec}\mathcal{X}(\Lambda)| = 1$ iff $\mathsf{Log}(Lambda)$ is a join-splitting of $Ext\mathbf{K}_{\text{TYPE}}$. Otherwise, $|\mathsf{Spec}\mathcal{X}(\Lambda)| = 2^{\aleph_0}$.*

5.2 Degrees of $\omega\mathcal{C}$-incompleteness

Theorem 5.10 *If $\Lambda \in Ext\mathbf{K}_{\mathrm{TYPE}}$ is not a join-splitting, then*

$$|\mathsf{Spec}\omega\mathcal{C}(\Lambda)| = 2^{\aleph_0}.$$

Proof: Before we commence, we need a particular concept.

Definition 5.11 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be* BAO*s, $\bar{a} = \{a_i\}_{i<k}$, $\bar{a'} = \{a'_j\}_{j<n}$ be sequences of elements of $\mathfrak{A}$ and $\bar{b}$, $\bar{b'}$ be sequences of elements of $\mathfrak{B}$ s.t. the length of $\bar{b}$ is k and the length of $\bar{b'}$ is n. Finally, let $\bar{\Pi} = \{\Pi_i\}_{i<k}$ and $\bar{\Pi'} = \{\Pi'_j\}_{j<n}$ be two sequences of nonempty subsets of* TYPE. A connected $(\bar{\Pi}, \bar{\Pi'})$-product $\langle \mathfrak{A}, \bar{a}, \bar{a'}\rangle \veebar \langle \mathfrak{B}, \bar{b}, \bar{b'}\rangle$ *is the algebra whose boolean reduct is the direct product of $\mathfrak{A}$ and $\mathfrak{B}$ and operators are defined as*

$$\Diamond_\pi\langle x, y\rangle \leftrightharpoons \langle \Diamond_\pi x \vee \bigvee_{i\in I} a_i, \Diamond_\pi y \vee \bigvee_{j\in J} b'_j\rangle,$$

where I is the set of numbers $i < k$ s.t. $\pi \in \Pi_i$ and $y \wedge b_i \neq \bot$. Similarly, J is the set of numbers $j < n$ s.t. $\pi \in \Pi_j$ and $x \wedge a'_j \neq \bot$.

The goal of introducing such a construction can be made more intuitive by defining a corresponding notion for frames.

Definition 5.12 *Let $\mathfrak{F}_1 \leftrightharpoons \langle W_1, \{R^1_\pi\}_{\pi\in\mathrm{TYPE}}, A\rangle$ and $\mathfrak{F}_2 \leftrightharpoons \langle W_2, \{R^2_\pi\}_{\pi\in\mathrm{TYPE}}, B\rangle$ be general* TYPE*-frames, $\bar{a} = \{a_i\}_{i<k}$, $\bar{a'} = \{a'_j\}_{j<n}$ be sequences of elements of A (i.e., admissible subsets of W_1) and $\bar{b}$, $\bar{b'}$ be sequences of elements of B (i.e., admissible subsets of W_2) s.t. the length of $\bar{b}$ is k and the length of $\bar{b'}$ is n. Finally, let $\bar{\Pi} = \{\Pi_i\}_{i<k}$ and $\bar{\Pi'} = \{\Pi'_j\}_{j<n}$ be two sequences of nonempty subsets of* TYPE. A connected $(\bar{\Pi}, \bar{\Pi'})$-sum $\langle \mathfrak{F}_1, \bar{a}, \bar{a'}\rangle \veebar \langle \mathfrak{F}_2, \bar{b}, \bar{b'}\rangle$ *is a frame $\mathfrak{G} \leftrightharpoons \langle W, \{R_\pi\}_{\pi\in\mathrm{TYPE}}, C\rangle$ such that*

(1) W is the disjoint union of W_1 and W_2,

(2) For every $\pi \in$ TYPE, $R_\pi \leftrightharpoons R^1_\pi \cup R^2_\pi \cup \{\langle w_1, w_2\rangle | \exists i < k\, \pi \in \Pi_i, w_1 \in a_i, w_2 \in b_i\} \cup \{\langle w_2, w_1\rangle | \exists j < n\, \pi \in \Pi'_j, w_1 \in a'_j, w_2 \in b'_j\}$,

(3) $C \leftrightharpoons \{a \cup b | a \in A, b \in B\}$.

Intuitively, all points from a_i "see" all points from b_i via all operators in Π_i and all points from b_j "see" all points from a_j via all operators in Π_j. These are the only "points of contact" between the two frames. Thus, we call elements of $\bar{a}$ and $\bar{b'}$ *exit sets* (*exit elements*, in the algebraic case) and elements of $\bar{b}$ and $\bar{a'}$ — *entrance sets* (*entrance elements*). The following should now be easy to grasp.

Claim 1: A connected product of BAOs is a BAO and a connected sum of general frames is a general frame. In addition, the dual algebra of the connected sum of frames is isomorphic to the connected product of dual algebras whose exit and entrance elements are exit and entrance sets of respective frames.

Proof of claim: We keep the same notation as in the above definition. For BAOs: preservation of the bottom element by newly defined operators is straightforward. As for distributivity over finite joins, it is enough to observe that for every $j < n$, $(y_1 \vee y_2) \wedge b_j \neq \bot$ iff either $y_1 \wedge b_j \neq \bot$ or $y_2 \wedge b_j \neq \bot$ and similarly for all a_i's. For general frames: one needs to check that the set of all unions of admissible sets from A and B is closed under newly defined operators. Let $c \leftrightharpoons c_1 \cup c_2$, where c_1 is an arbitrary element from A and c_2 is an arbitrary element from B. $\Diamond_\pi^{\mathfrak{G}} c = \Diamond_\pi^{\mathfrak{G}} c_1 \cup \Diamond_\pi^{\mathfrak{G}} c_2 = \{x \in W_1 | \exists y \in c_1 x R_\pi^1 y\} \cup \bigcup\{a_i | \pi \in \Pi_i, \exists y \in b_i\, y \in c_2\} \cup \{x \in W_2 | \exists y \in c_2 x R_\pi^2 y\} \cup \bigcup\{b'_j | \pi \in \Pi'_j, \exists y \in a'_j\, y \in c_1\} = \Diamond_\pi^{\mathfrak{F}_1} c_1 \cup \bigcup\{a_i | \pi \in \Pi_i, b_i \cap c_2 \neq \emptyset\} \cup \Diamond_\pi^{\mathfrak{F}_2} c_2 \cup \bigcup\{b'_j | \pi \in \Pi'_j, a'_j \cap c_1 \neq \emptyset\} \in C$. This proves also the second statement of the claim. ⊣

No problem arises when one admits bottom among entrance or exit elements (empty exit or entrance sets), but it should be clear by now it does not make much mathematical sense. Therefore, from now on we exclude such a case for the same reason we excluded empty subsets of TYPE. A particular case arises when the sequence of entrance elements (sets) of one algebra (frame) is empty and hence so is the respective sequence of exit elements (sets) of the other algebra (frame). We call then such a connected product (union) *an oriented product (union)*, the component with empty entrance — *the initial component* and the component with empty exit — *the final component*. If both components are both initial and final, the algebra (frame) is simply a direct product (a disjoint union). We say then the connected product (union) is *degenerated*; we are interested only in non-degenerated ones. The following is immediate

Claim 2: The final component of an oriented product (union) is its complete homomorphic image (generated subframe).

To prove the main theorem, we use techniques closely related to those used in Chagrov and Zakharyaschev [11] or Chagrova [12]; actually, the reader is invited to consult these works to compare similarities and differences with our approach. Assume $\Lambda \in Ext\mathbf{K}_{\mathrm{TYPE}}$ is not a join-splitting and let Λ' be the smallest join-splitting contained in Λ. By Theorem 5.5, Λ' is $\mathcal{F}-\mathcal{FI}$-complete. As $\Lambda' \subsetneq \Lambda$, there must exist $\mathfrak{A}$ and $\varphi \in \Lambda$ s.t.

(1) $\mathfrak{A} \in \mathcal{FS}$ and there exists $x \in \mathfrak{A}$ s.t. $x \neq \bot$ and $x \leq \bigvee_{\pi \in \mathrm{TYPE}} \Diamond_\pi x$,

(2) $\mathfrak{A} \models \Lambda'$ and $\mathfrak{A} \not\models \varphi$.

In addition, $\mathfrak{A}$ may be chosen to be minimal, i.e., every proper homomorphic image of $\mathfrak{A}$ is in $\mathbb{V}(\Lambda)$. Let $\mathfrak{F} \risingdotseq \mathfrak{A}_*$. Let $m \risingdotseq md(\varphi)+1$ and r be the root of $\mathfrak{F}$. By the minimality of $\mathfrak{A}$, r is the only point where φ can be refuted.

Let $u \in \mathfrak{A}$ be a minimal $u \neq \bot$ s.t. $u \leq \bigvee\limits_{\pi \in \mathrm{TYPE}} \Diamond_\pi u$ (we use finiteness here). It follows that for every $y_i \in At\mathfrak{A}$ (i.e., $y_i \in \mathfrak{F}$ as $At\mathfrak{A}$ is the universe of $\mathfrak{F}$), $y_i \leq u$ implies there exists nonempty $\Pi_i \subseteq \textsc{type}$ s.t. $y_i \leq \Diamond_\pi u$ iff $\pi \in \Pi_i$ and if $\bigvee\{y_0, \ldots, y_{k-1}\} < u$, then there exists $i < k$ s.t. for every $\pi \in \textsc{type}$, $y_i \leq \Box_\pi \bigwedge\{\neg y_0, \ldots, \neg y_{k-1}\}$ (i.e., for no $\pi \in \textsc{type}$ and $j < k$, $y_i R_\pi y_j$). In short, u is a *minimal cycle*. Let $\{y_0, \ldots y_{n-1}\}$ be an enumeration of all atoms below u such that $y_i \leq \Diamond_\pi y_{i+1}$ for some $\pi \in \textsc{type}$ and let $\mathfrak{G}$ be a frame whose universe is $\{y_0, \ldots, y_{n-1}\} \times m$ and for every $\pi \in \textsc{type}$, $\langle y_i, j\rangle R_\pi \langle y_k, l\rangle$ iff $y_i R_\pi y_k$. Let

$$\Pi_{i,j} \risingdotseq \begin{cases} \{\pi \in \textsc{type} | y_i \leq \Diamond_\pi \neg u\} & : \quad i \neq n-1 \,\mathrm{OR}\, j \neq m-1, \\ \{\pi \in \textsc{type} | y_{n-1} \leq \Diamond_\pi \neg u \vee y_0\} & : \quad i = n-1 \& j = m-1. \end{cases}$$

Similarly, let

$$z_{i,j} \risingdotseq \begin{cases} \{z \in \mathfrak{F} - \{y_0, \ldots, y_{n-1}\} | \exists \pi \in \Pi_{i,j}\, y_i R_\pi z\} & : \quad i \neq n-1, j \neq m-1, \\ \{z \in \mathfrak{F} - \{y_1, \ldots, y_{n-1}\} | \exists \pi \in \Pi_{i,j}\, y_i R_\pi z\} & : \quad i = n-1, j = m-1. \end{cases}$$

Now let $\bar{\Pi} \risingdotseq \{\Pi_{i,j} | i < n, j < m\}$, $\bar{\Pi}' \risingdotseq \{\Pi_{n-1,m}\}$, $\mathfrak{F}'$ be the (Π, Π')-connected sum $\langle \mathfrak{G}, (\langle y_i, j\rangle)_{i<n, j<m}, (\langle y_0, 0\rangle)\rangle \veebar \langle \mathfrak{F}, (z_{i,j})_{i<n,j<m}, (y_{n-1})\rangle$ and $\mathfrak{A}' \risingdotseq \mathfrak{F}'^{+}$. This construction is called by Chagrov and Zakharyaschev [11] *extending the cycle* x.

Claim 3:

(1) $\mathfrak{F}$ is a bounded morphic image of $\mathfrak{F}'$ (or, equivalently, $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}'$) via mapping

$$f(x) \risingdotseq \begin{cases} x & : \quad x \in \mathfrak{F}, \\ y_i & : \quad x = \langle y_i, k\rangle. \end{cases}$$

Thus, (*) for any valuation $\mathfrak{V}$ in $\mathfrak{F}^*$, any $x \in \mathfrak{F}$ and every formula ψ, $x \in \mathfrak{V}(\psi)$ iff $x \in \mathfrak{V}'(\psi)$, where $\mathfrak{V}'(p)$ is a valuation in $\mathfrak{F}'$ defined as $\mathfrak{V}' \leftrightharpoons f^{-1}[\mathfrak{V}'(p)]$ for every variable p.

(2) If $md(\psi) < m$ and $x \in \mathfrak{F}$, then (*) holds with $\mathfrak{F}'$ replaced by any connected union having $\mathfrak{F}'$ as one of its components with $\langle y_{n-1}, m-1\rangle$ as the only exit point.

Proof of claim: (1) By straightforward checking of cases.

(2) By the fact that there is no path of length no longer than m from any point of $\mathfrak{F}$ to $\langle y_{n-1}, m-1\rangle$ (here is where we use the minimality of the cycle).

$\dashv$

By Theorem 5.8, there is either $\mathfrak{B} \in HSP(\mathfrak{A})$ s.t. $\mathfrak{B} \models \bigvee \blacklozenge^{\leq n}\Box_\pi\bot$ for some $\pi \in$ TYPE or for some $\mathfrak{B} \in HSP(\mathfrak{A})$, $\mathfrak{B} \models \Diamond_\pi\top$ for all $\pi \in$ TYPE and $|\mathfrak{B}| = 2$. Let us consider the former case first, $\mathfrak{G}$ be any general frame s.t. $\mathfrak{B} = \mathfrak{G}^+$, t be a minimal number s.t. $\mathfrak{B} \models \bigvee \blacklozenge^{\leq t}\Box_\pi\bot$ and let $\tau \leftrightharpoons \bigvee \blacklozenge^{t}\Box_\pi\bot \wedge \bigwedge \blacksquare^{\leq t-1}\Diamond_\pi\top$. Let also $w \leftrightharpoons |\mathfrak{F}'| + 1$.

For arbitrary $I \subseteq \omega$, let $v \leftrightharpoons max\{t+1, w\}$, $\mathfrak{C}_I^-$ be the (TYPE)-oriented union $\langle \mathfrak{F}ine'_I, (\{d_0\}), ()\rangle \veebar \langle \mathfrak{G}, \tau, ()\rangle$ and $\mathfrak{F}ine'_I \leftrightharpoons \langle W', \{R'^I_\pi\}_{\pi \in \text{TYPE}}, A'\rangle$ be a modified version of the frame $\mathfrak{F}ine_I$ the reader already knows from Section 4.2. Namely, $W' \leftrightharpoons W \cup \{d_i\}_{i \in v}$, $R'^I_\pi \leftrightharpoons R^I_> \cup \{\langle a_0, d_v\rangle, \langle d_{i+1}, d_i\rangle, \langle d_0, c_0\rangle | i \in v-1\}$, for every $\pi \in$ TYPE and A' be the set of finite and cofinite subsets of W'. It is straightforward to see that $\mathfrak{F}ine'_I$ is a general frame.

Finally, let $\mathfrak{C}_I \leftrightharpoons \langle V, \{R^I_\pi\}_{\pi \in \text{TYPE}}, A\rangle$ be (TYPE, TYPE)-connected union $\langle \mathfrak{C}_I^-, (\{d_0\}), (\{d_0\})\rangle \veebar \langle \mathfrak{F}', (\{r\}), (\{\langle y_{n-1}, m-1\rangle\})\rangle$. Define the following variable-free formulas:

$$\begin{aligned}
\underline{d_0} &\leftrightharpoons \bigwedge \blacklozenge \tau,\\
\underline{d_1} &\leftrightharpoons \neg \underline{d_0} \wedge \bigwedge \blacklozenge \underline{d_0},\\
\underline{d_{k+1}} &\leftrightharpoons \neg \underline{d_k} \wedge \bigwedge \blacklozenge \underline{d_k} \wedge \bigwedge_{i<k} \neg \bigvee \blacklozenge^{\leq 1} \underline{d_i} \quad (k \in \{1, \ldots, v-1\}),\\
\underline{a_0} &\leftrightharpoons \bigwedge \blacklozenge(\underline{d_v} \wedge \bigwedge \blacklozenge(\underline{d_{v-1}} \wedge \ldots \wedge \bigwedge \blacklozenge \underline{d_0}) \ldots) \wedge\\
&\quad \bigwedge \blacksquare^2 \neg(\underline{d_v} \wedge \bigwedge \blacklozenge(\underline{d_{v-1}} \wedge \ldots \wedge \bigwedge \blacklozenge \underline{d_0}) \ldots) \wedge \bigwedge \blacksquare^{\leq v} \bigwedge \blacklozenge \top.
\end{aligned}$$

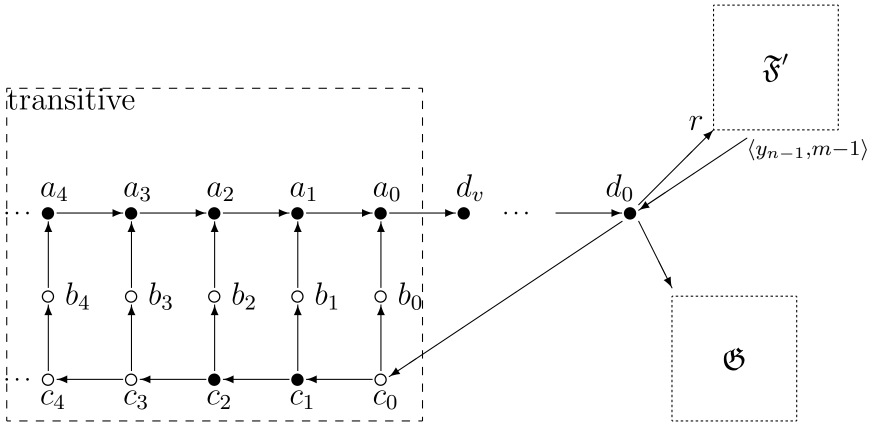

Figure 5.1: Frame $\mathfrak{C}_I$ for $I = \{0, 3, 4, \ldots\}$

Claim 4: In $\mathfrak{C}_I^+$, $\underline{a_0} = \{a_0\}$.

Proof of claim: First, observe that $\{a_0\} \subseteq \underline{a_0}$. It follows from the fact that

$$\{d_i\} \subseteq \underline{d_i} \wedge \bigwedge_{j \neq i} \neg \underline{d_j} \tag{5.1}$$

for every $i \leq v$. This also assures $\underline{a_0} \subseteq \neg\{d_0, \ldots, d_v\}$. As $W - \{a_0\} \subseteq \bigwedge \blacklozenge \underline{a_0}$ and $\underline{a_0} \leq \neg \bigwedge \blacklozenge \underline{a_0}$, $W - \{a_0\} \subseteq \neg \underline{a_0}$ as well. $\underline{a_0} \subseteq \bigwedge \blacksquare^{\leq v} \bigwedge \blacklozenge \top$ implies that for any $x \in \mathfrak{B}$, $x \subseteq \neg \underline{a_0}$. Assume then x is any element from the universe of $\mathfrak{F}'$. At most one d_i can be true at any point; thus, $x \in \underline{a_0}$ would imply the existence of π-chain of $v + 1$-points starting from x for some arbitrarily chosen $\pi \in$ TYPE. Observe that, by 5.1, the only element in the chain equal to $\{d_0\}$, the only entrance point of connected union, could be the last one. Thus, we would have a chain of v distinct elements in $\mathfrak{F}'$. But this is a contradiction with the fact that $v \geq w$. $\dashv$

Now, we use exactly the same technique as in Section 4.2. Choose arbitrary $\pi \in$ TYPE and define the following analogues of polynomials used in that section:

$$\begin{array}{lcl}\alpha_i^\pi(x) & \leftrightharpoons & \Diamond_\pi^i x \wedge \Box_\pi^{i+1}\neg x \wedge \Diamond_\pi \alpha_0^\pi(x)(i \geq 1),\\ \beta_i^\pi(x) & \leftrightharpoons & \Diamond_\pi^2 \alpha_i^\pi(x) \wedge \Box_\pi \neg \alpha_{i+1}^\pi(x) \wedge \Diamond_\pi \alpha_0^\pi(x),\\ \gamma_0^\pi(x) & \leftrightharpoons & \Diamond_\pi \beta_0^\pi(x) \wedge \Diamond_\pi \beta_1^\pi(x) \wedge \Diamond_\pi \alpha_0^\pi(x),\\ \gamma_j^\pi(x) & \leftrightharpoons & \neg\Diamond_\pi^+ \gamma_{j-1}^\pi(x) \wedge \Diamond_\pi \beta_j^\pi(x) \wedge \Diamond_\pi \beta_{j+1}^\pi(x) \wedge \Diamond_\pi \alpha_0^\pi(x) \quad (i \in \omega, j > 0).\end{array}$$

Define the following three sequences of variable-free formulas:

$$\underline{a_i} \leftrightharpoons \alpha_i^\pi(\underline{a_0}), \quad \underline{b_i} \leftrightharpoons \beta_i^\pi(\underline{a_0}), \quad \underline{c_i} \leftrightharpoons \gamma_i^\pi(\underline{a_0}) \quad (i \in \omega).$$

Claim 5: In $\mathfrak{C}_I^+$, $\underline{a_i} = \{a_i\}$, $\underline{b_i} = \{b_i\}$, $\underline{c_i} = \{c_i\}$ for arbitrary $i \in \omega$.

Proof of claim: See the proof of Theorem 4.4. ⊣

For arbitrary $I \subseteq \omega$, let $\Lambda_I \leftrightharpoons Th(\mathfrak{C}_I^+) \cap \Lambda$.

Claim 6: $\neg(\underline{c_i} \wedge \Diamond_\pi \underline{c_i} \wedge \Diamond_\pi^{v+3} \neg\varphi) \in \Lambda_I$ iff $i \notin I$.

Proof of claim: ($\Rightarrow$) The "only if" direction is proven by contraposition. If $i \in I$, then any valuation in $\mathfrak{C}_I^+$ refuting φ at r (there is such a valuation by (2) in Claim 3) refutes the formula in question, as guaranteed by Claim 5.

($\Leftarrow$). $\neg(\underline{c_i} \wedge \Diamond_\pi \underline{c_i} \wedge \Diamond_\pi^{v+3} \neg\varphi) \in \Lambda$ as $\varphi \in \Lambda$. By Claim 5, $i \in I$ implies $\mathfrak{C}_I^+ \models \neg(\underline{c_i} \wedge \Diamond_\pi \underline{c_i})$ and we are done. ⊣

Claim 7: For arbitrary $\mathfrak{D} \in \omega\mathcal{C}$, $\mathfrak{D} \in \mathbb{V}(\Lambda_I)$ iff $\mathfrak{D} \in \mathbb{V}(\Lambda)$.

Proof of claim: Λ_I is a sublogic of Λ, so it is enough to prove the "only if" direction. Assume then $\mathfrak{D} \in \mathbb{V}(\Lambda_I) \cap \omega\mathcal{C}$ and there is $\psi \in \Lambda$ s.t. $\mathfrak{D} \not\models \psi$. By the definition of Λ_I, it must be the case that $\mathfrak{C}_I^+ \not\models \psi$. Moreover, the point x where ψ is refuted cannot belong to $\mathfrak{G}$ (as $\Lambda \subseteq Th(\mathfrak{G})$) and if it belongs to $\mathfrak{F}'$, then some point from W' must be accessible from x in finitely many steps by some compound modality; otherwise, the rooted subframe generated by x would validate Λ. Thus, it can be proven there exists l s.t. for arbitrary $\sigma \in Th(\mathfrak{G})$, $\mathfrak{C}_I^+ \not\models \sigma$ iff $\bigvee \blacklozenge^{\leq l} \neg\sigma$ holds at x under some valuation; l can be taken to be $w + v + 4$ for the sake of definiteness. The following formulas hold then in $\mathfrak{C}_I^+$:

$$\begin{aligned}
\sigma &\leftrightharpoons \neg\psi \to \bigvee \blacklozenge^{\leq l}(\underline{c_0} \wedge \psi_0),\\
\chi_i &\leftrightharpoons \underline{c_i} \wedge \psi_0 \to \Diamond_\pi(\underline{c_{i+1}} \wedge \psi_0) \quad (i \in \omega),\\
\theta_{i,j} &\leftrightharpoons \underline{c_i} \wedge \psi_0 \to \neg \underline{c_j} \quad (i \neq j),\\
wgrz_\psi &\leftrightharpoons \neg(\psi_0 \wedge p \wedge \Box^+_\pi(\psi_0 \wedge p \to \Diamond_\pi(\psi_0 \wedge \neg p \wedge \Diamond_\pi(\psi_0 \wedge p)))),
\end{aligned}$$

where $\psi_0 \leftrightharpoons \Diamond_\pi \underline{a_0} \wedge \bigvee \blacklozenge^{\leq l}\neg\psi$. We assume that p does not appear in ψ; otherwise we replace it in $wgrz_\psi$ by the first variable which does not appear in ψ. That $\mathfrak{C}^+_I \models wgrz_\psi$ may be established by an argument analogous to the one in the proof of Theorem 4.4; the only additional fact one needs is that the universe of $\mathfrak{F}ine_I$ is defined by the formula $\Diamond_\pi \underline{a_0}$. In addition, as $\psi \in \Lambda$, all of σ, χ_i, $\theta_{i,j}$, $wgrz_\psi$ belong to Λ and thus to Λ_I.

By σ, $\mathfrak{V}(\underline{c_0} \wedge \bigvee \blacklozenge^{\leq l}\neg\psi) \neq \bot$ for some $\mathfrak{V}$ in $\mathfrak{D}$. Define $\mathfrak{V}'(p) \leftrightharpoons \bigvee_{i\in\omega} \mathfrak{V}(\underline{c_{2i}} \wedge \psi_0)$. If p actually appears in ψ, replace it by the first variable which does not appear in ψ. By $\{\theta_{i,j}\}_{i\neq j}$, $\bigvee_{i\in\omega} \mathfrak{V}(\underline{c_{2i}} \wedge \psi_0) \leq \neg \bigvee_{i\in\omega} \mathfrak{V}(\underline{c_{2i+1}} \wedge \psi_0)$ and by $\{\chi_i\}_{i\in\omega}$, $\bigvee_{i\in\omega} \mathfrak{V}(\underline{c_{2i}} \wedge \psi_0) \leq \Diamond_\pi \bigvee_{i\in\omega} \mathfrak{V}(\underline{c_{2i+1}} \wedge \psi_0)$ and $\bigvee_{i\in\omega} \mathfrak{V}(\underline{c_{2i+1}} \wedge \psi_0) \leq \Diamond_\pi \bigvee_{i\in\omega} \mathfrak{V}(\underline{c_{2i}} \wedge \psi_0)$. Thus, $\mathfrak{V}'$ refutes $wgrz_\psi$, a contradiction. See proof of Theorem 4.4 for a slightly simpler version of this argument.

$\dashv$

What is left is the situation when the only maximal consistent extension of Λ is the one determined by the two-element algebra with $\Diamond_\pi \top = \top$ for every $\pi \in$ TYPE, i.e., with no non-trivial variable-free formulas. Let $\mathfrak{G}$ be the single point frame where all R_π are reflexive. This time, we set $v \leftrightharpoons |\mathfrak{F}| + 1$. The definition of $\mathfrak{F}ine'_I$ for arbitrary $I \subseteq \omega$ remains unchanged. Again, $\mathfrak{C}^-_I$ is the (TYPE)-oriented union $\langle \mathfrak{F}ine'_I, (\{d_0\}), ()\rangle \veebar \langle \mathfrak{G}, \tau, ()\rangle$, where τ is this time the singleton universe of $\mathfrak{G}$. Also, $\mathfrak{C}_I \leftrightharpoons \langle V, \{R^I_\pi\}_{\pi\in \text{TYPE}}, A\rangle$ is (TYPE, TYPE)-connected union $\langle \mathfrak{C}^-_I, (\{d_0\}), (\{d_0\})\rangle \veebar \langle \mathfrak{F}', (\{r\}), (\{\langle y_{n-1}, m-1\rangle\})\rangle$ and, as before, for arbitrary $I \subseteq \omega$ let $\Lambda_I \leftrightharpoons Th(\mathfrak{C}^+_I) \cap \Lambda$.

This time, we cannot use variable-free formulas, but this obstacle may be overcome by using of a trick analogous to the one used in Chagrov and Zakharyaschev [11] or Chagrova [12]. Let $\delta \leftrightharpoons \bigvee \blacklozenge^{\leq 2*v+4}\neg s \wedge \bigvee \blacklozenge^{\leq 2*v+4} s$, where s is an arbitrary variable which does not occur in φ and define the following sequence of polynomials in s:

$$\begin{aligned}
\delta_0 &\leftrightharpoons (\bigwedge \blacklozenge \bigwedge \blacksquare^{\leq 2*v+4}\neg s \wedge \neg \bigwedge \blacksquare^{\leq 2*v+4}\neg s) \vee \\
&\quad (\bigwedge \blacklozenge \bigwedge \blacksquare^{\leq 2*v+4} s \wedge \neg \bigwedge \blacksquare^{\leq 2*v+4} s), \\
\delta_1 &\leftrightharpoons \neg\delta_0 \wedge \bigwedge \blacklozenge \delta_0, \\
\delta_{k+1} &\leftrightharpoons \neg\delta_k \wedge \bigwedge \blacklozenge \delta_k \wedge \bigwedge_{i<k} \neg \bigvee \blacklozenge^{\leq 1}\delta_i \quad (k \in \{1, \ldots, v-1\}), \\
\alpha'_0 &\leftrightharpoons \bigwedge \blacklozenge(\delta_v \wedge \bigwedge \blacklozenge(\delta_{v-1} \wedge \ldots \wedge \bigwedge \blacklozenge \delta_0)\ldots) \wedge \\
&\quad \bigwedge \blacksquare^2 \neg(\delta_v \wedge \bigwedge \blacklozenge(\delta_{v-1} \wedge \ldots \wedge \bigwedge \blacklozenge \delta_0)\ldots).
\end{aligned}$$

Claim 8: In $\mathfrak{C}_I^+$, if $\mathfrak{V}(\delta) \neq \bot$ for some $\mathfrak{V}$, then $\mathfrak{V}(\alpha'_0) = \{a_0\}$.

Proof of claim: Let $\mathfrak{V}$ be as assumed. It follows that there is at least one point in $\mathfrak{C}_I$ which belongs to $\mathfrak{V}(s)$ and at least one point in $\mathfrak{C}_I$ which belongs to $\mathfrak{V}(\neg s)$. Recall also that every point of $\mathfrak{C}_I$ is accessible from arbitrary d_i in less than $2 * v + 4$ steps. As before, the fact that $\{a_0\} \subseteq \mathfrak{V}(\alpha'_0)$ follows from $\{d_i\} \subseteq \mathfrak{V}(\delta_i \wedge \bigwedge_{j \neq i} \neg\delta_j)$ for every $i \leq v$. This also assures $\mathfrak{V}(\alpha'_0) \subseteq \neg\{d_0, \ldots, d_v\}$. As $W - \{a_0\} \subseteq \mathfrak{V}(\bigwedge \blacklozenge \alpha'_0)$, $W - \{a_0\} \subseteq \mathfrak{V}(\neg\alpha'_0)$ as well. As $\tau \in \mathfrak{V}(\neg \bigwedge \blacklozenge \delta_i)$ for arbitrary i, $\tau \in \mathfrak{V}(\neg\alpha'_0)$. The rest of the argument is exactly as in the proof of Claim 4. ⊣

Define now three sequences of polynomials

$$\alpha'_i \leftrightharpoons \alpha_i^\pi(\alpha'_0), \quad \beta'_i \leftrightharpoons \beta_i^\pi(\alpha'_0), \quad \gamma'_i \leftrightharpoons \gamma_i^\pi(\alpha'_0) \quad (i \in \omega),$$

where α_i^π, β_i^π and γ_i^π are as above.

Claim 9: Assume $\mathfrak{V}(\delta) \neq \bot$ for some $\mathfrak{V}$ in $\mathfrak{C}_I^+$. Then $\mathfrak{V}(\alpha'_i) = \{a_i\}$, $\mathfrak{V}(\beta'_i) = \{b_i\}$, $\mathfrak{V}(\gamma'_i) = \{c_i\}$ for arbitrary $i \in \omega$. What follows, $\neg(\gamma'_i \wedge \Diamond_\pi \gamma'_i \wedge \Diamond_\pi^{v+3} \neg\varphi) \in \Lambda_I$ iff $i \notin I$.

Proof of claim: Follows from Claim 8 by using the same reasoning as in the proofs of Claims 5 and 6. ⊣

Claim 10: For arbitrary $\mathfrak{D} \in \omega\mathcal{C}$, $\mathfrak{D} \in \mathbb{V}(\Lambda_I)$ iff $\mathfrak{D} \in \mathbb{V}(\Lambda)$.

Proof of claim: Again, it is enough to prove the "only if" direction. Assume $\mathfrak{D} \in \mathbb{V}(\Lambda_I) \cap \omega\mathcal{C}$ and there is $\psi \in \Lambda$ s.t. $\mathfrak{D} \not\models \psi$. By the definition of Λ_I, it must be the case that $\mathfrak{C}_I^+ \not\models \psi$. As in the proof of Claim 7, the point x where ψ is refuted cannot belong to $\mathfrak{G}$ (as $\Lambda \subseteq Th(\mathfrak{G})$) and if it belongs to $\mathfrak{F}'$, then some point from $\mathfrak{G}$ must be accessible from x in less than by v steps by some compound modality. The following formulas hold then in $\mathfrak{C}_I^+$:

$$
\begin{aligned}
\rho' &\leftrightharpoons \neg\psi \to \bigvee \blacklozenge^{\leq 2*v+4}\psi, \\
\sigma' &\leftrightharpoons \neg\psi \wedge \delta' \to \bigvee \blacklozenge^{\leq 2*v+4}(\gamma_0' \wedge \psi_0'), \\
\chi_i &\leftrightharpoons \gamma_i' \wedge \psi_0' \to \Diamond_\pi(\gamma_{i+1}' \wedge \psi_0') \quad (i \in \omega), \\
\theta_{i,j} &\leftrightharpoons \gamma_i' \wedge \psi_0' \to \neg\gamma_j' \quad (i \neq j), \\
wgrz_\psi &\leftrightharpoons \neg(\psi_0' \wedge p \wedge \Box_\pi^+(\psi_0' \wedge p \to \Diamond_\pi(\psi_0' \wedge \neg p \wedge \Diamond_\pi(\psi_0' \wedge p)))),
\end{aligned}
$$

where $\psi_0' \leftrightharpoons \Diamond_\pi\alpha_0' \wedge \bigvee \blacklozenge^{\leq 2*v+4}\neg\psi \wedge \delta'$. We assume that neither p nor s appear in ψ. Thus, if $\mathfrak{V}(\psi) \neq \top$ for some $\mathfrak{V}$, then by ρ', $\mathfrak{V}'(s) = \mathfrak{V}(\neg\psi)$ implies $\mathfrak{V}'(\neg\psi) \leq \mathfrak{V}'(\delta')$. Therefore, σ' gives that $\mathfrak{V}'(\psi_0' \wedge \gamma_0') \neq \bot$ for any such $\mathfrak{V}'$ and we can repeat the reasoning from the proof of Claim 7. $\dashv$

$\dashv$

5.3 Degrees of $\mathcal{AV} \cup \mathcal{T}$-incompleteness

Theorem 5.13 *If $\Lambda \in Ext\mathbf{K}_{\mathrm{TYPE}}$ is not a join-splitting, then*

$$|\mathsf{Spec}(\mathcal{AV} \cup \mathcal{T})(\Lambda)| = 2^{\aleph_0}.$$

Sketch of proof: For $\mathcal{T}$-BAOs and the lattice of unimodal logics, this observation can be found in Zakharyaschev et al. [60]. The technique used there is to replace $\mathfrak{F}ine_I$ from the proof in the previous section by a suitable modification of the algebra $\mathfrak{v}B$ from Section 4.4. The previous section have shown how to generalize this argument to the lattice of all modal logic in a given finite similarity type. Finally, by exactly the same argument as in Section 4.4, the continuum of logics obtained in the proof share with Λ not only the same class of $\mathcal{T}$-BAOs, but also the same class of $\mathcal{AV}$-BAOs. One may actually use the modification of $\mathfrak{v}B$ as given in original Blok's proof [4] instead of those used in [60], but this is a technical detail of minor importance. ⊣

5.4 Notes

The notion of *degree of (Kripke) incompleteness* of a given logic was introduced by Fine [18]. Blok [4] was an answer to Fine's challenge: given a logic Λ, characterize its degree of $\mathcal{CAV}$-incompleteness. His work dealt with the lattice of unimodal logics. Dziobiak [16] tried to generalize The Blok Alternative for degrees of $\mathcal{CA}$-completeness. He obtained some significant results, but failed to prove it in the general case by the use of Blok's methods. The reasons of this problem are discussed in Chapter 6.

For many years, Blok's result circulated as a technical report. It was finally published in Chagrov and Zakharyachev [11]. Their proof was different from that of Blok himself and easier to generalize. Chagrova [12] used their method to prove the Blok Alternative for degrees of $\mathcal{CA}$-completeness, twenty years after Dziobiak's attempt. Zakharyaschev et al. [60] contained an observation that The Blok Alternative is provable for degrees of $\mathcal{T}$-completeness.

All the papers mentioned above dealt with the lattice of unimodal logics. Kracht [36], [37] observed that the "positive" part of Blok's result, i.e., the characterization of join-splittings of $Ext\mathbf{K}_{\mathrm{TYPE}}$ and the proof that they have f.m.p. (he did not mention f.e.p.), can be generalized to arbitrary similarity types. Zakharyaschev et al. [60] contains the observation that The Blok Alternative itself can be generalized to arbitrary similarity types. The observation is given without a proof and without references to the literature. One can guess that the tool the authors had in mind must have been related to the Maximal Logic Theorem 5.8. Nevertheless, the present author was not able to find the Maximal Logic Theorem in the existing literature. At any rate, it is a corollary of Theorem 5.6 by Kowalski et al. [35] The proof in Kracht [37] was mistaken, but the one in the above-mentioned reference appears to be correct.

To the author's knowledge, The Blok Alternative has never been formulated and proven in such a generality as in the above chapter. Nevertheless, in view of what was written above, author's claims to originality are relatively restricted. The observation that The Blok Alternative can be generalized to degrees of $\mathcal{AV}$-incompleteness seems new, but the frame used in the proof is van Benthem [53] frame: the same as in the proofs in Blok [4], Chagrov et al. [11] and Zakharyaschev et al. [60]. Thus, the main new achievement of this chapter is perhaps the The Blok Alternative for degrees of $\omega\mathcal{C}$-completeness. It generalizes significantly the $\mathcal{CA}$-result of Chagrova [12]. In fact, the structure used in the present proof is also simpler that that of Chagrova.

It is an open question whether one can prove The Blok Alternative for degrees of $\mathcal{A}$-completeness. It could be possible by the use of construction in Venema [55] or the one from Section 4.1. On the other hand, Theorem 2.18 may suggest that $\mathcal{A}$-completeness behaves somewhat better than other completeness notions.

Needless to say, as we don't have any example of $\mathcal{V}$-incomplete logic, the question whether The Blok Alternative can be generalized for degrees of $\mathcal{V}$-completeness is also open.

The results of the present chapter are going to be published in Litak [VI].

Chapter 6

Subdirectly irreducible algebras

6.1 Subdirect indeterminacy of $\mathcal{C}$

As was mentioned, our proof of The Generalized Blok Alternative follows rather Chagrov-Zakharyaschev style proofs (cf. [11], [60] etc.) than the original proof of Blok [4]. The reason lies in his heavy use of the fact that for every logic Λ, the class of Kripke frames in $\mathbb{V}(\Lambda)$ is determined by its subdirectly irreducible elements; in symbols, $HSP(\mathbb{V}(\Lambda)\cap\mathcal{CAV}) = HSP(\mathbb{V}(\Lambda)\cap\mathcal{SCAV})$. In yet another notation, $\mathsf{Log}\mathcal{CAV}(\Lambda) = \mathsf{Log}(\mathcal{CAV}\cap\mathcal{S})(\Lambda)$. There is, in general, no reason why this should hold for arbitrary class of algebras. In fact, Dziobiak [16] admitted that this is the source of difficulties one encounters in attempts to generalize The Blok Alternative to $\mathcal{CA}$-algebras by using Blok's methods. The question whether $\mathsf{Log}\mathcal{CA}(\Lambda) = \mathsf{Log}(\mathcal{CA}\cap\mathcal{S})(\Lambda)$ posed there remained unanswered. Here, we are going to answer it in the negative and the answer provides us with an interesting example of an incomplete logic in the basic modal similarity type over **4**. This is the type we are going to work with in this section.

Theorem 6.1 *There is a $\mathcal{CA}$-BAO $\mathfrak{S}IC$ s.t.*

$$Th(\mathfrak{S}IC) \subsetneq \mathsf{Log}(\omega\mathcal{C}\cap\mathcal{S})(Th(\mathfrak{S}IC)).$$

Proof: The proof of the theorem is the first occasion for us to actually use *normal neighbourhood frames*, although the reader met them disguised in Section 4.4. Recall that a neighbourhood frame $\mathfrak{F} \leftrightharpoons \langle W, \mathbb{N}ghb\rangle$ consists of a universe W and a function $\mathbb{N}ghb$ from W to 2^{2^W}; $\mathbb{N}ghb(w)$ is usually denoted as $\mathbb{N}ghb_w$. The dual algebra of $\mathfrak{G}$ is the powerset algebra of W with the operator defined as $\Box_{\mathbb{N}ghb}X \leftrightharpoons \{x \in W | X \in \mathbb{N}ghb_x\}$. In general, the only restriction imposed on $\mathbb{N}ghb$ is that $\mathbb{N}ghb_x$ is a filter in the powerset algebra; if one's goal is to provide semantics for non-normal modal logic, even that assumption is dropped. In the case we are interested, W is defined as

$$W \leftrightharpoons \{a_n, b_n \,|\, n \in \omega\} \cup \{c_{ij} \,|\, j \in \omega - \{0\}, 1 \leq i \leq j\} \cup \{d\}.$$

Ξ is the set of all cofinite subsets of the set of natural numbers without 0.

$$N \in \mathbb{N}ghb_x \quad \text{iff} \quad \begin{cases} N \supseteq \{a_i | 0 \leq i \leq n-1\} & : \quad x = a_n; \\ N \supseteq \{b_n\} \cup \{a_i | i \leq n\} & : \quad x = b_n; \\ N \supseteq \{c_{kj} | i \leq k \leq j\} \cup & \\ \cup \{b_k | i \leq k \leq j\} \cup \{a_k | k \leq j\} & : \quad x = c_{ij}; \\ \exists K \in \Xi \; N \supseteq \{d\} \cup & \\ \cup \{c_{ik} | k \in K, 1 \leq i \leq k\} \cup \{a_n, b_n | n \in \omega\} & : \quad x = d. \end{cases}$$

Figure 6.1 provides some insight concerning the structure of $\mathfrak{G}IC$.
Let $\mathfrak{G}IC$ be the dual of $\mathfrak{F}$.

Claim 1: $\mathfrak{G}IC$ is not subdirectly irreducible.

Proof of claim: Note that, for example, $A \leftrightharpoons W - \{d\}$ and $B \leftrightharpoons W - \{c_{11}\}$ are both coatoms of 2^W such that $A = \Box_{\mathbb{N}ghb}A$ and $B = \Box_{\mathbb{N}ghb}B$. $\dashv$

In the remaining part of the proof, we drop the subscript $\mathbb{N}ghb$ from the modal operator. The following formulas are of importance.

$$\begin{aligned} \alpha_0 &\leftrightharpoons \Box\bot, \qquad \alpha_{n+1} \leftrightharpoons \Diamond\alpha_n \wedge \neg\Diamond\Diamond\alpha_n, \\ \beta_n &\leftrightharpoons \Diamond\Diamond\alpha_n \wedge \neg\Diamond\alpha_{n+1}, \\ \gamma_{n+1} &\leftrightharpoons \neg\Diamond\beta_n \wedge \Diamond\beta_{n+1} \wedge \Diamond\beta_{n+2}, \\ \delta &\leftrightharpoons \Diamond\beta_0 \wedge \Diamond\beta_1, \\ \zeta_{n+1} &\leftrightharpoons \delta \to \Diamond\gamma_{n+1} \wedge \Box^+(\gamma_{n+1} \to \Diamond\gamma_{n+2}) \qquad (n \in \omega), \\ \eta &\leftrightharpoons \Box^+(\Diamond^+\delta \to p) \vee \Box^+(\Diamond^+\delta \to \neg p) \end{aligned}$$

(recall that $\Box^+\varphi \leftrightharpoons \varphi \wedge \Box\varphi$ and $\Diamond^+\varphi \leftrightharpoons \varphi \vee \Diamond\varphi$). α's and β's look familiar; indeed, we could actually define them by means of polynomials from Section 4.2. In analogy with that section, too, one may prove

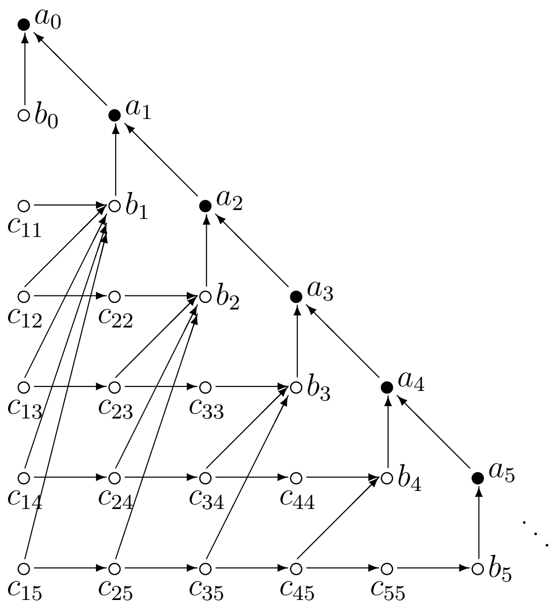

Figure 6.1: A relational frame corresponding to the upper part of $\mathfrak{G}IC$

Claim 2: In $\mathfrak{G}IC$, $\alpha_n = \{a_n\}$, $\beta_0 = \{b_0\}$, $\beta_m = \{b_m, c_{im} | 1 \leq i \leq m\}$, $\gamma_m = \{c_{mj} | j > m\}$, $\Diamond\gamma_n = \{d\} \cup \{c_{ij} | i \leq m, j > m\}$ and $\delta = \Diamond\delta = \{d\}$, for any $n \in \omega$ and any $m \geq 1$.

Claim 3: $\mathfrak{G}IC \models \mathbf{4}$.

Claim 4: For any $n \geq 1$, $\mathfrak{G}IC \models \zeta_n$.

Proof of claim: We have shown that $\delta = \{d\} \subseteq \Diamond\gamma_n$ and $\neg\{c_{nn+1}\} = \neg\gamma_n \vee \Diamond\gamma_{n+1}$. The set $W - \{c_{nn+1}\}$ belongs to $\mathbb{N}ghb_d$ and hence $\{d\} \subseteq \Box^+(\gamma_n \to \Diamond\gamma_{n+1})$. $\dashv$

Claim 5: $\mathfrak{G}IC \models \eta$.

Proof of claim: As follows from Claim 2, for any valuation $\mathfrak{V}$ on $\mathfrak{G}IC$, $\mathfrak{V}(\Diamond^+\delta) = \{d\}$. Therefore, $\mathfrak{V}(\Diamond^+\delta) \subseteq \mathfrak{V}(p)$ or $\mathfrak{V}(\Diamond^+\delta) \subseteq -\mathfrak{V}(p)$. $\dashv$

Claim 6: $\mathfrak{G}IC \models \mathbf{wgrz}$.

Proof of claim: For any $x \in W - \{d\}$ and any valuation $\mathfrak{V}$, it may be proven that $x \in \mathfrak{V}(\mathbf{wgrz})$ in the same way as for relational structures.

Assume now that for some $\mathfrak{V}$, $d \in \mathfrak{V}(\neg\mathbf{wgrz})$. This means that $d \in \mathfrak{V}(q)$. Let $M \leftrightharpoons \mathfrak{V}(p \to \Diamond(\neg p \wedge \Diamond p))$. Then $M \in \mathbb{N}ghb(d)$ and $M \cap \mathfrak{V}(\neg p \wedge \Diamond p) \neq \emptyset$. But $\mathfrak{V}(\neg p \wedge \Diamond p) \cap \mathfrak{V}(p) = \bot$ and hence there is $u \in (M - \{d\}) \cap \mathfrak{V}(\neg p \wedge \Diamond p)$. It follows that there is $v \in M - \{d\}$ such that $u \in \Diamond v$ and $v \in \mathfrak{V}(q)$. But $M \in \mathbb{N}ghb_v$ and hence $v \in \mathfrak{V}(\neg\mathbf{wgrz})$, which contradicts the observation from the beginning of our proof. ⊣

Now assume $\mathfrak{A} \in \omega\mathcal{C} \cap \mathcal{S}$, $\mathfrak{A} \vDash \mathbf{4} \wedge \eta \wedge (\delta \leftrightarrow \Diamond\delta)$, $\mathfrak{A} \vDash \zeta_n$ for any $n \geq 1$ and $\mathfrak{A} \nvDash \neg\delta$. Choose any $* \neq \top$ as in Lemma 2.6; as we are above **4**, it is equivalent to assuming that for any $x \neq \top$, $x \wedge \Box x \leq *$.

Claim 7: In $\mathfrak{A}$, $\delta = \Diamond\delta$ is an atom.

Proof of claim: Assume $\bot < x < \delta$. Let $\mathfrak{V}(p) \leftrightharpoons x$. Then, as $\Diamond^+\delta \nleq x$, $\mathfrak{V}(\Box^+(\Diamond^+\delta \to x)) \leq *$; in the same way, we prove $\mathfrak{V}(\Box^+(\Diamond^+\delta \to \neg x)) \leq *$ and thus $\mathfrak{V}(\eta) \leq *$, a contradiction. ⊣

Claim 8: $\mathfrak{A} \vDash \gamma_n \to \Diamond\gamma_{n+1}$, for arbitrary $n \geq 1$.

Proof of claim: Assume for some n and some $\mathfrak{V}$, $c \leftrightharpoons \mathfrak{V}(\gamma_n \to \Diamond\gamma_{n+1}) \neq \top$. $\neg\delta$ is a coatom whose principal filter is open and $\neg c \leq \neg\delta$. On the other hand, ζ_n guarantees that the open filter generated by c is contained in the principal filter generated by δ. Thus, we have two non-trivial open filters whose intersection is $\{\top\}$ and $*$ can belong to at most one of them; a contradiction. ⊣

The stage has been set for the same sort of trick we already saw in the proof of Theorem 4.4 and Theorem 5.10. The reader may guess that $\bigvee_{n\in\omega} \gamma_{2n} \leq \neg \bigvee_{n\in\omega} \gamma_{2n+1} \wedge \Diamond \bigvee_{n\in\omega} \gamma_{2n+1}$, $\bigvee_{n\in\omega} \gamma_{2n+1} \leq \neg \bigvee_{n\in\omega} \gamma_{2n} \wedge \Diamond \bigvee_{n\in\omega} \gamma_{2n}$ and thus

Claim 9: $\mathfrak{A} \nvDash \mathbf{wgrz}$.

The theorem follows. ⊣

6.2 Positive results

Thus, the question arises: why are $\mathcal{CAV}$-BAOs so well-behaved? Can we generalize the Birkhoff theorem for some other interesting classes of algebras? Here again, the class $\mathcal{AV}$ turns out to be the good guy.

Theorem 6.2 *Any $\mathfrak{A} \in \mathcal{AV}$ is a subdirect product of its complete morphic images.*

From that, one immediately obtains

Corollary 6.3 *Let $\mathcal{X} \subseteq \mathcal{AV}$ be closed under complete morphisms. Then* $\mathsf{Log}\mathcal{X}(\Lambda) = \mathsf{Log}(\mathcal{X} \cap \mathcal{S})(\Lambda)$, *i.e., the class of $\mathcal{X}$-BAOs in any variety is determined by its subdirectly irreducible members.*

Proof of Theorem 6.2: Let $x \in \mathfrak{A}$. As $\mathfrak{A} \in \mathcal{V}$, $\mathsf{Root}(x)$ is a complete filter by Fact 2.9 and $\mathfrak{A}/\mathsf{Root}(x)$ is a complete morphic image of $\mathfrak{A}$.

Claim 1: If $x \in At\mathfrak{A}$, $\mathfrak{A}/\mathsf{Root}(x)$ is subdirectly irreducible.

Proof of claim: We claim that $\neg x/\mathsf{Root}(x)$ is an opremum of $\mathfrak{A}/\mathsf{Root}(x)$. Assume $y/\mathsf{Root}(x) \neq \top/\mathsf{Root}(x)$. It is equivalent to $y \notin \mathsf{Root}(x)$, which in turn is equivalent to existence of $n \in \omega$ s.t. $x \not\leq \bigwedge \blacksquare^{\leq n} y$. As x is an atom, it is equivalent to $\bigwedge \blacksquare^{\leq n} y \leq \neg x$, as desired. $\dashv$

Thus, to prove that the mapping $f : \mathfrak{A} \mapsto \prod_{x \in At\mathfrak{A}} \mathfrak{A}/\mathsf{Root}(x)$ defined by $f(y)(x) \leftrightharpoons y/\mathsf{Root}(x)$ is a subdirect embedding, it is enough to show that for any $y \neq \bot$, $f(y) \neq \bot$. But for any atom x below y, $f(y)(x) \neq \bot$. $\dashv$

6.3 $\mathcal{S}$ and master modality based on well-order

This section proves a somewhat surprising result about subdirectly irreducible BAOs in varieties which have a universal modality based on a well-order. To make this statement precise, say that a logic Λ in TYPE has *a linear-order-based (LOB) universal modality* if there are $>, < \in$ TYPE s.t.

- $\Diamond_>$ and $\Diamond_<$ are conjugates;

- $\bigvee_{\rho \in \mathrm{TYPE}} \Diamond_\rho p \to \Diamond_> p \vee p \vee \Diamond_< p$;
- $\mathbf{Lin}^{>}_{<} \in \Lambda$.

All logic studied in Chapter 8 have a LOB universal modality. We say that Λ has *a well-order-based (WOB) universal modality* if it has a LOB universal modality and, in addition, $\mathbf{GL}_> \in \Lambda$. In the unimodal similarity type, there is only one logic with a WOB universal modality: it is **Ver**. But in polymodal similarity types, this notion becomes far more interesting and complicated. We have already encountered an example of a logic with a WOB universal modality: it was the logic discussed in Section 4.3. Logics with a WOB universal modality form an important subclass of logics studied in Chapter 8

Lemma 6.4 *Assume Λ has a linear-order based universal modality and $\mathfrak{A} \in \mathbb{V}(\Lambda) \cap \mathcal{S}$ (i.e., $\mathfrak{A}$ is a subdirectly irreducible algebra for Λ), $a \in \mathfrak{A}$ and $\bot \neq a < \Box_> \neg a$. Then a is an atom.*

Proof: Let $\mathfrak{A}$ and a be as in the assumptions of the theorem. Assume a is not an atom; it is equivalent to existence of $b \neq \bot$ and $c \neq \bot$ s.t. $b \vee c \leq a$ and $b \wedge c = \bot$. As $a \leq \Box_> \neg a \wedge \Box_< \neg a$ (by properties of conjugates) and $\Box_> \neg a \wedge \Box_< \neg a \leq \Box_> \neg b \wedge \Box_< \neg b$ and thus $c \leq \neg(\Diamond_> b \vee b \vee \Diamond_< b)$. It means $o \neq \top$, where $o \leftrightharpoons \Diamond_> b \vee b \vee \Diamond_< b$. On the other hand, $b \leq o$ and thus $o \neq \bot$. It is enough to show now that the principal filter generated by o is open to obtain a contradiction; as we are in a discriminator variety, any subdirectly irreducible algebra is simple. Because universal modality is LOB, we only have to prove $o \leq \Box_> o$ and $o \leq \Box_< o$. We show only the first inequality, the second being entirely symmetric. $\Diamond_< b \leq \Box_> \Diamond^2_< b \leq$ (by transitivity) $\Box_> \Diamond_< b \leq \Box_> o$. $\Diamond_> b \leq \Box_> o$ iff $\Diamond_< \Diamond_> b \leq o$; the latter is true by linearity. Finally, $b \leq \Box_> \Diamond_< b \leq \Box_> o$. Thus, $o \leq \Box_> o$. $\dashv$

We are ready now for the main

Theorem 6.5 *Assume Λ has a well-order based universal modality and $\mathfrak{A} \in \mathbb{V}(\Lambda) \cap \mathcal{S} \cap \mathcal{C}$ (i.e., $\mathfrak{A}$ is a complete and subdirectly irreducible algebra for Λ). Then $\mathfrak{A} \in \mathcal{CA}$. If, in addition, every operator has a conjugate, then $\mathfrak{A} \in \mathcal{CAV}$, i.e. it is the dual algebra of a Kripke frame.*

Proof: We need a slight modification of the Ordinal Slicing Lemma (Lemma 4.5). Namely, we show that for every $\mathfrak{A}$ satisfying the assumptions of the theorem there is a cardinal κ no greater than the cardinality of $\mathfrak{A}$ and a sequence $\{a_\lambda\}_{\lambda\in\kappa}$ of elements of $\mathfrak{A}$ satisfying the following conditions:

1. $\forall\lambda \in \kappa$, $a_\lambda \neq \bot$ and $a_\lambda \wedge \Diamond_{>} a_\lambda = \bot$;
2. $\forall\lambda \in \kappa$, $n \geq 1$, $a_\lambda \leq \Diamond_{<} a_{\lambda+n}$ if $\lambda + n < \kappa$;
3. $\forall\mu < \lambda \in \kappa$, $a_\mu \wedge (a_\lambda \vee \Diamond_{>} a_\lambda) = \bot$;
4. $\forall\lambda \in \kappa$, $a_\lambda \leq \Box_{>} \bigvee\limits_{\mu\in\lambda} a_\mu$
5. $\bigvee\limits_{\lambda\in\kappa} a_\lambda = \top$.

The difference now is that we don't assume κ is a limit ordinal, as we don't have **D**. This entails the difference in formulation of 2; this is also the reason why we cannot use simply the Ordinal Slicing Lemma in its original shape. Of course, another option would be simply to stipulate that the logic contains $\mathbf{D}_<$, i.e., to restrict our attention to serial-well-ordered based similarity types. It would have no negative consequences for intended applications of the theorem but it would unnecessarily limit its generality.

Once the described modification of the Ordinal Slicing Lemma is proven, the theorem follows. For, by Lemma 6.4, every element of the sequence is an atom (by 1) and thus, by 5 there is a family of atoms whose supremum is $\top$, i.e., the algebra is atomic.

The necessary modifications in the proof of the new version of Lemma 4.5 are few. In the formulation of the second claim one needs to assume in addition that $x \wedge \Diamond_{<}\top \neq \bot$; as by Lemma 6.4, x is an atom, $x \leq \Diamond_{<}\top$ and the proof of the claim goes through as before. In the transfinite induction, we accept the possibility the whole construction may stop after the initial step; then we simply set $\kappa \leftrightharpoons 1$. In the successor step, we may proceed only if $a_\iota \wedge \Diamond_{<}\top \neq \bot$. But it is not true, then we may stop the induction and set $\kappa = \iota'$. For this means that $a_\mu \leq \Box_{<}\bot$ and thus $\Diamond_{>} a_\mu = \bot$. By simplicity of $\mathfrak{A}$, $\top = \Diamond_{<} a_\iota \vee a_\iota \vee \Diamond_{>} a_\iota = \Diamond_{<} a_\iota \vee a_\iota$. But by 4, $\Diamond_{<} a_\iota \leq \bigvee\limits_{\mu\in\iota} a_\mu$ and thus we get $\top = \bigvee\limits_{\mu\leq\iota} a_\mu$, as desired. $\dashv$

The intuitive meaning of the above theorem is that if a logic in a conjugated similarity type has a well-order based universal modality, any lattice-complete and subdirectly irreducible algebra is the dual of a relational structure whose universe may be taken to be an ordinal with transitive closure of $\in$ interpreting $\Diamond_<$ and its converse interpreting $\Diamond_>$. To appreciate that this is by no means a common situation, consider the following simple counterexample.

Example 6.6 *Let $\mathfrak{O}$ be as in Section 4.1, i.e., the boolean algebra of regular open sets from* $\mathbf{I}$ *and let $\Diamond$ be the unary boolean discriminator on $\mathfrak{O}$. It is a subdirectly irreducible $\mathcal{CV}$-BAO — indeed, even a simple one — with a LOB universal modality but with no atoms at all.*

Thus, Theorem 6.5 seems to be of some independent mathematical interest. But there was a particular reason to prove it. It is going to become clear in the next chapter.

6.4 Notes

All results in the present chapter seem new. Section 6.1 answers in the negative Dziobiak's [16] question whether the following conjecture holds:

> A logic is complete with respect to neighbourhood semantics iff it is complete with respect to a class of subdirectly irreducible neighbourhood algebras.

The logic determined by $\mathfrak{S}IC$ is an extension of **4** which is $\mathcal{CA}$-complete, but $\mathcal{CAV}$-incomplete. The first example of such a logic was found by Gerson [24].

Theorem 6.2 and Corollary 6.3 generalize Lemma 4.1 in Blok [5].

Chapter 7

Strong completeness

Until now, we have been concerned mainly with the notion of weak $\mathcal{X}$-completeness. This chapter studies the notion of strong $\mathcal{X}$-completeness.

7.1 Local and global consequence

Recall that in Section 2.3 we introduced a number of completeness notions: $\mathcal{X}$-complexity, strong $\mathcal{X}$-completeness, strong local $\mathcal{X}$-completeness, (weak) $\mathcal{X}$-completeness. In general, those notions are pairwise non-equivalent, but every two of them are comparable.

Theorem 7.1 *(1) $\mathcal{X}_\kappa$ complexity implies strong global $\mathcal{X}_\kappa$-completeness.*

(2) Strong global $\mathcal{X}_\kappa$-completeness implies strong local $\mathcal{X}_\kappa$-completeness.

(3) Strong local $\mathcal{X}_\kappa$-completeness implies weak $\mathcal{X}$-completeness for any $\kappa \geq \omega$.

Proof: (1) It was proven in Section 2.3.

(2) Follows by Fact 2.4.

(3) Obvious.

$\dashv$

The natural question to answer now is for what classes of algebras $\mathcal{X}$ one is allowed to reverse the above implications.

Theorem 7.2 *(1) If $\mathcal{X}$ is closed under products, then strong $\mathcal{X}_\kappa$-completeness implies $\mathcal{X}_\kappa$-complexity.*

(2) If $\mathcal{X} \subseteq \mathcal{V}$ and $\mathcal{X}$ is closed under complete-morphic images, then strong local $\mathcal{X}_\kappa$-completeness implies strong global $\mathcal{X}_\kappa$-completeness.

(3) If $\mathcal{X}$ is closed under ultraproducts, then weak $\mathcal{X}$-completeness implies strong global $\mathcal{X}_\kappa$-completeness.

Proof: (1) Already proven in Section 2.3.

(2) Assume $\alpha \notin \mathsf{MN}_\kappa(\Lambda, \Gamma)$, where $\Lambda, \Gamma \subseteq \text{FORM}_\kappa$ and $\alpha \in \text{FORM}_\kappa$. By Fact 2.2, this is equivalent to $\alpha \notin \mathsf{M}_\kappa(\Lambda, \blacksquare^{\leq\omega}\Gamma)$ and this again to $\bot \notin \mathsf{M}_\kappa(\Lambda, \blacksquare^{\leq\omega}\Gamma \cup \{\neg\alpha\})$. By strong local completeness, there is an algebra $\mathfrak{A}$, a valuation $\mathfrak{V}$ and $x \in \mathfrak{A}$ s.t. $x \leq \blacksquare^{\leq\omega}\Gamma \cup \{\neg\alpha\})$. By Fact 2.9, complete additivity implies that $\mathsf{Root}(x)$ is a complete filter; by closure of $\mathcal{X}$ under complete morphisms we obtain that $\mathfrak{A}/\mathsf{Root}(x)$ is a $\mathcal{X}$-BAO with a valuation which sends Γ to $\top$ and α to an element distinct from the $\top$.

(3) An analogue of compactness theorem for first-order logic; cf. Corollary 4.1.11 in Chang and Keisler [13].

⊣

In this way, we obtained an explanation of Theorem 1.4.1 in [57], which states that for relational structures, strong local completeness, strong global completeness and complexity are the same thing.

Corollary 7.3 (Wolter [57], Theorem 1.4.1) *Strong local $\mathcal{CAV}$-completeness is equivalent to $\mathcal{CAV}$-complexity.*

Proof: The class of $\mathcal{CAV}$-BAOs is a class of $\mathcal{V}$-BAOs closed under products and complete morphic images. ⊣

The gap between weak Kripke completeness and strong Kripke completeness, exhibited by Theorems 2.14 and 2.15, is obviously caused by the fact that the class of $\mathcal{CAV}$-BAOs is not closed under ultraproducts. However, there is a class of BAOs for which all those notions collapse.

Corollary 7.4 *A logic is weakly $\mathcal{AV}$-complete iff it is $\mathcal{AV}$-complex.*

Proof: $\mathcal{AV}$ is a class of $\mathcal{AV}$-BAOs closed under products, complete morphisms and ultraproducts (the last by its being first-order definable). ⊣

Thus, we obtained another proof of a theorem proven by ten Cate, Litak [VII] that for discrete frames, strong and local completeness coincide. The above corollary seems to be interesting in its own right; in contrast to many results proven before, it seems to be a definitely 'positive' one. It comes as no surprise that yet again it is $\mathcal{AV}$-completeness which allows us to obtain positive results. To see how $\mathcal{AV}$-completeness differs here from other notions, let us consider the following

Theorem 7.5 (Shehtman [45], Theorem 3.1) *Every $\mathcal{CAV}$-complete logic in the basic modal similarity type which is an extension of* **4** *is strongly locally $\mathcal{CA}$-complete.*

Corollary 7.6 *For any $\mathcal{X}$ between $\mathcal{C}$ and $\mathcal{CA}$,* **GL** *is strongly locally $\mathcal{X}$-complete but not strongly $\mathcal{X}$-complete. The same holds for any Kripke complete logic between* **GL** *and* **GL.3***.*

Proof: Strong local $\mathcal{X}$-completeness by Theorem 7.5. Failure of strong $\mathcal{X}$-completeness by Theorem 2.15. $\dashv$

7.2 Application: modal definability for discrete frames

This section ties together several threads. We are going to apply the duality theory developed in Chapter 3 and characterization of strong completeness to obtain an algebraic proof of an analogue of the Goldblatt-Thomason Theorem (cf. Theorem 5.54 in Blackburn et al. [3]) for discrete frames. It is telling that the result reported below seems actually more elegant than its original version for Kripke frames. Once again, it can be concluded that the class of $\mathcal{AV}$-BAOs is a *very* well-behaved one. The result contained here were proven first in ten Cate, Litak [VII] by non-algebraic methods.

The crucial result may be actually formulated in a purely algebraic language; later, we use duality to obtain a frame-theoretical corollary in the spirit of the Amsterdam school. Say that a class K of $\mathcal{X}$-BAOs is *definable* in $\mathcal{X}$ if there is a set of formulas Λ s.t. $K = \mathbb{V}(\Lambda) \cap \mathcal{X}$. It is easy to see that this condition is in fact equivalent to $K = \mathbb{V}(Th(K)) \cap \mathcal{X}$.

Theorem 7.7 *Assume that $\mathcal{X} \subseteq \mathcal{V}$ is closed under complete-morphic images, products and ultraproducts. A class K of $\mathcal{X}$-BAOs is definable in $\mathcal{X}$ iff it is closed under the same operations and, in addition, $S(K) \cap \mathcal{X} \subseteq K$.*

Proof: The "only if" direction is straightforward; all varieties satisfy those closure conditions. Conversely, assume $\mathfrak{A} \in HSP(K) \cap \mathcal{X}$. By assumptions on K and Theorem 7.2, $HSP(K) = S(K)$. Hence, $\mathfrak{A} \in S(K) \cap \mathcal{X}$. But then $\mathfrak{A} \in K$. ⊣

Thus, this is in fact a straightforward corollary of Theorem 7.2. But by using duality, it may be turned into a nice result about modal definaility of classes of discrete frames. Say that a class of discrete frames K is *elementary* if there is a set of sentences Γ in two-sorted first order language supplied with binary relational constants $\{R_\pi\}_{\pi \in \mathrm{TYPE}}$ s.t. for every discrete frame $\mathfrak{F}$, $\mathfrak{F} \in K$ iff $\mathfrak{F} \vDash \Gamma$ (the definition of satisfiaction here being, of course, the standard first-order satisfaction). Analogously, a class of discrete frames K is *modally definable* if there is a set of modal formulas Λ s.t. for every discrete frame $\mathfrak{F}$, $\mathfrak{F} \in K$ iff $\mathfrak{F}^* \vDash \Lambda$; in other words, the dual class of $\mathcal{AV}$-BAOs is definable in $\mathcal{AV}$.

Theorem 7.8 (with ten Cate) *Every modally definable class of discrete frames is elementary.*

Proof: Cf. ten Cate, Litak [VII]. Sketch: use *the standard translation* of modal formulas (cf. Definition 2.45 in Blackburn et al. [3]) and prefix the output with universal quantifers for individual and set variables (remember, this is a two-sorted first-order language, not the second-order one). ⊣

Elementarity is, obviously, intimately related to closure under ultraproducts. An *ultraproduct* of general frames is an ultraproduct in the sense of many-sorted first-order logic. Every elementary class is closed under ultraproducts. It may be proven that $(\prod_U \mathfrak{F}_i)^+$ is isomorphic to $\prod_U \mathfrak{F}_i^+$, where the latter is, as may be expected, ultraproduct of algebraic structures. Similarly, it may be proven that the dual of a disjoint union of general frames is isomorphic to the product of respective dual algebras. Thus, we have the following analogue of Goldblatt-Thomason theorem:

Theorem 7.9 (with Balder ten Cate) *A class of discrete frames is modally definable iff it is elementary, closed under point-generated subframes, dimorphic images and disjoint unions.*

Proof: Follows from Theorems 7.7, 7.8 and $(\,)^\circ$,$(\,)_\circ$-duality between:

- point-generated subframes and complete-morphic images;
- disjoint unions and products;
- di-morphic images and subalgebras.

$\dashv$

7.3 Second-order character of $\mathcal{C}$-consequence

The Polish school of logic often views a logic as a (structural) consequence relation rather than a set of theorems. To avoid confusion, in this paragraph we denote the former as *logic*^{P} and the latter *logic*^{S}. Thus, with every system which is a *logic*^{S} in our sense — i.e., set of sentences closed under MN and substitution — one may associate two *logics*^{P} in the sense of the Polish school; the one given by the induced M-consequence ($\mathsf{M}_\Lambda(\Gamma) = \mathsf{M}(\Lambda, \Gamma)$) and the one given by the induced MN-consequence ($\mathsf{MN}_\Lambda(\Gamma) = \mathsf{MN}(\Lambda, \Gamma)$). Nevertheless, it should be clear by now that these are by no means the only possible ones. Given a *logic*^{S} Λ, every class of algebras $\mathcal{X}$ induces two *logics*^{P} over Λ (in the language with countably many variables): $\mathcal{X}_\Lambda(\Gamma) \coloneqq \mathcal{X}(\Lambda, \Gamma)$ and $\mathcal{X}^l_\Lambda(\Gamma) \coloneqq \mathcal{X}^l(\Lambda, \Gamma)$.

Of course, if Λ is strongly $\mathcal{X}$-complete (strongly locally $\mathcal{X}$-complete), then $\mathcal{X}_\Lambda(\,)$ ($\mathcal{X}^l_\Lambda(\,)$) coincides with $\mathsf{MN}_\lambda(\,)$. But it by no means always the case, as was already shown by Theorem 2.15. The aim of this section is to prove that some of those algebraically motivated *logics*^{P} associated with a given *logic*^{S} are very strong; indeed, in fact as strong as second-order logic. To make the meaning of the above precise, let us say that second-order logic in the language with countably many individual and monadic variables and a single binary relation constant is *embeddable* into $\mathcal{X}_\Lambda(\,)$ (or $\mathcal{X}^l_\Lambda(\,)$) iff there is a recursive function f from the set of second-order formulas into the set of modal formulas in a given similarity type s.t. $\alpha \in \mathsf{SO}(\Gamma)$ (where SO is the closure of Γ under second-order consequence, Γ is a set of second-order formulas and α — a second-order formula) iff $f(\alpha) \in \mathcal{X}_\Lambda(f[\Gamma])$ ($f(\alpha) \in \mathcal{X}^l_\Lambda(f[\Gamma])$).

A well-known paper of S.K. Thomason [52] provides a reduction of second-order logic to modal logic over Kripke frames. What Thomason had in mind, is not exactly the translation into $\mathcal{CAV}$-consequence, neither global nor local. He showed that there exists a finitely axiomatizable logic Λ in a similarity type with 30 operators and a translation f s.t. $\alpha \in \mathsf{SO}(\Gamma)$ iff $f(\alpha) \in \mathsf{Log}\mathcal{CAV}(\Lambda \cup f[\Gamma])$. Nevertheless, building on his construction one can show that

Theorem 7.10 *There is a finitely axiomatizable logic Λ s.t. the second-order consequence is embeddable into $\mathcal{SC}_\Lambda()$ (or $\mathcal{SC}^l_\Lambda()$).*

Sketch of proof: A careful examination of Thomason [52] reveals three surprising facts. First, his translation uses only variable-free formulas. Second, all operators in Λ are conjugate. Third, his logic has a WOB universal modality. These facts allow us to

(1) move from validity in a frame/algebras for Λ to satisfaction in a frame/algebra for Λ and

(2) use Theorem 6.5 and move from $\mathcal{CAV}$ or $\mathcal{CA}$ to $\mathcal{SC}$.

A detailed proof is moved to Appendix B for two reasons. First, the author's own contribution is summed up adequately by the above short description; the rest belongs to Thomason. Second, the proof requires many pages of axioms, auxiliary formulas, definitions and abbrevations, which would spoil the continuity of the thesis.

$\dashv$

Note here that for any $\mathcal{X}$ satisfying the assumptions of Corollary 6.3, there is no difference between strong $\mathcal{X}$-completeness and strong $\mathcal{SX}$-completeness; $\mathcal{X}$-consequence and $\mathcal{SX}$-consequence coincide for any logic Λ. Nevertheless, it does not have to be true for, e.g., $\mathcal{C}$ even if all operators are conjugated, i.e., all algebras in $\mathbb{V}(\Lambda)$ are $\mathcal{T}$-BAOs; at least the author has no idea how to prove it even if a logic has a WOB universal modality. It would be very nice if one could remove subdirect irreducibility from the formulation of Theorem 7.10.

An obvious consequence of Theorem 7.10 is that the logic Λ is $\mathcal{SC}$-incomplete in a very strong sense. Deductions in Λ are compact; second-order logic (and thus $\mathsf{Log}\mathcal{SC}(\Lambda)$) has no reasonable notion of deduction, the set of theorems is far beyond the whole arithmetical hierarchy …

Chapter 8

Tense logics of linear time flows

8.1 Motivation

The preceding chapters, in particular Chapter 5 have shown that even for classes of algebras significantly greater than $\mathcal{CAV}$ one cannot hope for general completeness results — that is, as long as we consider the mammoth lattice $Ext\mathbf{K}_{\mathrm{TYPE}}$. But we have also learned, again and again, that $\mathcal{AV}$ is a very well-behaved class of algebras. Thus, it would be nice if we could prove a general $\mathcal{AV}$-completeness result for some smaller lattice of logics. And it would be even better still if:

- some of logics in the lattice were not Kripke complete — thus showing that studying the notion of $\mathcal{AV}$-completeness can give wider completeness results that studying Kripke completeness;
- we could prove some other nice general result (e.g., concerning decidability or complexity) about this lattice of logics — thus proving once again that $\mathcal{AV}$-completeness tends to go hand in hand with other desirable properties.

And, almost magically, there are indeed such classes of logics and such results. In the lattice of extensions of *tense logic of linear time* (i.e., extensions of **Lin**):

(1) all logics are $\mathcal{AV}$-complete, even though many of them are $\mathcal{C}$-inconsistent (and thus, a fortiori, Kripke-incomplete);

(2) all finitely axiomatizable logics and all prime logics display very good computational behaviour: they are coNP-complete.

None of results in the present chapter could have been obtained with Frank Wolter. Some were obtained jointly; others rely crucially on his techniques and methods.

8.2 $\mathcal{AV}$-completeness result

This section is based on results from Wolter [59] and uses the notation of Zakharyaschev et al. [60] and Litak, Wolter [IV]: the reader is invited to consult the latter work. In the whole chapter, we set TYPE $\leftrightharpoons \{<,>\}$. All logics we are interested in are taken to be extensions of $\mathbf{Lin}^{>}_{<}$; for simplicity, we drop both $<$ and $>$. Instead of proving $\mathcal{AV}$-completeness directly, we use duality results and prove that all extensions of **Lin** are complete with respect to discrete frames.

Lemma 8.1 *Take any* $\mathfrak{A} \in \mathbb{V}(\mathbf{Lin}) \cap \mathcal{S}$ *and let* $\langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}}, A\rangle \leftrightharpoons \mathfrak{A}_+$. $R_<$ *(and thus* $R_>$*) is* a linear ordering*: i.e.,* $R_<$ *is transitive and connected:* $\forall x \forall y (xRy \vee yRx \vee x = y)$. *If, in addition,* $\mathfrak{A} \in \mathcal{AV}$*, then* $\mathfrak{A}_*$ *is linearly ordered as well.*

Proof: Transitivity of $\mathfrak{A}_+$ follows from canonicity of **4**. As for linearity, $\mathbb{V}(\mathbf{Lin})$ is discriminator, thus every subdirectly irreducible algebra is simple; hence, for every $a \neq \bot$, $\Diamond_< a \vee a \vee \Diamond_> a = \top$. Assume $\Gamma, \Delta \in Uf\mathfrak{A}$ and NOT $(\Gamma R_< \Delta \,\mathrm{OR}\, \Gamma = \Delta)$. It means there is $\delta_0 \in \Delta$ s.t. $\Diamond_< \delta_0 \notin \Gamma$. Take any $\delta_1 \in \Delta - \Gamma$. Thus, $\delta_0 \wedge \delta_1 \in \Delta - \Gamma$ and $\Diamond_<(\delta_0 \wedge \delta_1) \notin \Gamma$. By simplicity of $\mathfrak{A}$, $\Diamond_>(\delta_0 \wedge \delta_1) \in \Gamma$. Assume there is any $\delta_2 \in \Delta$ s.t. $\Diamond_> \delta_2 \notin \Delta$ and let $\delta \leftrightharpoons \delta_0 \wedge \delta_1 \wedge \delta_2$. $\delta \in \Delta$ and yet $\Diamond_> \delta \vee \delta \vee \Diamond_< \delta \notin \Gamma$, a contradiction. We have proven that if NOT $(\Gamma R_< \Delta \,\mathrm{OR}\, \Gamma = \Delta)$, then $\Gamma R_> \Delta$. As $R_>$ and $R_<$ are converses of each other, the theorem follows. The proof for $\mathfrak{A}_*$ is straightforward. ⊣

Thus, we may restrict our attention to linear general frames (with $R_<$ and $R_>$ being converses of each other); unless stated otherwise, it is tacitly assumed in the whole chapter. Given a finite sequence

$$\overline{\mathfrak{F}} \leftrightharpoons \left\langle \mathfrak{F}_i \leftrightharpoons \left\langle W_i, \{R^i_<, R^i_>\}, A_i \right\rangle : 1 \leq i \leq n \right\rangle$$

of disjoint frames, we denote by $[\overline{\mathfrak{F}}] \leftrightharpoons \mathfrak{F}_1 \lhd \cdots \lhd \mathfrak{F}_n$ the *ordered sum* of them, i.e., the frame $\langle W, R, A \rangle$ in which

$$W \leftrightharpoons \bigcup_{i=1}^{n} W_i,\ R^i_< \leftrightharpoons \bigcup_{i=1}^{n} R^i_< \cup \bigcup_{1 \le i < j \le n} (W_i \times W_j),\ R^i_> \leftrightharpoons (R^i_<)^{-1}$$

and $P \leftrightharpoons \{X_1 \cup \cdots \cup X_n : X_i \in A_i\}$. Each finite frame can be represented then as the ordered sum $C_1 \lhd \cdots \lhd C_n$ of its clusters (i.e., sets of the form $\{w\} \cup \{v \mid wRv \text{ and } vRw\}$). A cluster is called *degenerate* iff it consists of a single irreflexive point. Note here that the ordered sum of two frames is a particular example of *a connected sum* introduced in Chapter 5.

Fix two symbols m and j. A *cluster assignment* is a mapping $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ from the set of clusters of a finite frame into $\{\mathsf{m}, \mathsf{j}\} \times \{\mathsf{m}, \mathsf{j}\}$ such that every cluster consisting of an irreflexive point is mapped to (m, m) and no cluster consisting of more than one point is mapped to (j, j).

With every finite linear frame $\mathfrak{F} \leftrightharpoons \langle W, R \rangle \leftrightharpoons C_1 \lhd \cdots \lhd C_n$ with cluster assignment $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ we associate the formula

$$\alpha(\mathfrak{F}, \mathbf{t}) \leftrightharpoons \neg\overline{\delta}(\mathfrak{F}, \mathbf{t}),$$

where

$$\overline{\delta}(\mathfrak{F}, \mathbf{t}) \leftrightharpoons \delta(\mathfrak{F}, \mathbf{t}) \wedge \Box_< \delta(\mathfrak{F}, \mathbf{t}) \wedge \Box_> \delta(\mathfrak{F}, \mathbf{t})),$$

and

$$\begin{aligned}
\delta(\mathfrak{F}, \mathbf{t}) \quad = \quad & \bigwedge\{p_x \to \neg p_y \mid x \neq y\} \wedge \\
& \bigwedge\{p_x \to \Diamond_< p_y \mid (xR_< y) \& \text{NOT}\,(yR_< x) \\
& \bigwedge\{p_x \to \neg\Diamond_> p_y \mid \text{ NOT}\,(yR_< x)\} \wedge \\
& \bigwedge\{p_x \to \Diamond_< p_y \mid \exists i \le n\ (\mathbf{t}_1 C_i = \mathsf{m} \& x, y \in C_i \& xR_< y)\} \wedge \\
& \bigwedge\{p_x \to \Diamond_> p_y \mid \exists i \le n\ (\mathbf{t}_2 C_i = \mathsf{m} \& x, y \in C_i \& xR_> y)\} \wedge \\
& \bigvee\{p_y \mid y \in W\} \wedge \\
& \bigwedge\{\Diamond_< p_y \vee p_y \vee \Diamond_> p_y \mid y \in W\},
\end{aligned}$$

where $p_x \neq p_y$ if $x \neq y$. Observe that this is a variant of Jankov formulas.

The following result is a straightforward extension of Theorem 1 in Wolter [59]; cf. Theorem 3.1 in Litak, Wolter [IV].

Theorem 8.2 *There is an algorithm which, given a formula φ, returns formulas*

$$\alpha(\mathfrak{F}_1, \mathbf{t}_1), \ldots, \alpha(\mathfrak{F}_n, \mathbf{t}_n)$$

such that $\mathsf{Log}(\{\mathbf{Lin}, \varphi\}) = \mathsf{Log}(\{\mathbf{Lin}, \alpha(\mathfrak{F}_1, \mathbf{t}_1), \ldots, \alpha(\mathfrak{F}_n, \mathbf{t}_n)\})$ *and* $l(\mathfrak{F}_i) \leq 2l(\varphi) + 1$ *and* $cl(\mathfrak{F}_i) \leq l(\varphi)$.

Now we are going to define a family of descriptive frames which allows for a general completeness result. All frames in the family turn out to be of the form $(\mathcal{F}^+)_+$, for a discrete frame $\mathcal{F}$. To use the terminology of ten Cate [49], they are *strongly descriptive frames*. Thus, it follows that all **Lin**-logics are complete with respect to discrete frames.

Definition 8.3 (Atomic blocks) *(1) Denote by* $\textcircled{k}$ *the non-degenerate cluster with* $k > 0$ *points, by* $\bullet$ *the irreflexive point and by* $\circ$ *(or* $\textcircled{1}$ *) the reflexive point.*

(2) Let $\omega^<(0)$ *be the strictly ascending chain* $\langle \omega, < \rangle$ *of natural numbers,* $\omega^<(1)$ *the chain* $\langle \omega, \leq \rangle$, $\omega^<(2)$ *the ascending chain of natural numbers in which the even points are the reflexive ones,* $\omega^<(3)$ *the chain in which a point is reflexive iff it is a multiple of* 3 *and so on;* $\omega^>(n)$ *is the mirror image of* $\omega^<(n)$.

(3) For $0 < n < \omega$, $k \geq 1$ *and* $\textcircled{k} = \{a_0, \ldots, a_{k-1}\}$, *let*

$$\mathfrak{C}(n, \textcircled{k}) \leftrightharpoons \langle \omega^<(n) \lhd \textcircled{k}, A \rangle,$$

where $\textcircled{k} = \{a_0, \ldots, a_{k-1}\}$ *and* A *is generated using intersection and complementation from the set of all finite subsets of* $\omega^<(n)$ *together with the sets* $\{X_i : 0 \leq i \leq k-1\}$, *for* $X_i \leftrightharpoons \{a_i\} \cup \{kj + i : j \in \omega\}$, $0 \leq i \leq k-1$.

We call the cluster $\textcircled{k}$ *in* $\mathfrak{C}(n, \textcircled{k})$ *the* (m, j)*-cluster of* $\mathfrak{C}(n, \textcircled{k})$.

(4) For $n < \omega$, *let* $\mathfrak{C}(\textcircled{k}, n)$ *denote the mirror image of* $\mathfrak{C}(n, \textcircled{k})$.

(5) Let $\mathfrak{C}(0, \textcircled{1}, 0) = \langle \omega^<(0) \lhd \textcircled{1} \lhd \omega^>(0), A \rangle$, *where* A *consists of all finite sets which do not contain* $\textcircled{1}$ *and their complements.*

We call the cluster $\textcircled{1}$ *in* $\mathfrak{C}(0, \textcircled{1}, 0)$ *the* (j, j)*-cluster of* $\mathfrak{C}(0, \textcircled{1}, 0)$.

By Blocks we denote the family of all of frames of the form (1) or (3)-(5) in Definition 8.3. Blocks^* is the class of all finite sequences $\overline{\mathfrak{G}}$ from Blocks. In addition, $\mathsf{Blocks}_0 \leftrightharpoons \mathsf{Blocks} - \{\mathfrak{C}(0, \textcircled{1}, 0)\}$.

The following result is Theorem 12 of Wolter [59]; cf. also Theorem 3.3 in Litak, Wolter [IV].

Theorem 8.4 *Every $\bigcap$-irreducible tense logic is determined by a frame of the form $[\overline{\mathfrak{G}}]$ for some $\overline{\mathfrak{G}} \in \mathsf{Blocks}^*$.*

Given a sequence $\overline{\mathfrak{G}}$ in Blocks^*, we denote by $w(\overline{\mathfrak{G}})$ the maximal n such that a frame of the form $\mathfrak{C}(n, \textcircled{k})$ or $\mathfrak{C}(\textcircled{k}, n)$ occurs in $\overline{\mathfrak{G}}$. If no such frame occurs in $\overline{\mathfrak{G}}$, then set $w(\overline{\mathfrak{G}}) \risingdotseq 0$.

The following completeness result is Wolter's [59], Theorem 12 (i):

Theorem 8.5 *Every $\Lambda \in Ext\mathbf{Lin}$ is complete with respect to some family of frames from $[\mathsf{Blocks}^*]$, i.e., a family of ordered unions of frames from Blocks.*

For finitely axiomatizable logics we can prove even more.

Theorem 8.6 *Suppose*

$$\Lambda = \mathsf{Log}(\{\mathbf{Lin}, \alpha(\mathfrak{F}_1, \mathbf{t}_1), \ldots, \alpha(\mathfrak{F}_n, \mathbf{t}_n)\})$$

and $\varphi \notin \Lambda$. Then $\mathfrak{G}^+ \nvDash \varphi$, where $\mathfrak{G} \risingdotseq [\mathfrak{G}_1 \lhd \cdots \lhd \mathfrak{G}_m] \in [\mathsf{Blocks}_0^*]$ such that

(a) $\mathfrak{G}^+ \vDash \Lambda$;

(b) $m \leq (2l(\varphi) + 1) * (2 + \max\{l(\mathfrak{F}_i) \mid 1 \leq i \leq n\})$;

(c) $w(\mathfrak{G}) \leq 2 + \max\{l(\mathfrak{F}_i) \mid 1 \leq i \leq n\}$;

(d) $cl(\mathfrak{G}) \leq l(\varphi)$.

Proof: See Wolter [59], Theorem 14 (i) or Theorem 3.4 of Litak, Wolter [IV]. $\dashv$

It is fairly easy to deduce from Theorem 8.5 that all logics in **Lin** are $\mathcal{AV}$-complete. First observe that duals of all frames in Blocks are atomic algebras. As we are dealing with logics with conjugated operators, those duals must be in fact $\mathcal{AV}$-BAOs or even $\mathcal{AT}$-BAOs. This, together with the fact that an ordered sum of discrete frames is a discrete frame implies

Corollary 8.7 *All logics in Ext**Lin** are $\mathcal{AT}$-complete.*

As all frames in Blocks are quite simple, we can describe more exactly how suitable discrete frames — or $\mathcal{AT}$-BAOs — look like.

Theorem 8.8 *(1) For arbitrary n and k, $\mathfrak{C}(n, \textcircled{\scriptsize k}\,) = (\mathfrak{D}(n, \textcircled{\scriptsize k}\,)^+)_+$, where*

$$\mathfrak{D}(n, \textcircled{\scriptsize k}\,) \leftrightharpoons \langle \omega^<(n), A \rangle,$$

is a discrete frame, where A is generated using intersection and complementation from the set of all finite subsets of $\omega^<(n)$ together with the sets $\{X'_i : 0 \leq i \leq k-1\}$, for $X'_i \leftrightharpoons \{kj + i : j \in \omega\}$, $0 \leq i \leq k-1$.

(2) For arbitrary n and k, $\mathfrak{C}(\textcircled{\scriptsize k}\,, n) = (\mathfrak{D}(\textcircled{\scriptsize k}\,, n)^+)_+$, where $\mathfrak{D}(\textcircled{\scriptsize k}\,, n)$ is a mirror image of $\mathfrak{D}(n, \textcircled{\scriptsize k}\,)$.

(3) $\mathfrak{C}(0, \textcircled{\scriptsize 1}\,, 0) = (\mathfrak{D}(0, \textcircled{\scriptsize 1}\,, 0)^+)_+$, where $\mathfrak{D}(0, \textcircled{\scriptsize 1}\,, 0) \leftrightharpoons \langle \omega^<(0) \lhd \omega^>(0), A \rangle$, where A consists of all sets which are either finite or cofinite.

Proof: We are going to prove only (1), as the proofs for (2) and (3) are similar. All we have to show is that $\mathfrak{D}(n, \textcircled{\scriptsize k}\,)^+$ has only k non-principal ultrafilters, which form a cluster in $\mathfrak{D}(n, \textcircled{\scriptsize k}\,)$ and for every principal ultrafilter is a $R_<$-precedessor of all non-principal ultrafilters but no non-principal ultrafilter is a $R_<$ precedessor of any principal ultrafilter. This all follows from

Claim 1: Every nonprincipal ultrafilter of $\mathfrak{D}(n, \textcircled{\scriptsize k}\,)$ is of the form $\Xi_i \leftrightharpoons \{Y \in \omega^<(n) | |X'_i - Y| < \omega\}$ for some $i < k$.

Proof of claim: The proof for $k = 1$ is trivial, so we can assume $k > 1$. First, we have to show that every Ξ_i is an ultrafilter. It is closed under finite intersections and upward closed: if $Y \subseteq Z$ and $|X_i - Y| < \omega$, then of course $|X_i - Z| < \omega$ belong to Z. Assume now $Y \notin \Xi_i$; it means that $|X_i - Y| = \omega$. Y is either finite (in which case $\neg Y \in \Xi_i$) or else $Y = Z \cup \bigcup_{l \leq k} (Y_{l1} - Y_{l2})$ for some $k \neq 0$ s.t. Z and all Y_{l2}'s are finite, and for all l, Y_{l1} is equal either to $\neg X_i$ or to X_j for some $j \neq i$. For every l, $(Y_{l1} - Y_{l2})$ contains only finitely many elements of X_i. Thus, $\neg Y \in \Xi_i$. Ξ_i is an ultrafilter.

Assume now Γ is any filter extending the filter of all cofinite sets and that for every i there exists $Y_i \in \Gamma - \Xi_i$. By the above, for every i, $\neg Y_i \in \Xi_i$. Thus for every $i \in k$, $Z = \bigcup_{j \in k} \neg Y_j \in \Xi_i$. But this means Z is cofinite and yet $\neg Z \in \Gamma$; thus, Γ is not a proper filter. $\dashv$

$\dashv$

If DiscreteBlocks is the family consisting of an irreflexive point, finite clusters and discrete frames defined in points (1)-(3) of Theorem 8.8, then Theorems 8.5 and 8.8 imply that all logics in $Ext\mathbf{Lin}$ are complete with respect to some class of finite ordered sums of frames from DiscreteBlocks.

8.3 $\omega\mathcal{C}$-inconsistency once again

In this section, we are going to prove new ωC-incompleteness results by sharpening results of Wolter [59].

Theorem 8.9 *Assume* $\mathfrak{A} \leftrightharpoons [\overline{\mathfrak{G}}]^+$ *for some* $\overline{\mathfrak{G}} \in \mathsf{Blocks}^*$ *and* $Th(\mathfrak{A})$ *has a WOB universal modality, i.e.* $\mathfrak{A} \vDash \mathbf{GL}_\pi$*, for some* $\pi \in \{<, >\}$*. Then either* $\mathfrak{A}$ *is finite or* $Th(\mathfrak{A})$ *is* $\omega\mathcal{C}$*-inconsistent.*

The author is convinced that theorem can be proven without restricting the additional assumption about a WOB universal modality. Nevertheless, technical difficulties similar to those encountered in an attempt to prove Theorem 7.10 without the assumption of subdirect irreducibility has prevented him from proving it in full generality.

Proof: Without loss of generality, assume $\mathfrak{A} \vDash \mathbf{GL}_>$. Then in the sequence $\overline{\mathfrak{G}}$, there can be no reflexive points, no clusters, no frames of the form $\mathfrak{C}(ⓚ, n)$ for arbitrary $k, n \in \omega$ and no frames $\mathfrak{C}(n, ⓚ)$ for $n \geq 1$. Thus, if $\mathfrak{A}$ is not finite, there must be a chain $\{a_n\}_n \in \omega$ of irreflexive points in $[\overline{\mathfrak{G}}]$ s.t. a_0 has no $R_<$-precedessor and for every i and a_{i+1} is the immediate successor of a_i. Define a sequence of variable-free formulas, called *numerals*: $\underline{i}_\pi \leftrightharpoons \Diamond^i_\pi \top \wedge \Box^{i+1}_\pi \bot$ $(i \in \omega)$. The following is immediate.

Claim 1: (1) $\mathfrak{A} \vDash \underline{0}_> \vee \Diamond_> \underline{0}_>$,
(2) $\mathfrak{A} \vDash \underline{i}_> \to \Diamond_< \underline{i+n}_>$ $\quad (i \in \omega, n > 0)$,
(3) $\underline{i}_> \to \neg \underline{j}_> \in \mathsf{Log}(\mathbf{Lin})$ for every $i \neq j$.

Let k be maximal s.t. $\mathfrak{G}(0, ⓚ) \in \overline{\mathfrak{G}}$ and let $n \leftrightharpoons k+1$.

Claim 2: $\mathfrak{A} \vDash \mathbf{nocycle}^n_<$.

Let $\mathbf{InfAsc}_>$ be the logic axiomatized by the variable-free formulas from statements (1), (2) and (3) of Claim 1. By Claim 1, $\mathfrak{A} \models \mathbf{InfAsc}_>$.

Claim 3: Let $\mathfrak{B} \in \mathbb{V}(\mathbf{InfAsc}_>) \cap \omega\mathcal{C}$. $\mathfrak{B} \not\models \mathbf{nocycle}^m_<$, for any $m \in \omega$.

Proof of claim: Similar proofs appeared several times in this thesis. By Claim 1, for every $i < j$, $\underline{i}_> \neq \bot$ and $\underline{i}_> \leq \neg\underline{j}_> \wedge \Diamond_< \underline{j}_>$. For $i \leq m$, let $\mathfrak{V}(p_i) = \bigvee_{l \in \omega} \bigvee_{j<m, j\neq i} \underline{m * l + j}_>$. It follows that for every $i < m$, $\mathfrak{V}(\bar{p}^i_m) = \bigvee_{l\in\omega} \underline{m * l + i}_>$ and hence $\mathfrak{V}(\bar{p}^i_m \to \Diamond_< \bar{p}^{i+1}_m) = \top$. Thus, $\mathfrak{V}(\mathbf{nocycle}^m_<) = \neg\mathfrak{V}(\bar{p}^0_m) \neq \top$. ⊣

The theorem follows. ⊣

It should be observed that the theorem holds for any $\mathfrak{A} \leftrightharpoons [\overline{\mathfrak{G}}]$ s.t. $\overline{\mathfrak{G}} \in \mathsf{Blocks}^*$ and $\mathfrak{A} \models \mathbf{InfAsc}_>$. It is always the case when the first (leftmost) element of the sequence $\overline{\mathfrak{G}}$ is either $\mathfrak{C}(0, \textcircled{k})$ or $\mathfrak{C}(0, \textcircled{1}, 0)$. The theorem above covers, for example, all nonfinite maximal consistent extensions of **Lin**.

The theorem can be used to prove the following analogue of the Blok Theorem, which strengthens Proposition 16 in Wolter [59].

Theorem 8.10 *Assume $\Lambda \leftrightharpoons Th(\mathfrak{A})$ for $\mathfrak{A} \in \mathbb{V}(\mathbf{Lin}) \cap \mathcal{S}$, i.e., Λ is determined by a single subdirectly irreducible (simple, as $\mathbb{V}(\mathbf{Lin})$ is discriminator) algebra — for example, $\mathfrak{A}$ is a dual of a rooted Kripke frame or Λ is a $\bigcap$-irreducible logic. There exist continually many logics in $\mathsf{Spec}_{\omega\mathcal{C}}(\Lambda) \cap Ext\mathbf{Lin}$; i.e., the degree of $\omega\mathcal{C}$-incompleteness of Λ with respect to $Ext\mathbf{Lin}$ is equal to continuum.*

Proof: For every $m \in \omega$, define the Kripke frame $m^<$ as $m + 1$ with $\in$ interpreting $R_<$ and $\in^{-1}$ interpreting $R_>$. Define also $\mathbf{E}\varphi \leftrightharpoons \Diamond_< \varphi \vee \varphi \vee \Diamond_> \varphi$ and $\mathbf{A}\varphi \leftrightharpoons \neg\mathbf{E}\neg\varphi$. There are two cases we have to consider.

(I) For every $n \in \omega$, $\mathbf{E}\underline{n}_> \in \Lambda$. For arbitrary $M \subseteq \omega$, let $\Lambda_M \leftrightharpoons \Lambda \cap Th(\{\mathfrak{C}(0, \textcircled{k}) \lhd m^< | m \in M\})$. As for every $m \in \omega$, $\mathbf{E}\underline{m}_> \to \mathbf{E}\underline{m+1}_> \in \Lambda_M$ iff $m \notin M$, there are continually many distinct Λ_M. $\mathbb{V}(\Lambda) \subseteq \mathbb{V}(\Lambda_M)$. Assume $\omega\mathcal{C}$-BAO $\mathfrak{B}$ belongs to $\mathbb{V}(\Lambda_M) - \mathbb{V}(\Lambda)$. Then there is $\varphi \in \Lambda$ s.t. $\mathfrak{B} \not\models \varphi$, hence $\varphi \notin \Lambda_M$.

Claim 1: (1) $\neg\varphi \to \mathbf{A}(\underline{0}_> \vee \lozenge_> \underline{0}_>) \in \Lambda_M$,

(2) $\neg\varphi \to \mathbf{A}(\underline{i}_> \to \lozenge_< \underline{i+n}_>) \in \Lambda_M$, for $(i \in \omega, n > 0)$,

(3) $\mathbf{E}\neg\varphi \to \mathbf{nocycle}^2_< \in \Lambda_M$, where all variables in $\mathbf{nocycle}^2_<$ are substituted s.t. $\mathbf{nocycle}^2_<$ has no variables in common with φ.

(4) $\neg\varphi \to \mathbf{A}(\underline{i}_> \to \mathbf{E}\neg\varphi) \in \Lambda_M$

(5) $\underline{i}_> \to \neg\underline{j}_> \in \mathsf{Log}(\mathbf{Lin})$ for every $i \neq j$.

Observe that in point (3), we could use $\mathbf{wgrz}_<$ instead, as everywhere until now. Let $\mathfrak{V}$ be any valuation in $\mathfrak{B}$ s.t. $\mathfrak{V}(\varphi) \neq \top$ and let $\mathfrak{C} \leftrightharpoons \mathfrak{B}/\mathsf{Root}(\mathfrak{V}(\neg\varphi))$. As we are working in a conjugated modal similarity type, root filters are complete and thus $\mathfrak{C} \in \omega\mathcal{C}$. By Claim 1, $\mathfrak{C} \models \mathbf{InfAsc}_<$. Let $\mathfrak{V}'$ be a valuation defined as $\mathfrak{V}(p)/\mathsf{Root}(\mathfrak{V}(\varphi))$ for p's appearing in φ and in the same way as in the proof of Theorem 8.9 for p's appearing in $\mathbf{nocycle}^2_<$. Again, it may proven as in the proof of Theorem 8.9 that $\mathfrak{V}'(\mathbf{nocycle}^2_<) = \neg\mathfrak{V}'(\bar{p}^0_m)$. But $\mathfrak{V}(\bar{p}^0_m) \not\leq \neg\mathfrak{V}(\mathbf{E}\neg\varphi)$ by (4) and thus $\mathfrak{V}'(\mathbf{E}\neg\varphi) \not\leq \mathfrak{V}'(\mathbf{nocycle}^2_<)$, a contradiction with (3).

(II) For some $n \in \omega$, $\mathbf{E}\underline{n}_> \notin \Lambda$. As $\Lambda = Th(\mathfrak{A})$, it means $\mathfrak{A} \not\models \mathbf{E}\underline{n}_>$. But as $\mathfrak{A}$ is a discriminator algebra, it means that $\mathbf{A}\neg\underline{n}_> \in \Lambda$. Then for arbitrary $M \subseteq \omega - \{0, \ldots, n\}$ and every $m \in \omega - \{0, \ldots, n\}$, $\mathbf{E}\underline{m}_> \to \mathbf{E}\underline{m+1}_> \in \Lambda_M$ iff $m \notin M$ and the proof can be carried out as in the case of (I). $\dashv$

8.4 Aside on computational complexity

In this section, we are going to provide the promised computational complexity result. We prove that:

- given some $[\overline{\mathfrak{G}}]$ with $\overline{\mathfrak{G}} \in \mathsf{Blocks}^*$, it is decidable in non-deterministic polynomial time whether a formula φ is satisfiable in $[\overline{\mathfrak{G}}]$;
- it can be checked in non-deterministic polynomial time whether a formula φ is satisfiable in a frame $\mathfrak{G}$ satisfying constraints (b)-(d) of Theorem 8.6;
- it can be checked in polynomial time (in the length of φ) whether a frame $\mathfrak{G}$ satisfying the constraints (b)-(d) validates a finitely axiomatizable logics.

Given a finite frame $\mathfrak{F} = C_1 \lhd \cdots \lhd C_n$, we write

$$((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n))$$

for $(\mathfrak{F}, \mathbf{t})$. In addition, if

$$(\mathfrak{F}, \mathbf{t}) = \overbrace{(\mathfrak{G}, \mathbf{t}') \lhd \cdots \lhd (\mathfrak{G}, \mathbf{t}')}^{n \text{ times}},$$

we denote it by $(\mathfrak{F}, \mathbf{t}) = [(\mathfrak{G}, \mathbf{t}')]^n$.

Now define *the m-reduct* $r_m(\mathfrak{G})$ of $\overline{\mathfrak{G}} \in \mathsf{Blocks}^*$ as follows.

Definition 8.11 (m-reduct, canonical embedding) *(1) For $\mathfrak{F}$ a cluster, $r_m(\mathfrak{F}) \leftrightharpoons (\mathfrak{F}, \mathbf{t})$, where $\mathbf{t}$ assigns (m, m) to the cluster.*

(2) $r_m(\mathfrak{C}(0, ⓚ)) \leftrightharpoons [(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1} \lhd (ⓚ, (\mathsf{m}, \mathsf{j}))$. The embedding r_m from the underlying frame of $r_m(\mathfrak{C}(0, ⓚ))$ to $\mathfrak{C}(0, ⓚ)$ which maps $[(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1}$ onto an initial segment of $\omega^{<}(0)$ and ⓚ onto ⓚ is called the canonical embedding *of $r_m(\mathfrak{C}(0, ⓚ))$ into $\mathfrak{C}(0, ⓚ)$.*

(3) For $n > 0$, $r_m(\mathfrak{C}(n, ⓚ)) \leftrightharpoons [(\circ, (\mathsf{m}, \mathsf{m})) \lhd [(\bullet, (\mathsf{m}, \mathsf{m}))]^{n-1}]^{m+1} \lhd (ⓚ, (\mathsf{m}, \mathsf{j}))$. The canonical embedding r_m from $r_m(\mathfrak{C}(n, ⓚ))$ into $\mathfrak{C}(n, ⓚ)$ is defined as in (2) above.

(4) For $n \geq 0$, the m-reduct of $\mathfrak{C}(ⓚ, n)$ as well as the canonical embedding r_m are the the mirror images of the definition under (2) and (3).

(5) $r_m(\mathfrak{C}(0, ①, 0)) \leftrightharpoons [(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1} \lhd (\circ, (\mathsf{j}, \mathsf{j})) \lhd [(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1}$. The canonical embedding r_m from the underlying frame of $r_m(\mathfrak{C}(0, ①, 0))$ to $\mathfrak{C}(0, ①, 0)$ maps the initial $[(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1}$ to the initial segment of $\mathfrak{C}(0, ①, 0)$, ① to ①, and the final $[(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1}$ to the final segment of $\mathfrak{C}(0, ①, 0)$.

(6) For $\overline{\mathfrak{F}} = \langle \mathfrak{F}_1, \ldots, \mathfrak{F}_n \rangle \in \mathcal{B}^*$, the m-reduct is defined as follows:

$$r_m(\overline{\mathfrak{F}}) \leftrightharpoons r_m(\mathfrak{F}_1) \lhd \cdots \lhd r_m(\mathfrak{F}_m).$$

The canonical embedding *r_m of $r_m(\overline{\mathfrak{F}})$ into $[\overline{\mathfrak{F}}]$ is defined componentwise.*

Definition 8.12 (Good valuation) *Consider a formula φ and a finite frame $\mathfrak{F} = C_1 \lhd C_2 \cdots \lhd C_n$ with a type assignment $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$. A valuation $\mathfrak{V}$ in $\mathfrak{F}^+$ is called* good for φ and $\mathbf{t}$ *if the following hold for every $\psi \in sub(\varphi)$ and every cluster C_i of $\mathfrak{F}$:*

- *if* $\mathbf{t}_1 C_i = \mathsf{j}$ *and* $C_i \cap \mathfrak{V}(\psi) \neq \emptyset$, *then there exists* C_j *such that* $j > i$ *and* $C_j \cap \mathfrak{V}(\psi) \neq \emptyset$;
- *if* $\mathbf{t}_2 C_i = \mathsf{j}$ *and* $C_i \cap \mathfrak{V}(\psi) \neq \emptyset$, *then there exists* C_j *such that* $j < i$ *and* $C_j \cap \mathfrak{V}(\psi) \neq \emptyset$.

If φ *is of the form* $\overline{\delta}(\mathfrak{F}, \mathbf{s})$ *for some finite* $\mathfrak{F}$, *then* $\mathfrak{V}$ *is called* canonically good *for* φ *if those two conditions hold for every* propositional variable *in* φ.

Lemma 8.13 *(a) The following conditions are equivalent, for every* ψ, $\overline{\mathfrak{G}} \in \mathsf{Blocks}^*$, *and* $m \geq 2l(\psi) + 1$:

- ψ *is satisfiable in* $[\overline{\mathfrak{G}}]^+$;
- ψ *is satisfiable in* $r_m(\overline{\mathfrak{G}})^+$ *under a valuation which is good for* ψ *and the type assignment of* $r_m(\overline{\mathfrak{G}})$.

(b) The following conditions are equivalent, for every $\overline{\delta}(\mathfrak{F}, \mathbf{t})$, $\overline{\mathfrak{G}} \in \mathsf{Blocks}^*$, *and* $m \geq 2cl(\mathfrak{F})l(\mathfrak{F})$:

- $\overline{\delta}(\mathfrak{F}, \mathbf{t})$ *is satisfiable in* $[\overline{\mathfrak{G}}]^+$;
- $\overline{\delta}(\mathfrak{F}, \mathbf{t})$ *is satisfiable in* $r_m(\overline{\mathfrak{G}})^+$ *under a valuation which is canonically good for* $\overline{\delta}(\mathfrak{F}, \mathbf{t})$ *and the type assignment of* $r_m(\overline{\mathfrak{G}})$.

Proof: (a) ($\Rightarrow$). Suppose $w \in \mathfrak{V}(\psi)$ for some valuation $\mathfrak{V}$ in $[\overline{\mathfrak{G}}]^+$. Select for any $\chi \in sub(\psi)$ s.t. $\mathfrak{V}(\chi) \neq \bot$ a maximal point x^χ_{max} and a minimal point x^χ_{min} such that $\{x^\chi_{max}, x^\chi_{min}\} \subseteq \mathfrak{V}(\chi)$. Set $X \leftrightharpoons \{x^\chi_{min}, x^\chi_{max} | \chi \in sub(\psi)\}$. Then $|X| \leq 2l(\psi)$. It is readily seen that we may find an embedding e of the underlying frame of $r_m(\overline{\mathfrak{G}})$ into $[\overline{\mathfrak{G}}]$ such that

- any cluster C of $r_m(\overline{\mathfrak{G}})$ of type t is mapped onto a t-cluster D of $[\overline{\mathfrak{G}}]$;
- X is a subset of the range $ran(e)$ of e.

Now take a valuation $\mathfrak{V}'$ in $r_m(\overline{\mathfrak{G}})^+$ defined as $\mathfrak{V}'(p) \leftrightharpoons e^{-1}(\mathfrak{V}(p))$. It is not difficult to prove by induction for every $\chi \in sub(\psi)$ and point x in $r_m(\overline{\mathfrak{G}})$, $x \in \mathfrak{V}'(\chi)$ iff $e(x) \in \mathfrak{V}(\chi)$. Therefore, $\mathfrak{V}'$ is good for the type assignment of $r_m(\overline{\mathfrak{G}})$ and ψ. Moreover, let x be a maximal point in $\mathfrak{V}(\psi)$. Then $e^{-1}(x) \subseteq \mathfrak{V}'(\psi)$.

($\Leftarrow$). Suppose that $w \in \mathfrak{V}(\psi)$ for some valuation $\mathfrak{V}$ in $r_m(\overline{\mathfrak{G}})$ which is good for ψ and the type assignment of $r_m(\overline{\mathfrak{G}})$. Take the canonical embedding

r_m of $r_m(\overline{\mathfrak{G}})$ into $[\overline{\mathfrak{G}}]$. We define a valuation $\mathfrak{V}'$ in $[\overline{\mathfrak{G}}]^+$ as follows: for every v in the domain of $[\overline{\mathfrak{G}}]$ and propositional variable p set $v \in \mathfrak{V}'(p)$ iff there exists $v' \in \mathfrak{V}(p)$ such that $v = r_m(v')$ or v is not in the range of r_m but

- there exists $\mathfrak{C}(n, \textcircled{k})$ in $\overline{\mathfrak{G}}$ and $a_i \in \textcircled{k}$ such that $v \in X_i$ and $a_i \in \mathfrak{V}(p)$, or

- there exists $\mathfrak{C}(\textcircled{k}, n)$ in $\overline{\mathfrak{G}}$ and $a_i \in \textcircled{k}$ such that $v \in X_i$ and $a_i \in \mathfrak{V}(p)$, or

- there exists $\mathfrak{C}(0, \textcircled{1}, 0)$ in $\overline{\mathfrak{G}}$ with v in its domain such that $\textcircled{1} \in \mathfrak{V}(p)$.

Using the condition that $\mathfrak{V}$ is good for the type assignment of $r_m(\overline{\mathfrak{G}})$, it is not difficult to prove by induction for every $\chi \in sub(\psi)$ and point x in $r_m(\overline{\mathfrak{G}})$, $x \in \mathfrak{V}(\chi)$ iff $r_m(x) \in \mathfrak{V}'(\chi)$. Therefore, $r_m(w) \in \mathfrak{V}'(\psi)$.

(b) can be proved analogously. ⊣

We now provide a lemma from which it follows that validity of a formula $\alpha(\mathfrak{F}, \mathbf{t})$ in $[\overline{\mathfrak{G}}]^+$, where $\overline{\mathfrak{G}} \in \mathsf{Blocks}^*$, can be checked in polynomial time in the parameters of $\overline{\mathfrak{G}}$.

Definition 8.14 (type morphism) *Assume $(\mathfrak{F}, \mathbf{t})$, $\mathfrak{F} = \langle W, R\rangle$, and $(\mathfrak{F}', \mathbf{t}')$, $\mathfrak{F}' = \langle W', R'\rangle$, are two frames with types and f is a mapping from the set of clusters of $\mathfrak{F}$ onto set of clusters of $\mathfrak{F}'$. We say that f is a type morphism if the following conditions are satisfied:*

forth *For any two clusters C_1 and C_2 in $\mathfrak{F}$, $C_1 R C_2$ implies $f(C_1) R' f(C_2)$.*

degenerate *If D is a degenerate cluster in $\mathfrak{F}'$, then $f^{-1}(D)$ contains only a single degenerate cluster.*

future m *If $D = \textcircled{k}$ is a cluster in $\mathfrak{F}'$, $\mathbf{t}'_1 D = \mathsf{m}$ and C is the R-largest cluster in $f^{-1}(D)$, then C is a cluster $\geq k$ and $\mathbf{t}_1 C = \mathsf{m}$.*

past m *If $D = \textcircled{k}$ is a cluster in $\mathfrak{F}'$ and $\mathbf{t}'_2 D = \mathsf{m}$ and C is the R-smallest cluster in $f^{-1}(D)$, then C is a cluster $\geq k$ and $\mathbf{t}_2 C = \mathsf{m}$.*

future j *Suppose $D = \textcircled{k}$ is a cluster in $\mathfrak{F}'$, $\mathbf{t}'_1 D = \mathsf{j}$, and $\bigcup f^{-1} D = C_1 \lhd C_2 \cdots \lhd C_n$. Then the maximal final sequence $C_m \lhd C_{m+1} \lhd \cdots \lhd C_n$ such that $\mathbf{t}_1 C_i = \mathsf{m}$ for $m \leq i \leq n$ contains at least k points.*

past j *Suppose* $D = \textcircled{k}$ *is a cluster in* $\mathfrak{F}'$, $\mathbf{t}'_2 D = \mathsf{j}$, *and* $\bigcup f^{-1}D = C_1 \lhd C_2 \cdots \lhd C_n$. *Then the maximal initial sequence* $C_1 \lhd C_2 \lhd \cdots \lhd C_m$ *in* $\bigcup f^{-1}D$ *such that* $\mathbf{t}_2 C_i = \mathsf{m}$ *for* $1 \leq i \leq m$ *contains at least* k *points.*

Lemma 8.15 *The following conditions are equivalent, for every* $\overline{\mathfrak{G}} \in \mathsf{Blocks}^*$, *every finite frame with types* $(\mathfrak{F}, \mathbf{t})$, *and every* $m \geq 2l(\mathfrak{F})cl(\mathfrak{F})$*:*

- $[\overline{\mathfrak{G}}]$ *validates* $\alpha(\mathfrak{F}, \mathbf{t})$*;*
- *There does not exist a type morphism from* $r_m(\overline{\mathfrak{G}})$ *onto* $(\mathfrak{F}, \mathbf{t})$.

Proof: By Lemma 8.13 (b) it is sufficient to show that the following two conditions are equivalent:

- $\overline{\delta}(\mathfrak{F}, \mathbf{t})$ is satisfiable under a canonically good valuation in $r_m(\overline{\mathfrak{G}})$;
- there exists a type morphism from $r_m(\overline{\mathfrak{G}})$ onto $(\mathfrak{F}, \mathbf{t})$.

Suppose first that $\overline{\delta}(\mathfrak{F}, \mathbf{t})$ is satisfied in $r_m(\overline{\mathfrak{G}})^+$ under a canonically good valuation $\mathfrak{V}$. Define a mapping f from the set of clusters of $r_m(\overline{\mathfrak{G}})$ onto the set of clusters of $\mathfrak{F}$ as follows: $f(C) = D$ iff there exists $x \in D$ and $y \in C$ such that $y \in \mathfrak{V}(p_x)$. We claim that f is the required type morphism:

- f is a partial function: suppose there are x_1 and x_2 in different clusters from $\mathfrak{F}$ such that there are y_1 and y_2 in the same cluster of $r_m(\overline{\mathfrak{G}})$ and p_{x_1} is satisfied in y_1 and p_{x_2} is satisfied in y_2. If $y_1 = y_2$, then this contradicts the first conjunct of $\delta(\mathfrak{F}, \mathbf{t})$. Otherwise $p_{x_1} \wedge \Diamond_P p_{x_2}$ and $p_{x_2} \wedge \Diamond_P p_{x_1}$ are both satisfied. This contradicts the second conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.
- f is defined for every cluster in $r_m(\overline{\mathfrak{G}})$: this follows from the fifth conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.
- f is surjective: this follows from the last conjunct of $\delta(\mathfrak{F}, \mathbf{t})$ since each p_x is satisfied in at least one point of $r_m(\overline{\mathfrak{G}})$.
- Condition **forth** follows from the second conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.
- Condition **degenerate** follows from the second conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.
- Condition **future** m follows from the third conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.

- Condition **past** m follows from the fourth conjunct of $\delta(\mathfrak{F}, \mathbf{t})$.

- Conditions **future** j and **past** j follow from the fact that $\mathfrak{V}$ is a canonically good valuation.

Conversely, suppose f is a type morphism from $r_m(\overline{\mathfrak{G}})$ onto $(\mathfrak{F}, \mathbf{t})$. Denote by $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2)$ the type assignment of $r_m(\overline{\mathfrak{G}})$.

Define a valuation $\mathfrak{V}$ in $r_m(\overline{\mathfrak{G}})^+$ as follows: let D be a cluster in $\mathfrak{F}$ and let $C_1 \lhd C_2 \lhd \cdots \lhd C_n$ be the interval consisting of the clusters mapped to D. If $D = \{x\}$, for some x, then set $\mathfrak{V}(p_x) = \bigcup_{1 \leq i \leq n} C_i$. Otherwise we distinguish three cases:

- $\mathbf{t}D = (\mathsf{m}, \mathsf{m})$. In this case, by conditions **future** m and **past** m, C_1 and C_n are of cardinality $\geq |D|$. Hence define $\mathfrak{V}(p_x)$, $x \in D$, in such a way that they form a partition of C_1, C_2 and $\bigcup_{1 \leq i \leq n} C_i$.

- $\mathbf{t}D = (\mathsf{m}, \mathsf{j})$. In this case, by condition **future** m, C_n is of cardinality $\geq |D|$. By condition **past** j, there exists $m \leq n$ such that $C_1 \lhd C_2 \lhd \cdots \lhd C_m$ has cardinality $\geq |D|$ and $\mathbf{s}_2 C_i = \mathsf{m}$ for all $1 \leq i \leq m$. But then define $\mathfrak{V}(p_x)$, $x \in D$, in such a way that they form a partition of C_n, $C_1 \lhd C_2 \lhd \cdots \lhd C_m$, and $C_1 \lhd C_2 \lhd \cdots \lhd C_n$.

- $\mathbf{t}D = (\mathsf{j}, \mathsf{m})$. Take the mirror image of the definition above using **past** m and **future** j.

It is not difficult to see that $\mathfrak{V}$ is a canonically good valuation and that $\overline{\delta}(\mathfrak{F}, \mathbf{t})$ is satisfied in $r_m(\overline{\mathfrak{G}})$. ⊣

Lemma 8.16 *Assume that* $(\mathfrak{G}, \mathbf{t})$ *is of size* m *and* $(\mathfrak{G}', \mathbf{t}')$ *is of size* n. *It can be checked in time polynomial in* m *whether there exists a type morphism from* $(\mathfrak{G}, \mathbf{t})$ *onto* $(\mathfrak{G}', \mathbf{t}')$

Proof: As follows from the definition of type morphism, every type morphism induces a distinct partition of $\mathfrak{G}$ into $\leq n$ intervals. So it is enough to examine all such divisions of $\mathfrak{G}$. But the number of these partitions is bounded by m^n. ⊣

Theorem 8.17 *For every* $\bigcap$*-prime logic* $\Lambda \in Ext\mathbf{Lin}$ *and every formula* φ, *it is decidable in non-deterministic polynomial time whether* $\varphi \notin \Lambda$.

Proof: Assume $\varphi \fallingdotseq \neg\psi$. As was already observed, every $\bigcap$-irreducible tense logic is determined by $[\overline{\mathfrak{G}}]$, for some $\overline{\mathfrak{G}} \in \mathcal{B}^*$. Suppose now that $\overline{\mathfrak{G}} \in \mathsf{Blocks}^*$ is given and we want to check whether ψ is satisfiable in $[\overline{\mathfrak{G}}]$. Let $m \fallingdotseq 2(l(\psi) + 1)$. By Lemma 8.13, ψ is satisfiable in $[\overline{\mathfrak{G}}]$ iff it is satisfiable under a ψ-good valuation in $r_m(\overline{\mathfrak{G}})$. The cardinality of $r_m(\overline{\mathfrak{G}})$ is polynomial in $l(\psi)$. Now choose non-deterministically any valuation $\mathfrak{V}$ in $r_m(\overline{\mathfrak{F}})$ and check in polynomial time whether it is good for ψ and whether ψ is satisfied under $\mathfrak{V}$. ⊣

Theorem 8.18 *Suppose*

$$\Lambda \fallingdotseq \mathsf{Log}(\{\mathbf{Lin}, \alpha(\mathfrak{F}_1, \mathbf{t}_1), \ldots, \alpha(\mathfrak{F}_n, \mathbf{t}_n)\}).$$

For every formula φ, it is decidable in non-deterministic polynomial time whether $\varphi \notin \Lambda$

Proof: Assume $\varphi \fallingdotseq \neg\psi$. Set $r_1 \fallingdotseq \max\{l(\mathfrak{F}_i) \mid 1 \leq i \leq n\}$ and $r_2 \fallingdotseq \max\{cl(\mathfrak{F}_i) \mid 1 \leq i \leq n\}$. Choose non-deterministically $\overline{\mathfrak{G}} \in \mathsf{Blocks}_0^*$ satisfying the constraints (b)-(d) of Theorem 8.6 and a valuation $\mathfrak{V}$ in $r_m(\overline{\mathfrak{G}})$, where

$$m \fallingdotseq \max\{2r_1r_2, 2l(\psi) + 1\}.$$

Notice that the size of $r_m(\overline{\mathfrak{G}})$ is polynomial in ψ. Now it can be checked in polynomial time whether $\mathfrak{V}$ is good for ψ and ψ is satisfied. Moreover, by Lemma 8.16 it can be checked in polynomial time whether any of $(\mathfrak{F}_i, \mathbf{t}_i)$ is a type-morphic image of $r_m(\overline{\mathfrak{G}})$. In other words, by Lemma 8.15 and 8.13, we can decide in non-deterministic polynomial time whether ψ is satisfiable in a frame satisfying the constraints (a)-(d) of Theorem 8.6. ⊣

8.5 The gap between $\omega\mathcal{C}$-completeness and $\mathcal{C}$-completeness

Recall that in Section 4.3 we proved that for every κ, there exists a logic which in $\kappa\mathcal{C}$-complete but $\mathcal{C}$-inconsistent. It was the logic of $\mathfrak{S}ucc_\kappa$; recall that $\mathfrak{S}ucc_\kappa \in \mathbb{V}(\mathbf{Lin})$. In this section, we complete the proof of Theorem 4.6 by showing that

Theorem 8.19 *For every cardinal κ, $Th(\mathfrak{S}ucc_\kappa) = $* **Succ**. *Thus* **Succ** *is $\kappa\mathcal{C}$-complete for every cardinal κ, but — by Theorem 4.6 — $\mathcal{C}$-inconsistent.*

Proof: It has been observed before that $Th(\mathfrak{S}ucc_\kappa) \supseteq$ **Succ**. The important inclusion is the reverse one. We can provide an alternative axiomatization by the use of Wolter's canonical formulas introduced above; it makes our task much easier. The following convention is useful:

$$\alpha(-, (C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n)) \leftrightharpoons \alpha((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n)) \wedge$$
$$\wedge \alpha((\circ, (\mathsf{j}, \mathsf{j})) \lhd (C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n)).$$

$\alpha((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n), -)$ is defined dually. The canonical axiomatization mentioned above is

$$\mathbf{Succ} = \mathsf{Log}(\{\mathbf{Lin}, \alpha(② , (\mathsf{m}, \mathsf{j})), \alpha(-, (\circ, (\mathsf{j}, \mathsf{m})), \alpha(-, (\bullet, (\mathsf{m}, \mathsf{m})))\}).$$

Now assume $\varphi \leftrightharpoons \neg\psi \notin$ **Succ**. Our goal is to show that $\mathfrak{S}ucc_\kappa \nvDash \varphi$. We may assume that $\varphi \leftrightharpoons \alpha(\mathfrak{F}, \mathbf{t})$ and thus $\psi \leftrightharpoons \overline{\delta}(\mathfrak{F}, \mathbf{t})$. By Theorem 8.6, there is $[\overline{\mathfrak{G}}] \leftrightharpoons [\mathfrak{G}_1 \lhd \ldots \lhd \mathfrak{G}_n] \in [\mathsf{Blocks}_0^*]$ s.t. $[\overline{\mathfrak{G}}]^+ \vDash$ **Succ** and $[\overline{\mathfrak{G}}]^+ \nvDash \varphi$. Set $m \leftrightharpoons \max\{2cl(\mathfrak{F})l(\mathfrak{F}), 4\}$. By Lemma 8.13, $[\overline{\mathfrak{G}}]^+ \nvDash \varphi$ iff it fails in $r_m(\overline{\mathfrak{G}})^+$ under a canonically good valuation; by the same result $[\overline{\mathfrak{G}}]^+ \vDash$ **Succ** iff for no canonically good valuation, $r_m(\overline{\mathfrak{G}})^+ \nvDash$ **Succ**. By Lemma 8.15, the latter in turn is equivalent to non-existence of type morphism from $r_m(\overline{\mathfrak{G}})$ onto any of the following frames:

$$(② , (\mathsf{m}, \mathsf{j})), (\circ, (\mathsf{j}, \mathsf{m})), (\circ, (\mathsf{j}, \mathsf{j})) \lhd (\circ, (\mathsf{j}, \mathsf{m})), (\bullet, (\mathsf{m}, \mathsf{m})), (\circ, (\mathsf{j}, \mathsf{j})) \lhd (\bullet, (\mathsf{m}, \mathsf{m})).$$

Thus, we can describe the structure of $r_m(\overline{\mathfrak{G}})$, or even of $[\overline{\mathfrak{G}}]$ itself, quite accurately.

Claim 1: (i) There is no non-degenerate cluster C (not even a reflexive point) s.t. $\mathbf{t}_2 C = \mathsf{m}$. In consequence, the only frames which appear in the sequence $\overline{\mathfrak{G}_i}$ are either irreflexive points or frames of the from $\mathfrak{C}(0, ⓚ)$ for some $k \in \omega$.

(ii) The final cluster of $r_m(\overline{\mathfrak{G}})$ is $(\circ, (\mathsf{m}, \mathsf{j}))$; thus $\mathfrak{G}_n = \mathfrak{C}(0, ①)$.

Proof of claim: (i) Assume there is a non-degenerate cluster C in $r_m(\overline{\mathfrak{G}})$ s.t. $\mathbf{t}_2 = \mathsf{m}$. Define a mapping from the set of all clusters of $r_m(\overline{\mathfrak{G}})$ as follows: if C is the initial cluster of $r_m(\overline{\mathfrak{G}})$, then map all clusters to $(\circ, (\mathsf{j}, \mathsf{m}))$; otherwise, map C and its successors to $(\circ, (\mathsf{j}, \mathsf{m}))$ and the remaining ones to $(\circ, (\mathsf{j}, \mathsf{j}))$. It can be easily verified that we defined a type morphism onto

$(\circ, (\mathsf{j}, \mathsf{m}))$ in the first case, and a type morphism onto $(\circ, (\mathsf{j}, \mathsf{j})) \lhd (\circ, (\mathsf{j}, \mathsf{m}))$ in the second. The second statement easily follows: m-reducts of all remaining frames from Blocks_0 contain a non-degenerate cluster C s.t. $\mathbf{t}_2 C = \mathsf{m}$.

(ii) It can be proven in a similar way to (i) that if the final cluster is a degenerate cluster than we can define either a type morphism onto $(\bullet, (\mathsf{m}, \mathsf{m}))$ or onto $(\circ, (\mathsf{j}, \mathsf{j})) \lhd (\bullet, (\mathsf{m}, \mathsf{m}))$. So, the final cluster C must be a non-degenerate one. By (i), $\mathbf{t}_2 C = \mathsf{j}$; of course, $\mathbf{t}_1 C = \mathsf{m}$. If $|C| \geq 2$, the we could define a type morphism onto $(\circ, (\mathsf{m}, \mathsf{j}))$. ⊣

As a finite sequence of irreflexive points followed by $\mathfrak{C}(0, ⓚ)$ is exactly the frame $\mathfrak{C}(0, ⓚ)$ itself, we may in fact assume that $\overline{\mathfrak{G}}$ is a finite sequence of frames of the form $\mathfrak{C}(0, ⓚ)$ and the last element of the sequence is $\mathfrak{C}(0, ①)$. By the same token, $r_m(\overline{\mathfrak{G}})$ is a finite sequence of $[(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1} \lhd (ⓚ, (\mathsf{m}, \mathsf{j}))$[1] and the last element of the sequence is $[(\bullet, (\mathsf{m}, \mathsf{m}))]^{m+1} \lhd (①, (\mathsf{m}, \mathsf{j}))$. It is clear now how to define an admissible valuation refuting φ in $\mathfrak{S}ucc_\kappa$ using the canonically good valuation refuting φ in $r_m(\overline{\mathfrak{G}})$. ⊣

8.6 Notes

The present chapter builds on Wolter [59]. That work introduced the family [Blocks^*] and the canonical formulas for finite frames with types defined in Section 8.2. By those means, he proved a number of essential results concerning decidability, axiomatizability and completeness of logics in $Ext\mathbf{Lin}$. This work was continued in Wolter [58], where it was shown that using those tools one can provide algorithms deciding whether a finitely axiomatizable logic is complete or has the finite model property. Our presentation follows rather that of Zakharyaschev et al. [60] or Litak, Wolter [IV] than the original paper of Wolter [59], which does not make an easy reading. The observation that using those results one may prove $\mathcal{AV}$-completeness of all logics in $Ext\mathbf{Lin}$ seems new, but — as the reader probably noticed — it is not very hard to derive anyhow.

Theorem 8.9 strengthens the first $\omega\mathcal{C}$-inconsistency result in Litak [V]. It was observed by Wolter [59] himself that those logics have no Kripke frames, but our theorem is stronger, especially in view of Theorem 4.6. Similarly,

[1] We should attach index to k; we have not done so for typographic reasons

Theorem 8.10 generalizes Proposition 16 in Wolter [59] both in terms of scope (it covers more logics) and strength (it deals with degrees of $\omega\mathcal{C}$-completeness, not just Kripke completeness).

Computational complexity is not the main subject of this thesis, but results discussed in Section 8.4 are too important to be left out. The whole section is based on Litak, Wolter [IV]. Earlier, some particular systems in *Ext***Lin** were shown to be coNP-complete (see Ono and Nakamura [42] or Vardi [17]). Our result is incomparably more general. Its special character is made even more explicit by Section 8.3. Other general low-complexity results, like Spaan [47] (actually, our result generalizes that of Spaan) or Bezhanishvili et al. [2, 1] were based on polynomial finite model property of logics under consideration. Here, we deal with a lattice containing many logics without any adequate finite structures, Kripke frames or even $\omega\mathcal{C}$-BAOs at all. It supports our main thesis: Kripke completeness of a given system is neither neither a necessary nor a sufficient condition of its nice behaviour. Thus, it makes sense to study weaker completeness notions.

Chapter 9

Open problems

Problem 1 *Are there any $\mathcal{V}$-incomplete logics?*

A natural candidate would be the logic of van Benthem [53], but in the proof of Theorem 4.8 additional assumptions of atomicity and/or existence of residuals were used in a crucial way.

Problem 2 *Can the Blok Theorem be proven for degrees of $\mathcal{A}$-incompleteness?*

As was mentioned, it does not seem impossible. But results like Theorem 2.18 may suggest that $\mathcal{A}$-completeness behaves in a slightly different way than other completeness notions.

Problem 3 *Is there any logic above* **4** *with a nonconservative minimal tense extension or a minimal hybrid extension?*

The variety corresponding to van Benthem logic [53] has EDPC. That would seem to suggest that producing a $\mathcal{T}$-incomplete or an $\mathcal{AV}$-incomplete transitive logic is possible. But appearances may be misleading here; in fact, the author is tempted to cojecture (tentatively) that such logics do not exist. In a way, this is a question whether one may combine two famous examples of incomplete logics: that of van Benthem and that of Fine [18].

Problem 4 *Are all finitely axiomatizable unimodal/tense logics of transitive frames of finite width decidable? If so, what is their complexity?*

This is a hard problem, which would require a lot of technical work. Nevertheless, we suppose that it should be possible to obtain positive results.

Finally, two problems related to the notion of strong completeness:

Problem 5 *Is there a strongly $\mathcal{C}$-complete logic which is not strongly Kripke complete?*

Problem 6 *Can we remove the assumption of subdirect irreducibility from Theorem 7.10?*

Appendix A: Algebraic and model-theoretical preliminaries

The material presented below is not original. It is based on Burris and Sankappanavar [8], Chang and Keisler [13] and Kracht [37].

Familiarity with classical first-order predicate calculus is assumed in the thesis; our approach is the same as, e.g., that of Chang and Keisler [13]. A *(first-order) similarity type* σ is a set of relation and function constants s.t. every symbol has fixed finite arity. For a given σ, notions of *term*, *identity*, *first-order formula*, *model* (in the sense of first-order model theory) are defined in the standard way. Recall than *a quasi-equation* is an universal formula $\forall \bar{x}(\varphi \to \psi)$ s.t. φ is a conjunction of identities (possibly empty) and ψ is a single identity. An *algebra* is, as usual, a model for a similarity type without relation constants. An algebra is called *trivial* if its universe consists of one element. Interpretations of relation constants are called *predicates*, interpretations of function symbols are called *operations*. If $\sigma \subseteq \sigma'$, $\mathfrak{M}$ is a model for σ and $\mathfrak{M}'$ is a model for σ' s.t. the universe of $\mathfrak{M}'$ and the interpretation of all symbols from σ is the same as that of $\mathfrak{M}$. we say that $\mathfrak{M}'$ is *an expansion* of $\mathfrak{M}$ and that $\mathfrak{M}$ is *a reduct* of $\mathfrak{M}'$. If $\mathfrak{M}'$ and $\mathfrak{M}$ are models for σ, we say that $\mathfrak{M}$ is a *submodel* of $\mathfrak{M}'$ if the universe of $\mathfrak{M}$ is a subset of $\mathfrak{M}'$ and the interpretations of all symbols of σ in $\mathfrak{M}$ are restrictions of interpretations of respective symbols in $\mathfrak{M}'$; we denote it as $\mathfrak{M} \subseteq \mathfrak{M}'$. In the particular case of $\mathfrak{M}'$ being an algebra, we say that $\mathfrak{M}$ is *a subalgebra* of $\mathfrak{M}'$. Note that the definition forces $\mathfrak{M}$ to be closed under all operations. If this is not the case, i.e., if for some arguments values of some operations do not belong to $\mathfrak{M}$, we say $\mathfrak{M}$ is *a partial subalgebra of* $\mathfrak{M}'$. A closure of a class K of models under submodels is denoted as $S(K)$. *The direct product* $\prod_{i \in I} \mathfrak{M}_i$ of a family of models (algebras) $\{\mathfrak{M}_i\}_{i \in I}$ in the same similarity type is a structure whose universe is set-theoretical product of universes of all $\mathfrak{M}_i$'s and all predicates and operations are defined componentwise. If $I = \{0, \ldots, n-1\}$, $\prod_{i \in I} \mathfrak{M}_i$ is sometimes written as $\mathfrak{M}_0 \times \ldots \times \mathfrak{M}_{n-1}$. A closure of a class of models under direct products is denoted as $P(K)$. For every $i \in I$, *projection* $\pi_i : \prod_{i \in I} \mathfrak{M}_i \mapsto \mathfrak{M}_i$ is defined in the standard way. If U is an *ultrafilter* over the index set I — i.e., a nonempty upward closed family of subsets of I s.t. for every $X \subseteq I$ $|U \cap \{X, I - X\}| = 1$ — and $\mathfrak{M}$ is the direct product of $\{M_i\}_{i \in I}$, then $\mathfrak{M}_U$ is *the ultraproduct of* $\{M_i\}_{i \in I}$ *over* U if the universe of $\mathfrak{M}_U$ consists of equivalence classes $[\bar{m}]_U = \{\bar{m'} | \{i \in I | m_i = m'_i\} \in U\}$ and predicates and operations on equivalence classes are defined in the standard way. Closure of a class of models K under ultraproducts is denoted as $P_U(K)$. From now on, we restrict our attention to algebras. *A congruence* on $\mathfrak{A}$ is any equivalence relation θ on the universe of $\mathfrak{A}$ which respects all operations, i.e., for any n-ary $\eta \in \sigma$ and n-tuples $\bar{a}$, $\bar{b}$ of elements of $\mathfrak{A}$, if $a_i \theta b_i$ for every $i < n$, then $\eta\bar{a} = \eta\bar{b}$. For every $a, b \in \mathfrak{A}$, *the principal congruence generated by a and b* is the smallest congruence $\theta(a, b)$ s.t. $a\theta(a, b)b$. If $\mathfrak{A}$ and $\mathfrak{B}$ are two algebras in the same similarity type, then *a homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is any function $f : \mathfrak{A} \mapsto \mathfrak{B}$ s.t. for every $\eta \in \sigma$ and $\bar{a} \in \mathfrak{A}$, $\eta f(\bar{a}) = f(\eta\bar{a})$. Closure of a class of algebras K under homomorphic images is denoted as $H(K)$. It may be established that a composition of homomorphisms is again a homomorphism. An injective homomorphism is called *an embedding*. *An isomorphism* is an embedding which is onto. If there is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* and write $\mathfrak{A} \cong \mathfrak{B}$. A closure of a class of algebras K under isomorphisms is denoted as $I(K)$. For any homomorphism f, we define *kernel* f: a congruence relation θ_f s.t. $a\theta_f b$ iff $f(a) = f(b)$. Conversely, for any congruence θ, *the quotient algebra* $\mathfrak{A}/\theta$ is an algebra whose universe consists of equivalence classes of elements of a and all operations on equivalence classes are defined in the standard way. If $\theta \subseteq \psi$, then $\mathfrak{A}/\psi$ is a homomorphic image of $\mathfrak{A}/\theta$. The onto homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\theta$ defined by $f_\theta(a) \leftrightharpoons [a]_\theta$ is called *the natural homomorphism* determined by θ. For every θ, $\theta = \theta_{f_\theta}$ and for every onto homomorphism $f : \mathfrak{A} \mapsto \mathfrak{B}$, $\mathfrak{A}/\theta_f \cong \mathfrak{B}$. Thus, the mathematical meaning of the notions of *homomorphism onto* and *a congruence* is the same. Analogously, if $\mathfrak{A} \subseteq \mathfrak{B}$, then the identity mapping on $\mathfrak{A}$ is an embedding into $\mathfrak{B}$ and if $f : \mathfrak{A} \mapsto \mathfrak{B}$ is an embedding, then $f[\mathfrak{A}]$ is a subalgebra of $\mathfrak{B}$. Thus, the mathematical meaning of the notions of *embedding* and *subalgebra* is the same.

Theorem 9.1 *A class of algebras K is definable by a set of equations iff*

$K = HSP(K)$. A class of algebras is definable by a set of quasi-equations iff $K = ISPP_U(K)$. A class of algebras (or models) K is definable by a set of universal sentences iff $K = SP_U(K)$.

For any class of algebras K, $HSP(K)$ is called *the variety generated by* K and $ISPP_U(K)$ is called *the quasi-variety generated by* K. $\mathfrak{A}$ is *directly irreducible* if there are no non-trivial algebras $\mathfrak{B}$, $\mathfrak{C}$ s.t. $\mathfrak{A} \cong \mathfrak{B} \times \mathfrak{C}$. An algebra $\mathfrak{A}$ is *a subdirect product* of $\{\mathfrak{A}_i\}_{i\in I}$ if $\mathfrak{A} \subseteq \prod_{i\in I} \mathfrak{A}_i$ and for every $i \in I$, $\pi_i[\mathfrak{A}] = \mathfrak{A}_i$. If $f : \mathfrak{A} \mapsto \prod_{i\in I} \mathfrak{A}_i$ is an embedding s.t. $f[\mathfrak{A}]$ is a subdirect product of $\prod_{i\in I} \mathfrak{A}_i$, we say f is a *subdirect embedding.* An algebra is *subdirectly irreducible* if for every I, every $\{\mathfrak{A}_i\}_{i\in I}$ and every $f : \mathfrak{A} \mapsto \prod_{i\in I} \mathfrak{A}_i$, if f is a subdirect embedding, then there is $i \in I$ s.t. $\mathfrak{A} \cong \mathfrak{A}_i$. An algebra $\mathfrak{A}$ is *simple* if every nontrivial homomorphic image of $\mathfrak{A}$ is isomorphic to $\mathfrak{A}$. Every simple algebra is subdirectly irreducible and every subdirectly irreducible algebra is directly irreducible.

Theorem 9.2 (Birkhoff) *Every algebra is isomorphic to a subdirect product of its subdirectly irreducible homomorphic images.*

For every class of algebras K, denote by $Eq(K)$ ($QuEq(K)$, $Univ(K)$) the equational (quasi-equational, universal) theory of K — i.e., the set of all equations (quasi-equations, universal sentences) which hold in algebras from K. Conversely, for every set of equations (quasi-equations, universal sentences) Λ, $Mod(\Lambda)$ is the class of all algebras satisfying Λ. Thus, $HSP(K) = Mod(Eq(K))$, $ISPP_U(K) = Mod(QuEq(K))$, $SP_U(K) = Mod(Univ(K))$. Sometimes, we write $\mathfrak{A} \models \Lambda$ instead of $\mathfrak{A} \in Mod(\Lambda)$. It follows from Birkhoff's theorem that for every variety K, $Eq(K) = Eq(K_{SI})$, where K_{SI} is the class of subdirectly irreducible elements of K. But in every variety, there is actually a single algebra which determines its equational theory. For arbitrary similarity type σ with no relation constants, *the absolutely free algebra of terms in κ variables* $\textsc{form}_\kappa$ is the algebra of all terms in κ variables with operations defined in the standard way (i.e., $\eta^{\mathfrak{A}}t \leftrightharpoons \eta t$ for any term t and any function symbol η). For any set of equations Λ, *the Lindenbaum algebra of* Λ *in κ variables* $\textsc{form}_\kappa/\Lambda$ is the algebra whose elements are equivalence classes of terms $[t]_\Lambda \leftrightharpoons \{t' | \Lambda \vdash t = t'\}$. It is a homomorphic image of the absolutely free algebra. For any $\kappa \geq \omega$ and any Λ, $Mod(\Lambda) = HSP(\textsc{form}_\kappa/\Lambda)$.

A class of algebras K has the *finite embeddability property* if every finite partial subalgebra of an algebra from K can be embedded into an algebra from K. A set of elements X of $\mathfrak{A}$ *generates* $\mathfrak{A}$ if the smallest subalgebra of $\mathfrak{A}$ containing X is $\mathfrak{A}$ itself. A variety V is *locally finite* if every finitely generated algebra in V is finite. Every locally finite variety has the finite embeddability property. A variety generated by a finite family of finite algebras is locally finite. A variety V has *finite model property* if $V = HSP(V_F)$, where V_F is the class of all finite algebras from V. A quasivariety V has the *(quasi-variety) finite model property* if $V = ISPP_U(V_F)$. Observe here that a variety with the finite model property does not have to possess the quasi-variety finite model property, i.e., it may be the case that $V = HSP(V_F)$, but $ISPP_U(V_F) \subsetneq V$. However, if the variety has the finite embeddability property, then it has the quasi-variety finite model property. The quasi-variety finite model property is sometimes called *strong finite model property* (cf. Blok et al. [6]). As was mentioned in the thesis, the name can be misleading in the present context.

A class of algebras K in a given similarity type has *equationally definable principal congruences (EDPC)* if there is a conjunction of equations in four variables $\psi(x, y, z, u)$ s.t. for every $\mathfrak{A} \in K$ and every $a, b, c, d \in \mathfrak{A}$, $\langle a, b\rangle \in \theta(c, d)$ iff ψ holds in $\mathfrak{A}$ with variables evaluated as a, b, c and d. For varieties with EDPC in a finite similarity type, finite model property, quasi-variety finite model property and finite embeddability property are equivalent. In fact, the meaning of EDPC is that, in a sense, we may reduce quasi-equations to equations. A variety V is *discriminator* if there is a ternary term $t(x, y, z)$ in three variables s.t. for every subdirectly irreducible algebra $\mathfrak{A} \in V_{SI}$ and for every $a, b, c \in \mathfrak{A}$, $t^{\mathfrak{A}}(a, b, c) = a$ iff $a = b$ and $t^{\mathfrak{A}}(a, b, c) = c$ otherwise. In a discriminator variety, every subdirectly irreducible algebra is simple. Every discriminator variety has EDPC. In fact, the meaning of being discriminator is that, in a sense, we may reduce universal sentences to equations. A class of algebras in K has the congruence extension property (CEO) if for every congruence θ of $\mathfrak{A} \in K$ and every $\mathfrak{B} \in K$, $\mathfrak{A} \subseteq \mathfrak{B}$ implies there is a congruence θ' of $\mathfrak{B}$ s.t. $\theta = \theta' \cap \mathfrak{A}^2$. If a variety has CEP, then every subalgebra of a subdirectly irreducible algebra is subdirectly irreducible and every subalgebra of a simple algebra is simple.

A *lattice* is an algebra $\mathfrak{L}$ a similarity type σ containing two binary connectives $\wedge, \vee$ (called, respectively, *meet* and *join*) s.t. corresponding operations in $\mathfrak{L}$ are idempotent, associative, commutative (i.e. $x * x = x$, $x * (y * z) = (x * y) * z$, $x * y = y * x$ for $* \in \{\wedge, \vee\}$) and satisfy *the absorption law*: $x \wedge (x \vee y) = x$. The definition implies that the class of all lattices in a given similarity type is a variety. It is easy to see that the relation $\leq$ given by $x \leq y$ iff $y = x \vee y$ is a weak partial order and that this definition is equivalent to $x \leq y$ iff $x = x \wedge y$. If t is a term in a lattice similarity type containing no other connectives than $\wedge$ and $\vee$, then *the dual* of t is any term obtained from t by replacing all occurrences of $\wedge$ with $\vee$ and the other way around. *A lattice filter* F in a lattice $\mathfrak{L}$ is a nonempty family of elements satisfying $x, y \in F$ iff $x \wedge y \in F$. If, in addition, $x \vee y \in F$ implies either $x \in F$ or $y \in F$, we say F is *a prime filter*. For every x in $\mathfrak{L}$, the set $x\!\uparrow\, \leftrightharpoons \{y \in F | x \wedge y = x\}$ is *the principal lattice filter generated by* x. We say that filter is *proper* if it is not equal to the whole lattice; unless stated otherwise, by *a filter* we mean a proper filter. Notions of *ideal*, *prime ideal*, *principal ideal* and *proper ideal* are defined dually. If the lattice $\mathfrak{L}$ has an element $\top$ s.t. for every x, $x \wedge \top = x$, then for any homomorphism $f : \mathfrak{L} \mapsto \mathfrak{L}'$, $[\top]_{\theta_f}$ is a filter. A lattice $\mathfrak{L}$ is *complete* if for every family of elements X there is an element $\bigvee X$ called *supremum* of X s.t for every $x \in X$, $x \leq \bigvee X$ and for every y, if $y \geq x$ for all $x \in X$, then $y \geq \bigvee X$. The notion of *infimum* is defined dually. A lattice is called κ-complete if every family of elements whose cardinality is no greater than κ has supremum and infimum. Note that we stick here to definition of κ-completeness from, e.g., Sikorski [46] rather than the one from Koppelberg [33]. A homomorphism is *κ-complete* if it preserves all existing suprema and infima of families of elements whose cardinality is no greater than κ; i.e., if $|X| \leq \kappa$ and $\bigvee X$ exists, then $\bigvee f[X]$ exists and is equal to $f(\bigvee X)$. A homomorphism is *complete* if it is κ-complete for arbitrary κ. A filter F is κ-complete if it is closed under all existing infima of families of elements of F whose cardinality is no greater than κ. A filter is *complete* if it is κ-complete for arbitrary κ. If f is a κ-complete homomorphism whose domain is a lattice with the greatest element $\top$, then $[\top]_{\theta_f}$ is a κ-complete filter. If a lattice is complete, then every complete filter is principal. A lattice is *upper-continuous* if for every family of elements Y s.t. $\bigvee Y$ exists and for every x,$\bigvee\{x \wedge y | y \in Y\}$ exists and is equal to $x \wedge \bigvee Y$. Definition of *lower continuity* is dual. If a lattice is upper- or lower-continuous for finite Y, we say it is *distributive*.

For every algebra $\mathfrak{A}$, the algebra $\mathrm{Con}(\mathfrak{A})$ of all congruences of $\mathfrak{A}$ with set-theoretical join as $\wedge$ and $\theta \vee \psi \leftrightharpoons \bigcap\{\phi \in \mathrm{Con}(\mathfrak{A}) | \theta \cup \psi \subseteq \phi\}$ is a complete lattice. If it is a distributive lattice, we say $\mathfrak{A}$ is *congruence-distributive.* A class of algebras is congruence-distributive if all its members are congruence-distributive. Varieties of lattices are congruence-distributive.

A splitting pair in a lattice $\mathfrak{L}$ is a pair $\langle a_1, a_2 \rangle \in \mathfrak{L}^2$ s.t. every element of

$\mathfrak{L}$ belongs either to the principal ideal generated by a_1 or the principal filter generated by a_2 and none belongs to both. In such a case, we say that a_1 *splits* $\mathfrak{L}$. a_2 is called the *splitting* of $\mathfrak{L}$ by a_1 and denoted by $\mathfrak{L}/a_1$. It is immediate that $\mathfrak{L}/a_1$ is determined uniquely. An element a in a complete lattice is called *meet-prime* if (**p**) $a \geq \bigwedge_{i\in I} a_i$ implies $a \geq a_i$ for some $i \in I$. a is called *meet-irreducible* if (**p**) is satisfied after replacing $\geq$ by $=$. It is straightforward that meet-primeness implies meet-irreducibility. The converse holds if the lattice is lower-continuous, but is false in the general case. Definitions of join-primeness and join-irreducibility are dual. Again, join-primeness implies join-irreducibility, but the converse may fail to hold without upper continuity.

Theorem 9.3 *Let $\mathfrak{L}$ be a complete lattice. a splits $\mathfrak{L}$ iff it is meet-prime in $\mathfrak{L}$.*

Proof: ($\Rightarrow$). By contraposition. Let $\langle a, b\rangle$ be a splitting pair and assume $a \not\geq a_i$ for all a_i. By definition of a splitting pair, $b \geq a_i$ for all a_i. Thus, $b \leq \bigwedge_{i\in I} a_i$ and $a \not\geq \bigwedge_{i\in I} a_i$.

($\Leftarrow$). Let $b \leftrightharpoons \bigwedge\{x | a \not\geq x\}$. By definition, $a \not\geq x$ implies $b \leq x$, so it is enough to prove $b \not\leq a$ to show (a, b) is a splitting pair. But this follows straightforwardly from the definition of a prime element $\dashv$

An element is *join-splitting* or *union-splitting* of a complete lattice $\mathfrak{L}$ if it is a join of a family of splittings. If $F \leftrightharpoons \{a_i\}_{i\in I}$ is a family s.t. a_i splits $\mathfrak{L}$ for every $i \in I$, then we denote the respective join-splitting of $\mathfrak{L}$ by $\mathfrak{L}/F$. It is straightforward to see that for any $a \in \mathfrak{L}$, $a \geq \mathfrak{L}/F$ iff $a \not\leq a_i$ for all $i \in I$.

For any variety V, all its subvarieties form a complete lattice $SubVarV$. In many cases, it is more convenient to deal with its dual $ExtV$, the lattice of all equational theories extending the equational theory of V. This lattice is always complete.

Theorem 9.4 *If Γ splits $ExtV$, it is always of the form $Eq(\mathfrak{A})$ for some finitely generated subdirectly irreducible $\mathfrak{A}$.*

Proof: It is enough to observe that $Eq(V)$ is always determined by a family of finitely generated subdirectly irreducible algebras. Now use Theorem 9.3. $\dashv$

A boolean algebra $\mathfrak{A}$ is a distributive lattice in any similarity type containing lattice connectives, unary connective $\neg$ and a binary connective $\rightarrow$ and two constants $\top$, $\bot$ s.t. elements corresponding to $\top$ and $\bot$ are, respectively, the greatest and the smallest element of $\mathfrak{A}$, for every $a, b \in \mathfrak{A}$, $a \wedge \neg a = \bot$, $a \vee \neg a = \top$ and $a \rightarrow b = \neg a \vee b$. A prime filter of a Boolean algebra is also called *an ultrafilter*. Every congruence θ of a boolean algebra $\mathfrak{A}$ is uniquely determined by its *congruence filter*: $F_\theta \leftrightharpoons \{a \in \mathfrak{A} | a\theta\top\}$. It can be proven that for every $a, b \in \mathfrak{A}$, $a\theta b$ iff $a \leftrightarrow b \in F_\theta$, where $a \leftrightarrow b \leftrightharpoons (a \rightarrow b) \wedge (b \rightarrow a)$. Congruence ideals can be defined dually, with equivalence replaced by symmetric difference. Thus, when dealing with boolean algebras, we may choose freely whether we prefer to deal with congruences, congruence filters or congruence ideals. In this thesis, we always work with congruence filters. It can be proven that a boolean algebra is subdirectly irreducible iff there is smallest congruence filter distinct from $\{\top\}$ and that a boolean algebra is simple iff $\{\top\}$ is the only proper congruence filter. In a subdirectly irreducible algebra, every element in the smallest nontrivial congruence filter is called *an opremum*. As in every boolean algebra $a = b$ iff $a \leftrightarrow b = \top$, we can identify $Eq(\mathfrak{A})$, the set of equations which hold in $\mathfrak{A}$ under all valuations, with $Th(\mathfrak{A})$: the set of all terms which are sent to $\top$ under all valuations. Also, for any set of terms Γ, we define $\mathbb{V}(\Gamma) \leftrightharpoons Mod(\{\gamma = \top | \gamma \in \Gamma\})$. Similarly, $\text{FORM}_\kappa/\Gamma \leftrightharpoons \text{FORM}_\kappa/\{\gamma = \top | \gamma \in \Gamma\}$

A boolean algebra with (unary) operators (BAO) $\mathfrak{A}$ is a Boolean algebra in a finite similarity type s.t. every non-Boolean connective $\Diamond$ is a unary connective s.t. $\Diamond\bot = \bot$ and for every $a, b \in \mathfrak{A}$, $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$. A filter F in a BAO $\mathfrak{A}$ is *open* if for every $a \in F$, $\Box a \leftrightharpoons \neg\Diamond\neg a \in F$. A filter in a BAO is a congruence filter iff it is open. Boolean algebras with operators have the congruence extension property: if F is a filter of $\mathfrak{A} \subseteq \mathfrak{B}$, then there is a filter F' of $\mathfrak{B}$ s.t. $F = F' \cap \mathfrak{A}^2$. It has been observed by Jipsen [30] that a BAO $\mathfrak{A}$ is discriminator iff it has *unary discriminator term*; i.e., a term t s.t. $t^{\mathfrak{A}}(a) = \bot$ iff $a = \bot$ and $t^{\mathfrak{A}}(a) = \top$ otherwise.

Appendix B: set-theoretical preliminaries

In this work, we use a very strong underlying set theory, stronger than the standard Zermelo-Fraenkel system, the von Neumann-Bernays system or even the Kelley-Morse system. The reason, as explained in Section 2.2 is that we want to deal with collections of *proper classes*. The system we use, then, is the one discussed in section 7.5 of Fraenkel et al. [20] under the name of ST_2. The author is not a specialist in set theory and is not very well aware whether this system has received some attention in more recent literature; it seems likely, though. From our point of view it matters only that the system serves its purpose. In case of any difficulties, there is a number of books the reader may consult. Except for Fraenkel et al. [20], a very good, concise and precise presentation of set theory may be found in Kunen [40]. Devlin [14] is a very accessible text, whereas Kuratowski et al. [41], old-fashioned as it is, deserves the name of a classic.

The language has one binary constant $\in$ and one additional unary predicate Set of sethood. We use upper-case letters $X, Y, Z, X_1, Y_1, Z_2 \ldots$ to denote variables; the elements of any model of the theory — i.e., entities over which the variables are intended to range — are called *classes*. Formulas $\forall x \psi$ and $\exists y \psi$ are to be read as *relativizations* of ψ to $Set(x)$, i.e., $\forall X Set(X) \Rightarrow \psi$

and $\exists X Set(X) \wedge \psi$, respectively. Other abbreviations are:

$$
\begin{aligned}
X \subseteq Y &\equiv \forall Z(Z \in X \Rightarrow Z \in Y);\\
\emptyset = X &\equiv \forall Z \, \mathrm{NOT}\,(Z \in Y);\\
\emptyset \in X &\equiv \exists Y(Y \in X \& \forall Z \, \mathrm{NOT}\,(Z \in Y));\\
A \cup \{B\} \in C &\equiv \exists X(X \in C \& \forall Y(Y \in X \Leftrightarrow (Y = B) \, \mathrm{OR}\,(Y \in A)));\\
A = \langle B, C\rangle &\equiv \forall X(X \in A \Leftrightarrow \forall Y(Y \in X \Leftrightarrow Y = A) \, \mathrm{OR} \, \forall Y(Y \in X \Leftrightarrow Y = A \, \mathrm{OR} \, Y = B)\\
\langle A, B\rangle \in X &\equiv \exists Y(Y \in X \& Z = \langle A, B\rangle);\\
Rel(R) &\equiv \forall X(X \in R \Rightarrow \exists Y, Z(X = \langle Y, Z\rangle);\\
D = Dom(R) &\equiv \forall X(X \in D \Leftrightarrow \exists B\langle X, Y\rangle \in R);\\
A \in Dom(R) &\equiv \exists X\langle A, X\rangle \in R;\\
E = Range(R) &\equiv \forall B(B \in E \Leftrightarrow \exists A\langle A, B\rangle \in R);\\
Func(F) &\equiv Rel(F) \& \forall X, Y, Z(\langle X, Y\rangle \in F \& \langle X, Z\rangle \in F \Rightarrow Y = Z).
\end{aligned}
$$

It would be even more proper if instead of *defining*, e.g., $A = \langle B, C\rangle$ we rather used notation $IsAPair(A, B, C)$, but both pedantry of the present author and patience of the reader have its limits. The axioms of the theory are (universal closures of) the following formulas:

sethood $\forall x(A \in x \Rightarrow Set(A))$;

extensionality $\forall X(X \in A \Leftrightarrow X \in B) \Rightarrow A = B$;

pairing for sets $\forall a, b \exists x \forall y(y \in x \Leftrightarrow (y = a \, \mathrm{OR} \, y = b))$;

pairing for classes $\exists X \forall Y(Y \in X \Leftrightarrow (Y = A \, \mathrm{OR} \, Y = B))$;

union for sets $\forall a \exists b \forall x(x \in b \Leftrightarrow \exists c(c \in a \& x \in c))$;

union for classes $\exists X \forall Y(Y \in X \Leftrightarrow \exists Z(Z \in A \& Y \in Z))$;

power set $\forall x \exists y \forall z(z \subseteq x \Leftrightarrow z \in y)$;

power class $\exists X \forall Y(Y \subseteq A \Leftrightarrow Y \in X)$;

infinity $\exists x(\emptyset \in x \& \forall y(y \in x \Rightarrow y \cup \{y\} \in x))$;

subsets $\forall x \exists y \forall Z(Z \in y \Leftrightarrow (Z \in A \wedge Z \in x))$;

foundation $A \neq \emptyset \Rightarrow \exists X(X \in A \& \forall Y(Y \in A \Rightarrow Y \notin X))$;

replacement $\forall a(Func(F) \Rightarrow \exists x \forall y(y \in x \Leftrightarrow \exists zz \in a \& \langle z, y\rangle \in F))$;

replacement for classes $\forall U, V, W(U \in A \& (\varphi(U, V) \& \varphi(U, W) \Rightarrow U = W) \Rightarrow \exists Y \forall X(X \in Y \Leftrightarrow \exists Z(Z \in A \& \varphi(Z, X)))$ where U, V, W, Y are not free in $\varphi(Z, X)$;

impredicative comprehension $\exists X \forall y(y \in X \Leftrightarrow \varphi(y))$;

choice $\exists F(Func(F) \& \forall X(X \in A \& X \neq \emptyset \Rightarrow \exists Y(Y \in X \& \langle x, y\rangle \in F)))$.

Axiomatization could be made slightly more concise if we didn't keep apart axioms for classes and sets so rigorously. For example, instead of separate power set and power class axioms, we could formulate a single axiom:

$$\forall X \exists Y \forall Z((Z \subseteq X \Leftrightarrow Z \in Y) \& (Set(X) \Rightarrow Set(Y))).$$

Regardless of what way of writing down the axioms is chosen, what should be made explicit is that sets are closed under all standard operations, including union and powerset. The axiomatization we used here makes *the two-tier* (Fraenkel et al. [20, p. 143]) character of this theory very clear. What we mean is that axioms governing the interaction between sets and classes are those of Kelley-Morse set theory, whereas the axioms governing behaviour of all classes are those of ZF itself - with "set" replaced by "class". We are going to see soon why this procedure does not lead to any inconsistency (assuming consistency of other well-known mathematical theories) and how a model for such a strong theory may look like. Another point to make that as, in case of ZF, some of the axioms (e.g., pairing) are redundant.

Classes in the extensions of Set are called *sets* (thus, the class of all sets is just the extension of Set; such a class exists by comprehension); the remaining ones are called *proper classes*. Classes in the extensions of defined predicates Rel and $Func$ are called *class relations* and *class functions*, respectively. A class relation (class function) which is a set is called *a relation* (*a function*). Instead of writing $\langle X, Y\rangle \in R$, we usually write $R(X, Y)$; if R is a class function, it is written instead as $R(X) = Y$. The converse of class relation R, denoted as R^{-1}, is the class of all $\langle Y, X\rangle$ s.t. $\langle X, Y\rangle \in R$. The *domain of class relation* R is $Dom(R)$: the class of all X s.t. for some Y, $\langle X, Y\rangle \in R$. The *range* of R is $Range(R)$: the domain of its converse. If R is a class relation and X is a subclass of its domain, then the R-image of X, denoted as $R[X]$, is the class of all elements Y s.t. for some $Z \in X$, $\langle Z, Y\rangle \in R$.

A class function F is *injective* or 1–1 if its converse is a class function. F is *onto* X if the image of the domain of F contains X. X is *finite* if no function whose domain is a proper subset of X is onto X. If $X \subseteq Dom(R)$, then *a restriction* of R to X is defined as $R|_X \leftrightharpoons X \times R[X] \cap R$.

$X \cap Y$, $X \cup Y$ and $\bigcup X$ are defined in the standard way. If all elements of X are sets, we may also define $-X$: it is the class of all sets which are not in X. *A singleton* is any one-element class $\{X\}$ and a doubleton is any two-element class $\{X, Y\}$. The class of all doubletons from X is denoted as $[X]^2$. The class of all ordered pairs $\langle X, Y \rangle$ from A and B is denoted as $A \times B$ and is called *the cartesian product* of A and B. The cartesian product of X with itself is denoted as X^2. A generalization of the notion of a pair is *an n-tuple* $\langle \dots \langle\langle X_1, X_2 \rangle X_3 \rangle \dots X_{n-1} \rangle$; the class of all n-tuples of elements from X is denoted as X^*. *The generalized product* Y^X is the class of all functions from X to Y; to avoid confusion, sometimes the notation ${}^XY$ is used. *The power class* of X, denoted by 2^X is the collection of all $Y \subseteq X$. The axioms guarantee that for all X and Y, $X \cup Y$, $X \cap Y$, $\bigcup X$, X^2, $[X]^2$, $X \times Y$, X^*, Y^X and 2^X exist. In addition, *Set* is closed with respect to all those operations; i.e., if x and y are sets, then $x \cup y$, $x \cap y$, $\bigcup x$, x^2, $[x]^2$, $x \times y$, x^*, y^x and 2^y are sets, too. On the other hand, $-x$ is *never* a set. In general, for any X and Y a set we may define *relative complement* $X - y$ as the class of all elements of X which are not in Y. It is provable that if X and Y are sets, then $X - Y$ is a set, but if X is a proper class and y a set, then $X - y$ is a proper class. The class of all sets for which $\varphi(x)$ holds is written as $\{x | \varphi(x)\}$.

A class relation R is called *a (strict) partial order* on A if its restriction to A is *transitive* (i.e. for all $X, Y, Z \in A$, $R(X, Y)$ and $R(Y, Z)$ imply $R(X, Z)$) and *irreflexive* (i.e., for no $X \in A$, $R(X, X)$). It is called a *total order* if it is linear, i.e., for all $X, Y \in A$ either $R(X, Y)$ or $R(Y, X)$ hold. A total order is *a well-order* on A if it is *well-founded*, i.e., every subclass of the domain has an R-minimal element. $Y \in A$ is *a supremum of* $X \subseteq A$ if for all $Z \in X$, $Z \neq Y$ implies $R(Z, Y)$. Y is *a maximal element* of A if for no $Z \in A$, $R(A, Z)$.

X is called $\in$-*transitive* if for every $Y \in X$ and $Z \in Y$, $Z \in X$. For every class X, there exists the smallest $\in$-transitive class Y containing all elements of X. It is called the *transitive closure* of X. A class is called *a class ordinal* if it is $\in$-transitive and well-ordered by $\in$. A class ordinal is *an ordinal number* if it is a set. The (proper) class of all ordinal numbers is denoted by *Ord*. It is itself well-ordered by $\in$. We write $\alpha \leq \beta$ to denote that $\alpha \in \beta$ or $\alpha = \beta$. Every well-ordered set is isomorphic to an ordinal number and every well-ordered class is isomorphic to a class ordinal, i.e., if R is a well-order on X, then there is a class ordinal α (an ordinal number, if X is a set) and a class function F such that $X \subseteq Dom(F)$, $F|_X$ is injective and onto α (i.e., F is a *bijection* between X and α)and for every $Y, Z \in X$, $R(Y, Z)$ if $f(Y) \in f(Z)$. In addition, the axiom of choice implies that for every class X there exist a class ordinal α (an ordinal number if X is a set) and a class function F s.t. F is a bijection between X and α. Two well-known consequences of this fact are

Theorem 9.5 (Zermelo) *For every class X, there is a class relation R which well-orders X.*

Lemma 9.6 (Kuratowski, Zorn) *If R is a partial order on $X \neq \emptyset$ s.t. for every nonempty subclass of X which is linearly ordered by R has a supremum, then every $A \in X$ is below a maximal element.*

As we are interested mostly in sets, from now on we discuss only ordinal numbers, relations, function and sets unless stated otherwise. We have learned that for every x there exists an ordinal number α s.t. there is a bijection between x and α. As the class Ord is itself well-ordered by $\in$, for every x there is smallest α which is such a property. It is called *the cardinality* of x and denoted by $|x|$. The class of all ordinal numbers which are cardinalities of some sets is called the class of *cardinal numbers* and denoted by $Card$. It is straightforward to see that for every ordinal number α, $|\alpha| \leq \alpha$. On the other hand, we have the following

Theorem 9.7 (Cantor) *For every x, $|x| \in |2^x|$.*

Thus, every ordinal number is below some cardinal number. Finite cardinal numbers are identified with natural numbers and denoted as 0, 1, 2, ... (thus, we identify $\emptyset$ and 0). The set of all finite ordinals is denoted as ω. Every finite ordinal is a cardinal. Cantor's theorem implies there are infinitely many infinite cardinals. For every ordinal α, there exists the smallest ordinal α' s.t. $\alpha = \alpha'$. β is called *a successor ordinal* if there exists α s.t. $\beta = \alpha'$ and *a limit ordinal* is a nonempty ordinal which is not a successor. The class of all limit ordinals is denoted by Lim. All limit ordinals are infinite, but the converse does not hold. Similarly, all cardinal numbers are limit ordinals, but the converse does not hold.

A crucial property of ordinals is highlighted by the following:

Theorem 9.8 (Definition by transfinite induction) *For every class function F s.t. $Set \subseteq Dom(F)$, there is a unique function G whose domain is Ord and $G(\alpha) = F(G|_\alpha)$.*

We can order now all cardinals numbers in a sequence indexed by ordinal numbers. For arbitrary ordinal number α, define α^+ to be the smallest cardinal number strictly greater than α.

$$\aleph_0 \leftrightharpoons \omega, \quad \aleph_{\alpha'} \leftrightharpoons (\aleph_\alpha)^+; \quad \aleph_\alpha \leftrightharpoons \bigcup_{\beta\in\alpha} \aleph_\beta (\alpha \in Lim).$$

Thus, every cardinal number is equal to $\aleph_\alpha$ for some ordinal α. $\aleph_\alpha$ is called *a successor (limit) cardinal* if α is a successor (limit) ordinal. The Transfinite Induction Theorem also allows us to formulate definitions of addition, multiplication and exponentiation of ordinals as follows:

$$\alpha + 0 \leftrightharpoons \alpha, \quad \alpha + \beta' \leftrightharpoons (\alpha + \beta)'; \quad \alpha + \beta \leftrightharpoons \bigcup_{\gamma\in\beta} (\alpha + \gamma)(\beta \in Lim); \tag{9.1}$$

$$\alpha * 0 \leftrightharpoons 0, \quad \alpha * \beta' \leftrightharpoons \alpha * \beta + \alpha; \quad \alpha * \beta \leftrightharpoons \bigcup_{\gamma\in\beta} (\alpha * \gamma)(\beta \in Lim); \tag{9.2}$$

$$\alpha^0 \leftrightharpoons 1, \qquad \alpha^{\beta'} \leftrightharpoons \alpha^\beta * \alpha; \qquad \alpha^\beta \leftrightharpoons \bigcup_{\gamma\in\beta} (\alpha^\gamma)(\beta \in Lim). \tag{9.3}$$

Ordinal addition and multiplication are associative but not commutative. Every limit ordinal λ can be represented as $\omega * \kappa$, for some $\kappa \le \lambda$. Ordinal exponentiation is different from *cardinal exponentiation.* If κ and λ are cardinals, then $\kappa^\lambda \leftrightharpoons |{}^\kappa\lambda|$. The operation of cardinal exponentiation is in general more important than ordinal exponentiation. Therefore, if not stated otherwise, if $\kappa, \lambda \in Card$, then κ^λ is taken to be the result of the cardinal exponentiation. $\mathfrak{c}$ is $2^{\aleph_0}$.

Assume $\alpha, \beta \in Ord$. A function $f : \alpha \mapsto \beta$ is called *cofinal* if for every $\delta \in \beta$ there exists $\gamma \in \alpha$ s.t. $\delta \le f(\gamma)$. The *cofinality* of α, $cf(\alpha)$ is the smallest ordinal β s.t. there exists a cofinal function from β to α. It is provable that $cf(\alpha) \le \alpha$ (identity mapping is cofinal) and $cf(cf(\alpha)) = cf(\alpha)$. An ordinal α is *regular* if it is a limit ordinal and $cf(\alpha) = \alpha$. Every regular ordinal is a cardinal. Every successor cardinal is regular. It follows from a consequence of the axiom of choice: if κ is an infinite cardinal, $|x| < \kappa$ and for every $y \in x$, $|y| < \kappa$, then $|\bigcup x| < \kappa$. An example of an infinite cardinal which is not regular is $\aleph_\omega$: clearly, $|\aleph_\omega| = \omega$. A regular cardinal number α greater than ω is called *(strongly) inaccessible* if for all $\beta \in \alpha$, $2^\beta \in \alpha$.

The class of all sets x s.t. the cardinality of the transitive closure of x is stricitly smaller than any inaccessible cardinal is closed under all standard set-theoretical operations introduced in the beginning. It may be taken to be an universe of a model of Zermelo-Fraenkel set theory ZFC. This theory, most popular among mathematicians, is obtained from the present one by relativization of all formulas to Set and taking as axioms extensionality, pairing and union for sets, power set, infinity, restriction of foundation, choice and replacement for classes to sets. Consistency of ZFC is provable in the present set theory ST_2; thus, ST_2 is strictly stronger than ZFC as consistency of ZF is not provable in ZFC itself. Nevertheless, our theory is consistent relative to ZFC enriched with an additional axiom which may be informally expressed as "there exist an inaccessible ordinal" (cf. Fraenkel et al. [20, pp. 141–143]. At the present stage of development of set theory, this is a noncontroversial assumption.

If we assume that *a model* or *an algebra* is a structure whose universe is always a set, our theory turns out to be very convenient for model theory and universal algebra. Many collections one encounters in those fields — for example, the class of all algebras in a given similarity type — are proper classes. The only way to handle proper classes in ZFC is to treat them as *linguistic objects*: i.e., to identify them with (extensions of) formulas of ZFC. This is not very elegant; and, from this point of view, the notion of a class being a member of some other class does not make much sense. But sometimes, as in the present work, we want to handle classes whose elements are proper classes; similar problems arise in category theory (cf. Fraenkel et al. [20, pp. 143–145]), which is only briefly touched upon in this thesis. ST_2 seems much more natural from this point of view.

Of course, the theory under consideration is not any magical tool which solves all foundational problems. If we want to study very large collections of classes, such as the collection of all class ordinals or simply the collection of classes, we end up with problems analogous to problems arising with large classes of sets in ZFC. Argument analogous to Russell paradox ensures that there does not exist the class of all classes. Thus, we have to treat such collections the same way one treats proper classes in ZFC — by identifying them with formulas. Nevertheless, such problems arise only when ones studies the metatheory of ST_2 itself. For all foundational applications, including universal algebra, model theory and category theory, the system serves its purposes well. Thus, the claim that mathematics is reducible to ST_2 seems more justifiable than analogous claim for ZFC without additional axioms concerning the existence of large cardinal numbers.

Appendix C: Second-order logic and strong consequence

We provide here a proof of Theorem 7.10. Set

$$\text{TYPE} \leftrightharpoons \{1F, 1P, \dots, 15F, 15P\}.$$

Proof: It may be easier to give first an intuitive description of Thomason's ideas. Before providing a proper translation of the second-order language, we need to provide an arithmetization of a sort. This may be carried out in various ways, but in order to make references easier, we stick to the original arithmetization of Thomason. Given a formula α of the monadic second-order language with a single binary constant $\lhd$, we identify it with a triple $\tau(\alpha) \leftrightharpoons \langle \tau_1(\alpha), \tau_2(\alpha), \tau_3(\alpha) \rangle$. The peculiarity of Thomason's translation is that elements of the triple can be either a number or formulas themselves, albeit of a strictly lower complexity than α (thus, we could in principle reduce everything to triples of numbers — or even to natural numbers themselves).

$$\tau(x_i \lhd x_j) \leftrightharpoons \langle 0, i, j \rangle, \quad \tau(x_i = x_j) \leftrightharpoons \langle 1, i, j \rangle, \quad \tau(X_j(x_i)) = \langle 2, i, j \rangle,$$

$$\tau(\neg\alpha) \leftrightharpoons \langle 3, 3, \alpha \rangle, \quad \tau(\alpha \vee \beta) \leftrightharpoons \langle 4, \alpha, \beta \rangle,$$
$$\tau(\exists x_i \alpha) \leftrightharpoons \langle 5, i, \alpha \rangle, \quad \tau(\exists X_i \alpha) \leftrightharpoons \langle 6, i, \alpha \rangle.$$

Given a structure $\mathfrak{M} \leftrightharpoons \langle M, \lhd \rangle$ which is a model of monadic second-order logic, we turn it into a relational structure for our modal language $\mathsf{Modal}(\mathfrak{M}) \leftrightharpoons \langle W, \{R_\pi\}_{\pi \in \text{TYPE}} \rangle$. $W \leftrightharpoons U \cup V \cup N \cup A \cup S$, where

- U consists of countably many disjoint copies of M itself — $U \leftrightharpoons \bigcup_{i \in \omega} U_i$;

- V consists of countably many disjoint copies of the powerset of M — $V \leftrightharpoons \bigcup_{i\in\omega} V_i$;
- N is a copy of natural numbers, used for technical reasons;
- A is the set of all monadic second-order formulas in the similarity type determined by $\mathfrak{M}$;
- S is the set of all valuations of second-order formulas in $\mathfrak{M}$.

The 30 relations $\{R_{1F}, R_{1P}, \ldots, R_{15F}, R_{15P}\}$ are interpreted as follows:

- $R_{iP} \leftrightharpoons R_{iF}^{-1}$;
- $R_{1F} \leftrightharpoons \lhd$;
- R_{2F} is an irreflexive well-ordering of W (if necessary, obtained by Axiom of Choice);
- R_{3F} is irreflexive well-ordering of N, i.e., the standard strict order of natural numbers;
- R_{4F} and R_{5F} are functions from, respectively, U and V onto N s.t. the inverse image of $\{i\}$ is, respectively, U_i and V_i;
- R_{6F} and R_{7F} are coding the $\in$ relation between U_0 and V_0 and its complement — i.e. $u \in v$ iff $uR_{6F}v$ iff NOT $uR_{7F}v$;
- R_{8F} is a function from U onto U_0, assigning to every element of U_n its U_0-counterpart;
- R_{9F} is a function from V onto V_0, assigning to every element of V_n its V_0-counterpart;
- R_{10P} is a function from A to N assigning to every formula its degree of complexity defined in a standard way;
- R_{11P}, R_{12P} and R_{13P} are functions from A to $N \cup A$ which assign to α $\tau_1(\alpha)$, $\tau_2(\alpha)$ and $\tau_3(\alpha)$, respectively;
- $R_{14F} \subseteq (U \cup V) \times S$ is a relationship s.t. for any $u \in U_n$, $uR_{14F}s$ iff s assigns x_n to u and for any $v \in V_n$, $vR_{14F}s$ iff s assigns X_n to v;

- $R_{15F} \subseteq A \times S$ is the satisfaction relation, i.e., $\alpha R_{15F}\sigma$ iff σ satisfies α in $\mathfrak{M}$.

The longish list of definitions and axioms below is due to Thomason himself modulo some minor changes and improvements. The axioms marked with $\Leftarrow$ are those of essentially second-order character.

- $\mathbf{E}\alpha \leftrightharpoons \alpha \vee \Diamond_{2F}\alpha \vee \Diamond_{2P}\alpha, \qquad \mathbf{A}\alpha \leftrightharpoons \neg\mathbf{E}\neg\alpha;$
- $Unit(\alpha) \leftrightharpoons \mathbf{E}\alpha \wedge \mathbf{A}(\alpha \to \neg(\Diamond_{2F}\alpha \vee \Diamond_{2P}\alpha));$
- $\underline{N} \leftrightharpoons \Diamond_{3F}\top, \qquad Num(\alpha) \leftrightharpoons Unit(\alpha) \wedge \mathbf{A}(\underline{N} \wedge \alpha);$
- $Succ(\alpha) \leftrightharpoons \underline{N} \wedge (\Diamond_{3P}\alpha \wedge \Box_{3P}(\alpha \vee \Diamond_{3F}\alpha);$
- $\underline{0} \leftrightharpoons \underline{N} \wedge \Box_{3P}\bot, \qquad \underline{n+1} \leftrightharpoons Succ(\underline{n});$
- $\underline{U} \leftrightharpoons \Diamond_{4F}\underline{N}, \qquad \underline{U}_\alpha \leftrightharpoons \Diamond_{4F}\alpha;$
- $\underline{V} \leftrightharpoons \Diamond_{4F}\underline{N}, \qquad \underline{V}_\alpha \leftrightharpoons \Diamond_{4F}\alpha;$
- $Indvl(\alpha) \leftrightharpoons Unit(\alpha) \wedge \mathbf{E}(\underline{U_0} \wedge \alpha), \qquad Pred(\alpha) \leftrightharpoons Unit(\alpha) \wedge \mathbf{E}(\underline{V_0} \wedge \alpha);$
- $\underline{FO} \leftrightharpoons \Diamond_{10P}\underline{N}, \qquad \underline{FO}_\alpha \leftrightharpoons \Diamond_{10P}\alpha;$
- $Frml(\alpha) \leftrightharpoons Unit(\alpha) \wedge \mathbf{E}(\underline{FO} \wedge \alpha);$
- $Frml_\beta(\alpha) \leftrightharpoons Unit(\alpha) \wedge \mathbf{E}(\underline{FO} \wedge \alpha) \wedge \Diamond_{10P}\beta;$
- $\underline{S} \leftrightharpoons \Diamond_{14P}\top, \qquad Asgm \leftrightharpoons Unit(\alpha) \wedge \mathbf{E}(\underline{S} \wedge \alpha);$
- $\alpha''_x\beta \leftrightharpoons \underline{U_0} \wedge \Diamond_{8P}(\Diamond_{4F}\beta \wedge \Diamond_{14F}\alpha);$
- $\alpha''_X\beta \leftrightharpoons \underline{V_0} \wedge \Diamond_{9P}(\Diamond_{5F}\beta \wedge \Diamond_{14F}\alpha);$
- $\tau_1(\alpha) \leftrightharpoons \Diamond_{11F}\alpha, \qquad \tau_2(\alpha) \leftrightharpoons \Diamond_{12F}\alpha, \qquad \tau_3(\alpha) \leftrightharpoons \Diamond_{13F}\alpha;$
- $\alpha \lhd \beta \leftrightharpoons \mathbf{E}(\alpha \wedge \Diamond_{1F}\beta);$
- $\alpha\, EQTO\, \beta \leftrightharpoons \mathbf{E}(\alpha \wedge \beta), \qquad \alpha\, ELOF\, \beta \leftrightharpoons \mathbf{E}(\alpha \wedge \Diamond_{6F}\beta);$
- $Stsf(\alpha) \leftrightharpoons Frml(\alpha) \wedge \underline{S} \wedge \Diamond_{15P}\alpha;$
- $Stsf(\alpha, \beta) \leftrightharpoons Asgm(\alpha) \wedge \mathbf{E}(\alpha \wedge Stsf(\beta));$

Thom1. $\bigwedge_{i\leq 15}((p \to \Box_{iP}\Diamond_{iF}p) \land (p \to \Box_{iF}\Diamond_{iP}p))$;

Thom2. $\bigvee_{i\leq 15}\Diamond_{iF}p \to p \lor \Diamond_{2F}p\Diamond_{2P}p$;

Thom3. $\mathbf{Lin}_{2F}^{2P}$;

Thom4. $\mathbf{GL}_{2P}$; $(\Leftarrow)$

Thom5. $\mathbf{E}\underline{N}$;

Thom6. $\underline{N} \land \mathbf{E}(\underline{N} \land p) \to p \lor \Diamond_{3F}p \lor \Diamond_{3P}p$;

Thom7. $\Diamond_{3P}\underline{N} \to \underline{N}$;

Thom8. $\mathbf{Lin}_{3F}^{3P}$;

Thom9. $\mathbf{GL}_{3P}$; $(\Leftarrow)$

Thom10. $\mathbf{A}(p \to \underline{N}) \land \mathbf{E}p \land \mathbf{A}(p \to \Box_{3F}p) \land \mathbf{A}(p \to \Diamond_{3F}p) \to \mathbf{A}(\underline{N} \to p)$; $(\Leftarrow)$

Thom11. $\Diamond_{4F}\top \leftrightarrow \underline{U}$;

Thom12. $\Diamond_{4P}\top \leftrightarrow \underline{N}$;

Thom13. $\Diamond_{5F}\top \leftrightarrow \underline{V}$;

Thom14. $\Diamond_{5P}\top \leftrightarrow \underline{N}$;

Thom15. $\mathbf{Func}_{4F}$;

Thom16. $\mathbf{Func}_{5F}$;

Thom17. $\mathbf{E}\underline{U_0}$;

Thom18. $\Diamond_{6F}\top \leftrightarrow \underline{U_0}$;

Thom19. $\Diamond_{6P}\top \to \underline{V_0}$;

Thom20. $\Diamond_{7F}\top \leftrightarrow \underline{U_0}$;

Thom21. $\Diamond_{7P}\top \to \underline{V_0}$;

Thom22. $Indvl(p) \wedge Pred(q) \rightarrow \mathbf{A}(p \rightarrow (\Diamond_{6F} q \leftrightarrow \neg \Diamond_{7F} q));$

Thom23. $Pred(p) \wedge Pred(q) \wedge \mathbf{E}(p \wedge \Box_{6H} r \wedge \Box_{7H} \neg r) \wedge \mathbf{E}(q \wedge \Box_{6H} r \wedge \Box_{7H} \neg r);$

Thom24. $\mathbf{E}(\underline{V_0} \wedge \Box_{6P} p \wedge \Box_{7P} \neg p);$ $(\Leftarrow)$

Thom25. $\Diamond_{8P} \top \leftrightarrow \underline{U_0};$

Thom26. $\Diamond_{8F} \top \leftrightarrow \underline{U};$

Thom27. $Num(p) \wedge \underline{U_0} \rightarrow \Diamond_{8P} \underline{U_p};$

Thom28. $\underline{U} \rightarrow \Diamond_{8F} \underline{U_0};$

Thom29. $Num(p) \wedge \Diamond_{8P}(\underline{U_p} \wedge q) \rightarrow \Box_{8P}(\underline{U_p} \rightarrow q);$

Thom30. $\mathbf{Func}_{8F};$

Thom31. $\underline{U_0} \rightarrow \mathbf{Triv}_{8F};$

Thom32. $\Diamond_{9P} \top \leftrightarrow \underline{V_0};$

Thom33. $\Diamond_{9F} \top \leftrightarrow \underline{V};$

Thom34. $Num(p) \wedge \underline{V_0} \rightarrow \Diamond_{9P} \underline{V_p};$

Thom35. $\underline{V} \rightarrow \Diamond_{9F} \underline{V_0};$

Thom36. $Num(p) \wedge \Diamond_{9P}(\underline{V_p} \wedge q) \rightarrow \Box_{9P}(\underline{V_p} \rightarrow q);$

Thom37. $\mathbf{Func}_{9F};$

Thom38. $\underline{V_0} \rightarrow \mathbf{Triv}_{9F};$

Thom39. $\Diamond_{10P} \top \leftrightarrow \underline{FO};$

Thom40. $\Diamond_{10F} \top \leftrightarrow \underline{N};$

Thom41. $\mathbf{Func}_{10P};$

Thom42. $\underline{FO} \leftrightarrow \Diamond_{11F} \top \wedge \Diamond_{12F} \top \wedge \Diamond_{13F} \top;$

Thom43. $\Diamond_{11P} p \rightarrow \Box_{11P}(p \wedge (\underline{N} \vee \underline{FO}));$

Thom44. $\Diamond_{12P}p \to \Box_{12P}(p \wedge (\underline{N} \vee \underline{FO}))$;

Thom45. $\Diamond_{13P}p \to \Box_{13P}(p \wedge (\underline{N} \vee \underline{FO}))$;

Thom46. $Unit(p) \wedge Unit(q) \wedge Unit(r) \wedge Frml(s) \wedge Frml(t) \wedge \mathbf{E}(s \wedge \Diamond_{11P}p \wedge \Diamond_{12P}q \wedge \Diamond_{13P}r) \wedge \mathbf{E}(t \wedge \Diamond_{11P}p \wedge \Diamond_{12P}q \wedge \Diamond_{13P}r) \to \mathbf{A}(s \leftrightarrow t)$;

Thom47. $Num(p) \wedge Num(q) \to \mathbf{E}(\underline{FO}_{\underline{0}} \wedge \Diamond_{11P}\underline{0} \wedge \Diamond_{12P}p \wedge \Diamond_{13p}q)$;

Thom48. $Num(p) \wedge Num(q) \to \mathbf{E}(\underline{FO}_{\underline{0}} \wedge \Diamond_{11P}\underline{1} \wedge \Diamond_{12P}p \wedge \Diamond_{13p}q)$;

Thom49. $Num(p) \wedge Num(q) \to \mathbf{E}(\underline{FO}_{\underline{0}} \wedge \Diamond_{11P}\underline{2} \wedge \Diamond_{12P}p \wedge \Diamond_{13P}q)$;

Thom50. $Num(p) \wedge Frml_p(q) \to \mathbf{E}(\underline{FO}_{Succ(p)} \wedge \Diamond_{11P}\underline{3} \wedge \Diamond_{12P}\underline{3} \wedge \Diamond_{13P}q)$;

Thom51. $Num(p) \wedge Frml_{p \vee \Diamond_{3F}p}(q) \wedge Frml_{p \vee \Diamond_{3F}p}(r) \wedge (Frml_p(q) \vee Frml_p(r)) \to \mathbf{E}(\underline{FO}_{Succ(p)} \wedge \Diamond_{11P}\underline{4} \wedge \Diamond_{12P}q \wedge \Diamond_{13P}r)$;

Thom52. $Num(p) \wedge Num(q) \wedge Frml_p(r) \to \mathbf{E}(\underline{FO}_{Succ(p)} \wedge \Diamond_{11P}\underline{5} \wedge \Diamond_{12P}q \wedge \Diamond_{13P}r)$;

Thom53. $Num(p) \wedge Num(q) \wedge Frml_p(r) \to \mathbf{E}(\underline{FO}_{Succ(p)} \wedge \Diamond_{11P}\underline{6} \wedge \Diamond_{12P}q \wedge \Diamond_{13P}r)$;

Thom54. $\underline{FO}_{\underline{0}} \to (\Diamond_{11P}\underline{0} \vee \Diamond_{11P}\underline{1} \vee \Diamond_{11P}\underline{2}) \wedge \Diamond_{12P}\underline{N} \wedge \Diamond_{13P}\underline{N}$;

Thom55. $Num(p) \wedge \underline{FO}_{Succ(p)} \to (\Diamond_{11P}\underline{3} \wedge \Diamond_{12P}\underline{3} \wedge \Diamond_{13P}\underline{FO}_p) \vee (\Diamond_{11P}\underline{4} \wedge \Diamond_{12P}\underline{FO}_{p \vee \Diamond_{3F}p} \wedge \Diamond_{13P}\underline{FO}_{p \vee \Diamond_{3F}p} \wedge (\Diamond_{12P}\underline{FO}_p \vee \Diamond_{13P}\underline{FO}_p)) \wedge (\Diamond_{11P}\underline{5} \wedge \Diamond_{12P}\underline{N} \wedge \Diamond_{13P}\underline{FO}_p) \wedge (\Diamond_{11P}\underline{6} \wedge \Diamond_{12P}\underline{N} \wedge \Diamond_{13P}\underline{FO}_p)$;

Thom56. $\underline{S} \to \Box_{14P}(\underline{U} \wedge \underline{V})$;

Thom57. $Asgm(p) \wedge Num(q) \to Unit(\Diamond_{14F}p \wedge \Diamond_{4F}q) \wedge Unit(\Diamond_{14F}p \wedge \Diamond_{5F}q)$;

Thom58. $\mathbf{A}(\underline{N} \to \Diamond_{4P}p \wedge \Diamond_{5P}p) \to \mathbf{E}(\underline{S} \wedge \Box_{14P}p)$;

Thom59. $Asgm(p) \wedge Asgm(q) \wedge \mathbf{E}(p \wedge \Box_{14P}\Diamond_{14F}q) \to \mathbf{A}(p \leftrightarrow q)$;

Thom60. $\Diamond_{15P}\top \leftrightarrow \underline{S}$;

Thom61. $\Diamond_{15F}\top \to \underline{FO}$;

Thom62. $Asgm(p) \wedge Frml(q) \wedge (\tau_1(q)\, EQTO\, \underline{0}) \to (Stsf(p,q) \leftrightarrow p''_x\tau_2(q) \lhd p''_x\tau_3(q))$;

Thom63. $Asgm(p) \wedge Frml(q) \wedge (\tau_1(q)\, EQTO\, \underline{1}) \rightarrow (Stsf(p,q) \leftrightarrow p''_x\tau_2(q)\, EQTO\, p''_x\tau_3(q))$;

Thom64. $Asgm(p) \wedge Frml(q) \wedge (\tau_1(q)\, EQTO\, \underline{2}) \rightarrow (Stsf(p,q) \leftrightarrow p''_x\tau_2(q)\, ELOF\, p''_X\tau_3(q))$;

Thom65. $Asgm(p) \wedge Frml(q) \wedge (\tau_1(q)\, EQTO\, \underline{3}) \rightarrow (Stsf(p,q) \leftrightarrow \neg Stsf(p,\tau_3(q))$;

Thom66. $Asgm(p) \wedge Frml(q) \wedge (\tau_1(q)\, EQTO\, \underline{4}) \rightarrow (Stsf(p,q) \leftrightarrow Stsf(p,\tau_2(q)) \vee Stsf(p,\tau_3(q)))$;

Thom67. $Asgm(p) \wedge Frml(q) \wedge (\tau_1(q)\, EQTO\, \underline{5}) \rightarrow (Stsf(p,q) \leftrightarrow \mathbf{E}Stsf(\tau_3(q)) \wedge \Box_{14P}(\underline{U}_{\tau_2(q)} \vee \Diamond_{14F}p)))$; $(\Leftarrow)$

Thom68. $Asgm(p) \wedge Frml(q) \wedge (\tau_1(q)\, EQTO\, \underline{6}) \rightarrow (Stsf(p,q) \leftrightarrow \mathbf{E}Stsf(\tau_3(q)) \wedge \Box_{14P}(\underline{V}_{\tau_2(q)} \vee \Diamond_{14F}p)))$; $(\Leftarrow)$

Let Λ be the logic axiomatized by Thom1 — Thom68. Observe that Λ has a WOB universal modality and the relevant operators are $\Diamond_{2P}$ and $\Diamond_{2P}$; moreover, by Thom1, every operator has a conjugate. Thus, if $\mathfrak{A} \in \mathcal{SC} \cap \mathbb{V}(\Lambda)$, we may use Theorem 6.5 and prove that $\mathfrak{A}$ is isomorphic to the dual algebra of a Kripke frame based on an ordinal, with transitive closure of $\in$ interpreting $\Diamond_{2F}$. Now we can prove the following

Claim 1: Assume $\mathfrak{A} \in \mathcal{SC} \cap \mathbb{V}(\Lambda)$ and let $\mathfrak{F}$ be arbitrary Kripke frame s.t. $\mathfrak{A} = \mathfrak{F}^+$. Then $\mathfrak{F}$ is isomorphic to $\mathsf{Modal}(\mathfrak{M})$ for some $\mathfrak{M}$ — a model for monadic second-order logic.

Proof of claim: (sketch) Exactly the same as in Thomason [52]. It is a useful exercise to prove it using axioms Thom1 — Thom68 and in this way understand their semantic meaning. Briefly speaking, U is the set of points verifying $\underline{U}$ and $U_i = \underline{U_i}$, V is the set of points verifying $\underline{V}$ and $V_i = \underline{V_i}$, N is the set of points verifying $\underline{N}$, A is the set of points verifying $\underline{FO}$, S is the set of points verifying $\underline{S}$ and relations are exactly those interpreting modalities with relevant subscripts; again, it may be verified that relations behave exactly as postulated. We outline here only the most important part — that for $R_10, \ldots R_15$.

- $R_{10P} \ldots R_{13P}$. Axioms Thom39–Thom45 ensure that R_{10P} is a function from A to N (denoted by deg), R_{11P}, R_{12P}, R_{13P} are functions from A to $A \cup N$ denoted, respectively, as $(\,)_1$, $(\,)_2$, $(\,)_3$. Thom46 implies that if $(a)_1 = (a')_1$, $(a)_2 = (a')_2$, $(a)_3 = (a')_3$, then $a = a'$.

Thus, we may define a function from the set of monadic second-order formulas to A:

$$t(\alpha) = u \text{ iff } \begin{cases} (u)_1 = 0\,(u)_2 = i, (u)_3 = j & : \quad \alpha = x_i \lhd x_j; \\ (u)_1 = 1\,(u)_2 = i, (u)_3 = j & : \quad \alpha = x_i = x_j; \\ (u)_1 = 2\,(u)_2 = i, (u)_3 = j & : \quad \alpha = X_j(x_i); \\ (u)_1 = 3\,(u)_2 = 3, (u)_3 = t(\beta) & : \quad \alpha = \neg\beta; \\ (u)_1 = 4\,(u)_2 = t(\beta), (u)_3 = t(\gamma) & : \quad \alpha = \beta \vee \gamma; \\ (u)_1 = 5\,(u)_2 = i, (u)_3 = t(\beta) & : \quad \alpha = \exists x_i\beta; \\ (u)_1 = 6\,(u)_2 = i, (u)_3 = t(\beta) & : \quad \alpha = \exists X_i\beta. \end{cases}$$

– By induction on complexity of formulas, t is injective.
– By Thom47–Thom53 and induction on complexity of α, t is defined for every formula.
– By Thom54, Thom55 and induction on $dega$, t is onto.

Moreover, the degree of complexity of α is equal to $deg(t(\alpha))$.

- R_{14P}. Thom56 and Thom57 ensure that $R_{14P} \subseteq S \times (U \cup V)$ and every $s \in S$ has exactly one R_{14P}-successor in U_0, V_0, U_1, V_1, ... Thus, we may define a function $\sigma(\,)$ from S into the set of all second-order valuations as:

  – $\sigma(s)(x_i)$ is the unique R_8 successor in U_0 of the R_{14P}-precedessor of s in U_i;
  – $\sigma(s)(X_i)$ is the unique R_9 successor in V_0 of the R_{14P}-precedessor of s in V_i.

  Thom58 and Thom59 imply that $\sigma(\,)$ is onto and injective, respectively.

- R_{15F}. Thom60 and Thom61 ensure that $R_{15F} \subseteq A \times S$. It has been shown above how to identify points in A with formulas and points of S with assignments. It follows that by Thom62–Thom68, R_{15F} is indeed the satisfaction relation.

$\dashv$

Claim 2: For every $\mathfrak{M}$ — a model for monadic second-order logic, $\mathsf{Modal}(\mathfrak{M})^+ \in \mathbb{V}(\Lambda) \cap \mathcal{SC}$.

Proof of claim: It is straightforward, although tedious, to verify that for every $\mathfrak{M}$, $\mathsf{Modal}(\mathfrak{M})$ verifies all axioms Thom1 — Thom68, i.e.,

$$\mathsf{Modal}(\mathfrak{M})^+ \in \mathbb{V}(\Lambda).$$

$\mathsf{Modal}(\mathfrak{M})^+$ is complete, being the dual of a Kripke frame. And subdirect irreducibility of $\mathfrak{M}^+$ follows from the fact that R_{2F} is a total order. $\dashv$

Define a function g from the set of monadic second-order formulas in countably many individual and set variables to FORM$_\omega$:

$$g(\alpha) \leftrightharpoons \begin{cases} \Diamond_{11P}\underline{0} \wedge \Diamond_{12P}\underline{i} \wedge \Diamond_{13P}\underline{j} & : \quad \alpha = x_i \lhd x_j; \\ \Diamond_{11P}\underline{1} \wedge \Diamond_{12P}\underline{i} \wedge \Diamond_{13P}\underline{j} & : \quad \alpha = x_i = x_j; \\ \Diamond_{11P}\underline{2} \wedge \Diamond_{12P}\underline{i} \wedge \Diamond_{13P}\underline{j} & : \quad \alpha = X_j(x_i); \\ \Diamond_{11P}\underline{3} \wedge \Diamond_{12P}\underline{3} \wedge \Diamond_{13P}g(\beta) & : \quad \alpha = \neg\beta; \\ \Diamond_{11P}\underline{4} \wedge \Diamond_{12P}g(\beta) \wedge \Diamond_{13P}g(\gamma) & : \quad \alpha = \beta \vee \gamma; \\ \Diamond_{11P}\underline{5} \wedge \Diamond_{12P}\underline{i} \wedge \Diamond_{13P}g(\beta) & : \quad \alpha = \exists x_i\beta; \\ \Diamond_{11P}\underline{6} \wedge \Diamond_{12P}\underline{i} \wedge \Diamond_{13P}g(\beta) & : \quad \alpha = \exists X_i\beta. \end{cases}$$

Now, we can define

$$f(\alpha) \leftrightharpoons \mathbf{A}(\underline{S} \rightarrow \Diamond_{15P}g(\alpha)).$$

The proof of Claim 1 implies the following crucial

Claim 3: For arbitrary α — a formula of second-order logic and $\mathfrak{M}$ — a model of second-order logic, $\mathfrak{M} \models \alpha$ iff $\mathsf{Modal}(\mathfrak{M}) \models f(\alpha)$.

Remark 9.9 *Two things should be noted here. First, f uses only variable free formulas. Second, $f(\alpha)$ is preceded by universal modality. Thus, there is no difference between* local satisfaction *of $f(\alpha)$ (*satisfaction at a point *or satisfaction in a principal filter) and* global satisfaction *(satisfaction in the whole algebra/frame). Moreover, if the value of $f(\alpha)$ in* $\mathsf{Modal}(\mathfrak{M})$ *is distinct from $\bot$ for some valuation (valuations are irrelevant for variable-free formulas), then it means that* $\mathsf{Modal}(\mathfrak{M}) \in \mathbb{V}(f(\alpha))$*. Thus, Thomason translation can be used to prove several seemingly different results: either an embedding of the lattice of second-order theories into the lattice of $\mathcal{SC}_\Lambda(\,)$ -closed sets (i.e., $\mathcal{SC}$-theories of Λ),* **even in the variable free language***, or into the lattice of* $\mathsf{Log}\mathcal{SC}(\,)$*-closed sets over* $\mathsf{Log}\mathcal{SC}(\Lambda)$ *(i.e., the lattice of $\mathcal{SC}$ logics over Λ). Thomason chose the second option; we choose the first. The interesting question now is whether one can show that any of those embeddings is an isomorphism, as in Kracht and Wolter [39].*

Assume now $\alpha \notin \mathsf{SO}(\Gamma)$. It is equivalent to the existence of $\mathfrak{M}$ s.t. $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \not\models \alpha$. By Claim 3, it is equivalent to $\mathsf{Modal}(\mathfrak{M}) \models f[\Gamma]$ and $\mathsf{Modal}(\mathfrak{M}) \not\models f(\alpha)$. Thus, $f(\alpha) \notin \mathcal{SC}_\Lambda(f[\Gamma])$. Conversely, assume $f(\alpha) \notin \mathcal{SC}_\Lambda(f[\Gamma])$. It means there $\mathfrak{A} \in \mathbb{V}(\Lambda) \cap \mathcal{SC}$ s.t. $\mathfrak{A} \models f[\Gamma]$ (as was noted above in 9.9, valuations are irrelevant here) and $\mathfrak{A} \not\models f(\alpha)$. By Claim 1, $\mathfrak{A}$ is isomorphic to the dual of some $\mathsf{Modal}(\mathfrak{M})$ and by Claim 3, $M \models \Gamma$ and $M \not\models \alpha$. The theorem is proven.

$\dashv$

Bibliography

[1] N. Bezhanishvili and I. Hodkinson. All normal extensions of **S5**-squared are finitely axiomatizable. *Studia Logica*, 78, to appear.

[2] N. Bezhanishvili and M. Marx. All proper normal extensions of **S5**-square have the polynomial size model property. *Studia Logica*, 73:367–382, 2003.

[3] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2001.

[4] W. Blok. On the degree of incompleteness of modal logic and the covering relation in the lattice of modal logics. Technical Report 78-07, University of Amsterdam, 1978.

[5] W. Blok. The lattice of modal logics: an algebraic investigation. *The Journal of Symbolic Logic*, 45:221–236, 1980.

[6] W.J. Blok and C.J. van Alten. The finite embeddability property for residuated lattices, pocrims, and BCK-algebras. *Algebra Universalis*, 48:253–271, 2002.

[7] R. A. Bull. That all normal extensions of **S4.3** have the finite model property. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 12:314–344, 1966.

[8] S. Burris and H.P. Sankappanavar. *A Course in Universal Algebra.* Graduate texts in mathematics. Springer-Verlag, 1981.

[9] W. Buszkowski. Embedding boolean structures into atomic boolean algebras. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 32:227–8, 1986.

[10] A.V. Chagrov. To the question of reverse mathematics of modal logic. In *Logical Investigations*, pages 224–243. Moscow, 2001. In Russian.

[11] A.V. Chagrov and M.V. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides*. Clarendon Press, Oxford, 1987.

[12] L. Chagrova. On the degree of neighbourhood incompleteness of normal modal logics. In *Advances In Modal Logic I*, pages 63–72. California, CSLI, 1998.

[13] C.C. Chang and H. J. Keisler. *Model Theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, North-Holland, third edition, 1990.

[14] K. Devlin. *Joy of Sets*. Undergraduate texts in mathematics. Springer-Verlag, second edition, 1993.

[15] K. Došen. Duality between modal algebras and neighbourhood frames. *Studia Logica*, 48:219–234, 1989.

[16] W. Dziobiak. A note on incompleteness of modal logics with respect to neighbourhood semantics. *Bulletin of the Section of Logic*, 9:185–190, 1978.

[17] K. Etessami, M. Vardi, and T.Wilke. First-order logic with two variables and unary temporal logic. *Information and Computation*, 179:279–295, 2002.

[18] K. Fine. An incomplete logic containing **S4**. *Theoria*, 40:23–29, 1974.

[19] K. Fine. Logics extending **K4**. *The Journal of Symbolic Logic*, 39:31–42, 1974.

[20] A. Fraenkel, Y. Bar-Hillel, and A. Levy. *Foundations of Set Theory*, volume 67 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, North-Holland, second revised edition, 1984.

[21] G. Gargov, S. Passy, and T. Tinchev. Modal environment for Boolean speculations. In D. Skordev, editor, *Mathematical Logic and its Applications. Proceedings of the Summer School and Conference dedicated to the 80th Anniversary of Kurt Gödel (Druzhba, 1986)*, pages 253–263. Plenum Press, 1987.

[22] George Gargov and Valentin Goranko. Modal logic with names. *Journal of Philosophical Logic*, 22:607–636, 1993.

[23] S.J. Garland. Second-order cardinal characterizability. In T. Jech, editor, *Axiomatic set theory*, volume 13 of *Proceedings of symposia in pure mathematics*. American Mathematical Society, Providence, Rhode Island, 1974.

[24] M.S. Gerson. An extension of **S4** complete for the neighbourhood semantics but incomplete for the relational semantics. *Studia Logica*, 34:333–342, 1975.

[25] M.S. Gerson. The inadequacy of the neighbourhood semantics for modal logics. *Journal of Symbolic Logic*, 40:141–148, 1975.

[26] R. I. Goldblatt. Mathematical modal logic: a view of its evolution. *Journal of Applied Logic*, 1, October 2003.

[27] R.I. Goldblatt. *Axiomatizing the logic of computer programming*, volume 130 of *Lecture Notes in Computer Science*. Springer-Verlag, 1982.

[28] R.I. Goldblatt. *Mathematics of modality*, volume 43 of *CSLI Lecture Notes*. CSLI Publications, 1993.

[29] V. Goranko and M. Otto. Model theory of modal logic. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*. To appear.

[30] P. Jipsen. Discriminator varieties of boolean algebras with operators. In *Algebraic Logic*, volume 28 of *Banach Center Publications*. 1993.

[31] B. Jónsson and A. Tarski. Boolean algebras with operators. Part I. *American Journal of Mathematics*, 73:891–939, 1952.

[32] B. Jónsson and A. Tarski. Boolean algebras with operators. Part II. *American Journal of Mathematics*, 74:127–162, 1952.

[33] S. Koppelberg. *Handbook of boolean algebras*, volume I. Elsevier, North-Holland, 1989.

[34] T. Kowalski. Varieties of tense algebras. *Reports on Mathematical Logic*, 32:53–95, 1998.

[35] T. Kowalski and M. Kracht. Semisimple varieties of modal algebras. *Studia Logica*, to appear. Special issue in memory of Wim Blok.

[36] M. Kracht. An almost general splitting theorem in modal logic. *Studia Logica*, 49:455–470, 1990.

[37] M. Kracht. *Tools and Techniques in Modal Logic*, volume 142 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 1999.

[38] M. Kracht and T. Kowalski. *Atomic incompleteness or how to kill one bird with two stones*. *Bulletin of the Section of Logic*, 30:71–78, 2001.

[39] M. Kracht and F. Wolter. Normal monomodal logics can simulate all others. *Journal of Symbolic Logic*, 64(1):99–138, 1999.

[40] K. Kunnen. *Set theory: an introduction to independence proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 1980.

[41] K. Kuratowski and A. Mostowski. *Set Theory : with An Introduction to Descriptive Set Theory*, volume 86 of *Studies in logic and the foundations of mathematics*. Elsevier, 1976.

[42] H. Ono and A. Nakamura. On the size of refutation Kripke models for some linear modal and tense logics. *Studia Logica*, 39:325–333, 1980.

[43] W. Rautenberg. Splitting lattices of logics. *Archiv für Mathematische Logik*, 20:155–159, 1980.

[44] S. Shapiro. *Foundations without foundationalism: A case for second-order logic*, volume 17 of *Oxford Logic Guides*. Clarendon Press, 1991.

[45] V. Shehtman. On strong neighbourhood completeness of modal and intermediate propositional logic. In J. Gerbrandy, M. Maarx, M. de Rijke, and Y. Venema, editors, *JFAK. Essays Dedicated to Johan van Benthem on the Occasion of his 50th Birthday*. Vossiuspers AUP, 1999. CD.

[46] R. Sikorski. *Boolean algebras*, volume 25 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1969.

[47] E. Spaan. *Complexity of Modal Logics*. PhD thesis, Department of Mathematics and Computer Science, University of Amsterdam, 1993.

[48] Y. Tanaka and H. Ono. Rasiowa-Sikorski Lemma and Kripke completeness of predicate and infinitary modal logics. In M. Zakharyaschev, K. Segerberg, M. de Rijke, and H. Wansing, editors, *Advances in Modal Logic*, volume 2 of *CSLI Lecture Notes*, pages 401–419. CSLI Publications, 2001.

[49] B. ten Cate. *Model theory for extended modal languages*. PhD thesis, ILLC, University of Amsterdam, 2005.

[50] S.K. Thomason. Semantic analysis of tense logics. *Journal of Symbolic Logic*, 37:150–158, 1972.

[51] S.K. Thomason. Categories of frames for modal logic. *Journal of Symbolic Logic*, 40:439–442, 1975.

[52] S.K. Thomason. Reduction of second-order logic to modal logic. *Zeitschrift für mathemathische Logik und Grundlagen der Mathematik*, 21:107–114, 1975.

[53] J.F.A.K. van Benthem. Syntactic aspects of modal incompleteness theorems. *Theoria*, 45:63–77, 1979.

[54] Y. Venema. Atom structures. In *Advances in Modal Logic*, volume I, pages 63–72. CSLI, California, 1998.

[55] Y. Venema. Atomless varieties. *Journal of Symbolic Logic*, 68:607–614, 2003.

[56] Y. Venema. Algebras and coalgebras. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*. To appear.

[57] F. Wolter. *Lattices of Modal Logic*. PhD thesis, Fachbereich Mathematik, Freien Universität Berlin, 1993.

[58] F. Wolter. Properties of tense logics. *Mathematical Logic Quarterly*, 42:481–500, 1996.

[59] F. Wolter. Tense logic without tense operators. *Mathematical Logic Quarterly*, 42:145–171, 1996.

[60] M. Zakharyaschev, F. Wolter, and A. Chagrov. Advanced modal logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 3, pages 86–266. Kluwer, 2nd edition, 2001.

Publications

Papers[1]

[I] *A continuum of incomplete intermediate logics*, **Reports on Mathematical Logic**, vol. 36, pp. 131–141, 2002

[II] *Some notes on the superinituitionistic logic of chequered subsets*, **Bulletin of the Section of Logic**, vol. 33, pp. 81–86, 2004

[III] *Modal incompleteness revisited*, **Studia Logica**, vol. 76, pp. 329-342, 2004

[IV] (with Frank Wolter) *All finitely axiomatizable tense logics of linear time flows are coNP-complete*, to appear in **Studia Logica**

[V] *On notions of completeness weaker than Kripke completeness*, to appear in **Advances in Modal Logic 5**, eds. H. Wansing et al., King's College Publications

[VI] *On The Blok Alternative*, in preparation

[VII] (with Balder ten Cate) *The importance of being discrete*, in preparation

Chosen refereed talks at international conferences, reverse chronological order:

[i] *On strong completeness*, Algebraic and Topological Methods in Non-Classical Logics II, Spain, Barcelona, June 15–18, 2005

[1] JAIST requires that author's own publication and talks are listed separately from other references.

[ii] *Blok incompleteness theorem: how to generalize it and why*, UNILOG 2005, Switzerland, Montreux, March 31–April 03, 2005

[iii] *On notions of completness weaker than Kripke completeness*, International Conference on Advances in Modal Logic, Manchester, UK, September 9–11, 2004.

[iv] *Subdirect irreducibility, complete* BAO*s and incompleteness*, International Conference on Algebraic and Topological Methods in Non-classical Logics, Georgia, Tbilisi 2003

[v] *Modal incompleteness revisited*, Logic Colloquium, ASL European Summer Meeting, Germany, Münster 2002

[vi] *A continuum of incomplete intermediate logics*, 4th International Tbilisi Symposium on Logic, Language and Computation, Georgia, Borjomi 2001