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Description	

# Construction of Common Unfolding of a Regular Tetrahedron and a Cube

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## Abstract

A procedure that produces a common unfolding of a regular tetrahedron and a cube is given. It is adaptable to the length of an edge. If we allow a small error of the length of an edge of the tetrahedron, the procedure certainly halts and generates a common unfolding of a cube and an almost regular tetramonohedron. If we wish to generate the common unfolding of them with accuracy, we conjecture that the procedure does not halt and we obtain the common unfolding in the limit as a set of infinitely many points. The procedure has a potential to design a fractal structure given in a continued fraction form.

## 1 Introduction

Recently, several polygons that can fold to two different polyhedra have been developed (Table 1). Such a polygon is called a *common unfolding* of the polyhedra<sup>1</sup>. Observing these impressive unfoldings, it is natural to ask that whether there is a common unfolding of two (or more) different Platonic solids. This question has arisen several times independently, and it is still open (see [3, Section 25.8.3]).

In this paper, we give a procedure that generates a common unfolding of a regular tetrahedron and a cube<sup>2</sup>. The procedure is adaptable to the length of an edge of the tetrahedron. More precisely, the procedure generates the common unfolding of a cube and an almost regular tetramonohedron. If we wish to generate the common unfolding of a cube and a regular tetrahedron, the procedure produces a set of an infinite number of points, and we obtain the common unfolding in the limit. In a sense, this procedure gives an affirmative answer to the open problem; there exists a common unfolding of two Platonic solids (if our conjecture based on experiments is true). When we admit some error, say  $\epsilon > 0$ , then the procedure always halts and it certainly generates a common unfolding of a cube and an almost regular tetramonohedron whose

edge lengths are within the interval  $[\ell - \epsilon, \ell + \epsilon]$ , where  $\ell = \sqrt{2\sqrt{3}}$  is the length of an edge of the regular tetrahedron of surface area 6. Experimentally, we obtain a common unfolding of a cube and an almost regular tetramonohedron with  $\epsilon < 2.89200 \times 10^{-1796}$ .

Although we obtain an unfolding close to the ideal one, the connectivity of the unfolding generated by the procedure is not guaranteed in general. From the experimental results, we conjecture some useful properties of the unfolding. Based on it, we can take arbitrarily small  $\epsilon > 0$  to obtain such an unfolding of a cube and an almost regular tetramonohedron.

The behavior of the procedure seems to rely on the continued fraction form of the length of an edge. On the other hand, it is well known that some real numbers have simple (but infinite) continued fraction forms. The conjecture would imply that our procedure is also useful to generate so called fractal curves based on these forms; see Figures 6 and 7 in Appendix.

## 2 Preliminaries

We first show some basic results about unfolding of a convex polyhedron.

**Lemma 1 ([3, Sec. 22.1.3])** *All vertices of a polyhedron  $X$  are on the edges of any unfolding of  $X$ .*

Let  $P$  be a polygon on the plane, and  $R$  be a set of three points (called *rotation centers*) on the boundary of  $P$ . Then  $P$  has a *tiling* called symmetry group  $p2$  if  $P$  fills the plane by the repetition of 2-fold rotations around the points in  $R$ . The filling should contain no gaps nor overlaps. The rotation defines an equivalence relation on the points in the plane. Two points  $p_1$  and  $p_2$  are mutually equivalent if  $p_1$  can be moved to  $p_2$  by the 2-fold rotations. More on the properties of  $p2$  tiling can be seen, e.g., in [5]. Based on the notion of  $p2$  tiling, any unfolding of a tetramonohedron<sup>3</sup> can be characterized as follows:

**Theorem 2 ([1, 2])**  *$P$  is an unfolding of a tetramonohedron if and only if (1)  $P$  has a  $p2$  tiling, (2) four of the rotation centers consist in the triangular lattice formed by the triangle faces of the tetramonohedron, (3) the four rotation centers are the lattice points, and (4) no two of the four rotation centers belong to the same equivalent class on the tiling.*

<sup>3</sup>A *tetramonohedron* is a tetrahedron that consists of four congruent triangles.

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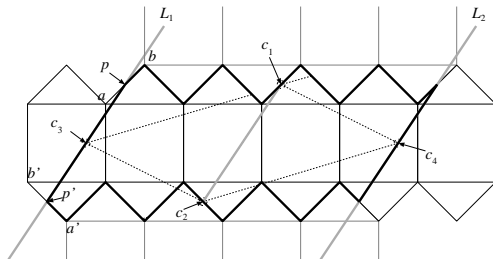
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<sup>1</sup>Note that an edge of an unfolding can pass through a flat face of the polyhedra. See an unfolding of a cube in Figure 1.

<sup>2</sup>In this paper, a *cube* always means a unit cube that is a box of size  $1 \times 1 \times 1$ .

A regular octahedron and a tetramonohedron	[3, Figure 25.50]
A regular tetrahedron and a box of size $1 \times 1 \times \sqrt{3} - 1/2 = 1.232$	[3, Figure 25.51]
A cube and a tetramonohedron of size $1 : \sqrt{34}/6 : \sqrt{34}/6 = 0.9718$	[6]
A regular octahedron and a tetramonohedron of size $1.0072 : 0.9965 : 0.9965$	[6]
A regular icosahedron and a tetramonohedron of size $1 : 1.145 : 1.25$	[4]

Table 1: Known common unfoldings of a regular polyhedron and another (nonregular) polyhedron.


 Figure 1: An initial unfolding  $P_1$  of a cube.

We start from an unfolding  $P_1$  of a cube in Figure 1. (The points  $c_1, c_2, p, p'$  are the centers of corresponding edges.) The thick lines in the figure gives an unfolding of a cube. Also,  $P_1$  satisfies the conditions of Theorem 2 with four points  $c_1, c_2, c_3, c_4$ ; that is,  $P_1$  is an unfolding of a tetramonohedron. We here note that  $c_1c_2$  is in parallel to the lines  $L_1$  and  $L_2$ , and the tiling is split by lines  $L_1$  and  $L_2$  into “parallel ribbons.” Thus we obtain infinitely many tilings by shifting these ribbons into any position along the lines. This fact implies that we can move the rotation centers  $c_3$  and  $c_4$  to *any* positions along  $L_1$  and  $L_2$ , respectively, as long as  $|c_1c_3| = |c_2c_4|$  (and hence  $|c_1c_4| = |c_2c_3|$ ) without changing the surface area of the tetramonohedron. That is,  $P_1$  is a common unfolding of a cube and an infinitely many tetramonohedra. (In the context of [3, Section 25],  $L_1$  and  $L_2$  make a *rolling belt* that is zipped to the edge  $c_3c_4$ .) Here we have  $|c_1c_2| = \sqrt{13}/2 = 1.80278$ , and each area of four congruent triangles is  $3/2$ . Thus taking  $|c_3c_1| = |c_3c_2| = |c_4c_1| = |c_4c_2|$  in Figure 1, we have the following lemma:

**Lemma 3** *There exists a common unfolding of a cube and a tetramonohedron with edge lengths  $\sqrt{13}/2 : \sqrt{745}/208 : \sqrt{745}/208 = 1.80278 : 1.89255 : 1.89255$ .*

The tetramonohedron in Lemma 3 is close to a regular tetrahedron. Our goal is to modify the edge lengths to the equal length  $\sqrt{2\sqrt{3}} = 1.86121$ .

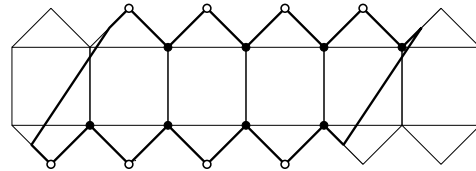


Figure 2: Fixed points on an unfolding.

### 3 Procedure for a common unfolding

Our procedure transforms  $P_1$ . More precisely, it moves  $c_1$  to the right and  $c_2$  to the left, respectively. Through these transformations, we keep two invariants for the polygon  $P_1$  that  $P_1$  is an unfolding of a cube, and  $P_1$  is an unfolding of a tetramonohedron with  $|c_1c_3| = |c_1c_4| = |c_2c_3| = |c_2c_4|$ . Hence, when  $|c_1c_2|$  becomes  $\sqrt{2\sqrt{3}}$ , we obtain a common unfolding of a cube and a regular tetrahedron. When we move  $c_1$  and  $c_2$ , a series of discrete processes occurs. It will not terminate if we try to change from  $|c_1c_2| = \sqrt{13}/2$  to  $|c_1c_2| = \sqrt{2\sqrt{3}}$  as shown later. Thus we have two choices; the common unfolding of a cube and the regular tetrahedron in the limit as the set of infinitely many points, or a common unfolding of a cube and an *almost regular tetramonohedron* for any given error  $\epsilon > 0$ . Here we define the *almost regular tetramonohedron* with error  $\epsilon > 0$  by a tetramonohedron with  $|c_1c_2| \in [\sqrt{2\sqrt{3}} - \epsilon, \sqrt{2\sqrt{3}} + \epsilon]$ .

Now we show how to stretch the distance between  $c_1$  and  $c_2$  and change  $|c_1c_2|$  from  $\sqrt{13}/2 = 1.80278$  to  $\sqrt{2\sqrt{3}} = 1.86121$ . Intuitively, we will slightly move the points  $c_1$  and  $c_2$  horizontally farther.

We here observe that, white circles in Figure 2 come to two “center points” in the top and bottom squares in the cube. If we remove these points, the squares have holes. On the other hand, if the unfolding contains these points inside, the squares have overlaps. Hence these points should be on the edge of  $P_1$ . Moreover, by Lemma 1, the vertices of the cube (black points in Figure 2) also should be on an edge of  $P_1$ . We call these immovable points *fixed points* of the unfolding.

Here we focus on the top edges of  $P_1$ . There are eight fixed points, and we have a rotation center  $c_1$

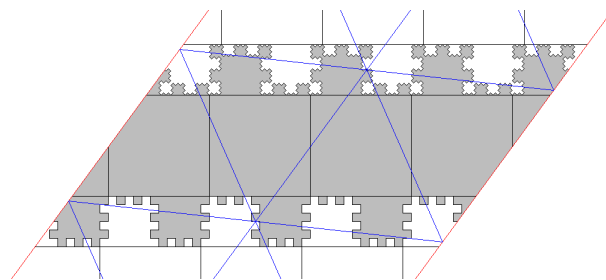


Figure 4: An example of an unfolding of a cube and a tetramonohedron.

<sup>5</sup>This “continued fraction form” is slightly different from the standard one. In the standard form,  $a_i \geq 1$  and all signs are “+.” In our continued fraction form, we use “−” and avoid the case  $a_i = 1$  for  $i > 1$ . The details are omitted in this draft.

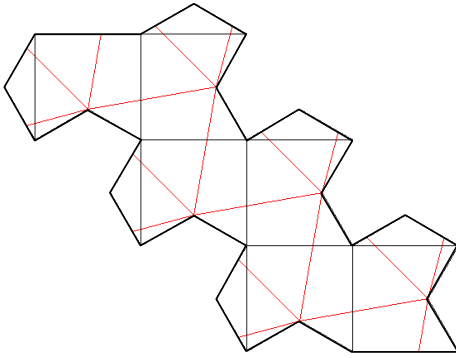


Figure 5: A common unfolding of a cube and a (non-regular) octahedron.

be closer  $\ell_1 + \ell_2 = \sqrt{2\sqrt{3} - 9/4} - 1$ :  $a_1 = 4, b_1 = 6, a_2 = 6, b_2 = -34, a_3 = -42, b_3 = -14, a_4 = -116, b_4 = -2146, a_5 = 4010, b_5 = -3316, a_6 = -4958, b_6 = 8684, a_7 = -7820, b_7 = 7082, a_8 = 2668, b_8 = -3684, a_9 = 4564, b_9 = 1662, a_{10} = 560, b_{10} = -158, \dots$ . We have obtained a connected unfolding for each  $i$ . When we use up to  $a_{10}$  and  $b_{10}$ , we obtain  $\epsilon < 4.63451 \times 10^{-56}$ , and the values up to  $a_{50}$  and  $b_{50}$  give us  $\epsilon < 2.89200 \times 10^{-1796}$ .

**Theorem 5** *There exists a common unfolding of a cube and an almost regular tetramonohedron with an error  $\epsilon < 2.89200 \times 10^{-1796}$ .*

We conjecture that this process can be repeated in arbitrary large, and we obtain a connected common unfolding of a cube and a regular tetrahedron in the limit:

**Conjecture 1** *There exists a series of points that converges to a connected common unfolding of a cube and a regular tetrahedron in the limit.*

## 4 Concluding Remarks

We give a procedure that generates a common unfolding of a cube and a regular tetrahedron. But this is not completed: the procedure does not halt, and it might produce a disconnected unfolding. The proof of the observation, or the characterization of the curve generated by the procedure is a future work. We also conjecture that a similar construction of a common unfolding of a regular octahedron and a tetramonohedron works.

It is a challenging problem to try to the other pair of Platonic solids rather than a regular tetrahedron; in the case, we cannot use tiling as a tool any more. However, recently, the first author finds a common unfolding of a cube and a (nonregular) octahedron (Figure 5; note that this octahedron consists of two regular

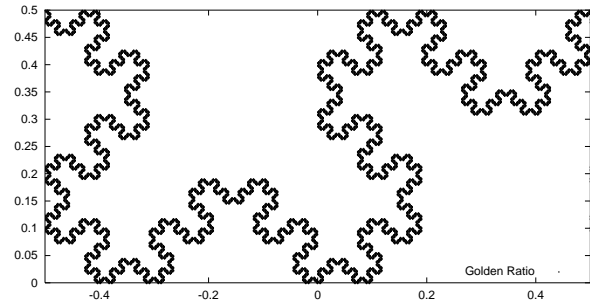


Figure 6: The set of the first 5000 points given by the golden ratio  $\phi = \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}$ .

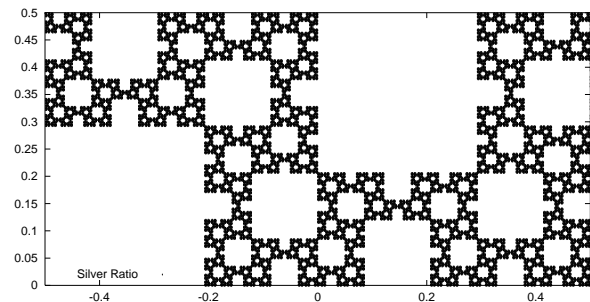


Figure 7: The set of the first 10000 points given by the silver ratio  $\phi = \sqrt{2} - 1 = \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}$ .

triangles of edge length  $\sqrt{2}$ , and six triangles of edge lengths  $\sqrt{2}, \sqrt{24 - 6\sqrt{3}/3}, \sqrt{24 - 6\sqrt{3}/3}$ ). Hence it is not so easy to give some negative results, and we might have a common unfolding of the other solids.

## Appendix

Based on the observation, we can design and generate some fractal pattern for a certain real number  $\phi$ . Some experimental results are given in Figures 6 and 7.

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